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Generalization of the addition and restriction theorems from free arrangements to the class of projective dimension one

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Abstract We study a generalized version of Terao's addition theorem for free arrangements to the category of those with projective dimension one. Namely, we give a formulation to determine the algebraic structure of the logarithmic derivation module of a hyperplane arrangement obtained by adding one hyperplane to a free arrangement under the assumption that the arrangement obtained by restricting onto that hyperplane is free too.

Also, we introduce a class of stair-strictly plus-one generated (SPOG) arrangements whose SPOGness depends only on the intersection lattice similar to the class of stair-free arrangements which satisfies Terao's conjecture.

1. Introduction

Let $\mathbb{K}$ be a field, $V = \mathbb{K}^\ell$, and $S = \mathrm{Sym}^*(V^*) = \mathbb{K}[x_1, \dots, x_\ell]$ be the coordinate ring of V. For the derivation module $\mathrm{Der}\, S := \oplus_{i=1}^{\ell} S\partial_{x_i}$ and a hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$, where H_i is defined as the zero locus of a non-zero linear form $\alpha_{H_i} \in V^*$, the **logarithmic derivation module** $D(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}\, S \mid \theta(\alpha_H) \in S\alpha_H \ (\forall H \in \mathcal{A})\}.$$

The module $D(\mathcal{A})$ is an S-graded reflexive module of rank ℓ, but not free in general. So we say that $\mathcal{A}$ is **free** with **exponents** $\exp(\mathcal{A}) = (d_1, \dots, d_\ell)$ if $D(\mathcal{A}) \simeq \oplus_{i=1}^{\ell} S[-d_i]$. In this article $\exp(\mathcal{A})$ is a **multiset**. Also in this article, we assume that all arrangements are **essential** unless otherwise specified, i.e., $\cap_{H\in\mathcal{A}} H = \{0\}$. If $\mathcal{A} \neq \varnothing$, then the submodule generated by the Euler derivation $\theta_E \in D(\mathcal{A})$ forms a direct summand of $D(\mathcal{A})$. Explicitly, $D(\mathcal{A}) = S\theta_E \oplus D_H(\mathcal{A})$, where $D_H(\mathcal{A}) := \{\theta \in D(\mathcal{A}) \mid \theta(\alpha_H) = 0\}$. So we may assume that $d_1 = 1 = \deg \theta_E \leqslant d_i$ for $i \geqslant 2$ when an essential arrangement $\mathcal{A}$ is free.

Free arrangements have been a central topic in the research of hyperplane arrangements. Among them, the most important problem is so called **Terao's conjecture** asking whether the freeness of $\mathcal{A}$ depends only on the intersection lattice

$$L(\mathcal{A}) := \{\cap_{H\in\mathcal{B}} H \mid \mathcal{B} \subset \mathcal{A}\}.$$

In other words, Terao's conjecture asks whether the freeness is combinatorial. This is completely open, but it was shown in [15] that the minimal free resolution of $D(\mathcal{A})$ is

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not combinatorial. To approach Terao's conjecture, one of the main tools is Terao's addition-deletion theorem. For the purpose of this article, we exhibit it in a slightly different way compared to its usual formulation:

Theorem 1.1 (Addition and restriction theorems, [11]). *Let $H \in \mathcal{A}$, $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ and let $\mathcal{A}^H := \{H \cap L \mid L \in \mathcal{A}'\}$. Assume that $\mathcal{A}'$ is free with $\exp(\mathcal{A}') = (1, d_2, \ldots, d_\ell)$. Then the following two conditions are equivalent:*

(1) *$\mathcal{A}$ is free.*
(2) *$\mathcal{A}^H$ is free and $|\exp(\mathcal{A}') \cap \exp(\mathcal{A}^H)| = \ell - 1$.*

If one of these two conditions holds, then for $d_i := \exp(\mathcal{A}') \setminus \exp(\mathcal{A}^H)$, it holds that

$$\begin{aligned} \exp(\mathcal{A}) &= (1, d_2, \ldots, d_{i-1}, d_i + 1, d_{i+1}, \ldots, d_\ell), \\ \exp(\mathcal{A}^H) &= (1, d_2, \ldots, d_{i-1}, d_{i+1}, \ldots, d_\ell). \end{aligned}$$

In general the condition $|\exp(\mathcal{A}') \cap \exp(\mathcal{A}^H)| = \ell - 1$ above is described as $\exp(\mathcal{A}') \supset \exp(\mathcal{A}^H)$. However, note that $|\exp(\mathcal{A}') \cap \exp(\mathcal{A}^H)| < \ell - 1$ often occurs even when both $\mathcal{A}'$ and $\mathcal{A}^H$ are free. So we have the first question in this article:

Problem 1.2. *Assume that $\mathcal{A}'$ is free. Then which condition of $D(\mathcal{A})$ makes $\mathcal{A}^H$ free? More precisely, are there any explicit condition for $D(\mathcal{A})$ to make $\mathcal{A}^H$ free when $\mathcal{A}'$ is free in terms of freeness, projective dimension, free resolution and so on?*

Also, recent developments show the following projective dimensional version of the addition theorem. Note that $\mathcal{A}$ is free if and only if $\operatorname{pd} \mathcal{A} = 0$, where $\operatorname{pd} \mathcal{A}$ denotes the projective dimension of the S-module $D(\mathcal{A})$.

Theorem 1.3 ([3, Theorem 1.11]). *(1) Assume that $\operatorname{pd} \mathcal{A}' = \operatorname{pd} \mathcal{A}^H = 0$. Then $\operatorname{pd} \mathcal{A} \leqslant 1$.*

(2) Assume that $\operatorname{pd} \mathcal{A}' = 0$ and $\operatorname{pd} \mathcal{A} \leqslant 1$. Then $\operatorname{pd} \mathcal{A}^H = 0$.

Now we have the second question in this article which is related to Problem 1.2:

Problem 1.4. *Can we describe the algebraic structure of $D(\mathcal{A})$ when $\mathcal{A}'$ and $\mathcal{A}^H$ are both free, but $|\exp(\mathcal{A}') \cap \exp(\mathcal{A}^H)| < \ell - 1$?*

Explicitly, we want to know the minimal free resolution of $D(\mathcal{A})$ under the above conditions. Contrary to these problems, when $\mathcal{A}$ is free, we can describe $D(\mathcal{A}')$, which was proved in [4]. To see this result, let us recall the definition of strictly plus-one generated (SPOG).

Definition 1.5 ([4]). *We say that $\mathcal{A}$ is **strictly plus-one generated (SPOG)** with $\operatorname{POexp}(\mathcal{A}) = (1, d_2, \ldots, d_\ell)$ and **level** d if there is a minimal free resolution of the following form:*

$$0 \to S[-d-1] \stackrel{(f_1, \ldots, f_\ell, \alpha)}{\to} \oplus_{i=1}^{\ell} S[-d_i] \oplus S[-d] \to D(\mathcal{A}) \to 0.$$

*Here $d_1 = 1, f_i \in S$ and $0 \neq \alpha \in V^*$. For the set of generators $\theta_E, \theta_2, \ldots, \theta_\ell, \theta$ with $\deg \theta_i = d_i$ and $\deg \theta = d$ for the SPOG module $D(\mathcal{A})$, $\theta_E, \theta_2, \ldots, \theta_\ell$ is called **the set of SPOG generators** and θ the **level element**.*

It was proved in [4] (see Theorem 2.4) that $\mathcal{A}'$ is SPOG if $\mathcal{A}$ is free and $\mathcal{A}'$ is not free. Interestingly, in this case the structure of $D(\mathcal{A}')$ is independent of that of $D(\mathcal{A}^H)$. However, in general $\mathcal{A}$ is neither free nor SPOG even if $\mathcal{A}'$ is free. The typical example is the case when $\mathcal{A}' : \prod_{i=1}^4 x_i = 0$ in $V = \mathbb{R}^4$. Then $\mathcal{A}$ is free with exponents $(1, 1, 1, 1)$. If you add $H : x_1 + x_2 + x_3 + x_4 = 0$ to $\mathcal{A}'$ to get $\mathcal{A}$, then it is well-known that $\mathcal{A}$ is neither free nor SPOG. In fact $\operatorname{pd}_S D(\mathcal{A}) = 2$ in this case.

