

ALGEBRAIC COMBINATORICS


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Volume 2, issue 2 (2019), p. 249-274.

http://alco.centre-mersenne.org/item/ALCO_2019__2_2_249_0

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Hook removal operators on the odd Young graph

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ABSTRACT In this article we consider hook removal operators on *odd* partitions, i.e., partitions labelling odd-degree irreducible characters of finite symmetric groups. In particular we complete the discussion, started in [8] and developed in [3], concerning the commutativity of such operators.

1. INTRODUCTION

Let n be a natural number and let χ be an irreducible character of odd degree of the symmetric group \mathfrak{S}_n . In [8] it was shown that whenever k is a non-negative integer such that $2^k \leq n$, then there exists a unique irreducible constituent $f_k^n(\chi)$ of $\chi_{\mathfrak{S}_{n-2^k}}$ of odd degree appearing with odd multiplicity. This result (together with [5, Theorem 1.1]) was the key ingredient in giving a purely algebraic interpretation of the canonical McKay bijection for symmetric groups first described combinatorially in [6, Theorem 4.3].

Let $\text{Irr}_{2'}(\mathfrak{S}_n)$ be the set of irreducible characters of \mathfrak{S}_n of odd degree. The main aim of this article is to continue the study, begun in [3], of the maps

$$f_k^n : \text{Irr}_{2'}(\mathfrak{S}_n) \longrightarrow \text{Irr}_{2'}(\mathfrak{S}_{n-2^k}), \quad \text{for all } n, k \in \mathbb{N}.$$

To ease the notation we will sometimes write f_k instead of f_k^n when the domain of the function is clear from the context.

A first natural question in order to understand the *compatibility* of these functions with respect to each other is: *when does $f_k^{n-2^k} f_k^n = f_{k+1}^n$ hold?* (in brief: $f_k f_k = f_{k+1}$?). We are able to fully answer this question. In order to formulate the precise statements of our theorems we need to introduce some specific combinatorial concepts and notation. This is done in details in Section 4. Here we restrict ourselves to summarise our main results.

FIRST MAIN RESULT. *Let n and k be natural numbers such that $2^{k+1} \leq n$. Then:*

- (i) *We completely describe the partitions labelling those irreducible characters χ such that $f_k^{n-2^k} f_k^n(\chi) = f_{k+1}^n(\chi)$. (This description is obtained in Corollary 4.17 together with Lemma 4.11 and Proposition 4.13.)*
- (ii) *As a corollary of (i), we give an explicit formula for the number of such irreducible characters. (This is contained in Theorem 4.8 and Corollary 4.16.)*

Manuscript received 24th November 2017, revised 23rd July 2018 and 25th August 2018, accepted 30th August 2018.

KEYWORDS. Symmetric groups, Young graph.

Surprisingly, we are also able to show that when $n/2^k$ is *small*, we only need to check the value of χ on one specified conjugacy class to see whether $f_k f_k(\chi) = f_{k+1}(\chi)$ or not. In fact, if $\lfloor n/2^k \rfloor \leq 4$ then there is a specified element $g \in \mathfrak{S}_n$, depending on n and k , such that for all $\chi \in \text{Irr}_{2'}(\mathfrak{S}_n)$ we have that $f_k f_k(\chi) = f_{k+1}(\chi)$ if and only if $|\chi(g)| = 1$ (Theorem 4.20).

In the second half of the article we focus on the commutativity problem. More precisely, given $\ell, k, n \in \mathbb{N}$ such that $2^k + 2^\ell \leq n$ we aim to determine the set $\mathcal{T}_{k,\ell}(n)$ consisting of those partitions of n that label irreducible characters χ of \mathfrak{S}_n such that $f_k f_\ell(\chi) = f_\ell f_k(\chi)$. Again, we postpone the description of the necessary combinatorial framework to Sections 5 and 6. Here we summarize our results.

SECOND MAIN RESULT. *Let $\ell, k, n \in \mathbb{N}$ be such that $2^k + 2^\ell \leq n$. Then:*

- (i) *We completely describe the set $\mathcal{T}_{k,\ell}(n)$. (In Lemma 5.5 we first reduce the problem to the case $k = 0$. The case $k = 0$ is fully analysed in Section 6.)*
- (ii) *As a corollary of (i), we give an explicit formula for $|\mathcal{T}_{k,\ell}(n)|$. (This is done in Theorem 5.10 together with Corollary 6.20.)*

We remark that our Second Main Result concludes the study of commutativity of the maps f_k . This work began in [8] and was subsequently developed in [3]. The characters of odd degrees on which we focus in this article are labelled by partitions which we call odd partitions; thus we focus on a subgraph of the Young graph which we call the odd Young graph. The hook removal operators that we discuss here act on this odd Young graph, and the main theme of this article is the relationship of the behavior of these operators on this graph.

2. BACKGROUND

We start by recalling some facts in the representation theory of symmetric groups. We refer the reader to [9], [10] or [12] for a more detailed account.

2.1. PARTITIONS AND HOOKS. A partition $\lambda = (u_1, u_2, \dots, u_\ell)$ is a finite weakly decreasing sequence of positive integers. We say that λ is a partition of $|\lambda| = \sum u_i$, written $\lambda \vdash |\lambda|$ and call the u_i 's the *parts* of λ . The Young diagram of λ is the set

$$[\lambda] = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq \ell, 1 \leq j \leq u_i\},$$

where we think of the diagram in matrix orientation, with the node $(1, 1)$ in the upper left corner. Given $(r, c) \in [\lambda]$, the corresponding *hook* $H_{(r,c)}(\lambda)$ is the set defined by

$$H_{(r,c)}(\lambda) = \{(r, y) \in [\lambda] \mid y \geq c\} \cup \{(x, c) \in [\lambda] \mid x \geq r\}.$$

We let $h_{r,c}(\lambda) = |H_{(r,c)}(\lambda)|$ be the *length* of the hook, and we denote by $\mathcal{H}(\lambda)$ the set of hooks in λ . When $h = H_{(r,c)}(\lambda)$ is a hook of λ , we say that we remove it from λ by removing the nodes on the rim of the Young diagram, starting at the hand (r, u_r) and ending at the foot of column c ; the partition thus obtained is denoted by $\lambda \setminus h$. Let $e \in \mathbb{N}$. A hook of length e in λ is called an *e-hook* of λ and we denote by $\mathcal{H}_e(\lambda)$ the set of hooks in λ whose lengths are divisible by e .

2.2. ODD DEGREE CHARACTERS AND ODD PARTITIONS. Let n be a natural number. We let $\text{Irr}(\mathfrak{S}_n)$ denote the set of irreducible characters of \mathfrak{S}_n and $\mathcal{P}(n)$ the set of partitions of n . There is a natural correspondence $\lambda \leftrightarrow \chi^\lambda$ between $\mathcal{P}(n)$ and $\text{Irr}(\mathfrak{S}_n)$. In this case we say then that λ labels χ^λ . We denote by $\text{Irr}_{2'}(\mathfrak{S}_n)$ the set of irreducible characters of \mathfrak{S}_n of odd degree. If $\chi^\lambda \in \text{Irr}_{2'}(\mathfrak{S}_n)$ we say that χ^λ is an *odd character*, we call λ an *odd partition* of n and write $\lambda \vdash_o n$. We denote by $\mathcal{O}(n)$ the set of odd partitions of n ; by convention, $\mathcal{O}(0) = \{\emptyset\}$.

The following key result was proved in [8, Theorem A and Proposition 4.2], generalizing the beautiful result obtained in [1, Theorem 1].

THEOREM 2.1. *Let n, k be such that $2^k \leq n$ and let $\lambda \vdash_o n$. Then there exists a unique 2^k -hook h in λ such that the partition $\mu = \lambda \setminus h$ obtained by removing h from λ is an odd partition of $n - 2^k$. Furthermore, we then have $f_k^n(\chi^\lambda) = \chi^\mu$.*

In the setting of Theorem 2.1 we abuse notation and consider f_k^n also as a map between the corresponding sets of partitions $\mathcal{O}(n)$ and $\mathcal{O}(n - 2^k)$. In particular we will write $f_k^n(\lambda) = \mu$.

DEFINITION 2.2. *Let $n \in \mathbb{N}$ and $\lambda \vdash_o n$. We call a hook h in λ odd if $\lambda \setminus h$ is again an odd partition. If $2^k \leq n$, we let $z_k(\lambda)$ and $s_k(\lambda)$ be the integers satisfying that $H_{z_k(\lambda), s_k(\lambda)}(\lambda)$ is the unique odd 2^k -hook in λ . Also let $r_k(\lambda) = h_{z_k(\lambda), 1}(\lambda)$, a first-column hook length.*

2.3. ABACUS CONFIGURATIONS, e -CORES, e -QUOTIENTS. As discovered by G. James, all of the operations on partitions concerning addition and removal of e -hooks may conveniently be recorded on an e -abacus. This is described in detail in [10, Section 2.7]. Since we are going to use abaci in some proofs we introduce our modification of this important object, which may somehow be more flexible and give some notational advantages.

An e -abacus consists of an odd number of rows, say $2a + 1$, labelled (from top to bottom) by the integers $-a, -a + 1, \dots, a - 1, a$, and e runners (or columns) labelled (from left to right) by $0, 1, \dots, e - 1$. For any $(x, y) \in \{-a, -a + 1, \dots, a - 1, a\} \times \{0, 1, \dots, e - 1\}$ we can place a bead in position (x, y) of the abacus. Any distribution of beads on an e -abacus is called an e -abacus configuration. Note that an e -abacus contains only finitely many beads in an e -abacus configuration. There is some freedom in the choice of a ; it just has to be sufficiently large so that we have enough beads and rows to perform all of the necessary operations.

A gap in an e -abacus configuration A is an empty position in A , i.e., a position not containing a bead. Numbering the positions in A from 0 to $e(2a + 1) - 1$, going from left to right and from top to bottom, we call the gap with the lowest position number the *first gap*. Listing the numbers of the positions with beads in A gives a set of non-negative integers, i.e., a sequence of β -numbers in the notation of [10, Section 2.7]. This is a generalization of a sequence of first-column hook lengths and uniquely identifies a partition λ . In fact, if c_1, c_2, \dots, c_s are the beads on A after the first gap, then the differences between the position numbers of these beads and the position number of the first gap gives the first-column hook lengths $h_{(i,1)}(\lambda)$ of a unique partition λ . In this case we say that A is an e -abacus configuration for λ (later we mostly call it simply an e -abacus for λ). Thus for a fixed a the e -abacus configurations for a given λ differ only by the number of beads before the first gap, i.e., by the position of the first gap. In particular, different e -abacus configurations for λ have different numbers of beads.

We define a *normalized e -abacus configuration for λ* as one having its first gap in position $(0, 0)$. Later we will have to deal with situations where a certain start configuration is normalized, and it will occasionally be necessary to move beads with a row label well below 0 on their runner to a row with a higher row label.

By [10, 2.7.13] the removal of an e -hook in λ is recorded in any sequence of β -numbers for λ by reducing one of its numbers by e .

We collect some of the above facts in a lemma for later reference.

LEMMA 2.3. *Let A be an e -abacus configuration for λ consisting of m beads.*

- (1) *The positions of the beads after the first gap in A determine the first-column hook lengths of λ . Each of these beads corresponds to one part of λ , which is equal to the number of gaps before the bead.*
- (2) *The removal or addition of an e -hook is recorded on the e -abacus by sliding a bead one step up or down to an empty position on a runner of A .*
- (3) *If A is normalized and if $|\lambda| < e$ then all beads after the first gap are in row 0 of A . If there are s such beads then their positions are $(0, h_{(i,1)}(\lambda))$ for $1 \leq i \leq s$.*
- (4) *Let B be an e -abacus configuration for λ . If B consists of m beads, then $A = B$.*

DEFINITION 2.4. *Given an e -abacus configuration A and a bead c in A , we denote by $A(y, \leftarrow c)$ the e -abacus obtained by moving bead c to its left y times, ending in a gap. More precisely, the position number of c in the abacus is reduced by y . Here we use left in a broad sense. In fact, in principle we could end up in a gap in a higher row. (This will not happen in our applications later in the article).*

DEFINITION 2.5. *Let n and k be natural numbers such that $2^k \leq n$ and let $\lambda \vdash_o n$. Let e be a natural number and let A be an e -abacus configuration for λ . We denote by $c_k(A)$ the unique bead in A such that $A(2^k, \leftarrow c_k(A))$ is an e -abacus configuration for $f_k(\lambda)$.*

We note that Theorem 2.1 guarantees that the $c_k(A)$'s in Definition 2.5 are well defined. The following is a consequence of Lemma 2.3(3).

LEMMA 2.6. *Let $2^k \leq n < e$ and $\lambda \vdash_o n$. Let A be a normalized e -abacus configuration for λ . Then the bead $c_k(A)$ is in position $(0, r_k(A))$ for some $r_k(A) \in \{1, \dots, n\}$. In fact, in the notation of Definition 2.2 we have $r_k(A) = h_{z_k(\lambda), 1}(\lambda) = r_k(\lambda)$, i.e., $r_k(A)$ is a first-column hook length of λ .*

Let λ be a partition of n and let A be an e -abacus configuration for λ . Denote by A_0, A_1, \dots, A_{e-1} the runners in A from left to right. For $j \in \{0, \dots, e-1\}$, denote by $|A_j|$ the number of beads on runner j . Moreover, we denote by A^\uparrow the e -abacus obtained from A by sliding all beads on each runner as high as possible. Similarly, we denote by $A_0^\uparrow, \dots, A_{e-1}^\uparrow$ the runners of A^\uparrow . Then A^\uparrow is an e -abacus for a partition $C_e(\lambda)$, known as the e -core of λ . Using Lemma 2.3(2) repeatedly, we see that $C_e(\lambda)$ does not contain any e -hook and is obtained from λ by removing a number of e -hooks (say $w_e(\lambda)$ e -hooks) from λ . We call $w_e(\lambda)$ the e -weight of λ .

Then $w_e(\lambda) = w_e(A)$, where $w_e(A)$ denotes the total number of up-slides needed to obtain A^\uparrow from A . Similarly, for $i \in \{0, \dots, e-1\}$ let $w_e(A_i)$ denote the number of those slides that were performed on runner i in the transition from A to A^\uparrow . Thus we have

$$w_e(\lambda) = w_e(A) = w_e(A_0) + \dots + w_e(A_{e-1}).$$

For $j \in \{0, \dots, e-1\}$, denote by $\mu_j = \mu_j(A)$ the partition corresponding to the runner A_j (considered as a 1-abacus configuration). We then obviously have that $w_e(A_j) = |\mu_j|$.

REMARK 2.7. Suppose that the e -abacus configuration A for the partition λ of n consists of m beads. The e -quotient $Q_e(\lambda) = (\lambda_0, \dots, \lambda_{e-1})$ of λ , as defined in [10, Section 2.7] and [12, Section 3] agrees with $(\mu_0(A), \dots, \mu_{e-1}(A))$ if $e \mid m$. If $e \nmid m$ then $(\mu_0, \dots, \mu_{e-1})$ is just a m' -cyclic shift of $(\lambda_0, \dots, \lambda_{e-1})$. Here m' is the smallest positive integer which is congruent to m modulo e . In particular, $Q_e(\lambda)$ is also determined by A .

It is well-known that a partition is uniquely determined by its e -core and e -quotient (see e.g. [12, Proposition 3.7]). This fact may also be deduced from the discussion above.

