ALGEBRAIC COMBINATORICS

David E. Roberson

Homomorphisms of strongly regular graphs

Volume 2, issue 4 (2019), p. 481-497.

<http://alco.centre-mersenne.org/item/ALCO_2019__2_4_481_0>

Algebraic Combinatorics
Volume 2, issue 4 (2019), p. 481–497
https://doi.org/10.5802/alco.50

Homomorphisms of strongly regular graphs

David E. Roberson

ABSTRACT We prove that if G and H are primitive strongly regular graphs with the same parameters and φ is a homomorphism from G to H, then φ is either an isomorphism or a coloring (homomorphism to a complete subgraph). Moreover, any such coloring is optimal for G and its image is a maximum clique of H. Therefore, the only endomorphisms of a primitive strongly regular graph are automorphisms or colorings. This confirms and strengthens a conjecture of Peter Cameron and Priscila Kazanidis that all strongly regular graphs are cores or have complete cores. The proof of the result is elementary, mainly relying on linear algebraic techniques. In the second half of the paper we discuss the idea underlying the proof, namely that it can be seen as a straightforward application of complementary slackness to a dual pair of semidefinite programs that define the Lovász theta function. We also consider implications of the result and show that essentially the same proof can be used to obtain a more general statement. We believe that one of the main contributions of the work is the novel proof technique, which is the first able to make use of the combinatorial regularity of a graph in order to obtain results about its endomorphisms/homomorphisms. Thus we expect this approach to have further applicability to the study of homomorphisms of highly regular graphs.

1. Introduction

A *homomorphism* from a graph G to H is an adjacency preserving map from the vertex set of G to the vertex set of H. When there are homomorphisms from G to H and from H to G, we say that the graphs are *homomorphically equivalent*. An important class of examples of homomorphisms are colorings: a c-coloring of G is equivalent to a homomorphism from G to the complete graph on c vertices, K_c. More generally, we will refer to any homomorphism whose image is a clique (complete subgraph) as a *coloring*.

Homomorphisms from a graph G to itself are called endomorphisms, and they are said to be *proper* if they are not an automorphism of G, or equivalently, their image is a proper subgraph of G. A graph with no proper endomorphisms is said to be a *core*, and these play a fundamental role in the theory of homomorphisms since every graph is homomorphically equivalent to a unique core. We refer to the unique core homomorphically equivalent to G as *the core of* G. It is known [11], and not difficult

Manuscript received 20th October 2017, revised 22nd November 2018, accepted 23rd November 2018.

KEYWORDS. Homomorphisms, strongly regular graphs, linear algebra, semidefinite programming, Lovász theta.

ACKNOWLEDGEMENTS. This work was done while the author was employed by University College London. The author was partially supported by Cambridge Quantum Computing Ltd. and the Engineering and Physical Sciences Research Council of the United Kingdom (EPSRC), as well as Simone Severini and Fernando Brandao.

to show, that the core of G is isomorphic to any vertex minimal induced subgraph of G to which G admits an endomorphism.

If the core of a graph G is a complete graph K_c, then G must contain a clique of size c and must also be c-colorable. Therefore, $\omega(G) = \chi(G) = c$. Conversely, if $\omega(G) = \chi(G) = c$, then the core of G is K_c. If a graph is either a core or has a complete graph as a core, then it is said to be *core-complete*. Many known results on cores are statements saying that all graphs in a certain class are core-complete [2, 10, 15], and often it remains difficult to determine whether a given graph in the class is a core or has a complete core.

For some classes of graphs, something stronger than core-completeness can be shown. A graph G is a *pseudocore* if every proper endomorphism of G is a coloring. It follows that such a graph either has no proper endomorphisms and is thus a core, or has some proper endomorphism to a clique and thus has a complete core. In other words, any pseudocore is core-complete, although the converse does not hold (consider a complete multipartite graph). Similarly, it is easy to see that any core is a pseudocore, but the converse does not hold in this case either (for instance the Cartesian product of two complete graphs of equal size at least three).

In this paper, we will focus on homomorphisms and cores of strongly regular graphs. An n-vertex k-regular graph is said to be *strongly regular* with parameters (n, k, λ, μ) if every pair of adjacent vertices have λ common neighbors, and every pair of distinct non-adjacent vertices have μ common neighbors. For short, we will call such a graph an $SRG(n, k, \lambda, \mu)$. A strongly regular graph is called *imprimitive* if either it or its complement is disconnected. In such a case, the graph or its complement is a disjoint union of equal sized complete graphs. Homomorphisms of these graphs are straightforward, and so we will only consider *primitive* strongly regular graphs here. Because of this, from now on when we consider a strongly regular graph, we will implicitly assume that it is primitive. In this case, we always have that $1 \leqslant \mu < k$, and that the diameter is two.

Cameron & Kazanidis [2] showed that a special class of strongly regular graphs, known as rank 3 graphs, are all core-complete. A graph is *rank 3* if its automorphism group acts transitively on vertices, ordered pairs of adjacent vertices, and ordered pairs of distinct non-adjacent vertices. The rank refers to the number of orbits on ordered pairs of vertices, and so after complete or empty graphs, rank 3 graphs are in a sense the graphs with the most symmetry. The proof of Cameron & Kazanidis exploits this symmetry by noting that either no pair of non-adjacent vertices can be identified (mapped to the same vertex) by an endomorphism of a rank 3 graph, or every such pair can. In the former case, the graph must be a core. In the latter, any endomorphic image that contains non-adjacent vertices cannot be minimal, and therefore the core must be complete.

Strongly regular graphs can be viewed as combinatorial relaxations of rank 3 graphs and, following their result, Cameron & Kazanidis (tentatively) conjectured that all strongly regular graphs are core-complete. Towards this, Godsil & Royle [10] showed that many strongly regular graphs constructed from partial geometries are core-complete. A partial geometry is simply a point-line incidence structure obeying certain rules. The point graph of a partial geometry has the points as vertices, such that two are adjacent if they are incident to a common line. The properties of partial geometries guarantee that their point graphs are strongly regular, and they are typically referred to as *geometric* graphs.

