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Abstract Motivated by a recent conjecture of R. P. Stanley we offer a lower bound for the sum of the coefficients of a Schubert polynomial in terms of 132-pattern containment.

1. Introduction

This paper is motivated by a conjecture of R. P. Stanley [8, Conjecture 4.1] concerning the *Schubert polynomials* of A. Lascoux and M.-P. Schützenberger [5]. A *permutation* is a bijection from the set $\{1, 2, \ldots, n\}$ to itself. We typically represent a permutation in one-line notation. For instance, $w = 25143$ is the permutation which maps 1 to 2, 2 to 5, 3 to 1, and so on. The *symmetric group* S_n consists of the set of permutations.

If $w_0 = n\, n-1\, \ldots\, 1$ is the longest permutation in S_n, define

$$\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

For any other $w \in S_n$, there is some i so that $w(i) < w(i+1)$. Then $\mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i}$, where $\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}$ and $s_i = (i, i+1)$ acts on f by exchanging the variables x_i and x_{i+1}. The ∂_i's satisfy the same braid and commutativity relations as the simple transpositions and so $\mathfrak{S}_w$ is well defined. The polynomial $\mathfrak{S}_w$ is called a *Schubert polynomial*. We will use an equivalent definition for Schubert polynomials as a weighted sum over *pipe dreams*. See Section 2 for these definitions.

We are interested in the following specialization: $\nu_w := \mathfrak{S}_w(1, 1, \ldots, 1)$. Let

$$P_{132}(w) := \{(i, j, k) : i < j < k \text{ and } w(i) < w(k) < w(j)\}. \tag{1}$$

Write $\eta_w := \# P_{132}(w)$. If $\eta_w \geqslant 1$ then w *contains* the pattern 132.

Example 1.1. Let $w = 25143$. Below, we list the elements of $P_{132}(w)$ by marking in bold the positions $i < j < k$ for which $(i, j, k) \in P_{132}(w)$.

251**4**3 **25**14**3** **2**51**43** 25**143**

As such, $\eta_w = 4$.

We prove that η_w provides a lower bound for ν_w.

Theorem 1.2 (The 132-bound). *For any $w \in S_n$, $\nu_w \geqslant \eta_w + 1$.*

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As a corollary, we obtain the following conjecture of R. P. Stanley [8, Conjecture 4.1].

Corollary 1.3. *$\nu_w = 2$ if and only if $\eta_w = 1$.*

Proof. Let $w \in S_n$. If $\eta_w = 0$ then $\nu_w = 1$ [6, Chapter 4]. If $\eta_w = 1$ then $\nu_w = 2$ [8, Section 4]. Otherwise, $\eta_w \geqslant 2$. Then we apply Theorem 1.2 and obtain

$$\nu_w \geqslant \eta_w + 1 \geqslant 3.$$

As such, $\nu_w = 2$ if and only if $\eta_w = 1$. □

2. Background on Permutations and Pipe Dreams

We will recall the necessary background on permutations and Schubert polynomials; our references are [7, Chapter 2] and [1] respectively. Each permutation has an associated *rank function* r_w, where

$$r_w(i,j) := \#\{k : 1 \leqslant k \leqslant i \text{ and } w(k) \leqslant j\}. \tag{2}$$

The pair (i,j) is an *inversion* of w if $i < j$ and $w(i) > w(j)$. Equivalently, each inversion corresponds to a 21-pattern in w. The *length* of a permutation is the number inversions,

$$\ell(w) := \#\{(i,j) : i < j \text{ and } w(i) > w(j)\}. \tag{3}$$

The *Rothe diagram* of $w \in S_n$ is the set

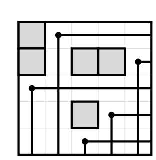
$$D(w) := \{(i,j) : 1 \leqslant i,j \leqslant n, w(i) > j, \text{ and } w^{-1}(j) > i\}. \tag{4}$$

Notice immediately from (4), we have

$$D(w^{-1}) = D(w)^t. \tag{5}$$

The diagram $D(w)$ is in bijection with the set of inversions of w by the map

$$(i,j) \mapsto (i, w^{-1}(j)). \tag{6}$$



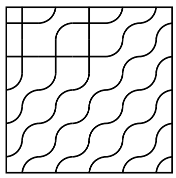
We may visualize $D(w)$ as follows. For each $i = 1, \ldots, n$, plot $(i, w(i))$. Then, strike out all boxes to the right and below each of the plotted points. The boxes which remain form $D(w)$. For example, $D(25143)$ is pictured to the right. Notice that we use matrix conventions; cell (i,j) sits in the ith row from the top and the jth column from the left.

