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Abstract Using critical conjugacy classes, we find a new criterion for constituents in Kronecker products of spin characters of the double covers of the symmetric and alternating groups. This is applied together with earlier results on spin characters to obtain constituents in Kronecker products of characters of the symmetric groups. Via this tool, we make progress on the Saxl conjecture; this claims that for a triangular number n, the square of the irreducible character of the symmetric group S_n labelled by the staircase contains all irreducible characters of S_n as constituents. With the new criterion we deduce a large number of constituents in this square which were not detected by other methods, notably all double-hooks. The investigation of Kronecker products of spin characters also inspires a spin variant of Saxl's conjecture.

1. Introduction

Even though tensor products of complex representations and the corresponding Kronecker products of characters of the symmetric group S_n have been studied for a long time, their decomposition is still an elusive central open problem in the field. A new benchmark for this is a conjecture by Heide, Saxl, Tiep and Zalesskii [12] which says that for any $n \neq 2, 4, 9$ there is always an irreducible character of S_n whose square contains all irreducible characters. For triangular numbers an explicit candidate was suggested by Saxl in 2012; denoting the irreducible character of S_n associated to a partition λ of n by $[\lambda]$, his conjecture is the following.

Saxl's Conjecture. *Let $\rho_k = (k, k-1, \ldots, 2, 1)$ be the staircase partition of $n = k(k+1)/2$. Then the Kronecker square $[\rho_k]^2$ contains all irreducible characters of S_n as constituents.*

This has inspired a lot of recent research. In particular, in the work of Pak, Panova and Vallejo [25] and Ikenmeyer [15] many constituents of the square $[\rho_k]^2$ have been identified, notably those to hooks and to partitions comparable to the staircase in dominance order.

Here, we will take a very different approach and show that results on the spin characters of a double cover $\tilde{S}_n$ of the symmetric group S_n can be fruitfully applied towards this problem. Indeed, it will turn out to be useful to consider also the spin characters of the double cover $\tilde{A}_n$ of the alternating group A_n. An important link is made as a consequence of an identity found in an earlier investigation of homogeneous

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Kronecker products. While in [4] it was shown that there are no nontrivial homogeneous products of S_n-characters, for the double covers $\tilde{S}_n$ of the symmetric groups nontrivial homogeneous spin products do occur for all triangular numbers n [5]. In [2], also nontrivial homogeneous mixed products of complex characters for the double covers $\tilde{S}_n$ were found, i.e. products of a non-faithful character of $\tilde{S}_n$ with a spin character. This information on nontrivial homogeneous Kronecker products is crucially used here towards obtaining many constituents in the square $[\rho_k]^2$.

We give a brief overview on the following sections. In Section 2, we collect the information and notation on the irreducible characters for the symmetric and alternating groups and their double covers to be used later. Here we already draw attention to special conjugacy classes of these groups, and we prove a general lemma that uses critical properties of conjugacy classes for identifying constituents in certain Kronecker products.

In Section 3 we first recall properties of the basic spin character as well as an important link between the faithful and non-faithful characters of $\tilde{S}_n$ labelled by the staircase partition ρ_k. This leads to crucial observations relating $[\rho_k]^2$ to spin characters and the sum of all hook characters. As an easy application we obtain all hook characters in $[\rho_k]^2$. We also set the scene for later applications by discussing products with the sum of all hook characters.

Section 4 starts with a short argument for the criterion given in [25, "Main Lemma"], and then moves on to the main results on constituents in certain spin character products (Theorems 4.3 and 4.12). These results are then applied towards the Saxl conjecture. Powerful new criteria for constituents in $[\rho_k]^2$ are given in Corollary 4.4; its usefulness is illustrated by providing several families of constituents. For characters to 2-part partitions this also involves the investigation of unimodality properties of the number of k-bounded partitions with distinct parts. As a main new family, characters to all double-hooks are shown to occur as constituents in Theorem 4.10.

The final Section 5 discusses a spin variant of Saxl's conjecture, involving the "spin staircase" $(2k-1, 2k-3, \ldots, 3, 1)$. Also spin variants of the conjecture by Heide, Saxl, Tiep and Zalesskii are presented.

Computational data for illustrating the efficiency of the new results or for finding conjectures have been obtained using Maple (together with Stembridge's SF package) and GAP [34].

2. Preliminaries

We denote by $P(n)$ the set of partitions of n, i.e. weakly decreasing sequences of nonnegative integers summing to n. For a partition $\lambda \in P(n)$, $l(\lambda)$ denotes its length, i.e. the number of positive parts of λ. The set of partitions of n into odd parts only is denoted by $O(n)$, and the set of partitions of n into distinct parts is denoted by $D(n)$. We write $D^+(n)$ resp. $D^-(n)$ for the sets of partitions λ in $D(n)$ with $n - l(\lambda)$ even resp. odd; the partition λ is then also called even resp. odd.

We write S_n for the symmetric group on n letters, and $\tilde{S}_n$ for one of its double covers; so $\tilde{S}_n$ is a non-split extension of S_n by a central subgroup $\langle z \rangle$ of order 2. It is well-known that the representation theory of these double covers is 'the same' for all representation theoretical purposes. For background on the properties of the double cover groups and its complex representations, the reader is referred to Stembridge's article [32]. In line with [32], we take the double cover with explicit presentation given as follows:

$$\tilde{S}_n = \left\langle t_1, \ldots, t_{n-1}, z \;\middle|\; \begin{array}{l} z^2 = 1, t_i^2 = z, \text{ for } 1 \leqslant i \leqslant n-1; \\ t_{i+1}t_it_{i+1} = t_it_{i+1}t_i, \text{ for } 1 \leqslant i \leqslant n-2; \\ t_it_j = zt_jt_i \text{ for } 1 \leqslant i,j \leqslant n-1, |i-j| \geqslant 2 \end{array} \right\rangle.$$

For $\lambda \in P(n)$, we write $[\lambda]$ for the corresponding irreducible character of S_n; this is identified with the corresponding non-faithful character of $\tilde{S}_n$. When we evaluate $[\lambda]$ on an element of S_n of cycle type $\mu \in P(n)$, we simply write $[\lambda](\mu)$ for the corresponding value. For background on the representation theory of the symmetric groups, the reader is referred to [16, 17]. The spin characters of $\tilde{S}_n$ are those that do not have z in their kernel. For details on the theory of spin characters resp. for some results we will need in the sequel, besides Stembridge's article [32] already recommended above, we refer to [1, 13, 20, 21, 30]. Below, we collect some of the necessary notation and some results from [32] that are crucial in later sections. For $n \leqslant 3$, the irreducible $\tilde{S}_n$-characters are lifted from S_n; hence for results on the double covers we will always assume that $n \geqslant 4$.

Let $\lambda \in P(n)$. Then the set C_λ of elements in $\tilde{S}_n$ projecting to elements in S_n of cycle type λ splits into two $\tilde{S}_n$-conjugacy classes if and only if $\lambda \in O(n) \cup D^-(n)$, otherwise it forms only one $\tilde{S}_n$-conjugacy class; the latter case happens exactly if each element x in the set is conjugate to xz (we then speak of a non-split class). When C_λ splits, a specific labelling $C_\lambda^\pm$ for the two $\tilde{S}_n$-classes in C_λ is made; we leave out the details here (see [30, 32]). Note that any spin character vanishes on the non-split classes, as the spin representations are $-\mathrm{id}$ on the central element z. Thus only the values on the split classes will be considered for spin characters; in this case, for a given spin character the values on the two classes differ only by a sign.