Godsil & Royle showed that the point graphs of generalized quadrangles are pseudocores, as are the block graphs of 2-$(v, k, 1)$ designs and orthogonal arrays with sufficiently many points. As they note, a result of Neumaier [19] is that for a fixed least eigenvalue, all but finitely many strongly regular graphs are the block graphs of 2-$(v, k, 1)$ designs or orthogonal arrays. Thus their result makes a significant step towards the conjecture of Cameron & Kazanidis. The main idea used in the proof of the Godsil & Royle result is that any endomorphism must map maximum cliques to maximum cliques. Starting with this simple observation, they show that if G is geometric, and the maximum cliques of G are exactly the lines of the underlying partial geometry, then G is a pseudocore. It then remains to show when this assumption on the maximum cliques holds true.

The main result of this paper is that if G and H are both strongly regular graphs with parameters (n, k, λ, μ), and φ is a homomorphism from G to H, then φ is either an isomorphism or a coloring. Letting $G = H$, this statement implies that all strongly regular graphs are pseudocores, thus proving and strengthening the conjecture of Cameron & Kazanidis. Using our main result and some previously known results, we also show that in the case where φ is a coloring, we must have $\chi(G) = \omega(H)$ and this value is equal to the Hoffman bound on chromatic number which depends only on (n, k, λ, μ). It follows from this that any strongly regular graph G falls into one of four classes depending on what subset of $\{\omega(G), \chi(G)\}$ meets the Hoffman bound. Using this we show that the homomorphism order of strongly regular graphs with a fixed parameter set has a simple description.

We also prove a generalization of our main result, where the strong regularity assumption on H is replaced by a strictly weaker algebraic condition. In this more general case, we are only able to conclude that any homomorphism from G to H is either a coloring or an isomorphism to an induced subgraph of H.

The original idea and the inspiration for the proof of the main result comes from the theory of vector colorings, which are a homomorphism-based formulation of the famous Lovász theta function. The author was aided greatly by a collaboration with Chris Godsil, Brendan Rooney, Robert Šamal, and Antonios Varvitsiotis which produced three papers [8, 7, 9] on vector colorings. In particular, the second paper [7] focused specifically on using vector colorings to restrict the possible homomorphisms between graphs. Note however that we will present an elementary proof of our main result which only requires basic knowledge of linear algebra and certain aspects of strongly regular graphs which we will review in Section 2. The connection between the proof techniques and vector colorings will not be discussed until Section 5.

Although the main concrete contribution of this paper is the resolution and strengthening of the Cameron & Kazanidis conjecture, we believe that the real significance of this work is the step it takes towards understanding how combinatorial regularity can impact the endomorphisms and core of a graph. Symmetry conditions, such as vertex- or distance-transitivity, often have easy-to-derive consequences for the endomorphisms and/or core of a graph. This is perhaps not surprising, since such symmetry conditions are assumptions about the automorphisms of a graph, which are just special cases of endomorphisms. However, it appears to be more challenging to make use of analogous regularity conditions, such as being strongly or distance regular. In fact, we believe that ours is the first example of such a result. Interestingly, by showing that strongly regular graphs are pseudocores, we establish a stronger result than was previously known even under the more stringent symmetry condition of being rank 3. Moreover, we know of no way to directly use the assumption of being rank 3 to show that a graph is a pseudocore. We believe that our proof technique has the potential to be applied to other classes of highly regular graphs to obtain similar results. In particular, it could be used to generalize and strengthen many homomorphism results based on symmetry to results based on combinatorial regularity, analogously to how it is used in this work. Here "highly regular graph" is intentionally imprecise, since it is not yet clear what regularity conditions will have an impact on properties of homomorphisms. However, one possibility would be to consider graphs in association schemes, since these enjoy many algebraic properties similar to those that we exploit here for strongly regular graphs.

1.1. Notation. We will denote the existence of a homomorphism from G to H by writing $G \to H$. Given a homomorphism φ from G to H, we will abuse terminology somewhat and refer to the subgraph of H induced by $\{\varphi(u) : u \in V(G)\}$ as the *image* of φ, and we will denote this by $\operatorname{Im}\varphi$.

Whenever we use θ and τ, we will be referring to the second largest and minimum eigenvalues of a strongly regular graph. This will sometimes be done without explicitly stating so. We will also use m_θ and m_τ to denote the multiplicities of these eigenvalues, and E_θ and E_τ will refer to the projections onto the corresponding eigenspaces.

The all ones matrix will be denoted by J. For a matrix M, we will use $\operatorname{sum}(M)$ to refer to the sum of the entries of M. For two matrices M and N with the same dimensions, $M \circ N$ will denote their Schur, or entrywise, product.

The complement of a graph G will be denoted by $\overline{G}$, and more generally we will add a bar over usual notation to refer to the analog in the complement. For instance, $\bar{\theta}$ will refer to the second largest eigenvalue of the complement of a given strongly regular graph.

We will use $u \sim v$ to mean that u and v are adjacent vertices. We will also use $u \not\sim v$ when u and v are not adjacent, which includes the case where $u = v$ since a vertex is not adjacent to itself. Sometimes we will need to exclude the $u = v$ case, and for this we will use $u \not\simeq v$. We will also refer to u and v as non-neighbors whenever $u \not\simeq v$. Lastly, note that $u \not\simeq v$ is equivalent to u and v being adjacent in the complement graph.

2. Properties of Strongly Regular Graphs

Here we will introduce some basic properties of strongly regular graphs that we will need later. We do not aim to give a full proof of every result, but rather enough explanation for the interested reader to work out the details. Most of these results are standard, and can be found in [11] or even on some widely used online sources that are not considered citable. Those familiar with strongly regular graphs can probably skip this section, with the possible exception of Lemma 2.1 and the definition of the cosines of a strongly regular graph at the end of Section 2.2.

