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Abstract It has been established in recent years how to approach acyclic cluster algebras of finite type using subword complexes. We continue this study by uniformly describing the c- and g-vectors, and by providing a conjectured description of the Newton polytopes of the F-polynomials. We moreover show that this conjectured description would imply that finite type cluster complexes are realized by the duals of the Minkowski sums of the Newton polytopes of either the F-polynomials or of the cluster variables, respectively. We prove this conjectured description to hold in type A and in all types of rank at most 8 including all exceptional types, leaving types B, C, and D conjectural.

1. Introduction

Let (W, S) be a finite crystallographic Coxeter system of rank n equipped with a fixed root system Φ, and let c W be a (standard) Coxeter element. It is well established how to associate to this data a cluster algebra $\mathsf{A}(W, c)$ of finite type with principal coefficients. We refer to Sections 2.1, 2.2 and 2.3 for detailed definitions and references.

Let $u(\mathbf{x}, \mathbf{y}) = p(\mathbf{x}, \mathbf{y}) / \mathrm{m}(\mathbf{x})$ $\mathsf{A}(W, c)$ be a cluster variable for a polynomial $p(\mathbf{x}, \mathbf{y})$ and a monomial $\mathrm{m}(\mathbf{x})$. Sending $u(\mathbf{x}, \mathbf{y})$ to the exponent vector of $\mathrm{m}(\mathbf{x})$ is a bijection between cluster variables in $\mathsf{A}(W, c)$ and almost positive roots in Φ. This means in particular that the denominator $\mathrm{m}(\mathbf{x})$ uniquely determines the cluster variable $u(\mathbf{x}, \mathbf{y})$. Even though this bijection is well-studied, to the best of our knowledge there has not been any successful attempt to describe the numerator $p(\mathbf{x}, \mathbf{y})$ from that perspective. This is, no explicit and non-iterative construction of the numerator of the cluster variable solely from its denominator for general finite type cluster algebras is known. The main aim of this paper is to

> *start the program of uniformly describing cluster variables in finite type cluster algebras in terms of combinatorial data from root systems.*

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Towards such a description, we follow the recently introduced subword complex approach to finite type cluster algebras which we recall in Sections 2.4 and 2.5. These subword complexes were originally considered by A. Knutson and E. Miller in the context of Gröbner geometry of Schubert varieties in [11, 12]. Their appearance in the context of finite type cluster algebras was established by C. Ceballos, J.-P. Labbé, V. Pilaud and the second author in various collaborations. In particular, they establish

- a description of the cluster complex of the cluster algebra [3, Theorem 2.2],
- a vertex and facet description of its polytopal realization [19, Theorem 6.4],
- a proof that the barycenter of this realization equals the barycenter of the corresponding permutahedron [20, Theorem 1.1],
- an explicit description of the principal parts of the exchange matrices of the clusters [19, Theorem 6.20],
- an explicit description of the d-vectors with respect to any initial seed (including cyclic seeds) [4, Corollary 3.4].

In the present paper, we extend this viewpoint by providing several constructions in terms of subword complexes towards uniformly describing further data of cluster variables and cluster seeds purely in terms of the combinatorial data from root systems.

All the following results can be found in detail in Section 2.6. We show in Theorem 2.9 that the c-vectors of a cluster seed of the cluster algebra $\mathcal{A}(W, c)$ are given by the root configuration of subword complexes defined in (3), and deduce in Corollary 2.10 that the g-vectors are given by the weight configuration of subword complexes defined in (4), We then start the development of uniformly describing the F-polynomials for $\mathcal{A}(W, c)$ in Conjectures 2.12 and 2.13 by conjecturally providing all their monomials and in particular their Newton polytopes. Both conjectures are then proven in type A and in all types of rank at most 8 including all exceptional types, see Theorems 2.14 and 2.16, leaving the other classical types B, C, and D conjectural. A combinatorial description of the coefficients of the monomials of the F-polynomials would therefore be the last step to provide a complete combinatorial description of the cluster algebra as it is well known how to recover the cluster variables from the g-vectors and the F-polynomials as we recall in Proposition 2.19. We then show in Theorem 2.20 that this conjectured description of the F-polynomials would imply that the c-associahedron is given by the Minkowski sum of the Newton polytopes of the F-polynomials and of the cluster variables, respectively.

Before we provide detailed definitions, examples and the main constructions in Section 2, we finish this introduction with three remarks about directly related previous work by other authors.

Remark 1.1. As we will later use, R. Schiffler gave in [25] an explicit description of the cluster variables of type A_n via *T-paths* on triangulations of the regular $(n+3)$-gon, and G. Musiker and R. Schiffler generalized that description in [16] to cluster variables for cluster algebras associated to *unpunctured surfaces* with arbitrary coefficients. Together with L. Williams, they extended in [17] the results also to arbitrary surfaces, allowing punctures. We refer to Section 4 for further details. We note that their combinatorial model for punctured surfaces might possibly be used to prove our combinatorial description of the Newton polytopes of F-polynomials for type D cluster complexes.

Remark 1.2. In her PhD thesis, T. Tran provides a case-by-case description of the F-polynomials in the classical types (A, B, C, and D) using root system data, see [27, Theorem 4.16]. In these types, her case-by-case results give explicit formulas for the F-polynomials. That approach is very different from the approach here in the sense that the description there is *not uniform*, while we conjecture a uniform description of their Newton polytopes in Conjecture 2.12. Moreover, [27, Theorem 4.16] does not provide any indication how such a description should lead to F-polynomials in the exceptional types, or towards a uniform description that does not treat every classification type individually. We again note that this combinatorial model for classical types might possibly be used to prove our combinatorial description of the Newton poltytopes of F-polynomials.

Remark 1.3. N. Reading and D. Speyer provide in [23] a general combinatorial framework for acyclic cluster algebras to obtain information about exchange matrices, and c- and g-vectors, see Remarks 2.8 and 2.11 for details. As we will discuss in those remarks, both approaches are closely related. The two main differences are that our approach has not been extended beyond finite types (see also [24]), while their approach only uses (their versions of) the root and the coroot configurations, but does not (seem to) provide information about F-polynomials and cluster variables.

2. Definitions and main results

In this section, we first recall the needed notions from finite root systems, from cluster algebra theory, and from the theory of subword complexes and their relations to finite type cluster algebras. The main results of this paper are then presented in Section 2.6, accompanied with a detailed example.

2.1. Root systems. Let (W, S) be a finite crystallographic Coxeter system acting essentially on a Euclidean vector space V of dimension n with inner product $\langle \cdot \mid \cdot \rangle$, we refer to [10, Part II] for definitions of Coxeter systems and their geometric representations. Let $\Delta = \{\alpha_s : s \in S\}$ be a choice of *simple roots* and set $\Delta^\vee = \{\alpha_s^\vee : s \in S\}$ to be the *simple coroots*. We then have that $\alpha_s^\vee = 2\alpha_s / \langle \alpha_s \mid \alpha_s \rangle$, that the *Cartan matrix* $A = (a_{st})_{s,t \in S}$ is given by $a_{st} = \langle \alpha_t \mid \alpha_s^\vee \rangle$, and that $s(\alpha_t) = \alpha_t - a_{st}\alpha_s$. The *fundamental weights* $\nabla = \{\omega_s : s \in S\}$ and *fundamental coweights* $\nabla^\vee = \{\omega_s^\vee : s \in S\}$ are the bases dual to the simple coroots and to the simple roots, respectively. That is, $\langle \omega_s \mid \alpha_t^\vee \rangle = \langle \alpha_s \mid \omega_t^\vee \rangle = \delta_{s=t}$. It is then easy to verify that

$$\alpha_s = \sum_{t \in S} a_{ts}\omega_t, \quad \alpha_t^\vee = \sum_{s \in S} a_{ts}\omega_s^\vee, \tag{1}$$

and that moreover, $s(\omega_t) = \omega_t - \delta_{s=t}\alpha_s$ for $s \in S$.

Let $c \in W$ be a *(standard) Coxeter element* for (W, S); i.e. $c = s_1 \dots s_n$ is the product of all elements in S in some order, and fix the reduced word $\mathrm{c} = \mathrm{s}_1 \dots \mathrm{s}_n$ of c. We then often write $\alpha_i = \alpha_{s_i}$, $\alpha_i^\vee = \alpha_{s_i}^\vee$, $\omega_i = \omega_{s_i}$, and $\omega_i^\vee = \omega_{s_i}^\vee$ to avoid double indices. Denote by $L = \mathbb{Z}\Delta$ the $\mathbb{Z}$-lattice spanned by Δ, by L^+ the nonnegative span $\mathbb{Z}_{\geq 0}\Delta$, and by $L^- = -L^+$ the nonpositive span. We call $\beta \in L$ *sign-coherent* if $\beta \in L^+ \cup L^-$. Denote moreover by $\Phi = W(\Delta) = \{w(\alpha_s) : w \in W, s \in S\} \subseteq L^+ \cup L^-$ the *root system* for (W, S), by $\Phi^+ = \Phi \cap L^+$ the *positive roots*, and by $\Phi_{\geq -1} = \Phi^+ \cup -\Delta \subseteq \Phi$ the *almost positive roots*. The set of reflections in W is denoted by

$$R = \{wsw^{-1} : w \in W, s \in S\}.$$

The map from Φ to R given by sending $\beta = w(\alpha_s) \in \Phi$ to $s_\beta = wsw^{-1} \in R$ is two-to-one and restricts to a bijection between Φ^+ and R. We often use the letter $N = |\Phi^+| = |R|$, so that $n + N = |\Phi_{\geq -1}|$.

Example 2.1. Let W be the symmetric group $A_2 = \mathfrak{S}_3$ with simple transpositions

$$S = \{\tau_1 = (12), \tau_2 = (23)\},$$

Coxeter element $c = \tau_1\tau_2 = (123)$, simple roots

$$\Delta = \{\alpha_1 = (1, -1, 0), \alpha_2 = (0, 1, -1)\},$$

and fundamental weights

$$= \{\omega_1 = (1, 0, 0), \omega_2 = (1, 1, 0)\}.$$

As this example lives in the space $\mathbb{R}^3/\mathbb{R}(1, 1, 1)$, we have that

$$\alpha_1 = 2\omega_1 - \omega_2, \quad \alpha_2 = 2\omega_2 - \omega_1,$$

in agreement with (1).

2.2. Coxeter elements and skew-symmetrizable Cartan matrices. For simple reflections $s \neq t$, we have that $a_{st} = 0$ if and only if $st = ts$. We can thus think of the Coxeter element c as an acyclic orientation of the corresponding Coxeter diagram by orienting an edge $s \to t$ if s comes before t in the given reduced word $\mathrm{c} = \mathrm{s}_1\mathrm{s}_2 \ldots \mathrm{s}_n$ (or, equivalently, in all reduced words for c). It is indeed the case that this mapping yields a one-to-one correspondence between Coxeter elements and acyclic orientations of the Coxeter diagram, and that the reduced words for a Coxeter element are given by all linear extensions of this orientation. The reason we work with the particular word $\mathrm{c} = \mathrm{s}_1 \ldots \mathrm{s}_n$ is that it will later simplify several notations, both in the indexing of variables, and in the definition of subword complexes (see Section 2.4).

For a given such orientation of the Coxeter diagram, define the skew-symmetrizable matrix $\mathsf{M}_c = (b_{st})_{s,t \in \mathcal{S}}$ by

$$b_{st} = \begin{cases} -a_{st} & \text{if } s \to t, \\ a_{st} & \text{if } s \leftarrow t, \\ 0 & \text{else.} \end{cases} \tag{2}$$

2.3. Cluster algebras and cluster complexes. To a skew-symmetrizable Cartan matrix M_c, one may associate an *initial (cluster) seed* $(\widetilde{\mathsf{M}}_c, \mathbf{x}, \mathbf{y})$, where $\widetilde{\mathsf{M}}_c$ is the *exchange matrix* $\left[\begin{smallmatrix}\mathsf{M}_c \\ \mathbb{1}_n\end{smallmatrix}\right]$ with principal part M_c and extended part $\mathbb{1}_n$ an identity matrix, and where $\mathbf{x} = (x_1, \ldots, x_n)$ are the cluster variables (the cluster of the seed), and $\mathbf{y} = (y_1, \ldots, y_n)$ are the frozen variables (the coefficients of the seed). Cluster seeds may now be mutated in directions $1, \ldots, n$, we refer to Section 3 for the technical definitions of cluster mutations. The *cluster algebra* $\mathcal{A}(W, c) = \mathcal{A}(\widetilde{\mathsf{M}}_c)$ is then the subalgebra of the ring of rational functions generated by all cluster variables obtained from the cluster variables in the initial seed by mutations, we refer to [6, 7, 8] for all needed background on cluster algebras. In the present context, one should think of the variables x_k and y_k as being indexed by α_k for $1 \leqslant k \leqslant n$, so they are in particular indexed in a way that is consistent with the order of the simple reflections in the given Coxeter element c.

It is known that every cluster variable $u(\mathbf{x}, \mathbf{y}) \in \mathcal{A}(W, c)$ is a Laurent polynomial in the ring $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}; y_1, \ldots, y_n]$, i.e. $u(\mathbf{x}, \mathbf{y}) = p(\mathbf{x}, \mathbf{y})/\,\mathrm{m}(\mathbf{x})$ where $p(\mathbf{x}, \mathbf{y})$ is a polynomial in $\mathbf{x}, \mathbf{y}$ with integer coefficients and $\mathrm{m}(\mathbf{x})$ is a monomial in $\mathbf{x}$, see [8, Proposition 3.6]. The *d-vector* $\mathsf{d}(u)$ of $u(\mathbf{x}, \mathbf{y})$ is the exponent vector of the denominator monomial $\mathrm{m}(\mathbf{x})$, i.e. $\mathsf{d}(u) = (d_1, \ldots, d_n)$ for $\mathrm{m}(\mathbf{x}) = x_1^{d_1} \ldots x_n^{d_n}$ and should be thought of as a vector in the basis Δ, i.e.

$$\mathsf{d}(u) = d_1\alpha_1 + \cdots + d_n\alpha_n.$$

Under this identification, it is shown in [7, Theorem 1.9] and [28], that the map $u \to \mathsf{d}(u)$ is a bijection between cluster variables in $\mathcal{A}(W, c)$ and almost positive roots $\Phi_{\geq -1}$ where one sets $\mathsf{d}(u) = -\alpha_i \in -\Delta$ for $u(\mathbf{x},\mathbf{y}) = x_i$. We will regularly use this bijection in indexing objects. For example, we denote by

$$\mathsf{F}_u(\mathbf{y}) = \mathsf{F}_\beta(\mathbf{y}) = u(\mathbf{1},\mathbf{y}) = p(\mathbf{1},\mathbf{y})$$

the *F-polynomial* associated to $u(\mathbf{x},\mathbf{y}) \in \mathcal{A}(W,c)$ and to $\beta \in \Phi_{\geq -1}$ with $\mathsf{d}(u) = \beta$. As $\mathsf{F}_\beta(\mathbf{y}) = 1$ for $\beta \in -\Delta$, one usually considers F-polynomials only for positive roots $\beta \in \Phi^+$. We also denote by $\mathsf{g}(u) = \mathsf{g}(\beta) = (g_1,\dots,g_n)$ the *g-vector* given by the exponent vector of $u(\mathbf{x},0)$. For reasons that will become clear later, we consider the g-vector to live inside the weight space, i.e. $\mathsf{g}(u) = g_1\omega_1 + \cdots + g_n\omega_n$.