Let sgn denote the sign character of $\tilde{S}_n$, inflated from the sign character of S_n. A character χ of $\tilde{S}_n$ is called *self-associate* if $\mathrm{sgn} \cdot \chi = \chi$, otherwise we have a pair of *associate* characters, $\chi \neq \mathrm{sgn} \cdot \chi$.

In 1911, Schur has proved the following classification result [30], giving a complete list of irreducible complex spin characters of $\tilde{S}_n$. For each $\lambda \in D^+(n)$ there is a self-associate spin character $\langle\lambda\rangle$, and for each $\lambda \in \mathcal{D}^-(n)$ there is a pair of associate spin characters $\langle\lambda\rangle_+$ and $\langle\lambda\rangle_-$. When we want to consider a spin character labelled by λ, and it is not specified whether λ is in D^+ or D^-, we write $\langle\lambda\rangle_{(\pm)}$. The spin characters labelled by $\lambda = (\ell_1, \ldots, \ell_m) \in D(n)$ take the following values on $\sigma_\alpha \in C_\alpha^+$:

$$\begin{aligned} \langle\lambda\rangle_+(\sigma_\alpha) &= \langle\lambda\rangle_-(\sigma_\alpha) && \text{for } \alpha \in O(n), \lambda \in D^-(n) \\ \langle\lambda\rangle_{(\pm)}(\sigma_\alpha) &= 0 && \text{for } \alpha \in D^-(n), \lambda \neq \alpha \\ \langle\lambda\rangle_+(\sigma_\lambda) &= -\langle\lambda\rangle_-(\sigma_\lambda) = i^{(n-m+1)/2}\sqrt{\frac{\prod_j \ell_j}{2}} && \text{for } \lambda \in D^-(n). \end{aligned}$$

The values $\langle\lambda\rangle_{(\pm)}(\sigma_\alpha)$, $\alpha \in O(n)$, are integers determined by a recursion rule akin to the Murnaghan–Nakayama rule, which is due to Morris [20, 21]. If $\alpha = (\alpha_1, \ldots, \alpha_k)$, then in this rule, instead of removing hooks of lengths α_j, $1 \leqslant j \leqslant k$, so-called *bars* of this length are removed from λ, and apart from signs, extra 2-powers appear. Similar as in the classical case of Schur functions, the values $\langle\lambda\rangle_{(\pm)}(\sigma_\alpha)$, $\alpha \in O(n)$, appear crucially in the expansion of the Schur Q-function Q_λ into the power sum functions p_α, $\alpha \in O(n)$.

For later purposes, it will turn out to be useful to define

$$\langle\hat{\lambda}\rangle := \begin{cases} \langle\lambda\rangle & \text{if } \lambda \in D^+(n) \\ \langle\lambda\rangle_+ + \langle\lambda\rangle_- & \text{if } \lambda \in D^-(n) \end{cases}$$

and we set

$$\langle\lambda\rangle^{(2)} := \langle\lambda\rangle_{(\pm)} \cdot \langle\hat{\lambda}\rangle.$$

An important role in the theory is taken by the *basic spin characters* $\langle n\rangle_{(\pm)}$. Their values on elements $\sigma_\alpha \in C_\alpha^+$ are given explicitly as follows (known already to Schur [30]).

When n is odd,

$$\langle n\rangle(\sigma_\alpha) = 2^{(l(\alpha)-1)/2} \quad \text{for } \alpha \in O(n)\ ,$$

and when $n = 2k$ is even,

$$\langle n\rangle_\pm(\sigma_\alpha) = \begin{cases} 2^{(l(\alpha)-2)/2} & \text{for } \alpha \in O(n) \\ \pm i^k\sqrt{k} & \text{for } \alpha = (n). \end{cases}$$

All other values of $\langle n\rangle_{(\pm)}$ are zero.

We will also need some information about the characters of the alternating groups A_n and their double covers $\tilde{A}_n$.

The classification of the irreducible A_n-characters is derived from that for S_n. We obtain all irreducible characters of A_n as constituents in the restriction of the characters $[\lambda]$ as follows (see [17]).

Let $\mu \in P(n)$, and let μ' be the transposed partition to μ. Let $h(\mu) = (h_1, \dots, h_d)$ be the partition of principal hook lengths $h_1, \dots, h_d$ in μ; then d is the Durfee length $d(\mu)$ of μ.

When $\mu \neq \mu'$, $[\mu]\downarrow_{A_n} = [\mu']\downarrow_{A_n} = \{\mu\} = \{\mu'\} \in \mathrm{Irr}(A_n)$.

When $\mu = \mu'$, $[\mu]\downarrow_{A_n} = \{\mu\}_+ + \{\mu\}_-$; the characters $\{\mu\}_\pm$ are conjugate irreducible characters of A_n. The two characters $\{\mu\}_\pm$ differ only on the two conjugacy classes of cycle type $h(\mu)$; note that $[\mu]$ takes the value $e_\mu = (-1)^{(n-d)/2}$ on elements of this cycle type. With $\sigma^\pm_{h(\mu)}$ being appropriate representatives of the conjugacy classes of A_n of this cycle type, the character values are given as

$$\{\mu\}_+(\sigma^\pm_{h(\mu)}) = \frac{1}{2}\left(e_\mu \pm \sqrt{e_\mu \prod_{j=1}^d h_j}\right),$$

and similarly, with interchanged signs, for $\{\mu\}_-$. In particular, these two conjugate characters are the only irreducible characters that differ on the elements $\sigma^\pm_{h(\mu)}$.

One also obtains the classification of irreducible spin characters of $\tilde{A}_n$ from that of the spin characters of $\tilde{S}_n$ (see [13, 30, 32]).

For each $\lambda \in D^-(n)$, the restriction to $\tilde{A}_n$ gives one irreducible spin character

$$\langle\lambda\rangle_+\downarrow_{\tilde{A}_n} = \langle\lambda\rangle_-\downarrow_{\tilde{A}_n} = \langle\langle\lambda\rangle\rangle\ .$$

Dually, for each $\lambda = (\ell_1, \dots, \ell_m) \in D^+(n)$, the restriction to $\tilde{A}_n$ gives two conjugate irreducible spin characters

$$\langle\lambda\rangle\downarrow_{\tilde{A}_n} = \langle\langle\lambda\rangle\rangle_+ + \langle\langle\lambda\rangle\rangle_-\ .$$

If $\sigma \in \tilde{A}_n$ projects to cycle type λ, then for the difference of the values on σ we have (with the sign depending on the choice of associates)

$$\Delta^\lambda(\sigma) = \langle\langle\lambda\rangle\rangle_+(\sigma) - \langle\langle\lambda\rangle\rangle_-(\sigma) = \pm i^{(n-m)/2}\sqrt{\prod_{j=1}^m \ell_j}\ .$$

If $\sigma \in \tilde{A}_n$ does not project to type λ, then $\Delta^\lambda(\sigma) = 0$ [30]. Let σ project to type λ. If $\lambda \in D \cap O(n)$, and $\tau \in \tilde{A}_n$ is $\tilde{S}_n$-conjugate to σ but not in $\sigma^{\tilde{A}_n}$, then $\Delta^\lambda(\sigma) = -\Delta^\lambda(\tau)$

(see [32, remark after 7.5]). Note that $\langle\lambda\rangle(\sigma) = 0$ if $\lambda \in D^+(n) \smallsetminus (D \cap O(n))$; thus we can compute all the character values also in the case where $\lambda \in D^+(n)$.