REMARK 2.8. Let $\mathcal{H}(Q_e(\lambda)) = \bigcup_{i=0}^{e-1} \mathcal{H}(\lambda_i)$ (considered as multisets). As explained in [12, Theorem 3.3], there is a bijection between $\mathcal{H}_e(\lambda)$ and $\mathcal{H}(Q_e(\lambda))$ mapping hooks in λ of length ex to hooks in the quotient of length x . Moreover, the bijection respects the process of hook removal. In particular we conclude that

$$|\mathcal{H}_e(\lambda)| = w_e(\lambda) \quad \text{and} \quad |\lambda| = |C_e(\lambda)| + ew_e(\lambda).$$

For later use we also introduce the following object when $e = 2$. We denote by $\mathcal{Q}_2(\lambda)$ the 2-quotient tower and by $\mathcal{C}_2(\lambda)$ the 2-core tower of a partition λ . These have rows numbered by $k \geq 0$. The k th row $\mathcal{Q}_2^{(k)}(\lambda)$ of $\mathcal{Q}_2(\lambda)$ contains 2^k partitions $\lambda_i^{(k)}$, $1 \leq i \leq 2^k$, and the k th row $\mathcal{C}_2^{(k)}(\lambda)$ of $\mathcal{C}_2(\lambda)$ contains the 2-cores of these partitions in the same order, i.e., $\mathcal{C}_2(\lambda_i^{(k)})$, $1 \leq i \leq 2^k$. The 0th row of $\mathcal{Q}_2(\lambda)$ contains $\lambda = \lambda_1^{(0)}$ itself, row 1 contains the partitions $\lambda_1^{(1)}, \lambda_2^{(1)}$ occurring in the 2-quotient $Q_2(\lambda)$, row 2 contains the partitions occurring in the 2-quotients of partitions occurring in row 1, and so on. Specifically we have $Q_2(\lambda_i^{(k)}) = (\lambda_{2i-1}^{(k+1)}, \lambda_{2i}^{(k+1)})$ for $i \in \{1, \dots, 2^k\}$. We remark that the 2^k partitions in $\mathcal{Q}_2^{(k)}(\lambda)$ are the same as those in the 2^k -quotient $Q_{2^k}(\lambda)$ of λ , but in a different order for $k \geq 2$. We also denote by $\mathcal{D}_2^{(k)}(\lambda)$ the k -data of λ . This table (first introduced in [3]) consists of $k+1$ rows. For all $j \in \{0, \dots, k-1\}$ we have that row j coincides with $\mathcal{C}_2^{(j)}(\lambda)$; the $(k+1)$ th (last) row equals $\mathcal{Q}_2^{(k)}(\lambda)$.

3. PRELIMINARY RESULTS

Having collected the combinatorial background material, we now turn again to characters and prove some preliminary results that will turn out to be useful in the later sections.

3.1. CHARACTER VALUES AND ODD PARTITIONS. We apply the Murnaghan–Nakayama formula (in short the MN-formula; see [10, 2.4.7]) for some easy but useful characterizations of odd partitions.

Suppose that $\chi^\lambda \in \text{Irr}(\mathfrak{S}_n)$ and that $\pi \in \mathfrak{S}_n$ is a 2-element. Then $\chi^\lambda(1)$ and $\chi^\lambda(\pi)$ have the same parity (see [4, (6.4)]). In particular odd characters have odd values on all 2-elements, and a character is odd if and only if it takes odd values on some (non-empty) set of 2-elements.

The 2-powers in the binary expansion of numbers under consideration will play a crucial role. When 2^a appears in such an expansion of n , we say that 2^a is a *binary digit* of n and write $2^a \subseteq_2 n$. Furthermore, we say that two natural numbers are *2-disjoint* if they do not have any common binary digit.

We fix some notation for the following results. We write $n = 2^{a_1} + \dots + 2^{a_r}$ where $a_1 > \dots > a_r \geq 0$. Let ω_n be an element in \mathfrak{S}_n of cycle type $(2^{a_1}, 2^{a_2}, \dots, 2^{a_r})$. A *2-chain* in $\lambda \vdash n$ is a sequence $\lambda = \lambda^1, \dots, \lambda^r$ where λ^{i+1} is obtained from λ^i by removing a 2^{a_i} -hook. If it exists, a 2-chain in λ is unique. We note that, as an immediate consequence of the main result in [2], the number of partitions of n that have a 2-chain is exactly $\prod_{i=1}^r 2^{a_i}$.

A part of the following result appeared in [1, Lemma 1] with a different proof; the criterion in part (3) will be applied repeatedly in later sections.

LEMMA 3.1. *Let $\lambda \vdash n$. Let π be a 2^{a_1} -cycle in \mathfrak{S}_n .*

- (1) *If λ contains a 2^{a_1} -hook h , and $\gamma \in \mathfrak{S}_n$ acts on the fixed points of π , then $|\chi^\lambda(\pi\gamma)| = |\chi^{\lambda \setminus h}(\gamma)|$.*

- (2) We have $\chi^\lambda(\pi) \neq 0$ if and only if λ contains a 2^{a_1} -hook h . In that case $\chi^\lambda(1)$ and $\chi^{\lambda \setminus h}(1)$ have the same parity.
- (3) The partition λ is odd if and only if λ contains a 2^{a_1} -hook h such that $\lambda \setminus h$ is odd.
- (4) If λ contains a 2-chain then $|\chi^\lambda(\omega_n)| = 1$.

Proof. (1) is an instance of the MN-formula.

(2) By (1), $\chi^\lambda(\pi) \neq 0$ if and only if λ contains a 2^{a_1} -hook h . If this is the case then $\chi^\lambda(\pi) = \pm \chi^{\lambda \setminus h}(1)$, so $\chi^\lambda(1)$ and $\chi^{\lambda \setminus h}(1)$ have the same parity.

(3) If λ is odd, then $\chi^\lambda(\pi)$ is odd, so that h exists and $\lambda \setminus h$ is odd by (2). If h exists and $\lambda \setminus h$ is odd, then λ is odd, by (2).

(4) is obtained by repeated application of (2). □

The next result is implicit in McKay’s paper [11], where he introduced his famous conjecture.

LEMMA 3.2. *Let $\lambda \vdash n$. The following statements are equivalent:*

- (1) λ is odd.
- (2) λ contains a 2-chain.
- (3) $|\chi^\lambda(\omega_n)| = 1$.

Proof. (1) \Leftrightarrow (2) follows by repeated use of Lemma 3.1 (3).

(1) \Rightarrow (3) is stated in Lemma 3.1 (4), and the converse holds as $\chi^\lambda(\omega_n)$ and $\chi^\lambda(1)$ are of the same parity. □

It is known that the Murnaghan–Nakayama formula may be iterated to get an interesting factorization formula for certain character values involving $C_e(\lambda)$ and $Q_e(\lambda)$; we recall this here (see [10, 2.7.33]).

LEMMA 3.3. *Let $\lambda \vdash n$, $Q_e(\lambda) = (\lambda_0, \dots, \lambda_{e-1})$. Assume that $w \geq w_e(\lambda)$, and let $\rho \in \mathfrak{S}_n$ be the product of w e -cycles; let $\gamma \in \mathfrak{S}_n$ act on the fixed points of ρ . Then*

$$\chi^\lambda(\rho\gamma) = \begin{cases} \pm \binom{w}{|\lambda_0|, |\lambda_1|, \dots, |\lambda_{e-1}|} \chi^{C_e(\lambda)}(\gamma) \prod_{i=0}^{e-1} \chi^{\lambda_i}(1) & \text{if } w = w_e(\lambda) \\ 0 & \text{if } w > w_e(\lambda). \end{cases}$$

For any $\lambda \vdash n$, $w_{2^k}(\lambda) \leq \lfloor \frac{n}{2^k} \rfloor$, by Remark 2.8. If $\lambda \vdash_o n$ then $w_{2^k}(\lambda) = \lfloor \frac{n}{2^k} \rfloor$, by Theorem 2.1. Applying Lemma 3.3 with $e = 2^k$ and ρ a product of $w = \lfloor \frac{n}{2^k} \rfloor$ 2^k -cycles we immediately obtain

COROLLARY 3.4. *Let $\lambda \vdash n$, with $Q_2^{(k)}(\lambda) = (\lambda_1^{(k)}, \dots, \lambda_{2^k}^{(k)})$. If ρ is a product of $w = \lfloor \frac{n}{2^k} \rfloor$ 2^k -cycles and $\gamma \in \mathfrak{S}_n$ acts on the fixed points of ρ , then*

$$\chi^\lambda(\rho\gamma) = \begin{cases} \pm \binom{w}{|\lambda_1^{(k)}|, |\lambda_2^{(k)}|, \dots, |\lambda_{2^k}^{(k)}|} \chi^{C_{2^k}(\lambda)}(\gamma) \prod_{i=1}^{2^k} \chi^{\lambda_i^{(k)}}(1) & \text{if } w = w_{2^k}(\lambda) \\ 0 & \text{if } w > w_{2^k}(\lambda). \end{cases}$$

COROLLARY 3.5. *Let $\lambda \vdash n$, ρ, γ be as in Corollary 3.4. Then $|\chi^\lambda(\rho\gamma)| = 1$ if and only if the following conditions are all fulfilled:*

- (1) There exists a unique $s \in \{1, 2, \dots, 2^k\}$ such that $\lambda_s^{(k)} \in \{(w), (1^w)\}$.
- (2) $|\chi^{C_{2^k}(\lambda)}(\gamma)| = 1$.
- (3) $w = w_{2^k}(\lambda)$.

As a consequence of the previous results we find all the character values ± 1 on certain 2-elements. We use the notation introduced before; in particular, $n = 2^{a_1} + \dots + 2^{a_r}$ where $a_1 > \dots > a_r \geq 0$.

COROLLARY 3.6. *Let $k \in \mathbb{N}$ with $2^k \leq n$. Set $w = \lfloor n/2^k \rfloor$, $m = n - w \cdot 2^k$. Let $\rho \in \mathfrak{S}_n$ be a product of w 2^k -cycles, $\gamma = \omega_m$ on the fixed points of ρ . Let $\lambda \vdash_o n$. Then $|\chi^\lambda(\rho\gamma)| = 1$ if and only if $\lambda_s^{(k)} \in \{(w), (1^w)\}$ for some $s \in \{1, 2, \dots, 2^k\}$.*

In particular, when $a_q \geq k > a_{q+1}$, there are exactly $2^{k'}$ $\prod_{i=q+1}^r 2^{a_i}$ irreducible characters with value ± 1 on $\rho\gamma$, where $k' = k$ when $w = 1$, and $k' = k + 1$ when $w > 1$.

Proof. Since $\lambda \vdash_o n$, $w_{2^k}(\lambda) = \lfloor \frac{n}{2^k} \rfloor$. Furthermore, $C_{2^k}(\lambda)$ is an odd partition of $m = n - w \cdot 2^k$. Hence $|C_{2^k}(\lambda)(\omega_m)| = 1$ by Lemma 3.2. Thus both conditions (2) and (3) of Corollary 3.5 are satisfied for λ , and the criterion for $|\chi^\lambda(\rho\gamma)| = 1$ follows from Corollary 3.5; note that the index s is unique as the total weight of $\mathcal{Q}_2^{(k)}(\lambda)$ is w .

For counting the $\lambda \vdash_o n$ satisfying $|\chi^\lambda(\rho\gamma)| = 1$, note first that there are exactly $\prod_{i=q+1}^r 2^{a_i}$ possible partitions $C_{2^k}(\lambda)$; then there are 2^k possible indices s such that $\lambda_s^{(k)} \in \{(w), (1^w)\}$, and either 1 or 2 possibilities for $\lambda_s^{(k)}$, when $w = 1$ or $w > 1$, respectively. \square

For ρ a product of $w_{2^k}(\lambda)$ 2^k -cycles, we know that λ is odd if and only if $\chi^\lambda(\rho)$ is odd. This happens if and only if the following numbers are all odd:

$$\left(|\lambda_1^{(k)}|, |\lambda_2^{(k)}|, \dots, |\lambda_{2^k}^{(k)}| \right), \quad \chi^{\lambda_i^{(k)}}(1), \text{ for all } i, \quad \chi^{C_{2^k}(\lambda)}(1).$$

Now $(|\lambda_1^{(k)}|, |\lambda_2^{(k)}|, \dots, |\lambda_{2^k}^{(k)}|)$ is odd if and only if the numbers $|\lambda_i^{(k)}|$, $1 \leq i \leq 2^k$, are pairwise 2-disjoint. Thus λ is odd if and only if the following conditions are all fulfilled:

- (a) $C_{2^k}(\lambda)$ is odd.
- (b) The partitions $\lambda_i^{(k)}$, $1 \leq i \leq 2^k$, are all odd.
- (c) The numbers $|\lambda_i^{(k)}|$, $1 \leq i \leq 2^k$, are pairwise 2-disjoint.

From this we may deduce the following theorem, which is also [3, Theorem 2.5]. It shows that the k -data encodes full information on the parity of the character degree $\chi^\lambda(1)$.

THEOREM 3.7. *Let $\lambda \vdash n$, and let $k \geq 0$ be fixed. Consider $\mathcal{Q}_2^{(k)}(\lambda) = (\lambda_1^{(k)}, \dots, \lambda_{2^k}^{(k)})$. Then λ is odd if and only if the following conditions are all fulfilled:*

- (1) $c_2^{(j)}(\lambda) \leq 1$ for all $j < k$.
- (2) The partitions $\lambda_i^{(k)}$, $1 \leq i \leq 2^k$, are all odd.
- (3) The numbers $|\lambda_i^{(k)}|$, $1 \leq i \leq 2^k$, are pairwise 2-disjoint.

In this case $\sum_{i \geq 1} |\lambda_i^{(k)}| = \lfloor \frac{n}{2^k} \rfloor$.

3.2. ODD PARTITIONS: R -OPERATORS, k -TYPES. Let $n, j \in \mathbb{N}$ be such that $2 \leq 2^j \leq n$. Let $\lambda \vdash_o n$ and let $h = H_{z_j(\lambda), s_j(\lambda)}(\lambda)$ be the unique 2^j -hook in λ whose removal leads to the odd partition $f_j^n(\lambda)$ of $n - 2^j$ (see Definition 2.2). Repeated applications of Remark 2.8 show that for any $k \in \{0, 1, \dots, j\}$ the removal of h from λ corresponds to the removal of a 2^{j-k} -hook \bar{h} from exactly one of the 2^k partitions involved in $\mathcal{Q}_2^{(k)}(\lambda)$. We denote this distinguished component of $\mathcal{Q}_2^{(k)}(\lambda)$ by $R_k^j(\lambda)$; note that this partition occurs only once in $\mathcal{Q}_2^{(k)}(\lambda)$, as for an odd partition λ the nonempty partitions in $\mathcal{Q}_2^{(k)}(\lambda)$ are of different sizes (by Theorem 3.7(3)). It is easy to see that for all $k \in \{0, 1, \dots, j\}$ the k -data of $f_j^n(\lambda)$ is obtained from the k -data of λ by replacing $R_k^j(\lambda)$ with $f_{j-k}^r(R_k^j(\lambda))$, where $r = |R_k^j(\lambda)|$.

We denote by $\mathcal{H}(n)$ the set of hook partitions of n . The following lemma shows that if λ belongs to a specific class of partitions, then it is very easy to explicitly describe $f_k(\lambda)$.

LEMMA 3.8. *Let $t, k, w \in \mathbb{N}$ be such that $0 \leq k < t$ and $0 \leq w \leq 2^t - 1$, and let $\lambda = (2^t - w, 1^w) \in \mathcal{H}(2^t)$. Then*

$$f_k(\lambda) = \begin{cases} (2^t - w, 1^{w-2^k}) & \text{if } 2^k \subseteq_2 w, \\ (2^t - w - 2^k, 1^w) & \text{if } 2^k \not\subseteq_2 w. \end{cases}$$

Proof. It is well known that if $\mu = (n - w, 1^w) \in \mathcal{H}(n)$, then $\chi^\mu(1) = \binom{n-1}{w}$. The statement of the lemma follows directly from this fact. \square

For arbitrary partitions λ of an arbitrary natural number n , it is not always as easy as it is for hook partitions of a power of 2 to understand the structure of $f_k^n(\lambda)$. For this reason we give an explicit arithmetic determination for $R_k^j(\lambda)$ which will turn out to be very useful.

LEMMA 3.9. *Let $n = 2^{a_1} + \dots + 2^{a_r}$ with $a_1 > a_2 > \dots > a_r \geq 0$. Let $\lambda \vdash_o n$ and $j \in \mathbb{N}$ with $2^j \leq n$. Let $k \in \mathbb{N}_0$ with $k \leq j$ and $\mathcal{Q}_2^{(k)}(\lambda) = (\lambda_1, \dots, \lambda_{2^k})$. Let $a_t = \min\{a_i \mid a_i \geq j\}$. Then 2^{a_t-k} is a binary digit of $|\lambda_s|$ for a unique $s \in \{1, \dots, 2^k\}$, and $R_k^j(\lambda) = \lambda_s$.*

Proof. By Theorem 3.7 we know that $\sum_{i=1}^{2^k} |\lambda_i| = \lfloor \frac{n}{2^k} \rfloor$ and that the numbers $|\lambda_i|$ are pairwise 2-disjoint. Hence there exists a unique $s \in \{1, \dots, 2^k\}$ such that 2^{a_t-k} is a binary digit of $|\lambda_s|$. Moreover, by definition we have that λ_s is an odd partition such that $2^{j-k} \leq |\lambda_s|$. Hence $f_{j-k}(\lambda_s)$ is a well defined odd partition whose size is 2-disjoint from $|\lambda_r|$ for all $r \in \{1, \dots, 2^k\} \setminus \{s\}$. Thus $R_k^j(\lambda) = \lambda_s$. \square

In particular, for the realization of the removal of a 2^k -hook from λ in its k -data, we look for the partition λ_s such that the smallest binary digit of $\lfloor n/2^k \rfloor$ is a digit of $|\lambda_s|$.