When $\operatorname{pd}\mathcal{A}' = \operatorname{pd}\mathcal{A}^H = 0$, then $\operatorname{pd}\mathcal{A} \leqslant 1$ by Theorem 1.3. Also Theorem 1.1 shows that one additional condition for exponents confirms that $\operatorname{pd}\mathcal{A} = 0$. So a weaker condition for exponents when $\operatorname{pd}\mathcal{A}' = \operatorname{pd}\mathcal{A}^H = 0$ could determine the minimal free resolution of $D(\mathcal{A})$. Namely, we can show the following, which answers Problems 1.2 and 1.4 partially:

Theorem 1.6. *Let $\mathcal{A}'$ be free with $\exp(\mathcal{A}') = (1, d_2, \ldots, d_\ell)_\leqslant$. Here for the set of integers $(a_1, \ldots, a_s)$, the notation $(a_1, \ldots, a_s)_\leqslant$ means that $a_1 \leqslant \ldots \leqslant a_s$. Let $d_j < d := d_i + d_j + |\mathcal{A}^H| - |\mathcal{A}'| \leqslant d_{j+1}$ for some $i < j$. Then the following two conditions are equivalent:*

(1) *$\mathcal{A}^H$ is free with $\exp(\mathcal{A}^H) = (1, d_2, \ldots, \hat{d}_i, \ldots, \hat{d}_j, \ldots, d_\ell) \cup (d)$*
(2) *$\mathcal{A}$ is SPOG with $\operatorname{POexp}(\mathcal{A}) = (1, d_2, \ldots, d_i + 1, \ldots, d_j + 1, \ldots, d_\ell)$ and level d.*

Theorem 1.6 can be regarded as an extension of the addition and restriction theorems (Theorem 1.1). Namely, Theorem 1.6 determines a minimal free resolution of $D(\mathcal{A})$ as an SPOG-module when $|\exp(\mathcal{A}') \cap \exp(\mathcal{A}^H)| = \ell - 2$. The condition $d \leqslant d_{j+1}$ is necessary, see Example 4.8 for details.

Now go back to Terao's conjecture. As we have seen, SPOG arrangements can be regarded as a close analogue of free arrangements. Thus to study Terao's conjecture by using an inductive approach, it is important to study combinatorial dependency of SPOG arrangements. For that purpose, let us introduce the following class of arrangements.

Definition 1.7. *We say that $\mathcal{A}$ is **stair-SPOG** if there is $H \in \mathcal{A}$ such that both $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ and $\mathcal{A}^H$ are stair-free (see Definition 2.6 and Theorem 2.7), and $\exp(\mathcal{A}')$, $\exp(\mathcal{A}^H)$ and $|\mathcal{A}'| - |\mathcal{A}^H|$ satisfy the conditions in Theorem 1.6. Let $\mathcal{S}_\ell$ denote the set of stair-SPOG arrangements in an ℓ-dimensional vector space and let*

$$\mathcal{S} := \bigcup_{\ell \geqslant 2} \mathcal{S}_\ell.$$

Theorem 1.8. *$\mathcal{A}$ is SPOG if $\mathcal{A} \in \mathcal{S}$. Moreover, if there are $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} \in \mathcal{S}$ and $L(\mathcal{A}) \simeq L(\mathcal{B})$, then $\mathcal{B}$ is SPOG too.*

The organization of this article is as follows. In §2 we introduce several results and definitions for the proof of the main results in this article. In §3 we prove some useful results on the cardinality of the set of minimal generators. In §4 we prove the main results of this article. Several examples are also exhibited in §4. §5 is devoted to investigate the relation between Ziegler restriction of an arrangement and their SPOGness by using the methods introduced in the previous sections.

2. Preliminaries

In this section let us introduce several definitions and results for the proof of the main results in this article. First recall some combinatorics of arrangements. For the intersection lattice $L(\mathcal{A})$, we can define the **Möbius function** $\mu : L(\mathcal{A}) \to \mathbb{Z}$ by $\mu(V) = 1$ and by

$$\mu(X) := - \sum_{X \subsetneq Y \subset V,\ Y \in L(\mathcal{A})} \mu(Y)$$

for $X \in L(\mathcal{A}) \setminus \{V\}$. The generating function of μ is called the **characteristic polynomial** of $\mathcal{A}$ defined by

$$\chi(\mathcal{A}; t) := \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X},$$

which is a combinatorial invariant. The absolute value of the coefficient of $t^{\ell-i}$ in $\chi(\mathcal{A};t)$ is called the **i-th Betti number** of $\mathcal{A}$ and denoted by $b_i(\mathcal{A})$. Next let us recall several useful results on $D(\mathcal{A})$.

Theorem 2.1 (Terao's polynomial B, [11]). *Let $\mathcal{C} \setminus \{H\} = \mathcal{C}'$. Then there is a homogeneous polynomial $B \in S$ of degree $|\mathcal{C}| - 1 - |\mathcal{C}^H|$ such that*

$$D(\mathcal{C}')(\alpha_H) := \{\theta(\alpha_H) \mid \theta \in D(\mathcal{C}')\} \subset (\alpha_H, B).$$

*We call such B a **polynomial** B of $(\mathcal{C}, H)$.*

Theorem 2.2 (Terao's factorization theorem, [12]). *Assume that $\mathcal{A}$ is free with* $\exp(\mathcal{A}) = (d_1, d_2, \ldots, d_\ell)$*, then*

$$\chi(\mathcal{A};t) = \prod_{i=1}^{\ell}(t - d_i).$$

For $H \in \mathcal{C}$ and $\mathcal{C}' := \mathcal{C} \setminus \{H\}$ we have the following **Euler exact sequence**

$$0 \to D(\mathcal{C}') \stackrel{\cdot\alpha_H}{\to} D(\mathcal{C}) \stackrel{\rho}{\to} D(\mathcal{C}^H). \tag{2.1}$$

Here for $\theta \in D(\mathcal{A})$ and the image $\overline{f} \in S/\alpha_H S$ of a polynomial $f \in S$ by the canonical surjection $S \to S/\alpha_H S$, $\rho(\theta)$ is defined by

$$\rho(\theta)(\overline{f}) := \overline{\theta(f)}.$$

The Euler exact sequence is not right exact in general, but it is so when $\mathcal{C}'$ is free as follows.

Theorem 2.3 (Free surjection theorem (FST), Theorem 1.13, [3]). *Let $\mathcal{C} = \mathcal{C}' \cup \{H\}$ and assume that $\mathcal{C}'$ is free. Then $\rho = \rho^H : D(\mathcal{C}) \to D(\mathcal{C}^H)$ is surjective.*

Next let us introduce the results on freeness and SPOGness.

Theorem 2.4 (Theorem 1.4, [4]). *Let $\mathcal{A}$ be free with* $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$*, $H \in \mathcal{A}$ and assume that $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ is not free. Then $\mathcal{A}'$ is SPOG with* $\mathrm{POexp}(\mathcal{A}') = (d_1, \ldots, d_\ell)$ *and level $d := |\mathcal{A}'| - |\mathcal{A}^H|$.*

Theorem 2.5 (Division theorem, Theorem 1.1, [1]). *Assume that $\mathcal{A}^H$ is free and $\chi(\mathcal{A}^H;t) \mid \chi(\mathcal{A};t)$. Then $\mathcal{A}$ is free. Thus if we can show the freeness of $\mathcal{A}$ by using the division theorem several times, then the freeness of $\mathcal{A}$ is combinatorial, and such a free arrangement is called a **divisionally free arrangement**.*

Definition 2.6 (Definition 4.2, [5]). *We say that $\mathcal{A}$ is **stair-free** if the freeness of $\mathcal{A}$ can be proved by using the addition and division theorems.*

Theorem 2.7 (Theorem 4.3, [5]). *If $\mathcal{A}$ is stair-free, then its freeness depends only on $L(\mathcal{A})$.*

Finally let us recall the fundamentals of the multiarrangement theory introduced by Ziegler in [16]. A pair $(\mathcal{A}, m)$ is a **multiarrangement** if $\mathcal{A}$ is an arrangement and $m : \mathcal{A} \to \mathbb{Z}_{>0}$. Multiarrangements were defined by Ziegler in [16] and used in several research of arrangements. We can define their logarithmic derivation module $D(\mathcal{A}, m)$ as follows:

$$D(\mathcal{A}, m) := \{\theta \in \operatorname{Der} S \mid \theta(\alpha_H) \in S\alpha_H^{m(H)}\ (\forall H \in \mathcal{A})\}.$$

Then their freeness and exponents can be defined in the same manner as for $D(\mathcal{A})$. For details, see [16]. We have a canonical way to construct a multiarrangement from $\mathcal{A}$. Let $H \in \mathcal{A}$. Then the **Ziegler multiplicity** $m^H : \mathcal{A}^H \to \mathbb{Z}$ is defined by

$$m^H(X) := |\{L \in \mathcal{A} \setminus \{H\} \mid H \cap L = X\}|.$$

The pair $(\mathcal{A}^H, m^H)$ is called the **Ziegler restriction** of $\mathcal{A}$ onto H. Also recall that for $H \in \mathcal{A}$, the submodule $D_H(\mathcal{A})$ of $D(\mathcal{A})$ is defined by

$$D_H(\mathcal{A}) := \{\theta \in D(\mathcal{A}) \mid \theta(\alpha_H) = 0\}.$$

Since $D(\mathcal{A}) = S\theta_E \oplus D_H(\mathcal{A})$ by the splitting exact sequence

$$0 \to S\theta_E \to D(\mathcal{A}) \to D_H(\mathcal{A}) \to 0$$

with the map $D(\mathcal{A}) \ni \theta \mapsto \theta - \frac{\theta(\alpha_H)}{\alpha_H}\theta_E \in D_H(\mathcal{A})$ and the canonical inclusion as a section, we know that $\mathcal{A}$ is free if and only if $D_H(\mathcal{A})$ is free. Then $D_H(\mathcal{A})$ is closely related to $D(\mathcal{A}^H, m^H)$ as we can see in the following several results:

Theorem 2.8 ([16]). *There is an exact sequence*

$$0 \to D_H(\mathcal{A}) \stackrel{\cdot\alpha_H}{\to} D_H(\mathcal{A}) \stackrel{\pi}{\to} D(\mathcal{A}^H, m^H). \tag{2.2}$$

*Here $\pi := \rho|_{D_H(\mathcal{A})}$. This is called the **Ziegler exact sequence**. Moreover, if $\mathcal{A}$ is free with exponents $(1, d_2, \ldots, d_\ell)$, then $(\mathcal{A}^H, m^H)$ is also free with exponents $(d_2, \ldots, d_\ell)$.*

Theorem 2.9 (Theorem 5.1, [7]). *Let $\pi : D_H(\mathcal{A}) \to D(\mathcal{A}^H, m^H)$ be the Ziegler restriction of $\mathcal{A}$ onto $(\mathcal{A}^H, m^H)$. Then the preimages of a set of generators for $Im(\pi)$ by π generate $D_H(\mathcal{A})$.*

Theorem 2.10 (Yoshinaga's criterion, Theorem 2.2, [13]). *$\mathcal{A}$ is free if and only if $\mathcal{A}$ is locally free along H (i.e., $\mathcal{A}_X$ is free for all $0 \neq X \in L(\mathcal{A}^H)$), and $(\mathcal{A}^H, m^H)$ is free.*

For a multiarrangement we can introduce the concept of SPOG multiarrangements as follows.