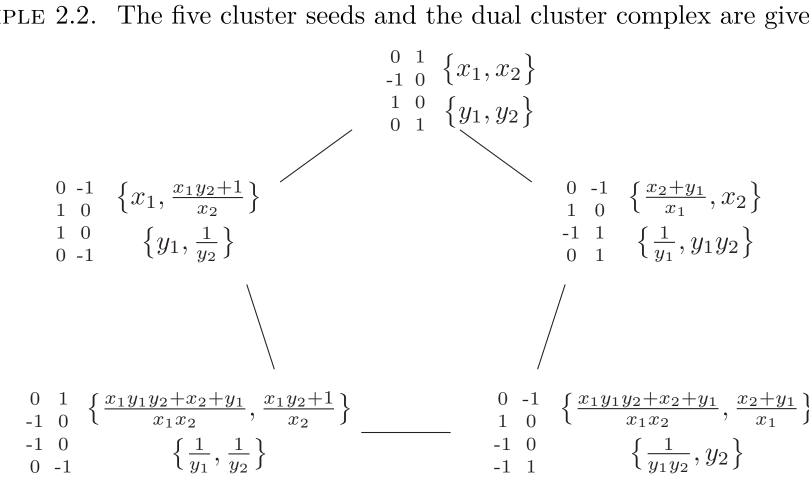
For the course of this paper, it will be natural to consider the vector notation in the exchange matrices inside the root space: we think of any exchange matrix $\widetilde{\mathsf{M}} = \left[\begin{smallmatrix}\mathsf{M}^{pr}\\ \mathsf{M}^{ex}\end{smallmatrix}\right]$ of a cluster seed of $\mathcal{A}(W,c)$ with cluster $(u_1,\dots,u_n)$ as being indexed as follows. Row and column i of M^{pr} are both indexed by the almost positive root $\mathsf{d}(u_i)$. Equally, column i of M^{ex} is indexed by this almost positive root, while row i of M^{ex} is indexed by the simple root α_i. The *c-vector* $\mathsf{c}(\widetilde{\mathsf{M}},u) = \mathsf{c}(\widetilde{\mathsf{M}},\beta)$ with $\beta = \mathsf{d}(u)$ *inside this cluster seed* is then given by the column vector of M^{ex} in the column indexed by the almost positive root β, written as a linear combination of the simple roots,

$$\mathsf{c}(\widetilde{\mathsf{M}},u) = \mathsf{c}(\widetilde{\mathsf{M}},\beta) = [\mathsf{M}^{ex}]_{\alpha_1,\beta}\alpha_1 + \cdots + [\mathsf{M}^{ex}]_{\alpha_n,\beta}\alpha_n.$$

Every cluster seed is uniquely determined by its cluster, and the *cluster complex* of $\mathcal{A}(W,c)$ is the simplicial complex with ground set being the set of cluster variables, and with facets being the clusters. Cluster complexes of finite type with the initial seed coming from a bipartite Coxeter element (i.e. those where every vertex in the corresponding orientation of the Coxeter diagram is either a sink or a source) were studied and completely described in terms of compatibility of d-vectors in [7]. Polytopal realizations of the cluster complex of type $\mathcal{A}(W,c)$ were first obtained by F. Chapoton, S. Fomin, and A. Zelevinsky in [5] for bipartite Coxeter elements, and by C. Hohlweg, C. Lange, and H. Thomas in [9] for general Coxeter elements.

For later comparison, we provide the following well understood running example of type A_2 with principal coefficients. *This example has to be treated with caution* as it does not show several difficulties that appear in types other than A_n, as here we have that Newton polytopes of F-polynomials have no inner lattice points, and all their monomials appear with coefficient 1.

Example 2.2. The five cluster seeds and the dual cluster complex are given by



where we refer to (6), (7), (8) on page 559 for the definition how to mutate exchange matrices, cluster variables and frozen variables.

Observe that between the two clusters $\left\{\frac{x_2+y_1}{x_1}, x_2\right\}$ and $\left\{\frac{x_1y_1y_2+x_2+y_1}{x_1x_2}, \frac{x_2+y_1}{x_1}\right\}$ we switched the position of the common variable in the sense that the two columns and the first two rows of the mutation matrices switched. As mentioned, we prefer to think of the columns and rows being indexed by almost positive roots and simple roots rather than such a linear listing. The cluster variables, their d- and g-vectors, and F-polynomials are thus given by

$u(\mathbf{x},\mathbf{y})$	$\mathsf{d}(u) \in \Phi_{\geq -1}$	$\mathsf{g}(u)$	$\mathsf{F}_u(\mathbf{y})$
$x_1 = \frac{1}{x_1^{-1}}$	$-\alpha_1$	ω_1	1
$x_2 = \frac{1}{x_2^{-1}}$	$-\alpha_2$	ω_2	1
$\frac{x_2+y_1}{x_1}$	α_1	$\omega_2 - \omega_1$	y_1+1
$\frac{x_1y_1y_2+x_2+y_1}{x_1x_2}$	$\alpha_1+\alpha_2$	$-\omega_1$	$y_1y_2+y_1+1$
$\frac{x_1y_2+1}{x_2}$	α_2	$-\omega_2$	y_2+1

2.4. Subword complexes. Let $\mathrm{Q} = \mathrm{q}_1 \dots \mathrm{q}_m$ be a word in the simple system $\mathcal{S}$ and let $\rho \in W$. The ***subword complex*** $\mathcal{SC}(\mathrm{Q},\rho)$ is the simplicial complex of (positions of) letters in Q whose complement contains a reduced word of ρ. These complexes were introduced by A. Knutson and E. Miller in [11], we refer to Example 2.24 on page 557 for a detailed example for $\rho = w_\circ \in W$ being the longest element. Observe that $\mathcal{SC}(\mathrm{Q},\rho)$ is pure by construction with ground set $[m] = \{1,\dots,m\}$ given by the indices of letters in Q. Its facets thus all have the same size and we consider them as sorted lists of integers, written in set notation. This is, $I = \{i_1 < \cdots < i_n\}$ is a facet of $\mathcal{SC}(\mathrm{Q},\rho)$ if and only if the word $\mathrm{q}_1 \dots \widehat{\mathrm{q}}_{i_1} \dots \widehat{\mathrm{q}}_{i_n} \dots \mathrm{q}_m$, with the letters $\mathrm{q}_{i_1},\dots,\mathrm{q}_{i_n}$ omitted, is a reduced word for $\rho \in W$.

Recall the following fundamental observation about subword complexes. It explains in all later constructions the independence of the chosen reduced word $\mathrm{c} = \mathrm{s}_1 \dots \mathrm{s}_n$ of the Coxeter element c.

Lemma 2.3 ([3, Proposition 3.8]). *Let* Q' *be a word in* $\mathcal{S}$ *that coincides with* Q *up to commutations of consecutive letters that commute in* W*. Then* $\mathcal{SC}(\mathrm{Q},\rho) = \mathcal{SC}(\mathrm{Q}',\rho)$*, and the isomorphism is given by the natural identification between letters in* Q *and in* Q'*.*

In this paper, we are only interested in the case that $\rho = w_\circ \in W$ is the unique longest element with respect to the weak order, and Q being one specific word constructed from the Coxeter element c. We thus write $\mathcal{SC}(\mathrm{Q})$ for $\mathcal{SC}(\mathrm{Q}, w_\circ)$ and assume that Q does indeed contain a reduced word for $w_\circ$. This immediately implies that $\mathcal{SC}(\mathrm{Q})$ is a simplicial sphere, see [11, Theorem 3.7]. Define I_{g} and I_{ag} to be the lexicographically first and last facets of $\mathcal{SC}(\mathrm{Q})$, respectively. These are called the ***greedy facet*** and the ***antigreedy facet***.

For $\mathrm{Q} = \mathrm{q}_1 \dots \mathrm{q}_m$, associate to any facet I of the subword complex $\mathcal{SC}(\mathrm{Q})$ a ***root function*** $\mathsf{r}(I,\cdot) : [m] \to W(\Delta)$ and a ***weight function*** $\mathsf{w}(I,\cdot) : [m] \to W(\nabla)$ defined by

$$\mathsf{r}(I,k) = \Pi\mathrm{Q}_{[k-1]\setminus I}(\alpha_{q_k}) \quad \text{and} \quad \mathsf{w}(I,k) = \Pi\mathrm{Q}_{[k-1]\setminus I}(\omega_{q_k}),$$

where $\Pi\mathrm{Q}_X$ denotes the product of the simple reflections $q_x \in \mathrm{Q}$, for $x \in X$, in the order given by Q. For later convenience, we as well define the ***coroot function*** $\mathsf{r}^\vee(I,\cdot) : [m] \to W(\Delta^\vee)$ and a ***coweight function*** $\mathsf{w}^\vee(I,\cdot) : [m] \to W(\nabla^\vee)$ by

$$\mathsf{r}^\vee(I,k) = \Pi\mathrm{Q}_{[k-1]\setminus I}(\alpha^\vee_{q_k}) \quad \text{and} \quad \mathsf{w}^\vee(I,k) = \Pi\mathrm{Q}_{[k-1]\setminus I}(\omega^\vee_{q_k}).$$

Observe that it is immediate from this definition that all of these functions are invariant under the isomorphism given in Lemma 2.3. The root function (and, equivalently, the coroot function) locally encodes the flip property in the subword complex: each facet adjacent to I in $\mathcal{SC}(\mathrm{Q})$ is obtained by exchanging an element $i \in I$ with the unique element $j \notin I$ such that $\mathsf{r}(I,j) \in \{\pm \mathsf{r}(I,i)\}$. If $i < j$ such a flip is called *increasing*, and it is called *decreasing* otherwise. Observe that the greedy facet and the antigreedy facet are the unique facets such that every flip is increasing and decreasing, respectively. After this exchange, the root function and the weight function are updated by an application of $s_{\mathsf{r}(I,i)}$ as recalled in Lemma 3.2 below. The root and the coroot functions are used to define the *root configuration* and the *coroot configuration* of the facet I as the ordered multisets

$$\mathsf{R}(I) = \{\!\{ \mathsf{r}(I,i) \;:\; i \in I \}\!\}, \quad \mathsf{R}^\vee(I) = \{\!\{ \mathsf{r}^\vee(I,i) \;:\; i \in I \}\!\}. \tag{3}$$

Similarly, the weight and the coweight functions are used to define the *weight configuration* and the *coweight configuration*

$$\mathsf{W}(I) = \{\!\{ \mathsf{w}(I,i) \;:\; i \in I \}\!\}, \quad \mathsf{W}^\vee(I) = \{\!\{ \mathsf{w}^\vee(I,i) \;:\; i \in I \}\!\}. \tag{4}$$

By ordered multiset we simply mean the ordered tuple written in set notation. For later convenience, we denote by

$$\mathsf{r}(I,i)_j = \left\langle\, \mathsf{r}(I,i) \,\middle|\, \omega_j^\vee \,\right\rangle$$

the coefficient of α_j in the root $\mathsf{r}(I,i)$.

Next, the weight function is used to define the *brick vector* of I as

$$\mathsf{B}(I) = \sum_{k \in [m]} \mathsf{w}(I,k),$$

and the *brick polytope* of Q is defined to be the convex hull of the brick vectors of all facets of the subword complex $\mathcal{SC}(\mathrm{Q})$,

$$\mathcal{B}(\mathrm{Q}) = \operatorname{conv}\left\{ \mathsf{B}(I) \;:\; I \text{ facet of } \mathcal{SC}(\mathrm{Q}) \right\}.$$

It is shown in [19] that the brick polytope $\mathcal{B}(\mathrm{Q})$ is the Minkowski sum of a family of Coxeter matroid polytopes in the sense of [1].

Theorem 2.4 ([19, Proposition 1.5]). *For any word* Q *in* S *of length* m *containing a reduced word for* $w_\circ$ *we have that*

$$\mathcal{B}(\mathrm{Q}) = \textstyle\sum_{k \in [m]} \mathcal{B}(\mathrm{Q},k)$$

where $\mathcal{B}(\mathrm{Q},k) = \operatorname{conv}\left\{ \mathsf{w}(I,k) \;:\; I \text{ facet of } \mathcal{SC}(\mathrm{Q}) \right\}$.

2.5. Cluster complexes as subword complexes. For the Coxeter element c with fixed reduced word $\mathrm{c} = \mathrm{s}_1 \dots \mathrm{s}_n$, the *Coxeter-sorting word* (or c*-sorting word*) $\rho(\mathrm{c})$ of an element $\rho \in W$ is given by the lexicographically first subword of c^∞ that is a reduced word for ρ. In particular, $\mathrm{w}_\circ(\mathrm{c})$ is the lexicographically first subword of c^∞ that is a reduced word for the longest element $w_\circ \in W$. Observe that the word $\rho(\mathrm{c})$ does also depend on the chosen reduced word. But instead, one should think of this sorting word as being associated to the Coxeter element c and being defined up to commutations of consecutive commuting letters. The notion of c-sorting words was defined by N. Reading in [22] and plays a crucial role in the combinatorial descriptions of finite type cluster algebras and in particular in the description of cluster complexes in terms of subword complexes. The main results in [3] provides the following description of the combinatorics of the cluster complex of $\mathcal{A}(W,c)$, where we observe from Lemma 2.3 that the subword complex does not depend on the chosen reduced word for c.