Because of this arithmetic description, the following definition will play an important role.

DEFINITION 3.10. *Let $n, k \in \mathbb{N}$, $2^k \leq n$. Let $\lambda \vdash n$, with $\mathcal{Q}_2^{(k)}(\lambda) = (\lambda_1, \dots, \lambda_{2^k})$. We say that λ is of k -type (n_1, \dots, n_{2^k}) (of size $\lfloor n/2^k \rfloor$) if $n_j = |\lambda_j|$ for $1 \leq j \leq 2^k$.*

We let $\mathcal{O}(n, \tau)$ denote the subset of $\mathcal{O}(n)$ of partitions of k -type τ .

Note that for odd partitions of n , their k -types are exactly the 2-disjoint weak compositions of $\lfloor n/2^k \rfloor$ of length 2^k . Moreover, for any $j \geq k$, the position of $R_k^j(\lambda)$ only depends on the k -type of λ , by Lemma 3.9.

4. COMPARING $f_k f_k$ WITH f_{k+1}

Let n and k be natural numbers such that $2^{k+1} \leq n$. The aim of this section is to completely describe those partitions $\lambda \in \mathcal{O}(n)$ such that $f_k f_k(\lambda) = f_{k+1}(\lambda)$. The strategy we adopt to achieve our goal is to reduce our problem to the study of the case where $k = 0$.

We define

$$\mathcal{G}_k(n) = \{\lambda \vdash_o n \mid f_k f_k(\lambda) = f_{k+1}(\lambda)\}$$

and set $G_k(n) = |\mathcal{G}_k(n)|$. For a k -type τ of size $\lfloor n/2^k \rfloor$, we let $\mathcal{G}_k(n; \tau)$ be the subset of $\mathcal{G}_k(n)$ of partitions of k -type τ , and set $G_k(n; \tau) = |\mathcal{G}_k(n; \tau)|$. Furthermore, we set $\mathcal{G}_k = \bigcup_{n \geq 2^{k+1}} \mathcal{G}_k(n)$.

The reduction to the case $k = 0$ is possible thanks to the following lemma.

LEMMA 4.1. *Let $k, n \in \mathbb{N}_0$ be such that $2^{k+1} \leq n$. Let $\lambda \vdash_o n$. Then $\lambda \in \mathcal{G}_k$ if and only if $R_k^k(\lambda) = R_k^{k+1}(\lambda) \in \mathcal{G}_0$.*

Proof. Suppose that $R_k^k(\lambda) = R_k^{k+1}(\lambda) =: \mu \in \mathcal{G}_0$. Then $R_k^k(f_k(\lambda)) = f_0(R_k^{k+1}(\lambda)) = f_0(\mu)$. Hence the k -data of $f_k f_k(\lambda)$ and $f_{k+1}(\lambda)$, respectively, are obtained from that of λ by replacing μ with $f_0 f_0(\mu)$ and $f_1(\mu)$, respectively. Since $\mu \in \mathcal{G}_0$, these are equal, and since the k -data uniquely determines the partition, we deduce that $f_k f_k(\lambda) = f_{k+1}(\lambda)$.

In order to prove the converse, we assume that $f_k f_k(\lambda) = f_{k+1}(\lambda)$. Then $R_k^k(\lambda) = R_k^{k+1}(\lambda) =: \mu$ as otherwise $\mathcal{Q}_2^{(k)}(f_k f_k(\lambda)) \neq \mathcal{Q}_2^{(k)}(f_{k+1}(\lambda))$. Furthermore the k -data of $f_k f_k(\lambda)$ and $f_{k+1}(\lambda)$, respectively, are obtained from that of λ by replacing μ with $f_0 f_0(\mu)$ and $f_1(\mu)$, respectively. Hence we deduce $\mu \in \mathcal{G}_0$. \square

We fix below some notation for the preparatory results towards the counting formula in Theorem 4.8.

NOTATION 4.2. Let $n, k \in \mathbb{N}_0$ be such that $2^{k+1} \leq n$. We write $n = 2^{a_1} + \dots + 2^{a_r}$, where $a_1 > a_2 > \dots > a_r$. We define $p, q \in \{1, \dots, r\}$ to be maximal such that $a_p \geq k+1$ and $a_q \geq k$, respectively. Set $b_j = a_j - k$, for $1 \leq j \leq q$, and $B = \{b_1, \dots, b_q\}$. Note that then either $b_p > b_q = 0$ or $b_p = b_q > 0$.

Using the definition and notation above, we may reformulate Lemma 4.1 as follows; this is a direct consequence of Lemma 3.9.

COROLLARY 4.3. *Let $\lambda \in \mathcal{O}(n; \tau)$ with $\mathcal{Q}_2^{(k)}(\lambda) = (\lambda_1, \dots, \lambda_{2^k})$, $\tau = (n_1, \dots, n_{2^k})$. Then $\lambda \in \mathcal{G}_k(n; \tau)$ if and only if there is an $s \in \{1, \dots, 2^k\}$ such that $2^{b_p}, 2^{b_q} \subseteq_2 n_s$ and $\lambda_s \in \mathcal{G}_0(n_s)$.*

We also immediately deduce the following description.

LEMMA 4.4. *Let $\tau = (n_1, \dots, n_{2^k})$ be a k -type of size $\lfloor n/2^k \rfloor$.*

- (1) *If $2^{b_p}, 2^{b_q}$ are binary digits of two different parts of τ , then $\mathcal{G}_k(n, \tau) = \emptyset$.*
- (2) *If $2^{b_p}, 2^{b_q}$ are binary digits of the same part of τ , say n_j , then*

$$\mathcal{G}_k(n; \tau) = \{\lambda \in \mathcal{O}(n; \tau) \mid \mathcal{Q}_2^{(k)}(\lambda)_j \in \mathcal{G}_0(n_j)\}.$$

We pause to illustrate the results obtained so far in a small example.

EXAMPLE 4.5. Keeping definitions and notation as in 4.2, we set $k = 1$ and $n = 6 = 2^2 + 2^1$. We observe that in this case we have $a_1 = 2, a_2 = 1, r = 2, p = 1$ and $q = 2$. Moreover, the set $\mathcal{O}(6)$ consists of the partitions

$$(6), (5, 1), (4, 2), (3^2), (2^3), (2^2, 1^2), (2, 1^4) \text{ and } (1^6).$$

Their 2-quotients are (in order) listed below:

$$(\emptyset, (3)), ((3), \emptyset), ((1), (2)), ((2), (1)), ((1), (1^2)), ((1^2), (1)), (\emptyset, (1^3)) \text{ and } ((1^3), \emptyset).$$

It follows that $\mathcal{O}(6; (0, 3)) = \{(6), (2, 1^4)\}$, $\mathcal{O}(6; (3, 0)) = \{(5, 1), (1^6)\}$, $\mathcal{O}(6; (2, 1)) = \{(3^2), (2^2, 1^2)\}$, and $\mathcal{O}(6; (1, 2)) = \{(4, 2), (2^3)\}$.

Since $b_p = 1$ and $b_q = 0$, we deduce from Lemma 4.4 that $\mathcal{G}_1(6; (2, 1)) = \mathcal{G}_1(6; (1, 2)) = \emptyset$. This could have been verified directly. For instance

$$f_1 f_1((4, 2)) = f_1((4)) = (2) \neq (1^2) = f_2((4, 2)).$$

Finally, since both $(3), (1^3) \in \mathcal{G}_0(3)$ by direct verification, we deduce from Lemma 4.4 that

$$\mathcal{G}_1(6) = \mathcal{G}_1(6; (3, 0)) \cup \mathcal{G}_1(6; (0, 3)) = \{(6), (5, 1), (2, 1^4), (1^6)\}.$$

In Example 4.12 below, we will calculate $\mathcal{G}_0(6)$.

From the results above we deduce that for $\lambda \in \mathcal{O}(n)$, the property $f_k f_k(\lambda) = f_{k+1}(\lambda)$ does not depend on the rows of the k -data corresponding to the cores. Hence towards counting these odd partitions we have:

LEMMA 4.6. *We set $\bar{n} = \sum_{i=1}^q 2^{a_i}$. For any k -type τ of size $\lfloor n/2^k \rfloor$ we have*

$$G_k(n; \tau) = \left(\prod_{j=q+1}^r 2^{a_j} \right) \cdot G_k(\bar{n}; \tau).$$

In particular,

$$G_k(n) = \left(\prod_{j=q+1}^r 2^{a_j} \right) \cdot G_k(\bar{n}).$$

We remark that Lemma 4.6 holds trivially if $r = q$, as $n = \bar{n}$ and $G_k(n) = G_k(\bar{n})$.

To state the reduction formula for $G_k(n)$ in a nice way, we introduce weights that are based on the counts for the case $k = 0$. For the numbers $G_0(n)$, we will derive explicit formulae later in this section. (A similar approach will be taken for the commutativity problem in later sections.)

DEFINITION 4.7. *We fix $k \in \mathbb{N}$. For $J \subseteq I = \{1, \dots, q\}$ we define its G -weight with respect to the set $B = \{b_1, \dots, b_q\}$ to be*

$$w_G(J) = \left(\prod_{i \in I \setminus J} 2^{b_i} \right) \cdot (2^k - 1)^{|I \setminus J|} \cdot G_0 \left(\sum_{j \in J} 2^{b_j} \right).$$

Then we have the following formula for the number $G_k(n)$, based on the values $G_0(u)$.

THEOREM 4.8. *With the notation introduced in 4.2, we have*

$$G_k(n) = \left(\prod_{j=q+1}^r 2^{a_j} \right) \cdot 2^k \cdot \sum_{\{p,q\} \subseteq J \subseteq I} w_G(J).$$

Proof. In view of Lemma 4.6, we focus on computing the numbers $G_k(\bar{n})$. We take the sum over all k -types of size $\lfloor n/2^k \rfloor$, and for each k -type τ we count the partitions in $\mathcal{G}_k(\bar{n}; \tau)$ according to Lemma 4.4.

This lemma tells us that we only get a contribution from those k -types τ where $2^{b_p}, 2^{b_q}$ are binary digits in the same part of τ . To construct these, take any $J \subseteq I$ containing p, q and define $n_J := \sum_{j \in J} 2^{b_j}$. We now consider the contributions from all k -types for this fixed J . We first have 2^k choices for the position j of the part of which $2^{b_p}, 2^{b_q}$ are binary digits. For each $i \in I \setminus J$ we have $2^k - 1$ choices to place 2^{b_i} as a binary digit of a part of the type τ under construction, different from the j -th. By Lemma 4.4, for such a $\tau = (n_1, \dots, n_{2^k})$, the partitions $\lambda \in \mathcal{G}_k(\bar{n}; \tau)$ are then those such that $\mathcal{Q}_2^{(k)}(\lambda)_i$ is an arbitrary odd partition of n_i for $i \neq j$, and $\mathcal{Q}_2^{(k)}(\lambda)_j \in \mathcal{G}_0(n_j)$ (and the earlier rows of the k -data contain only empty cores). Hence we get $|\mathcal{G}_k(\bar{n}; \tau)| = \left(\prod_{i \in I \setminus J} 2^{b_i} \right) \cdot G_0(\sum_{j \in J} 2^{b_j})$, and the total contribution for J is exactly $2^k w_G(J)$.

This proves the stated formula. □

In light of the reduction results obtained so far, we now turn to the case where $k = 0$.

The following lemma will be used several times and utilizes some of the concepts introduced in Sections 2 and 3. It is a generalization of [1, Main Lemma], and may also be found in [8] with a different proof.

LEMMA 4.9. *Let $n = 2^t + m$ with $m < 2^t$, and let $\ell \in \mathbb{N}_0$ be such that $2^\ell \leq m$. If $\lambda \vdash_o n$ then $f_\ell(C_{2^t}(\lambda)) = C_{2^t}(f_\ell(\lambda))$.*

Proof. Let B be a normalized 2^t -abacus for $\gamma = C_{2^t}(\lambda)$. By Remark 2.8, we have that λ contains a unique 2^t -hook and thus $\gamma \vdash_o m$ by Theorem 2.1. Using Definition 2.5 we see that if $c = c_\ell(B)$ then $B(2^\ell, \leftarrow c)$ is a 2^t -abacus configuration for $f_\ell(\gamma)$. Lemma 2.6 with $k = \ell$ and $m < e = 2^t$ shows that c is in position $(0, r_\ell(B))$ where $r_\ell(\gamma) = r_\ell(B) \in \{2^\ell, \dots, 2^t - 1\}$.

Let now A be the abacus configuration for λ such that $A^\uparrow = B$. By Lemma 2.3, A is obtained from B by sliding one bead down one row; in particular $w_{2^t}(A) = 1$. Let $x \in \{0, 1, \dots, 2^t - 1\}$ be such that $w_{2^t}(A_x) = 1$ and $w_{2^t}(A_y) = 0$ for all $y \neq x$. Let $d = c_\ell(A)$. Thus $A(2^\ell, \leftarrow d)$ is a 2^t -abacus for $f_\ell(\lambda)$. Put $j := r_\ell(B)$. If $x \notin \{j, j - 2^\ell\}$ then we have that d is in position $(0, j)$ of A (really $d = c$), by Lemma 2.6. Hence we have that $A(2^\ell, \leftarrow d)^\uparrow = B(2^\ell, \leftarrow c)$, and therefore we have that $f_\ell(C_{2^t}(\lambda)) = C_{2^t}(f_\ell(\lambda))$, as desired. On the other hand, if $x = j$ then d is in position $(1, j)$ and again we deduce that $A(2^\ell, \leftarrow d)^\uparrow = B(2^\ell, \leftarrow c)$. A similar argument is used for the case where $x = j - 2^\ell$. \square

LEMMA 4.10. *Let $n = 2^t + m$ with $2 \leq m < 2^t$. Then $\lambda \in \mathcal{G}_0(n)$ implies $C_{2^t}(\lambda) \in \mathcal{G}_0(m)$.*

Proof. This is a consequence of Lemma 4.9. \square

For the notation in the following results we refer to Definition 2.5 and Lemma 2.6.

LEMMA 4.11. *Let $2 \leq m < 2^t$ and let $\gamma \vdash_o m$. Let B be a normalized 2^t -abacus for γ . Then $\gamma \in \mathcal{G}_0(m)$ if and only if $r_0(B) = r_1(B)$ or $r_0(B) = r_1(B) - 1$.*

Proof. If $r_0(B) = r_1(B) =: r$ then there is a bead in position $(0, r)$ of B (call it c) and there are gaps in positions $(0, r - 2)$ and $(0, r - 1)$ of B . Moreover, by definition we have that $B^0 := B(1, \leftarrow c)$ and $B^1 := B(2, \leftarrow c)$ are 2^t -abaci for $f_0(\gamma)$ and $f_1(\gamma)$ respectively. Clearly $B^1 = B^0(1, \leftarrow c)$. Since B^1 is a 2^t -abacus for an odd partition we deduce from Theorem 2.1 that this partition is $f_0 f_0(\gamma)$. Hence $f_0 f_0(\gamma) = f_1(\gamma)$.

If $r := r_0(B) = r_1(B) - 1$ then we have beads c and d in positions $(0, r)$ and $(0, r + 1)$ of B respectively, and there is a gap in position $(0, r - 1)$. By construction

$$B^1 := B(2, \leftarrow d) = B^0(1, \leftarrow d'),$$

where $B^0 := B(1, \leftarrow c)$ and d' is the bead in position $(0, r + 1)$ of B^0 . Arguing as in the previous case we deduce that $f_0 f_0(\gamma) = f_1(\gamma)$.