Definition 2.11 ([6]). *We say that $(\mathcal{A}, m)$ is **SPOG** with* $\mathrm{POexp}(\mathcal{A}, m) = (d_1, d_2, \ldots, d_\ell)$ *and **level** d if there is a minimal free resolution of the following form:*

$$0 \to S[-d-1] \stackrel{(f_1,\ldots,f_\ell,\alpha)}{\to} \oplus_{i=1}^{\ell} S[-d_i] \oplus S[-d] \to D(\mathcal{A}, m) \to 0$$

with $0 \neq \alpha \in V^*$.

3. Cardinality of minimal sets of generators

In this section we show some new results on the cardinality of a minimal set of generators for $D(\mathcal{A})$, which will play a key role to prove our main theorem.

Definition 3.1. *For an arrangement $\mathcal{A}$, let $g(\mathcal{A})$ denote the cardinality of a minimal set of generators for $D(\mathcal{A})$. Clearly it is independent of the choice of the set of minimal generators.*

Moreover for $\mathcal{A} = \mathcal{A}' \cup \{H\}$ we define an integer for a free arrangement $\mathcal{A}'$ that measures how far $D(\mathcal{A}')$ is from being tangent to H.

Definition 3.2. *Let $\mathcal{A} = \mathcal{A}' \cup \{H\}$ and assume that $\mathcal{A}'$ is free. Let $FB(\mathcal{A}')$ be the set of all homogeneous basis for $D(\mathcal{A}')$ and for each $\boldsymbol{B} := \{\theta_1, \ldots, \theta_\ell\} \in FB(\mathcal{A}')$ define*

$$NT(\boldsymbol{B}) := |\{i \mid 1 \leqslant i \leqslant \ell,\ \theta_i \notin D(\mathcal{A})\}|,$$

and define

$$SNT(\mathcal{A}') := \min\{NT(\boldsymbol{B}) \mid \boldsymbol{B} \in FB(\mathcal{A}')\}.$$

First we record the following easy facts.

Lemma 3.3. *Assume that $\mathcal{A}'$ and $\mathcal{A}^H$ are both free. Then $g(\mathcal{A}) \leqslant 2\ell - 2$.*

Proof. By Theorem 2.3, we can choose $\theta_E, \theta_2, \dots, \theta_{\ell-1} \in D(\mathcal{A})$ as preimages of the basis for $D(\mathcal{A}^H)$ by ρ. Let $\theta_E, \varphi_2, \dots, \varphi_\ell$ be a basis for $D(\mathcal{A}')$. Then the Euler exact sequence (2.1) shows that

$$\theta_E, \theta_2, \dots, \theta_{\ell-1}, \alpha_H\varphi_2, \dots, \alpha_H\varphi_\ell$$

generate $D(\mathcal{A})$, hence $g(\mathcal{A}) \leqslant \ell - 1 + \ell - 1 = 2\ell - 2$. □

On $g(\mathcal{A})$ the following proposition is fundamental.

Proposition 3.4. *Let $\mathcal{A}'$ be free with $SNT(\mathcal{A}') = s$. Let $\theta_E, \theta_2, \dots, \theta_\ell$ form a basis for $D(\mathcal{A}')$ such that $\theta_i \notin D(\mathcal{A})$ $(2 \leqslant i \leqslant s+1)$ and $\theta_i \in D(\mathcal{A})$ $(i \geqslant s+2)$. Then $g(\mathcal{A}) \geqslant \ell + s - 1$.*

Proof. Let $\alpha_H = x_1$. By the assumption on $\theta_E, \theta_2, \dots, \theta_\ell$, it is clear that we can choose derivations $\varphi_j \in D(\mathcal{A})$ $(j = 1, \dots, k)$ of the form $\sum_{i=2}^{s+1} f_i^j \theta_i$ such that $f_i^j \in S' := \mathbb{K}[x_2, \dots, x_\ell]$ are of positive degrees, and

$$\theta_E, \alpha_H\theta_2, \dots, \alpha_H\theta_{s+1}, \theta_{s+2}, \dots, \theta_\ell$$

together with the derivations φ_j $(j = 1, \dots, k)$ form a minimal set of generators for $D(\mathcal{A})$. Since their images by ρ generate the rank $(\ell-1)$-module $D(\mathcal{A}^H)$ due to Theorem 2.3, we can compute

$$|\{\theta_E, \theta_{s+2}, \dots, \theta_\ell, \varphi_1, \dots, \varphi_k\}| = k + 1 + \ell - s - 1 = \ell + k - s \geqslant \ell - 1.$$

So $g(\mathcal{A}) = \ell + k \geqslant \ell + s - 1$. □

Next let us show a key result to prove Theorem 1.6.

Proposition 3.5. *Let $1 \leqslant i < j \leqslant \ell$, and let $\mathcal{A}'$ be free with $SNT(\mathcal{A}') = 2$. Let $\theta_E, \theta_2, \dots, \theta_\ell$ be a basis for $D(\mathcal{A}')$ such that $\theta_k \in D(\mathcal{A})$ $(k \neq i, j)$. Let $\theta_i(\alpha_H) = f_i\alpha_H + g_iB$, $\theta_j(\alpha_H) = f_j\alpha_H + g_jB$ by Theorem 2.1. Then $(g_i, g_j) = 1$ and $D(\mathcal{A})$ is generated by $\{\theta_k\}_{k \neq i,j} \cup \{\alpha_H\theta_i, \alpha_H\theta_j, g_j\theta_i - g_i\theta_j\}$. In particular, $\mathcal{A}$ is SPOG.*

Proof. The proof is essentially the same as that of [4, Theorem 1.9] (see [10, Proposition 3.6] too). For the completeness, let us give a sketch of the proof. We may assume that $\alpha_H = x_1$ and let $(g_i, g_j) = g \in S$. Let $g_i = gh_i$, $g_j = gh_j$ with $(h_i, h_j) = 1$. Then clearly $x_1\theta_i, x_1\theta_j, \varphi := h_j\theta_i - h_i\theta_j \in D(\mathcal{A})$. By definition, we can choose $g_i, g_j \in \mathbb{K}[x_2, \dots, x_\ell] =: S'$ so we may assume that $h_i, h_j, g \in S'$. First let us prove that $\{\theta_k\}_{k \neq i,j} \cup \{x_1\theta_i, x_1\theta_j, \varphi\}$ generate $D(\mathcal{A})$. Let $\theta \in D(\mathcal{A})$. Since $D(\mathcal{A}) \subset D(\mathcal{A}')$, there are $a_k \in S$ such that

$$\theta = \sum_{k=1}^{\ell} a_k\theta_k.$$

Since $\{\theta_k\}_{k \neq i,j} \subset D(\mathcal{A})$, it suffices to show that $\theta - \sum_{k \neq i,j} a_k\theta_k = a_i\theta_i + a_j\theta_j$ is expressed as a linear combination of $x_1\theta_i, x_1\theta_j, \varphi$. Let us replace $\theta - \sum_{k \neq i,j} a_k$ by θ. Then we can express

$$\theta = b_i(x_1\theta_i) + b_j(x_1\theta_j) + c_i\theta_i + c_j\theta_j$$

for some $b_i, b_j \in S$, $c_i, c_j \in S'$. Thus

$$\theta(x_1) = b_ix_1\theta_i(x_1) + b_jx_1\theta_j(x_1) + c_i\theta_i(x_1) + c_j\theta_j(x_1).$$

Taking the modulo $x_1 = \alpha_H$ combined with Theorem 2.1 (see the proof of [4, Theorem 1.9] for details), we know that

$$c_i\theta_i + c_j\theta_j = c(h_j\theta_i - h_i\theta_j) + c'x_1$$

for some $c, c' \in S$. As a conclusion, it holds that $\theta \in S\theta_i + S\theta_j + S\varphi$, and

$$D(\mathcal{A}') = \langle \{\theta_k\}_{k \neq i,j} \cup \{x_1\theta_i, x_1\theta_j, \varphi\} \rangle_S.$$

