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Abstract In this paper, we simplify and generalize formulas for the expansion of rank 2 cluster variables. In particular, we prove an equivalent, but simpler, description of the colored Dyck subpaths framework introduced by Lee and Schiffler. We then prove the conjectured bijectivity of a map constructed by Feiyang Lin between collections of colored Dyck subpaths and compatible pairs, objects introduced by Lee, Li, and Zelevinsky to study the greedy basis. We use this bijection along with Rupel's expansion formula for quantum greedy basis elements, which sums over compatible pairs, to provide a quantum generalization of Lee and Schiffler's colored Dyck subpaths formula.

1. Introduction

The theory of *cluster algebras*, introduced twenty years ago by Fomin and Zelevinsky [5], gives us a combinatorial framework for understanding the previously opaque nature of certain algebras. Each cluster algebra is generated by its *cluster variables*, which can be obtained via the recursive process of *mutation*. The *Laurent phenomenon* says that each cluster variable in a rank-n cluster algebra can be expressed as a Laurent polynomial in the n *initial* cluster variables. While in general, finding explicit formulas for the Laurent expansions of arbitrary cluster variables is difficult, there has been significant progress in understanding the expansions in low-rank cluster algebras. In this work, we attempt to unify and simplify some existing expansion formulas for rank-2 cluster variables and their quantum generalizations.

In 2011, Lee and Schiffler provided the first combinatorial formula for the Laurent expansion of arbitrary skew-symmetric rank-2 cluster variables [10, Theorem 9]. They expressed the coefficients as sums over certain collections of non-overlapping *colored subpaths* of a *maximal Dyck path*. This established the positivity of the Laurent expansion in skew-symmetric rank 2 cluster algebras, and later for arbitrary rank [11, 12]. Lee and Schiffler [9, Theorem 11] and Rupel [15, Theorem 6] then generalized this formula (in the skew-symmetric and skew-symmetrizable cases, respectively) to the non-commutative rank-2 setting, giving each collection a weight expressed as an ordered product of two non-commuting initial cluster variables. In 2012, Lee, Li, and Zelevinsky [8] defined the *greedy basis* for rank-2 cluster algebras, which includes the cluster variables. They provided a combinatorial formula for the Laurent expansion of each greedy basis elements as a sum over *compatible pairs*, certain collections of edges

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of a maximal Dyck path [8, Theorem 11]. Rupel later gave a non-commutative analogue of this formula, which specializes to a formula for the coefficients in the *quantum rank-2 cluster algebra* setting [16, Corollary 5.4]. In particular, each compatible pair is weighted by a corresponding power of a quantum parameter q, where the exponent is computed as a sum over all pairs of edges in the maximal Dyck path. Rupel [14, Theorem 1.2] has also provided a quantum analogue to the Caldero–Chapoton expansion formula [2] for rank-2 cluster variables expressed as a sum over indecomposable valued-quiver representations.

To summarize, there are two different combinatorial formulas for the Laurent expansion of skew-symmetric rank-2 cluster variables: one in terms of collections of colored Dyck subpaths [10] and the other in terms of compatible pairs [8]. The combinatorics of collections of colored Dyck subpaths and compatible pairs are somewhat similar, suggesting that there might be a nice correspondence between them. A correspondence was known to Lee, Li, and Zelevinsky [8] and they suggested in their 2012 paper that they planned to provide details in the future, but this has not yet appeared in the literature. In 2021, Feiyang Lin constructed a map between a superset of the collections of colored Dyck subpaths and compatible pairs and conjectured that the map restricts to a bijection in [13, Conjecture 3]. Lin made partial progress toward proving this, reducing the conjecture to a technical statement [13, Conjecture 4].

In this work, we start by providing a simplification of Lee–Schiffler's formula for rank-2 cluster variables in terms of colored Dyck subpath conditions. We then use our simpler formula to prove Lin's conjectures [13, Conjectures 3 & 4] that the map constructed between collections of colored Dyck subpaths and compatible pairs is indeed a bijection. (Our methods do not rely on the technical reformulation presented by Lin.) This bijection gives an efficient method for generating all compatible pairs in the cluster variable case. We then use the bijection along with Rupel's quantum weighting of compatible pairs [16] to provide a quantum version of Lee and Schiffler's rank-2 expansion formula for cluster variables. This new formula has the advantage of requiring less computation than that in [16, Corollary 5.4] and explicitly calculating the coefficients in the quantum case, rather than expressing each term as an ordered product as in [15]. It is also more elementary than the expansion formula in [14], which is based on the theory of valued quiver representations.

The paper is organized as follows. In Section 2, we give an overview of the results, culminating in the main results, Theorem 2.8 and Theorem 2.11. Section 3 contains some preliminaries concerning maximal Dyck paths. The proof of the simplification of Lee and Schiffler's [10] colored Dyck subpath conditions is the focus of Section 4. Section 5 contains the proof of Lin's conjectures [13, Conjectures 3 & 4], establishing a bijection between collections of colored Dyck subpaths from the Lee–Schiffler [10] setting and compatible pairs from the Lee–Li–Zelevinsky [8] setting. This bijection is applied to Rupel's [16] quantum weighting on compatible pairs to yield a quantum analogue of Lee–Schiffler's [10] expansion formula in Section 6. We conclude with a discussion of further directions in Section 7.

2. Statement of results

For a positive integer r and variables X_1, X_2, we consider the sequence $\{X_n\}_{n\in\mathbb{Z}}$ of expressions recursively defined by

$$X_{n+1} = \frac{X_n^r + 1}{X_{n-1}}\,. \tag{1}$$

This sequence is precisely the set of variables of the rank-2 cluster algebra $\mathcal{A}(r,r)$ associated to the r-Kronecker quiver, which consists of two vertices with r arrows between them. The sequence is periodic when $r=1$, and otherwise all X_n are distinct. For background on cluster algebras, see [4].

Definition 2.1. *For nonnegative integers a and b, the* maximal Dyck path $\mathcal{P}(a,b)$ *is the path proceeding by unit north and east steps from $(0,0)$ to (a,b) that is closest to the line segment between $(0,0)$ and (a,b) without crossing strictly above it. For two vertices u,w along such a path, let $s(u,w)$ denote the slope of the line segment between them.*

Let $\{c_n\}_{n=1}^{\infty}$ be the sequence of non-negative integers defined recursively by:

$$c_1=0,\ c_2=1,\ \text{and } c_n=rc_{n-1}-c_{n-2} \text{ for } r\geqslant 2\,. \tag{2}$$

Let $\mathcal{C}_n=\mathcal{P}(c_{n-1},c_{n-2})$ and $\mathcal{D}_n=\mathcal{P}(c_{n-1}-c_{n-2},c_{n-2})$. We label the leftmost vertex at each height of $\mathcal{D}_n$ by v_i for $i=0,\dots,c_{n-2}$, with the subindex increasing from south to north; such vertices are called *northwest corners*. Let $\gamma(i,k)$ be the subpath spanning from v_i to v_k for any $0\leqslant i<k\leqslant b$. An example of the maximal Dyck path $\mathcal{D}_5=\mathcal{P}(5,3)$ when $r=3$ is shown in Figure 1.

Lemma 2.2. *Let $t(i)$ be the minimum integer greater than i such that $s(v_i,v_{t(i)})>s$. Then we have $t(i)-i=c_m-wc_{m-1}$ for a unique choice of $2\leqslant w\leqslant r-1$ and $3\leqslant m\leqslant n-2$.*

This result allows us to simplify the expansion formula of Lee and Schiffler, which is briefly described below. The next few definitions and Corollary 2.4 emulate the results of Lee and Schiffler, except for slight modifications due to the simplified coloring conditions above.

Definition 2.3 (cf. Definition 4.1). *For any $0\leqslant i<k\leqslant c_{n-2}$, let $\gamma(i,k)$ be the subpath of $\mathcal{D}_n$ from v_i to v_k, which is assigned a color as follows:*

(1) *If $s(v_i,v_t)\leqslant s$ for all t such that $i<t\leqslant k$, then $\gamma(i,k)$ is blue.*
(2*) *Otherwise, let m,w be chosen with respect to i as in Lemma 2.2. Then we say $\gamma(i,k)$ is (m,w)-brown.*(1)

A *subpath* of $\mathcal{D}_n$ is a path of the form $\gamma(i,k)$ or a single edge α_i. We denote the set of such subpaths by $\mathcal{P}'(\mathcal{D}_n)$. We define the set $\mathcal{F}'(\mathcal{D}_n)$ to contain any collection of subpaths in $\mathcal{P}'(\mathcal{D}_n)$ satisfying that

(i) no two subpaths share an edge,
(ii) two subpaths share a vertex only if at least one of them is a single edge, and
(iii) at least one of the $c_{m-1}-wc_{m-2}$ edges preceding each (m,w)-brown subpath is contained in another subpath.

Given $\beta\in\mathcal{F}'(\mathcal{D}_n)$, the quantity $|\beta|_1$ is defined additively over the subpaths, taking value $k-i$ on $\gamma(i,k)$ and value 0 on single edges. The quantity $|\beta|_2$ is the total number of edges in β. This yields the following expansion formula for the cluster variables.

Corollary 2.4 (analogue of [10, Theorem 9]). *Consider the cluster algebra $\mathcal{A}(r,r)$ with cluster variables X_i for $i\in\mathbb{Z}$. For $n\geqslant 3$, we have*

$$X_n=X_1^{-c_{n-1}}X_2^{-c_{n-2}}\sum_{\beta\in\mathcal{F}'(\mathcal{D}_n)}X_1^{r|\beta|_1}X_2^{r(c_{n-1}-|\beta|_2)}$$

and

$$X_{3-n}=X_2^{-c_{n-1}}X_1^{-c_{n-2}}\sum_{\beta\in\mathcal{F}'(\mathcal{D}_n)}X_2^{r|\beta|_1}X_1^{r(c_{n-1}-|\beta|_2)}\,.$$

(1)The color brown was chosen because it is the combination of Lee and Schiffler's red and green cases.

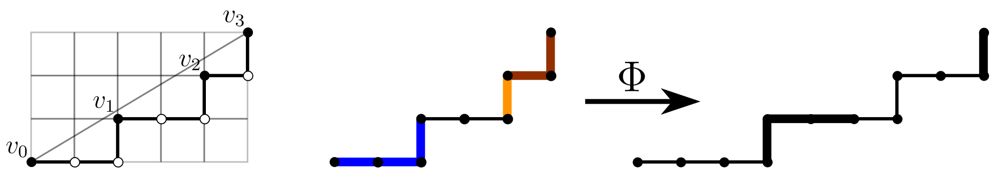

Figure 1. The leftmost image shows the maximal Dyck path $\mathcal{D}_5$ for $r = 3$, along with the corresponding 5×3 grid and main diagonal. The northwest corners are labeled and depicted as filled vertices. The center image is the collection β of colored Dyck subpaths in $\mathcal{F}'(\mathcal{D}_5)$ from Example 2.5, consisting of the blue subpath $\gamma(0,1)$, the single edge α_6 (shown in orange), and the $(3,2)$-brown subpath $\gamma(2,3)$. The rightmost image is the compatible pair on $\mathcal{C}_5$ that β maps to under Φ, where an edge is thickened whenever it is included in the compatible pair.

A generalization of the above expansion result to the case of skew-symmetric rank-2 cluster algebras with coefficients is presented at the end of Subsection 4.3 (see Corollary 4.19).

Example 2.5. For $r = 3$, the collection of colored subpaths $\beta = \{\gamma(0,1), \alpha_6, \gamma(2,3)\}$ is in $\mathcal{F}'(\mathcal{D}_5)$. This collection is depicted in Figure 1 by thickened paths of the corresponding colors, where the single vertical edge α_6 is represented by a thick orange edge. Note that in this case, we have $|\beta|_1 = 2$ and $|\beta|_2 = 6$.