Schubert polynomials can be written as a sum over *pipe dreams*. Pipe dreams appear in the literature under various names; they are the *pseudo-line configurations* of S. Fomin and A. N. Kirillov [3] and the *RC-graphs* of N. Bergeron and S. C. Billey [1]. They were studied from a geometric perspective by A. Knuston and E. Miller [4].

Let $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ be the semi-infinite grid, starting from the northwest corner. A *pipe dream* is a tiling of this grid with ⌟⌜'s (elbows) and a finite number of +'s (pluses). For simplicity, we will often draw the elbows as dots. We freely identify each pipe dream with a subset of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ by recording the coordinates of the pluses. Associate a weight monomial to $\mathcal{P}$:

$$\mathtt{wt}(\mathcal{P}) = \prod_{(i,j)\in\mathcal{P}} x_i.$$

Equivalently, the exponent of x_i counts the number of pluses which appear in row i of $\mathcal{P}$.



We may interpret $\mathcal{P}$ as a collection of overlapping strands, using the rule that a strand never bends at a right angle. The +'s indicate the positions where two strands cross. Each row on the left edge of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is connected by some strand to a unique column along the top, and vice versa. If the ith row is connected to the jth column, let $w_{\mathcal{P}}(i) := j$. There exists some n so that $w_{\mathcal{P}}(i) = i$ for all $i > n$, so $w_{\mathcal{P}} \in S_\infty$. In practice, we identify $w_{\mathcal{P}}$ with its representative in some finite symmetric group. For example, if $\mathcal{P}$ is the pipe dream pictured above, then we write $w_{\mathcal{P}} = 25143$.

If $\#\mathcal{P} = \ell(w_{\mathcal{P}})$ then $\mathcal{P}$ is *reduced.* Let

$$\mathtt{RP}(w) := \{\mathcal{P} : w_{\mathcal{P}} = w \text{ and } \mathcal{P} \text{ is reduced}\}.$$

Theorem 2.1 ([1, 3]).

$$\mathfrak{S}_w = \sum_{\mathcal{P} \in \mathtt{RP}(w)} \mathtt{wt}(\mathcal{P}). \tag{7}$$

Recall, $\nu_w := \mathfrak{S}_w(1, 1, \ldots, 1)$. Immediately from (7), $\nu_w = \#\mathtt{RP}(w)$.

Example 2.2. The reduced pipe dreams for $w = 25143$ are pictured below.

$$\begin{bmatrix} + & \cdot & + & + & \cdot \\ + & \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} + & \cdot & + & \cdot & \cdot \\ + & + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} + & \cdot & + & + & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} + & \cdot & \cdot & + & \cdot \\ + & + & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\begin{bmatrix} + & \cdot & \cdot & \cdot & \cdot \\ + & + & + & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} + & \cdot & + & + & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} + & \cdot & \cdot & + & \cdot \\ + & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} + & \cdot & \cdot & \cdot & \cdot \\ + & + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Therefore,

$$\mathfrak{S}_w = x_1^3x_2^2 + x_1^2x_2^3 + x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1x_2^3x_3 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1x_2^3x_4$$

and $\nu_w = 8$.

There are two pipe dreams which have an explicit description in terms of w. Let

$$m_i(w) = \#\{j : (i, j) \in D(w)\}. \tag{8}$$

Then the *bottom pipe dream* is

$$\mathcal{B}_w = \{(i, j) : j \leqslant m_i(w)\}. \tag{9}$$

Graphically, $\mathcal{B}_w$ is obtained from $D(w)$ by replacing each box with a plus and then left justifying within each row. We define the *top pipe dream* as the transpose of the bottom pipe dream of w^{-1}:

$$\mathcal{T}_w := \mathcal{B}^t_{w^{-1}}.$$

By (5), $\mathcal{T}_w$ is obtained from $D(w)$ by top justifying pluses within columns.