For elements projecting to other types, the values of the two conjugate characters $\langle\langle\lambda\rangle\rangle_\pm$ are the same; in particular, the characters $\langle\langle\lambda\rangle\rangle_\pm$ vanish on classes projecting to a cycle type $\mu \neq \lambda$ that is not in $O(n)$. We emphasize that when $\lambda \in D \cap O(n)$, both of these two conjugate characters have different values on the two $\tilde{A}_n$-classes that come from one $\tilde{S}_n$-class projecting to cycle type λ, and they are the only irreducible spin characters of $\tilde{A}_n$ with this property.

We also note the following special situation. When $\lambda = \lambda'$, the principal hook length partition $h(\lambda)$ is in $D \cap O(n)$, and then the two doubling classes of $\tilde{S}_n$ projecting to cycle type $h(\lambda)$ each split a second time, into two classes of $\tilde{A}_n$. As pointed out above, the spin characters $\langle\langle h(\lambda)\rangle\rangle_+, \langle\langle h(\lambda)\rangle\rangle_-$ differ on the two classes of $\tilde{A}_n$ contained in one $\tilde{S}_n$-class but projecting onto different A_n-classes of cycle type $h(\lambda)$. Note that also the non-faithful characters $\{\lambda\}_\pm$ differ on these classes.

The alternating groups A_n and the double covers $\tilde{S}_n$ and $\tilde{A}_n$ exhibit a special phenomenon that we want to highlight here.

Definition 2.1. *Let G be a finite group, $I \subset \operatorname{Irr}(G)$. A pair of conjugacy classes x^G and y^G is said to be* critical *for I if $\chi(x) \neq \chi(y)$, for $\chi \in I$, but $\chi(x) = \chi(y)$ for all $\chi \in \operatorname{Irr}(G) \smallsetminus I$.*

We call a critical pair x^G and y^G for $I = \{\chi_1, \chi_2\} \subset \operatorname{Irr}(G)$ a pair of detecting classes *for χ_1, χ_2 if $\chi_1(x) - \chi_1(y) = \chi_2(y) - \chi_2(x)$.*

Of course, the set I above should be taken of small size to give interesting information.

Remarks and Examples 2.2.

(i) If there is a pair of detecting classes for χ_1, χ_2, then $\chi_2 = \overline{\chi_1}$, or the characters are both real.

(ii) For A_n, for each symmetric partition μ of n, the pair of classes $\sigma^\pm_{h(\mu)}$ is a detecting pair for $\{\mu\}_\pm$.

(iii) For $\tilde{S}_n$, for each $\lambda \in D^-(n)$, the pair $C^\pm_\lambda$ of classes of elements projecting to cycle type λ is a detecting pair for $\langle\lambda\rangle_\pm$.

(iv) For $\tilde{A}_n$, for each $\lambda \in D^+(n)$, the two pairs of classes of elements of $\tilde{A}_n$ projecting to cycle type λ and belonging to one $\tilde{S}_n$ class are detecting pairs for $\langle\langle\lambda\rangle\rangle_\pm$.

(v) While the situations above are the ones used in the later sections, it should be noted that there are many more such instances, and we mention just a few examples. For $G = \mathrm{GL}(3,2)$, the pair of classes of elements of order 7 is detecting for the pair of characters of degree 3. For $G = \mathrm{PSL}(2,11)$, the pair of classes of elements of order 5 (resp. order 11) is detecting for the pair of characters of degree 12 (resp. degree 5). For $G = M_{11}$, the pair of classes of elements of order 8 (resp. order 11) is detecting for the conjugate pair of characters of degree 10 (resp. degree 16).

The reason for the notions of critical and detecting classes is the following easy but very useful result on Kronecker products. It originates with the usage of detecting classes in [3], its variations in [25, 23], and the idea to consider detecting classes for pairs of spin characters of the double cover groups; we will follow this up in later sections.

Lemma 2.3. *Let G be a finite group, $x, y \in G$. Let ψ be a character of G with $\psi(x) = \psi(y) \neq 0$.*

(1) *Assume that x^G, y^G is a critical pair for $I \subset \operatorname{Irr}(G)$. Then for any $\chi \in I$, $\psi \cdot \chi$ has a constituent in I.*
In particular, if ψ is irreducible and $\chi \in I$, then $\chi \cdot \sum_{\nu \in I} \nu$ contains ψ as a constituent.

(2) *Assume that x^G, y^G is a detecting pair for $\chi_1, \chi_2 \in \operatorname{Irr}(G)$. Set $m_j = \langle \psi \cdot \chi_1, \chi_j \rangle$, $j = 1, 2$. Then*
$$\psi(x) = m_1 - m_2 \,.$$
Furthermore,
$$\max(m_1, m_2) \geqslant |\psi(x)| > 0 \,.$$

Proof. (1) By our assumption on ψ and Definition 2.1, $\psi \cdot \chi(x) \neq \psi \cdot \chi(y)$, for any $\chi \in I$. Since the values of all characters in $\operatorname{Irr}(G) \smallsetminus I$ coincide on x, y, the product $\psi \cdot \chi$ must have a constituent in I.

As I is closed under complex conjugation, we then deduce $\langle \psi, \chi \cdot \sum_{\nu \in I} \nu \rangle = \langle \psi \cdot \bar{\chi}, \sum_{\nu \in I} \nu \rangle > 0$, for all $\chi \in I$.

For the claim in (2), we compute the difference of the values of $\psi \cdot \chi_j$ on the two classes. Set $t := \chi_1(x) - \chi_1(y) = \chi_2(y) - \chi_2(x)$; note that $t \neq 0$. First we have
$$\psi\chi_1(x) - \psi\chi_1(y) = \psi(x)(\chi_1(x) - \chi_1(y)) = \psi(x)t.$$
On the other hand, $\psi\chi_1 = m_1\chi_1 + m_2\chi_2 + \sum_{\chi \neq \chi_1, \chi_2} m_\chi \chi$, and hence (as the pair x^G, y^G is detecting for χ_1, χ_2) we obtain
$$\psi\chi_1(x) - \psi\chi_1(y) = m_1(\chi_1(x) - \chi_1(y)) + m_2(\chi_2(x) - \chi_2(y)) = (m_1 - m_2)t.$$
As $t \neq 0$, we deduce $\psi(x) = m_1 - m_2$. The assertion $\max(m_1, m_2) \geqslant |\psi(x)|$ now follows immediately. □

3. Spin characters, hooks, and a link to the Saxl conjecture

Of central interest in the representation theory of the symmetric groups are the Kronecker coefficients $g(\lambda, \mu, \nu)$ appearing as expansion coefficients in the Kronecker products
$$[\lambda][\mu] = \sum_\nu g(\lambda, \mu, \nu)[\nu] \,.$$
Using this notation, Saxl's conjecture may be rephrased as saying the following for the staircase partition $\rho_k = (k, k-1, \ldots, 2, 1)$ of $n = k(k+1)/2$:
$$g(\rho_k, \rho_k, \lambda) > 0 \quad \text{for all partitions } \lambda \text{ of } n \,.$$

As we will show in the following, products of spin characters or mixed products of a spin character and a non-faithful character can play an important role towards finding constituents in the square $[\rho_k]^2$.

For the product of any ordinary character $[\lambda]$ of S_n with the basic spin characters $\langle n \rangle_{(\pm)}$, Stembridge has provided an efficient combinatorial formula in [32]. It was already observed in [2] that as an immediate consequence of this formula the following result is obtained.