We now define the compatible pairs in $\mathcal{C}_n$, introduced by Lee–Li–Zelevinsky [8], and we will show that these are in bijection with collections of colored Dyck subpaths of $\mathcal{D}_n$. Given two vertices u, w in a maximal Dyck path $\mathcal{P}(a,b)$, let $\overrightarrow{uw}$ denote the subpath proceeding east from u to w, continuing cyclically around $\mathcal{P}(a,b)$ if u is to the east of w. Let $|uw|_1$ (resp. $|uw|_2$) denote the number of horizontal (resp. vertical) edges of $\overrightarrow{uw}$. Given a set of horizontal edges S_1 and vertical edges S_2 in $\mathcal{P}(a,b)$, the pair (S_1, S_2) is *compatible* if, for every edge in S_1 with left vertex u and every edge S_2 with top vertex w, there exists a lattice point $t \neq u, w$ in the subpath $\overrightarrow{uw}$ such that

$$|tw|_1 = r|\overrightarrow{tw} \cap S_2|_2 \text{ or } |ut|_2 = r|\overrightarrow{ut} \cap S_1|_1\,.$$

Let the horizontal (resp. vertical) edges of $\mathcal{C}_n$ be labeled by η_i (resp. ν_i), increasing to the east (resp. north). Lin defined the following map Φ from $\mathcal{F}'(\mathcal{D}_n)$ to pairs (S_1, S_2) in $\mathcal{C}_n$. Note that while we define Φ as a map on blue/brown colored subpaths, it was originally defined for Lee–Schiffler's blue/green/red colored subpaths, and the two definitions are essentially identical.

Definition 2.6 ([13]). *Given $\beta \in \mathcal{F}'(\mathcal{D}_n)$, let $\Phi(\beta) = (\Phi_1(\beta), \Phi_2(\beta))$, where*

$$\Phi_1(\beta) = \{\eta_j : \alpha_j \text{ is not a part of any subpath of } \beta\}\,,$$
$$\Phi_2(\beta) = \{\nu_j : \gamma(i,k) \in \beta \text{ for some } i < j \leqslant k\}\,.$$

Example 2.7. The compatible pair $(\{\eta_4, \eta_5\}, \{\nu_1, \nu_3\})$ obtained by applying Φ to the collection of subpaths $\beta \in \mathcal{F}'(\mathcal{D}_5)$ from Example 2.5 is shown in Figure 1.

Theorem 2.8. *The map Φ is a bijection between collections of colored subpaths in $\mathcal{F}'(\mathcal{D}_n)$ and compatible pairs in $\mathcal{C}_n$.*

Switching to the quantum setting, we will now work inside the quantum torus $\mathcal{T} := \mathbb{Z}[q^{\pm 1}]\langle Z_1^{\pm 1}, Z_2^{\pm 1} : Z_1 Z_2 = q^2 Z_2 Z_1\rangle$. The quantum rank-2 r-Kronecker cluster algebra $\mathcal{A}_q(r,r)$ is the $\mathbb{Z}[q^{\pm 1}]$ subalgebra of the skew field of fractions of $\mathcal{T}$ generated by the quantum cluster variables $\{Z_n\}_{n\in\mathbb{Z}}$, which follow the recursion $Z_{n+1}Z_{n-1} = q^{-r}Z_n^r + 1$ (cf. Equation 1). We use the bijectivity of Φ along with Rupel's quantum weighting on compatible pairs [16] to construct a quantum weighting of collections of colored subpaths.

For $\beta = \{\beta_1, \dots, \beta_t\} \in \mathcal{F}'(\mathcal{D}_n)$, where β_i appears to the left of β_{i+1}, we define the set of *complimentary subpaths* $\overline{\beta_0}, \overline{\beta_1}, \dots, \overline{\beta_t}$ such that $\overline{\beta_i}$. For $1 \leqslant i \leqslant t-1$, let $\overline{\beta_i}$ contain all edges and northwest corners between the end of β_i and the start of β_{i+1}, where a northwest corner on the boundary of a path is included in $\overline{\beta_i}$ unless it is the right endpoint of a brown or blue subpath. We set $\overline{\beta_0}$ to be the portion of the path before β_1 excluding v_0, and we set $\overline{\beta_t}$ to be the portion of the path after β_t including $v_{c_{n-2}}$. Note that it is possible for some $\overline{\beta_i}$ to be empty. Let $|\overline{\beta_i}|_1$ (resp. $|\overline{\beta_i}|_2$) denote the number of northwest corners (resp. edges) in $\overline{\beta_i}$.

Definition 2.9. *For $\beta = \{\beta_1, \dots, \beta_t\} \in \mathcal{F}'(\mathcal{D}_n)$, we let*

$$w_q(\beta) = (c_{n-1} + c_{n-2} - 1) + \sum_{j=0}^{t} r|\overline{\beta_j}|_2 \left(\sum_{i=1}^{t} (-1)^{\mathbb{1}_{i<j}} |\beta_i|_2\right) + \left(r|\overline{\beta_j}|_1 - r^2|\overline{\beta_j}|_2\right) \left(\sum_{i=1}^{t} (-1)^{\mathbb{1}_{i<j}} |\beta_i|_1\right),$$

where $\mathbb{1}_{i<j}$ takes value 1 when $i<j$ and 0 otherwise. We then set

$$u_q(\beta) = w_q(\beta) - (c_{n-1} + c_{n-2} - 1) + (c_{n-1} - r|\beta|_1)(c_n - r|\beta|_2).$$

Example 2.10. For the collection $\beta \in \mathcal{F}'(\mathcal{D}_5)$ from Example 2.5, we have $\overline{\beta_0} = \overline{\beta_3} = \varnothing$, $\overline{\beta_1} = \{\alpha_4, \alpha_5\}$, and $\overline{\beta_2} = \{v_2\}$. We thus have $w_q(\beta) = 10$.

We prove that the quantum cluster variable Laurent coefficients can be expressed as a sum over weighted collections of subpaths in $\mathcal{F}'(\mathcal{D}_n)$.

Theorem 2.11. *Consider the quantum cluster algebra $\mathcal{A}_q(r,r)$ with quantum cluster variables Z_i for $i \in \mathbb{Z}$. For $n \geqslant 4$, we have*

$$Z_n = Z_1^{-c_{n-1}} Z_2^{-c_{n-2}} \sum_{\beta \in \mathcal{F}'(\mathcal{D}_n)} q^{u_q(\beta)} Z_1^{r|\beta|_1} Z_2^{r(c_{n-1} - |\beta|_2)}$$

and

$$Z_{3-n} = Z_2^{-c_{n-1}} Z_1^{-c_{n-2}} \sum_{\beta \in \mathcal{F}'(\mathcal{D}_n)} q^{u_q(\beta)} Z_2^{r|\beta|_1} Z_1^{r(c_{n-1} - |\beta|_2)}.$$

3. Preliminaries

Let $r \geqslant 2$ be fixed throughout this paper. Both the Lee–Schiffler and Lee–Li–Zelevinsky expansion formulas involve sums over certain collections of edges in a maximal Dyck path. Moreover, the width and height of these Dyck paths have certain recursive properties that are used in the proofs of both formulas. We begin by setting up a framework for studying these paths and describing their recursive behavior.

Recall the sequence c_n defined recursively by Equation 2. While the indexing of the sequence $\{c_n\}_{n\geqslant 1}$ is identical to the indexing in the work of Lee and Schiffler [10], the indexing is shifted by one from that defined by Lin [13], i.e. it is equivalent to $\{c_{n-1}\}_{n\geqslant 1}$ in Lin's work. It is straightforward to check that for $n > 1$, the quantities c_n and c_{n+1} are relatively prime, hence so are c_n and $c_{n+1} - c_n$. Thus, the only vertices of $\mathcal{D}_n$ and $\mathcal{C}_n$ that lie on the main diagonal are the first and last.

Fix $a, b \in \mathbb{N}$. Consider a rectangle with vertices $(0,0)$, $(0,b)$, $(a,0)$, and (a,b) having a designated diagonal from $(0,0)$ to (a,b).

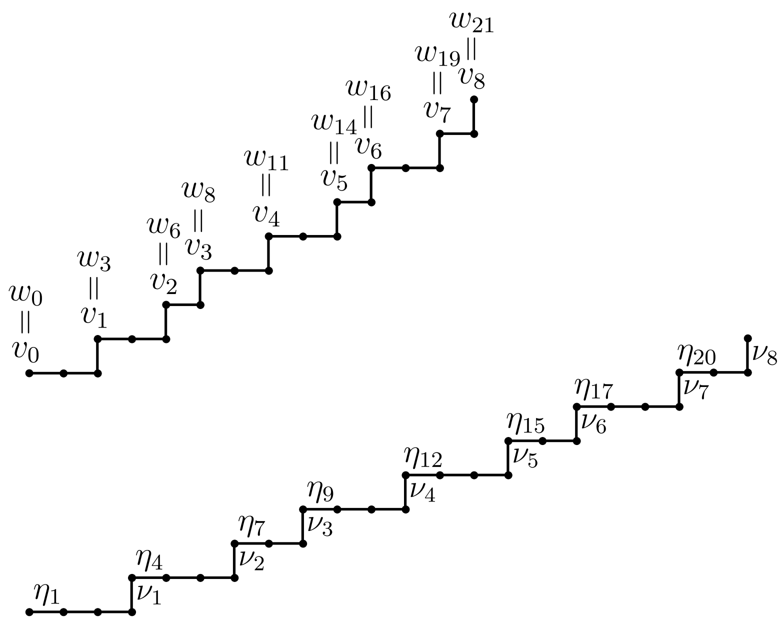

Figure 2. The maximal Dyck paths $\mathcal{D}_6$ (above) and $\mathcal{C}_6$ (below) are shown with some of their vertex and edge labels for $r = 3$. Each northwest corner of $\mathcal{D}_6$ is labeled with both its corresponding w_i and v_j label. Some edges of $\mathcal{C}_6$ are labeled; the ν_i's refer to the vertical edge left of the label, and the η_j's refer to the horizontal edge below the label.

Definition 3.1. *A* Dyck path *is a lattice path in $\mathbb{Z}^2$ starting at $(0,0)$ and ending at a lattice point (a,b) where $a,b \geqslant 0$, proceeding by only unit north and east steps and never passing strictly above the diagonal. Given a Dyck path P, we denote the number of east steps by $|P|_1$ and the number of north steps by $|P|_2$. The* length *of the Dyck path P is the quantity $|P|_1 + |P|_2$. We denote the set of lattice points contained in the Dyck path P, ordered from left to right and including both endpoints, by $V(P) = \{w_0, w_1, \ldots, w_{|P|_1+|P|_2}\}$.*

The Dyck paths from $(0,0)$ to (a,b) form a partially ordered set by comparing the heights at all vertices. The maximal Dyck path $\mathcal{P}(a,b)$, as defined in Definition 2.1, is the maximal element under this partial order. We focus on the following two classes of maximal Dyck paths, defined for $n \geqslant 3$, $\mathcal{C}_n = \mathcal{P}(c_{n-1}, c_{n-2})$ and $\mathcal{D}_n = \mathcal{P}(c_{n-1} - c_{n-2}, c_{n-2})$.

Recall that a vertex of a maximal Dyck path P is a *northwest corner* if there are no vertices directly north of (equivalently, to the east of) it. In $\mathcal{D}_n$, these are precisely the vertices labeled by v_i for some $0 \leqslant i \leqslant c_{n-2}$.

When a and b are relatively prime, as is the case for $\mathcal{C}_n$ and $\mathcal{D}_n$, we can associate to this Dyck path the *(lower) Christoffel word* of slope $\frac{b}{a}$ on the alphabet $\{E, N\}$. This word can be constructed by reading the edges of the maximal Dyck path from $(0,0)$ to (a,b), recording an E for each east step and an N for each north step. For further background on Christoffel words, see [1].

Example 3.2. Let $r = 3$. The Christoffel words corresponding to the maximal Dyck paths $\mathcal{D}_6$ and $\mathcal{C}_6$ depicted in Figure 2 are

$$E^2NE^2NENE^2NE^2NENE^2NEN \text{ and } E^3NE^3NE^2NE^3NE^3NE^2NE^3NE^2N\,,$$

respectively.

Remark 3.3. The Christoffel word corresponding to $\mathcal{C}_n$ is obtained by applying the morphism $\theta = \{E \mapsto E, N \mapsto EN\}$, i.e. the map that replaces each instance of the letter N with the string EN, to the Christoffel word corresponding to $\mathcal{D}_n$. This follows directly from, for example, [1, Lemma 2.2].