Example 2.3. Let $w = 25143$.

$$\mathcal{B}_w = \begin{bmatrix} + & \cdot & \cdot & \cdot & \cdot \\ + & + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad \mathcal{T}_w = \begin{bmatrix} + & \cdot & + & + & \cdot \\ + & \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Pictured above are the bottom and top pipe dreams for w.

N. Bergeron and S. C. Billey gave a procedure to obtain any pipe dream in $\mathtt{RP}(w)$ algorithmically, starting from $\mathcal{B}_w$. A *ladder move* is an operation on pipe dreams which produces a new pipe dream by a replacement of the following type.

$$\begin{array}{cc} \cdot & \cdot \\ + & + \\ + & + \\ \vdots & \vdots \\ + & + \\ + & \cdot \end{array} \quad \mapsto \quad \begin{array}{cc} \cdot & + \\ + & + \\ + & + \\ \vdots & \vdots \\ + & + \\ \cdot & \cdot \end{array}$$

In the above picture, the columns and rows are consecutive. If $\mathcal{P} \mapsto \mathcal{P}'$ is a ladder move, then $\mathcal{P} \in \mathtt{RP}(w)$ if and only if $\mathcal{P}' \in \mathtt{RP}(w)$. In other words, $\mathtt{RP}(w)$ is closed under ladder moves [1]. Furthermore, any element of $\mathtt{RP}(w)$ can be reached by some sequence of ladder moves from the bottom pipe dream.

Theorem 2.4 ([1, Theorem 3.7]). *If $\mathcal{P} \in \mathtt{RP}(w)$, then $\mathcal{P}$ can be obtained by a sequence of ladder moves from $\mathcal{B}_w$.*

We will mostly focus on a special type of ladder move. A *simple ladder move* is a replacement of the following form.

$$\begin{array}{cc} \cdot & \cdot \\ + & \cdot \end{array} \quad \mapsto \quad \begin{array}{cc} \cdot & + \\ \cdot & \cdot \end{array}$$

The outline of the proof is as follows. In Lemma 3.6, we show that any sequence of ladder moves connecting $\mathcal{B}_w$ to $\mathcal{T}_w$ must use only simple ladder moves. The exact number of pipe dreams in any such sequence, is $\eta_w + 1$. Since each pipe dream in the sequence is distinct, this provides a lower bound for $\nu_w = \#\mathtt{RP}(w)$.

3. Proof of Theorem 1.2

We start by interpreting η_w as a weighted sum over $D(w)$. The "32" in each 132-pattern of w corresponds to a box $(i,j) \in D(w)$. The "1" contributes to the rank function $r_w(i,j)$.

Lemma 3.1.
$$\eta_w = \sum_{(i,j)\in D(w)} r_w(i,j).$$

Proof. Suppose $(i,j,k) \in P_{132}(w)$. Then $w(j) > w(k)$ and $w^{-1}(w(k)) = k > j$. By (4), we have $(j, w(k)) \in D(w)$. Furthermore, $i \leqslant j$ and $w(i) \leqslant w(k)$. Then by (2),

$$\#\{\ell : (\ell, j, k) \in P_{132}(w)\} \leqslant \#\{\ell : \ell \leqslant j \text{ and } w(\ell) \leqslant w(k)\} = r_w(j, w(k)).$$