Proposition 3.1. *For the spin product $\langle n \rangle^{(2)}$ we have:*
$$\langle n \rangle^{(2)} = \langle n \rangle_{(\pm)} \cdot \widehat{\langle n \rangle} = \sum_{j=0}^{n-1} [n-j, 1^j] =: \chi_{hook}.$$

Regev [26] showed that the sum of all hook characters has particular easy values; Taylor recently gave a different proof of this fact [33]. Here, we point out that this is a direct consequence of the proposition above and the knowledge of the values of the basic spin characters stated in Section 2:

Corollary 3.2. *Let $\sigma \in S_n$ be of cycle type α. Then*

$$\chi_{hook}(\sigma) = \begin{cases} 2^{l(\alpha)-1} & \text{if } \alpha \in O(n) \\ 0 & \text{otherwise.} \end{cases}$$

For our purpose of finding constituents in the square of the staircase character the following result on spin products for $\tilde{S}_n$ turns out to be important; it was obtained in the context of classifying the homogeneous spin products [5]. Here, it provides the crucial link to the Saxl conjecture.

Proposition 3.3. *Let $n = k(k+1)/2$, $\rho_k = (k, k-1, \dots, 2, 1)$. Then*

$$\langle n \rangle_{(\pm)} \cdot \langle \rho_k \rangle_{(\pm)} = 2^{a(k)}[\rho_k]$$

where $a(k)$ is given by

$$a(k) = \begin{cases} \frac{k-2}{2} & \text{if } k \text{ is even} \\ \frac{k-1}{2} & \text{if } k \equiv 1 \mod 4 \\ \frac{k-3}{2} & \text{if } k \equiv 3 \mod 4. \end{cases}$$

The proposition above is the key for finding new constituents in $[\rho_k]^2$; this is due to the following observations on the connections between $[\rho_k]^2$, χ_{hook} and $\langle \rho_k \rangle^{(2)} = \langle \rho_k \rangle_{(\pm)} \cdot \widehat{\langle \rho_k \rangle}$ which are crucial for our new contributions to Saxl's conjecture.

Lemma 3.4. *Let λ be a partition of $n = k(k+1)/2$.*

(1) *The products $\chi_{hook} \cdot \langle \rho_k \rangle^{(2)}$ and $[\rho_k]^2$ have the same constituents (apart from multiplicities).*
(2) *If $[\lambda]$ is a constituent of $\langle \rho_k \rangle^{(2)}$, then all constituents of $\chi_{hook} \cdot [\lambda]$ are constituents of $[\rho_k]^2$. In particular, all constituents of $\langle \rho_k \rangle^{(2)}$ are constituents of $[\rho_k]^2$.*
(3) *The character $[\lambda]$ is a constituent of $[\rho_k]^2$ if and only if $\chi_{hook} \cdot [\lambda]$ and $\langle \rho_k \rangle^{(2)}$ have a common constituent.*

Proof. (1) By Proposition 3.3 we have for any choice of associates

$$\langle n \rangle_{(\pm)} \langle n \rangle_{(\pm)} \cdot \langle \rho_k \rangle_{(\pm)} \cdot \langle \rho_k \rangle_{(\pm)} = 2^{2a(k)}[\rho_k]^2 .$$

Now the assertion follows immediately from Proposition 3.1.
(2) is an immediate consequence of (1).
(3) By (1), $[\lambda]$ is a constituent of $[\rho_k]^2$ if and only if

$$\langle [\lambda], \chi_{\text{hook}} \cdot \langle \rho_k \rangle^{(2)} \rangle = \langle [\lambda] \cdot \chi_{\text{hook}}, \langle \rho_k \rangle^{(2)} \rangle > 0.$$

Clearly, this is equivalent to $[\lambda] \cdot \chi_{\text{hook}}$ and $\langle \rho_k \rangle^{(2)}$ having a common constituent. □

Computational data suggest that while the character $\langle \rho_k \rangle^{(2)}$ does not contain all irreducible characters as constituents (see also [2, Theorem 3.5]), the number of missing irreducible characters is relatively small. In the next section we will investigate $\langle \rho_k \rangle^{(2)}$ in more detail.

As an immediate contribution towards the Saxl conjecture we obtain the following consequence of Lemma 3.4, which was proved by very different methods by Ikenmeyer [15] and a weaker asymptotic version by Pak, Panova and Vallejo [25]:

Corollary 3.5. *All hook characters $[n-j,1^j]$ are constituents in $[\rho_k]^2$.*

Proof. For any $\mu \in D(n)$, $\overline{\langle\mu\rangle_{(\pm)}}$ is one of the characters $\langle\mu\rangle_{(\pm)}$, so the character $\langle\mu\rangle^{(2)}$ always contains $[n]$. Hence, applying this to $\mu = \rho_k$, the assertion follows immediately from Lemma 3.4. □

We now take a closer look at the products with the character χ_{hook}. Note that formulae for the products with a hook character are available (see [8], [11], [19]), but we will take a simpler route here. To prepare for our next result, we recall a useful formula due to Dvir. This determines the constituents with maximal first part in a product $[\lambda]\cdot[\mu]$, assuming that we already know the decomposition of the product of the skew characters $[\lambda/\lambda\cap\mu]$ and $[\mu/\lambda\cap\mu]$ (by the Littlewood–Richardson rule, the decomposition of the skew characters is considered to be known).

For a partition $\nu = (\nu_1,\nu_2,\dots,\nu_m)$ we set $\hat{\nu} = (\nu_2,\nu_3,\dots,\nu_m)$, and when $t \geqslant \nu_1$, we denote by (t,ν) the partition obtained by adjoining the first part t to ν.

Theorem 3.6 ([9, Theorems 1.6 and 2.4]). *Let λ and μ be partitions of n. Then*

$$\max\{\nu_1 \mid g(\lambda,\mu,\nu) > 0\} = |\lambda\cap\mu|.$$

Furthermore, when ν is a partition of n with $\nu_1 = |\lambda\cap\mu|$, we have

$$g(\lambda,\mu,\nu) = \langle[\lambda/\lambda\cap\mu]\cdot[\mu/\lambda\cap\mu],[\hat{\nu}]\rangle\ .$$

Moreover, if $[\alpha]$ is a constituent of $[\lambda/\lambda\cap\mu]\cdot[\mu/\lambda\cap\mu]$, then $[|\lambda\cap\mu|,\alpha]$ is a constituent in $[\lambda]\cdot[\mu]$ of multiplicity $\langle[\lambda/\lambda\cap\mu]\cdot[\mu/\lambda\cap\mu],[\alpha]\rangle$.

This is now applied to prove the following product property of χ_{hook}.

Proposition 3.7. *Let λ be a partition. Then, the maximal constituent in the product $\chi_{hook}\cdot[\lambda]$, with respect to lexicographic ordering of the labelling partitions, is $[h(\lambda)]$ and its multiplicity is $2^{d(\lambda)-1}$.*

Proof. Let λ be a partition of n, $\ell := \ell(\lambda)$. We set $\chi_n := \chi_{\text{hook}}$ at level n.

We prove the assertion by induction on $d = d(\lambda)$. If $d = 1$, λ is a hook, so $h(\lambda) = (n)$, and clearly $\langle[n],\chi_n\cdot[\lambda]\rangle = \langle[\lambda],\chi_n\rangle = 1$.

Now assume $d > 1$; let $h(\lambda) = (h_1,\dots,h_d)$, and let λ^* be the partition of $n-h_1 > 0$ obtained by removing the first principal hook H_{11} from λ.