Observation 3.4. It is straightforward to calculate that

$$V(\mathcal{C}_3) = \{(0,0),(1,0)\} \text{ and } V(\mathcal{C}_4) = \{(0,0),(1,0),\dots,(r,0),(r,1)\}$$

As we shall see, both of these families of Dyck paths have a recursive structure. The following lemma is a special case of a result of Rupel.

Lemma 3.5 ([15, Lemma 3]). *For all $n \geqslant 4$, the maximal Dyck path $\mathcal{D}_n$ consists of $r-1$ copies of D_{n-1} followed by a copy of D_{n-1} with a prefix D_{n-2} removed. In particular, $\mathcal{D}_{n-1}$ (resp. $\mathcal{C}_{n-1}$) is a subpath of $\mathcal{D}_n$ (resp. $\mathcal{C}_n$).*

This allows us to define the limit of these paths, which can be realized by taking a union of finite subpaths.

Definition 3.6. *Let $\mathcal{C}$ (resp. $\mathcal{D}$) be the infinite path on $\mathbb{Z}^2$ formed by the union $\bigcup_{n\geqslant 3} \mathcal{C}_n$ (resp. $\bigcup_{n\geqslant 3} \mathcal{D}_n$). We identify the paths $\mathcal{C}_n$ and $\mathcal{D}_n$ with the prefix of the same length of $\mathcal{C}$ and $\mathcal{D}$, respectively. Thus, the vertices of $\mathcal{D}$ are labeled by w_i and the northwest corners by v_j, as described after Definition 2.1 in Section 2. Similarly, horizontal edges of $\mathcal{C}$ are labeled by η_i and the vertical edges are labeled by ν_j.*

4. Simplification of the colored Dyck subpaths conditions

4.1. Lee–Schiffler expansion formula. We first recall the original expansion formula given by Lee and Schiffler [10] for rank-two skew-symmetric cluster variables. This requires us to set up the language of colored subpaths in a Dyck path via Lee and Schiffler's conventions, which differs from that in Section 2. We then describe the map between certain non-overlapping collections of colored subpaths, namely from $\mathcal{F}(\mathcal{D}_n)$ as defined by Lee–Schiffler to the set $\mathcal{F}'(\mathcal{D}_n)$ which we defined after Definition 2.3 in Section 2.

Let s denote the slope of the main diagonal of $\mathcal{D}_n$, so $s = \frac{c_{n-2}}{c_{n-1}-c_{n-2}}$.

Definition 4.1 ([10], cf. Definition 2.3). *For any $0 \leqslant i < k \leqslant c_{n-2}$, let $\alpha(i,k)$ be the subpath of $\mathcal{D}_n$ defined as follows:*

(1) *If $s(v_i, v_t) \leqslant s$ for all t such that $i < t \leqslant k$, then $\alpha(i,k)$ is defined to be the subpath from v_i to v_k; each such subpath is called blue.*
(2) *If $s(v_i, v_t) > s$ for some $i < t \leqslant k$, then*
   (2-a) *if the smallest such t is of the form $i + c_m - wc_{m-1}$ for some $3 \leqslant m \leqslant n-2$ and $1 \leqslant w \leqslant r-2$, then $\alpha(i,k)$ is defined to be the subpath from v_i to v_k; each such subpath is called (m,w)-green.*
   (2-b) *otherwise, $\alpha(i,k)$ is set to be the subpath from the vertex immediately below v_i to v_k; each such subpath is called red.*

Each such pair i,k corresponds to precisely one subpath of $\mathcal{D}_n$. Denote the single edges of $\mathcal{D}_n$ be by $\alpha_1, \dots, \alpha_{c_{n-1}}$ proceeding from southwest to northeast, and let

$$\mathcal{P}(\mathcal{D}_n) = \{\alpha(i,k) : 0 \leqslant i < k \leqslant c_{n-2}\} \cup \{\alpha_1, \dots, \alpha_{c_{n-1}}\}.$$

The formula involves sums over collections of subsets of $\mathcal{P}(\mathcal{D}_n)$ satisfying certain non-overlapping requirements. In particular, Lee and Schiffler set

$$\begin{aligned}\mathcal{F}(\mathcal{D}_n) = \{\{\beta_1,\dots,\beta_t\} : t \geqslant 0, \beta_j \in \mathcal{P}(\mathcal{D}_n) \text{ for all } 1 \leqslant j \leqslant t,\\ \text{if } j \neq j' \text{ then } \beta_j \text{ and } \beta_{j'} \text{ have no common edge,}\\ \text{if } \beta_j = \alpha(i,k) \text{ and } \beta_{j'} = \alpha(i',k') \text{ then } i \neq k' \text{ and } i' \neq k,\\ \text{and if } \beta_j \text{ is } (m,w)\text{-green then at least one of the}\\ (c_{m-1} - wc_{m-2}) \text{ edges preceding } v_i \text{ is contained}\\ \text{in some } \beta_{j'}\}\,.\end{aligned}$$

For any collection of subpaths β, we associate two non-negative integers $|\beta|_1$ and $|\beta|_2$. The first quantity $|\beta|_1$ is defined to be 0 on single edges and $k-i$ on $\alpha(i,k)$, then extended additively on unions of these subpaths. The second quantity $|\beta|_2$ is the total number of edges α_i covered by the subpaths in β. We can now state the original formulation of Lee and Schiffler's expansion result.

Theorem 4.2 ([10, Theorem 9]). *For $n \geqslant 3$, we have*

$$X_n = X_1^{-c_{n-1}} X_2^{-c_{n-2}} \sum_{\beta \in \mathcal{F}(\mathcal{D}_n)} X_1^{r|\beta|_1} X_2^{r(c_{n-1}-|\beta|_2)}$$

and

$$X_{3-n} = X_2^{-c_{n-1}} X_1^{-c_{n-2}} \sum_{\beta \in \mathcal{F}(\mathcal{D}_n)} X_2^{r|\beta|_1} x_1^{r(c_{n-1}-|\beta|_2)}\,.$$

In order to show that our expansion formula Corollary 2.4 is equivalent to Theorem 4.2, we define the map χ which connects the colored subpaths in both settings.

Definition 4.3. *We define a map $\chi : \mathcal{F}(\mathcal{D}_n) \to \mathcal{F}'(\mathcal{D}_n)$ that modifies the colored subpaths of $\beta \in \mathcal{F}(\mathcal{D}_n)$ via the following rules:*

- *each red subpath is split into two subpaths: its leftmost edge (viewed as a single edge) and the remainder of the path, which is a $(3, r-1)$-brown subpath;*
- *each (m,w)-green subpath changes color to become an (m,w)-brown subpath if $w > 1$ or an $(m+1, r-1)$-subpath if $w = 1$;*
- *the blue subpaths and single edges remain unchanged.*

Note that the set of edges covered by β is preserved under χ. Lemma 4.18 establishes that χ is well-defined and is in fact a weight-preserving bijection with respect to $|\beta|_1$ and $|\beta|_2$. The statement and proof of Lemma 4.18 appear in Subsection 4.3.

4.2. Vertices and slopes in $\mathcal{D}_n$.

We now prove several results relating to the position of vertices in $\mathcal{D}_n$ and the slopes of the line segments between northwest corners of $\mathcal{D}_n$. To help illuminate the recursive structure of the infinite path $\mathcal{D}$, which contains each $\mathcal{D}_n$ as a prefix, we define a map taking vertices to northwest corners in $\mathcal{D}$.

Definition 4.4. *Let $\mu : V(\mathcal{D}) \to V(\mathcal{D})$ be the map sending w_i, the i^{th} vertex of $\mathcal{D}$, to v_i, the i^{th} northwest corner of $\mathcal{D}$.*

The following result describes the behavior of μ in terms of coordinates.

Lemma 4.5. *If $w_i \in V(\mathcal{D})$ has coordinates (x,y), then we have*

$$\mu(w_i) = ((r-1)x + (r-2)y, x+y)\,.$$

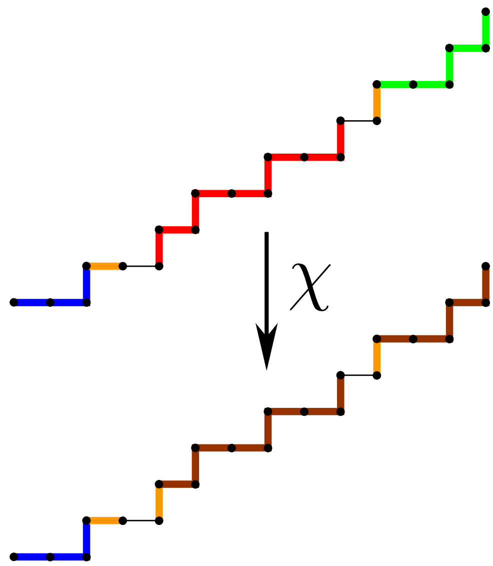

Figure 3. The top image depicts a collection of colored Dyck subpaths in $\mathcal{F}(\mathcal{D}_6)$, consisting of the blue path $\alpha(0,1)$, the single edge α_4, the red path $\alpha(2,5)$, the single edge α_{16}, and the $(3,1)$-green path $\alpha(6,8)$. By applying the map χ to the top collection, we obtain the collection of colored Dyck subpaths in $\mathcal{F}'(\mathcal{D}_6)$ depicted in the bottom image. This collection consists of the blue path $\gamma(0,1)$, the single edges α_4 and α_6, the $(3,2)$-brown path $\gamma(2,5)$, the single edge α_{16}, and the $(4,2)$-brown path $\gamma(6,8)$.

Proof. Fix n large enough such that $(x,y) \in V(\mathcal{D}_n)$. For each $(x,y) \in V(\mathcal{D}_n)$, the claim is equivalent to showing that the following inequalities hold:

$$\frac{x+y}{(r-1)x+(r-2)y} \leqslant \frac{c_{n-1}}{c_n - c_{n-1}} < \frac{x+y+1}{(r-1)x+(r-2)y}\,.$$

In order to prove the first inequality, we first note that

$$\frac{c_{n-2}}{c_{n-1}-c_{n-2}} \geqslant \frac{y}{x}\,,$$

which holds since $(x,y) \in V(\mathcal{D}_n)$. Cross multiplying and adding yc_{n-2} to both sides yields

$$\frac{c_{n-2}}{c_{n-1}} \geqslant \frac{y}{x+y}\,.$$

Hence we have

$$\begin{aligned}\frac{c_{n-1}}{c_n-c_{n-1}} &= \frac{c_{n-1}}{(r-1)c_{n-1}-c_{n-2}}\\ &= \frac{1}{(r-1)-\frac{c_{n-2}}{c_{n-1}}}\\ &\geqslant \frac{1}{(r-1)-\frac{y}{x+y}}\\ &= \frac{x+y}{(r-1)x+(r-2)y}\,,\end{aligned}$$

as desired.

We can prove the second inequality similarly. Since we have $\frac{c_{n-2}}{c_{n-1}-c_{n-2}} < \frac{y+1}{x}$, then cross multiplying and adding $(y+1)c_{n-2}$ to both sides yields

$$\frac{c_{n-2}}{c_{n-1}} < \frac{y+1}{x+y+1}\,.$$

Thus, we can conclude

$$\frac{c_{n-1}}{c_n - c_{n-1}} = \frac{1}{(r-1) - \frac{c_{n-2}}{c_{n-1}}} < \frac{1}{(r-1) - \frac{y+1}{x+y+1}} \leqslant \frac{x+y+1}{(r-1)x+(r-2)y}\,. \qquad \square$$

We now study the slopes of the line segments between northwest corners of $\mathcal{D}$, as these play a central role in the definition of colored Dyck subpaths in both our setting and the Lee–Schiffler setting. We will utilize several classical results concerning maximal Dyck paths and Christoffel words, which can be found, for example, in [1].