2.1. Eigenvalues. Here we review some basic algebraic properties of strongly regular graphs. For more details we refer the reader to Chapter 10 of [11]. Let G be an $SRG(n, k, \lambda, \mu)$ with adjacency matrix A. Since G is a connected k-regular graph, k is a simple eigenvalue of A with the all-ones vector as its unique (up to scalar) eigenvector. Considering the entries of A^2, it is easy to show that

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J, \tag{1}$$

where J is the all ones matrix. It follows from this that all eigenvalues of A other than k must satisfy the equation $x^2 + (\mu - \lambda)x + (\mu - k) = 0$. Therefore, the other two distinct eigenvalues of an $SRG(n, k, \lambda, \mu)$, denoted θ and τ, depend only on the parameters and are given as follows:

$$\theta = \frac{1}{2}\left[(\lambda - \mu) + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right]$$
$$\tau = \frac{1}{2}\left[(\lambda - \mu) - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right].$$

Note that these eigenvalues satisfy $k > \theta > 0 > \tau$. The multiplicities, m_θ and m_τ, of θ and τ can also be expressed in terms of the parameters n, k, λ, μ, but we will not need to make their values explicit. A key point to take away from this is that the eigenvalues, including their multiplicities, of a strongly regular graph depend only on the parameters, not on the specific graph.

2.2. Projections onto eigenspaces. It follows from Equation (1) that any polynomial in A is contained in $\operatorname{span}\{I, A, J\}$ or, equivalently, $\operatorname{span}\{I, A, \bar{A}\}$ where $\bar{A} = J - I - A$ is the adjacency matrix of the complement. A standard result of linear algebra is that the projections onto the eigenspaces of a real symmetric matrix are polynomials in that matrix. Therefore, denoting by E_θ and E_τ the projections onto the θ- and τ-eigenspaces of A respectively, we have that both of these projections are contained in the span of $\{I, A, \bar{A}\}$. This means that E_θ and E_τ have three distinct entries: those corresponding to vertices, edges, and non-edges of G. Therefore, using the identity $\operatorname{Tr}(M^T N) = \operatorname{sum}(M \circ N)$, we can compute the entries of E_τ on, say, the edges of G as

$$\frac{1}{nk}\operatorname{sum}(A \circ E_\tau) = \frac{1}{nk}\operatorname{Tr}(AE_\tau) = \frac{1}{nk}\operatorname{Tr}(\tau E_\tau) = \frac{\tau m_\tau}{nk},$$

since the trace of a projection is equal to its rank. Similarly, we can show

$$(E_\tau)_{uv} = \begin{cases} m_\tau/n & \text{if } u = v \\ \tau m_\tau/nk & \text{if } u \sim v \\ (-\tau - 1)m_\tau/n(n-k-1) & \text{if } u \not\simeq v. \end{cases}$$

One can also determine the entries of E_θ in a similar manner, but we will not need this.

The proof of our main result makes use of the projection E_τ, but we will actually want to scale this matrix so that its diagonal entries are equal to one. Thus we define the *cosine matrix* of a strongly regular graph G, denoted E_G, to be the matrix given as follows:

$$(E_G)_{uv} = \frac{n}{m_\tau}(E_\tau)_{uv} = \begin{cases} 1 & \text{if } u = v \\ \tau/k & \text{if } u \sim v \\ (-\tau - 1)/(n-k-1) & \text{if } u \not\simeq v. \end{cases}$$

The key properties of E_G that we will make use of are that it is positive semidefinite and that $(A - \tau I)E_G = 0$, both of which follow from the fact that it is a positive multiple of E_τ.

Since the matrix E_G is positive semidefinite with ones on the diagonal, it is the Gram matrix of some unit vectors that we can consider as being assigned to the vertices of the graph. The off diagonal entries of E_G are then the cosines of the angles between these vectors, thus motivating the term "cosine matrix". We refer to the values τ/k and $(-\tau - 1)/(n - k - 1)$ as the *adjacency and non-adjacency cosines* of a strongly regular graph, respectively. Note that for a primitive strongly regular graph G, its adjacency cosine is always contained in the interval $(-1, 0)$, and its non-adjacency cosine is contained in the interval $(0, 1)$. The latter follows from the fact, presented in the next section, that $n - k - 1$ and $-\tau - 1$ are the largest and second largest eigenvalues of the complement of G respectively.

Note that the parameters of a strongly regular graph determine its adjacency and non-adjacency cosines, but the converse is not true. Indeed, strongly regular graphs with parameter sets $(16, 10, 6, 6)$, $(26, 15, 8, 9)$, or $(36, 20, 10, 12)$ all have adjacency and non-adjacency cosines equal to $-1/5$ and $1/5$ respectively.

2.3. Complements and some combinatorial properties. It is easy to check that if G is a strongly regular graph with parameters (n, k, λ, μ), then the complement of G, denoted $\overline{G}$, is also a strongly regular graph with parameters $(n, \bar{k}, \bar{\lambda}, \bar{\mu})$ where

$$\begin{aligned}\bar{k} &= n - k - 1\\ \bar{\lambda} &= n - 2k - 2 + \mu\\ \bar{\mu} &= n - 2k + \lambda.\end{aligned}$$

The eigenvalues of $\overline{G}$ are denoted by $\bar{k} > \bar{\theta} > \bar{\tau}$. The latter two can be computed from the parameters of $\overline{G}$ using the identities in Section 2.1, but it is easier to use the fact that the adjacency matrix of $\overline{G}$ is equal to $J - I - A$, where A is the adjacency matrix of G. From this it follows that

$$\begin{aligned}\bar{\theta} &= -\tau - 1\\ \bar{\tau} &= -\theta - 1.\end{aligned}$$

The last property of strongly regular graphs that we will need concerns the second neighborhoods of vertices. The *second neighborhood* of a vertex v, denoted $N_2(v)$, is the set of vertices at distance exactly two from v. The following result was proven in [5]:

Lemma 2.1. *Let G be a primitive strongly regular graph. For any $v \in V(G)$, the subgraph of G induced by $N_2(v)$ is connected.*

3. Homomorphisms Between SRGs

In this section we prove our main result that any homomorphism between strongly regular graphs with the same parameters is either an isomorphism or a coloring. However, we will first need to introduce the following construction:

Definition 3.1. *Suppose that M is a symmetric matrix with rows and columns indexed by some finite set T. For any set S and function $\varphi : S \to T$, let M^{φ} denote the matrix indexed by S and defined entrywise as $(M^{\varphi})_{uv} = M_{\varphi(u)\varphi(v)}$.*

It turns out that this construction preserves positive semidefiniteness:

Lemma 3.2. *Suppose M is a positive semidefinite matrix indexed by some set T and let $\varphi : S \to T$ for some set S. Then M^{φ} is positive semidefinite.*

Proof. Since M is positive semidefinite, it is the Gram matrix of some multiset of vectors $\{p_w : w \in T\}$. In other words, $M_{ww'} = p_w^T p_{w'}$. But then we have that $M^{\varphi}_{uv} = M_{\varphi(u)\varphi(v)} = p_{\varphi(u)}^T p_{\varphi(v)}$. Thus M^{φ} is the Gram matrix of the multiset of vectors $\{p_{\varphi(u)} : u \in S\}$, and is therefore positive semidefinite. □

Using the above, we can prove the following which will be instrumental in proving our main result.

Lemma 3.3. *Suppose G and H are strongly regular graphs with the same adjacency cosines. Let A be the adjacency matrix of G and τ its least eigenvalue. If φ is a homomorphism from G to H, then $(A - \tau I)E_H^{\varphi} = 0$.*

Proof. First, recall that $(A - \tau I)E_G = 0$. Since φ is a homomorphism and G and H have the same adjacency cosines, we have that E_G and E_H^{φ} agree on their diagonals and entries corresponding to the edges of G. Therefore,

$$\begin{aligned}\operatorname{Tr}\left((A - \tau I)E_H^{\varphi}\right) &= \operatorname{sum}\left((A - \tau I) \circ E_H^{\varphi}\right)\\ &= \operatorname{sum}\left((A - \tau I) \circ E_G\right)\\ &= \operatorname{Tr}\left((A - \tau I)E_G\right) = 0.\end{aligned}$$

Since both $A-\tau I$ and E_H^φ are positive semidefinite (using Lemma 3.2 for the latter), the above implies that $(A-\tau I)E_H^\varphi = 0$. $\square$

Suppose that G and H are strongly regular graphs with equal adjacency cosines α and non-adjacency cosines β and β' respectively. If φ is a homomorphism from G to H, define the *homomorphism matrix* of φ to be $X := E_H^\varphi - E_G$. Then

$$X_{uv} = \begin{cases} 1-\beta & \text{if } u \not\simeq v \ \& \ \varphi(u) = \varphi(v) \\ \alpha - \beta & \text{if } u \not\simeq v \ \& \ \varphi(u) \sim \varphi(v) \\ \beta' - \beta & \text{if } u \not\simeq v \ \& \ \varphi(u) \not\simeq \varphi(v) \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\alpha \in (-1,0)$ and $\beta, \beta' \in (0,1)$. Therefore we have that $1-\beta > 0$ and $\alpha-\beta < 0$. The noteworthy property of the homomorphism matrix is that $(A-\tau I)X = 0$ where A is the adjacency matrix of G and τ its least eigenvalue. This follows immediately from the fact that $(A-\tau I)E_G = 0$ and $(A-\tau I)E_H^\varphi = 0$ by Lemma 3.3. The other important property of the homomorphism matrix is that it contains many zeros. This allows us to prove our main result:

Theorem 3.4. *Let G and H be primitive strongly regular graphs with the same adjacency cosines, and non-adjacency cosines equal to β and β' respectively. Suppose φ is a homomorphism from G to H. Then the following hold:*

(1) *If $\beta > \beta'$, then φ is a coloring.*
(2) *If $\beta = \beta'$, then φ is either a coloring or an isomorphism to an induced subgraph of H.*

Proof. Let X be the homomorphism matrix of φ. Suppose that φ is not a coloring. Then there exist vertices $u, v \in V(G)$ such that $\varphi(u) \not\simeq \varphi(v)$. Note that this implies that $u \not\simeq v$. For notational purposes, define the following sets:

$$\begin{aligned} C_1 &= \{w \in V(G) : w \sim u, w \not\simeq v, \varphi(w) = \varphi(v)\} \\ C_2 &= \{w \in V(G) : w \sim u, w \not\simeq v, \varphi(w) \sim \varphi(v)\} \\ C_3 &= \{w \in V(G) : w \sim u, w \not\simeq v, \varphi(w) \not\simeq \varphi(v)\}. \end{aligned}$$

Note that the sets C_1, C_2, C_3 partition the set of all neighbors of u contained in $N_2(v)$. Since φ is a homomorphism, $w \in C_1$ implies that $\varphi(u) \sim \varphi(w) = \varphi(v)$ which contradicts our assumption that $\varphi(u) \not\simeq \varphi(v)$. Thus C_1 must be empty. Now let A be the adjacency matrix of G, τ its least eigenvalue, and α the common adjacency cosine of G and H. Then $(A-\tau I)X = 0$ and therefore

$$\begin{aligned} 0 = ((A-\tau I)X)_{uv} &= \sum_{w \in V(G)} (A-\tau I)_{uw} X_{wv} \\ &= -\tau X_{uv} + \sum_{w \sim u} X_{wv} \\ &= -\tau(\beta'-\beta) + (1-\beta)|C_1| + (\alpha-\beta)|C_2| + (\beta'-\beta)|C_3| \\ &= -\tau(\beta'-\beta) + (\alpha-\beta)|C_2| + (\beta'-\beta)|C_3|. \end{aligned}$$

Now $\alpha - \beta < 0$, and $-\tau > 0$. Therefore, if $\beta > \beta'$ then every summand above is non-positive, and the first term is strictly negative. This is a contradiction and so in this case no homomorphism that is not a coloring can exist. This proves the first claim.