In order to prove the converse let us suppose that $f_0 f_0(\gamma) = f_1(\gamma)$ and let $r := r_1(B) \in \{2, \dots, 2^t - 1\}$ be such that $r_0(B) \notin \{r, r - 1\}$. Let c be the bead in position $(0, r)$ of B , and observe that position $(0, r - 2)$ of B is empty. Hence $B^1 := B(2, \leftarrow c)$ is the 2^t -abacus configuration for $f_1(\gamma)$ and has a bead in position $(0, r - 2)$.

Suppose that position $(0, r - 1)$ of B is empty. Since $r_0(B) \notin \{r, r - 1\}$ we deduce that positions $(0, r - 2)$ and $(0, r - 1)$ are empty in $B^0 = B(1, \leftarrow d)$, where d is the bead in position $(0, r_0(B))$ and B^0 is the 2^t -abacus for $f_0(\gamma)$. It follows that the abacus configuration $B^{0,0}$ for $f_0 f_0(\gamma)$ obtained from B^0 by sliding one bead to its left has a gap in position $(0, r - 2)$ and therefore we have $B^{0,0} \neq B^1$ and hence $f_0 f_0(\gamma) \neq f_1(\gamma)$.

We still have to consider the case where there is a bead in position $(0, r - 1)$ of B . Again we deduce that $f_0 f_0(\gamma) \neq f_1(\gamma)$ by arguing in a completely similar fashion as done above. \square

We use the example below to illustrate, in a small concrete case, the main combinatorial ideas used in Lemma 4.11.

EXAMPLE 4.12. Let $n = 14 = 2^3 + 2^2 + 2$, and let $m = 6$. As already shown in Example 4.5, $\mathcal{O}(6)$ consists of $(6), (5, 1), (4, 2), (3^2)$ and their conjugates. Let B be the normalized 8-abacus configuration of (6) . Since the first gap is in position $(0, 0)$, it is easy to see that the only bead (say c) in a non-negative row lies in position $(0, 6)$. In particular we have that $r_0(B) = r_1(B) = 6$. Clearly $B^0 := B(1, \leftarrow c)$ is an 8-abacus for $(5) = f_0((6))$. Moreover, $B^0(1, \leftarrow c') = B(2, \leftarrow c)$ is an 8-abacus for the partition (4) , that is equal to $f_0(f_0((6)))$ and to $f_1((6))$.

On the other hand, let us analyze the partition $(5, 1)$. Let B be its normalized 8-abacus configuration. After the first gap in position $(0, 0)$ we have a bead c in position $(0, 1)$ and a bead d in position $(0, 6)$. Hence $r_0(B) = 1$ while $r_1(B) = 6$. It follows that $B(2, \leftarrow d)$ is an 8-abacus configuration for $f_2((5, 1))$. In fact $f_2((5, 1)) = (3, 1)$. On the other hand $B(1, \leftarrow c)$ is an 8-abacus configuration for $f_0((5, 1))$. Hence $f_0((5, 1)) = (5)$ and therefore $f_0 f_0((5, 1)) = (4) \neq f_2((5, 1))$.

Repeating a similar analysis with the remaining 6 partitions in $\mathcal{O}(6)$ we deduce that $\mathcal{G}_0(6) = \{(6), (3, 3), (2, 2, 2), (1^6)\}$.

For an odd partition γ , we denote by $\mathcal{E}(\gamma, 2^t)$ the set of all odd partitions that are obtained from γ by adjoining a 2^t -hook to γ . Recall that if $|\gamma| < 2^t$ then $|\mathcal{E}(\gamma, 2^t)| = 2^t$, by the main result of [2].

PROPOSITION 4.13. *Let $n = 2^t + m$ with $2 \leq m < 2^t$. Let $\gamma \in \mathcal{G}_0(m)$ and let $\lambda \in \mathcal{E}(\gamma, 2^t)$. Let B be a normalized 2^t -abacus for γ , and let A be the 2^t -abacus for λ such that $A^\uparrow = B$. Let $x \in \{0, \dots, 2^t - 1\}$ be such that $w_{2^t}(A_x) = 1$.*

- (1) *If $r_0(B) = r_1(B) =: r$, then $\lambda \in \mathcal{G}_0(n)$ if and only if $x \notin \{r - 2, r - 1\}$.*
- (2) *If $r_0(B) = r_1(B) - 1 =: r$, then $\lambda \in \mathcal{G}_0(n)$ if and only if $x \notin \{r, r + 1\}$.*

Proof. Suppose we are in case (1) where $r_0(B) = r_1(B) =: r$, and let us assume that $x \notin \{r - 2, r - 1, r\}$. Let c be the bead in position $(0, r)$ of A . Using Lemma 3.1 (3) we get that $A^0 := A(1, \leftarrow c)$ and $A^1 := A(2, \leftarrow c)$ are 2^t -abacus configurations for $f_0(\lambda)$ and $f_1(\lambda)$, respectively. Let c' be the bead in position $(0, r - 1)$ of A^0 . It follows that $A^{0,0} := A^0(1, \leftarrow c')$ is a 2^t -abacus configuration for $f_0 f_0(\lambda)$. Since $A^{0,0} = A^1$ we deduce that $f_0 f_0(\lambda) = f_1(\lambda)$. If $x = r$, we let c be the bead in position $(1, r)$ of A . Arguing exactly as above, we deduce that $f_0 f_0(\lambda) = f_1(\lambda)$, also in this case.

Suppose now that $x = r - 1$. Let c be the bead in position $(0, r)$ of A and let d be the bead in position $(-1, r)$ of A . Using Lemma 3.1 (3), we observe that $A^1 := A(2, \leftarrow c)$ is a 2^t -abacus configuration for $f_1(\lambda)$. On the other hand we notice that $A^0 := A(1, \leftarrow d)$ is a 2^t -abacus configuration for $f_0(\lambda)$. It follows that $A^{0,0} := A^0(1, \leftarrow e)$ is a 2^t -abacus configuration for $f_0 f_0(\lambda)$, where we denoted by e the bead in position $(0, r - 1)$ of A^0 . Since $A^{0,0} \neq A^1$ are two abacus configurations with the same number of beads, we deduce that $f_0 f_0(\lambda) \neq f_1(\lambda)$, by Lemma 2.3. With very similar ideas we deal with the case where $x = r - 2$.

In order to prove the statement in case (2) we use a very similar argument to the one used above to deal with case (1). □

EXAMPLE 4.14. Let $n = 14$ and $m = 6$, so that $n = 2^3 + m$. As shown in Example 4.12, $(6) \in \mathcal{G}_0(6)$. As explained right above the statement of Proposition 4.13, the set $\mathcal{E}((6), 2^3)$ consists of all those partitions of 14 obtained by adjoining an 8-hook to (6) . Thus,

$$\mathcal{E}((6), 2^3) = \{(14), (7^2), (6^2, 1^2), (6, 5, 1^3), (6, 4, 1^4), (6, 3, 1^5), (6, 2, 1^6), (6, 1^8)\}.$$

Following Proposition 4.13, we will now determine $\mathcal{G}_0(14) \cap \mathcal{E}((6), 2^3)$.

Let B be the normalized 8-abacus configuration for (6) , as described in Example 4.12. We observed that $r_0(B) = r_1(B) = 6$. Using the notation of Proposition 4.13,

for $x \in \{0, 1, \dots, 7\}$ we let A be the abacus configuration obtained from B by sliding the lowest bead on runner x one row down. Clearly, while x runs from 0 to 7, A runs through the 8-abacus configurations for the partitions in $\mathcal{E}((6), 2^3)$. Let $x = 5$. Then $x = 6 - 1 = r_0(B) - 1$. We observe that for $x = 5$, A is an 8-abacus configuration for $(6^2, 1^2)$. From Proposition 4.13, we expect $f_0 f_0((6^2, 1^2)) \neq f_1((6^2, 1^2))$. Let c be the bead in position $(0, 6)$ of A . Then $A(2, \leftarrow c)$ is an 8-abacus configuration for $(5^2, 1^2)$. It is easy to verify that $f_1((6^2, 1^2)) = (5^2, 1^2)$. On the other hand, $f_0 f_0((6^2, 1^2)) = (6, 5, 1)$.

Similarly one can observe that choosing $x = 4 = r_0(B) - 2$, we obtain an 8-abacus configuration for the partition $(6, 5, 1^3)$. Hence Proposition 4.13 gives

$$\mathcal{G}_0(14) \cap \mathcal{E}((6), 2^3) = \mathcal{E}((6), 2^3) \setminus \{(6^2, 1^2), (6, 5, 1^3)\}.$$

For $n \in \mathbb{N}$ of the form $n = 2^t$ or $n = 2^t + 1$, the numbers $G_0(n)$ may easily be computed, and these numbers will provide crucial input in the general case.

PROPOSITION 4.15. *We have $G_0(1) = 1$, $G_0(2) = 2 = G_0(3)$.*

For $t > 1$, the following holds.

- (1) *Let $\lambda = (2^t - u, 1^u) \vdash_o 2^t$, for some $u \in \{0, 1, \dots, 2^t - 1\}$. Then $\lambda \in \mathcal{G}_0(2^t)$ if and only if $u \equiv 0$ or $u \equiv 3 \pmod 4$. In particular, $G_0(2^t) = 2^{t-1}$.*
- (2) *Let $\lambda \vdash_o 2^t + 1$. Then $\lambda \in \mathcal{G}_0(2^t + 1)$ if and only if λ or λ' is $(2^t + 1)$ or $(2^t - 1, 2)$. In particular, $G_0(2^t + 1) = 4$.*

Proof. We may assume that $t > 1$.

- (1) Using Lemma 3.8 we compute for $\lambda = (2^t - u, 1^u)$:

$$f_0 f_0(\lambda) = \begin{cases} (2^t - u - 2, 1^u) & \text{for } u \equiv 0 \pmod 4 \\ (2^t - u - 1, 1^{u-1}) & \text{for } u \equiv 1 \text{ or } 2 \pmod 4 \\ (2^t - u, 1^{u-2}) & \text{for } u \equiv 3 \pmod 4 \end{cases}$$

and

$$f_1(\lambda) = \begin{cases} (2^t - u - 2, 1^u) & \text{for } u \equiv 0 \text{ or } 1 \pmod 4 \\ (2^t - u, 1^{u-2}) & \text{for } u \equiv 2 \text{ or } 3 \pmod 4 \end{cases}$$

This implies the claim.

- (2) Note that $f_0(\lambda)$ is a hook. In the stated four cases, it is easy to see that $\lambda \in \mathcal{G}_0(2^t + 1)$. When we are not in one of these cases, we have $\lambda = (2^t - u, 2, 1^{u-1})$ for some $u \in \{2, \dots, 2^t - 3\}$, and $f_1(\lambda)$ is not a hook, so cannot coincide with $f_0 f_0(\lambda)$. \square

Thus the following gives an explicit formula for $G_0(n)$ which then allows to compute all values $G_k(n)$.

COROLLARY 4.16. *Let $n = 2^{a_1} + 2^{a_2} + \dots + 2^{a_r} + \varepsilon \geq 2$, where $a_1 > \dots > a_r > 0$ and $\varepsilon \in \{0, 1\}$. Then*

$$G_0(n) = G_0(2^{a_r} + \varepsilon) \cdot \prod_{j=1}^{r-1} (2^{a_j} - 2).$$

In particular, $\mathcal{G}_0(n) = \mathcal{O}(n)$ only holds for $n \in \{2, 3, 5\}$, and $\mathcal{G}_0(n) = \{(n), (1^n)\}$ only holds for $n \in \{2, 3, 4\}$.

COROLLARY 4.17. *Let $n \in \mathbb{N}$, $k > 0$ with $2^{k+1} \leq n$. Then $\mathcal{G}_k(n) = \mathcal{O}(n)$ if and only if $n = 2^{k+1} + m$, with $m < 2^k$, i.e., if and only if $\lfloor n/2^k \rfloor = 2$.*

Proof. Clearly, $\mathcal{G}_k(n) = \mathcal{O}(n)$ if and only if $\mathcal{G}_k(n; \tau) = \mathcal{O}(n; \tau)$ for all k -types τ of size $w = \lfloor n/2^k \rfloor$. With the notation in 4.2, we get $b_p = b_q$ by Lemma 4.4, as otherwise there is a k -type τ with $\mathcal{G}_k(n; \tau) = \emptyset$. This implies $b_q > 0$ and hence w is even. Taking a k -type with w as its only non-zero part, Lemma 4.4 and Corollary 4.16 then imply $w = 2$. \square

We also note the following implication of Corollary 4.3 where the bound for w is sharp by Corollary 4.16:

COROLLARY 4.18. *Assume $w = \lfloor n/2^k \rfloor \in \{2, 3, 4\}$. Let $\lambda \vdash_o n$ with $\mathcal{Q}_2^{(k)}(\lambda) = (\lambda_1, \dots, \lambda_{2^k})$. Then $\lambda \in \mathcal{G}_k(n)$ if and only if there is an $s \in \{1, \dots, 2^k\}$ such that $\lambda_s \in \{(w), (1^w)\}$.*

EXAMPLE 4.19. We illustrate the efficiency of the formulae for $G_k(n)$ obtained in this section by considering an example.

Let $n = 24 = 2^4 + 2^3$ and let us consider $k = 2$. Referring to Notation 4.2, we have $p = q = r = 2$. Hence, Theorem 4.8 implies

$$G_2(24) = 2^2(w_G(\{2\}) + w_G(\{1, 2\})) = 2^2(2^2 \cdot 3 \cdot G_0(2) + G_0(2 + 2^2)).$$

Now we use Corollary 4.16 and Proposition 4.15 to obtain the necessary values of G_0 ; here, we have $G_0(2) = 2$ and $G_0(6) = G_0(2)(2^2 - 2) = 4$. Thus we conclude $G_2(24) = 112$.

It is illustrative to also go through the arguments again that tell us how to detect whether a certain $\lambda \in \mathcal{O}(24)$ is such that $f_2 f_2(\lambda) = f_3(\lambda)$. By Lemma 4.1, we just need to analyze the 2-data of λ . In particular, in this case we just need to look at $\mathcal{Q}_2^{(2)}(\lambda) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Referring to Notation 4.2, we observe that $b_p = b_q = 3 - 2 = 1$ in this case. Therefore we can either have two non-empty odd partitions λ_i, λ_j in $\mathcal{Q}_2^{(2)}(\lambda)$ of size 2 and 4, respectively, such that $\lambda_i \in \mathcal{G}_0(2)$ (this gives us $(2^2 \cdot G_0(2)) \cdot (3 \cdot 2^2)$ possibilities for λ) or we can have a unique non-empty partition λ_i in $\mathcal{Q}_2^{(2)}(\lambda)$ of size 6, such that $\lambda_i \in \mathcal{G}_0(6)$ (this gives us $2^2 \cdot G_0(6)$ possibilities for λ).

We conclude the section by proving Theorem 4.20 (below), as promised in the introduction. It shows that when $\lfloor n/2^k \rfloor \leq 4$, then for $\lambda \vdash_o n$ the information on whether $f_k f_k(\lambda) = f_{k+1}(\lambda)$ holds is encoded in the character table of \mathfrak{S}_n .