Theorem 2.5 ([3, Theorem 2.2]). *The cluster complex of the cluster algebra $\mathcal{A}(W,c)$ is isomorphic to the subword complex $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$.*

We thus refer to $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ as the *c-cluster complex*. We moreover remark that the abstract simplicial complex $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ does not depend on the chosen Coxeter element c, while its combinatorics in the sense of root and weight functions does depend on c. This is, $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big) = \mathcal{SC}\big(\mathrm{c}'\mathrm{w}_\circ(\mathrm{c}')\big)$ for any two Coxeter elements c and c' with reduced words c and c', respectively, see [3, Theorem 2.6].

One identifies positions in $\mathrm{cw}_\circ(\mathrm{c})$ and almost positive roots by sending the k^{th} letter s_k $(1 \leqslant k \leqslant n)$ of the initial copy of $\mathrm{c} = \mathrm{s}_1 \dots \mathrm{s}_n$ to the negative simple root $-\alpha_k = -\alpha_{s_k}$, and the k^{th} letter q_k $(1 \leqslant k \leqslant N)$ of $\mathrm{w}_\circ(\mathrm{c}) = \mathrm{q}_1 \dots \mathrm{q}_N$ to the positive root $q_1 \dots q_{k-1}(\alpha_{q_k})$. See Lemma 3.2 (1) on page 560 that this indeed is a bijection, and observe that this equals in the natural way the root function of the greedy facet. In symbols, this is for $1 \leqslant k \leqslant N$

$$q_1 \dots q_{k-1}(\alpha_{q_k}) = \mathsf{r}(I_{\mathrm{g}}, n+k). \tag{5}$$

That identification yields the isomorphism in Theorem 2.5 by sending a cluster to the positions inside the word $\mathrm{cw}_\circ(\mathrm{c})$ corresponding to the almost positive roots of the d-vectors of the cluster. To make this explicit, we use the following notation. Let $I = \{i_1 < \dots < i_n\}$ be a facet of the cluster complex $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$. We then denote by $S(I) = \big(\widetilde{\mathsf{M}}(I), \mathbf{u}(I), \mathbf{f}(I)\big)$ with

$$\begin{aligned} \widetilde{\mathsf{M}}(I) &= \begin{bmatrix} \mathsf{M}^{pr}(I) \\ \mathsf{M}^{ex}(I) \end{bmatrix} \\ \mathbf{u}(I) &= \big(u_{i_1}(I), \dots, u_{i_n}(I)\big) \\ \mathbf{f}(I) &= \big(f_{i_1}(I), \dots, f_{i_n}(I)\big) \end{aligned}$$

the cluster seed of $\mathcal{A}(W,c)$ corresponding to I under the given isomorphism between cluster variables, almost positive roots, and positions in the word $\mathrm{cw}_\circ(\mathrm{c})$. The columns of $\widetilde{\mathsf{M}}(I)$ are then also indexed by the positions $i_1, \dots, i_n$ of I as are the rows of $\mathsf{M}^{pr}(I)$, while the rows of $\mathsf{M}^{ex}(I)$ are indexed by the positions $1, \dots, n$ (which are the positions of the greedy facet $I_{\mathrm{g}} = \{1, \dots, n\}$). We also denote by $\mathsf{c}(I,i)$ the c-vector coming from column $i \in I$ of $\mathsf{M}^{ex}(I)$, and by $\mathsf{g}(I,i)$ the g-vector of the entry $u_i(I)$.

Polar polytopal realizations of the cluster complex were first obtained by F. Chapoton, S. Fomin, and A. Zelevinsky in [5]. C. Hohlweg, C. Lange, and H. Thomas then constructed in [9] a generalization depending on a Coxeter element c, that reduces for bipartite c to the construction in [5]. As one obtains for type A_n classical constructions of associahedra, such polytopal realizations are called *c-associahedra*. The subword complex approach and the brick polytope construction provide an explicit and straightforward construction of these.

Theorem 2.6 ([19, Theorem 4.9]). *The cluster complex of $\mathcal{A}(W,c)$ is realized by the polar of the brick polytope $\mathcal{B}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$.*

This polytopal realization turns out to be equal to the construction in [9] up to a translation, see [19, Corollary 6.10]. Its main advantage is that it provides a vertex description that yields a very simple proof of Theorem 2.6. This construction lives inside the weight space, while its natural translation by $\mathsf{B}(I_{\mathrm{ag}})$ lives inside the root space. We will conjecturally see in Conjecture 2.12 that this is closely related to the F-polynomials also "living inside the root space".

Next, we recall how the indexing of the principal part of the exchange matrix is chosen, and why one can think of it as a *matrix of scalars*.

THEOREM 2.7 ([19, Theorem 6.40]). *Let I be a facet of $SC(\mathrm{cw}_\circ(\mathrm{c}))$. The principal part of the exchange matrix $\widetilde{\mathsf{M}}(I)$ is then given for $i, j \quad I$ by*

$$[\mathsf{M}^{pr}(I)]_{ij} = \begin{cases} -\left\langle \mathsf{r}(I,j) \,\middle|\, \mathsf{r} \;\; (I,i) \right\rangle & \text{if } i < j, \\ \left\langle \mathsf{r}(I,j) \,\middle|\, \mathsf{r} \;\; (I,i) \right\rangle & \text{if } i > j, \\ 0 & \text{if } i = j. \end{cases}$$

Observe that one could directly express this as well in terms of the skew-symmetric bilinear form defined by Equation (2).

The following remark starts to clarify the connection between the subword complex approach to finite type cluster algebras and the approach using N. Reading and D. Speyer's combinatorial frameworks [23].

REMARK 2.8. The central structures in their combinatorial frameworks are the *labels* and *colabels*. It follows from [19, Proposition 6.20] that the labels in finite types are the root configurations defined in (3), and we obtain by duality that the colabels in finite types are the coroot configurations also defined in (3). Given this connection in finite types, we immediately obtain that Theorem 2.7 is the same description of the principal part of the exchange matrix as given in [23, Theorem 3.25]. See Remark 2.11 below for the relation of the subword complex approach and [23, Theorem 3.26].

2.6. MAIN RESULTS. In this section, we provide and discuss the main results, we refer to the end of this section for a detailed example. The first result of this paper shows the close relationship between the c-vectors in finite type cluster algebras and the root function of the corresponding subword complex.

THEOREM 2.9. *Let I be a facet of the c-cluster complex $SC(\mathrm{cw}_\circ(\mathrm{c}))$ corresponding to the seed $S(I)$ in the cluster algebra $A(W,c)$. Then the columns of $\mathsf{M}^{ex}(I)$ are given by the root configuration, i.e.*

$$\mathsf{c}(I,i) = \mathsf{r}(I,i)$$

for all $i \quad I$. In particular, $\mathsf{c}(I,i) \quad \Phi$ is sign-coherent and $\{\mathsf{c}(I,i) : i \quad I\}$ forms a lattice basis of the root lattice.

We will prove this theorem in Section 3. To emphasize the similarity of this result with Theorem 2.7, we rewrite this result in terms of the frozen variables and the extended part of the mutation matrix as a *matrix of scalars* and obtain $\mathbf{f}_i(I) = y_1^{\mathsf{r}(I,i)_1} \dots y_n^{\mathsf{r}(I,i)_n}$, which means that the extended part of the exchange matrix $\widetilde{\mathsf{M}}(I)$ is given for $i \quad I_{\mathrm{g}}$ and $j \quad I$ by

$$[\mathsf{M}^{ex}(I)]_{ij} = \left\langle \mathsf{r}(I,j) \,\middle|\, \mathsf{w} \;\; (I_{\mathrm{g}},i) \right\rangle.$$

This explains why we think of the columns of $\mathsf{M}^{ex}(I)$ as being indexed by the almost positive roots in a facet, while we think of the rows as being indexed by the simple roots.

COROLLARY 2.10. *In the situation of Theorem 2.9, we also obtain that the g-vectors are given by the weight configuration,*

$$\mathsf{g}(I,i) = \mathsf{w}(I,i)$$

for all $i \quad I$. In particular, $\{\mathsf{g}(I,i) : i \quad I\}$ forms a lattice basis of the weight lattice.

Proof. Given a cluster algebra $A(W,c)$ as considered in Theorem 2.9, it was proven in [18, Theorem 1.2] that the g-matrix, whose columns consist of the g-vectors of $A(W,c)$, is equal to the transpose inverse of the c-matrix, whose columns consist of the c-vectors of $A(W,c)$.

This fact along with Theorem 2.9 tells us that the g-vectors form the dual basis to the coroot configuration, see also [23, Section 3.1]. The statement thus follows with a stronger version of Lemma 3.2 (5) below which was established in [19, Proposition 6.6].

As an example, one may compare the g-vectors displayed in Example 2.2 with the weight configuration displayed in Example 2.24.

Remark 2.11. We have seen in Remark 2.8 how the description of the mutation matrix through subword complexes relates to the description through N. Reading and D. Speyer's combinatorial frameworks. Indeed, Theorem 2.9 is the subword complex counterpart of [23, Theorem 3.26]. Theorem 2.9 is then the same as Theorem 3.26(1) and (2), and Corollary 2.10 implies Theorem 3.26(3) and (4). We also remark that [23, Theorem 5.39] provides a way of computing the g-vectors in finite types using their combinatorial framework. Theorem 3.26(5) then states that all F-polynomials in finite type have constant term 1, this follows by the same argument via [8, Proposition 5.6].

We have now seen how to obtain properties from the root and coroot configurations, and also from the weight configuration. Indeed, we have *not* used the root and coroot functions outside of the facets to derive information. This does not seem very surprising in light of Lemma 3.2 (1) which recalls that the root function on the complement of a given facet is always the complete set of positive roots.

Next, we *look at properties of the cluster algebra that can be studied using the weight function*, this time outside of a given facet. Indeed, we will see that we can conjecturally obtain further desired information about the cluster algebra from this weight function, see Conjecture 2.12, Corollaries 2.17 and 2.18, and Theorem 2.20 below.

To state the main conjecture of this paper, we define the *Newton polytope* of an F-polynomial $\mathsf{F}_\beta(\mathbf{y})$ as the convex hull of its exponent vectors in the root basis Δ. That is,

$$\mathrm{Newton}\big(\mathsf{F}_\beta(\mathbf{y})\big) = \mathrm{conv}\big\{\lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n \ :\ y_1^{\lambda_1}\dots y_n^{\lambda_n} \text{ monomial in } \mathsf{F}_\beta(\mathbf{y})\big\}.$$

Conjecture 2.12. *Let $\mathsf{F}_\beta(\mathbf{y})$ be the F-polynomial associated to the positive root β for the cluster algebra $A(W,c)$. Let k be the unique index k $\{n+1,\dots,n+N\}$ such that $\mathsf{r}(I_{\mathrm{g}},k)=\beta$ associated to β in Equation (5). Then*

$$\mathrm{Newton}\big(\mathsf{F}_\beta(\mathbf{y})\big) = \mathrm{conv}\big\{\mathsf{w}(I,k) - \mathsf{w}(I_{\mathrm{ag}},k) \ :\ I \text{ facet of } \mathit{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)\big\}.$$

We moreover conjecture that knowing the Newton polytope of an F-polynomial $\mathsf{F}_\beta(\mathbf{y})$ in a finite type cluster algebra is enough to recover all monomials in $\mathsf{F}_\beta(\mathbf{y})$.

Conjecture 2.13. *The exponent vectors of the monomials of the F-polynomial $\mathsf{F}_\beta(\mathbf{y})$ are given by all lattice points inside its Newton polytope,*

$$\big\{\lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n \ :\ y_1^{\lambda_1}\dots y_n^{\lambda_n} \text{ monomial in } \mathsf{F}_\beta(\mathbf{y})\big\} = \mathrm{Newton}\big(\mathsf{F}_\beta(\mathbf{y})\big) \cap L^+.$$

We emphasize that we currently do not have an explicit conjecture what the coefficients of the monomials look like—it is planned to investigate this in future research.

Theorem 2.14. *Conjectures 2.12 and 2.13 hold for $A(W,c)$ with W of type A_n.*

This theorem will be proved in Section 4 by relating it to the combinatorial model of type A_n cluster algebras of R. Schiffler [25] using its description given by G. Musiker and R. Schiffler in [16].

Remark 2.15. The combinatorial model for F-polynomials for cluster algebras from punctured surfaces can as well be used to provide the F-polynomials for type D

cluster complexes via the description given for example by C. Ceballos and V. Pilaud in [4]. We expect that constructions similar to those we use in Section 4 to derive Theorem 2.14 can as well be given to prove Conjecture 2.12 in type D. We plan to also further investigate this explicit combinatorial approach.

Theorem 2.16. *Conjectures 2.12 and 2.13 hold for $A(W,c)$ with W being of rank at most 8. In particular, they hold in all exceptional types.*

Proof. This was obtained via explicit computer explorations(1).

Given Theorems 2.14 and 2.16, the two conjectures remains at this point open in types B, C, and D. Whenever Conjecture 2.12 holds, we obtain very simple combinatorial proofs of many properties of finite type cluster algebras that were already conjectured in [8]. Since then, all those properties were proven in acyclic finite types, but often using rather intricate machinery, while the proofs here are elementary once the needed combinatorial properties of subword complexes are established. We refer to [23, Section 3.3] and in particular the table at the end of that section for references to proofs of these properties.

The first two corollaries describe how to obtain properties of the F-polynomial from Conjecture 2.12, and we then describe what is missing to obtain the actual cluster variables from the root and the weight function.

Corollary 2.17 ([8, Conjecture 7.17]). *Assume that Conjecture 2.12 holds for $A(W,c)$. For any β Φ^+, we then have that the F-polynomial $\mathsf{F}_\beta(\mathbf{y})$ has a unique monomial of maximal degree whose exponent vector equals β, and such that any of its monomials divides this monomial of maximal degree.*

Proof. This directly follows from Lemmas 3.6 and 3.7 below. First, Conjecture 2.12 implies that $\mathsf{F}_\beta(\mathbf{y})$ has a monomial with exponent vector given by $\mathsf{w}(I_{\mathrm{g}},k) - \mathsf{w}(I_{\mathrm{ag}},k)$ where k be the unique index k $\{n+1,\dots,n+N\}$ such that $\mathsf{r}(I_{\mathrm{g}},k) = \beta$. Lemma 3.7 then shows that the exponent vector of this monomial is $\mathsf{r}(I_{\mathrm{g}},k) = \beta$. Any given facet I of $\mathit{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ is obtained from the greedy facet I_{g} by a sequence of increasing flips, and the antigreedy facet I_{ag} is obtained from I again by a sequence of increasing flips. Lemma 3.6 thus shows that every monomial of $\mathsf{F}_\beta(\mathbf{y})$ divides the monomial given by $\mathsf{w}(I_{\mathrm{g}},k) - \mathsf{w}(I_{\mathrm{ag}},k)$. Observe here that we can deduce this property for all monomials in $\mathsf{F}_\beta(\mathbf{y})$ from the same property for the monomials corresponding to the vertices of the Newton polytope of $\mathsf{F}_\beta(\mathbf{y})$.