Then

$$\eta_w \leqslant \sum_{(i,j)\in D(w)} r_w(i,j). \tag{10}$$

On the other hand, suppose $(i,j) \in D(w)$. Then

$$w(i) > j = w(w^{-1}(j)) \text{ and } w^{-1}(j) > i.$$

Take

$$k \in \{k : k \leqslant i \text{ and } w(k) \leqslant j\}.$$

Since $(i,j) \in D(w)$, we must have $k < i$ and $w(k) < j$. Then

$$k < i < w^{-1}(j) \text{ and } w(k) < w(w^{-1}(j)) < w(i)$$

and so

$$(k, i, w^{-1}(j)) \in P_{132}(w).$$

As such, if $(i, j) \in D(w)$,

$$\#\{\ell : (\ell, i, w^{-1}(j)) \in P_{132}(w)\} \geqslant r_w(i, j).$$

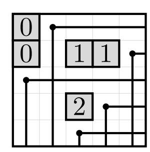
Therefore,

$$\eta_w \geqslant \sum_{(i,j)\in D(w)} r_w(i, j). \tag{11}$$

As such,

$$\eta_w = \sum_{(i,j)\in D(w)} r_w(i, j).$$ □

Example 3.2. Again, let $w = 25143$. Below, we label each box $(i, j) \in D(w)$ with $r_w(i, j)$.



As such,

$$\sum_{(i,j)\in D(w)} r_w(i, j) = 4.$$

In Example 1.1, we found that $\eta_w = 4$. This agrees with Lemma 3.1.

If $\mathcal{P} \in \mathtt{RP}(w)$, let $\mathbf{a}_{\mathcal{P}} := (a_{\mathcal{P}}(1), \ldots, a_{\mathcal{P}}(n))$ where

$$a_{\mathcal{P}}(k) = \#\{(i, j) \in \mathcal{P} : i + j - 1 = k\}. \tag{12}$$

Equivalently, $a_{\mathcal{P}}(k)$ is the number of pluses that occur in the kth antidiagonal of $\mathcal{P}$. Notice if $\mathcal{P} \mapsto \mathcal{P}'$ is a simple ladder move, then $\mathbf{a}_{\mathcal{P}} = \mathbf{a}_{\mathcal{P}'}$. If $\mathcal{P} \mapsto \mathcal{P}'$ is a ladder move which is not simple, then $\mathbf{a}_{\mathcal{P}} \neq \mathbf{a}_{\mathcal{P}'}$.

Example 3.3. Let $\mathcal{P}, \mathcal{P}' \in \mathtt{RP}(25143)$ be as pictured below.

$$\mathcal{P} = \begin{bmatrix} + & \cdot & \cdot & \cdot & \cdot \\ + & + & + & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad \mathcal{P}' = \begin{bmatrix} + & \cdot & + & \cdot & \cdot \\ + & + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Although $\mathcal{P}'$ can be obtained from $\mathcal{P}$ by a ladder move, it is not a simple ladder move. Indeed, $\mathbf{a}_{\mathcal{P}} = (1, 1, 1, 2, 0)$ and $\mathbf{a}_{\mathcal{P}'} = (1, 1, 2, 1, 0)$. Therefore, $\mathbf{a}_{\mathcal{P}} \neq \mathbf{a}_{\mathcal{P}'}$.

This idea extends to sequences of ladder moves.

Lemma 3.4. *Suppose there is a path of ladder moves from $\mathcal{P}$ to $\mathcal{Q}$:*

$$\mathcal{P} = \mathcal{P}_0 \mapsto \mathcal{P}_1 \mapsto \cdots \mapsto \mathcal{P}_N = \mathcal{Q}. \tag{13}$$

Each ladder move in (13) *is simple if and only if* $\mathbf{a}_{\mathcal{P}} = \mathbf{a}_{\mathcal{Q}}$.

Proof.

($\Rightarrow$) Assume each $\mathcal{P}_i \mapsto \mathcal{P}_{i+1}$ is a simple ladder move. Then $\mathcal{P}_{i+1}$ is obtained from $\mathcal{P}_i$ by moving a single plus to a new position in the same antidiagonal. As such, $\mathbf{a}_{\mathcal{P}_i} = \mathbf{a}_{\mathcal{P}_{i+1}}$ for each i. Therefore $\mathbf{a}_{\mathcal{P}} = \mathbf{a}_{\mathcal{Q}}$.