THEOREM 4.20. *Let $k, n \in \mathbb{N}$ with $2^{k+1} \leq n$ and $w = \lfloor n/2^k \rfloor \leq 4$; set $m = n - w \cdot 2^k$. Let $\rho \in \mathfrak{S}_n$ be a product of w 2^k -cycles, $\gamma = \omega_m$ on the fixed points of ρ ; set $g = \rho\gamma \in \mathfrak{S}_n$. Let $\lambda \vdash_o n$. Then $f_k f_k(\lambda) = f_{k+1}(\lambda)$ if and only if $|\chi^\lambda(g)| = 1$.*

Proof. Let $\mathcal{Q}_2^{(k)}(\lambda) = (\lambda_1, \dots, \lambda_{2^k})$. By Corollary 3.6, $|\chi^\lambda(g)| = 1$ if and only if $\lambda_s \in \{(w), (1^w)\}$ for some $s \in \{1, 2, \dots, 2^k\}$. As $w \leq 4$, by Corollary 4.18 this is equivalent to $\lambda \in \mathcal{G}_k(n)$, thus proving the claim. \square

COROLLARY 4.21. *Let $\lambda \vdash_o n$. Let $n = 2^{a_1} + 2^{a_2} + \dots + 2^{a_r}$ where $a_1 > a_2 > \dots > a_r$. Choose $k = a_i - 1$ for some $i \in \{1, \dots, r\}$. Let $\omega_{n,i} \in \mathfrak{S}_n$ be an element of cycle type $(2^{a_1}, 2^{a_2}, \dots, 2^{a_{i-1}}, 2^k, 2^k, 2^{a_{i+1}}, \dots, 2^{a_r})$. If $f_k f_k(\lambda) = f_{k+1}(\lambda)$ then $|\chi^\lambda(\omega_{n,i})| = 1$.*

Proof. Let $\lambda = \lambda^1, \dots, \lambda^r$ be the 2-chain in λ . Applying Lemma 3.1 $i - 1$ times we get that $|\chi^\lambda(\omega_{n,i})| = |\chi^{\lambda^i}(\omega_{n,i}^*)|$ where $\omega_{n,i}^*$ is an element of cycle type $(2^k, 2^k, 2^{a_{i+1}}, \dots, 2^{a_r})$. We get from Theorem 4.20 that $|\chi^\lambda(\omega_{n,i})| = 1$ if and only if $f_k f_k(\lambda^i) = f_{k+1}(\lambda^i)$. By Lemma 4.9 $f_k f_k(\lambda) = f_{k+1}(\lambda)$ implies $f_k f_k(\lambda^i) = f_{k+1}(\lambda^i)$. \square

The following example shows that the converse of Corollary 4.21 does not hold.

EXAMPLE 4.22. Let $n = 12 = 2^3 + 2^2$ and take $k = 1$. The cycle type of $\omega_{12,2}$ is $(8, 2, 2)$. Let $\lambda = (10, 1^2) \vdash_o 12$. We have that $|\chi^\lambda(\omega_{12,2})| = 1$. However, $f_1 f_1(\lambda) = (8)$ and $f_2(\lambda) = (6, 1^2)$.

5. COMMUTATIVITY

Let n, k, ℓ be nonnegative integers such that $k < \ell$ and $2^k + 2^\ell \leq n$. We want to investigate for which odd partitions $\lambda \in \mathcal{O}(n)$ the commutativity relation $f_k f_\ell(\lambda) = f_\ell f_k(\lambda)$ holds. We define

$$\mathcal{T}_{k,\ell}(n) = \{\lambda \vdash_o n \mid f_k f_\ell(\lambda) = f_\ell f_k(\lambda)\}, \mathcal{F}_{k,\ell}(n) = \{\lambda \vdash_o n \mid f_k f_\ell(\lambda) \neq f_\ell f_k(\lambda)\}$$

and set $T_{k,\ell}(n) = |\mathcal{T}_{k,\ell}(n)|$, $F_{k,\ell}(n) = |\mathcal{F}_{k,\ell}(n)|$. Furthermore, we set $\bar{\mathcal{T}}_{k,\ell} = \bigcup_{n \geq 2^k + 2^\ell} \mathcal{T}_{k,\ell}(n)$ and $\bar{\mathcal{F}}_{k,\ell} = \bigcup_{n \geq 2^k + 2^\ell} \mathcal{F}_{k,\ell}(n)$.

Write $n = 2^t + m$ with $0 \leq m < 2^t$. By [3, Theorem B] we know that $\bar{\mathcal{T}}_{k,\ell}(n) = \mathcal{O}(n)$ if and only if $m < 2^k$, or $\ell = t$, or $n = 6$ and $(k, \ell) = (0, 1)$. Therefore we may from now on assume that $n > 6$, and that $\ell, k \in \mathbb{N}_0$ are such that $k < \ell < t$ and $2^k \leq m$.

We fix below some notation that will be used throughout this section.

NOTATION 5.1. We write $n = 2^{a_1} + \dots + 2^{a_r}$, where $a_1 > a_2 > \dots > a_r$, and define $p, q \in \{1, \dots, r\}$ to be maximal such that $a_p \geq \ell$ and $a_q \geq k$, respectively. Set $b_j = a_j - k$, for $1 \leq j \leq q$, and $B = \{b_1, \dots, b_q\}$.

Using a strategy similar to the one used in Section 4, we start by observing that we can reduce the study of our problem to the case where the smaller parameter k is equal to 0. This is shown in the following two lemmas.

LEMMA 5.2. *Let $\lambda \vdash_o n$ be such that $R_k^k(\lambda) \neq R_k^\ell(\lambda)$. Then $\lambda \in \bar{\mathcal{T}}_{k,\ell}(n)$.*

Proof. Let $\mathcal{Q}_2^{(k)}(\lambda) = (\lambda_1, \dots, \lambda_{2^k})$. Suppose that $R_k^k(\lambda) = \lambda_i \neq \lambda_j = R_k^\ell(\lambda)$, for some $i, j \in \{1, \dots, 2^k\}$. By Lemma 3.9, $2^{b_q} \subseteq_2 |\lambda_i|$ and $2^{b_p} \subseteq_2 |\lambda_j|$. Since $|f_0(\lambda_i)| = |\lambda_i| - 1$ and $|f_{\ell-k}(\lambda_j)| = |\lambda_j| - 2^{\ell-k}$, we deduce that

$$\{ |f_0(\lambda_i)|, |f_{\ell-k}(\lambda_j)|, |\lambda_s| \mid s \in \{1, \dots, 2^k\} \setminus \{i, j\} \},$$

is a set consisting of pairwise 2-disjoint natural numbers. Therefore we have that $f_k f_\ell(\lambda) = f_\ell f_k(\lambda)$ is the partition with k -data obtained from the k -data of λ by replacing λ_i and λ_j by $f_0(\lambda_i)$ and $f_{\ell-k}(\lambda_j)$, respectively. \square

LEMMA 5.3. *Let $\lambda \vdash_o n$ be such that $R_k^k(\lambda) = R_k^\ell(\lambda)$. Then $\lambda \in \bar{\mathcal{T}}_{k,\ell}(n)$ if and only if $R_k^k(f_\ell(\lambda)) = f_{\ell-k}(R_k^k(\lambda))$ and $R_k^k(\lambda) \in \mathcal{T}_{0,\ell-k}$.*

Proof. Let $\mathcal{Q}_2^{(k)}(\lambda) = (\lambda_1, \lambda_2, \dots, \lambda_{2^k})$ and $R_k^k(\lambda) = R_k^\ell(\lambda) = \lambda_i$, for some $i \in \{1, \dots, 2^k\}$.

Suppose first that $\lambda \in \bar{\mathcal{T}}_{k,\ell}(n)$. Comparing row k of the k -data of $f_k(\lambda)$ with row k of the k -data of $f_\ell(\lambda)$ we deduce that the two stated conditions must hold.

Conversely, let ρ be the partition of $n - 2^k - 2^\ell$ whose k -data is obtained from the k -data of λ by replacing λ_i with $f_0 f_{\ell-k}(\lambda_i)$. Since $R_k^k(f_\ell(\lambda)) = f_{\ell-k}(R_k^k(\lambda))$, we deduce that $\rho = f_k f_\ell(\lambda)$. As $\lambda_i = R_k^k(\lambda) \in \mathcal{T}_{0,\ell-k}$, the k -data of ρ equals the k -data of $f_\ell f_k(\lambda)$. Hence we deduce $\rho = f_\ell f_k(\lambda)$, as required. \square

With our arithmetical understanding of the removal process (Lemma 3.9), the condition $R_k^k(f_\ell(\lambda)) = f_{\ell-k}(R_k^k(\lambda))$ is not difficult to study; it is equivalent to say that $|R_k^k(\lambda)| > 2^{\ell-k}$ and $|f_0 f_{\ell-k}(R_k^k(\lambda))|$ is 2-disjoint from the sizes of the other $2^k - 1$ partitions in $\mathcal{Q}_2^{(k)}(\lambda)$.

Lemmas 5.2 and 5.3 provide a reduction to the case where $k = 0$. We will first discuss the reduction to this situation in detail, and then investigate the situation when $f_0 f_\ell(\lambda) = f_\ell f_0(\lambda)$, for any given $\lambda \vdash_o n = 2^t + m$ where $1 \leq m$ and $\ell < t$ (and $n > 6$).

DEFINITION 5.4. Let $n, k, \ell \in \mathbb{N}$, $k < \ell$, and $2^k + 2^\ell \leq n$. For a given k -type τ of size $\lfloor n/2^k \rfloor$, we let $\mathcal{T}_{k,\ell}(n; \tau)$, $\mathcal{F}_{k,\ell}(n; \tau)$ denote the subsets of $\mathcal{T}_{k,\ell}(n)$, $\mathcal{F}_{k,\ell}(n)$, respectively, of partitions of k -type τ . By $T_{k,\ell}(n; \tau)$, $F_{k,\ell}(n; \tau)$ we denote the cardinalities of the corresponding sets.

Using Lemmas 3.9, 5.2 and 5.3, we easily deduce the following description that reduces the problem for parameters (k, ℓ) to that for $(0, \ell - k)$:

LEMMA 5.5. We let $\tau = (n_1, \dots, n_{2^k})$ be a k -type, of size $\lfloor n/2^k \rfloor$.

- (1) If $2^{b_p}, 2^{b_q}$ are binary digits of two different parts of τ , then $\mathcal{T}_{k,\ell}(n, \tau) = \mathcal{O}(n, \tau)$.
- (2) Assume that $2^{b_p}, 2^{b_q}$ are binary digits of the same part of τ , say n_j .
If $b_p = b_q > \ell - k$, or if $b_p > b_q$, then

$$\mathcal{T}_{k,\ell}(n; \tau) = \{\lambda \in \mathcal{O}(n; \tau) \mid Q_{2^k}(\lambda)_j \in \mathcal{T}_{0,\ell-k}(n_j)\}.$$

- (3) Assume $b_p = b_q = \ell - k$.
If $2^{b_{p-1}}, 2^{b_p}$ are binary digits of the same part of τ , say n_j , then

$$\mathcal{T}_{k,\ell}(n; \tau) = \{\lambda \in \mathcal{O}(n; \tau) \mid Q_{2^k}(\lambda)_j \in \mathcal{T}_{0,\ell-k}(n_j)\},$$

otherwise $\mathcal{T}_{k,\ell}(n, \tau) = \emptyset$.

Together with [3, Theorem B] this result tells us when for all odd partition of a fixed k -type commutativity holds.

COROLLARY 5.6. We let $\tau = (n_1, \dots, n_{2^k})$ be a k -type, of size $\lfloor n/2^k \rfloor$. Then $\mathcal{T}_{k,\ell}(n, \tau) = \mathcal{O}(n; \tau)$ if and only if one of the following holds:

- (1) $2^{b_p}, 2^{b_q}$ are binary digits of two different parts of τ .
- (2) $2^{b_p}, 2^{b_q}$ are binary digits of the same part $n_j > 2^{\ell-k}$ of τ , and $2^{\ell-k} = 2^{b_p}$ is the largest binary digit of n_j , or $n_j = 2^{b_p}$, or $n_j = 6$ and $(b_p, b_q, \ell - k) \in \{(2, 2, 1), (2, 1, 1), (1, 1, 1)\}$.

Proof. By [3, Theorem B] we know that $\mathcal{T}_{0,\ell-k}(n_j) = \mathcal{O}(n_j)$ if and only if $2^{\ell-k}$ is the largest binary digit of n_j , or n_j is a 2-power, or $n_j = 6$ and $\ell - k = 1$. The description of $\mathcal{T}_{k,\ell}(n; \tau)$ in Lemma 5.5 then implies the stated assertion. \square

We use the example below to give an overview of the ideas introduced in the first part of Section 5.

EXAMPLE 5.7. Let $n = 10 = 2^3 + 2^1$ and let $\ell = 2$ and $k = 1$. Using Notation 5.1, we see that $a_1 = 3$, $a_2 = 1$, $p = 1$ and $q = 2$. Moreover, $b_1 = 2$ and $b_2 = 0$. There are 16 odd partitions of 10. We partition the set $\mathcal{O}(10)$ according to the possible 1-types. In this case we have four possible 1-types, namely $(5, 0)$, $(0, 5)$, $(4, 1)$ and $(1, 4)$. For instance,

$$\mathcal{O}(10; (4, 1)) = \{(7, 3), (5, 3, 1^2), (3^2, 1^4), (2^2, 1^6)\}.$$

From Lemma 5.5(1), we expect $\mathcal{O}(10; (4, 1)) \subseteq \mathcal{T}_{1,2}(10)$.

Here we verify that $\lambda := (5, 3, 1^2) \in \mathcal{T}_{1,2}(10)$. Observe that $Q_2^{(1)}(\lambda) = ((3, 1), (1))$. Hence $R_1^1(\lambda) = (1)$ and $R_1^2(\lambda) = (3, 1)$. It follows that $f_1(\lambda) = (5, 1^3)$, where $Q_2^{(1)}((5, 1^3)) = ((3, 1), \emptyset)$. On the other hand, $f_2(\lambda) = (2^2, 1^2)$, where $Q_2^{(1)}((2^2, 1^2)) = ((1, 1), (1))$. It is now easy to observe that

$$f_2 f_1(\lambda) = f_2((5, 1^3)) = (1^4) = f_1((2^2, 1^2)) = f_1 f_2(\lambda).$$

Similarly one sees that $\mathcal{O}(10; (1, 4))$ is contained in $\mathcal{T}_{1,2}(10)$, as asserted by Lemma 5.5(1). Let us now look at odd partitions of 1-type $(5, 0)$. We observe that

$$\mathcal{O}(10; (5, 0)) = \{(9, 1), (5, 2^2, 1), (3, 2^2, 1^3), (1^{10})\}.$$

We have $\mathcal{Q}_2^{(1)}((9, 1)) = ((5), \emptyset)$. Since clearly $(5) \in \mathcal{T}_{0,1}(5)$, Lemma 5.5(2) predicts that $(9, 1) \in \mathcal{T}_{1,2}(10)$. This can easily be verified, as $f_1(9, 1) = (7, 1)$, $f_2((9, 1)) = (5, 1)$ and $\mathcal{Q}_2^{(1)}((7, 1)) = ((4), \emptyset)$, $\mathcal{Q}_2^{(1)}((5, 1)) = ((3), \emptyset)$. As expected, it follows that

$$f_2 f_1((9, 1)) = f_2((7, 1)) = (3, 1) = f_1((5, 1)) = f_1 f_2((9, 1)).$$

Let us now consider $\lambda := (5, 2^2, 1)$. We have $\mathcal{Q}_2^{(1)}(\lambda) = ((3, 2), \emptyset)$. We observe that $(3, 2) \notin \mathcal{T}_{0,1}(5)$, since $f_0 f_1((3, 2)) = (2) \neq (1^2) = f_1 f_0((3, 2))$. Hence by Lemma 5.5(2) we should have that $\lambda \notin \mathcal{T}_{1,2}(10)$. We verify this fact, by observing that $f_2(\lambda) = (5, 1)$, $f_1(\lambda) = (5, 1^3)$ where $\mathcal{Q}_2^{(1)}((5, 1)) = ((3), \emptyset)$ and $\mathcal{Q}_2^{(1)}((5, 1^3)) = ((3, 1), \emptyset)$. We conclude that

$$f_1 f_2(\lambda) = f_1((5, 1)) = (3, 1) \neq (1^4) = f_2((5, 1^3)) = f_2 f_1(\lambda).$$

Similarly one sees that $\mathcal{O}(10; (5, 0)) \cap \mathcal{T}_{1,2}(10) = \{(9, 1), (1^{10})\}$. Since $\mathcal{O}(10; (0, 5))$ consists of the conjugates of the partitions in $\mathcal{O}(10; (5, 0))$, we conclude that

$$\mathcal{T}_{1,2}(10) = \mathcal{O}(10; (4, 1)) \cup \mathcal{O}(10; (1, 4)) \cup \{(10), (9, 1), (2, 1^8), (1^{10})\}.$$

From Lemmas 5.2 and 5.3 we also deduce that for $\lambda \in \mathcal{O}(n)$, the commutativity property does not depend on the rows of the k -data corresponding to the cores. Hence we have:

LEMMA 5.8. *We let $\bar{n} = \sum_{i=1}^q 2^{a_i}$. For any k -type τ of size $\lfloor n/2^k \rfloor$ we have*

$$T_{k,\ell}(n; \tau) = \left(\prod_{j=q+1}^r 2^{a_j} \right) \cdot T_{k,\ell}(\bar{n}; \tau)$$

In particular,

$$T_{k,\ell}(n) = \left(\prod_{j=q+1}^r 2^{a_j} \right) \cdot T_{k,\ell}(\bar{n}).$$

Analogous statements hold for $F_{k,\ell}(n; \tau), F_{k,\ell}(n)$.