By comparing the second Betti number of $\mathcal{A}$ calculated combinatorially and algebro-geometrically (see the proof of [4, Theorem 1.9] for details), we can see that h is a unit. Thus in fact

$$D(\mathcal{A}') = \langle \{\theta_k\}_{k \neq i,j} \cup \{x_1\theta_i, x_1\theta_j, g_j\theta_i - g_i\theta_j\} \rangle_S.$$

A minimal free resolution of $D(\mathcal{A})$ is easily obtained by the form of this minimal set of generators, which completes the proof. □

An immediate corollary of Proposition 3.4 is as follows:

Corollary 3.6. *Let $\mathcal{A}'$ be free and assume that $g(\mathcal{A}) = \ell + 1$. Then $\mathcal{A}$ is SPOG.*

Proof. By Proposition 3.4, $g(\mathcal{A}) = \ell + 1$ only when $s \leqslant 2$. $s = 0$ cannot occur, and $s = 1$ implies that $\mathcal{A}$ is free by the addition theorem, thus $g(\mathcal{A}) = \ell$. So the rest case is when $s = 2$. In this case, Proposition 3.5 shows that $\mathcal{A}$ is SPOG. □

Since the freeness of $\mathcal{A}^H$ and $\mathcal{A}'$ implies that $\operatorname{pd} \mathcal{A} \leqslant 1$ by Theorem 1.3, it is natural to study which condition on $g(\mathcal{A})$ makes the arrangement $\mathcal{A}^H$ free.

Theorem 3.7. *Let $\ell \geqslant 3$ and $\mathcal{A}'$ be free. Then $\mathcal{A}^H$ is free if $g(\mathcal{A}) \leqslant \ell + 2$.*

Proof. By the addition and restriction theorems, Theorem 2.3, Corollary 3.6 and the explicit form of the set of SPOG generators as in Proposition 3.5, the statement follows if $g(\mathcal{A}) \leqslant \ell + 1$. Assume that $g(\mathcal{A}) = \ell + 2$. Then by Proposition 3.4, we have a basis $\theta_E, \theta_2, \ldots, \theta_\ell$ for $D(\mathcal{A}')$ such that $\theta_i \notin D(\mathcal{A})$ for $i \geqslant \ell - 2$ and $\theta_i \in D(\mathcal{A})$ for $i \leqslant \ell - 3$. We may assume that $\alpha_H = x_1$. Then clearly $D(\mathcal{A})$ has a minimal set of generators with cardinality $\ell + 2$ of the form

$$\theta_E, \theta_2, \ldots, \theta_{\ell-3}, x_1\theta_{\ell-2}, x_1\theta_{\ell-1}, x_1\theta_\ell, \varphi_1, \varphi_2,$$

by the same argument as in the proof of Proposition 3.4 or 3.5, where φ_j is a linear combination of $\theta_{\ell-2}, \theta_{\ell-1}, \theta_\ell$ over $\mathbb{K}[x_2, \ldots, x_\ell]$. By Theorem 2.3, the images of $\theta_E, \theta_2, \ldots, \theta_{\ell-3}, \varphi_1, \varphi_2$ by ρ have to generate $D(\mathcal{A}^H)$. Since $\operatorname{rank}_{S/\alpha_H S} D(\mathcal{A}^H) = \ell - 1$, it holds that $\mathcal{A}^H$ is free. □

By using results above, we can show the following proposition which is fundamental on the relation between free and SPOG arrangements.

Proposition 3.8. *Let $\mathcal{A}$ be SPOG, $H \in \mathcal{A}$ and $\mathcal{A}' := \mathcal{A} \setminus \{H\}$. If $\mathcal{A}'$ is free, then there are a set of SPOG generators $\theta_1 = \theta_E, \theta_2, \ldots, \theta_\ell$, a level element φ and two distinct integers $1 < s < t \leqslant \ell$ such that*

$$\theta_E, \theta_2, \ldots, \theta_{s-1}, \theta_s/\alpha_H, \theta_{s+1}, \ldots, \theta_{t-1}, \theta_t/\alpha_H, \theta_{t+1}, \ldots, \theta_\ell$$

form a free basis for $D(\mathcal{A}')$.

Proof. By Proposition 3.4 and the assumption that $g(\mathcal{A}) = \ell + 1$, for the basis $\theta_E, \varphi_2, \ldots, \varphi_\ell$ for $D(\mathcal{A}')$, we may assume that $\varphi_i \in D(\mathcal{A})$ $(i \leqslant \ell - 2)$ and $\varphi_{\ell-1}, \varphi_\ell \notin D(\mathcal{A})$. Then

$$\theta_E, \varphi_2, \ldots, \varphi_{\ell-2}, \alpha_H\varphi_{\ell-1}, \alpha_H\varphi_\ell, f_\ell\varphi_{\ell-1} - f_{\ell-1}\varphi_\ell$$

form a minimal set of generators for $D(\mathcal{A})$, where $\varphi_i(\alpha_H) = f_i B$ modulo α_H for $i = \ell - 1, \ell$ and B is Terao's polynomial, which give the required set of generators for $D(\mathcal{A})$. □

4. Proof of the main results

In this section let us prove Theorems 1.6 and 1.8. For that, let us introduce the following two results.

Lemma 4.1. *Let $N \subset M$ be S-graded free modules. Let $\theta_1, \ldots, \theta_n$ be a homogeneous basis for N with $\deg \theta_1 \leqslant \cdots \leqslant \deg \theta_n$ and $\varphi_1, \ldots, \varphi_{n+t}$ be a homogeneous basis for M with $\deg \varphi_1 \leqslant \cdots \leqslant \deg \varphi_{n+t}$. If $\deg \theta_i = \deg \varphi_i$ for $1 \leqslant i \leqslant n$, then we may choose $\theta_1, \ldots, \theta_n, \varphi_{n+1}, \ldots, \varphi_{n+t}$ as a basis for M.*

Proof. This is essentially the same as Theorem 4.42 in [9], but we give a proof for the completeness. Let $d_i := \deg \theta_i = \deg \varphi_i$. We prove by induction on $1 \leqslant i \leqslant n$. Let $\theta_1 \in N \subset M$. Let $\varphi_1, \ldots, \varphi_s$ be of degree d_1 and $d_{s+1} > d_1$. Then

$$\theta_1 = \sum_{i=1}^{s} c_i \varphi_i$$

for constants $c_1, \ldots, c_s \in \mathbb{K}$. We may assume that $c_1 \neq 0$. Then we can choose $\theta_1, \varphi_2, \ldots, \varphi_{n+t}$ as a basis for M.

Now assume that $\theta_i = \varphi_i$ for $1 \leqslant i \leqslant k-1$. Let us prove that we may choose $\theta_k = \varphi_k$. Again by $\theta_k \in N \subset M$, we can express

$$\theta_k = \sum_{i=1}^{k-1} f_i \theta_i + c_k \varphi_k + \cdots + c_s \varphi_s,$$

where $d_k = \cdots = d_s < d_{s+1}$ or $s = n+t$. If $c_i = 0$ for all i, then $\theta_1, \ldots, \theta_k$ are not independent over S. So we may assume that $c_k = 1$, and we can choose $\theta_1, \ldots, \theta_k, \varphi_{k+1}, \ldots, \varphi_{n+t}$ as a basis for M. □

Lemma 4.2. *Let $H \in \mathcal{A}$, $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ and assume that $\mathcal{A}'$ and $\mathcal{A}^H$ are both free. Let $\varphi_1 = \theta_E, \varphi_2, \ldots, \varphi_\ell$ form a homogeneous basis for $D(\mathcal{A}')$ with $\deg \varphi_i =: d_i \leqslant d_{i+1} = \deg \varphi_{i+1}$ for all i. Assume that there are an integer $1 \leqslant k \leqslant \ell$, finite subsets $I \subset \{1, \ldots, k-1\} =: [k-1]$, T, and derivations*

$$G := \{\varphi_i\}_{i \in I} \cup \{\phi_j := \alpha_H \varphi_j\}_{j \in [k-1] \setminus I} \cup \{\psi_t\}_{t \in T} \cup \{\eta_u\}_{u=k}^{\ell}$$

in $D(\mathcal{A})$ which satisfy the following conditions:

1. *ψ_t is a linear combination of $\{\varphi_j\}_{j \in [k-1] \setminus I}$ over S for all $t \in T$, and $\deg \eta_u = \deg \varphi_u = d_u$ for all $k \leqslant u \leqslant \ell$,*
2. $$\deg \eta_u \geqslant \max\{\deg \varphi_i, \deg \phi_j, \deg \psi_t \mid i \in I, j \in [k-1] \setminus I, t \in T\}$$
   for all $k \leqslant u \leqslant \ell$,
3. $$\Big(\bigoplus_{j \in [k-1] \setminus I} S\varphi_j\Big) \cap D(\mathcal{A}) = \sum_{j \in [k-1] \setminus I} S\phi_j + \sum_{t \in T} S\psi_t,$$
   and
4. *the image of $G \setminus \{\phi_j\}_{j \in [k-1] \setminus I}$ by the Euler restriction map ρ form a basis for $D(\mathcal{A}^H)$.*

Then the S-module generated by

$$G' := \{\varphi_i\}_{i \in I} \cup \{\phi_j\}_{j \in [k-1] \setminus I} \cup \{\psi_t\}_{t \in T} \cup \{\varphi_u\}_{u=k}^{\ell}$$

coincides with the S-module generated by G.