Using [8, Proposition 7.16], we can now also deduce how to compute the g-vector from the F-polynomial, and indeed from the weight function alone.

Corollary 2.18 ([8, Conjecture 6.11]). *Assume that Conjecture 2.12 holds for $A(W,c)$. For any β Φ^+, we then have that the g-vector $\mathsf{g}(\beta)$ is given by*

$$\mathsf{g}(\beta) = \max\big(\mathsf{F}_\beta(\hat{\mathbf{x}})\big) - \beta$$

where $\hat{\mathbf{x}} = (\hat{x}_1,\dots,\hat{x}_n)$ with $\hat{x}_i = \prod_j x_j^{-[\mathsf{M}_c]_{ji}}$, and $\max(\mathsf{F}_\beta(\hat{\mathbf{x}}))$ denotes the componenentwise maximum of the exponent vectors of the monomials in $\mathsf{F}_\beta(\hat{\mathbf{x}})$. Moreover, this maximum is obtained when only considering exponent vectors of monomials that correspond to vertices of $\mathrm{Newton}\big(\mathsf{F}_\beta(\mathbf{y})\big)$.

(1) The computations were performed using sage-8.2. The development was supported by the project "Combinatorial and geometric structures for reflection groups and groupoids" within the German Research Foundation priority program "Algorithmic and experimental methods in Algebra, Geometry, and Number Theory". The code and examples are available upon request.

Observe that one usually uses tropical notation to express $\max\big(\mathsf{F}_\beta(\hat{\mathbf{x}})\big)$ as briefly used in the second paragraph of Section 3 below. We chose to show it in the present form for simplicity in the given context.

Proof of Corollary 2.18. The equality follows from Corollary 2.17 via [8, Proposition 7.16]. The following simple fact about polytopes implies that it is enough to consider vertices of the Newton polytope. Let $P \subset \mathsf{R}^n$ be a polytope, and let $\hat{P}$ is the image of P under a linear map $\varphi : \mathsf{R}^n \to \mathsf{R}$. Then the maximum of $\hat{P}$ is obtained at a vertex of P. This is,

$$\max \hat{P} = \max\{\varphi(v) \ : \ v \text{ vertex of } P\}.$$

Applying this observation to every component yields the desired restriction to vertices of the Newton polytope.

To state the main implication of the conjecture, we recall in the following proposition how to recover the cluster variable from the g-vector and the F-polynomial.

Proposition 2.19 ([8, Corollary 6.3]). *For any $\beta \in \Phi^+$, we have that the cluster variable $\mathbf{u}_\beta(\mathbf{x},\mathbf{y})$ is given by*

$$\mathbf{u}_\beta(\mathbf{x},\mathbf{y}) = x_1^{g_1}\dots x_n^{g_n}\mathsf{F}_\beta(\hat{\mathbf{y}}),$$

where we use the g-vector $\mathsf{g}(\beta) = (g_1,\dots,g_n)$ and where we set $\hat{\mathbf{y}} = (\hat{y}_1,\dots,\hat{y}_n)$ with $\hat{y}_i = y_i\hat{x}_i^{-1} = y_i \cdot \prod_j x_j^{[\mathsf{M}_c]_{ji}}$.

Theorem 2.20. *Let W be a finite crystallographic Coxeter group acting on a vector space V, and let $c \in W$ be a Coxeter element. Assume that Conjecture 2.12 holds for $\mathsf{A}(W,c)$. Then the c-associahedron $\mathsf{B}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ coincides up to translation with the two Minkowski sums*

- $\sum_{\beta\in\Phi^+} \mathrm{Newton}\big(\mathsf{F}_\beta(\mathbf{y})\big) \subset V$, *or*
- $\sum_{\beta\in\Phi^+} \mathrm{Newton}\big(u_\beta(\mathbf{x},\mathbf{y})\big) \subset \varphi(V) \subset V \times V$

for a suitable affine embedding φ of V into $V \times V$.

Proof. The first description in terms of the F-polynomials follows from the Minkowski decomposition of any brick polytope into Coxeter matroid polytopes recalled in Theorem 2.4.

The second description then in terms of the cluster variables follows from the description in terms of the F-polynomials using Proposition 2.19 as this shows that the cluster variables depend affinely on the $\mathbf{y}$-variables, so that

$$\sum_\beta \mathrm{Newton}\big(u_\beta(\mathbf{x},\mathbf{y})\big) \subset V \times V$$

indeed lives inside the affine embedding $\varphi(V) \subset V \times V$ given in Proposition 2.19.

Remark 2.21. For the *linear Coxeter element* $c = (1,\dots,n+1)$ in type A_n, this description is equivalent to the description given by A. Postnikov in [21], see Corollary 8.2 and the following two paragraphs(2). There, it is shown in this case that the Minkowski sum of the Newton polytopes of the F-polynomials is exactly the realization given by J.-L. Loday in [15].

(2)We thank Vincent Pilaud for pointing out this connection.

Theorem 2.20 immediately suggests the following conjecture. Let $A(\widetilde{\mathsf{M}})$ be a finite type cluster algebra with principal coefficients and possibly cyclic initial seed $(\widetilde{\mathsf{M}}, \mathbf{x}, \mathbf{y})$. Define the $\widetilde{\mathsf{M}}$-*associahedron* to be the Minkowski sum of the Newton polytopes of the F-polynomials of the cluster algebra. This is the sum

$$\sum \text{Newton}\big(\mathsf{F}(\mathbf{y})\big) \quad V,$$

ranging over all F-polynomial $\mathsf{F}(\mathbf{y})$ of $A(\widetilde{\mathsf{M}})$.

Conjecture 2.22. *All $\widetilde{\mathsf{M}}$-associahedra of a given finite type have the same combinatorial type, i.e. they all have the same face lattice. In particular, the duals of the $\widetilde{\mathsf{M}}$-associahedra realize the cluster complex.*

In light of Proposition 2.19, one could use instead cluster variables in this definition for cluster algebras with principal coefficients. Indeed, this conjecture would even make sense (in a slightly weaker form) in infinite types. Taking the Minkowski sum corresponds to taking the finest common coarsening of the normal fans; one could thus also consider the finest (infinite) coarsening of the normal fans of the Newton polytopes of the F-polynomial in infinite types.

Remark 2.23. The description of the $\widetilde{\mathsf{M}}$-associahedron using the finest coarsening of the normal fans of the Newton polytopes of the cluster variables was already conjectured by D. Speyer and L. Williams in [26, Conjecture 8.1] via the language of tropical geometry. They consider the variety of a cluster algebra $A(\widetilde{\mathsf{M}})$ of finite type and the positive part of its tropicalization and conjecture that whenever the finite type mutation matrix $\widetilde{\mathsf{M}}$ has full rank (which is the case for principal coefficients), the common refinement of the normal fans of the Newton polytopes of the cluster variables should be the fan given by the cluster complex of $A(\widetilde{\mathsf{M}})$ as the d-vector fan. Thus, Theorem 2.20 would imply their conjecture in the case of acyclic finite type cluster algebras with principal coefficients.

The conjecture further states that if the mutation matrix does not have full rank, then the common refinement of the normal fans of the Newton polytopes of the cluster variables should be a coarsening of the fan dual to the cluster complex of $A(\widetilde{\mathsf{M}})$. C. Ceballos, J.-P. Labbé, and the first author proved this conjecture in type D_4 in [2].

After having presented the results of this paper, we explain them in great detail in the example of type A_2.

Example 2.24. This example shows the root and the weight function of type A_2, together with the construction of the c-cluster complex $\mathit{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$. It is presented in order to emphasize the *close similarity* to the type A_2 cluster algebra in Example 2.2. The word $\mathrm{Q} = \mathrm{cw}_\circ(\mathrm{c})$ is given by

$$\mathrm{q}_1\mathrm{q}_2\ \mathrm{q}_3\mathrm{q}_4\mathrm{q}_5 = \underbrace{\tau_1\tau_2}_{\mathrm{c}}\ \underbrace{\tau_1\tau_2\tau_1}_{\mathrm{w}_\circ(\mathrm{c})},$$

and the facets of $\mathit{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ are

$$\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{1,5\}.$$

The following table records the root function of $\mathsf{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ indexed both by almost positive roots and positions in the word $\mathrm{cw}_\circ(\mathrm{c})$:

I	$-\alpha_1$ 1	$-\alpha_2$ 2	α_1 3	$\alpha_1+\alpha_2$ 4	α_2 5
$I_{\mathrm{g}}=\{1,2\}$	α_1 $(1,-1,0)$	α_2 $(0,1,-1)$	α_1 $(1,-1,0)$	$\alpha_1+\alpha_2$ $(1,0,-1)$	α_2 $(0,1,-1)$
$\{2,3\}$	α_1 $(1,-1,0)$	$\alpha_1+\alpha_2$ $(1,0,-1)$	$-\alpha_1$ $(-1,1,0)$	$\alpha_1+\alpha_2$ $(1,0,-1)$	α_2 $(0,1,-1)$
$\{3,4\}$	α_1 $(1,-1,0)$	$\alpha_1+\alpha_2$ $(1,0,-1)$	α_2 $(0,1,-1)$	$-\alpha_1-\alpha_2$ $(-1,0,1)$	α_2 $(0,1,-1)$
$I_{\mathrm{ag}}=\{4,5\}$	α_1 $(1,-1,0)$	$\alpha_1+\alpha_2$ $(1,0,-1)$	α_2 $(0,1,-1)$	$-\alpha_1$ $(-1,1,0)$	$-\alpha_2$ $(0,-1,1)$
$\{1,5\}$	α_1 $(1,-1,0)$	α_2 $(0,1,-1)$	$\alpha_1+\alpha_2$ $(1,0,-1)$	α_1 $(1,-1,0)$	$-\alpha_2$ $(0,-1,1)$

Observe that the root configuration of a facet I (indicated in grey) written in simple roots coincides with the c-vectors of the corresponding cluster seed in Example 2.2. E.g. the facet $I=\{3,4\}$ corresponds to the cluster seed where the d-vectors are the almost positive roots $\big(\alpha_1,\alpha_1+\alpha_2\big)$. It has root configuration $\big(\alpha_2,-\alpha_1-\alpha_2\big)$ which corresponds to the two c-vectors $\mathsf{c}(I,3)=\alpha_2$ and $\mathsf{c}(I,4)=-\alpha_1-\alpha_2$. This phenomenon is explained in all finite types in Theorem 2.9.

Similarly, the following table records the weight function of $\mathsf{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$:

I	$-\alpha_1$ 1	$-\alpha_2$ 2	α_1 3	$\alpha_1+\alpha_2$ 4	α_2 5	$\mathsf{B}(I)$
$I_{\mathrm{g}}=\{1,2\}$	ω_1 $(1,0,0)$	ω_2 $(1,1,0)$	ω_1 $(1,0,0)$	ω_2 $(1,1,0)$	$\omega_2-\omega_1$ $(0,1,0)$	$\omega_1+3\omega_2$ $(4,3,0)$
$\{2,3\}$	ω_1 $(1,0,0)$	ω_2 $(1,1,0)$	$\omega_2-\omega_1$ $(0,1,0)$	ω_2 $(1,1,0)$	$\omega_2-\omega_1$ $(0,1,0)$	$-\omega_1+4\omega_2$ $(3,4,0)$
$\{3,4\}$	ω_1 $(1,0,0)$	ω_2 $(1,1,0)$	$\omega_2-\omega_1$ $(0,1,0)$	$-\omega_1$ $(0,1,1)$	$\omega_2-\omega_1$ $(0,1,0)$	$-2\omega_1+3\omega_2$ $(2,4,1)$
$I_{\mathrm{ag}}=\{4,5\}$	ω_1 $(1,0,0)$	ω_2 $(1,1,0)$	$\omega_2-\omega_1$ $(0,1,0)$	$-\omega_1$ $(0,1,1)$	$-\omega_2$ $(0,0,1)$	$-\omega_1+\omega_2$ $(2,3,2)$
$\{1,5\}$	ω_1 $(1,0,0)$	ω_2 $(1,1,0)$	ω_1 $(1,0,0)$	$\omega_1-\omega_2$ $(1,0,1)$	$-\omega_2$ $(0,0,1)$	$3\omega_1-\omega_2$ $(4,1,2)$

This yields that the brick polytope is given by

$$\begin{aligned}
B\big(\mathrm{cw}_\circ(\mathrm{c})\big) &= \operatorname{conv}\big\{430,340,241,232,412\big\}\\
&= \operatorname{conv}\{\omega_1\}+\operatorname{conv}\{\omega_2\}+\operatorname{conv}\{\omega_1,\omega_2-\omega_1\}\\
&\qquad+\operatorname{conv}\{\omega_2,-\omega_1,\omega_1-\omega_2\}+\operatorname{conv}\{\omega_2-\omega_1,-\omega_2\}.
\end{aligned}$$

There are multiple things to be observed in this table which are conjectured/explained in this paper. First, the weight configuration I (again indicated in grey) written in fundamental weights coincides with the g-vectors of the corresponding cluster seed in Example 2.2. E.g. the facet $\{3,4\}$ has weight configuration $(\omega_2-\omega_1,-\omega_1)$ which corresponds to the two g-vectors $\mathsf{g}(\alpha_1)=(-1,1)=\omega_2-\omega_1$ and $\mathsf{g}(\alpha_1+\alpha_2)=(-1,0)=-\omega_1$. This phenomenon is explained in all finite types in Corollary 2.10. Moreover, the weights inside a column are all equal within the entries inside the facets (the entries in grey) and these weights also coincide with the weight in the row of the antigreedy facet. This will be explained in Lemma 3.5 and the following paragraph.