($\Leftarrow$) We prove the contrapositive. Suppose there is a nonsimple ladder move in the sequence (13). It acts by removing a plus from the ith antidiagonal and replacing it in the jth antidiagonal with $i < j$. In particular, we may pick j to be the maximum such label. By the maximality, no plus moves into the jth antidiagonal from a different antidiagonal. Then $a_{\mathcal{P}}(j) > a_{\mathcal{Q}}(j)$ and so $\mathbf{a}_{\mathcal{P}} \neq \mathbf{a}_{\mathcal{Q}}$. ☐

Fix an indexing set I. A labeling of a pipe dream is an injective map $\mathcal{L}_{\mathcal{P}} : \mathcal{P} \to I$. Suppose $\mathcal{P} \mapsto \mathcal{P}'$ is a simple ladder move. Then $\mathcal{P}'$ inherits a labeling from $\mathcal{P}$ as follows:

$$\mathcal{L}_{\mathcal{P}'}(i,j) = \begin{cases} \mathcal{L}_{\mathcal{P}}(i,j) & \text{if } (i,j) \in \mathcal{P} \\ \mathcal{L}_{\mathcal{P}}(i+1,j-1) & \text{otherwise.} \end{cases}$$

Since $\mathcal{P} \mapsto \mathcal{P}'$ is a simple ladder move, $\mathcal{P}'$ is obtained from $\mathcal{P}$ by adding some (i,j) to $\mathcal{P}$ and removing $(i+1, j-1)$. Therefore $\mathcal{L}_{\mathcal{P}'}$ is well defined. If there is a path of simple ladder moves from $\mathcal{P}$ to $\mathcal{Q}$, then $\mathcal{Q}$ inherits the labeling $\mathcal{L}_{\mathcal{Q}}$ from $\mathcal{L}_{\mathcal{P}}$ inductively.

Lemma 3.5. *Let $L_{\mathcal{P}}$ be a labeling. Suppose $\mathcal{Q}$ can be reached from $\mathcal{P}$ by simple ladder moves. Then $\mathcal{Q}$ inherits the same labeling from $\mathcal{P}$ regardless of the choice of sequence.*

Proof. Suppose $\mathcal{P} \mapsto \mathcal{P}'$ is a simple ladder move. Then within any antidiagonal, both pipe dreams have the same set of labels in the same relative order. Iterate this argument along a path of simple ladder moves from $\mathcal{P}$ to $\mathcal{Q}$. Then, in each antidiagonal, $\mathcal{P}$ and $\mathcal{Q}$ have the same set of labels, still in the same relative order. As such, the labeling is uniquely determined and independent of the choice of path. ☐

Lemma 3.6.

(i) *The map*

$$(i,j) \mapsto (i, j - r_w(i,j))$$

is a bijection between $D(w)$ and $\mathcal{B}_w$.

(ii) *The map*

$$(i,j) \mapsto (i - r_w(i,j), j)$$

is a bijection between $D(w)$ and $\mathcal{T}_w$.

(iii) *$\mathcal{B}_w$ and $\mathcal{T}_w$ are connected by simple ladder moves.*

Proof.

(i). Suppose $\ell > i$ and $w(\ell) < w(i)$. Since $w^{-1}(w(\ell)) = \ell > i$ and $w(i) > w(\ell)$, by (4), we have $(i, w(\ell)) \in D(w)$. Therefore,

$$w(\ell) \in \{j : (i, w(j)) \in D(w)\}.$$

By (8), the ith row of $D(w)$ has as many boxes as there are pluses in the ith row of $\mathcal{B}_w$. Let

$$j_1 < j_2 < \cdots < j_{m_i(w)}$$

be the sequence obtained by sorting the set $\{j : (i,j) \in D(w)\}$. Then

$$\begin{aligned} j_\ell - r_w(i, j_\ell) &= j_\ell - \#\{k : k \leqslant i \text{ and } w(k) \leqslant j_\ell\} \\ &= \#\{k : k > i \text{ and } w(k) \leqslant j_\ell\} \\ &= \#\{j : (i,j) \in D(w) \text{ and } j \leqslant j_\ell\} \\ &= \ell. \end{aligned}$$

Therefore $(i, j_\ell) \mapsto (i, \ell)$. Since $1 \leqslant \ell \leqslant m_i(w)$ the map is well defined. This holds for any

$$\ell \in \{1, \ldots, m_i(w)\}$$

so the map is surjective. By definition, $j_\ell = j_{\ell'}$ if and only if $\ell = \ell'$, giving injectivity. As such, this is a bijection.