Proof. Let $J := [k-1] \setminus I$. We prove by induction on $k \leqslant u \leqslant \ell$. Assume that $\eta_a = \theta_a$ for $k \leqslant a \leqslant u-1$, and let us show that we can choose $\eta_u = \varphi_u$. Since $\eta_u \in D(\mathcal{A}) \subset D(\mathcal{A}')$, we can express

$$\eta_u = \sum_{a=1}^{\ell} g_a \varphi_a$$

for some $g_a \in S$. Since we are interested in sets of generators, we may replace $\eta_u - \sum_{i \in I} g_i \varphi_i$ by η_u to get an expression

$$\eta_u = \sum_{j \in J} g_j \varphi_j + \sum_{a=k}^{u-1} g_a \eta_a + \sum_{a=u}^{\ell} g_a \varphi_a.$$

Let $d_u = \cdots = d_b < d_{b+1}$ or $b = \ell$. Then replacing $\eta_u - \sum_{a=k}^{u-1} g_a \eta_a$ by η_u, we obtain

$$\eta_u = \sum_{j \in J} g_j \varphi_j + \sum_{a=u}^{b} g_a \varphi_a.$$

Assume that $\sum_{a=u}^{b} g_a \varphi_a \neq 0$, say $g_u = 1$ by the reason of degrees in the conditions (1) and (2). Then replacing $\sum_{j \in J} g_j \varphi_j + \sum_{a=u}^{b} g_a \varphi_a$ by φ_u, we can choose η_u as φ_u. So assume that all $g_a = 0$ for $u \leqslant a \leqslant b$. Then

$$\eta_u = \sum_{j \in J} g_j \varphi_j.$$

Since $\eta_u \in D(\mathcal{A})$, the condition (3) shows that

$$\eta_u = \sum_{j \in J} h_j \phi_j + \sum_{t \in T} h_t \psi_t$$

for some $h_j, h_t \in S$. Sending it by ρ, the Euler exact sequence shows that

$$\rho(\eta_u) - \sum_{t \in T} h_t \rho(\psi_t) = 0,$$

contradicting the independency of the basis for $D(\mathcal{A}^H)$ in the condition (4), which completes the proof. □

Proof of Theorem 1.6. First we prove (2) ⇒ (1). It is easy to see that, for the basis $\theta_E, \varphi_2, \ldots, \varphi_\ell$ for $D(\mathcal{A}')$ with $\deg \varphi_k = d_k$, the set of SPOG generators for $D(\mathcal{A})$ is of the form $\{\theta_k\}_{k \neq i,j} \cup \{\alpha_H \theta_k\}_{k=i,j} \cup \{f_j \theta_i - f_i \theta_j\}$ by Proposition 3.8. Thus Theorem 2.3 shows that $D(\mathcal{A}^H)$ is generated by the image of $\{\theta_k\}_{k \neq i,j} \cup \{f_j \theta_i - f_i \theta_j\}$. Since $\operatorname{rank}_S D(\mathcal{A}^H) = \ell - 1$, it follows that $\mathcal{A}^H$ is free with the given exponents.

Next we prove (1) ⇒ (2). Assume that $\mathcal{A}^H$ is free with the given exponents above. In this assumption, Terao's addition theorem shows that $\mathcal{A}$ is not free since $\exp(\mathcal{A}^H) \not\subset \exp(\mathcal{A}')$. By Theorem 2.3, the Euler restriction map $\rho^H : D(\mathcal{A}) \to D(\mathcal{A}^H)$ is surjective. Thus there are S-independent derivations $\theta_E, \theta_2, \ldots, \hat{\theta}_i, \ldots, \hat{\theta}_j, \ldots, \theta_\ell, \theta \in D(\mathcal{A})$ such that $\deg \theta_k = d_k$, $\deg \theta = d$, and their images by ρ^H form a basis for $D(\mathcal{A}^H)$. Let $\theta_E = \varphi_1, \varphi_2, \ldots, \varphi_\ell$ be a basis for $D(\mathcal{A}')$ with $\deg \varphi_k = d_k$. We may assume that $d_i < d_{i+1}$ and $d_j < d_{j+1}$ or $j = \ell$. Since $D(\mathcal{A}) \subset D(\mathcal{A}')$, it holds that

$$\bigoplus_{k=1}^{i-1} S\theta_k \subset \bigoplus_{k=1}^{i-1} S\varphi_k.$$

So by Lemma 4.1, we can choose $\varphi_k = \theta_k$ for $k < i$.

Next let us show that $\varphi_i \notin D(\mathcal{A})$. Assume that $\varphi_i \in D(\mathcal{A})$. Recall that $d_i < d_{i+1}$. So $\rho(\varphi_i) \in D(\mathcal{A}^H)_{d_i}$ is a linear combination of $\rho(\theta_1), \ldots, \rho(\theta_{i-1})$. Hence

$$\varphi_i - \sum_{s=1}^{i-1} f_s\theta_s \in \alpha_H D(\mathcal{A}')$$

for some $f_s \in S$. Thus replacing $\varphi_i - \sum_{s=1}^{i-1} f_s\theta_s$ by φ_i, it holds that $\varphi_i/\alpha_H \in D(\mathcal{A}')$, contradicting the minimality of the basis $\theta_E, \varphi_2, \ldots, \varphi_\ell$ for $D(\mathcal{A}')$. So $\varphi_i \notin D(\mathcal{A})$. Hence $D(\mathcal{A}) \cap (S\varphi_i) = S\alpha_H\varphi_i$. Now apply Lemma 4.2 to obtain that $\theta_s = \varphi_s$ for $1 \leqslant s \leqslant i-1,\ i+1 \leqslant s \leqslant j-1$. Moreover, the same proof as that of $\varphi_i \notin D(\mathcal{A})$ shows that $\varphi_j \notin D(\mathcal{A})$.

Now express θ in the following form by using the fact that $d = \deg\theta \leqslant d_{j+1}$:

$$\theta = \sum_{i\neq k=1}^{j-1} f_k\theta_k + \sum_{k=j+1}^{p} a_k\varphi_k + f_i\varphi_i + f_j\varphi_j. \tag{4.1}$$

Here $d_{j+1} = \cdots = d_p = d < d_{p+1}$ and $a_k \in \mathbb{K}$. First assume that $a_k = 0$ for all k. Then replacing $\theta - \sum_{i\neq k=1}^{j-1} f_k\theta_k$ by θ, we have

$$\theta = f_i\varphi_i + f_j\varphi_j. \tag{4.2}$$

By the independency of images of θ_k and θ by ρ, at least one of f_i, f_j is not zero. Assume that only one of them is not zero, say $f_i \neq 0$ and $f_j = 0$. Then (4.2) combined with the fact that $\varphi_i \notin D(\mathcal{A})$ shows that $\alpha_H \mid f_i$. So $\rho^H(\theta) = 0$, a contradiction. Thus both f_i and f_j are not zero. Recall that $\varphi_i, \varphi_j \notin D(\mathcal{A})$. Thus the same proof as Proposition 3.5 shows that, letting $\varphi_i(\alpha_H) = g_iB$, $\varphi_j(\alpha_H) = g_jB$ modulo α_H, $(g_i, g_j) = h$, $g_i = hh_i$, $g_j = hh_j$ and $(h_i, h_j) = 1$, it holds that $\varphi := h_j\varphi_i - h_i\varphi_j \in D(\mathcal{A})$. Note that $g_i, g_j, h_i, h_j, h \in S' := \mathbb{K}[x_2, \ldots, x_\ell]$. By the construction, $\deg\varphi \leqslant d = \deg\theta = d_i + d_j - |\mathcal{A}'| + |\mathcal{A}^H| = d_i + d_j - \deg B$. Assume that $\deg\varphi < d$. Send φ by ρ, then we have

$$\varphi = h_j\varphi_i - h_i\varphi_j = \sum_{i\neq s=1}^{j-1} b_s\varphi_s + \alpha_H(b_i\varphi_i + b_j\varphi_j)$$

for some $b_s \in S$. Thus $\alpha_H \mid h_i$ and $\alpha_H \mid b_j$, contradicting $h_i, h_j \in S'$. Thus we may assume that $h = 1$ and $\deg(g_i\varphi_j - g_j\varphi_i) = d = \deg\theta$. Hence the equation (4.2) shows that

$$\theta = g_i\varphi_j - g_j\varphi_i$$

modulo α_H.