If $\beta = \beta'$, then the above implies that C_2 is empty, and we already noted that C_1 is empty. Let us consider what this means. Since $C_1 \cup C_2 \cup C_3$ is the set of all neighbors of u in $N_2(v)$, this implies that all such vertices w satisfy $\varphi(w) \not\simeq \varphi(v)$.

In other words, if $\varphi(u) \not\simeq \varphi(v)$, then φ preserves non-adjacency between v and every neighbor of u in $N_2(v)$.

Now we can apply the above argument again, replacing u with any neighbor of u in $N_2(v)$. Since $N_2(v)$ is connected by Lemma 2.1, iterating this argument implies that φ preserves non-adjacency between v and every vertex of $N_2(v)$. But now, for any $w \in N_2(v)$, we have that $\varphi(v) \not\simeq \varphi(w)$ and thus it must follow that φ preserves non-adjacency between w and every vertex of $N_2(w)$. Iterating again, and using the fact that $\overline{G}$ is connected, we see that φ must preserve all non-adjacencies, i.e. it is an isomorphism to an induced subgraph of H. □

As a corollary, we immediately obtain the following:

Corollary 3.5. *If G and H are primitive strongly regular graphs with the same parameters, then any homomorphism from G to H is either a coloring or an isomorphism.*

Proof. In this case we have that $\beta = \beta'$ in Theorem 3.4, and therefore any such homomorphism is a coloring or an isomorphism to an induced subgraph of H. However, since they have the same parameters, G and H have the same number of vertices. Therefore, any isomorphism to an induced subgraph of H is simply an isomorphism to H. □

Finally, we obtain a strengthening of the Cameron and Kazanidis conjecture:

Corollary 3.6. *Every primitive strongly regular graph is a pseudocore.*

In Section 6 we will see a generalization of Theorem 3.4 in which the graph H does not necessarily need to be strongly regular. Also, in Section 6.1, we will see that the conclusion of Theorem 3.4 does not hold in the case where $\beta < \beta'$, nor when the adjacency cosine of G is strictly less than that of H (regardless of how β and β' compare).

4. Cliques, Colorings, and the Homomorphism Order

Since we now know that all homomorphisms between strongly regular graphs with the same parameters are either isomorphisms or colorings, it is worth considering the properties of the colorings. In order to distinguish them, we will refer to homomorphisms that are not also isomorphisms as *proper homomorphisms*. We will see that, for a fixed parameter set, the proper homomorphisms between strongly regular graphs are not only required to be colorings, but colorings with a fixed number of colors. For this we will need to introduce some known bounds on the clique and chromatic numbers of a strongly regular graph. In Section 5 we will see that these bounds actually coincide with the Lovász theta function of the complement. But for now we will not need this connection, and so we will present these bounds purely as spectral bounds which is historically how they were derived.

By results of Delsarte [3], later generalized by Hoffman [14], it is known that any strongly regular graph G satisfies

$$\omega(G) \leqslant 1 - \frac{k}{\tau} \leqslant \chi(G), \tag{2}$$

where $\omega(G)$ and $\chi(G)$ are the clique and chromatic numbers of G respectively. Cliques meeting the bound are often referred to as *Delsarte cliques*, and colorings meeting the bound are referred to as *Hoffman colorings*. We also say that a coclique of G meeting the above bound for $\overline{G}$ is a *Delsarte coclique*. We remark that Hoffman colorings appear to be quite special. Indeed, it is known that for a fixed $c \in \mathbb{N}$, only finitely many strongly regular graphs have Hoffman colorings with c colors [12].

Importantly for us, the above simultaneous bound on the clique and chromatic numbers of a strongly regular graph depends only on the parameters, not the specific graph. We are therefore able to prove the following:

Lemma 4.1. *Let G and H both be $SRG(n,k,\lambda,\mu)$'s. There exists a proper homomorphism from G to H if and only if*

$$\chi(G) = 1 - \frac{k}{\tau} = \omega(H),$$

i.e. G has a Hoffman coloring and H contains a Delsarte clique.

Proof. Suppose there exists a proper homomorphism from G to H. By Corollary 3.5, this homomorphism must be a coloring. Therefore, using Equation (2), we have that

$$1 - \frac{k}{\tau} \leqslant \chi(G) \leqslant \omega(H) \leqslant 1 - \frac{k}{\tau}.$$

The converse is trivial. □

Note that the above lemma implies that if G and H are non-isomorphic $SRG(n,k,\lambda,\mu)$'s, then $G \to H$ if and only if $\chi(G) = 1 - k/\tau = \omega(H)$. We also obtain the following corollary giving an if and only if condition for when a strongly regular graph is a core:

Corollary 4.2. *If G is a strongly regular graph, then G is NOT a core if and only if*

$$\omega(G) = 1 - \frac{k}{\tau} = \chi(G).$$

In this case the core of G is a complete graph of size $1 - \frac{k}{\tau}$.

4.1. Types and the homomorphism order. The result of Lemma 4.1 suggests a useful partition of strongly regular graphs of a fixed parameter set. Namely, to classify them according to which subset of $\{\omega(G), \chi(G)\}$ meet the Hoffman bound. We therefore propose the following four "types" of strongly regular graphs:

- Type A: $\omega(G) < 1 - \frac{k}{\tau} = \chi(G)$;
- Type B: $\omega(G) = 1 - \frac{k}{\tau} = \chi(G)$;
- Type C: $\omega(G) = 1 - \frac{k}{\tau} < \chi(G)$;
- Type X: $\omega(G) < 1 - \frac{k}{\tau} < \chi(G)$.