By Dvir's formula, for the constituents $[\nu]$ of maximal width in $\chi_n\cdot[\lambda]$ we have $\nu_1 = h_1$, and they all occur in the products of $[\lambda]$ with hooks containing $(\lambda_1,1^{\ell-1})$, i.e. in $\displaystyle\sum_{k=0}^{n-h_1}[\lambda_1+k,1^{n-k-\lambda_1}]\cdot[\lambda]$. Here, the skew characters in Dvir's formula are easily determined, as for $0 \leqslant k \leqslant n-h_1$ the intersection $(\lambda_1+k,1^{n-k-\lambda_1})\cap\lambda$ is exactly $H_{11} = (\lambda_1,1^{\ell-1})$. Thus from the k-th summand we have the contribution

$$([k]\circ[1^{n-h_1-k}])\cdot[\lambda^*] = ([k,1^{n-h_1-k}]+[k+1,1^{n-h_1-k-1}])\cdot[\lambda^*],$$

where $\circ$ denotes the outer product of two characters; note that on the right hand side the first summand is 0 for $k=0$, and the second is 0 for $k = n-h_1$. Hence, by Theorem 3.6, the constituents $[\alpha]$ in $2\chi_{n-h_1}\cdot[\lambda^*]$ give us the constituents $[h_1,\alpha]$ in $[\lambda]\cdot[\mu]$ with maximal first part. Now by induction applied to λ^*, $\chi_{n-h_1}\cdot[\lambda^*]$ has the maximal constituent $[h(\lambda^*)] = [h_2,\dots,h_d]$, with multiplicity $2^{d(\lambda^*)-1} = 2^{d(\lambda)-2}$. Hence the maximal constituent in $\chi_n\cdot[\lambda]$ is $[h(\lambda)] = [h_1,\dots,h_d]$, with multiplicity $2^{d(\lambda)-1}$, as claimed. □

We can now deduce the following connections between $\langle\rho_k\rangle^{(2)}$ and $[\rho_k]^2$:

Corollary 3.8. *Let $k \in \mathbb{N}$, λ a partition of $n = k(k+1)/2$. Then the following holds:*

(1) *If $[\lambda]$ is a constituent of $\langle\rho_k\rangle^{(2)}$, $[h(\lambda)]$ is a constituent of $[\rho_k]^2$.*
(2) *If $[h(\lambda)]$ is a constituent of $\langle\rho_k\rangle^{(2)}$, $[\lambda]$ is a constituent of $[\rho_k]^2$.*

Proof. Both assertions follow immediately from Lemma 3.4 and Proposition 3.7. ☐

Example 3.9. In [2, Theorem 3.5] it was shown that $[n-3,3]$ is a constituent of $\langle\rho_k\rangle^{(2)}$. By Corollary 3.8 (1) we then obtain $[n-2,2]$ as a constituent in $[\rho_k]^2$. When $n > 6$, Corollary 3.8 (2) gives all $[\lambda]$ as constituents in $[\rho_k]^2$ that satisfy $h(\lambda) = (n-3,3)$, i.e. all double-hooks where the smaller principal hook is of size 3.

Remark 3.10. We emphasize the strength of the corollary above. In the case of $[\rho_k]^2$, we know by Ikenmeyer's criterion that all characters to partitions with distinct parts appear as constituents. Given a similar (or even weaker) property for $\langle\rho_k\rangle^{(2)}$, Saxl's conjecture would follow immediately from Corollary 3.8 (2). Unfortunately, in general $\langle\rho_k\rangle^{(2)}$ does not contain all the characters to partitions of type $h(\lambda)$ (i.e. those with part differences at least 2).

4. Constituents in spin products and the Saxl conjecture

As mentioned earlier, the common theme in the character theories of the alternating groups and the double covers of the symmetric and alternating groups that is crucial in the applications here is the existence of critical or detecting classes.

For example, by using the characters of the alternating groups, it was shown in [3] that for a symmetric partition λ, the character $[\lambda]$ is always a constituent in its own square $[\lambda]^2$. The idea was to use the pair of conjugacy classes in the alternating groups that detect the characters $\{\lambda\}_\pm$, namely the ones of cycle type $h(\lambda)$. This was taken up by Pak, Panova and Vallejo in [25] to provide a criterion for constituents in $[\lambda]^2$; we give a very short argument here.

Lemma 4.1 ([25, "Main Lemma"]). *Let λ be a symmetric partition. Let μ be a partition with $[\mu](h(\lambda)) \neq 0$. Then $[\mu]$ is a constituent of $[\lambda]^2$.*

Proof. The restriction $\chi = [\mu]\downarrow_{A_n}$ takes the same (non-zero) values on the two classes of cycle type $h(\lambda)$, which are critical for the pair of characters $\{\lambda\}_\pm$. Hence by Lemma 2.3 the character $\chi \cdot \{\lambda\}_+$ contains one of $\{\lambda\}_\pm$, and thus the covering product $[\mu]\cdot[\lambda]$ contains $[\lambda]$, i.e. $g(\lambda,\lambda,\mu) > 0$, as required. ☐

Towards the Saxl conjecture this immediately gives:

Corollary 4.2 ([25]). *Let μ be a partition with $[\mu](h(\rho_k)) \neq 0$. Then $[\mu]$ is a constituent of $[\rho_k]^2$.*

Unfortunately, many irreducible characters vanish on the class of cycle type $h(\rho_k)$; this was already analyzed in [25]. It turns out that the idea of critical detecting classes is more powerful in the context of spin characters. For constituents in a spin product $\langle\lambda\rangle^{(2)}$ we have the following new criterion, which is based again on non-vanishing character values.

Theorem 4.3. *Let $\lambda \in D(n)$, $n \geqslant 4$. Let μ be a partition with $[\mu](\lambda) \neq 0$. Then $[\mu]$ is a constituent of $\langle\lambda\rangle^{(2)}$.*

Proof. For $\lambda \in D^-(n)$, this follows immediately from Lemma 2.3 (1), as the two $\tilde{S}_n$-classes projecting to cycle type λ are a critical pair for the pair of characters $\langle\lambda\rangle_\pm$.

For $\lambda \in D^+(n)$, we use spin characters of the double covers of the alternating groups. First assume $\lambda \in D^+(n) \smallsetminus (D \cap O(n))$. We recall from Section 2 that the two classes of $\tilde{A}_n$ projecting to cycle type λ are critical for the two spin characters of $\tilde{A}_n$ labelled by λ. Hence one of $\langle\langle\lambda\rangle\rangle_\pm$ is a constituent of $[\mu]\downarrow_{\tilde{A}_n} \cdot\langle\langle\lambda\rangle\rangle_\pm$; thus $\langle\lambda\rangle$ is a constituent of $[\mu]\cdot\langle\lambda\rangle$, and the claim follows. Now take $\lambda \in D \cap O(n)$; note that we then have four $\tilde{A}_n$-classes projecting to type λ. We consider non-conjugate elements $\sigma_1, \sigma_2 \in \tilde{A}_n$ that belong to the same $\tilde{S}_n$-class, projecting to cycle type λ. Recall that $\langle\langle\lambda\rangle\rangle_\pm$ are the only irreducible spin characters that differ on σ_1, σ_2. Now $[\mu]\downarrow_{\tilde{A}_n} \cdot\langle\langle\lambda\rangle\rangle_\pm$ is a spin character with different values on σ_1, σ_2, hence it must have one of $\langle\langle\lambda\rangle\rangle_\pm$ as a constituent, implying the claim as before. □

This leads to powerful new criteria in the context of the Saxl conjecture:

Corollary 4.4. *Let μ be a partition of $n = k(k+1)/2$. Then the following holds:*

(1) *If $[\mu](\rho_k) \neq 0$, then all constituents of $\chi_{hook}\cdot[\mu]$ are constituents of $[\rho_k]^2$. In particular, $[\mu]$ and $[h(\mu)]$ are constituents of $[\rho_k]^2$.*
(2) *If $[h(\mu)](\rho_k) \neq 0$, then $[\mu]$ is a constituent of $[\rho_k]^2$.*