Definition 4.6. *For $n \geqslant 3$, we define the function $\pi_n : \{0,1,\ldots,c_{n-1}\} \to \mathbb{Z}$ by*

$$\pi_n(i) := xc_{n-2} - (i-x)(c_{n-1} - c_{n-2}) \text{ where } w_i = (x, i-x) \in \mathcal{D}_n\,.$$

Remark 4.7. Note that we have $\pi_n(0) = \pi_n(c_{n-1}) = 0$. It is a standard result from the theory of Christoffel words (see, for example, [1, Lemma 1.3]) that the sequence $\pi_n(1), \pi_n(2), \ldots, \pi_n(c_{n-1})$ is a permutation of the elements $\{0, 1, \ldots, c_{n-1}-1\}$, and this is order-isomorphic to the sequence of distances from each vertex to the line segment between w_0 and $w_{c_{n-1}}$. Thus, $s(w_i, w_j) \geqslant s = s(w_0, w_{c_{n-1}})$ in $\mathcal{D}_n$ if and only if $\pi_n(j) \leqslant \pi_n(i)$.

Example 4.8. For $r = 3$, the values $\pi_6(0), \pi_6(1), \ldots, \pi_6(21)$ are given by the following sequence:

$$0, 8, 16, 3, 11, 19, 6, 14, 1, 9, 17, 4, 12, 20, 7, 15, 2, 10, 18, 5, 13, 0\,.$$

We now prove some relations between the sequences $\{\pi_n(i)\}_{i=0}^{c_{n-1}}$ and $\{\pi_{n-1}(i)\}_{i=0}^{c_{n-2}}$.

Lemma 4.9. *For any vertex with coordinates (x, y) in $\mathcal{D}_{n-1}$ where $n \geqslant 4$, we have*

$$\pi_n(rx + (r-1)y) = \pi_{n-1}(x+y)\,.$$

Proof. By Lemma 3.5 and Lemma 4.5, both (x, y) and $\mu(w_{x+y}) = ((r-1)x + (r-2)y, x+y)$ are vertices of $\mathcal{D}_n$. Hence, we can expand the right-hand side using Definition 4.6 and apply the relation $c_n = rc_{n-1} - c_{n-2}$ to obtain

$$\begin{aligned}
\pi_n(rx + (r-1)y) &= ((r-1)x + (r-2)y)c_{n-2} - (x+y)(c_{n-1} - c_{n-2}) \\
&= ((r-1)x + (r-2)y)c_{n-2} - (x+y)((r-1)c_{n-2} - c_{n-3}) \\
&= -yc_{n-2} + (x+y)c_{n-3} \\
&= xc_{n-3} - y(c_{n-2} - c_{n-3})\,.
\end{aligned}$$

Comparing with Definition 4.6, we see that the final quantity is precisely $\pi_{n-1}(x+y)$, as desired. $\square$

Lemma 4.10. *For $n \geqslant 3$, the set $\{(x, y) \in V(\mathcal{D}_n) : \pi_n(x+y) \in \{0, \ldots, c_{n-2} - 1\}\}$ is precisely the set of northwest corners of $\mathcal{D}_n$.*

Proof. We proceed inductively on n. The base case $n = 3$ is straightforward to check. Remark 4.7 implies that $\pi_n^{-1}(0) = \{v_0, v_{c_{n-1}}\}$ and that $\pi_n(i)$ takes each value of $\{1, 2, \ldots, c_{n-1}-1\}$ exactly once for $i \in \{2, 3, \ldots, c_{n-1}\}$. For each vertex $(x, y) \in \mathcal{D}_{n-1}$, we can evaluate

$$\pi_n(rx + (r-1)y) = \pi_{n-1}(x+y) \in \{0, 1, 2, \ldots, c_{n-2} - 1\}\,.$$

Moreover, the unique vertex $(x', y') \in \mathcal{D}_n$ such that $x' + y' = rx + (r-1)y$ is $\mu(w_{x+y})$. Since the image of μ applied to the set $V(\mathcal{D}_{n-1})$ is the set of northwest corners of $\mathcal{D}_n$, the desired set equality holds. $\square$

Example 4.11. We can see the statement of Lemma 4.10 illustrated in comparing the path $\mathcal{D}_6$ at the top of Figure 2 and Example 4.8. The northwest corners of $\mathcal{D}_6$ are the vertices w_0, w_2, w_3, w_6, w_8, w_{11}, w_{14}, w_{16}, w_{19}, and w_{21}. The subscripts of these northwest corner are precisely the inputs $i \in \{0, 1, \dots, 21\}$ such that $0 \leqslant \pi_6(i) \leqslant 7 = c_4 - 1$, as seen in Example 4.8.

Corollary 4.12. *Applying the morphism $\lambda = \{E \mapsto E^{r-1}N, N \mapsto E^{r-2}N\}$ to the Christoffel word for $\mathcal{D}_n$ yields the Christoffel word for $\mathcal{D}_{n+1}$.*

Applying the morphism $\theta \circ \lambda \circ \theta^{-1}$ to the Christoffel word for $\mathcal{C}_n$ yields the Christoffel word for $\mathcal{C}_{n+1}$, where θ is the morphism $\{E \mapsto E, N \mapsto EN\}$.

Proof. Let w_i denote the Christoffel word for $\mathcal{D}_i$. Applying the map μ to $V(\mathcal{D}_n)$ takes a vertex (x, y) to $((r-1)x + (r-2)y, x + y)$. Hence, by Lemma 4.10, the index j of the k^{th} vertical edge α_j in $\mathcal{D}_{n+1}$, i.e. the positions of y's in w_{n+1}, is equal to

$$r|\{\alpha_i^n : i < j, \alpha_i^n \text{ is horizontal}\}| + (r-1)|\{\alpha_i^n : i < j, \alpha_i^n \text{ is vertical}\}| + 1\,.$$

This is precisely the position of y's in the word $\lambda(w_n)$, hence we can conclude that $\lambda(w_n) = w_{n+1}$. The second statement follows from Remark 3.3. $\square$

Finally, we can determine the rest of the values of π_n from its values on the northwest corners.

Observation 4.13. Suppose that $(i, j) \in \mathcal{D}_n$ is not a northwest corner. Let (i', j) be the corner vertex immediately preceding (i, j). Then it follows from the definition of π_n that

$$\pi_n(i + j) = \pi_n(i' + j) + (i - i')c_{n-2}\,.$$

4.3. Proof of the simplification. We now show that the map χ (see Definition 4.3) is well-defined and weight-preserving with respect to $|\beta|_1$ and $|\beta|_2$. This shows that our definition of $\mathcal{F}'(\mathcal{D}_n)$ is, in a sense, equivalent to that of Lee–Schiffler.

Observation 4.14. Lemma 4.9 states that the values of π_n on the northwest corners of $\mathcal{D}_n$ lie in the set $\{0, 1, \dots, c_{n-2} - 1\}$. Since π_n is injective on the interior of $\mathcal{D}_n$, this implies that the values of π_n of the vertices which are not northwest corners constitute the set $\{c_{n-2}, c_{n-2} + 1, \dots, c_{n-1} - 1\}$.

The colors of paths in Lee and Schiffler's setting depend on the number of northwest corners that one needs to traverse from the starting endpoint of the path until the slope to the northwest corner was at least the slope of the diagonal. We also consider the slopes between vertices on $\mathcal{D}_n$ that are not necessarily northwest corners, which will help us to derive results about the northwest corners. We in turn use this to determine which vertex $w_{d(i)}$ to the east of a given vertex w_i is the first such that the slope between w_i and $w_{d(i)}$ is at least that of the diagonal.

Definition 4.15. *For $0 \leqslant i < c_{n-1}$, we define*

$$\begin{aligned} d(i) &= \min\left(\{j \in \{i+1, i+2, \dots, c_{n-1}\} : s(w_i, w_j) \geqslant s)\}\right) \\ &= \min\left(\{j \in \{i+1, i+2, \dots, c_{n-1}\} : \pi_n(j) \leqslant \pi_n(i)\}\right)\,. \end{aligned}$$

Note that $d(i)$ is well-defined since $s(w_i, w_{c_{n-1}}) \geqslant s$ for all $0 \leqslant i < c_{n-1}$, and the fact that the two expressions are equivalent follows directly from Remark 4.7. We now show some properties of how the functions d and μ interact.

Corollary 4.16. *Suppose w_i is a northwest corner of $\mathcal{D}_n$ for $n \geqslant 4$. Let $w_i = \mu(w_{i'})$ for some $w_{i'} \in \mathcal{D}_{n-1}$, and let $(x, y) = w_{d(i')}$ in $\mathcal{D}_{n-1}$ with $d(i') - i' = mc_{n-2} - wc_{n-3}$. Then we have $w_{d(i)} = ((r-1)x + (r-2)y, x+y)$, and $d(i) - i = mc_{n-1} - wc_{n-2}$.*

If w_i is not a northwest corner, then $w_{d(i)}$ is the northwest corner immediately following w_i. In particular, $d(i) - i \leqslant r - 1$.

Proof. By Observation 4.13, the function $\pi_n(i)$ increases when the step preceding w_i is an east step. Since $d(i) > i$ is chosen minimally such that $\pi_n(d(i)) \leqslant \pi_n(i)$, we must have that $w_{d(i)}$ is preceded by a north step, i.e. $w_{d(i)}$ is a northwest corner. Lemma 4.10 implies that $w_{d(i)}$ must then be the image of $w_{d(i')}$ under the correspondence between vertices of $\mathcal{D}_{n-1}$ and northwest corners of $\mathcal{D}_n$. It is straightforward to check that the distances follow the formula described under this correspondence. The second claim follows directly from Observation 4.14. □

From this, we can determine that the values $d(i) - i$ are of a particular form.

Lemma 4.17. *For all positive integers i, we have $d(i) - i = c_m - wc_{m-1}$ for a unique choice of $2 \leqslant w \leqslant r-1$ and $3 \leqslant m \leqslant n-2$.*

Proof. We prove this via induction on n. The base case $n = 3$ is straightforward to check. Suppose the statement holds on $\mathcal{C}_{n-1}$. Whenever w_i is not a northwest corner of $\mathcal{C}_n$, then we have $d(i) - i < r$ by Corollary 4.16, so $d(i) - i = c_3 - wc_2$ for some $2 \leqslant w \leqslant r-1$ or $d(i) = r - 1 = c_4 - (r-1)c_3$. Otherwise, if w_i is a northwest corner of $\mathcal{C}_n$, then $w_i = \mu(w_{i'})$ for some $w_{i'} \in \mathcal{C}_{n-1}$. By assumption, $d(i') - i' = c_m - wc_{m-1}$ for some appropriate choice of m, w. Applying Corollary 4.16, we can conclude that $d(i) - i = c_{m+1} - wc_m$.

The fact that this representation is unique follows immediately from the fact that, for all $m \geqslant 2$, we have $c_m - (r-1)c_{m-1} > c_{m-1} - 2c_{m-2}$. □

We are now ready to establish that the value $t(i) - i$, as given in Lemma 2.2, can also be represented as $c_m - wc_{m-1}$ for an appropriate choice of m, w.

Proof of Lemma 2.2. We prove this via induction on n, with the straightforward base case $n = 3$. Fix a northwest corner $v_i = w_j \in \mathcal{C}_n$, and let $t(i)$ be the minimum positive integer such that $s(v_i, v_{t(i)}) \geqslant s$. Since v_i is a northwest corner, we have $v_i = \mu(w_{i'})$ for some $w_{i'} \in \mathcal{C}_{n-1}$. Applying Lemma 4.17, we have that $d(i') - i' = c_m - wc_{m-1}$ for a unique choice of $2 \leqslant w \leqslant r-1$ and $3 \leqslant m \leqslant n-2$. By Remark 4.7, we have that $t(i) - i = d(i') - i'$. Thus, $t(i) - i$ is of the desired form. □

The following lemma establishes that χ preserves weights.