Similarly as we have done it in Section 4, we introduce suitable weights that can be computed from the numbers for the $(0, \ell - k)$ -case.

DEFINITION 5.9. *For $J \subseteq I = \{1, \dots, q\}$ we define its F -weight with respect to the set $B = \{b_1, \dots, b_q\}$ to be*

$$w_F(J) = \left(\prod_{i \in I \setminus J} 2^{b_i} \right) \cdot (2^k - 1)^{|I \setminus J|} \cdot F_{0,\ell-k} \left(\sum_{j \in J} 2^{b_j} \right).$$

Using in part similar arguments as before, we have the following formulae for the numbers $F_{k,\ell}(n)$, based on the values $F_{0,\ell-k}(u)$ (and hence also such formulae for $T_{k,\ell}(n)$).

THEOREM 5.10. *With the notation introduced in 5.1, the following holds.*

If $a_p > a_q$ or if $a_p = a_q > \ell$, then

$$F_{k,\ell}(n) = \left(\prod_{j=q+1}^r 2^{a_j} \right) \cdot 2^k \cdot \sum_{\{p,q\} \subseteq J \subseteq I} w_F(J).$$

If $a_p = a_q = \ell$, then

$$F_{k,\ell}(n) = \left(\prod_{j=q+1}^r 2^{a_j} \right) \cdot \left(2^k \cdot \left(\sum_{\{p-1,p\} \subseteq J \subseteq I} w_F(J) \right) + 2^{k(q-1)}(2^k - 1) \prod_{i \in I} 2^{b_i} \right).$$

Proof. In view of Lemma 5.8, we focus on computing the numbers $F_{k,\ell}(\bar{n})$. We take the sum over all k -types of size $\lfloor n/2^k \rfloor$, and for each k -type τ we count the partitions in $\mathcal{F}_{k,\ell}(\bar{n}; \tau)$ according to Lemma 5.5.

When $a_p > a_q$, we only get a contribution from those k -types τ where $2^{b_p}, 2^{b_q}$ are binary digits in the same part of τ . To construct these, we take any $J \subseteq I$ containing p, q and define $n_j := \sum_{j \in J} 2^{b_j}$. We now consider the contributions from all k -types for this fixed J . We first have 2^k choices for the position j of the part of which $2^{b_p}, 2^{b_q}$ are binary digits. For each $i \in I \setminus J$ we have $2^k - 1$ choices to place 2^{b_i} as a binary digit of a part of the τ under construction, different from the j -th. By Lemma 5.5, for such a $\tau = (n_1, \dots, n_{2^k})$, the partitions $\lambda \in \mathcal{F}_{k,\ell}(\bar{n}; \tau)$ are then those such that $\mathcal{Q}(\lambda)_i$ is an arbitrary odd partition of n_i for $i \neq j$, and $\mathcal{Q}(\lambda)_j \in \mathcal{F}_{0,\ell-k}(n_j)$ (and the earlier rows of the k -data contain only empty cores). Hence we get $|\mathcal{F}_{k,\ell}(\bar{n}; \tau)| = (\prod_{i \in I \setminus J} 2^{b_i}) \cdot F_{0,\ell-k}(\sum_{j \in J} 2^{b_j})$, and the total contribution for J is exactly $2^k w_F(J)$.

When $a_p = a_q$, the construction of the relevant k -types τ is verbatim the same as before, noticing that now $\{p, q\} = \{p\}$.

In the situation where $a_p = a_q > \ell$, for each k -type τ thus constructed the number $|\mathcal{F}_{k,\ell}(\bar{n}; \tau)|$ is determined as before, giving again the total contribution $2^k w_F(J)$ for each $J \subseteq I$ with $p \in J$.

In the remaining situation, where $a_p = a_q = \ell$, we have to distinguish between the k -types τ where 2^{b_p} and $2^{b_{p-1}}$ are binary digits of the same part of τ , and those where this does not hold. In the first case, take any $J \subseteq I$ containing $\{p-1, p\}$, define $n_j := \sum_{j \in J} 2^{b_j}$ and argue as before to construct the k -types to J . Applying Lemma 5.5, the total contribution for J is again exactly $2^k w_F(J)$. To construct the k -types in the second case, note that having placed 2^{b_p} (in 2^k ways) there are only $2^k - 1$ choices to place $2^{b_{p-1}}$ as a binary digit, but for each of the other $q - 2$ binary digits 2^{b_i} there are 2^k choices. For any τ constructed in this way, we know by Lemma 5.5 that $\mathcal{F}_{k,\ell}(\bar{n}; \tau) = \mathcal{O}(\bar{n}; \tau)$, hence $|\mathcal{F}_{k,\ell}(\bar{n}; \tau)| = \prod_{i \in I} 2^{b_i}$. Thus the total contribution over all k -types of the second form is exactly $2^{k(q-1)}(2^k - 1) \prod_{i \in I} 2^{b_i}$. This proves the stated formula. \square

EXAMPLES 5.11. We illustrate the efficiency of the formula above by giving some examples. Here we assume that we already know the values $F_{0,\ell-k}(u)$; these – or their counterparts $T_{0,\ell-k}(u)$, respectively – are computed explicitly in the next section.

- (1) Let $n = 24 = 2^4 + 2^3$, $k = 2$, $\ell = 3$. Then $F_{2,3}(24) = 2^2 \cdot 2^3 \cdot 3 + F_{0,1}(2^2 + 2^1)$. As $F_{0,1}(6) = 0$, $F_{2,3}(24) = 96$.
- (2) Let $n = 36 = 2^5 + 2^2$, $k = 2$, $\ell = 3$. Then $F_{2,3}(36) = 2^2 F_{0,1}(2^3 + 2^0)$. As $F_{0,1}(9) = 6$, $F_{2,3}(36) = 24$.

6. THE BASE CASE $\mathcal{T}_{0,\ell-k}(x)$

We now turn to the discussion of the situation when $f_0 f_\ell(\lambda) = f_\ell f_0(\lambda)$, for any given $\lambda \vdash_o n = 2^t + m$ where $1 \leq m < 2^t$ and $\ell < t$.

DEFINITION 6.1. For $n \in \mathbb{N}$ we denote by $\Omega(\ell; n) = \mathcal{T}_{0,\ell}(n)$ the set consisting of all odd partitions λ of n such that $f_0 f_\ell(\lambda) = f_\ell f_0(\lambda)$; for $n \leq 2^\ell$, the set $\Omega(\ell; n)$ then is the set of all odd partitions. When $n = 2^t + m$ as above, and $\gamma \vdash_o m$, we set

$$\Omega(\ell; n \mid \gamma) := \{\lambda \in \Omega(\ell; n) \mid C_{2^t}(\lambda) = \gamma\}.$$

STRATEGY. For any natural number $n \geq 2^{\ell+1}$ we write $n = 2^{t_1} + \dots + 2^{t_a} + m$ where $t_1 > t_2 > \dots > t_a > \ell$ and $m \leq 2^\ell$. We will first describe the set $\Omega(\ell; n)$ in terms of $\Omega(\ell; n - 2^{t_1})$.

Afterwards, we will study the base cases, namely the set $\Omega(\ell; 2^t + m)$ for any $0 \leq m \leq 2^\ell$ and $t \geq \ell$. To do this we will first analyse the case where $m \neq 2^\ell$. Separately we will deal with the remaining case: namely $m = 2^\ell$ and $t > \ell$.

This will provide us with a complete recursive description of the set $\Omega(\ell; n)$. As an application of this, we derive a closed formula for $|\Omega(\ell; n)|$.

LEMMA 6.2. Let $n = 2^t + m$ where $2^\ell + 1 \leq m < 2^t$, and let $\lambda \vdash_o n$. If $\lambda \in \Omega(\ell; n)$ then $C_{2^t}(\lambda) \in \Omega(\ell; m)$. Thus,

$$\Omega(\ell; n) = \bigcup_{\gamma \in \Omega(\ell; m)} \Omega(\ell; n | \gamma).$$

Proof. This follows directly from Lemma 4.9. □

6.1. INDUCTIVE STEP. In light of Lemma 6.2 we will now fix $\gamma \in \Omega(\ell; m)$ and explicitly describe which and how many of the 2^t odd extensions of γ lie in $\Omega(\ell; n | \gamma)$. In the following we are referring to Definition 2.2 and we are using Lemma 2.6.

DEFINITION 6.3. Let n and ℓ be natural numbers such that $2^\ell \leq n$. Let $\lambda \vdash_o n$. We say that λ is of type (1+) if $z_0(\lambda) = z_\ell(\lambda)$, and λ is of type (1-) if $s_0(\lambda) = s_\ell(\lambda)$ (i.e., the odd 1-hook and odd 2^ℓ -hook of λ have the same hand or foot, respectively). Otherwise, we say that λ is of type (2).

When $\lambda \in \Omega(\ell; n)$, we write $\lambda \in \Omega^\varepsilon(\ell; n)$ if λ is of type $\varepsilon \in \{(1+), (1-), (2)\}$. Clearly, $\Omega(\ell; n) = \Omega^{1+}(\ell; n) \cup \Omega^{1-}(\ell; n) \cup \Omega^2(\ell; n)$. We set $\Omega^1(\ell; n) := \Omega^{1+}(\ell; n) \cup \Omega^{1-}(\ell; n)$. We will sometimes say that λ is of type (1) to indicate that $\lambda \in \Omega^1(\ell; n)$.

NOTATION 6.4. Let n and $\lambda \vdash_o n$ as above. Assume that $n < 2^s$; let A be a normalized 2^s -abacus for λ , i.e., the beads after the first gap on A correspond to the first column hook lengths in λ . Let $c = c_0(A)$ and $d = c_\ell(A)$. By Lemma 2.6, c and d are in positions $(0, r_0(\lambda))$ and $(0, r_\ell(\lambda))$ of A , respectively.

Note that λ is of type (1+) if $r_0(\lambda) = r_\ell(\lambda)$, i.e., we have the situation of the *same start bead*, while λ is of type (1-) if $r_\ell(\lambda) = r_0(\lambda) + 2^\ell - 1$, i.e., the beads have the *same target position*.

For an illustration of this definition as well as for providing a basic case, we consider the situation where n is a 2-power. Note that in this case $\Omega(\ell; n) = \mathcal{H}(n)$.

LEMMA 6.5. Let $n = 2^t$ and $0 < \ell \leq t$. Let $\lambda \in \Omega(\ell; n)$, say $\lambda = (2^t - w, 1^w)$ with $0 \leq w \leq 2^t - 1$.

For $\ell < t$, λ is of type (1+) if $1, 2^\ell \not\subseteq_2 w$, it is of type (1-) if $1, 2^\ell \subseteq_2 w$, and it is of type (2) otherwise. In particular,

$$|\Omega^{1\pm}(\ell; 2^t)| = 2^{t-2}, \quad |\Omega^2(\ell; 2^t)| = 2^{t-1}.$$

For $\ell = t$, λ is of type (1+) if w is even, and it is of type (1-) if w is odd. In particular,

$$|\Omega^{1\pm}(t, 2^t)| = 2^{t-1}, \quad |\Omega^2(t, 2^t)| = 0.$$

Proof. Let $r_0 = r_0(\lambda)$, $r_\ell = r_\ell(\lambda)$ be defined as above. First we assume $\ell < t$. By Lemma 3.8 we have $r_0 = r_\ell$ if and only if $1, 2^\ell \not\subseteq_2 w$, and $r_\ell = r_0 + 2^\ell - 1$ if and only if $1, 2^\ell \subseteq_2 w$. Similarly, in the case $\ell = t$, we have $r_\ell = r_0$ if w is even and $r_\ell = r_0 + 2^\ell - 1$ if w is odd. □

LEMMA 6.6. Let $n = 2^t + m$ where $2^\ell + 1 \leq m < 2^t$, and let $\gamma \in \Omega^2(\ell; m)$. Then

$$\Omega(\ell; n | \gamma) = \mathcal{E}(\gamma, 2^t) \subseteq \Omega^2(\ell; n).$$

Proof. Let $\lambda \in \mathcal{E}(\gamma, 2^t)$. Let B be a normalized 2^t -abacus for γ and let A be the 2^t -abacus for λ such that $A^\uparrow = B$. Let $c := c_0(B)$ and $d := c_\ell(B)$. Thus $B(1, \leftarrow c)$ is a 2^t -abacus for $f_0(\gamma)$ and $B(2^\ell, \leftarrow d)$ is a 2^t -abacus for $f_\ell(\gamma)$. Following Notation 6.4 we have that c is in position $(0, r_0(\gamma))$ and d is in position $(0, r_\ell(\gamma))$. Since γ is of type (2) we have that $r_\ell(\gamma) \notin \{r_0(\gamma), r_0(\gamma) + 2^\ell - 1\}$. Moreover, since $f_0 f_\ell(\gamma) = f_\ell f_0(\gamma)$ we deduce that $r_0(f_\ell(\gamma)) = r_0(\gamma)$ and that $r_\ell(f_0(\gamma)) = r_\ell(\gamma)$. More precisely we have that

$$[B(1, \leftarrow c)](2^\ell, \leftarrow d) = [B(2^\ell, \leftarrow d)](1, \leftarrow c),$$

is a 2^t -abacus for $f_0 f_\ell(\gamma) = f_\ell f_0(\gamma)$.

Let $x \in \{0, \dots, 2^t - 1\}$ be such that $w_{2^t}(A_x) = 1$. If $x \notin \{r_0(\gamma) - 1, r_0(\gamma), r_\ell(\gamma) - 2^\ell, r_\ell(\gamma)\}$ then $c_0(A)$ and $c_\ell(A)$ are in positions $(0, r_0(\gamma))$ and $(0, r_\ell(\gamma))$ of A respectively, by Lemma 3.1(3). Moreover, positions $(0, r_0(\gamma) - 1)$ and $(0, r_\ell(\gamma) - 2^\ell)$ of A are empty, because they were empty in B . By Lemma 3.1(3), we deduce that

$$A^{\ell, 0} := [A(2^\ell, \leftarrow c_\ell(A))](1, \leftarrow c_0(A)),$$

is a 2^t -abacus for $f_\ell f_0(\lambda)$. Similarly $A^{0, \ell} := [A(1, \leftarrow c_0(A))](2^\ell, \leftarrow c_\ell(A))$ is a 2^t -abacus for $f_0 f_\ell(\lambda)$. Since $A^{0, \ell}$ is equal to $A^{\ell, 0}$ we deduce that $\lambda \in \Omega(\ell; n | \gamma)$.

Suppose now that $x \in \{r_0(\gamma) - 1, r_0(\gamma), r_\ell(\gamma) - 2^\ell, r_\ell(\gamma)\}$. For instance let $x = r_0(\gamma)$. Then $c_0(A)$ is the bead in position $(1, r_0(\gamma))$ of A , by Lemma 3.1(3). On the other hand $c_\ell(A) = c_\ell(B)$ is the bead in position $(0, r_\ell(\gamma))$ of A . It is now clear that the 2^t -abacus configuration obtained from A by moving bead $c_0(A)$ to position $(1, r_0(\gamma) - 1)$ and bead $c_\ell(A)$ to position $(0, r_\ell(\gamma) - 2^\ell)$ is a 2^t -abacus configuration for both $f_0 f_\ell(\lambda)$ and $f_\ell f_0(\lambda)$. Hence they coincide. A very similar argument applies to the cases where $x \in \{r_0(\gamma) - 1, r_\ell(\gamma) - 2^\ell, r_\ell(\gamma)\}$. We conclude that $\Omega(\ell; n | \gamma) = \mathcal{E}(\gamma, 2^t)$.