Second assume that $a_k \neq 0$ for some k in the equation (4.1). Then we may assume that $a_{j+1} = 1$ and the equation (4.2) in this case is as follows:

$$\theta = f_i\varphi_i + f_j\varphi_j + \varphi_{j+1} + \sum_{k=j+2}^{p} a_k\varphi_k. \tag{4.3}$$

Replacing $f_i\varphi_i + f_j\varphi_j + \varphi_{j+1} + \sum_{k=j+2}^{p} a_k\varphi_k$ by φ_{j+1}, we may assume that $\theta = \varphi_{j+1}$. Continue this for $\theta_{j+1}, \ldots, \theta_p$, then we obtain either $\theta_k = \varphi_{k+1}$, or $\theta_k = g_i\varphi_j - g_j\varphi_i$ modulo α_H for $j+1 \leqslant k \leqslant p$. So exchanging an appropriate θ_k by θ, we obtain that $\theta = g_j\theta_i - g_i\theta_j$ modulo α_H. Hence in both cases,

$$D(\mathcal{A}) \cap (S\varphi_i \oplus S\varphi_j) = S\alpha_H\varphi_i + S\alpha_H\varphi_j + S\theta.$$

Thus applying Lemma 4.2, we obtain that $\varphi_s = \theta_s$ for all $1 \leqslant s \leqslant \ell$ with $s \neq i, j$ and $\varphi_i, \varphi_j \notin D(\mathcal{A})$. Therefore, Proposition 3.5 shows that $\mathcal{A}$ is SPOG with $\mathrm{POexp}(\mathcal{A}) = (1, d_2, \ldots, d_i + 1, \ldots, d_j + 1, \ldots, d_\ell)$ and level $d = d_i + d_j - |\mathcal{A}'| + |\mathcal{A}^H|$. $\square$

The following case is the most practical to apply Theorem 1.6.

Corollary 4.3. *Let $\mathcal{A}'$ be free with* $\exp(\mathcal{A}') = (1, d_2, \dots, d_\ell)_\leqslant$ *and* $d := d_i + d_\ell + |\mathcal{A}^H| - |\mathcal{A}'| > d_\ell$ *for some* i*. Then the following two conditions are equivalent:*

(1) $\mathcal{A}^H$ *is free with* $\exp(\mathcal{A}^H) = (1, d_2, \dots, \hat{d}_i, \dots, d_{\ell-1}) \cup (d)$
(2) $\mathcal{A}$ *is SPOG with* $\operatorname{POexp}(\mathcal{A}) = (1, d_2, \dots, d_{i-1}, d_i + 1, d_{i+1}, \dots, d_{\ell-1}, d_\ell + 1)$ *and level* d*.*

Proof. Clear by the proof of Theorem 1.6. □

Let us apply Theorem 1.6 to some examples.

Example 4.4. Let $\mathcal{A}$ be the Weyl arrangement of the type A_4 defined by

$$Q(\mathcal{A}) = \prod_{i=1}^{4} x_i \prod_{1 \leqslant i < j \leqslant 4} (x_i - x_j) = 0.$$

$\mathcal{A}$ is well-known to be free with $\exp(\mathcal{A}) = (1, 2, 3, 4)$. Let $\mathcal{A} \not\ni H : x_1 - x_2 + 2x_3 - 2x_4 = 0$ and let $\mathcal{B} := \mathcal{A} \cup \{H\}$. Then $|\mathcal{B}^H| = 9 < 10 = |\mathcal{A}|$. It is easy to show that $\mathcal{B}^H$ is free with $\exp(\mathcal{B}^H) = (1, 4, 4)$. Note that

$$d := 2 + 3 - |\mathcal{A}| + |\mathcal{B}^H| = 5 - 1 = 4.$$

In this setup, from $\exp(\mathcal{A}) = (1, 2, 3, 4)$, the integers 2 and 3 are removed and $d = 4$ coincides with the remaining integer 4. Hence we can apply Theorem 1.6 to obtain that $\mathcal{B}$ is SPOG with $\operatorname{POexp}(\mathcal{B}) = (1, 3, 4, 4)$ and level 4.

Note that $Q(\mathcal{B}^H)$ is

$$x_2 x_3 x_4 (x_2 - x_3)(x_2 - x_4)(x_3 - x_4)(x_2 - 2x_3 + 2x_4)(x_2 - 3x_3 + 2x_4)(x_2 - 2x_3 + x_4).$$

Let $L : x_2 = 0$ and let $\mathcal{C} := \mathcal{B}^{H \cap L}$. Then it is easy to show that $\chi(\mathcal{C}; t) = (t-1)(t-4)$ and $\chi(\mathcal{B}^H; t) = (t-1)(t-4)^2$. Thus $\mathcal{B}^H$ is divisionally free as in Theorem 2.5. Since $\mathcal{A}$ is divisionally free too, by Theorem 2.2, the freeness and exponents of $\mathcal{A}$ and $\mathcal{B}^H$ are both combinatorial. Thus Theorem 1.6 shows that the SPOGness of $\mathcal{B}$ is combinatorially determined.

Proof of Theorem 1.8. Clear by Theorems 1.6 and 2.7. □

We can use Theorem 1.6 to show the combinatorial freeness of arrangements by using a non-free but SPOG arrangements. Let us check it by the following example:

Example 4.5. Let $\mathcal{A}$ be the Weyl arrangement of the type B_4 defined by

$$Q(\mathcal{A}) = \prod_{i=1}^{4} x_i \prod_{1 \leqslant i < j \leqslant 4} (x_i^2 - x_j^2) = 0.$$

Let $\mathcal{A}' := \mathcal{A} \setminus \{H\}$, where $H : x_1 = 0$. Let $L : x_1 + x_2 + x_3 = 0$. We know that $\mathcal{A}'$ is divisionally free with $\exp(\mathcal{A}') = (1, 3, 5, 6)$, and $(\mathcal{A}' \cup \{L\})^L$ is also divisionally free with $\exp(\mathcal{A}' \cup \{L\})^L) = (1, 5, 7)$. Thus Theorem 1.6 confirms that $\mathcal{B} := \mathcal{A}' \cup \{L\}$ is SPOG with $\operatorname{POexp}(\mathcal{B}) = (1, 4, 5, 7)$ and level 7. Next let $\mathcal{C} := \mathcal{B} \cup \{H\}$. Then $|\mathcal{C}^H| = 9$, so $|\mathcal{B}| - |\mathcal{C}^H| = 16 - 9 = 7$. Thus the set of generators of degrees $1, 4, 5$ for $D(\mathcal{B})$ are in $D(\mathcal{C})$ too, and we may assume that one of two generators of degree 7 is in $D(\mathcal{C})$ by Theorem 2.1. Since the relations in $D(\mathcal{B})$ are among three derivations of degrees $4, 7, 7$ by the explicit construction of the set of SPOG generators and a level element in Proposition 3.5, we know that these 4-basis elements in $D(\mathcal{C})$ are S-independent. So $\mathcal{C}$ is combinatorially free with $\exp(\mathcal{C}) = (1, 4, 5, 7)$ since the SPOGness of $\mathcal{B}$ and $|\mathcal{C}|$ are both combinatorial.

Unfortunately, there are cases in which $\mathcal{A}'$ and $\mathcal{A}^H$ are free, $\mathcal{A}$ is SPOG but Theorem 1.6 cannot be applied.

Example 4.6. Let $\mathcal{A}$ be the Weyl arrangement of the type $\mathcal{A}_4$ and let $H : x_1+x_2+x_3 = 0$. Say that $\mathcal{B} := \mathcal{A} \cup \{H\}$. Then $\mathcal{B}^H$ is free with $\exp(\mathcal{B}^H) = (1,4,5) \not\subset (1,2,3,4) = \exp(\mathcal{A})$. We can check that $\mathcal{B}$ is SPOG with $\mathrm{POexp}(\mathcal{B}) = (1,3,4,4)$ and level 5 by using Macaulay2 in [8], but we cannot apply Theorem 1.6.

Problem 4.7. *Generalize Theorem* 1.6 *to all cases when $\mathcal{A}'$ and $\mathcal{A}^H$ are free, $\ell - 2 \leqslant |\exp(\mathcal{A}') \cap \exp(\mathcal{A}^H)| \leqslant \ell - 1$ and $\mathcal{A}$ is not free.*

In fact, to apply Theorem 1.6 the condition $d \leqslant d_{j+1}$ is necessary. Let us see the following example.

Example 4.8. Let $\mathcal{A}'$ be an arrangement in $\mathbb{R}^4$ defined by

$$Q(\mathcal{A}') = (x_1+x_2+x_3+x_4)\prod_{i=1}^{4} x_i \prod_{i=2}^{4}(x_1+x_i)\prod_{i=2}^{4}(x_1+x_2+x_3+x_4-x_i).$$

Then $\mathcal{A}'$ is free with $\exp(\mathcal{A}') = (1,3,3,4)$. Let $H = x_2+x_3+x_4$ and let $\mathcal{A} := \mathcal{A}' \cup \{H\}$. Then $\mathcal{A}^H$ is free with $\exp(\mathcal{A}^H) = (1,4,5)$. So $\exp(\mathcal{A}') \cap \exp(\mathcal{A}^H) = \{1,4\}$. However, Macaulay2 in [8] shows that $D_0(\mathcal{A}) := D(\mathcal{A})/S\theta_E$ is not SPOG but has a following minimal free resolution:

$$0 \to S[-5] \oplus S[-6] \to S[-4]^3 \oplus S[-5]^2 \to D_0(\mathcal{A}) \to 0.$$

Since $d = 5 = 3 + 3 - |\mathcal{A}'| + |\mathcal{A}^H| > 4$, the condition $d \leqslant d_{j+1}$ in Theorem 1.6 is necessary.