Lemma 4.18. *For all $n \geqslant 3$, we have $\beta \in \mathcal{F}(\mathcal{D}_n)$ if and only if $\chi(\beta) \in \mathcal{F}'(\mathcal{D}_n)$. Moreover, we have $|\beta|_i = |\chi(\beta)|_i$ for $i = 1, 2$.*

Proof of Lemma 4.18. Fix $\beta \in \mathcal{F}(\mathcal{D}_n)$. Suppose $\alpha(i,k) \in \beta$. By Lemma 2.2, either we have that $\alpha(i,k)$ is (m,w)-green, or we have that $t(i) - i = 1$. In this case, condition (2^*) of Definition 2.3 enforces that the edge immediately preceding $\alpha(i,k)$ is contained in β_j. By the non-overlapping condition for membership in $\mathcal{F}(\mathcal{D}_n)$, we have $\beta_j \neq \alpha(i', k')$ for any i', k'. Thus, $\beta_j = \alpha'_i$ for some i'. In particular, it does not contribute to $|\beta|_1$ and contributes 1 to $|\beta|_2$, which is the same as if we had considered $\alpha(i,k)$ to contain this preceding edge. □

We can now combine the results about the map χ in order to prove the expansion formula in our setting.

Proof of Corollary 2.4. This modified expansion formula follows immediately from Lemma 2.2 and from the expansion formula (Theorem 4.2) given by Lee and Schiffler [10]. $\square$

We now furthermore discuss a generalization of Corollary 2.4 to the setting of the *framed r-Kronecker cluster algebra with principal coefficients*. While there is a more general theory of cluster algebras with coefficients (see, for example, [6]), we will give a brief explicit description of this cluster algebra here. For a positive integer r, initial cluster variables X_1, X_2 and coefficient variables Y_1, Y_2, we consider the sequence $\{\widetilde{X}_n\}_{n\in\mathbb{Z}}$ and $\{\widetilde{Y}_n\}_{n\in\mathbb{Z}}$ defined by

$$\widetilde{Y}_{n+1} = \frac{\widetilde{Y}_n^r}{\widetilde{Y}_{n-1}}\,,\ \text{where } \widetilde{Y}_1 = Y_1 \text{ and } \widetilde{Y}_2 = Y_1^r Y_2,$$

$$\widetilde{X}_{n+1} = \frac{\widetilde{X}_n^r + \widetilde{Y}_{n-1}}{\widetilde{X}_{n-1}}\,,\ \text{where } \widetilde{X}_1 = X_1 \text{ and } \widetilde{X}_2 = X_2\,.$$

The use of tildes is to distinguish the settings with and without coefficients. Let $\mathbb{P}$ be the tropical semifield $\mathrm{Trop}[Y_1, Y_2]$. The *framed r-Kronecker cluster algebra* $\widetilde{\mathcal{A}}(r,r)$ *with principal coefficients* is the $\mathbb{ZP}[\widetilde{X}_1, \widetilde{X}_2]$ algebra generated by the cluster variables $\{\widetilde{X}_n\}_{n\in\mathbb{Z}}$. Note that when we specialize to the case $Y_1 = Y_2 = 1$, we recover the classical r-Kronecker cluster algebra.

Corollary 4.19. *Consider the framed r-Kronecker cluster algebra $\widetilde{\mathcal{A}}(r,r)$ with principal coefficients, having initial cluster variables X_1, X_2 and coefficient variables Y_1, Y_2. For $n \geqslant 4$, the cluster variable $\widetilde{X}_n$ is given by*

$$\widetilde{X}_n = X_1^{-c_{n-1}} X_2^{-c_{n-2}} \sum_{\beta\in\mathcal{F}'(\mathcal{D}_n)} X_1^{r|\beta|_1} X_2^{r(c_{n-1}-|\beta|_2)} Y_1^{|\beta|_2} Y_2^{|\beta|_1}$$

and

$$X_{3-n} = X_2^{-c_{n-1}} X_1^{-c_{n-2}} \sum_{\beta\in\mathcal{F}'(\mathcal{D}_n)} X_2^{r|\beta|_1} X_1^{r(c_{n-1}-|\beta|_2)} Y_2^{-|\beta|_2} Y_1^{-|\beta|_1}\,.$$

Proof. Setting $\deg(X_i) = e_i$ and $\deg(Y_i) = (-1)^{i+1} r e_{3-i}$, it is known that the cluster variable $\widetilde{X}_n$ is a homogeneous Laurent polynomial by [6, Proposition 6.1]. Moreover, this degree is readily calculated from the recurrence relations on *g-vectors* to be $-c_{n-1}e_1 + c_n e_2$ for $n \geqslant 2$ and $c_{-n+3}e_1 - c_{-n+2}e_2$ for $n < 2$ (see, for example, [13, Subsection 4.1]). This determines the powers of Y_1 and Y_2 that must appear in each monomial term, yielding the above expansion formula directly from Corollary 2.4. $\square$

5. Bijection between compatible pairs and colored subpaths of Dyck paths

In this section, we prove a conjecture of Feiyang Lin that the map Φ, constructed by Lin and described in Definition 2.6, is a bijection between the collections of colored subpaths introduced by Lee–Schiffler [10] and the compatible pairs introduced by Lee–Li–Zelevinsky [8]. This shows the correspondence between the objects summed over by each set of authors in their expansion formulas for rank-2 cluster algebras. Specifically, we show that Lin's map is a bijection between collections β of colored Dyck subpaths in $\mathcal{F}'(\mathcal{D}_n)$ with a fixed $|\beta|_1$ and $|\beta|_2$ and compatible pairs on $\mathcal{C}_n$ consisting of $|\beta|_1$ vertical edges and $(c_{n-1} - |\beta|_2)$ horizontal edges.

5.1. Compatible pairs and Lin's map. The rank-2 cluster expansion formula given by Lee–Li–Zelevinsky [8] sums over certain subsets of edges of a maximal Dyck path, known as compatible pairs, which we discuss here. We will mainly work over $\mathcal{C}_n$, though sometimes we work in more generality. Let S_1 be a subset of the vertical edges of a maximal Dyck path $\mathcal{P}(a,b)$, and let S_2 be a subset of the horizontal edges of $\mathcal{P}(a,b)$.

In order to study compatible pairs, Li, Lee, and Zelevinsky [8] introduced the notion of the "shadow" of a set of edges. While they only defined shadows of subsets of vertical edges, we extend this notion to subsets of horizontal edges as well. These notions will be used throughout our construction of the bijection between collections of colored Dyck subpaths and compatible pairs.

Definition 5.1. *For a vertical edge $\nu \in S_2$ with upper endpoint w, we define its* local vertical shadow*, denoted $\mathrm{sh}(\nu; S_2)$, to be the set of horizontal edges in the shortest subpath $\overrightarrow{tw}$ of $\mathcal{P}(a,b)$ such that $|tw|_1 = r|\overrightarrow{tw} \cap S_2|_2$. Analogously, for a horizontal edge $\eta \in S_1$ with left endpoint u, we define its* local horizontal shadow*, denoted $\mathrm{sh}(\eta, S_2)$, to be the set of vertical edges in the shortest subpath $\overrightarrow{ut}$ of $\mathcal{P}(a,b)$ such that $|ut|_2 = r|\overrightarrow{ut} \cap S_1|_1$. If there is no such subpath $\overrightarrow{tw}$ or $\overrightarrow{ut}$, respectively, then we define the local vertical (resp. horizontal) shadow to be the entire set of horizontal (resp. vertical) edges in $\mathcal{P}(a,b)$.*

For $S \subseteq S_i$ where $i \in \{1,2\}$, let $\mathrm{sh}(S; S_i) = \bigcup_{\alpha \in S} \mathrm{sh}(\alpha; S_i)$, and write $\mathrm{sh}(S_i) := \mathrm{sh}(S_i; S_i)$.

Observation 5.2. It is a straightforward consequence of Definition 5.1 that for $S \subseteq S_1$, we have $|\,\mathrm{sh}(S)| = \min(b, r|S|)$. Similarly, for $S \subseteq S_2$, we have $|\,\mathrm{sh}(S)| = \min(a, r|S|)$

The expansion formula for cluster variables given by Lee, Li, and Zelevinsky has monomials corresponding to compatible pairs on $\mathcal{C}_n$. Their expansion formula works in the more general setting of elements of the greedy basis, which contains the cluster variables. For further details on the greedy basis, see [8]. We present their formula in the special case of cluster variables.

Theorem 5.3. [8, Theorem 1.11] *For each $n \geqslant 1$, the cluster variable X_n in $\mathcal{A}(r,r)$ is given by*

$$X_n = x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum_{(S_1,S_2)} x_1^{r|S_2|} x_2^{r|S_1|},$$

where the sum is over all compatible pairs (S_1, S_2) in $\mathcal{C}_n$.

Lin's map from collections of colored Dyck subpaths in $\mathcal{D}_n$ to compatible pairs in $\mathcal{C}_n$ is described in Definition 2.6. An example is shown in Figure 4. Lin conjectured that the map Φ is a bijection between the desired sets [13, Conjecture 3], which essentially involves showing that $\Phi(\beta)$ is indeed a compatible pair for every $\beta \in \mathcal{F}'(\mathcal{D}_n)$. We verify this claim in the next subsection. Lin made partial progress toward this conjecture by showing that it was sufficient to consider only compatible pairs satisfying a certain irreducibility condition [13, Proposition 4.8.4, Conjecture 4]. We proceed by a different approach than Lin, so our methods do not rely on this simplification.

In order to show the correspondence between the Lee–Schiffler and Lee–Li–Zelevinsky expansion formulas, one needs to show not only bijectivity between the sets summed over, but also that the resulting monomials correspond. Lin defined a weight function for collections of colored subpaths and for compatible pairs that keeps track of the exponents associated to these monomials.

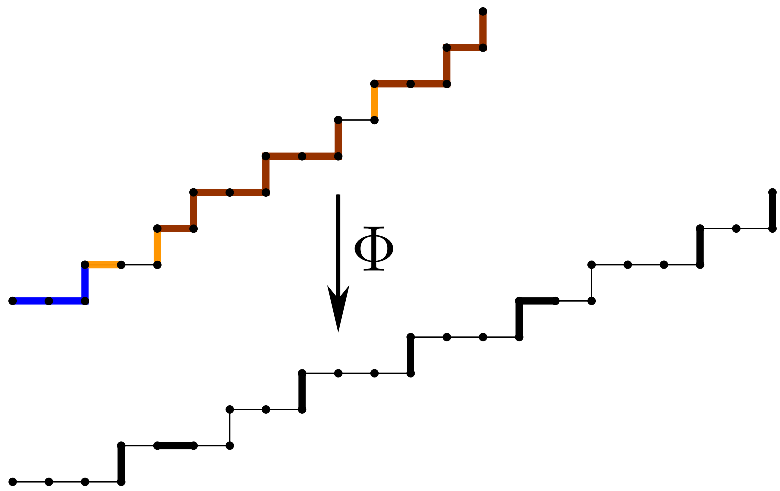

Figure 4. The top image depicts a collection of colored Dyck subpaths in $\mathcal{F}'(\mathcal{D}_6)$, identical to that at the bottom of Figure 3. The bottom image depicts the corresponding compatible pair (S_1, S_2) in $\mathcal{C}_6$, obtained by applying the map Φ to the collection of colored Dyck subpaths. The edges that are contained in either S_1 or S_2 are depicted by bold edges in the lower image. In particular, we have $S_1 = \{\eta_5, \eta_{15}\}$ and $S_2 = \{\nu_1, \nu_3, \nu_4, \nu_5, \nu_7, \nu_8\}$. Thus $\mathrm{wt}((S_1, S_2)) = X_1^{18} X_2^{6}$.

Definition 5.4. *We define the* weight *of a compatible pair* (S_1, S_2) *by*

$$\mathrm{wt}((S_1, S_2)) = X_1^{r|S_2|} X_2^{r|S_1|}$$

and the weight *of a collection of colored subpaths* $\beta \in \mathcal{F}'(\mathcal{D}_n)$ *by*

$$\mathrm{wt}_n(\beta) = X_1^{r|\beta|_1} X_2^{r(c_{n-2} - |\beta|_2)}.$$

Lin showed that Φ is a weight-preserving map from a superset of $\mathcal{F}(\mathcal{D}_n)$ to $\mathcal{F}(\mathcal{C}_n)$, and conjectured that it restricted to a bijection between $\mathcal{F}(\mathcal{D}_n)$ and $\mathcal{F}(\mathcal{C}_n)$. We prove this in the next subsection. We convert Lin's map into the setting of $\mathcal{F}'(\mathcal{D}_n)$ instead of $\mathcal{F}(\mathcal{D}_n)$ in order to make easier reference to the results of the previous section, though it is straightforward to show the equivalence between these two settings.