Proof. Assertion (1) follows immediately from Theorem 4.3, Lemma 3.4 and Corollary 3.8(1). The claim in (2) is a direct consequence of Theorem 4.3 and Corollary 3.8(2). □

Remark 4.5. Computational data indicate that using non-vanishing on the class ρ_k produces many more constituents than non-vanishing on the class $h(\rho_k)$! Also, the criterion detecting $[\mu]$ as a constituent in $[\rho_k]^2$ via non-vanishing of $[h(\mu)]$ at ρ_k is quite useful. Fortunately, the non-vanishing tests may be combined to give even more constituents. Here are the numerical values up to $k = 11$ (the last column in the table gives the percentage of irreducible characters $[\mu]$ found as constituents of $[\rho_k]^2$ by the combination of all three tests):

k	n	$p(n)$	$[\mu](h(\rho_k)) \neq 0$	$[\mu](\rho_k) \neq 0$	$[h(\mu)](\rho_k) \neq 0$	combined tests	%
1	1	1	1	1	1	1	100
2	3	3	3	2	3	3	100
3	6	11	5	6	6	11	100
4	10	42	21	24	21	33	78.6
5	15	176	45	114	91	148	84.1
6	21	792	231	524	441	712	89.9
7	28	3718	573	2408	2024	3205	86.2
8	36	17977	3321	12734	11149	16281	90.6
9	45	89134	9321	67462	63110	83442	93.6
10	55	451276	59091	370590	353523	436315	96.7
11	66	2323520	183989	2036486	1932462	2279648	98.1

Recall that Ikenmeyer's criterion in [15] gives all characters to partitions comparable to ρ_k in dominance order as constituents in $[\rho_k]^2$; in the region above, the corresponding percentage is decreasing, and already below 50% at $k = 9$.

We want to illustrate the usefulness of our new criterion by finding new families of constituents of $[\rho_k]^2$, in particular all characters to double-hooks (i.e. partitions of Durfee length 2). We start by a discussion of the characters to 2-part partitions. Note that by the criterion which uses the value on the class to $h(\rho_k)$, asymptotically, the characters to 2-part partitions are found to be constituents in [25, Corollary 6.4]; on the other hand, the constituents to 2-part partitions are also obtained by Ikenmeyer's result [15].

We set the scene for the following theorem and discuss first how the non-vanishing of the character values $[n-j,j](\rho_k)$ is connected to a unimodality question for a sequence of certain partition numbers.

We consider a character to a 2-part partition $[n-j,j]$, with $1 \leqslant j \leqslant n/2$. By the Littlewood–Richardson rule, we have

$$[n-j,j] = [n-j] \circ [j] - [n-j+1] \circ [j-1].$$

For $k \leqslant n$, we set

$$d_k(n) = |\{\lambda = (\ell_1, \dots) \in D(n) \mid \ell_1 \leqslant k\}| \,.$$

As we want to apply the criterion given in Theorem 4.3, we have to evaluate $[n-j,j](\rho_k)$, for $n = k(k+1)/2$. By the above, we obtain the value

$$[n-j,j](\rho_k) = d_k(j) - d_k(j-1) \,.$$

Thus the critical set of characters $[n-j,j]$ vanishing at the class of cycle type ρ_k is determined by the exceptional set

$$\mathcal{E}_k := \{j \in \{1, \dots, \lfloor n/2 \rfloor\} \mid d_k(j) = d_k(j-1)\}.$$

The partition numbers $d_k(m)$, $0 \leqslant m \leqslant n = k(k+1)/2$, are easily seen to form a symmetric sequence. They are also the coefficients in the expansion

$$\prod_{i=1}^{k}(1+x^i) = \sum_{m=0}^{n} d_k(m) x^m \,.$$

This polynomial is known to be unimodal, by quite different and intricate proofs due to Hughes [14], and Odlyzko and Richmond [22]; see also Stanley's article [31] for more on this and related unimodal sequences. Here, based on a result by Odlyzko and Richmond [22], we find that the sequence of numbers $d_k(m)$, $0 \leqslant m \leqslant n = k(k+1)/2$, is in fact *almost strictly* unimodal; there are only few instances where equality holds in the sequence, and these can be described explicitly.

Theorem 4.6. *Let $k \in \mathbb{N}$, $k > 1$. Then the exceptional sets*

$$\mathcal{E}_k = \{j \in \{1, \dots, \lfloor k(k+1)/4 \rfloor\} \mid d_k(j) = d_k(j-1)\}$$

are as follows:

k	$\mathcal{E}_k$
2	1
3	1, 2
4	1, 2, 4, 5
5	1, 2, 4, 6, 7
6	1, 2, 4, 7, 8, 10
7	1, 2, 4, 8, 11, 13, 14
8	1, 2, 4, 16, 17
9	1, 2, 4, 19, 22
10	1, 2, 4, 26
11	1, 2, 4, 32
$\geqslant 12$	1, 2, 4

All other equalities in the sequences $(d_k(m))_{0 \leqslant m \leqslant k(k+1)/2}$ are deduced by their symmetry (including an equality for the two middle terms when $k \equiv 1$ or $2 \mod 4$).

Proof. For $k \leqslant 11$, the sets are easily obtained by computation. For any $k \geqslant 4$, we have $d_k(0) = d_k(1) = d_k(2) = 1$ and $d_k(3) = d_k(4) = 2$, so $\{1,2,4\} \subseteq \mathcal{E}_k$.

Now let $k \geqslant 12$. It remains to show $d_k(j-1) < d_k(j)$ for all $j \in \{5, \dots, \lfloor k(k+1)/4 \rfloor\}$.

Considering partitions of m into distinct parts with largest part being smaller than k or equal to k, respectively, gives the recursion

$$d_k(m) = d_{k-1}(m) + d_{k-1}(m-k)\,.$$

This will now be applied to show our claim in an induction on k.

For $12 \leqslant k < 60$, this claim holds by direct computation. So we assume now that $k \geqslant 60$. Then, using unimodality of the sequences and induction, for $5 \leqslant j \leqslant k(k-1)/4$ we obtain

$$d_k(j) - d_k(j-1) = d_{k-1}(j) - d_{k-1}(j-1) + d_{k-1}(j-k) - d_{k-1}(j-1-k) > 0\,.$$

For the range $k(k-1)/4 \leqslant j \leqslant k(k+1)/4$, the inequality $d_k(j-1) < d_k(j)$ is provided by [22, Theorem 4]. Hence we are done. ☐

As discussed above, $[n-j,j](\rho_k) = d_k(j)-d_k(j-1)$, so we now deduce immediately:

Corollary 4.7. *Let $k \in \mathbb{N}$, $n = k(k+1)/2$. Then $[n-j,j](\rho_k) > 0$ for $0 \leqslant j \leqslant n/2$, $j \notin \mathcal{E}_k$ (with $\mathcal{E}_k$ given explicitly in Theorem 4.6).*

Our criterion in Theorem 4.3 now implies:

Corollary 4.8. *Let $k \in \mathbb{N}$, $n = k(k+1)/2$, $0 \leqslant j \leqslant n/2$, $j \notin \mathcal{E}_k$. Then $[n-j,j]$ is a constituent of $\langle\rho_k\rangle^{(2)} = \langle\rho_k\rangle_{(\pm)} \cdot \widehat{\langle\rho_k\rangle_{(\pm)}}$.*

Towards the Saxl conjecture we deduce:

Corollary 4.9. *Let $k \in \mathbb{N}$, $n = k(k+1)/2$.*

(1) *Let $0 \leqslant j \leqslant n/2$, $j \notin \mathcal{E}_k$. Then all constituents of $\chi_{hook} \cdot [n-j,j]$ are constituents of $[\rho_k]^2$.*
(2) *All characters $[n-j,j]$, $0 \leqslant j \leqslant n/2$, are constituents of $[\rho_k]^2$.*

Proof. The first assertion follows from Corollary 4.8 and Lemma 3.4.