If we remove the assumption on $\exp(\mathcal{A}')$ and $\exp(\mathcal{A}^H)$ in Theorem 1.6, we have an example related to Problem 4.7.

Example 4.9. Let

$$Q(\mathcal{A}') = \prod_{i=1}^{4} x_i \prod_{i=1}^{3}(x_i^2 - x_4^2)(x_i^2 - 4x_4^2)\prod_{i=2}^{3}(x_i^2 - 9x_4^2)(x_3^2 - 16x_4^2).$$

Then $\mathcal{A}'$ is free with $\exp(\mathcal{A}') = (1,5,7,9)$. Let $H_1 : x_2 + x_3 + 7x_4 = 0$, $H_2 : x_1 + x_2 + x_3 = 0$, and let $\mathcal{A}_i := \mathcal{A}' \cup \{H_i\}$. Then $\mathcal{A}_1$ is SPOG with $\mathrm{POexp}(\mathcal{A}_1) = (1,5,8,10)$ and level 15. So Theorem 1.6 shows that $\mathcal{A}_1^{H_1}$ is free with exponents $(1,5,15)$ and vice versa. On the other hand, $\mathcal{A}_2^{H_2}$ is free with exponents $(1,10,11)$, and $D(\mathcal{A}_2)$ has a minimal free resolution

$$0 \to S[-11] \oplus S[-12] \to S[-6] \oplus S[-8] \oplus S[-10]^2 \oplus S[-11] \to D_0(\mathcal{A}_2) \to 0.$$

So in general, it can happen that $\mathcal{A}$ is of projective dimension one, is not SPOG, but $\mathcal{A}'$ and $\mathcal{A}^H$ are both free. Note that the freeness of $\mathcal{A}^H$ follows from $g(\mathcal{A}) \leqslant 6$ and the freeness of $\mathcal{A}'$ by Theorem 3.7.

5. Ziegler restrictions and SPOG arrangements

Let us study a method to check whether $\mathcal{A}$ is SPOG or not by using Ziegler restrictions, i.e., a theory of multiarrangements. First recall the following two results which we will use later.

Theorem 5.1 (Theorem 2.3, [13]). *Let E be a reflexive sheaf on $\mathbf{P}^n$ ($n \geqslant 3$) and assume that E is locally free except for a finite number of points in $\mathbf{P}^n$. Then $H^1(E(e)) = 0$ for all $e << 0$.*

Proposition 5.2 (Proposition 2.5, [6] and the equation (1.5), [14]). *Let $\mathcal{A}$ be an arrangement in V and m be a multiplicity on $\mathcal{A}$. Then*

$$\oplus_{e\in\mathbb{Z}} H^0(\widetilde{D(\mathcal{A},m)}(e)) = D(\mathcal{A},m),$$

where $\widetilde{D(\mathcal{A},m)}$ is a sheaf on $\mathbf{Proj}(V)$ obtained as the coherent sheaf associated to the module of $D(\mathcal{A},m)$.

Next, we prove the characterization of SPOG arrangements in terms of that of the Ziegler restrictions as follows:

Proposition 5.3. *Assume that $\pi : D_H(\mathcal{A}) \to D(\mathcal{A}^H, m^H)$ is surjective, $\mathcal{A}$ is not free and $D(\mathcal{A}^H, m^H)$ is SPOG with* $\mathrm{POexp}(\mathcal{A}^H, m^H) = (d_2,\ldots,d_{\ell-1},d_\ell)$ *and level d. Then $\mathcal{A}$ is SPOG with* $\mathrm{POexp}(\mathcal{A}) = (1, d_2,\ldots,d_\ell)$ *and level d.*

Proof. Since π is surjective, there are $\theta_2,\ldots,\theta_\ell,\theta \in D_H(\mathcal{A})$ such that $\pi(\theta_2),\ldots,\pi(\theta_\ell)$ form a set of SPOG generators for $D(\mathcal{A}^H, m^H)$ with a level element $\pi(\theta)$. For $\varphi \in D_H(\mathcal{A})$ let $\overline{\varphi}$ denote its image by the Ziegler restriction map π. Since π is surjective, Theorem 2.9 shows that $\theta_2,\ldots,\theta_\ell,\theta$ together with θ_E generate $D(\mathcal{A})$. Let

$$\overline{\alpha}\overline{\theta} = \sum_{i=2}^{\ell} \overline{f}_i \overline{\theta}_i$$

be the unique relation in the SPOG module $D(\mathcal{A}^H, m^H)$, where $\alpha_H \neq \alpha \in V^*$ and $f_i \in S$. Then its preimages are of the form

$$\alpha\theta - \sum_{i=2}^{\ell} f_i\theta_i \in \alpha_H D_H(\mathcal{A})$$

by the Ziegler exact sequence (2.2). Since $D_H(\mathcal{A}) = \langle \theta_2,\ldots,\theta_\ell,\theta\rangle_S$ which is a minimal set of generators because of the non-freeness of $D_H(\mathcal{A})$ and $\mathrm{rank}_S D_H(\mathcal{A}) = \ell - 1$, it holds that

$$\alpha\theta - \sum_{i=2}^{\ell} f_i\theta_i = \alpha_H(\sum_{i=2}^{\ell} g_i\theta_i + c\theta)$$

for some $g_i \in S$ and $c \in \mathbb{K}$. Since $\overline{\alpha} \neq 0$, we have a relation

$$(\alpha - c\alpha_H)\theta - \sum_{i=2}^{\ell}(f_i + \alpha_H g_i)\theta_i = 0 \tag{5.1}$$

in $D(\mathcal{A})$.

On the other hand, assume that there is a relation

$$h_1\theta_E + \sum_{i=2}^{\ell} h_i\theta_i + h\theta = 0$$

in $D(\mathcal{A})$. Since we have a decomposition $D(\mathcal{A}) = S\theta_E \oplus D_H(\mathcal{A})$, we may assume that $h_1 = 0$. Since we have to determine the second syzygy, take a free module

$$M := Se + \oplus_{i=2}^{\ell} Se_i$$

such that by the map $G : M \to D_H(\mathcal{A})$ defined by

$$G(e_i) = \theta_i \ (i = 2,\ldots,\ell),\ G(e) = \theta,$$

M becomes the first syzygy of $D_H(\mathcal{A})$. Since $\theta_2,\ldots,\theta_\ell,\theta$ form a minimal set of generators for $D_H(\mathcal{A})$ and their images by π form a minimal set of generators for

$D(\mathcal{A}^H, m^H)$, by the nine-lemma, we have an exact commutative diagram as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, "\cdot\alpha_H"] & M \arrow[r, "G"] \arrow[d, "\cdot\alpha_H"] & D_H(\mathcal{A}) \arrow[r] \arrow[d, "\cdot\alpha_H"] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & M \arrow[r, "G"] \arrow[d, "\pi"] & D_H(\mathcal{A}) \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & \overline{S}[-d-1] \arrow[r] \arrow[d] & \overline{M} \arrow[r, "\overline{G}"] \arrow[d] & D(\mathcal{A}^H, m^H) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

So what we are assuming is that

$$\sum_{i=2}^{\ell} h_i e_i + he \in \ker(G) = K. \tag{5.2}$$

Sending this by π, the commutativity shows that

$$\sum_{i=2}^{\ell} \overline{h_i}\overline{e_i} + \overline{h}\overline{e} \in \ker \overline{G}.$$

Since $D(\mathcal{A}^H, m^H)$ is SPOG, $\ker \overline{G}$ is generate by the unique relation

$$\overline{\alpha e} - \sum_{i=2}^{\ell} \overline{f_i}\overline{e_i}$$

of degree $d+1$. Thus the exactness of the middle column in the diagram shows that

$$\sum_{i=2}^{\ell} h_i e_i + he = F(\alpha e - \sum_{i=2}^{\ell} f_i e_i) + \alpha_H \varphi \tag{5.3}$$

for some $F \in S$ and $\varphi \in M$. Rewrite (5.3) into the following way:

$$\sum_{i=2}^{\ell} h_i e_i + he - F\{(\alpha - c\alpha_H)e - \sum_{i=2}^{\ell}(f_i + \alpha_H g_i)e_i\} = \alpha_H(Fce + F\sum_{i=2}^{\ell} g_i e_i + \varphi).$$

By (5.1), the left hand side of the above is in $\ker(G)$. So is the right hand side. Since $\alpha_H \neq 0$ in S and $D_H(\mathcal{A})$ is torsion free, we know that

$$Fce + F\sum_{i=2}^{\ell} g_i e_i + \varphi \in \ker(G).$$

So we have a new relation among $\theta_2, \ldots, \theta_\ell, \theta$ but the degrees of this relation is lower than the original relation (5.2). Since the lowest degree relation in $D(\mathcal{A}^H, m^H)$ is at degree $d+1$ by the assumption, the lowest degree relation among $\theta_2, \ldots, \theta_\ell, \theta$ in $D_H(\mathcal{A})$ is (5.1), which is of degree $d+1$. Hence applying the same argument to this new relation continuously, we can show that all the relations among $\theta_2, \ldots, \theta_\ell, \theta$ are generated by the unique relation (5.1), i.e., $K \simeq S[-d-1]$. Therefore, $\mathcal{A}$ is SPOG with the desired exponents and level. $\square$

To introduce the main result in this section let us recall some definitions and facts on the freeness.