5.2. Proof of bijectivity. We now proceed to show that Lin's map Φ indeed takes every collection of colored subpaths in $\mathcal{F}'(\mathcal{D}_n)$ to a unique compatible pair on $\mathcal{C}_n$. It then follows from the work of Lee–Schiffler [10] and Lee–Li–Zelevinsky [8] that Φ is a bijection. For $2 \leqslant w \leqslant r-1$ and $m \geqslant 3$, we define $a_{m,w}$ to be the quantity $c_m - wc_{m-1}$. We can use the quantities $a_{m,w}$ to define the size of images of atomic colored paths under Φ, as well as their shadows.

Observation 5.5. It is readily deduced from the recursive definition of the sequence c_n that for $w, m \geqslant 1$, we have $ra_{m,w} = a_{m+1,w} + a_{m-1,w}$.

In order to establish that the conditions for compatibility correspond to the conditions for membership in $\mathcal{F}'(\mathcal{D}_n)$ via Φ, we first show that this is true for certain simple colored subpaths.

Definition 5.6. *We call a subpath of* $\mathcal{D}_n$ atomic *if it consists of a single edge, is blue, or is an* (m, w)*-brown path of the form* $\gamma(i, i + a_{m,w})$.

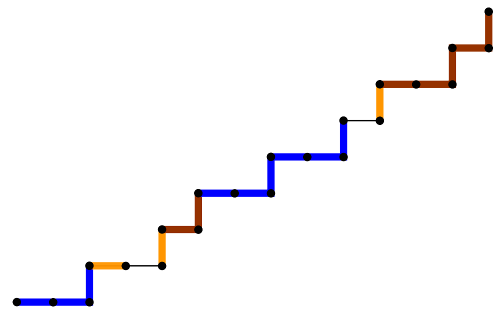

Figure 5. The figure depicts the decomposition of the collection β of subpaths from Figure 4 into its atomic components. The difference between this atomic decomposition and the original collection β is that the $(3,2)$-brown path $\gamma(2,5)$ has been decomposed into the $(3,2)$-brown path $\gamma(2,3)$ and the blue path $\gamma(3,5)$, which meet at the vertex v_3. Since these subpaths meet at a vertex, this collection of subpaths is not in $\mathcal{F}'(\mathcal{D}_n)$.

Lemma 5.7. *Any subpath of $\mathcal{D}_n$ of the form $\gamma(i,k)$ can be written uniquely as a union of atomic components meeting only at vertices such that*

(i) *every component except the last is (m,w)-brown for some choice of m and w, and*

(ii) *the last component is either blue or (m,w)-brown.*

Proof. If $\gamma(i,k)$ is blue, then we are done. Otherwise, $\gamma(i,k)$ is (m,w)-brown for some appropriate choice of m and w. In this case, we split the path into the atomic (m,w)-brown path $\gamma(i,i+a_{m,w})$ and the path $\gamma(i+a_{m,w},k)$. We can then repeat this process on the remaining portion $\gamma(i+a_{m,w},k)$ until the path is decomposed as desired. ☐

Note that when we decompose a path into its atomic components, adjacent components will necessarily overlap at a vertex. Thus, after this decomposition, the set of paths may no longer be non-overlapping, and hence not in $\mathcal{F}'(\mathcal{D}_n)$. An example is shown in Figure 5.

We now study the structure of the (m,w)-brown paths as Christoffel words. Recall the morphism $\lambda = \{E \mapsto E^{r-1}N, N \mapsto E^{r-2}N\}$.

Lemma 5.8. *The Christoffel word for an atomic (m,w)-brown path is given by $\lambda^{(m-2)}(E^{r-w-1}N)$, where λ^0 is the identity map, which has length $a_{m+1,w}$.*

Proof. Let $\gamma(i,k)$ be an atomic (m,w)-brown path, and let ρ denote the corresponding Christoffel word. Fix $i',k' \in \mathbb{Z}$ such that $v_i = \mu(w_{i'})$ and $v_k = \mu(w_{k'})$.

First note that if $w_{i'}$ is not a northwest corner, then by Observation 4.14 we have $k'-i' \leqslant r$. Thus, we have $m=3$ and $\rho = \lambda(E^{r-w-1}N)$ where $r-w-1 = k'-i'-1$, which is of the desired form.

Now suppose that $w_{i'} = v_{i''}$ is a northwest corner. We automatically have that $w_{k'} = v_{k''}$ is a northwest corner from Observation 4.14. We aim to show that $\gamma(i'',k'')$ is an $(m-1,w)$-brown path. Thus, the statement would follow from induction, since the Christoffel word corresponding to $\gamma(i,k)$ is given by applying λ to the word corresponding to $\gamma(i'',k'')$.

In order to show that $\gamma(i'',k'')$ is an $(m-1,w)$-brown path, we study the slopes from $s(w_{i'},w_{i'+j})$ for $1 \leqslant j \leqslant k'-i' = k-i$. It readily follows from the recurrence for the sequence c_n and the formula for μ given in Lemma 4.5 that

$$s(w_{i'},w_{i'+j}) - s(v_0,v_{c_n}) = s(\mu(w_{i'}),\mu(w_{i'+j})) - s(v_0,v_{c_{n+1}}) = s(v_i,v_{i+j}) - s(v_0,v_{c_{n+1}})\,.$$

Since $s(v_i, v_{i+j}) - s(v_0, v_{c_{n+1}}) < 0$ for $1 \leqslant j \leqslant k-i$ and $s(v_i, v_k) \geqslant s(v_0, v_{c_{n+1}})$, the same holds for $s(w_{i'}, w_{i'+j}) - s(v_0, v_{c_n})$. That is, the slope from $w_{i'}$ to any vertex on $\gamma(i'', k'')$ does not exceed that of the diagonal, except the slope from $w_{i'}$ to $w_{k'}$.

We now just need to determine $k'' - i''$, or equivalently, the number of vertical edges in $\gamma(i'', k'')$. Since $\gamma(i, k)$ as $a_{w,m}$ vertical edges, then we know $\gamma(i'', k'')$ has $a_{w,m}$ total edges. Since the $a_{w,m}$ uniquely determine w, m, we can conclude from the inductive hypothesis that $\gamma(i'', k'')$ is $(m-1, w)$-brown. Moreover, by Observation 5.5, we have

$$(r-1)a_{w-1,m} + r(a_{w,m} - a_{w-1,m}) = a_{w+1,m}\,,$$

so we can conclude that $\gamma(i, k)$ has length $a_{w+1,m}$. □

We are interested in the portion of the path spanned by the vertical shadow of the image of an (m, w)-brown path. We determine the structure of this portion of the path with the following result.

Corollary 5.9. *The $a_{m-1,w}$ edges preceding an (m, w)-brown path form an $(m-2, w)$-brown path or, when $a_{m-1,w} < r$, a path corresponding to the Christoffel word $E^{a_{m-1,w}-1}N$.*

Proof. By definition, the edge preceding an (m, w)-brown path is vertical, so the latter statement follows immediately.

We prove the first claim via induction on m. For $m \leqslant 5$, we note that $a_{m-1,w} \leqslant r-1$, and hence the preceding path is of the form $E^{a_{m-1,w}-1}N$. Let ρ denote the Christoffel word formed by the $a_{m-2,w}$ edges preceding an $(m-1, w)$-brown path. Then, by Lemma 5.8, we can obtain the word corresponding to the $a_{m-1,w}$ edges preceding an (m, w)-brown path by applying λ to ρ. By the inductive hypothesis and Lemma 5.8, we can conclude that $\lambda(\rho)$ is an $(m-2, w)$-brown path. □

Corollary 5.10. *Let $\gamma(i, k)$ be an (m, w)-brown path in $\mathcal{D}_n$, and let $(S_1, S_2) = \Phi\left(\{\gamma(i, k)\}\right)$. Then the shadow of S_2 has length $a_{m+1,w} + a_{m-1,w}$.*

Proof. By Lemma 5.8, it follows that S_2 consists of $a_{m,w}$ consecutive vertical edges. Thus, by Observation 5.2, the shadow will contain $\min(ra_{m,w}, c_{n-1})$ horizontal edges. Applying Observation 5.5, we see that $ra_{m,w} = a_{m+1,w} + a_{m-1,w}$. We then have by the bounds on w and m that

$$ra_{m,w} = a_{m+1,w} + a_{m-1,w} \leqslant (c_{m+1} - 2c_m) + c_{m-1} \leqslant c_{m+1} \leqslant c_{n-1}\,.$$

So we can conclude the length of the shadow is $a_{m+1,w} + a_{m-1,w}$. □

We can now establish that the image under Φ of an atomic (m, w)-brown path is a compatible pair in $\mathcal{C}_n$. As we will later see, this encodes most of the complexity of the compatibility conditions on $\mathcal{C}_n$.

Lemma 5.11. *Let $\gamma(i, k)$ be an atomic (m, w)-brown path in $\mathcal{D}_n$ and γ_j be one of the $a_{m-1,w}$ edges preceding v_i. Then $\Phi(\{\gamma(i, k), \gamma_j\})$ is a compatible pair.*

Proof. Let $S_2 = \Phi_2(\{\gamma(i, k)\})$. Let ρ' denote the path formed by the $a_{m-1,w}$ edges preceding v_i, and let $\rho = \sigma(\rho')$. It follows from Corollary 5.10 and Corollary 5.9 the shadow of Φ_2 spans a path of type $\rho\lambda^2(\rho)$.

Let υ be the path composed of $a_{m,w}$ north steps and $a_{m-1,w}$ west steps starting from the vertex immediately below $\Phi(v_i)$, defined as follows: for $i \geqslant 2$, the i-th north step of υ is $(i-1)r$ units to the west of the $(i-1)$-st edge in S_2. By definition, υ forms the eastern border of the shadow of each edge of S_2 except the last. Moreover, the position of west steps in υ is determined by the occurrence of subwords $E^{r-1}N$ and E^rN in $\lambda^2(\rho)$. From the definition of λ, it follows that Christoffel word corresponding to the 90 degree clockwise rotation of υ is precisely $\lambda^{-1}(\lambda^2(\rho)) = \lambda(\rho)$.

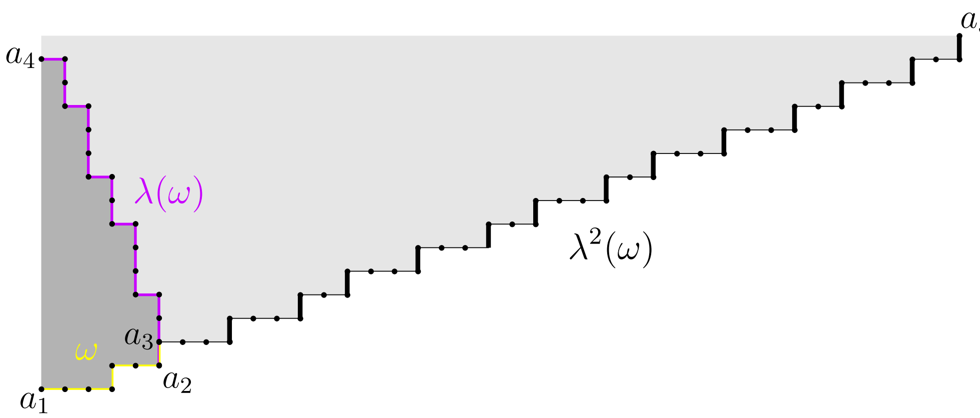

Figure 6. The yellow path from a_1 to a_3 and the black path from a_3 to a_5 constitute a portion of $\mathcal{D}$. The thick black edges depict those that are contained in $\Phi_2(\gamma)$, where γ is a $(6,2)$-brown path. The shadow of $\Phi_2(\gamma)$ contains all horizontal edges in the black and yellow paths. The purple path from a_2 to a_4 shows the left endpoint of the shadow of the thick vertical edges to its right. Note that the yellow and purple paths overlap in one edge.