The existence of a homomorphism between any two non-isomorphic $SRG(n,k,\lambda,\mu)$'s is determined by their types: Any graph of type A or B has homomorphisms to any graph of type B or C, and there are no other homomorphisms between non-isomorphic $SRG(n,k,\lambda,\mu)$'s. Furthermore, all graphs of type A, C, or X are cores, and all graphs of type B have complete graphs of size $1 - k/\tau$ as their cores. Summarizing these observations, the Hasse diagram of the homomorphism order of $SRG(n,k,\lambda,\mu)$'s is given in Figure 1 below.

Note that the type B graphs are represented by a single node in the above diagram since they are all homomorphically equivalent, whereas graphs of any other fixed type are incomparable (have no homomorphisms in either direction between them). The diagram suggests that any homomorphism from a type A graph G to a type C graph H can be "factored" into a homomorphism from G to a type B graph K and then from K to H. Since type B graphs are homomorphically equivalent to complete graphs, this is essentially what our main theorem says.

If the Hoffman bound is not an integer, then neither the clique nor chromatic number can meet this bound with equality, and therefore only graphs of type X can occur. This happens for conference graphs of non-square order, since these have τ equal to an irrational number. However, this can also occur for other parameter sets.

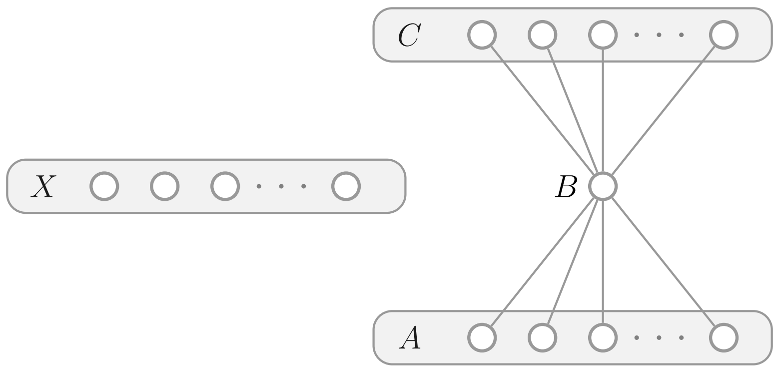

Figure 1. Homomorphism order of $SRG(n, k, \lambda, \mu)$'s.

Some examples include $(10, 3, 0, 1)$, $(16, 5, 0, 2)$, $(21, 10, 3, 6)$, $(26, 10, 3, 4)$, $(36, 14, 4, 6)$, and $(36, 21, 10, 15)$, for all of which there do exist strongly regular graphs. Also note that if the Hoffman bound of the complementary parameter set is not an integer, then there can be no Delsarte cocliques, and therefore no Hoffman colorings. Therefore, for such parameter sets, there will only be type C and/or X graphs.

Computations reveal that there are parameter sets which contain only graphs of a single type. Examples of this for each type, including an example for type X where the Hoffman bound is an integer, are given below:

- Type A - $(27, 16, 10, 8)$;
- Type B - $(49, 12, 5, 2)$;
- Type C - $(45, 32, 22, 24)$;
- Type X - $(16, 10, 6, 6)$.

On the other hand, there are also parameter sets having all four types. Some examples include $(36, 20, 10, 12)$, $(45, 12, 3, 3)$, and $(64, 18, 2, 6)$. In general, for the strongly regular graphs we performed computations on (obtained from Ted Spence's webpage [21]), almost all of them were either type C or X. This seems to indicate that having a Hoffman coloring is a rare property for a strongly regular graph, but having a Delsarte clique is not. The latter observation is perhaps not so surprising since it is known that all strongly regular graphs arising as point graphs of partial geometries have Delsarte cliques.

The computations for the above were done in Sage [23]. One only needs to determine if the given strongly regular graph has a clique of a certain size and/or coloring with certain number of colors. For the former, the built in clique number routine is very fast, and so there is no problem finding the clique number of all the strongly regular graphs from [21]. This is not the case for chromatic number. Sage's built in coloring routines seem to be far too slow to be of any use for this endeavor. However, there is a GAP package called Digraphs [16] developed by researchers at The University of St Andrews, and the coloring routine in this package works very quickly in comparison. In fact, it is hard to overstate how much faster it seems to be.

5. Vector Colorings and the Lovász ϑ Function

In this section we will see that some of the results of Section 3 are part of a more general theory involving semidefinite programs and the Lovász theta number of a graph [18].

For a graph G and a real number $t \geqslant 2$, a *strict vector t-coloring* of G is an assignment, $u \mapsto p_u$, of unit vectors to the vertices of G such that

$$p_u^T p_v = \frac{-1}{t-1} \text{ for all } u \sim v. \tag{3}$$

If we drop the "strict", then we only require that the inner product in (3) is bounded above by the righthand side. We note however that for strongly regular graphs, every optimal vector coloring is also a strict vector coloring [8]. For a non-empty graph G, its *strict vector chromatic number* is the minimum $t \geqslant 2$ such that G admits a strict vector t-coloring. For empty graphs, this parameter is defined to be equal to 1. The strict vector chromatic number was defined by Karger, Motwani, and Sudan [17], and they showed that it is equal to the Lovász theta number of the complement graph. The Lovàsz theta number is typically denoted by ϑ, and so we will use $\bar{\vartheta}(G) := \vartheta(\overline{G})$ to denote the strict vector chromatic number of G. We will give two of the more well known formulations of the Lovász theta number in Section 5.1.

By considering the Gram matrix of vectors in a strict vector coloring, it is easy to see that G has a strict vector t-coloring if and only if there exists a positive semidefinite matrix M indexed by the vertices of G such that

$$M_{uv} = \begin{cases} 1 & \text{if } u = v \\ \frac{-1}{t-1} & \text{if } u \sim v. \end{cases}$$

Using this interpretation, it is not difficult to see that a complete graph on n vertices has strict vector chromatic number equal to n. It is also now apparent that the matrices E_G and E_H^φ from Section 3 were Gram matrices of strict vector colorings.