From the discussion above it follows immediately that given any $\lambda \in \Omega(\ell; n | \gamma)$ and considering the 2^t -abacus configuration A for λ as described above, we have that $c_0(A)$ lies on runner $A_{r_0(\gamma)}$ and $c_\ell(A)$ lies on runner $A_{r_\ell(\gamma)}$. Since γ is of type (2), this is enough to deduce that λ is of type (2) and therefore that $\Omega(\ell; n | \gamma) \subseteq \Omega^2(\ell; n)$. \square

LEMMA 6.7. *Let $n = 2^t + m$ where $2^\ell + 1 \leq m < 2^t$, and let $\gamma \in \Omega^{1+}(\ell; m)$ and $\lambda \in \mathcal{E}(\gamma, 2^t)$. Let B be a normalized 2^t -abacus for γ and A be the 2^t -abacus for λ such that $A^\uparrow = B$. Let $x \in \{0, 1, \dots, 2^t - 1\}$ be such that $w_{2^t}(A_x) = 1$. Then $\lambda \in \Omega(\ell; n | \gamma)$ if and only if $x \notin \{r_0(\gamma) - 1, r_\ell(\gamma) - 2^\ell, r_\ell(\gamma) - 2^\ell - 1\}$. Moreover, $\Omega(\ell; n | \gamma) \subseteq \Omega^{1+}(\ell; n)$.*

Proof. Let γ be such that $r_0(\gamma) = r_\ell(\gamma) =: r$. Let $c := c_0(B) = c_\ell(B)$. It is easy to see that the 2^t -abacus E obtained from B by moving bead c from position $(0, r)$ to position $(0, r - 2^\ell - 1)$ is a 2^t -abacus for $f_0 f_\ell(\gamma) = f_\ell f_0(\gamma)$.

Suppose that $x \notin \{r_0(\gamma) - 1, r_\ell(\gamma) - 2^\ell, r_\ell(\gamma) - 2^\ell - 1\} = \{r - 1, r - 2^\ell, r - 2^\ell - 1\}$. If $x \neq r$, let c' ($= c$) be the bead in position $(0, r)$ of A , for $x = r$, let c' be the bead in position $(1, r)$ of A . We deduce from Lemma 3.1(3) that $A(1, \leftarrow c')$, $A(2^\ell, \leftarrow c')$ and $A(1 + 2^\ell, \leftarrow c')$ are 2^t -abacus configurations for odd partitions μ_1, μ_2 and μ_3 of $n - 1, n - 2^\ell$ and $n - 2^\ell - 1$ respectively. In particular $\mu_1 = f_0(\lambda)$ and $\mu_2 = f_\ell(\lambda)$. Moreover μ_3 is an odd partition of $n - 2^\ell - 1$ obtained from μ_2 by removing a 1-hook. Hence we deduce that $\mu_3 = f_0(\mu_2) = f_0 f_\ell(\lambda)$. On the other hand we also notice that μ_3 is obtained from μ_1 by removing a 2^ℓ -hook. Hence $\mu_3 = f_\ell(\mu_1) = f_\ell f_0(\lambda)$. We conclude that $\lambda \in \Omega^{1+}(\ell; n)$.

Suppose now that $x \in \{r - 2^\ell - 1, r - 2^\ell, r - 1\}$. We will now show that the 2^t -abaci for $f_0 f_\ell(\lambda)$ and $f_\ell f_0(\lambda)$ obtained from A by performing the appropriate bead moves are distinct. Suppose that $x = r - 1$ (the other cases are treated in a completely similar way). Then A is obtained from B by sliding down the bead in position $(-1, r - 1)$ to position $(0, r - 1)$. Hence position $(-1, r - 1)$ is empty in A . On the other hand, there are beads in positions $(-1, r)$ and $(-1, r - 1 - 2^\ell)$ of A , since A coincides with

B on every runner distinct from runner $r - 1$. Let d be the bead in position $(-1, r)$ of A . From Lemma 3.1 (3), we see that $A^0 := A(1, \leftarrow d)$ is a 2^t -abacus for $f_0(\lambda)$. Let $A^{0,\ell}$ be the 2^t -abacus configuration for $f_\ell f_0(\lambda)$ obtained from A^0 by moving a bead left. Since there is a bead in position $(-1, r - 1 - 2^\ell)$ of A^0 , we deduce that $A^{0,\ell}$ is not obtained by moving again bead d ; in particular, $A^{0,\ell}$ has a bead in position $(-1, r - 1)$. On the other hand, calling $e (= c)$ the bead in position $(0, r)$ of A , it follows from Lemma 3.1 (3) that $A^\ell := A(2^\ell, \leftarrow e)$ is the 2^t -abacus for $f_\ell(\lambda)$. Let e' be the bead in position $(0, r - 2^\ell)$ of A^ℓ . Arguing as above one observes that $A^{\ell,0} := A(2^\ell + 1, \leftarrow e) = A^\ell(1, \leftarrow e')$ is the 2^t -abacus configuration for $f_0 f_\ell(\lambda)$. It is clear that position $(-1, r - 1)$ of $A^{\ell,0}$ is empty. Hence $A^{\ell,0} \neq A^{0,\ell}$ and therefore $\lambda \notin \Omega(\ell; n)$. \square

We use the example below to illustrate the ideas involved in Lemma 6.7.

EXAMPLE 6.8. Let $n = 7, m = 3, t = 2$ and $\ell = 1$. Notice that the partition (3) lies in $\Omega^{1+}(1; 3)$. This can be seen by considering the normalized 4-abacus B for (3) . We observe that $r_0((3)) = r_1((3)) = 3$. Adjoining 4-hooks to (3) we observe that

$$\mathcal{E}((3), 4) = \{(7), (3^2, 1), (3, 2, 1^2), (3, 1^4)\}.$$

For any $\lambda \in \mathcal{E}((3), 4)$ we let A be the 4-abacus configuration for λ such that $A^\uparrow = B$ and we let $x \in \{0, 1, 2, 3\}$ be such that $w_4(A_x) = 1$.

If $\lambda = (7)$ then $x = 3 \notin \{r_0((3)) - 1, r_1((3)) - 2, r_1((3)) - 2 - 1\} = \{2, 1, 0\}$. By Lemma 6.7 we expect to have that $(7) \in \Omega(1; 7 | (3))$. This is true, since

$$f_1 f_0((7)) = f_1((6)) = (4) = f_0((5)) = f_0 f_1((7)).$$

On the other hand, if $\lambda = (3^2, 1)$ then $x = 2 = r_0((3)) - 1$. In this case we verify that $(3^2, 1) \notin \Omega(1; 7 | (3))$. In fact we have that

$$f_0 f_1((3^2, 1)) = f_0((2^2, 1)) = (2, 1^2) \neq (3, 1) = f_1((3^2)) = f_1 f_0((3^2, 1)).$$

In a completely similar fashion we observe that $(3, 2, 1^2), (3, 1^4) \notin \Omega(1; 7 | (3))$, and hence we deduce that $\Omega(1; 7 | (3)) = \{(7)\}$. To conclude the example we refer to Notation 6.4, and we let $s = 3$ so that $2^s = 8 > 7 = n$. Letting C be the normalized 8-abacus for (7) , we observe that $r_0((7)) = r_1((7)) = 7$. Therefore (7) is of type $(1+)$ and $\Omega(1; 7 | (3)) = \{(7)\} \subseteq \Omega^{1+}(1; 7)$, as prescribed by Lemma 6.7.

LEMMA 6.9. Let $n = 2^t + m$ where $2^\ell + 1 \leq m < 2^t$, and let $\gamma \in \Omega^{1-}(\ell; m)$ and $\lambda \in \mathcal{E}(\gamma, 2^t)$. Let B be a normalized 2^t -abacus for γ and A be the 2^t -abacus for λ such that $A^\uparrow = B$. Let $x \in \{0, 1, \dots, 2^t - 1\}$ be such that $w_{2^t}(A_x) = 1$. Then $\lambda \in \Omega(\ell; n | \gamma)$ if and only if $x \notin \{r_0(\gamma), r_\ell(\gamma), r_\ell(\gamma) + 1\}$. Moreover, $\Omega(\ell; n | \gamma) \subseteq \Omega^{1-}(\ell; n)$.

Proof. Let γ be such that $r_\ell(\gamma) = r_0(\gamma) + 2^\ell - 1$ and set $r := r_0(\gamma)$. Let $c := c_0(B)$. Since $f_0 f_\ell(\gamma) = f_\ell f_0(\gamma)$, it is easy to see that B has a bead d at position $(0, r + 2^\ell)$. Then the 2^t -abacus obtained from B by moving bead c from position $(0, r)$ to position $(0, r - 1)$ and bead d to position $(0, r)$ is a 2^t -abacus for $f_\ell f_0(\gamma)$.

The further arguments are entirely similar to the proof of Lemma 6.7. \square

We give a brief example to illustrate Lemma 6.9. This completes the analysis of $\Omega(1; 7)$ started in Example 6.8.

EXAMPLE 6.10. Let $n = 7, m = 3, t = 2$ and $\ell = 1$. Notice that the partition (1^3) lies in $\Omega^{1-}(1; 3)$. Let B be the normalized 4-abacus for (1^3) . We observe that $r_0((1^3)) = 1$ and that $r_1((1^3)) = 2$. Adjoining 4-hooks to (1^3) we have that

$$\mathcal{E}((1^3), 4) = \{(5, 1^2), (4, 2, 1), (3, 2^2), (1^7)\}.$$

For any $\lambda \in \mathcal{E}((1^3), 4)$ we let A be the 4-abacus configuration for λ such that $A^\uparrow = B$ and we let $x \in \{0, 1, 2, 3\}$ be such that $w_4(A_x) = 1$.

If $\lambda = (5, 1^2)$ then $x = 3 = r_1((1^3)) + 1$. By Lemma 6.9 we expect to have that $(5, 1^2) \notin \Omega(1; 7 | (1^3))$. This is true, since

$$f_1 f_0((5, 1^2)) = f_1((5, 1)) = (3, 1) \neq (4) = f_0((5)) = f_0 f_1((5, 1^2)).$$

Similarly we verify that $(4, 2, 1), (3, 2^2) \notin \Omega(1; 7 | (1^3))$. On the other hand, if $\lambda = (1^7)$ then $x = 0 \notin \{r_0((1^3)), r_1((1^3)), r_1((1^3)) + 1\} = \{1, 2, 3\}$. In fact we have that

$$f_0 f_1((1^7)) = (1^4) = f_1 f_0((1^7)).$$

We deduce that $\Omega(1; 7 | (1^3)) = \{(1^7)\} \subseteq \Omega^{1-}(1; 7)$. To conclude we observe that Examples 6.8 and 6.10 show that $\Omega(1; 7) = \{(7), (1^7)\}$.

COROLLARY 6.11. *Let $n = 2^t + m$ where $2^\ell + 1 \leq m < 2^t$. Then*

$$|\Omega^{1\pm}(\ell; n)| = (2^t - 3)|\Omega^{1\pm}(\ell; m)| \text{ and } |\Omega^2(\ell; n)| = 2^t |\Omega^2(\ell; m)|.$$

In particular,

$$T_{0,\ell}(n) = |\Omega(\ell; n)| = (2^t - 3)|\Omega^1(\ell; m)| + 2^t |\Omega^2(\ell; m)|.$$

Proof. Recalling that $|\mathcal{E}(\gamma, 2^t)| = 2^t$ whenever $|\gamma| < 2^t$ by the main result of [2], the statement follows directly from Lemmas 6.6, 6.7 and 6.9. \square

In order to obtain an explicit closed formula for $|\Omega(\ell; n)|$ for any given natural number n , it is now enough to compute the values $|\Omega^1(\ell; m)|$ and $|\Omega^2(\ell; m)|$ for any $m \in \mathbb{N}$ such that $2^\ell + 1 \leq m < 2^{\ell+1}$ or such that $m = 2^t + q$ for some $0 \leq q \leq 2^\ell$.

6.2. BASE CASES. In this subsection we let $n = 2^t + m$ with $m < 2^t$, where $t \geq \ell$ and $1 \leq m \leq 2^\ell$. The following setup and notation for the case where $m < 2^\ell$ will be used throughout most of this section, before we turn to the case $m = 2^\ell$.

NOTATION 6.12. Let $\lambda \vdash_o n$ and let $\gamma := C_{2^t}(\lambda) \vdash_o m$. Let B be a normalized 2^ℓ -abacus for γ . Since $m < 2^\ell$ we know from Lemma 2.6 that B does not have beads in rows $i \geq 1$. Let $c = c_0(B)$ so that $B^0 := B(1, \leftarrow c)$ is a 2^ℓ -abacus for $f_0(\gamma)$. Then c is in position $(0, b_0)$ of B , where $b_0 = r_0(B) \in \{1, \dots, 2^\ell - 1\}$.

Let A be the 2^ℓ -abacus for λ such that $A^\uparrow = B$. For $j \in \{0, \dots, 2^\ell - 1\}$ let μ_j be the partition represented on runner A_j (seen as a 1-abacus). As observed in Remark 2.7, we know that $(\mu_0, \dots, \mu_{2^\ell-1})$ equals $\mathcal{Q}_\ell(\lambda)$ (up to a cyclic shift of its components). Hence we deduce that there exists a unique $x \in \{0, \dots, 2^\ell - 1\}$ such that $\mu_x \in \mathcal{H}(2^{t-\ell})$ and such that $\mu_y = \emptyset$ for all $y \neq x$. In particular we have that $w_{2^\ell}(A) = w_{2^\ell}(A_x) = 2^{t-\ell}$ and $w_{2^\ell}(A_y) = 0$ for all $y \neq x$. Let A^ℓ be the 2^ℓ -abacus for $f_\ell(\lambda)$ such that $(A^\ell)^\uparrow = B$. The above discussion shows that A^ℓ is obtained from A by sliding a bead up one row on runner A_x , to obtain a 1-abacus configuration for $f_0(\mu_x)$. On the other hand a 2^ℓ -abacus for $f_0(\lambda)$ can be obtained from A by sliding a bead c' in position (i, b_0) of A to position $(i, b_0 - 1)$. This follows from Lemma 3.1 (3) because $w_{2^\ell}(A(1, \leftarrow c')) = w_{2^\ell}(A)$ (by direct verification, or more generally by [7, Lemma 2.5]) and $(A(1, \leftarrow c'))^\uparrow = B^0$. We call $A^0 := A(1, \leftarrow c')$.

LEMMA 6.13. *Let $\lambda \vdash_o n$ be as in Notation 6.12. If $x \notin \{b_0 - 1, b_0\}$ then $f_0 f_\ell(\lambda) = f_\ell f_0(\lambda)$. In this case, $\lambda \in \Omega^2(\ell; n)$.*

Proof. Let A^ℓ be the 2^ℓ -abacus for $f_\ell(\lambda)$ described in Notation 6.12. Let d be the bead in position $(0, b_0)$ of A^ℓ . (There is such a bead in A^ℓ because $(A^\ell)_y = B_y$ for all $y \neq x$.) We claim that $A^{\ell,0} := A^\ell(1, \leftarrow d)$ is a 2^ℓ -abacus for $f_0 f_\ell(\lambda)$. By Theorem 2.1, we observe that in order to show this it is enough to show that $A^{\ell,0}$ is a 2^ℓ -abacus for an odd partition of $n - 2^\ell - 1$. The previous statement holds by Theorem 3.7 because $w_{2^\ell}(A^{\ell,0}) = w_{2^\ell}(A^\ell)$ and $(A^{\ell,0})^\uparrow = B^0$.

Let A^0 be the 2^ℓ -abacus for $f_0(\lambda)$ described at the end of Notation 6.12. Since $x \notin \{b_0 - 1, b_0\}$, we observe that $w_{2^\ell}((A^0)_y) = 0$ for all $y \neq x$ and that $(A^0)_x = A_x$ (here we regard two runners equal if they coincide as 1-abaci). In particular $(A^0)_x$ is a 1-abacus for μ_x (as defined in Notation 6.12). This shows that a 2^ℓ -abacus $A^{0,\ell}$ for $f_\ell f_0(\lambda)$ can be obtained from A^0 by sliding a bead up one row on runner $(A^0)_x$ to obtain a 1-abacus for $f_0(\mu_x)$. Hence we deduce that $(A^{0,\ell})_x = (A^\ell)_x = (A^{\ell,0})_x$. Similarly we also have that $(A^{0,\ell})_y = (A^{\ell,0})_y$ for all $y \neq x$. \square

LEMMA 6.14. *Let $\lambda \vdash_o n$ be as in Notation 6.12, $u = 2^{t-\ell}$. If $x = b_0$ then $f_0 f_\ell(\lambda) = f_\ell f_0(\lambda)$ if and only if $\mu_x = (u)$. In this case, $\lambda \in \Omega^{1+}(\ell; n)$.*

Proof. Suppose that $\mu_x = (u)$. Let d be the bead in A lying in position (u, x) . It is easy to see that both the abaci $A^{\ell,0}$ and $A^{0,\ell}$ are obtained from A by moving bead d to position $(u - 1, x - 1)$. Hence they coincide and therefore we have that $f_0 f_\ell(\lambda) = f_\ell f_0(\lambda)$.