(ii). Let ϕ be the map defined by $(i,j) \mapsto (j,i)$. Restricted to $D(w)$, ϕ is a bijection between $D(w)$ and $D(w^{-1})$. By the definition of $\mathcal{T}_w$, the restriction

$$\phi : \mathcal{B}_{w^{-1}} \to \mathcal{T}_w$$

is also a bijection.

Let

$$\psi : \mathcal{P}(w^{-1}) \to \mathcal{B}_w$$

be the map in (i). Then the composition

$$D(w) \xrightarrow{\phi} D(w^{-1}) \xrightarrow{\psi} \mathcal{B}_{w^{-1}} \xrightarrow{\phi} \mathcal{T}_w$$

is a bijection. Computing directly,

$$\begin{aligned} \phi(\psi(\phi(i,j))) &= \phi(\psi(j,i)) \\ &= \phi(j, i - r_{w^{-1}}(j,i)) \\ &= (i - r_{w^{-1}}(j,i), j). \end{aligned}$$

Applying (2),

$$\begin{aligned} r_{w^{-1}}(j,i) &= \#\{k : k \leqslant j \ \text{ and } \ w^{-1}(k) \leqslant i\} \\ &= \#\{\ell : w(\ell) \leqslant j \ \text{ and } \ w^{-1}(w(\ell)) \leqslant i\} \\ &= \#\{\ell : \ell \leqslant i \ \text{ and } \ w(\ell) \leqslant j\} \\ &= r_w(i,j). \end{aligned}$$

Therefore,

$$\phi(\psi(\phi(i,j))) = (i - r_w(i,j), j).$$

(iii). By Theorem 2.4, there is a path of ladder moves from $\mathcal{B}_w$ to $\mathcal{T}_w$. Applying (12) and the bijections in parts (i) and (ii),

$$\begin{aligned} \mathbf{a}_{\mathcal{B}_w}(k) &= \#\{(i,j) \in D(w) : i + (j - r_w(i,j)) - 1 = k\} \\ &= \#\{(i,j) \in D(w) : (i - r_w(i,j)) + j - 1 = k\} \\ &= \mathbf{a}_{\mathcal{T}_w}(k). \end{aligned}$$

By Lemma 3.4, the path uses only simple ladder moves. □

In light of the previous lemma, we may label the pluses of $\mathcal{B}_w$ using the map $(i,j) \mapsto (i, j - r_w(i,j))$, i.e. we refer to the plus which is the image of (i,j) as $+_{(i,j)}$. Likewise we label $\mathcal{T}_w$ using the map $(i,j) \mapsto (i - r_w(i,j), j)$.

Lemma 3.7. *The above labeling of $\mathcal{T}_w$ is the same as the labeling it inherits from $\mathcal{B}_w$.*

Proof. It is enough to show that within any given antidiagonal the labels in $\mathcal{B}_w$ and $\mathcal{T}_w$ are the same and have the same relative order. If $(i,j) \in D(w)$, then $+_{(i,j)}$ is in position $(i, j - r_w(i,j))$ in $\mathcal{B}_w$ and in position $(i - r_w(i,j), j)$ in $\mathcal{T}_w$. Since

$$i + j - r_w(i,j) = i - r_w(i,j) + j,$$

they are in the same antidiagonal.

Now consider the rth antidiagonal in $\mathcal{B}_w$. Suppose the sorted list of pluses from top to bottom is

$$+_{(i_1,j_1)}, +_{(i_2,j_2)}, \cdots, +_{(i_k,j_k)}.$$

Since the map from $D(w)$ is by left justification, we must have $i_1 < i_2 < \cdots < i_k$. As $i_\ell + j_\ell - 1 = r$ for all ℓ, it follows that $j_1 > j_2 > \cdots > j_k$. Since the map from $D(w)$ to $\mathcal{T}_w$ is by top justification, the sorted list of pluses from top to bottom must also be

$$+_{(i_1,j_1)}, +_{(i_2,j_2)}, \cdots, +_{(i_k,j_k)}.$$

Therefore, the labeling which $\mathcal{T}_w$ inherits from $\mathcal{B}_w$ coincides with the labeling determined by the map $(i,j) \mapsto (i - r_w(i,j), j)$. □

We conclude with the proof of the 132-bound.