Next, and most importantly, one shifts all weights inside a column by the weight in the row of the antigreedy facet I_{ag} and expresses the result in terms of the simple roots to obtain in each column the exponent vectors of the monomials in the F-polynomials for the corresponding cluster variable:

	$-\alpha_1$	$-\alpha_2$	α_1	$\alpha_1+\alpha_2$	α_2	$\mathsf{B}(I)-\mathsf{B}(I_{\mathrm{ag}})$
$I_{\mathrm{g}}=\{1,2\}$	0	0	α_1	$\alpha_1+\alpha_2$	α_2	$2\alpha_1+2\alpha_2$
$\{2,3\}$	0	0	0	$\alpha_1+\alpha_2$	α_2	$\alpha_1+2\alpha_2$
$\{3,4\}$	0	0	0	0	α_2	α_2
$I_{\mathrm{ag}}=\{4,5\}$	0	0	0	0	0	0
$\{1,5\}$	0	0	α_1	α_1	0	$2\alpha_1$
$\mathsf{F}_\beta(\mathbf{y})$	1	1	1 y_1	1 y_1 y_1y_2	1 y_2	

We prove this phenomenon in type A_n, while we only conjecture generalizations thereof in general finite types which we verify in low ranks including all exceptional types.

Additional support for this conjecture is that the following properties of the columns perfectly match known properties of F-polynomials and hold for general finite type c-cluster complexes:

(1) When shifting all weights inside the columns by the entries of the antigreedy facet I_{ag}, all entries inside the facets become 0 and the row of the greedy facet I_{g} coincides with the row of I_{g} for the table of the root function in the positions corresponding to the positive roots (while the simple negative roots become 0 in this table), see Lemma 3.7.

(2) Every other entry is obtained from the entry of the greedy facet I_{g} (the antigreedy facet I_{ag}) by subtracting (adding) simple roots, see Lemma 3.6.

The first item corresponds to the facts that F-polynomials have constant term 1 and a monomial with exponent vector equal to the d-vector, and the second item corresponds to the fact that this monomial is the unique monomial of highest degree and is divided by every other monomial in the F-polynomial.

3. Proof of Theorem 2.9

In this section, we prove Theorem 2.9 and also provide several auxiliary results for general finite type c-cluster complexes, which will be used in Section 4 to show the close relationship between F-polynomials and weight functions in type A_n.

We start with recalling cluster mutations on cluster seeds. Let $S=\big(\widetilde{\mathsf{M}},\mathbf{u},\mathbf{f}\big)$ with $\widetilde{\mathsf{M}}=\left[\begin{smallmatrix}\mathsf{M}^{pr}\\ \mathsf{M}^{ex}\end{smallmatrix}\right]$ be a cluster seed as above. Given that we have indexed columns of $\widetilde{\mathsf{M}}$ and the rows of M^{pr} both by the d-vectors of the cluster variables $\mathbf{u}=(u_1,\dots,u_n)$, we now mutate S at $\beta \quad \Phi_{>-1}$ such that $\beta=\mathsf{d}(u_i)$. The ***seed mutation*** $\mu_i=\mu_\beta$ ***in direction*** β defines a new seed $\mu_i(S)=(\widetilde{\mathsf{M}}\ ,\mathbf{u}\ ,\mathbf{f}\ )$ defined by the following ***exchange relations***, written for better readability in the indices $\{1,\dots,n\}$ of $\{u_1,\dots,u_n\}$ rather than in their d-vectors:

- The entries of $\widetilde{\mathsf{M}}\ =(b_{k\ell})$ are given by

$$b_{k\ell}=\begin{cases}-b_{k\ell} & \text{if } k=i \text{ or } \ell=i\\ b_{k\ell}+b_{ki}b_{i\ell} & \text{if } b_{ki}>0 \text{ and } b_{i\ell}>0\\ b_{k\ell}-b_{ki}b_{i\ell} & \text{if } b_{ki}<0 \text{ and } b_{i\ell}<0\\ b_{k\ell} & \text{otherwise.}\end{cases} \tag{6}$$

- The cluster variables u_k of the cluster $\mathbf{u} = \{u_1, \ldots, u_n\}$ are given by $u_k = u_k$ for $k = i$ and

$$u_i = \frac{f_i \prod u_k^{\max\{b_{ki},0\}} + \prod u_k^{\max\{-b_{ki},0\}}}{(f_i \quad 1)u_i} \tag{7}$$

- The frozen variables f_ℓ of the coefficients $\mathbf{f} = \{f_1, \ldots, f_n\}$ are given by

$$f_\ell = \begin{cases} f_i^{-1} & \text{if } \ell = i \\ f_\ell f_i^{\max\{b_{i\ell},0\}}(f_i \quad 1)^{-b_{i\ell}} & \text{if } \ell = i. \end{cases} \tag{8}$$

As usual, we use the tropical notation in Equations (7) and (8), which is defined for monomials by $\left(\prod_i y_i^{a_i}\right) \quad \left(\prod_i y_i^{b_i}\right) = \prod_i y_i^{\min\{a_i,b_i\}}$. It is worthwhile to compare this with the notation used in Corollary 2.18.

A direct consequence of the definition is the following description of the frozen variables.

Lemma 3.1 ([8, Eq. (2.13)]). *The frozen variables are given by*

$$f_i = y_1^{[\mathsf{M}^{ex}]_{1i}} \cdots y_n^{[\mathsf{M}^{ex}]_{ni}}.$$

To prove Theorem 2.9, we will show that the entries in the root configuration behave as the c-vectors described in the matrix mutation in Equation (6). In order to properly set this up, it is convenient to extract the coefficient of α_j in the root $\mathsf{r}(I,i)$ using the inner product with the fundamental coweights, so that we aim to show that

$$\left[\mathsf{M}^{ex}(I)\right]_{ji} = \mathsf{r}(I,i)_j = \left\langle \mathsf{r}(I,i) \,\middle|\, \omega_j \right\rangle. \tag{9}$$

The argument follows the same lines as the proof of Theorem 2.7 in [19]. We frequently make use of the following properties of the root and the weight function.

Lemma 3.2 ([3, Lemma 3.3 & Lemma 3.6], [19, Lemma 3.3, Lemma 4.4 & Proposition 6.6]). *Let I and J be two adjacent facets of the subword complex $\mathcal{SC}(\mathrm{Q})$ with $I \smallsetminus i = J \smallsetminus j$. Then*

(1) *The map $\mathsf{r}(I,\cdot) : k \to \mathsf{r}(I,k)$ is a bijection between the complement of I and Φ^+.*
(2) *The position j is the unique position in the complement of I for which $\mathsf{r}(I,j)$ $\{\pm\mathsf{r}(I,i)\}$. Moreover, $\mathsf{r}(I,j) = \mathsf{r}(I,i) \quad \Phi^+$ if $i < j$, while $\mathsf{r}(I,j) = -\mathsf{r}(I,i)$ Φ^- if $j < i$.*
(3) *The map $\mathsf{r}(J,\cdot)$ is obtained from $\mathsf{r}(I,\cdot)$ by*

$$\mathsf{r}(J,k) = \begin{cases} s_{\mathsf{r}(I,i)}(\mathsf{r}(I,k)) & \text{if } \min\{i,j\} < k \leqslant \max\{i,j\}, \\ \mathsf{r}(I,k) & \text{otherwise}. \end{cases}$$

(4) *The map $\mathsf{w}(J,\cdot)$ is obtained from $\mathsf{w}(I,\cdot)$ by*

$$\mathsf{w}(J,k) = \begin{cases} s_{\mathsf{r}(I,i)}(\mathsf{w}(I,k)) & \text{if } \min\{i,j\} < k \leqslant \max\{i,j\}, \\ \mathsf{w}(I,k) & \text{otherwise}. \end{cases}$$

(5) *For $k \quad I$, we have for $k \quad I$ with $k = k$ that*

$$\mathsf{r}(I,k) \,|\, \mathsf{w}(I,k) = 0$$

and we have for $k \;/\; I$ that

$$\begin{cases} \mathsf{r}(I,k) \,|\, \mathsf{w}(I,k) > 0 & \text{if } k > k, \\ \mathsf{r}(I,k) \,|\, \mathsf{w}(I,k) \leqslant 0 & \text{if } k < k. \end{cases}$$

We first show that (9) holds for the initial seed, and second that it is preserved under mutations.

Proposition 3.3. *Let I_{g} be the greedy facet of $SC\big(\mathrm{cw}_\circ(\mathrm{c})\big)$. Then*

$$\left[\mathsf{M}^{ex}(I_{\mathrm{g}})\right]_{ji} = \mathsf{r}(I_{\mathrm{g}}, i)_j.$$

Proof. This is the case as both sides are clearly equal to $\langle \alpha_i \mid \omega_j \rangle = \delta_{i=j}$.

Proposition 3.4. *Let I, J be two faces of $SC\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ with $I \setminus i = J \setminus j$, and let $k \in I \setminus i$ and $\ell \in \{1, \dots, n\}$. Then $\mathsf{r}(J,j) = -\mathsf{r}(I,i)$ and*

$$\mathsf{r}(J,k)_\ell = \begin{cases} \mathsf{r}(I,k)_\ell + \mathsf{r}(I,i)_\ell \cdot \left[\mathsf{M}^{pr}(I)\right]_{ik} & \text{if } \mathsf{r}(I,i)_\ell \geq 0, \left[\mathsf{M}^{pr}(I)\right]_{ik} \geq 0, \\ \mathsf{r}(I,k)_\ell - \mathsf{r}(I,i)_\ell \cdot \left[\mathsf{M}^{pr}(I)\right]_{ik} & \text{if } \mathsf{r}(I,i)_\ell \leq 0, \left[\mathsf{M}^{pr}(I)\right]_{ik} \leq 0, \\ \mathsf{r}(I,k)_\ell & \text{otherwise.} \end{cases}$$

Proof. The property that $\mathsf{r}(J,j) = -\mathsf{r}(I,i)$ holds in general for facets $I \setminus i = J \setminus j$ in subword complexes. It is a direct consequence of Lemma 3.2 (2).

It thus remains to show that $\mathsf{r}(J,k)_\ell$ is obtained from $\mathsf{r}(I,k)_\ell$ as described. For simplicity, observe that we can assume that $i < j$ as every facet of any subword complex $SC(\mathrm{Q})$ can be obtained from the greedy facet by a sequence of increasing flips. This implies, again by Lemma 3.2 (2), that $\mathsf{r}(I,i) \in \Phi^+$ and thus $\mathsf{r}(I,i)_\ell \geq 0$. Even though this is not needed, we note that the case of a decreasing flip $i > j$ could also be computed in the exact same way.

The first case is $k \in \{i+1, \dots, j-1\}$. It follows from [19, Lemma 6.43] that also $\left[\mathsf{M}^{pr}(I)\right]_{ik} \geq 0$. And, as desired, we obtain

$$\begin{aligned} \mathsf{r}(J,k)_\ell &= \left\langle \Pi Q_{[k] \setminus J}(\alpha_{q_k}) \,\middle|\, \omega_\ell \right\rangle \\ &= \left\langle \Pi Q_{[i] \setminus I} \cdot q_i\big(\Pi Q_{[i,k] \setminus I}(\alpha_{q_k})\big) \,\middle|\, \omega_\ell \right\rangle \\ &= \left\langle \Pi Q_{[i] \setminus I} \cdot \Big(\Pi Q_{[i,k] \setminus I}(\alpha_{q_k}) - \left\langle \Pi Q_{[i,k] \setminus I}(\alpha_{q_k}) \,\middle|\, \alpha_{q_i} \right\rangle \alpha_{q_i}\Big) \,\middle|\, \omega_\ell \right\rangle \\ &= \left\langle \Pi Q_{[k] \setminus I}(\alpha_{q_k}) \,\middle|\, \omega_\ell \right\rangle - \left\langle \Pi Q_{[i,k] \setminus I}(\alpha_{q_k}) \,\middle|\, \alpha_{q_i} \right\rangle \cdot \left\langle \Pi Q_{[i] \setminus I}(\alpha_{q_i}) \,\middle|\, \omega_\ell \right\rangle \\ &= \mathsf{r}(I,k)_\ell + \mathsf{r}(I,i)_\ell \cdot \left[\mathsf{M}^{pr}(I)\right]_{ik}, \end{aligned}$$

where, as before, we write $\mathrm{cw}_\circ(\mathrm{c}) = \mathrm{q}_1 \dots \mathrm{q}_{n+N}$. The first and the last equalities are the definitions together with Theorem 2.7. The second equality is obtained as we do the flip from $i \in I$ to $j \in J$, the third equality is the definition of the application of the simple reflection q_i to $\Pi Q_{[i,k] \setminus I}(\alpha_{q_k})$, and the fourth equality is the linearity of the inner product.

The second case is $k \notin \{i, \dots, j\}$. It follows from [19, Lemma 6.43] that $\left[\mathsf{M}^{pr}\right]_{ik} \leq 0$, while $\mathsf{r}(I,i)_\ell \geq 0$. And indeed, the flip from i to j does not effect the root function at k by Lemma 3.2 (3), and we obtain that $\mathsf{r}(I,k)_\ell = \mathsf{r}(J,k)_\ell$, as desired.

We are now in the situation to deduce Theorem 2.9.

Proof of Theorem 2.9. It follows from Equation (6) and Proposition 3.4 that

$$\left[\mathsf{M}^{ex}(I)\right]_{\ell k} = \mathsf{r}(I,k)_\ell = \quad \left[\mathsf{M}^{ex}(J)\right]_{\ell k'} = \mathsf{r}(J,k')_\ell$$

for $I \setminus i = J \setminus j$ and either $(k,k') = (i,j)$ or $k = k' \neq i$. As Proposition 3.3 provides the equality for the initial mutation matrix, we obtain $\left[\mathsf{M}^{ex}(I)\right]_{\ell i} = \mathsf{r}(I,i)_\ell$ for all $i \in I$.

The property of the sign-coherence then follows as $\mathsf{R}(I) \subseteq \Phi$ for all facets I, and the fact that $\mathsf{R}(I)$ forms a basis of the root space is a direct consequence of its iterative description.

After this calculation, we present several general lemmas about cluster complexes that we then use in Section 4 in type A_n to deduce Theorem 2.14.