Suppose now that $\mu_x = (u - w, 1^w)$ for some $w \in \{1, \dots, u - 1\}$. Let us also assume that w is odd (the case w even is completely analogous). Then we have beads in positions (j, x) of A for all $j \in \{-w + 1, -w + 2, \dots, 0\}$ and a gap in position $(-w, x)$. Moreover, the highest gap on runner $x - 1$ of A is in position $(0, x - 1)$ since $A_{x-1} = B_{x-1}$. This shows that $A^0 = A(1, \leftarrow d)$ where d is the bead in position $(0, x)$ of A (this follows from Lemma 3.1(3) and Theorem 2.1). We observe that $(A^0)_x$ is a 1-abacus configuration for the partition $\overline{\mu_x} := (u - w + 1, 1^{w-1})$.

Let c' be the bead in position $(-w + 1, x)$ of A and let e be the bead in position $(u - w, x)$ of A .

By Lemma 3.8 we have that $f_0(\mu_x) = (u - w, 1^{w-1})$. Hence A^ℓ is obtained from A by sliding bead c' up to position $(-w, x)$. Since there is a bead in position $(-w, x - 1)$ of A we deduce that $A^{\ell,0}$ has a bead in position $(-w, x)$.

On the other hand also $f_0(\overline{\mu_x}) = (u - w, 1^{w-1})$, by Lemma 3.8. Therefore $A^{0,\ell}$ is obtained from A^0 by sliding bead e up to position $(u - w - 1, x)$. Hence $A^{0,\ell}$ has a gap in position $(-w, x)$. We conclude that $A^{0,\ell} \neq A^{\ell,0}$, as desired. \square

LEMMA 6.15. *Let $\lambda \vdash_o n$ be as in Notation 6.12. If $x = b_0 - 1$ then $f_0 f_\ell(\lambda) = f_\ell f_0(\lambda)$ if and only if $\mu_x = (1^{2^{t-\ell}})$. In this case, $\lambda \in \Omega^{1-}(\ell; n)$.*

Proof. In this case we observe that λ' satisfies the hypothesis of Lemma 6.14. Hence the statement follows. \square

THEOREM 6.16. *Let $\lambda \vdash_o n$; we assume Notation 6.12 and set $u = 2^{t-\ell}$. Then the following are equivalent.*

- (1) $f_0 f_\ell(\lambda) \neq f_\ell f_0(\lambda)$.
- (2) $x = b_0$ and $\mu_x \neq (u)$ or $x = b_0 - 1$ and $\mu_x \neq (1^u)$.
- (3) λ or λ' are such that $x = b_0$ and $\mu_x \neq (u)$.

Proof. This is a consequence of Lemmas 6.13, 6.14, 6.15. \square

We use the following example to illustrate in a concrete case the ideas involved in Notation 6.12, Lemmas 6.13, 6.14, 6.15 and Theorem 6.16.

EXAMPLE 6.17. Let $n = 11$, $t = m = 3$ and $\ell = 2$. Let $\gamma = (3)$ and let B be the normalized 4-abacus for (3) . We observe that $b_0 = 3$. We define the following subsets of $\mathcal{P}(11)$: $\mathcal{D}_0 := \{(4^2, 1^3), (3, 1^8)\}$, $\mathcal{D}_1 := \{(5, 4, 1^2), (3, 2, 1^6)\}$, $\mathcal{D}_2 := \{(6, 4, 1), (3^2, 1^5)\}$, $\mathcal{D}_3 := \{(11), (7, 4)\}$. We observe that

$$\mathcal{E}((3), 8) = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3.$$

For any $\lambda \in \mathcal{E}((3), 8)$ we let A be the 4-abacus for λ such that $A^\uparrow = B$. For $x \in \{0, 1, 2, 3\}$ let μ_x be the partition represented on runner x . Notice that $\lambda \in \mathcal{D}_x$ if and only if $w_4(A_x) = 2$.

From Lemma 6.13 we expect to have $\mathcal{D}_0 \cup \mathcal{D}_1 \subseteq \Omega(2; 11)$. Here we verify this directly. For instance we have that

$$f_2 f_0((5, 4, 1^2)) = f_2((5, 3, 1^2)) = (2^2, 1^2) = f_0((3, 2, 1^2)) = f_0 f_2((5, 4, 1^2)).$$

Let us now consider \mathcal{D}_2 . If $\lambda = (6, 4, 1)$ then $\mu_2 = (2)$ and Lemma 6.15 implies that $(6, 4, 1) \notin \Omega(2; 11)$. On the other hand, when $\lambda = (3^2, 1^5)$ we have that $\mu_2 = (1^2)$ and that $(3^2, 1^5) \in \Omega(2; 11)$. It follows that $\mathcal{D}_2 \cap \Omega(2; 11) = \{(3^2, 1^5)\}$. In fact,

$$f_0 f_2((3^2, 1^5)) = f_0((3^2, 1)) = (3, 3) = f_2((3^2, 1^4)) = f_2 f_0((3^2, 1^5)).$$

Similarly, one can verify directly that $\mathcal{D}_3 \cap \Omega(2; 11) = \{(11)\}$, as prescribed by Lemma 6.14.

The unique base case still missing is the following for which we have to adapt our notation, using again the 2^t -abacus as in Notation 6.4.

NOTATION 6.18. Let $n = 2^t + 2^\ell$, with $0 < \ell < t$, i.e., we now consider the case $m = 2^\ell$. Let $\lambda \vdash_o n$, and let $\gamma := C_{2^t}(\lambda) \vdash_o m$. Since $m = 2^\ell$, γ is a hook, say $\gamma = (m - w, 1^w)$, for some $w \in \{0, 1, \dots, m - 1\}$. Let B be a normalized 2^t -abacus for γ . Then B does not have beads in rows $i \geq 1$, by Lemma 2.6. Let $c = c_0(B)$ so that $B^0 := B(1, \leftarrow c)$ is a 2^t -abacus for $f_0(\gamma)$. Since γ is a hook, c is in position $(0, b_0)$ of B with $b_0 = 1$ or $b_0 = 2^\ell$. The bead $d = c_\ell(B)$ in B such that $B^\ell := B(2^\ell, \leftarrow d)$ is a 2^t -abacus for $f_\ell(\gamma) = \emptyset$ is clearly in position $(0, b_\ell)$ with $b_\ell = 2^\ell$.

Let A be the 2^t -abacus for λ such that $A^\uparrow = B$. For $j \in \{0, \dots, 2^t - 1\}$ let μ_j be the partition represented on runner A_j (seen as a 1-abacus). So there exists a unique $x \in \{0, \dots, 2^t - 1\}$ such that $\mu_x = (1)$ and $\mu_y = \emptyset$ for all $y \neq x$; we also allow $x = -1$ and identify $\mu_{-1} = \mu_{2^t-1}$ if needed. As before, we note that a 2^t -abacus for $f_0(\lambda)$ can be obtained from A by moving a bead c' one position to the left. We call $A^0 := A(1, \leftarrow c')$. Similarly, let $A^\ell := A(2^\ell, \leftarrow d')$ be the 2^t -abacus for $f_\ell(\lambda)$, obtained from A by moving a bead d' 2^ℓ positions to the left.

Since $f_k(C_{2^t}(\lambda)) = C_{2^t}(f_k(\lambda))$ for $k \leq \ell$ by Lemma 4.9, we know that c' and d' are on the same runners b_0 and b_ℓ , respectively, as the beads c and d of B .

LEMMA 6.19. Let $n = 2^t + 2^\ell$, with $0 < \ell < t$. Let $\gamma \in \Omega(\ell; 2^\ell) = \Omega^1(\ell; 2^\ell)$, $\gamma = (2^\ell - w, 1^w)$. Let $\lambda \in \mathcal{E}(\gamma, 2^t)$ and x as in Notation 6.18. Then $\lambda \in \Omega(\ell; n \mid \gamma)$ if and only if $x \in \{0, (-1)^{w-1}, 2^\ell, 2^\ell + (-1)^{w-1}\}$.

More precisely, when $\gamma \in \Omega^{1+}(\ell; 2^\ell)$, then $\lambda \in \Omega^{1+}(\ell; n \mid \gamma)$ for $x \in \{2^\ell - 1, 2^\ell\}$ and $\lambda \in \Omega^2(\ell; n \mid \gamma)$ for $x \in \{0, 2^t - 1\}$. When $\gamma \in \Omega^{1-}(\ell; 2^\ell)$, then $\lambda \in \Omega^{1-}(\ell; n \mid \gamma)$ for $x \in \{0, 2^\ell + 1\}$ and $\lambda \in \Omega^2(\ell; n \mid \gamma)$ for $x \in \{1, 2^\ell\}$. In particular,

$$|\Omega^2(\ell; n)| = 2^{\ell+1}, \quad |\Omega^{1\pm}(\ell; n)| = 2^\ell \quad \text{and} \quad T_{0,\ell}(n) = |\Omega(\ell; n)| = 2^{\ell+2}.$$

Proof. We consider the 2^t -abacus of λ using the notation introduced above. Clearly, as $b_0 \in \{1, 2^\ell\}$ and $b_\ell = 2^\ell$, the partition γ is always of type (1); it is of type $(1-)$, i.e., $b_0 = 1$, if and only if w is odd, by Lemma 6.5.

We discuss the case $b_0 = 1$, i.e., w odd, in detail; the case $b_0 = b_\ell$ is similar (and just corresponds to the conjugate partition).

First we want to show that $f_\ell f_0(\lambda) = f_0 f_\ell(\lambda)$ when $x \in \{0, 1, 2^\ell, 2^\ell + 1\}$.

For $x = 0$, $\lambda = (m - w, 1^{2^t+w})$, and we easily compute $f_\ell f_0(\lambda) = (m - w, 1^{2^t-m+w-1}) = f_0 f_\ell(\lambda)$; we also note that $\lambda \in \Omega^{1-}(\ell; n)$. For $x = 2^\ell + 1$, we use the 2^t -abacus to compute again $f_0 f_\ell(\lambda) = (m - w, 1^{2^t-m+w-1})$, which then also equals $f_\ell f_0(\lambda)$. Again, $\lambda \in \Omega^{1-}(\ell; n)$.

For $x = 1$, the bead c' is at position $(1, 1)$ in A , d' is at $(0, 2^\ell)$, and both target positions $(1, 0)$ and $(0, 0)$ are empty; then we obtain the 2^ℓ -abacus $A^{0,\ell} := [A(1, \leftarrow c')](2^\ell, \leftarrow d')$ for $f_\ell f_0(\lambda)$ and the 2^ℓ -abacus $A^{\ell,0} := [A(2^\ell, \leftarrow d')](1, c')$ for $f_0 f_\ell(\lambda)$, and these coincide. The case $x = 2^\ell$ is handled analogously. In both cases, we easily see that $\lambda \in \Omega^2(\ell; n)$.

Note that in the case where $\gamma \in \Omega^{1+}(\ell; m)$, i.e., $b_0 = b_\ell$, a similar argument gives two extensions $\lambda \in \Omega(\ell; n)$ of type $(1+)$ and two extensions of type (2) as claimed.

Continuing with the case $b_0 = 1$, we next want to show that $f_\ell f_0(\lambda) \neq f_0 f_\ell(\lambda)$ when $x \notin \{0, 1, 2^\ell, 2^\ell + 1\}$.

We have three types of extensions λ , corresponding to $x \in \{2, \dots, w\}$ if $w > 1$, $x \in \{w + 1, \dots, 2^\ell - 1\}$ if $w < 2^\ell - 1$, and $x \in \{2^\ell + 2, \dots, 2^\ell - 1\}$ if $(\ell; t) \neq (1, 2)$.

In all cases, the 2^ℓ -abacus A^ℓ has beads at positions $(0, 0)$ and $(0, 1)$, so also $A^{\ell,0}$ has this property. But A^0 has a gap at position $(0, 1)$, which can then not be filled in $A^{0,\ell}$ as A^0 has no bead in position $(0, 2^\ell + 1)$. So $A^{\ell,0} \neq A^{0,\ell}$, and the claim follows. \square

We collect all the counting formulae for $T_{0,\ell}(n)$ in the following result. Together with the reduction formula for the numbers $F_{k,\ell}(n)$ (so also for $T_{k,\ell}(n)$) in Theorem 5.10 we thus have a complete answer to our counting problem.

COROLLARY 6.20. *Let $n = 2^{a_1} + \dots + 2^{a_r}$ where $a_1 > a_2 > \dots > a_r$, and let $0 < \ell \leq a_1$.*

(1) *Assume $a_r > \ell$. Then*

$$T_{0,\ell}(n) = |\Omega(\ell; n)| = 2^{a_r - 1} \left(\prod_{j=1}^{r-1} (2^{a_j} - 3) + \prod_{j=1}^{r-1} 2^{a_j} \right).$$

(2) *Assume $\ell > a_r$, say $a_p \geq \ell > a_{p+1}$. Then*

$$T_{0,\ell}(n) = |\Omega(\ell; n)| = \left((2^\ell - 2) 2^{a_p - \ell} \prod_{j=1}^{p-1} 2^{a_j} + 2 \prod_{j=1}^{p-1} (2^{a_j} - 3) \right) \prod_{j=p+1}^r 2^{a_j}.$$

(3) *Assume $a_r = \ell$. Then*

$$T_{0,\ell}(n) = |\Omega(\ell; n)| = 2^{\ell+1} \left(\prod_{j=1}^{r-2} 2^{a_j} + \prod_{j=1}^{r-2} (2^{a_j} - 3) \right).$$

Proof. (1) Using Corollary 6.11 in an inductive argument, we obtain the claimed formula.

(2) The formula follows from Lemmas 6.13, 6.14, 6.15 and by Corollary 6.11.

(3) Together with our previous results, we deduce the claim using Lemma 6.19. \square

Acknowledgements. Part of this work was carried out while the second and third authors were visiting the Institute for Algebra, Number Theory and Discrete Mathematics at Leibniz Universität Hannover. They would like to thank everyone at the institute for the kind hospitality. Finally, we thank the referees for valuable comments and suggestions.

REFERENCES

- [1] Arvind Ayyer, Amritanshu Prasad, and Steven Spallone, *Odd partitions in Young's lattice*, Sémin. Lothar. Comb. **75** (2016), B75g (13 pages).
- [2] Christine Bessenrodt, *On hooks of Young diagrams*, Ann. Comb. **2** (1998), no. 2, 103–110.
- [3] Christine Bessenrodt, Eugenio Giannelli, and Jørn B. Olsson, *Restriction of odd degree characters of \mathfrak{S}_n* , SIGMA, Symmetry Integrability Geom. Methods Appl. **13** (2017), 070 (10 pages).
- [4] Walter Feit, *Characters of finite groups*, W. A. Benjamin, 1967.

- [5] Eugenio Giannelli, *Characters of odd degree of symmetric groups*, J. Lond. Math. Soc. **96** (2017), no. 1, 1–14.
- [6] Eugenio Giannelli, Alexander Kleshchev, Gabriel Navarro, and Pham Huu Tiep, *Restriction of odd degree characters and natural correspondences*, Int. Math. Res. Not. (2017), no. 20, 6089–6118.
- [7] Eugenio Giannelli, Stacey Law, and Stuart Martin, *On the p' -subgraph of the Young graph*, to appear in Algebr. Represent. Theory, 2018, <https://arxiv.org/abs/1707.00366>.
- [8] I. Martin Isaacs, Gabriel Navarro, Jørn B. Olsson, and Pham Huu Tiep, *Character restrictions and multiplicities in symmetric groups*, J. Algebra **478** (2017), 271–282.
- [9] Gordon James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics, vol. 682, Springer, 1978.
- [10] Gordon James and Adalbert Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and Its Applications, vol. 16, Addison-Wesley Publishing Co., 1981, With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson.
- [11] John McKay, *Irreducible representations of odd degree*, J. Algebra **20** (1972), 416–418.
- [12] Jørn B. Olsson, *Combinatorics and Representations of finite groups*, Vorlesungen aus dem Fachbereich Mathematik der Universität Essen, vol. 20, Univ. GH Essen, 1994.

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