We are now interested in the vertical distance from each horizontal edge η_i of ρ to that of $\lambda(\rho)$. This determines the maximum number M of horizontal edges to the right of and including η_i that can be included in S_1 while satisfying the compatibility conditions. Namely, this distance is the maximum height of the shadow at η_i, i.e. rM. Using the same inductive techniques as in Lemma 5.8, it is readily seen from the base case $\rho = E^wN$ that this distance is $r(a_{m-1,w} - i + 1)$ when $i > 1$. Hence, it is possible that any combination of these edges is contained in S_1. When $i = 1$, the distance is $ra_{m-1,w} - 1$. Since this is less than $ra_{m-1,w}$ but greater than $r(a_{m-1,w} - 1)$, it is not possible that all edges of ρ are contained in S_1, but it is possible that all but one are. That is, the pair (S_1, S_2) is compatible if and only if at least one edge of ρ is not contained in S_1. □

Example 5.12. Figure 6 illustrates the construction in the proof of Lemma 5.11. Observe that the purple path is a 90 degree counterclockwise rotation of a Dyck path. Letting $\rho = E^2NEN$ denote the Christoffel word for the yellow path, observe that the Christoffel word corresponding to the (90-degree clockwise rotation of the) purple path is given by $\lambda(\rho)$. Moreover, the Christoffel word corresponding to the black path is given by $\lambda^2(\rho)$. The vertical distance between the purple and yellow paths is precisely the maximum height of the shadow at each horizontal yellow edge such that the compatibility conditions are satisfied.

Lemma 5.13. *Let β consist of a single atomic path and, if β is (m,w)-brown, one of the $a_{m-1,w}$ edges preceding this path. Then $\Phi(\beta)$ is a compatible pair.*

Proof. We break into three cases based on the form of β. If β consists of a single edge, then $\Phi(\beta)$ has no vertical edges and hence is compatible. If β is (m,w)-brown, then this is precisely the result of Lemma 5.11.

Thus, the only remaining case is when $\gamma(i,k)$ is blue. Then it is the prefix of an atomic (m,w)-brown path, obtained by extending $\gamma(i,k)$ until its slope exceeds that of the diagonal. Let $\gamma(i,k')$ denote this atomic (m,w)-brown path, and let $\beta' = \{\gamma(i,k'), \eta_j\}$ where η_j is the (a_{m-1})-th edge preceding v_i. In the previous case, we have shown that $\Phi(\beta')$ is compatible. Note now that $\Phi_2(\beta) \subseteq \Phi_2(\beta')$ and $\mathrm{sh}(\Phi_2(\beta)) \subseteq \Phi_1(\beta')$. Therefore, the compatibility of $\Phi(\beta)$ follows directly from the compatibility of $\Phi(\beta')$. ☐

Now that we have handled the case of atomic paths, we show that we can determine the compatibility of the image of many colored subpaths paths by restricting to the compatibility conditions on each of its atomic components.

Definition 5.14. *Given a pair (S_1, S_2) on $\mathcal{P}(a_1, a_2)$, and let (S_1', S_2') be a pair on $\mathcal{P}(a_1', a_2')$. We define the* insertion of (S_1', S_2') into (S_1, S_2) at position (j_1, j_2) *to be the pair (S_1'', S_2'') on $\mathcal{P}(a_1 + a_1', a_2 + a_2')$ determined as follows:*

$$e_i \in S_k'' \iff \begin{cases} e_i \in S_k \text{ and } 1 \leqslant i \leqslant j_k\,, \text{ or} \\ e_{i-j_k} \in S_k' \text{ and } j_k < i \leqslant j_k + a_k'\,, \text{ or} \\ e_{i-a_k'} \in S_k \text{ and } j_k + a_k' < i \leqslant a_k + a_k'\,, \end{cases}$$

for $k \in \{1, 2\}$ and $(j_1, j_2) \in V(\mathcal{P}(a_1, b_1))$. Here, each e_i refers to the i^{th} horizontal or i^{th} vertical edge of the corresponding path, where the orientation of the edge is determined by the context.

Definition 5.15. *We say that a compatible pair (S_1, S_2) on $\mathcal{P}(a_1, a_2)$ has* non-spanning shadows *if*

$$|xw_{a+b}|_2 \geqslant r|xw_{a+b} \cap S_1| \text{ and } |w_0x|_1 \geqslant r|w_0x \cap S_2|$$

for all choices of vertices x in $\mathcal{P}(a_1, a_2)$.

Lemma 5.16. *Let (S_1, S_2) and (S_1', S_2') be compatible pairs on the paths $\mathcal{P}(a_1, a_2)$ and $\mathcal{P}(a_1', a_2')$, respectively, where (S_1', S_2') has non-spanning shadows. Then for any $(j_1, j_2) \in V(\mathcal{P}(a_1, a_2))$, the insertion (S_1'', S_2'') of (S_1', S_2') into (S_1, S_2) at position (j_1, j_2) is a compatible pair on $\mathcal{P}(a_1 + a_1', a_2 + a_2')$.*

Proof. Since (S_1', S_2') is a compatible pair on $\mathcal{P}(a_1', a_2')$ with non-spanning shadows, we have $rS_1' \leqslant a_2'$ and $rS_2' \leqslant a_1'$.

We can then see that for $e_i \in S_1''$ with $1 \leqslant i \leqslant j_1$, we have

$$|\,\mathrm{sh}(e_i; S_1'')| \leqslant |\,\mathrm{sh}(e_i; S_1)| + |\,\mathrm{sh}(e_1, S_1'')| < |\,\mathrm{sh}(e_i; S_1)| + a_2'\,.$$

Similarly, for $j_2 + a_2' < i \leqslant a_2 + a_2'$, we have

$$|\,\mathrm{sh}(e_i; S_2'')| \leqslant |\,\mathrm{sh}(e_{i-a_2'}; S_2)| + |\,\mathrm{sh}(e_{a_2'}, S_2'')| \leqslant |\,\mathrm{sh}(e_i; S_2)| + a_1'\,.$$

The lengths of the shadows at the other edges, with the corresponding shift in indices, is the same as in the original paths. Thus, the horizontal and vertical shadows will never intersect, so the pair is indeed compatible. ☐

This allows us to combine our results on atomic paths in order to handle any collection of subpaths in $\mathcal{F}'(\mathcal{D}_n)$

Theorem 5.17. *If $\beta \in \mathcal{F}'(\mathcal{D}_n)$, then $\Phi(\beta)$ is a compatible pair.*

Proof. Let $(S_1, S_2) = \Phi(\beta)$. We proceed by induction on the number of atomic components in β, which we denote by t. Note that if $t = 0$, then S_2 is empty and so (S_1, S_2) is compatible.

If β has one atomic component, the compatibility follows directly from Lemma 5.13.

If we add a singular edge to β, then the resulting pair is a subset of the original, and hence compatible. Otherwise, suppose we add an atomic component to β that appears to the left of all other atomic components. Then we can view the resulting path as the insertion of the atomic path (along with the portion to the left of it) into the previous compatible pair. It is straightforward to check for the atomic components that the corresponding compatible pair has non-spanning shadows. Since both paths involved in the insertion are compatible, we can conclude using Lemma 5.16 that $\Phi(\beta)$ is compatible. □

Lemma 5.18. *Every compatible pair in $\mathcal{C}_n$ is the image of some $\beta \in \mathcal{F}'(\mathcal{D}_n)$.*

Proof. We know from Theorem 5.17 that $\Phi(\beta)$ for $\beta \in \mathcal{F}'(\mathcal{D}_n)$ is a compatible pair in $\mathcal{C}_n$. Moreover, we know from Theorem 4.2 and Theorem 5.3 that $\mathcal{F}'(\mathcal{D}_n)$ and the set of compatible pairs in $\mathcal{C}_n$ are equinumerous, since both are the result of the substitution $X_1 = X_2 = 1$. Lastly, we know from the work of Lin [13, Proposition 4.7.3] that Φ is injective. Thus Φ is also surjective. □

Combining the results proven above along with work of Lee–Schiffler and Lee–Li–Zelevinsky, we can prove the bijectivity of the map Φ.

Proof of Theorem 2.8. By Theorem 5.17 and Lemma 5.18, we can see that Φ is a bijection from $\mathcal{F}'(\mathcal{D}_n)$ onto the set of compatible pairs in $\mathcal{C}_n$. Using the work of Lin [13, Proposition 4.7.3], we additionally see that Φ is weight-preserving. □

6. Quantization of colored Dyck subpaths

In Lee and Schiffler's work on expanding rank-2 cluster variables, they showed that the coefficients of the Laurent expansion could be obtained by taking sums over certain collections of colored Dyck subpaths. Each such collection was taken to have weight 1. In order to quantize this construction, we instead weight each collection by a power of a formal variable q. We then show that an analogous formula holds for rank-2 quantum cluster variables with an appropriate choice of q-weights, where setting $q = 1$ recovers Lee and Schiffler's formula.

As discussed by Lee–Li–Rupel–Zelevinsky [7, Section 3], the combinatorial formula for greedy basis elements of a rank-2 cluster algebra cannot be extended to the quantum setting by merely weighting compatible pairs by a power of q, since positivity of these elements can fail in general. However, Dylan Rupel [16, Corollary 5.4] established that the classical rank-2 formula for the quantum cluster variables, which are a proper subset of the greedy basis, given by Lee–Li–Zelevinsky [8] could be extended in this way. We proceed by applying the bijection established in the previous section to Rupel's expansion formula over weighted compatible pairs associated to quantum cluster variables.

An advantage of Theorem 2.11 is that it requires fewer computations compared to Rupel's formula in [16, Corollary 5.4]. Our formula requires $O(|\beta|^2)$ computations, where $|\beta|$ is the number of subpaths in $\beta \in \mathcal{F}'(\mathcal{D}_n)$. Rupel's formula requires $\binom{c_{n-1}+c_{n-2}}{2}$ computations, which is generally much larger. Moreover, without knowledge of the bijection between collections of colored subpaths and compatible pairs, it is unclear how to generate all compatible pairs. Naïvely, one must consider all collections of edges of $\mathcal{C}_n$ and check that the compatibility condition holds for each pair of edges. It thus seems more efficient to generate all collections in $\mathcal{F}'(\mathcal{D}_n)$ and compute their quantum weights using Definition 2.9 than to generate all compatible pairs on $\mathcal{C}_n$ and compute quantum weights using [16, Corollary 5.4].

In our proof of Theorem 2.11, we translate each compatible pair into a finite word so that we can refer to the language of combinatorics on words. We will work over the alphabet $A = \{h, v, H, V\}$, with A^* denoting the set of finite words on A. For the purposes of this section, we represent a compatible pair by a word in A^*. The letters h and H (resp. v and V) represent horizontal (resp. vertical) edges, with the capital letter denoting those edges in S_1 (resp. S_2).

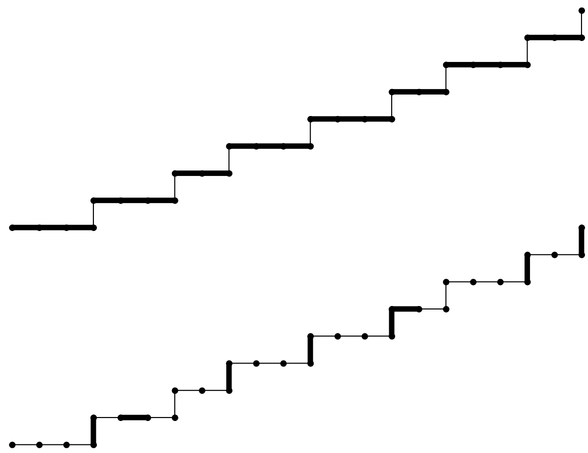

FIGURE 7. The top path depicts the compatible pair on $\mathcal{C}_6$ obtained by applying Φ to $\beta_\varnothing$ on $\mathcal{D}_6$, which has $S_1 = \{\eta_1, \eta_2, \ldots, \eta_{21}\}$ and $S_2 = \varnothing$. The bottom path depicts the compatible pair obtained by applying Φ to the collection of colored Dyck subpaths shown in Figure 4.