Lemma 3.5. *Let I, J be two facets of $\mathsf{SC}(\mathrm{cw}_\circ(\mathrm{c}))$ with $k \in I \cap J$. Then $\mathsf{w}(I,k) = \mathsf{w}(J,k)$.*

Proof. This is a direct consequence of Lemma 3.2 (4) and (5) and the observation that all facets of $\mathsf{SC}(\mathrm{cw}_\circ(\mathrm{c}))$ containing k are connected by flips (see [19, Corollary 3.11]). (Indeed, this property of the weight function was already used in the proof of [19, Proposition 6.8].) To see this, assume that $k \in I \mathsf{r} i = J \mathsf{r} j$. The first part of Lemma 3.2 (4) shows how the weight $\mathsf{w}(J,k)$ is obtained from $\mathsf{w}(I,k)$ by applying $s_{\mathsf{r}(I,i)}$ and the first part of (5) implies that $\mathsf{w}(I,k)$ is contained in the hyperplane fixed by $s_{\mathsf{r}(I,i)}$, implying

$$\mathsf{w}(J,k) = s_{\mathsf{r}(I,i)}(\mathsf{w}(I,k)) = \mathsf{w}(I,k).$$

This lemma implies as well that $\mathsf{w}(I,i) = \mathsf{w}(I_{\mathrm{ag}},i)$ for any facet I and any $i \in I$. If $i \in I_{\mathrm{ag}}$, this follows immediate. Otherwise, this follows from the observation that one can construct a facet I' with $i \in I'$ being minimal. Lemma 3.5 then implies that $\mathsf{w}(I,i) = \mathsf{w}(I',i)$ and Lemma 3.2 (4) implies that $\mathsf{w}(I',i) = \mathsf{w}(I_{\mathrm{ag}},i)$. We do not make further use of this additional information.

We next recall the following lemma.

Lemma 3.6 ([19, Lemma 4.5]). *Let $I \mathsf{r} i = J \mathsf{r} j$ with $i < j$ be two facets of $\mathsf{SC}(\mathrm{cw}_\circ(\mathrm{c}))$. For any $k \in \{1, \ldots, n+N\}$ we then have*

$$\mathsf{w}(J,k) = \mathsf{w}(I,k) - \lambda \mathsf{r}(I,i) \text{ for some } \lambda \in \mathbb{Z}_{>0} \text{ and } \mathsf{r}(I,i) \in \Phi^+.$$

In particular, $\mathsf{w}(I,k) - \mathsf{w}(J,k) \in L^+$.

Proof. This is a direct consequence of Lemma 3.2 (4) and (5).

The following lemma has not been considered before and will serve as the starting point of understanding F-polynomials in terms of the weight function.

Lemma 3.7. *For $k \in \{n+1, \ldots, n+N\}$, we have that*

$$\mathsf{w}(I_{\mathrm{g}},k) - \mathsf{w}(I_{\mathrm{ag}},k) = \mathsf{r}(I_{\mathrm{g}},k).$$

Observe that, as we have seen in Equation (5) on page 552, this is also closely related to the bijection relating cluster algebras and subword complexes.

Proof of Lemma 3.7. Starting with the greedy facet I_{g}, we flip the first position as long as we can without flipping into position k to obtain a facet I. Observe that along these flips, the facet always consists of a consecutive sequence of the n simple reflections. We therefore obtain, up to commutations of consecutive commuting letters, that $I = \{k-n, \ldots, k-1\}$ and

$$\mathsf{w}(I_{\mathrm{g}},k) = \mathsf{w}(I,k).$$

By the same argument, we flip the last position in I_{ag} until we flip into position k to obtain a facet J. Again up to commutations of consecutive commuting letters, we get that $J = \{k, \ldots, k+n-1\}$ and

$$\mathsf{w}(I_{\mathrm{ag}},k) = \mathsf{w}(J,k) = \mathsf{w}(J',k).$$

Here J' is the facet obtained from J by again flipping in J the last position $n-1$ times so that, up to commutations, we have $J' = \{k-n+1, \ldots, k\}$. The second equality is thus a direct consequence of Lemma 3.5 as $k \in J \cap J'$. With these observations, we finally obtain for $\mathrm{w}_\circ(\mathrm{c}) = \mathrm{q}_1 \dots \mathrm{q}_N$ that

$$\begin{aligned} \mathsf{w}(I_{\mathrm{g}}, k) - \mathsf{w}(I_{\mathrm{ag}}, k) &= \mathsf{w}(\{k-n, \dots, k-1\}, k) - \mathsf{w}(\{k-n+1 \dots, k\}, k) \\ &= q_1 \dots q_{k-1}(\omega_{q_k}) - q_1 \dots q_k(\omega_{q_k}) \\ &= q_1 \dots q_{k-1}(\omega_{q_k}) - q_1 \dots q_{k-1}(\omega_{q_k} - \alpha_{q_k}) \\ &= q_1 \dots q_{k-1}(\alpha_{q_k}) \\ &= \mathsf{r}(I_{\mathrm{g}}, k). \end{aligned}$$

We refer to the last table in Example 2.24 for an example of this correspondence in type A_2.

4. Proof of Theorem 2.14

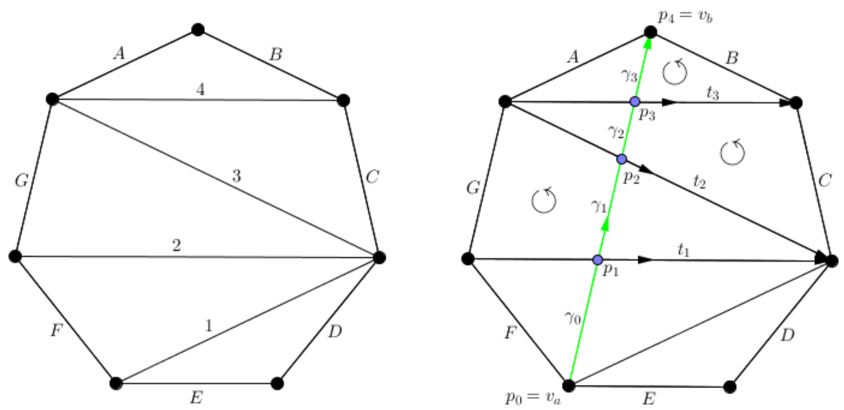
Before we recall the construction for the F-polynomials of type A_n to prove Theorem 2.14, we set the needed notations. The Coxeter group W is the symmetric group $\mathfrak{S}_{n+1}$ acting on $\mathsf{R}^{n+1}/\mathsf{R}(1, \dots, 1)$, whose simple system S is the set of simple transpositions $\mathsf{S} = \{\tau_1, \dots, \tau_n\}$ for $\tau_i = (i, i+1)$ interchanging e_i and e_{i+1}. Thus, the Coxeter element c is given by the product of all simple transpositions in some order. The simple roots are moreover given by $\Delta = \{e_i - e_{i+1} : 1 \leqslant i \leqslant n\}$, the positive roots by $\Phi^+ = \{e_i - e_{j+1} : 1 \leqslant i \leqslant j \leqslant n\}$, and the fundamental weights by $= \{e_1 + \dots + e_i : 1 \leqslant i \leqslant n\}$. We refer to Example 2.24 on page 557 for these notations in type A_2.

For consecutive simple transpositions, we write $\tau_i < \tau_{i-1}$ if τ_i appears to the left of τ_{i-1} in c, and $\tau_i > \tau_{i-1}$ if τ_i appears to the right of τ_{i-1}. We say that an element $\tau_{i_1} \dots \tau_{i_m}$ is a ***prefix of*** c if there is a reduced word c for c beginning with $\tau_{i_1}, \dots, \tau_{i_m}$. If all $\tau_{i_1}, \dots, \tau_{i_m}$ are inside the interval $\{\tau_i, \tau_{i+1}, \dots, \tau_j\}$ for $i \leqslant j$, we moreover say that it is a ***prefix of*** c ***restricted to*** $\{\tau_i, \dots, \tau_j\}$ if the prefix property holds after removing all letters not in $\{\tau_i, \dots, \tau_j\}$ from c. As an example, consider $c = \tau_1\tau_3\tau_2 = \tau_3\tau_1\tau_2 \quad \mathfrak{S}_4$. The prefixes of c are $-, \tau_1, \tau_3, \tau_1\tau_3 = \tau_3\tau_1, \tau_1\tau_3\tau_2$, and the prefixes of c restricted to $\{\tau_1, \tau_2\}$ are $-, \tau_1, \tau_1\tau_2$.

4.1. F-polynomials from T-paths. R. Schiffler derived in [25] an explicit formula for the cluster variables of type A_n via ***T-paths***. These are certain paths on the diagonals of triangulations of a regular $(n+3)$-gon. G. Musiker and R. Schiffler then extended that description and obtained in [16] an explicit formula for cluster variables in a similar fashion for cluster algebras with principal coefficients associated to unpunctured surfaces. Together with L. Williams, they extended in [17] the results also to arbitrary surfaces, allowing punctures. In this section, we review that construction for type A_n to establish the needed notions to relate the description to the weight function in order to derive Theorem 2.14. To present their results in a convenient way, we follow [16, Section 5] as they directly work with principal coefficients, except that we use slightly simplified notions of T-paths.

Let T be a triangulation of a regular $(n+3)$-gon, with boundary diagonals (or edges) labelled by $B_1, \dots, B_{n+3}$ and with proper diagonals labelled by $\tau_1, \dots, \tau_n$. An example can be found in Figure 1(a); we use $A, B, C, \dots$ instead of $B_1, B_2, B_3, \dots$ and $1, 2, 3, \dots$ instead of $\tau_1, \tau_2, \tau_3, \dots$ in examples for better readability.

Let $\gamma \,/\, T$ be another proper diagonal connecting non-adjacent vertices v_a and v_b, oriented from v_a to v_b. Denote the intersection points of γ with the diagonals in T along its orientation by $p_1, \dots, p_d$, and the corresponding diagonals in T by $t_1, \dots, t_d$. Let γ_k denote the segment of γ from point p_k to point p_{k+1}, where we use $p_0 = v_a$ and $p_{d+1} = v_b$. Each γ_k lies in exactly one triangle $\;_k$, and we orient the diagonal t_k in T by the orientation induced from the counterclockwise orientation of $\;_k$. Note



(a) (b)

Figure 1. A path in a triangulation of the 7-gon.

that if one considers the opposite path γ^{-1} from v_b to v_a, then the segment γ_i would become γ_{d+1-i} and t_i would become t_{d+1-i}. Moreover, the induced orientations of all t_k would change.

A *T-path* ζ from v_a to v_b in T is a path $\zeta = (\zeta_1, \ldots, \zeta_{2d+1})$ in T where each ζ_i is either a diagonal in T or a boundary edge and which uses the oriented diagonals in the even positions, i.e. $\zeta_{2k} = t_k$ for $1 \leqslant k \leqslant d$. Observe that such a T-path is uniquely determined by the directions in which the diagonals $t_1, \ldots, t_d$ in the even position are followed. If the direction of ζ coincides along the diagonal t_k with the direction induced by the counterclockwise orientation of the triangle Δ_k, we write that ζ travels t_k in *positive direction*, and it travels t_k in *negative direction* otherwise. It is not hard to see that there is always a unique T-path that travels all t_k's in positive direction. We call this path the *greedy T-path*, and denote it by ζ_{g}. Similarly, we denote by ζ_{ag} the *antigreedy T-path* that travels all t_k's in negative direction. (Note that these appeared in [16] as $\widetilde{\alpha}_{P_+}$ and $\widetilde{\alpha}_{P_-}$ in the paragraph before Theorem 5.1.) For instance, the greedy T-path in Figure 1(b) is $(F, 2, 3, 3, 3, 4, B)$ and the antigreedy T-path is $(1, 2, 2, 3, 4, 4, A)$.

We say that two T-paths ζ and ζ' are *flipped* if ζ and ζ' only differ in two odd positions $2k-1$ and $2k+1$ for some k. In other words, t_k is the unique diagonal which is traveled by ζ and ζ' in opposite directions, while all others are traveled in the same direction. We thus also say that t_k *is flipped* between ζ and ζ'. In Figure 1(b), flipping $\zeta_6 = t_3$ in the T-path $(F, 2, 3, 3, 3, 4, B)$ yields the T-path $(F, 2, 3, 3, C, 4, A)$.

To a T-path $\zeta = (\zeta_1, \ldots, \zeta_{2d+1})$, one associates the monomial $m[\zeta]$ given by the product of variables y_ℓ such that $\zeta_{2k} = t_k = \tau_\ell$ is traveled in positive direction. For instance, the greedy T-path $\zeta_{\mathrm{g}} = (F, 2, 3, 3, 3, 4, B)$ yields the monomial $m[\zeta_{\mathrm{g}}] = y_2y_3y_4$, while the antigreedy T-path $\zeta_{\mathrm{ag}} = (1, 2, 2, 3, 4, 4, A)$ yields $m[\zeta_{\mathrm{ag}}] = 1$. Moreover, all monomials obtained from T-paths for the diagonal γ in the example are given by

ζ	$m[\zeta]$
$F2^+3\,3^+3\,4^+B$	$y_2y_3y_4$
$F2^+3\,3^+C4^-A$	y_2y_3
$1\,2^-G3^+3\,4^+B$	y_3y_4
$1\,2^-G3^+C4^-A$	y_3
$1\,2^-2\,3^-4\,4^-A$	1

where we labelled the even steps $\zeta_{2k} = t_k$ for $1 \leqslant k \leqslant d$ with a "$+$" if it is traveled in positive direction and with a "$-$" if it is traveled in negative direction. Also observe how $m[\zeta]$ changes under flips. If t_i is flipped between ζ and ζ' then $m[\zeta] = m[\zeta'] \cdot y_\ell$ if $t_i = \tau_\ell$ is traveled in ζ in positive direction (and thus ζ' in negative direction).

As we have noted above, the orientations of the t_k's depend on the orientation of γ, while the monomial $m[\zeta]$ *does not* depend on this orientation, but only on the two unordered endpoints $\{v_a, v_b\}$. This combinatorial model now provides a description of the F-polynomials for the cluster algebra where the initial datum is the fixed given triangulation T of a regular $(n+3)$-gon. It is well-known that F-polynomials for this cluster algebra are indexed by (unoriented) diagonals $\gamma \notin T$, see [16].

Theorem 4.1 ([16, Theorem 5.1]). *Let T be a triangulation of the regular $(n+3)$-gon, and let $\gamma \notin T$ with endpoints $\{v_a, v_b\}$. The F-polynomial $F_\gamma(\mathbf{y})$ of γ is then given by*

$$F_\gamma(\mathbf{y}) = \sum m[\zeta],$$

where the sum ranges over all T-paths ζ from v_a to v_b.

Next, we recall how to associate a triangulation T_c of the regular $(n+3)$-gon to a Coxeter element c in type A_n.