Definition 5.4 (Proposition 3.6, [1]). *For $H \in \mathcal{A}$,* ***the b_2-inequality*** *for $(\mathcal{A}, H)$ is the inequality*

$$b_2(\mathcal{A}) \geqslant b_2(\mathcal{A}^H) + |\mathcal{A}^H|(|\mathcal{A}| - |\mathcal{A}^H|).$$

Moreover, we say that the ***b_2-equality*** *holds for $(\mathcal{A}, H)$ if*

$$b_2(\mathcal{A}) = b_2(\mathcal{A}^H) + |\mathcal{A}^H|(|\mathcal{A}| - |\mathcal{A}^H|).$$

Theorem 5.5 (Theorem 3.6, [2]).
(1) *The b_2-inequality holds for all $\mathcal{A}$ and $H \in \mathcal{A}$.*
(2) *$\mathcal{A}$ is free if the b_2-equality holds for $(\mathcal{A}, H)$ and $\mathcal{A}^H$ is free.*

Theorem 5.6 (Theorem 3.1, [2]). *Let $H \in \mathcal{A}$ and assume that the b_2-equality holds for $(\mathcal{A}, H)$. If $\theta_E, \theta_2, \ldots, \theta_\ell$ form a minimal set of generators for $D(\mathcal{A}^H)$, then we may assume that $\theta_2, \ldots, \theta_\ell \in D(\mathcal{A}^H, m^H)$ and*

$$\frac{Q(\mathcal{A}^H, m^H)}{Q(\mathcal{A}^H)} \theta_E, \theta_2, \ldots, \theta_\ell$$

form a set of generators for $D(\mathcal{A}^H, m^H)$. Moreover, they form a minimal set of generators unless $(\mathcal{A}^H, m^H)$ is free.

Now we have the following result for SPOGness.

Theorem 5.7. *Let $\ell \geqslant 5$. Let $\mathcal{A} \ni H$ and assume that $\mathcal{A}$ is not free, $\mathcal{A}^H$ is SPOG, the b_2-equality holds for $(\mathcal{A}, H)$ and $\mathcal{A}$ is locally free along H, i.e., $\mathcal{A}_X$ is free for all $X \in L(\mathcal{A}^H) \setminus \{0\}$. Then $\mathcal{A}$ is SPOG with* $\mathrm{POexp}(\mathcal{A}) = \mathrm{POexp}(\mathcal{A}^H) \cup (|\mathcal{A}| - |\mathcal{A}^H|)$ *and the same level as $\mathcal{A}^H$.*

Proof. By Proposition 5.3, it suffices to show that π is surjective and $(\mathcal{A}^H, m^H)$ is SPOG. First, let us show that $\mathrm{pd}_{\overline{S}} D(\mathcal{A}^H, m^H) \leqslant 1$. Since the b_2-equality holds, Theorem 5.6 shows that, for a set of SPOG generator and the level element $\theta_E, \theta_2, \ldots, \theta_{\ell-1}, \theta$ for $D(\mathcal{A}^H)$ with $\theta_2, \ldots, \theta_{\ell-1}, \theta \in D(\mathcal{A}^H, m^H)$, we know that $Q'\theta_E, \theta_2, \ldots, \theta_{\ell-1}, \theta$ form a set of generators for $D(\mathcal{A}^H, m^H)$. Here $Q' := Q(\mathcal{A}^H, m^H)/Q(\mathcal{A}^H)$. If $(\mathcal{A}^H, m^H)$ is free, then clearly $\mathrm{pd}_{\overline{S}} D(\mathcal{A}^H, m^H) = 0$. So assume that $(\mathcal{A}^H, m^H)$ is not free. Since $\mathrm{rank}_{\overline{S}} D(\mathcal{A}^H, m^H) = \ell - 1$, it holds that $Q'\theta_E, \theta_2, \ldots, \theta_{\ell-1}, \theta$ form a minimal set of generators by Theorem 5.6. Since $\mathcal{A}^H$ is SPOG, letting $\theta_\ell := \theta$ as a level element, there are a non-zero $\alpha \in V^*$ and $\overline{f_i} \in \overline{S}$ such that

$$\overline{f_1}\theta_E + \sum_{i=2}^{\ell-1} \overline{f_i}\theta_i + \overline{\alpha}\theta_\ell = 0 \tag{5.4}$$

which is the unique relation in $D(\mathcal{A}^H)$. Again by Theorem 5.6, θ_i are in $D(\mathcal{A}^H, m^H)$. Thus $Q' \mid \overline{f_1}$. Hence (5.4) is also a relation among a minimal set of generators in $D(\mathcal{A}^H, m^H)$ obtained above. Since every relation among this minimal set of generators for $D(\mathcal{A}^H, m^H)$ is also a relation in $D(\mathcal{A}^H)$, the fact that $\mathcal{A}^H$ is SPOG shows that $D(\mathcal{A}^H, m^H)$ also has the unique relation (5.4). Hence in this case $(\mathcal{A}^H, m^H)$ is SPOG. So in each case, $\mathrm{pd}_{\overline{S}} D(\mathcal{A}^H, m^H) \leqslant 1$. In particular, since $H \simeq \mathbf{P}^{\ell-2}$ and $\ell - 2 \geqslant 3$, it holds that $H^1(\widetilde{D(\mathcal{A}^H, m^H)}(e)) = 0$ for all $e \in \mathbb{Z}$.

Second, let us prove the surjectivity of π. Since π is locally free along H, Theorem 2.3 shows that π is locally surjective. So we have the sheaf exact sequence

$$0 \to \widetilde{D_H(\mathcal{A})} \xrightarrow{\cdot\alpha_H} \widetilde{D_H(\mathcal{A})} \xrightarrow{\pi} \widetilde{D(\mathcal{A}^H, m^H)} \to 0.$$

Since $H^1(D(\widetilde{\mathcal{A}^H, m^H})(e)) = 0$ for all $e \in \mathbb{Z}$ as above, the map

$$H^1(\widetilde{D_H(\mathcal{A})}(e-1)) \stackrel{\cdot \alpha_H}{\to} H^1(\widetilde{D_H(\mathcal{A})}(e))$$

is surjective. Note that there are at most finite number of non-local free points of $\widetilde{D_H(\mathcal{A})}$. Assume not, then there is $X \in L(\mathcal{A})$ such that $\mathcal{A}_X$ is not free and $\dim X \geqslant 2$. Then it has the intersection with H of dimension at least one, contradicting the local freeness of $\mathcal{A}$ along H. Thus Theorem 5.1 shows that $H^1(\widetilde{D_H(\mathcal{A})}(e)) = 0$ for all $e \in \mathbb{Z}$. By using Proposition 5.2, it holds that π is surjective.

Finally let us show that $D(\mathcal{A}^H, m^H)$ is SPOG. By Yoshinaga's criterion (Theorem 2.10) and the surjectivity of π, it holds that $(\mathcal{A}^H, m^H)$ is not free. Thus the first investigation of the generators for $D(\mathcal{A}^H, m^H)$ shows that $(\mathcal{A}^H, m^H)$ is SPOG. $\square$

Example 5.8. Let

$$\mathcal{A}_1 := \prod_{i=1}^{5} x_i \prod_{1 \leqslant i < j \leqslant 5} (x_i - x_j) = 0.$$

Then define

$$\mathcal{A} := \mathcal{A}_1 \setminus \{x_1 = 0, x_5 = 0, x_2 = x_3, x_1 = x_2\}.$$

Let $\{x_1 = x_5\} = H \in \mathcal{A}$. Then by choosing appropriate coordinates x, y, z for H^*, $\mathcal{A}^H$ is isomorphic to

$$xyz(x-w)(y-w)(z-w)(x-z)(y-z) = 0.$$

Let $\{y = w\} = X \in \mathcal{A}^H$. Then $\mathcal{A}^H \setminus \{X\} =: \mathcal{B}$ is easily checked to be divisionally free by Theorem 2.5, with exponents $(1,2,2,2)$ and $\mathcal{A}^X$ is free with exponents $(1,2,3)$, which is also divisionally free. Thus Theorem 1.6 shows that $\mathcal{A}^H$ is SPOG with $\mathrm{POexp}(\mathcal{A}^H) = (1,2,3,3)$ and level 3, which is combinatorial by Theorem 1.8. Now we can show by case-by-case argument that $\mathcal{A}$ is locally free along H and these local freeness depends only on $L(\mathcal{A})$. Also, since $b_2(\mathcal{A}) = 48$ and $b_2(\mathcal{A}^H) = 24$, the b_2-equality holds for $(\mathcal{A}, H)$. Thus the SPOGness of $\mathcal{A}^H$ combined with local freeness along H and Theorem 5.7 shows that $\mathcal{A}$ is SPOG with $\mathrm{POexp}(\mathcal{A}) = (1,2,3,3,3)$ and level 3, here $3 = |\mathcal{A}| - |\mathcal{A}^H| = 12 - 9$. Also the SPOGness of $\mathcal{A}$ is combinatorial by this argument.

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