For the second assertion, we only have to check the characters $[n-j,j]$ for the few exceptional values $j \in \mathcal{E}_k$. For $j=1$, we have a hook, a case already dealt with. By direct computation for $k \leqslant 11$, we may assume $k > 11$. Applying Corollary 3.8 (1) to $\lambda = (n-j-1, j+1)$ when $j+1 \leqslant n/2$ and $j+1 \notin \mathcal{E}_k$, reduces the amount of computation and also deals with the cases $j=2$ and $j=4$ for all $k \geqslant 12$. (Note also that for $j=1, k \geqslant 2$, or $j=2, k\geqslant 3$, or $j=4, k \geqslant 4$, the assertion $g(\rho_k,\rho_k,(n-j,j)) > 0$ is already known by work of Saxl [29].) ☐

Products of hook characters and characters to 2-part partitions have been studied by Remmel [27] and Rosas [28]. By their formulae, the extra constituents that we obtain from Corollary 4.9 (1) are all labelled by double hooks. Indeed, going beyond the families of hooks and 2-part characters, we illustrate the power of our new criterion by dealing with all characters labelled by double-hooks. We note that Corollary 4.8 together with Corollary 3.8 (2) already provide all double-hook characters $[\mu]$ as constituents in $[\rho_k]^2$ where $h(\mu) = (n-j,j)$ satisfies $j \notin \mathcal{E}_k$. As we want to use some of the known formulae for the exceptional j in any case, we will prove the following result applying these together only with Corollary 4.9.

Theorem 4.10. *Let $k \in \mathbb{N}$, $n = k(k+1)/2$. Let μ be a partition of n with $d(\mu)=2$. Then $[\mu]$ is a constituent of $[\rho_k]^2$.*

Proof. Since 2-part partitions are already dealt with, we may assume (conjugating, if necessary) that our partition has the form $\mu = (v,u,2^t,1^s)$, where $v \geqslant u$, $v \geqslant 3$, $u \geqslant 2$, $s,t \geqslant 0$, and $s+t \geqslant 1$. By Corollary 4.9 (1), it suffices to show that $[\mu]$ is a constituent in $\chi_{\text{hook}} \cdot [n-j,j]$ with $j \notin \mathcal{E}_k$.

We recall from [27, Theorem 2.2(iii)(b)] that for any m with $2 \leqslant m \leqslant n/2$, the multiplicity of $[\mu]$ in the product $[n-m,m]\cdot[u+v,1^{s+2t}]$ is 1 if $u \leqslant m-t \leqslant \min(v,u+s)$, and 0 otherwise.

Choosing $m = u+t$ then implies that $[\mu]$ is a constituent in the product $[v+t+s,u+t]\cdot[u+v,1^{s+2t}]$. Thus, if $u+t \notin \mathcal{E}_k$, we are done.

Again, the few exceptional cases for $k \leqslant 11$ may be done by direct computation (or by a similar reasoning as follows below). So we now assume that $k \geqslant 12$; as $u+t \geqslant 2$, we only have to deal with the cases $u+t=2$ and $u+t=4$.

When $u+t=2$, we have $u=2$ and $t=0$, i.e. $\mu=(v,2,1^s)$, with $v \geqslant 3$ and $s \geqslant 1$. For such μ, we take $m := 3$ and find that $[\mu]$ is a constituent of $[n-3,3]\cdot[u+v,1^{s+2t}]$. So we are done in this case.

Now assume $u+t=4$. Conjugating, if necessary, we may assume that $v-u \geqslant s$. If $s \geqslant 1$, we take $m := u+t+1 = 5$ and find again that $[\mu]$ is a constituent of $[n-5,5]\cdot[u+v,1^{s+2t}]$ by [27, Theorem 2.2(iii)(b)]. If $s=0$, we have $\mu=(v,u,2^t)$ with $u \leqslant 3$, $t = 4-u \leqslant 2$; as $n \geqslant 10$, we have $v > u$. In this case we use [27, Theorem 2.2(iii)(c)]; this implies that for $2 \leqslant m \leqslant n/2$, $[\mu]$ is a constituent in $[n-m,m]\cdot[u+v-1,1^{s+2t+1}]$ if $u \leqslant m-t \leqslant \min(v,u+s+1)$ holds but $m-t=v=u+s$ is not true. We choose $m=u+t+1=5$ and find that $[\mu]$ is a constituent of $[n-5,5]\cdot[u+v-1,1^{s+2t+1}]$, completing the proof. □

Remark 4.11. We note that Pak and Panova have considered special cases of double-hooks in [23, Theorem 7.1]. They sketch a proof that $[\rho_k]^2$ has constituents to partitions of the form $(n-\ell-m,m,1^\ell)$, $2 \leqslant m \leqslant 10$, when ℓ, k are sufficiently large.

Already in [3], for symmetric λ the result on the multiplicity $g(\lambda,\lambda,\lambda)$ was made more precise; from the calculation of the scalar products on the level of the alternating groups a congruence mod 4 was found for the Kronecker coefficient. Indeed, a similar calculation was used to find the value $|[\mu](h(\lambda))|$ as a lower bound for the Kronecker coefficient $g(\lambda,\lambda,\mu)$ in [23]; we have seen a general version of this in Lemma 2.3.

Also in the spin case we can make the result in Theorem 4.3 more precise.

Theorem 4.12. *Let $\lambda \in D(n)$, $n \geqslant 4$. Let μ be a partition such that $[\mu](\lambda) \neq 0$.*

(1) *Let $\lambda \in D^-(n)$. Set $m_\pm = \langle \langle\lambda\rangle_\pm, [\mu]\cdot\langle\lambda\rangle_+\rangle$. Then*

$$m_+ - m_- = [\mu](\lambda)\,.$$

In particular, $0 < |[\mu](\lambda)| \leqslant \max(m_+,m_-) \leqslant \langle [\mu], \langle\lambda\rangle^{(2)}\rangle\,.$

(2) *Let $\lambda \in D^+(n)$. Set $m_\pm = \langle \langle\langle\lambda\rangle\rangle_\pm, [\mu]\downarrow_{\tilde{A}_n}\cdot\langle\langle\lambda\rangle\rangle_+\rangle$. Then*

$$m_+ - m_- = [\mu](\lambda)$$

and

$$m_+ + m_- = 2m_- + [\mu](\lambda) = \langle [\mu]\langle\lambda\rangle, \langle\lambda\rangle\rangle\,.$$

In particular,

$$0 < |[\mu](\lambda)| \leqslant \max(m_+,m_-) \leqslant \langle [\mu]\langle\lambda\rangle, \langle\lambda\rangle\rangle$$

and

$$\langle [\mu]\langle\lambda\rangle, \langle\lambda\rangle\rangle \equiv [\mu](\lambda) \mod 2\,.$$

Proof. The D^- case is an immediate consequence of Lemma 2.3.

For the D^+ case, we compute the difference of the values of $[\mu]\downarrow_{\tilde{A}_n}\cdot\langle\langle\lambda\rangle\rangle_+$ on the two $\tilde{A}_n$-classes contained in one $\tilde{S}_n$ class projecting to cycle type λ, similarly as in the proof of Lemma 2.3. Note that this character is a linear combination of irreducible spin characters, and restricted to the spin characters, the two $\tilde{A}_n$-classes are critical.