Suppose that G and H are graphs and that $w \mapsto p_w$ for $w \in V(H)$ is a strict vector t-coloring of H. If φ is a homomorphism from G to H, then it is easy to see that $u \mapsto p_{\varphi(u)}$ for $u \in V(G)$ is a strict vector t-coloring of G (note that this is the exact construction used in the proof of Lemma 3.2 to show that M^φ is positive semidefinite). It follows that if $G \to H$, then $\bar{\vartheta}(G) \leqslant \bar{\vartheta}(H)$, i.e. the strict vector chromatic number is *homomorphism monotone*. In particular, using the fact that $\bar{\vartheta}(K_n) = n$, this implies the well known "sandwich theorem":

$$\omega(G) \leqslant \bar{\vartheta}(G) \leqslant \chi(G).$$

5.1. Semidefinite programming. One of the many useful properties of the Lovász theta number is that it can be written as a semidefinite program that satisfies strong duality. This provides us with both a minimization and maximization program for this parameter:

$$\begin{array}{rlll} & \text{PRIMAL} & & \text{DUAL} \\ \bar{\vartheta}(G) = & \min t & = & \max \operatorname{sum}(B) \\ & \text{s.t. } M_{uu} = t-1 \text{ for } u \in V(G) & & \text{s.t. } B_{uv} = 0 \text{ for } u \not\simeq v \\ & \quad M_{uv} = -1 \text{ for } u \sim v & & \quad \operatorname{Tr}(B) = 1 \\ & \quad M \succeq 0 & & \quad B \succeq 0 \end{array}$$

Note that a feasible solution of value t for the primal program above is exactly $(t-1)$ times the Gram matrix of a strict vector t-coloring of G, and so we see that these are equivalent definitions of $\bar{\vartheta}$.

Suppose that M and B are feasible solutions to the above primal and dual formulations of $\bar{\vartheta}$ with objective values P and D respectively. Then,

$$\operatorname{Tr}(MB) = \operatorname{sum}(M \circ B) = (P-1)\operatorname{Tr}(B) - [\operatorname{sum}(B) - \operatorname{Tr}(B)] = P - D.$$

It thus follows that if M and B are feasible solutions for the primal and dual programs respectively, then they are both optimal if and only if $\operatorname{Tr}(MB) = 0$ if and only if $MB = 0$. This is in fact just the complementary slackness condition for these semidefinite programs.

For any graph G with adjacency matrix A and least eigenvalue τ, the matrix $A-\tau I$ meets the first and third conditions for the dual program above. If we let B be the positive scaling of $A-\tau I$ that has trace one, then B is a feasible solution to the dual. If G is strongly regular, then we have seen in Section 2.2 that the cosine matrix of G, E_G, is constant on the diagonal, and is a negative constant on entries corresponding to edges of G. Therefore, up to a scalar multiple, this is a feasible solution to the primal program for $\bar{\vartheta}(G)$. If we let M denote this scalar multiple of E_G, then it is obvious that $MB = 0$. Therefore these are both optimal solutions to their respective programs. It is then only a matter of arithmetic to show that $\bar{\vartheta}(G)$ is equal to our old friend the Hoffman bound for any strongly regular graph G. In particular, this means that for strongly regular G with adjacency cosine α, we have $\bar{\vartheta}(G) = 1 - \frac{1}{\alpha}$. Note that this is monotonically increasing with α.

We can now see Lemma 3.3 for what it is:(1) The strongly regular graph G has feasible solutions E_G and $A-\tau I$ to the primal and dual respectively, and these must be optimal since their product is 0. Similarly, the cosine matrix E_H is an optimal primal solution for H. Furthermore, E_H^{φ} is the Gram matrix of the strict vector coloring of G obtained by composing φ with the strict vector coloring of H whose Gram matrix is E_H. Since both graphs are strongly regular with the same parameters, they have the same strict vector chromatic number and therefore E_H^{φ} is an optimal primal solution for G. Finally, since $A-\tau I$ was already shown to be an optimal dual solution for G, we have that $(A-\tau I)E_H^{\varphi} = 0$.

Of course, a similar technique can be applied to any homomorphism between graphs with the same strict vector chromatic number. But the primal and dual solutions for the two graphs will likely not be as nice as in the strongly regular case. The key feature of the primal solutions we used is that their entries depend only on whether the corresponding vertices are equal, adjacent, or non-adjacent. Most graphs will not have an optimal primal solution of this form.

On the other hand, distance regular graphs also have E_τ and $A-\tau I$ as optimal primal and dual solutions, and the uv-entry of the matrix E_τ only depends on the distance between vertices u and v. Thus, distance regular graphs are a natural choice for attempting to generalize our main theorem. Indeed, strongly regular graphs are exactly distance regular graphs of diameter two. However, the analysis seems more difficult in this case, since the matrix $E_H^{\varphi} - E_G$ will potentially have a different nonzero entry for every way in which the homomorphism φ can change the distance between two vertices. This is actually the same for our case, but for us there are only two such possibilities for how φ can change the distance between two vertices.

Another possible route for generalization would be to consider *directed* strongly regular graphs. These were introduced in [4] and have been given a fair amount of attention in the literature. Since homomorphisms extend naturally to directed graphs, and many of the algebraic properties of strongly regular graphs have analogs in the directed case [6], it seems plausible that our main result could be generalized to this larger class of graphs.

6. A Generalization

We did not make extensive use of the fact that H was a strongly regular graph in the proof of our main result, nor the lemmas leading up to it. If we let G be an $SRG(n,k,\lambda,\mu)$, then the only thing we required of H in our arguments is that the

(1)All instances of the phrase "up to a scalar" have been removed from the following for brevity.

matrix $I + \alpha A_H + \beta' \overline{A}_H$, where α is the adjacency cosine of G and β' is at most the non-adjacency cosine of G, is positive semidefinite. The proof of the main result now proceeds exactly as before.