(1) Pick a fixed vertex of the $(n+3)$-gon, labelled v_1, and draw an edge connecting the two vertices adjacent to v_1. Label the new edge by τ_1.
(2) For each $i = 2, \dots, n$, label the vertex

$$\begin{cases} \text{clockwise} & \text{if } \tau_i < \tau_{i-1} \\ \text{counterclockwise} & \text{if } \tau_i > \tau_{i-1} \end{cases}$$

from v_{i-1} by v_i, draw an edge connecting the two vertices adjacent to v_i and different from v_{i-1}, and label the new edge τ_i.

An example of type A_3 with $c = \tau_1\tau_3\tau_2$ is shown in Figure 2. Moreover, the triangulation in Figure 1 corresponds to the Coxeter element $c = \tau_3\tau_2\tau_1\tau_4$ in type A_4.

Lemma 4.2. *Let c be a Coxeter element in type A_n, and let $1 \leqslant i \leqslant j \leqslant n$. Then there is a unique diagonal $\gamma \notin T_c$ that crosses exactly the diagonals labelled by $\tau_i, \tau_{i+1}, \dots, \tau_j$, and every diagonal not in T_c can be obtained this way.*

Proof. The triangulations that can be obtained (up to rotational symmetry) from a Coxeter element by the procedure are exactly the triangulations that do not have inner triangles, i.e. no triangles for which all three sides are proper diagonals. As the diagonals are labelled consecutively, the statement follows.

Example 4.3. Figure 2 shows for $c = \tau_1\tau_3\tau_2$ in type A_3 and each positive root $\beta = e_i - e_{j+1}$, the unique diagonal $\gamma \notin T_c$ crossing the diagonals $\tau_i, \dots, \tau_j$, and all T_c-paths for this γ.

We have the following corollary of the above Theorem 4.1, which we will then use to deduce Theorem 2.14.

Corollary 4.4. *Let c be a Coxeter element in type A_n, let $\beta = e_i - e_{j+1}$ be a positive root, and let $\gamma \notin T_c$ be the unique diagonal crossing exactly the diagonals labelled $\tau_i, \dots, \tau_j$ in T_c. Let the endpoints of γ be $\{v_a, v_b\}$. The F-polynomials for the cluster algebra $A(W, c)$ associated to β is then given by*

$$\mathsf{F}_\beta(\mathbf{y}) = \sum m[\zeta],$$

where the sum ranges over all T-paths ζ from v_a to v_b.

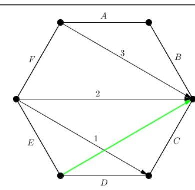
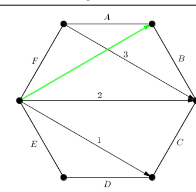
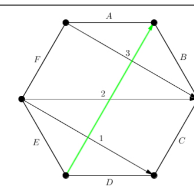
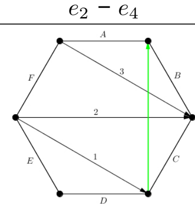
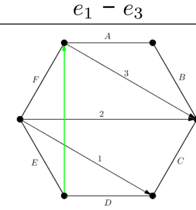
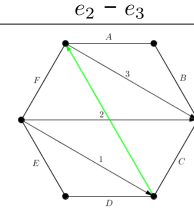
β	$e_1 - e_2$	$e_3 - e_4$	$e_1 - e_4$
γ			
T_c-paths	$E1^+C$ $D1^-2$	$2\,3^+A$ $F3^-B$	$E1^+1\,2^+3\,3^+B$ $E1^+C2^-F3^+B$ $D1^-2\,2^-F3^+B$ $E1^+C2^-2\,3^-A$ $D1^-2\,2^-2\,3^-A$
$\mathsf{F}_\beta(\mathbf{y})$	y_1+1	y_3+1	$y_1y_2y_3+y_1y_3+y_3+y_1+1$
β	$e_2 - e_4$	$e_1 - e_3$	$e_2 - e_3$
γ			
T_c-paths	$1\,2^+3\,3^+B$ $C2^-F3^+B$ $C2^-2\,3^-A$	$E1^+1\,2^+3$ $E1^+C2^-F$ $D1^-2\,2^-F$	$1\,2^+3$ $C2^-F$
$\mathsf{F}_\beta(\mathbf{y})$	$y_2y_3+y_3+1$	$y_1y_2+y_1+1$	y_2+1

Figure 2. T_c-paths for $c=\tau_1\tau_3\tau_2$ in type A_3.

Proof. This follows from the well known connection between $A(W,c)$ and the triangulation T_c described above.

Example 4.5. For $c=\tau_1\tau_3\tau_2$ in type A_3, Figure 2 also provides all $\mathsf{F}_\beta(\mathbf{y})$ for $\beta \quad \Phi^+$ using the construction of T_c-paths.

We finally need the following proposition regarding possible flips in triangulations with respect to a Coxeter element c.

Proposition 4.6. *Let T_c be the triangulation associated to a Coxeter element c, and let $\gamma \,/\, T_c$ be the unique diagonal oriented from v_a to v_b which crosses exactly the diagonals $\tau_i,\dots,\tau_j$ in this order. Then for any prefix $\tau_{i_1}\dots\tau_{i_m}$ of c restricted to $\{\tau_i,\dots,\tau_j\}$, one can flip the diagonals labelled $\tau_{i_1},\dots,\tau_{i_m}$ in this order in the anti-greedy T_c-path from v_a to v_b. Moreover, every T_c-path from v_a to v_b is obtained this way for a unique prefix.*

Proof. We explicitly describe the four possible restrictions for directions in which T_c-paths can travel. To this end, consider the situation that t_{i-1} and t_i are oriented towards their shared vertex in T_c, or, equivalently, that $\tau_i < \tau_{i-1}$ (see the path for $e_2 - e_4$ in Figure 2). Then, any T_c-path ζ from v_a to v_b

- that travels the diagonal $\zeta_{2i-2} = t_{i-1}$ in *positive* direction must also travel $\zeta_{2i} = t_i$ in *positive* direction, and
- that travels the diagonal $\zeta_{2i} = t_i$ in *negative* direction must also travel $\zeta_{2i-2} = t_{i-1}$ in *negative* direction.

The situation where t_{i-1} and t_i are oriented away from their shared vertex in T_c, or, equivalently, that $\tau_i > \tau_{i-1}$ is the same with the roles of positive and negative direction interchanged (see the path for $e_1 - e_3$ in Figure 2).

Clearly, these are the only restrictions on T_c-paths. This means that a given sequence of orientations of the t_i's corresponds to a T_c-path if and only if these restrictions are satisfied. It then directly follows that every T_c-path is uniquely obtained from the antigreedy T_c-path ζ_{ag} (which travels all the diagonals in negative direction) by flipping diagonals labelled $\tau_{i_1}, \dots, \tau_{i_m}$ in this order for a prefix $\tau_{i_1} \dots \tau_{i_m}$ of c restricted to $\{\tau_i, \dots, \tau_j\}$, as desired. Observe here that if one considers two different words for the same prefix, then both sequences of flips yield the same T_c-path, as expected.

We refer to Figure 2 for several examples, and also to Example 4.8 below for two concrete computations.

Corollary 4.7. *In the situation of Corollary 4.4, we have that*

$$\mathsf{F}_\beta(\mathbf{y}) = \sum y_{i_1} \dots y_{i_m}$$

where the sum ranges over all prefixes $\tau_{i_1} \dots \tau_{i_m}$ of c restricted to $\{\tau_i, \dots, \tau_j\}$.

Proof. Proposition 4.6 implies that $\mathsf{F}_\beta(\mathbf{y}) = \sum m[\zeta_{i_1,\dots,i_m}]$ where the sum ranges over all prefixes of c restricted to $\{\tau_i, \dots, \tau_j\}$. Moreover, the resulting T-path $\zeta_{i_1,\dots,i_m}$ is obtained from the antigreedy T-path ζ_{ag} by flipping all the diagonals $\tau_{i_1}, \dots, \tau_{i_m}$ in this order. The definition of $m[\zeta]$ thus implies that $m[\zeta_{i_1,\dots,i_m}] = y_{i_1} \dots y_{i_m}$, as desired.

Example 4.8. Following the two examples of the two positive roots $e_1 - e_4 = \alpha_1 + \alpha_2 + \alpha_3$ and $e_1 - e_3 = \alpha_1 + \alpha_2$ in type A_3 with $c = \tau_1\tau_3\tau_2$ above, we obtain that the prefixes of c are $-, \tau_1, \tau_3, \tau_1\tau_3, \tau_1\tau_3\tau_2$ yielding

$$\mathsf{F}_{e_1-e_4}(\mathbf{y}) = 1 + y_1 + y_3 + y_1y_3 + y_1y_2y_3,$$

and that the prefixes of c restricted to $\{\tau_1, \tau_2\}$ are $-, \tau_1, \tau_1\tau_2$ yielding

$$\mathsf{F}_{e_1-e_3}(\mathbf{y}) = 1 + y_1 + y_1y_2.$$

Both cases can be checked in Figure 2.

4.2. F-polynomials from subword complexes. We now use Corollary 4.7 to obtain the F-polynomials from the weight vectors of $\mathsf{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ by providing the analogous property of the weight vectors in Theorem 4.12. We start with the following explicit description of all different weights $\mathsf{w}(I,k)$ that occur for the various facets for a fixed position k. For $1 \leqslant i \leqslant j \leqslant n$, consider the positive root $e_i - e_{j+1}$. We have seen in Lemma 3.2 that there is a unique $k \in \{n+1, \dots, n+N\}$ such that $\mathsf{r}(I_{\mathrm{g}}, k) = e_i - e_{j+1}$, and we have then seen in Lemma 3.7 that

$$\mathsf{w}(I_{\mathrm{g}}, k) - \mathsf{w}(I_{\mathrm{ag}}, k) = \mathsf{r}(I_{\mathrm{g}}, k) = e_i - e_{j+1}. \tag{10}$$

This yields the following lemma.

Lemma 4.9. *We have*

$$\mathsf{w}(I_{\mathrm{g}}, k) = (\epsilon_1, \dots, \epsilon_{i-1},\ 1,\ \epsilon_{i+1}, \dots, \epsilon_j,\ 0,\ \epsilon_{j+2}, \dots, \epsilon_{n+1}) \tag{11}$$

$$\mathsf{w}(I_{\mathrm{ag}}, k) = (\epsilon_1, \dots, \epsilon_{i-1},\ 0,\ \epsilon_{i+1}, \dots, \epsilon_j,\ 1,\ \epsilon_{j+2}, \dots, \epsilon_{n+1}) \tag{12}$$

for fixed $\epsilon_i \in \{0,1\}$ with $i \in \{1, \dots, n+1\} \smallsetminus \{i, j+1\}$.

Proof. This follows from (10) as all W-orbits of fundamental weights in type A_n consist of $(0,1)$-vectors.

I	α_1 4	α_3 5	$\alpha_1+\alpha_2+\alpha_3$ 6	$\alpha_2+\alpha_3$ 7	$\alpha_1+\alpha_2$ 8	α_2 9
$I_{\text{g}} = \{1, 2, 3\}$	$(1,0,0,0)$	$(1,1,1,0)$	$(1,1,0,0)$	$(0,1,0,0)$	$(1,1,0,1)$	$(0,1,0,1)$
$\{1, 2, 9\}$	$(1,0,0,0)$	$(1,1,1,0)$	$(1,0,1,0)$	$(0,0,1,0)$	$(1,0,1,1)$	$(0,0,1,1)$
$\{1, 3, 5\}$	$(1,0,0,0)$	$(1,1,0,1)$	$(1,1,0,0)$	$(0,1,0,0)$	$(1,1,0,1)$	$(0,1,0,1)$
$\{1, 5, 7\}$	$(1,0,0,0)$	$(1,1,0,1)$	$(1,0,0,1)$	$(0,0,0,1)$	$(1,1,0,1)$	$(0,1,0,1)$
$\{1, 7, 9\}$	$(1,0,0,0)$	$(1,1,0,1)$	$(1,0,0,1)$	$(0,0,0,1)$	$(1,0,1,1)$	$(0,0,1,1)$
$\{2, 3, 4\}$	$(0,1,0,0)$	$(1,1,1,0)$	$(1,1,0,0)$	$(0,1,0,0)$	$(1,1,0,1)$	$(0,1,0,1)$
$\{2, 4, 8\}$	$(0,1,0,0)$	$(1,1,1,0)$	$(0,1,1,0)$	$(0,1,0,0)$	$(0,1,1,1)$	$(0,1,0,1)$
$\{2, 8, 9\}$	$(0,1,0,0)$	$(1,1,1,0)$	$(0,1,1,0)$	$(0,0,1,0)$	$(0,1,1,1)$	$(0,0,1,1)$
$\{3, 4, 5\}$	$(0,1,0,0)$	$(1,1,0,1)$	$(1,1,0,0)$	$(0,1,0,0)$	$(1,1,0,1)$	$(0,1,0,1)$
$\{4, 5, 6\}$	$(0,1,0,0)$	$(1,1,0,1)$	$(0,1,0,1)$	$(0,1,0,0)$	$(1,1,0,1)$	$(0,1,0,1)$
$\{4, 6, 8\}$	$(0,1,0,0)$	$(1,1,0,1)$	$(0,1,0,1)$	$(0,1,0,0)$	$(0,1,1,1)$	$(0,1,0,1)$
$\{5, 6, 7\}$	$(0,1,0,0)$	$(1,1,0,1)$	$(0,1,0,1)$	$(0,0,0,1)$	$(1,1,0,1)$	$(0,1,0,1)$
$\{6, 7, 8\}$	$(0,1,0,0)$	$(1,1,0,1)$	$(0,1,0,1)$	$(0,0,0,1)$	$(0,1,1,1)$	$(0,1,0,1)$
$I_{\text{ag}} = \{7, 8, 9\}$	$(0,1,0,0)$	$(1,1,0,1)$	$(0,1,0,1)$	$(0,0,0,1)$	$(0,1,1,1)$	$(0,0,1,1)$

Figure 3. The weights of the facets appearing in the unique shortest chain of increasing flips from the greedy to the antigreedy facet for $c = \tau_1\tau_3\tau_2$ in type A_3.

Using this observation, one explicitly obtains these weights for a given Coxeter element c as described next.