EXAMPLE 6.1. The word in A^* corresponding to the top compatible pair in Figure 7 is

$$H^3vH^3vH^2vH^3vH^3vH^2vH^3vH^2v\,.$$

The word in A^* corresponding to the bottom compatible pair in Figure 7 is

$$h^3VhHhvh^2Vh^3Vh^3VHhvh^3Vh^2V\,.$$

We define a morphism $f : \mathbb{Z}A^* \to \mathbb{Z}$, where $\mathbb{Z}A^*$ is the group of formal $\mathbb{Z}$-sums of words in A^*. The function w_q is defined on words of length 2 in A^* as follows:

$$w_q(hv) = w_q(Hv) = w_q(hV) = 1\,, \quad w_q(Hh) = w_q(vV) = r\,, \quad w_q(VH) = r^2 - 1\,,$$

and for $x, y \in A$, $w_q(xy) = -w_q(yx)$. Note that in particular, this implies that $w_q(hh) = w_q(HH) = w_q(vv) = w_q(VV) = 0$. The function w_q naturally extends to formal $\mathbb{Z}$-sums of words of length 2 on A. It is then extended to words of larger length by taking the formal sum over all length 2 (not necessarily contiguous) subwords with multiplicity, and again extended naturally to formal $\mathbb{Z}$-sums of any words on A. We sometimes apply w_q to a compatible pair; in this case, we interpret w_q as being applied to the corresponding word on A. We refer to w_q as the *quantum weight* of a word or compatible pair. Computing the value of w_q on the word associated to a compatible pair in this way is essentially calculating Rupel's weighting on compatible pairs associated to quantum cluster variables [16].

EXAMPLE 6.2. Let t_1 (resp. t_2) denote the word in A^* corresponding to the top (resp. bottom) compatible pair in Figure 7, computed in Example 6.1. Note that here we have $r = 3$. Looking at all the length-2 subwords of t_1, we can see that t_1 has 98 instances of the subword Hv and 70 instances of the subword vH. The only other length-2 subwords of t_1 are HH and vv, and we have $w_q(HH) = w_q(vv) = 0$. Thus, we can compute

$$w_q(t_1) = 98w_q(Hv) + 70w_q(vH) = 98 \cdot 1 + 70 \cdot (-1) = 28\,.$$

Via a similar computation, we have

$$\begin{aligned} w_q(t_2) &= 69w_q(hV) + 45w_q(Vh) + 21w_q(Hh) + 17w_q(hH) + 7w_q(vV) + 5w_q(Vv) \\ &\quad + 5w_q(VH) + 7w_q(HV) + 19w_q(hv) + 19(vh) + 3w_q(Hv) + w_q(vH) \\ &= 69 - 45 + 21 \cdot 3 - 17 \cdot 3 + 7 \cdot 3 - 5 \cdot 3 + 5 \cdot 8 - 7 \cdot 8 + 19 - 19 + 3 - 1 \\ &= 28 \end{aligned}$$

As we will show, there are more efficient methods for computing these weights than looking at all length-2 subwords.

For a word $u \in A^*$ and a letter $x \in A$, let $(u)_x$ denote the number of instances of x in u. In order to find a compact formula for the weights corresponding to collections of subpaths

Lemma 6.3. *Let $\beta_\varnothing \in \mathcal{F}'(\mathcal{D}_n)$ be the empty collection of colored Dyck subpaths on $\mathcal{D}_n$. For all $n \geqslant 3$, we have*

$$w_q\left(\Phi_n(\beta_\varnothing)\right) = c_{n-1} + c_{n-2} - 1\,.$$

Proof. We proceed by induction on n. For the base case $n = 3$, we have

$$w_q\left(\Phi_3(\beta_\varnothing)\right) = w_q(H^r v) = r w_q(Hv) + \binom{r}{2} w_q(HH) = r = c_3 + c_2 - 1\,.$$

We now proceed to the inductive step. Let ψ be the morphism $\{H \mapsto Hv, v \mapsto v\}$. Observe that for a word $u \in \{H, v\}^*$ that starts with H, ends with v, and has no consecutive instances of v, we have

$$w_q(\psi(u)) = w_q(u) + (u)_H\,.$$

Let u_j be the word associated to $\Phi_j(\beta_\varnothing)$. Then, by Remark 3.3 and Lemma 4.5, u_{n+1} can be obtained by applying the morphism $\chi = \{H \mapsto H^r v, v \mapsto H^{r-1} v\}$ to $\psi^{-1}(u_n)$. We can readily calculate

$$\begin{aligned} w_q(\chi(HH)) &= w_q(H^r v H^r v) = 2r = w_q(HH) + 2w_q(\chi(H))\,, \\ w_q(\chi(vv)) &= w_q(H^{r-1} v H^r v) = 2r - 2 = w_q(vv) + 2w_q(\chi(v))\,, \\ w_q(\chi(Hv)) &= w_q(H^r v H^{r-1} v) = 2r = w_q(Hv) + w_q(H^r v) + w_q(H^{r-1} v)\,, \\ w_q(\chi(vH)) &= w_q(H^{r-1} v H^r v) = 2r - 2 = w_q(vH) + w_q(H^{r-1} v) + w_q(H^r v)\,. \end{aligned}$$

Moreover, $\psi^{-1}(u_n)$ has $c_{n-1} - c_{n-2}$ instances of H and c_{n-2} instances of v. Therefore, we have

$$\begin{aligned} w_q(\Phi_{n+1}(\beta_\varnothing) &= w_q(\chi(\psi^{-1}(u_n) \\ &= w_q(\psi^{-1}(u_n)) + (c_{n-1} - c_{n-2}) w_q(\chi(H)) + c_{n-2} w_q(\chi(v)) \\ &= (\psi(u_n) - c_{n-2}) + r(c_{n-1} - c_{n-2}) + (r-1)c_{n-2} \\ &= (c_{n-1} + c_{n-2} - 1 - c_{n-2}) + c_n \\ &= c_n + c_{n-1} + 1\,. \end{aligned}$$

□

Having calculated the weight of the empty collection of paths in $\mathcal{F}'(\mathcal{D}_n)$, we wish to calculate the quantum weight when we add in colored subpaths. Note that for the word corresponding to the compatible pair, this involves swapping out certain instances of H for h and v for V. We now calculate how such a substitution affects the quantum weight of the word.

Lemma 6.4. *Let $t_1, u_1, t_2, u_2, \ldots, t_m, u_m$ be words corresponding to compatible pairs. Furthermore, suppose that $t_i, u_i \in \{H, v\}$. Let σ be the morphism $\{H \mapsto h, v \mapsto V\}$. Then we have*

$$w_q\left(\prod_{i=1}^{s} t_i\sigma(u_i)\right) = w_q\left(\prod_{i=1}^{s} t_iu_i\right) + \sum_{i=1}^{s}\sum_{j=1}^{s}(-1)^{\mathbb{1}_{i<j}}\left(r(t_j)_(u_i)_h + \big(r(t_j)_v - r^2(t_j)_H\big)(u_i)_V\right).$$

Proof. Since $w_q(hV) = w_q(Hv)$, we have $w_q(u_i) = w_q(\sigma(u_i))$ for all i. Hence, we need only to calculate the change in its value under w_q after applying σ to the length-2 subwords with one letter from a u_i and the other from a t_j.

Let U_i denote the value under w_q of the sum over all length-2 subwords of $\prod_{i=1}^{s} t_i\sigma(u_i)$ with one letter in $\sigma(u_i)$ and the other from some t_j. We then have

$$\begin{aligned} U_i = \sum_{j=1}^{s}(-1)^{\mathbb{1}_{i<j}}\Big(&(t_j)_H(u_i)_h\left(w_q(Hh) - w_q(HH)\right) \\ &+ (t_j)_H(u_i)_V\left(w_q(HV) - w_q(Hv)\right) \\ &+ (t_j)_v(u_i)_h\left(w_q(vh) - w_q(vH)\right) \\ &+ (t_j)_v(u_i)_V\left(w_q(vV) - w_q(vv)\right)\Big) \\ = \sum_{j=1}^{s}(-1)^{\mathbb{1}_{i<j}}&\left(r\big((t_j)_H(u_i)_h + (t_j)_v(u_i)_V\big) - r^2(t_j)_H(u_i)_V\right). \end{aligned}$$

□

Applying the previous general result about compatible pairs to the specific case of $\mathcal{C}_n$ and using the connection between $\mathcal{F}'(\mathcal{D}_n)$ and compatible pairs on $\mathcal{C}_n$, we can now prove the quantum cluster variable expansion formula.

Proof of Theorem 2.11. Adding a path to $\beta \in \mathcal{F}'(\mathcal{D}_n)$ corresponds to applying the morphism σ from Lemma 6.4 to the appropriate portion of the associated compatible word. Note that $|\beta_i|_2 = (\beta_i)_h$ and $|\beta_i|_1 = (\beta_i)_V$. Similarly, $|\overline{\beta_j}|_2 = (\overline{\beta_j})_H$ and $|\overline{\beta_j}|_1 = (\overline{\beta_j})_v$. It follows from Lemma 6.4, Lemma 6.3, and the definition of w_q for words in A^* that $w_q(\beta) = \gamma_\omega + \beta_\omega$, where the terms on the right-hand side are those in [16, Corollary 5.4]. Thus, the expansion formula can be deduced directly from [16, Corollary 5.4]. □

Example 6.5. Let $\beta = \{\beta_1, \ldots, \beta_6\} \in \mathcal{F}'(\mathcal{D}_6)$ be the collection of colored Dyck subpaths shown in Figure 4. Then we have $\overline{\beta_0} = \overline{\beta_1} = \overline{\beta_6} = \varnothing$, $\overline{\beta_2} = \{\alpha_4\}$, $\overline{\beta_3} = \{v_2\}$, $\overline{\beta_4} = \{\alpha_{14}\}$, and $\overline{\beta_5} = \{v_6\}$. Applying Theorem 2.11, we have

$$\begin{aligned} w_q(\beta) &= (c_5 + c_4 - 1) + 3(15-4) - 9(5-1) + 3(5-1) + 3(6-13) \\ &\quad - 9(2-4) + 3(2-4) \\ &= 28\,. \end{aligned}$$

which confirms the second calculation in Example 6.2.

7. Further directions

Many of our methods rely on the highly structured nature of the maximal Dyck paths $\mathcal{C}_n$ and $\mathcal{D}_n$. We use this to better understand the conditions for a set of edges to form a compatible pair on $\mathcal{C}_n$, in particular deriving a criterion for compatibility in terms of the sequences of consecutive vertical edges. While $\mathcal{C}_n$ is the relevant choice of maximal

Dyck path for the cluster variables, compatible pairs over arbitrary maximal Dyck paths were studied by Lee, Li, and Zelevinsky [8] in their work on the greedy basis. One way to study compatibility is in terms of forbidden edge sets, i.e. the minimal subsets of edges which violate the compatibility condition for compatible pairs. It is easy to verify that for the staircase Dyck path $\mathcal{P}(a,a)$ and $r=2$, a set of edges is compatible if and only if it does not contain a horizontal edge and the vertical edge immediately following it. From the proof of the bijectivity of Φ, it follows that on $\mathcal{C}_n$ the forbidden edge sets are the images under Φ of an (m,w)-brown path and the $c_{m-1}-wc_{m-2}$ edges preceding it. It could be interesting to study whether the criterion for compatibility can also be reduced for other families of maximal Dyck paths.

Having established that Lin's map Φ is a bijection, we now have an explicit connection between the combinatorial objects in two of the manifestly positive formulas for rank-two cluster variables. A third manifestly positive formula was given by Cheung–Gross–Muller–Musiker–Rupel–Stella–Williams [3] and sums over broken lines in a rank-two scattering diagram. In order to complete the unification of these known rank-two objects, it would be interesting to find a bijection between compatible pairs and broken lines, as suggested by the authors of [3]. This problem is still open even in the cluster variable case, but could be considered more generally for the entire greedy basis.

While positivity fails in general for the quantum greedy basis [7, Section 3], it would be interesting to know under what conditions positivity holds. Rupel's formula [16, Corollary 5.4] establishes the positivity property for the quantum cluster variables, but perhaps this is a special case of a more general phenomenon. If so, these elements may also admit a quantum weighting of the associated compatible pairs, similar to that given by Rupel.

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