Proof of Theorem 1.2. By Lemma 3.6, there is a path of simple ladder moves connecting $\mathcal{B}_w$ to $\mathcal{T}_w$, say

$$\mathcal{B}_w = \mathcal{P}_0 \mapsto \mathcal{P}_1 \mapsto \cdots \mapsto \mathcal{P}_N = \mathcal{T}_w. \tag{14}$$

Let $n_{i,j} = \#\{k : \mathcal{P}_k \mapsto \mathcal{P}_{k+1} \text{ moves } +_{(i,j)}\}$. By definition, $\mathcal{P}_k \mapsto \mathcal{P}_{k+1}$ moves exactly one plus, labeled by an element of $D(w)$. Therefore,

$$N = \sum_{(i,j)\in D(w)} n_{i,j}. \tag{15}$$

Claim 3.8. *If $(i,j) \in D(w)$ then $n_{i,j} = r_w(i,j)$.*

Proof. By Lemma 3.7, $+_{(i,j)}$ must move from position $(i, j - r_w(i,j))$ in $\mathcal{B}_w$ to position $(i - r_w(i,j), j)$ in $\mathcal{T}_w$. At each step $+_{(i,j)}$ remains stationary or it moves up a row and one column to the right. As such, $+_{(i,j)}$ must move exactly $i - (i - r_w(i,j)) = r_w(i,j)$ times to go from row i to row $i - r_w(i,j)$. □

Then

$$\begin{aligned} \eta_w &= \sum_{(i,j)\in D(w)} r_w(i,j) && \text{(by Lemma 3.1)} \\ &= \sum_{(i,j)\in D(w)} n_{i,j} && \text{(by Claim 3.8)} \\ &= N && \text{(by (15)).} \end{aligned}$$

Each $\mathcal{P}_i$ in the sequence (14) is distinct. As such, $\#\mathtt{RP}(w) \geqslant N + 1$. Therefore

$$\nu_w = \#\mathtt{RP}(w) \geqslant N + 1 = \eta_w + 1.$$

□

Example 3.9. Let $w = 25143$. Below, we give a sequence of simple ladder moves connecting $\mathcal{B}_w$ to $\mathcal{T}_w$. The last row and column of each pipe dream has been omitted.

$$\begin{bmatrix} +_{(1,1)} & \cdot & \cdot & \cdot \\ +_{(2,1)} & +_{(2,3)} & +_{(2,4)} & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ +_{(4,3)} & \cdot & \cdot & \cdot \end{bmatrix} \mapsto \begin{bmatrix} +_{(1,1)} & \cdot & \cdot & +_{(2,4)} \\ +_{(2,1)} & +_{(2,3)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ +_{(4,3)} & \cdot & \cdot & \cdot \end{bmatrix} \mapsto \begin{bmatrix} +_{(1,1)} & \cdot & \cdot & +_{(2,4)} \\ +_{(2,1)} & +_{(2,3)} & \cdot & \cdot \\ \cdot & +_{(4,3)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\mapsto \begin{bmatrix} +_{(1,1)} & \cdot & +_{(2,3)} & +_{(2,4)} \\ +_{(2,1)} & \cdot & \cdot & \cdot \\ \cdot & +_{(4,3)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \mapsto \begin{bmatrix} +_{(1,1)} & \cdot & +_{(2,3)} & +_{(2,4)} \\ +_{(2,1)} & \cdot & +_{(4,3)} & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Notice for each $(i,j) \in D(w)$, the plus $+_{(i,j)}$ moves $r_w(i,j)$ times. For instance, $r_w(4,3) = 2$ and $+_{(4,3)}$ moves twice. This follows from Claim 3.8. The above sequence from $\mathcal{B}_w$ to $\mathcal{T}_w$ uses $\eta_w + 1 = 5$ pipe dreams in total. This agrees with the 132-bound, $\nu_w \geqslant \eta_w + 1$.

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