Proposition 4.10. *In the situation of Lemma 4.9, we have the following properties of the ϵ_i's (where the cases in (2) are only considered either if $i > 1$, or if $j < n$, respectively):*

(1) $\epsilon_1 = \cdots = \epsilon_{i-1}$ *and* $\epsilon_{j+2} = \cdots = \epsilon_{n+1}$*;*

(2) $\epsilon_{i-1} = \begin{cases} 0 & \text{if } \tau_{i-1} < \tau_i \\ 1 & \text{if } \tau_{i-1} > \tau_i \end{cases}$ *and* $\epsilon_{j+2} = \begin{cases} 0 & \text{if } \tau_{j+1} < \tau_{j+2} \\ 1 & \text{if } \tau_{j+1} > \tau_{j+2}; \end{cases}$

(3) *for* $i < \ell \leqslant j$, $\epsilon_\ell = \begin{cases} 1 \text{ if } \tau_{\ell-1} < \tau_\ell \\ 0 \text{ if } \tau_{\ell-1} > \tau_\ell. \end{cases}$

Proof. One can flip the letters in the initial copy of c inside $\mathrm{cw}_\circ(\mathrm{c})$ from right to left to obtain a sequence of increasing flips from I_{g} to I_{ag}. (We indeed obtain exactly the shortest sequences of increasing flips from the greedy to the antigreedy facet.) As the root configuration of I_{g} is given by all simple roots $\{e_1 - e_2, \ldots, e_n - e_{n+1}\}$, Lemma 3.2(3) implies that along the above sequence of flips from I_{g} to I_{ag}, every pattern $(\epsilon_\ell, \epsilon_{\ell+1})$ of consecutive indices of $\mathsf{w}(I_{\text{g}}, k)$ is updated exactly once (where we set $\epsilon_i = 1$ and $\epsilon_{j+1} = 0$ as in (11). Moreover, Lemma 3.6 implies that along this procedure, either $\epsilon_\ell = \epsilon_{\ell+1}$ and the application of τ_ℓ does not change the weight, or $(\epsilon_\ell, \epsilon_{\ell+1}) = (1, 0)$, and this application moves the 1 in position ℓ into position $\ell + 1$. As such a move is not reversible again by Lemma 3.6, we directly obtain the second property. The first property is obtained with the additional observation that those entries coincide in I_{g} and in I_{ag}. The last property finally follows with the observation that every τ_ℓ for $i \leqslant \ell \leqslant j$ must indeed move a 1 one position to the right to obtain I_{ag} from I_{g} this way.

Example 4.11. We again consider $c = \tau_1\tau_3\tau_2$ in type A_3. Figure 3 shows all weights $\mathsf{w}(I, k)$ in this case, the weights for I_{g} and I_{ag} can be computed as described in Lemma 4.9 and Proposition 4.10.

To state the main observation towards the proof of Theorem 2.14, define the following set of weights as an "interval" in the weights,

$$\mathrm{Int}\left(\mathsf{w}(I_{\mathrm{g}},k),\ \mathsf{w}(I_{\mathrm{ag}},k)\right)=\left\{\omega \in W(\ ) \,:\, \mathsf{w}(I_{\mathrm{g}},k)-\omega,\omega-\mathsf{w}(I_{\mathrm{ag}},k) \in L^{+}\right\}.$$

Using this notion, we deduce the following theorem from Proposition 4.10, for which we need one additional observation. We may flip from the greedy to the antigreedy facet in $\mathcal{SC}\left(\mathrm{cw}_\circ(\mathrm{c})\right)$ in n flips. For a fixed expression $c=s_1\dots s_n$, these flips are exactly given by sequences of flips in positions $i_n,\dots,i_1$ such that $c=s_{i_1}\dots s_{i_n}$. We denote by $\mathcal{I}$ the set of facets of $\mathcal{SC}\left(\mathrm{cw}_\circ(\mathrm{c})\right)$ that lie on such a shortest sequence of increasing flips and remark that $\mathcal{I}$ is given by construction by all facets I that are obtained from I_{g} by flipping *suffixes* of the Coxeter element c, compare Example 4.13.

Theorem 4.12. *We then have for* $k \in \{1,\dots,n+N\}$ *that*

$$\left\{\mathsf{w}(I,k) \,:\, I \text{ facet of } \mathcal{SC}\left(\mathrm{cw}_\circ(\mathrm{c})\right)\right\}=\left\{\mathsf{w}(I,k) \,:\, I \in \mathcal{I}\right\}.$$

Proof. The first inclusion in

$$\begin{aligned}\left\{\mathsf{w}(I,k) \,:\, I \in \mathcal{I}\right\} &\subseteq \left\{\mathsf{w}(I,k) \,:\, I \text{ facet of } \mathcal{SC}\left(\mathrm{cw}_\circ(\mathrm{c})\right)\right\}\\ &\subseteq \mathrm{Int}\left(\mathsf{w}(I_{\mathrm{g}},k),\ \mathsf{w}(I_{\mathrm{ag}},k)\right).\end{aligned}$$

is trivial, while the second is a direct consequence of Lemma 3.6 as every facet lies on a (not necessarily shortest) sequence of increasing flips from the greedy to the antigreedy facet. On the other hand, Proposition 4.10 implies that $\{\mathsf{w}(I,k) \,:\, I \in \mathcal{I}\} = \mathrm{Int}\left(\mathsf{w}(I_{\mathrm{g}},k),\ \mathsf{w}(I_{\mathrm{ag}},k)\right)$ as follows. For $\mathsf{r}(I_{\mathrm{g}},k)=e_i-e_{j+1}$, we have that the interval $\mathrm{Int}\left(\mathsf{w}(I_{\mathrm{g}},k),\ \mathsf{w}(I_{\mathrm{ag}},k)\right)$ is given by starting with $\mathsf{w}(I_{\mathrm{g}},k)$ and replacing consecutive pattern $(\epsilon_\ell,\epsilon_{\ell+1})=(1,0)$ by $(0,1)$ in all possible ways so that every consecutive pattern $(\epsilon_\ell,\epsilon_{\ell+1})$ with $i \leqslant \ell \leqslant j$ is modified exactly once. Now observe that Proposition 4.10(3) gives that $(\epsilon_\ell,\epsilon_{\ell+1})=(1,0)$ in $\mathsf{w}(I_{\mathrm{g}},k)$ if and only if $\tau_{\ell-1}<\tau_\ell>\tau_{\ell+1}$ for $i<\ell<j$ and $\tau_\ell>\tau_{\ell+1}$ for $\ell=i$ and $\tau_{\ell-1}<\tau_\ell$ for $\ell=j$. This means that the possible ways to replace consecutive patterns $(\epsilon_\ell,\epsilon_{\ell+1})=(1,0)$ by $(0,1)$ in $\mathsf{w}(I_{\mathrm{g}},k)$ is given by doing this replacements at all indices corresponding to *suffixes* of the Coxeter element c when restricted to $\tau_i,\dots,\tau_j$.

Example 4.13. In the example in Figure 3, we have two shortest chains

$$\begin{array}{ccccc} & & \nearrow \{1,7,9\} \searrow & & \\ \{1,2,3\} \longrightarrow \{1,2,9\} & & & & \{7,8,9\} \\ & & \searrow \{2,8,9\} \nearrow & & \end{array}$$

of facets from $I_{\mathrm{g}}=\{1,2,3\}$ to $I_{\mathrm{ag}}=\{7,8,9\}$ (corresponding to the two reduced words $c=\tau_1\tau_3\tau_2=s_1s_2s_3=s_2s_1s_3$). As highlighted in Figure 3 one observes that these weights indeed yield the complete desired intervals.

To clarify the procedure in the previous proof, we also give a larger example in type A_8 with $c=\tau_3\tau_7\tau_2\tau_4\tau_5\tau_6\tau_8\tau_1$ Let $\beta=e_1-e_9=\alpha_1+\dots+\alpha_8$, so $i=1$ and $j=8$. For the appropriate k with $\mathsf{r}(I_{\mathrm{g}},k)=\beta$, we have according to Proposition 4.10 that

$$\begin{aligned}\mathsf{w}(I_{\mathrm{g}},k)&=(1,0,0,1,1,1,0,1,0)\\ \mathsf{w}(I_{\mathrm{ag}},k)&=(0,0,0,1,1,1,0,1,1).\end{aligned}$$

As expected, one obtains $\mathrm{Int}\left(\mathsf{w}(I_{\mathrm{g}},k),\ \mathsf{w}(I_{\mathrm{ag}},k)\right)$ by replacing consecutive pattern $(\epsilon_\ell,\epsilon_{\ell+1})=(1,0)$ by $(0,1)$ at every index $1 \leqslant \ell \leqslant 8$ exactly once (so that the 1 in position 8 goes to position 9, the 1 in position 6 goes to position 8 and so on). Now, the three indices $1,6,8$ are exactly the initial positions ℓ such that $(\epsilon_\ell,\epsilon_{\ell+1})=(1,0)$ and as well the three indices that τ_ℓ is a suffix of c. The suffix corresponding to above expresssion for c yields that we are allowed to flip positions $1, 8, 6, 5, 4, 2, 7, 3$ in this order. $\square$

Given this theorem, we obtain indeed all weights of facets of $\mathsf{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$.

Proof of Theorem 2.14. Let $\mathcal{A}(W, c)$ be the cluster algebra of type A_n for a given Coxeter element c, and let $\beta = e_i - e_{j+1}$ be a positive root. By Corollary 4.7, we have that

$$\mathsf{F}_\beta(\mathbf{y}) = \sum y_{i_1} \ldots y_{i_m}$$

where the sum ranges over all prefixes $\tau_{i_1} \ldots \tau_{i_m}$ of c restricted to $\{\tau_i, \ldots, \tau_j\}$. Let k be the unique index such that $\mathsf{r}(I_{\mathrm{g}}, k) = \beta$. We aim to show that this sum is also given by $\sum \mathbf{y}^\gamma$ where the sum ranges over all γ in

$$\big\{ \mathsf{w}(I, k) - \mathsf{w}(I_{\mathrm{ag}}, k) \ : \ I \in \mathcal{I} \big\}$$

expressed in the root basis Δ. Together with Theorem 4.12, this implies Conjecture 2.12 for $\mathcal{A}(W, c)$. Conjecture 2.13 is then trivial as all exponent vectors of monomials in $\mathsf{F}_\beta(\mathbf{y})$ are $(0, 1)$-vectors in type A.

By construction, the set $\mathcal{I}$ in Theorem 4.12 is obtained from the greedy facet I_{g} by flipping all possible sequences of *suffixes* of the Coxeter element c. Let $i_1, \ldots, i_n$ be such that $c = \tau_{i_1} \ldots \tau_{i_n}$, and let I_m be the facet obtained from I_{g} by flipping the letters $i_n, i_{n-1}, \ldots, i_m$ in this order. Then flipping the letter i_n in I_{g} yields I_n, flipping the letter i_m in I_{m+1} yields I_m and $I_1 = I_{\mathrm{ag}}$. Moreover,

$$\mathsf{r}(I_{\mathrm{g}}, i_m) = \mathsf{r}(I_{m+1}, i_m) = \alpha_{i_m} \tag{13}$$

because I_{g} and I_{m+1} coincide until position i_m. This means that $\mathsf{w}(I_{m+1}, k) - \mathsf{w}(I_m, k)$ is either 0 or α_{i_m}. Since $\mathsf{w}(I_{\mathrm{g}}, k) - \mathsf{w}(I_{\mathrm{ag}}, k) = \mathsf{r}(I_{\mathrm{g}}, k)$, this implies that $\mathsf{w}(I_m, k) - \mathsf{w}(I_{\mathrm{ag}}, k)$ is obtained from $\mathsf{r}(I_{\mathrm{g}}, k)$ by replacing all 1's by 0's in positions $\{i_m, \ldots, i_n\}$. This is because every position that is 1 in $\mathsf{r}(I_{\mathrm{g}}, k)$ must become 0 within the sequence $I_{\mathrm{g}}, I_n, \ldots, I_1 = I_{\mathrm{ag}}$ of increasing flips, and (13) means that the only flip that may modify a position $1 \leqslant a \leqslant n$ is the flip of the index i_m with $i_m = a$.

In total, this gives that $\sum \mathbf{y}^\gamma$ ranging over all $\gamma \in \{\mathsf{w}(I, k) - \mathsf{w}(I_{\mathrm{ag}}, k) \ : \ I \in \mathcal{I}\}$ equals $\sum \mathbf{y}^\beta / (y_{i_1} \ldots y_{i_m})$ ranging over all suffixes of c restricted to $\{\tau_i, \ldots, \tau_j\}$. The later is clearly equal to the desired sum expression for $\mathsf{F}_\beta(\mathbf{y})$.

The given description of the F-polynomials in type A_n has the consequence that there is a generalization of Loday's realization of the classical associahedron mentioned in Remark 2.21 to all c-associahedra of type A_n.

Corollary 4.14. *The type A_n c-associahedron is given by*

$$\sum \mathrm{conv}\{e_{i_1} + \cdots + e_{i_m}\}$$

where the sum ranges over all pairs $1 \leqslant i \leqslant j \leqslant n$ and each convex hull is over all prefixes $i_m \ldots i_1$ of the Coxeter element c restricted to $\{\tau_i, \ldots, \tau_j\}$.

We remark that C. Lange obtained in [13] a different Minkowski decomposition of the c-associahedron into sums and differences of simplicies. We refer to [13, Theorem 4.3] and also to [14, Section 4] for details.

Example 4.15. We have seen that the prefixes of $c = \tau_1\tau_3\tau_2$ are given by

$$-, \tau_1, \tau_3, \tau_1\tau_3, \tau_1\tau_3\tau_2,$$

so that we obtain that the associahedron is given by

$$\begin{aligned} &\mathrm{conv}\{000, 100\} + \mathrm{conv}\{000, 100, 110\} + \mathrm{conv}\{000, 100, 001, 101, 111\} \\ &\qquad\qquad + \mathrm{conv}\{000, 010\} + \mathrm{conv}\{000, 001, 011\} + \mathrm{conv}\{000, 001\} \end{aligned}$$

where the summands correspond to the intervals $\{\tau_i, \ldots, \tau_j\}$ for $1 \leqslant i \leqslant j \leqslant 3$ in the order $\{\tau_1\}, \{\tau_1, \tau_2\}, \{\tau_1, \tau_2, \tau_3\}, \{\tau_2\}, \{\tau_2, \tau_3\}, \{\tau_3\}$. For example, the second summand is given by the convex hull of $\{000, 100, 110\}$. These are the different indicator vectors of prefixes $-, \tau_1, \tau_1\tau_2$ of c where the letter τ_3 is deleted.

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