This gives the first assertion. Observing that $\langle\langle\langle\lambda\rangle\rangle_\pm, [\mu]\downarrow_{\tilde{A}_n} \cdot\langle\langle\lambda\rangle\rangle_-\rangle = m_\mp$ then gives the second assertion.

In both cases the additional claims are an immediate consequence. □

Remark 4.13. There is a different way to obtain a character $[\mu]$ as a constituent in $[\rho_k]^2$ by using its character value on the class of cycle type ρ_k. As the criterion to be described now is much weaker than the one given in Theorem 4.3, we only sketch the main arguments without going into the necessary background in detail; again, it is based on using a special detection property of the class of cycle type ρ_k.

By the results in [6] and [7], each partition $\alpha \in O(n)$ is special for the spin character(s) $\langle\beta\rangle_{(\pm)}$ to its Glaisher correspondent $\beta \in D(n)$, with respect to the condition that the 2-power in the spin character value of $\langle\beta\rangle_{(\pm)}$ on elements projecting to cycle type α is the smallest (among the 2-powers in spin character values on this class). The spin character(s) to $\beta = \rho_k$ are indeed unique (up to associates) with this property on their special class, to the Glaisher correspondent α of ρ_k. Thus, when we multiply $\langle\rho_k\rangle_{(\pm)}$ with a character $[\mu]$ that has odd value on α, the product has to contain $\langle\rho_k\rangle_{(\pm)}$ as a constituent. Now it follows from a general character-theoretic fact [10, (6.4)] that the values $[\mu](\rho_k)$ and $[\mu](\alpha)$ are congruent modulo 2. Hence, whenever $[\mu](\rho_k)$ is odd, we deduce that $[\mu]$ is a constituent of $\langle\rho_k\rangle_{(\pm)}\widehat{\langle\rho_k\rangle}$, and hence also of $[\rho_k]^2$.

5. Spin variants of Saxl's conjecture

The product of two spin characters for the double cover groups decomposes into non-faithful irreducible characters, while a mixed product of a spin character and a non-faithful character decomposes into irreducible spin characters. Thus in a variant of Saxl's conjecture for the double cover groups we cannot expect to find an irreducible character whose square contains all irreducible characters but have to be more modest.

We start with a result obtained in the context of classifying homogeneous mixed products; it is a special product appearing in [2, Theorem 3.2]:

Proposition 5.1. *For $n = k^2 \geqslant 4$ we have*

$$\langle n\rangle_{(\pm)} \cdot [k^k] = 2^{\lfloor\frac{k-1}{2}\rfloor}\langle 2k-1, 2k-3, \dots, 3, 1\rangle\,.$$

We denote by $\tau_k = (2k-1, 2k-3, \dots, 3, 1) \in D(k^2)$ the "spin staircase" of length k; note that $\tau_k \in D^+(k^2)$ and $\tau_k = h((k^k))$.

Arguing similarly as for Lemma 3.4, we deduce from Proposition 5.1, using also Proposition 3.7:

Lemma 5.2. *Let μ be a partition of k^2.*

(1) *The products $\chi_{hook} \cdot [k^k]^2$ and $\langle\tau_k\rangle^2$ have the same constituents (apart from multiplicities).*
(2) *If $[\mu]$ is a constituent of $[k^k]^2$, then all constituents of $\chi_{hook}\cdot[\mu]$ are constituents of $\langle\tau_k\rangle^2$; in particular, $\langle\tau_k\rangle^2$ contains $[\mu]$ and $[h(\mu)]$.*
(3) *The character $[\mu]$ is a constituent of $\langle\tau_k\rangle^2$ if and only if $\chi_{hook} \cdot [\mu]$ and $[k^k]^2$ have a common constituent.*

The lemma immediately implies:

Corollary 5.3. *All hook characters $[k^2-j, 1^j]$, $j \in \{0, 1, \dots, k^2-1\}$ are constituents of $\langle\tau_k\rangle^2$.*

Similarly as before we obtain the following criteria for constituents of $\langle\tau_k\rangle^2$.

Corollary 5.4. *Let μ be a partition such that $[\mu](\tau_k) \neq 0$ or $[h(\mu)](\tau_k) \neq 0$. Then $[\mu]$ is a constituent of $\langle\tau_k\rangle^2$.*

Proof. First assume $[\mu](\tau_k) \neq 0$. As $\tau_k = h((k^k))$, Lemma 4.1 implies that $[\mu]$ is a constituent of $[k^k]^2$. Hence Lemma 5.2 (2) yields that $[\mu]$ is a constituent of $\langle\tau_k\rangle^2$.

Now assume $[h(\mu)](\tau_k) \neq 0$. Again, Lemma 4.1 implies that $[h(\mu)]$ is a constituent of $[k^k]^2$. By Proposition 3.7, $[h(\mu)]$ is also a constituent of $\chi_{\text{hook}} \cdot [\mu]$, thus Lemma 5.2 (3) implies that $[\mu]$ is a constituent of $\langle\tau_k\rangle^2$. □

Remark 5.5. In fact, for $2 \leqslant k \leqslant 5$, $\langle\tau_k\rangle^2$ contains all characters $[\mu]$, $\mu \in P(k^2)$. So as a spin variant of Saxl's conjecture we may ask whether this holds for all $k \geqslant 2$.

From the criteria in Corollary 5.4 above, already many constituents of $\langle\tau_k\rangle^2$ are obtained; here are the numerical values up to $k = 8$ (the penultimate column gives the number of partitions where at least one of the two non-vanishing criteria holds, the percentage refers to the number of constituents $[\mu]$ of $\langle\tau_k\rangle^2$ found by the combined tests):

k	n	$p(n)$	$[\mu](\tau_k) \neq 0$	$[h(\mu)](\tau_k) \neq 0$	combined tests	%
1	1	1	1	1	1	100
2	4	5	3	4	5	100
3	9	30	15	23	28	93.3
4	16	231	93	97	148	64.1
5	25	1958	755	754	1240	63.3
6	36	17977	7185	7554	11860	65.9
7	49	173525	75430	85750	124418	71.7
8	64	1741630	851522	961907	1338428	76.8

Computations with GAP [34] led to the following conjecture, adding to the conjectures made by Heide, Saxl, Tiep and Zalesskii [12] on character squares; note that for $\lambda \in D^-(n)$, always one of $[n]$ or $[1^n]$ is missing from the square $\langle\lambda\rangle^2_\pm$.

Conjecture 5.6. *For any $n \geqslant 4$, $n \neq 5$, there is a spin character $\langle\lambda\rangle$, $\lambda \in D^+(n)$, whose square $\langle\lambda\rangle^2$ contains all $[\mu]$, $\mu \in P(n)$.*

It was also conjectured in [12] that for $n \geqslant 4$ there is always an irreducible character of the alternating group A_n whose square contains all irreducible A_n-characters; in fact, a quick check with GAP up to $n = 25$ shows that for growing n a large percentage of irreducible A_n-characters has this property. For the double cover groups $\tilde{A}_n$, data computed with GAP [34] up to $n = 25$ provides evidence for the following conjecture; again, for growing n, it seems that a large percentage of irreducible $\tilde{A}_n$ spin characters has the property considered here.

Conjecture 5.7. *For any $n \geqslant 5$ there is a spin character $\chi \in \mathrm{Irr}(\tilde{A}_n)$ whose square χ^2 contains all non-faithful $\psi \in \mathrm{Irr}(\tilde{A}_n)$.*

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