The assumption that $I + \alpha A_H + \beta' \overline{A}_H$ is positive semidefinite implies that H admits a strict vector coloring of value $1 - 1/\alpha = \bar{\vartheta}(G)$. Since we also assumed that $G \to H$, this must be an optimal strict vector coloring of H. This inspires the following definition. For real numbers α and β, we say that H is an (α, β)-graph if $I + \alpha A_H + \beta \overline{A}_H$ is the Gram matrix of an optimal strict vector coloring of H. Note that this implies that $\alpha \in [-1, 0)$. We can now succinctly state the above discussed generalization of our main result:

Theorem 6.1. *Suppose G is a strongly regular graph with adjacency and non-adjacency cosines α and β respectively, and that H is an (α, β')-graph. Let φ be a homomorphism from G to H. Then the following hold:*

(1) *If $\beta > \beta'$, then φ is a coloring.*
(2) *If $\beta = \beta'$, then φ is either a coloring or an isomorphism to an induced subgraph of H.*

Note that in the case of a coloring, the image of φ must be a maximum clique of H of size $1 - \frac{k}{\tau}$. In either case, the image of φ must have strict vector chromatic number equal to that of both G and H, namely $1 - \frac{k}{\tau}$.

As an example, let H be the complement of the line graph of the Petersen graph. Note that this is *not* strongly regular. If we let G be the unique $SRG(25, 16, 9, 12)$ (this is the complement of the line graph of $K_{5,5}$) then both G and H are $(-\frac{1}{4}, \frac{3}{8})$-graphs. It then follows from Theorem 6.1 that any homomorphism from G to H is a coloring or an isomorphism to an induced subgraph of H. However, H has fewer vertices than G, so there can only be colorings. There do exist such colorings since $\chi(G) = 5 = \omega(H)$, but there can be no other homomorphisms from G to H by Theorem 6.1.

6.1. Generalizations that fail.

Let G and H be primitive strongly regular graphs with adjacency cosines α, α' and non-adjacency cosines β, β' respectively. Theorem 3.4 characterizes the homomorphisms from G to H in the case where $\alpha = \alpha'$ and $\beta \geqslant \beta'$. But we might hope that the conclusion of this theorem holds in other cases as well. We will see that such generalizations do not hold except in one trivial case.

If $\alpha > \alpha'$, then $\bar{\vartheta}(G) > \bar{\vartheta}(H)$ since the strict vector chromatic number of a strongly regular graph is monotonically increasing with its adjacency cosine. Furthermore, since $\bar{\vartheta}$ is homomorphism monotone, in this case we have that there are no homomorphisms from G to H. So we can generalize our result to the case where $\alpha > \alpha'$, but this is really just an instance of the homomorphc monotonicity of $\bar{\vartheta}$, and thus is nothing new.

In the case of $\alpha = \alpha'$, Theorem 3.4 covers both the $\beta = \beta'$ and $\beta > \beta'$ subcases. For $\beta < \beta'$, we can let G be the Shrikhande graph and H be the complement of the line graph of $K_{4,4}$. These are an $SRG(16, 6, 2, 2)$ and an $SRG(16, 9, 4, 6)$ respectively, and we have that $\alpha = \alpha' = -1/3$ and $\beta = 1/9 < 1/3 = \beta'$ for these graphs. One can find, for instance with the GAP package Digraphs, a homomorphism from G to H whose image is a 6-vertex subgraph of H formed by gluing together two K_4's along an edge. Obviously, this homomorphism is neither a coloring nor an isomorphism to an induced subgraph, and so we see that Theorem 3.4 cannot be generalized to this case.

For the case of $\alpha < \alpha'$, we will fix G to be the Petersen graph, which is an $SRG(10, 3, 0, 1)$ and has $\alpha = -2/3$ and $\beta = 1/6$. We will also always have $\alpha' = -1/4 > \alpha$ for the counterexamples in this case.

Let H be the $SRG(45, 12, 3, 3)$ with graph6 string given below[(2)]:

```
l~}CKMF_C?oB_FPCGaICQOaH@DQAHQ@Ch?aJHAQ@GP_CQAIGcAJGO‘IcGOY‘@IGaGHGaKS
CDI?gGDgGcE_@OQAg@PCSO_hOa‘GIDADAD@XCIASDKB?oKOo@_SHCc?SGcGd@A‘B?bOOHG
QH?ROQOW‘?XOPa@C_hcGo‘CGJK
```

This is also the first graph in the list of $SRG(45, 12, 3, 3)$'s given on [21]. For this graph we have $\alpha' = -1/4$ and $\beta' = 1/16 < 1/6 = \beta$. One can find a homomorphism from G to H whose image is a 6-vertex subgraph of H isomorphic to the graph constructed from a K_5 by adding a vertex adjacent to just one of its vertices.

For the $\beta' = \beta$ case, we let H be the line graph of K_6. This is an $SRG(15, 8, 4, 4)$ for which we have $\alpha' = -1/4$ and $\beta' = 1/6 = \beta$. One can check that G has a homomorphism to H whose image is a 6-vertex subgraph of H that can be constructed by gluing together a K_3 and 4-cycle along an edge and then adding a sixth vertex adjacent to all others.

Finally, let H be the unique $SRG(25, 16, 9, 12)$ (this is the complement of the line graph of $K_{5,5}$). For this graph we have $\alpha' = -1/4$ and $\beta' = 3/8 > 1/6 = \beta$. However, there is a homomorphism from G to H whose image is a 6-vertex subgraph of H that can be constructed by gluing three K_3's together along a single edge and then adding a vertex adjacent to the three vertices not incident to the merged edge. Thus we see that Theorem 3.4 cannot be generalized to this case either.

The above examples show that the cases dealt with by Theorem 3.4 are exactly the cases where the conclusion does in fact hold (except the trivial case of $\alpha > \alpha'$).

7. Discussion

The main purpose of this work was to prove the conjecture of Cameron & Kazanidis. However, our results have several other implications and raise certain questions. We will discuss some of these here.

Since all but finitely many strongly regular graphs with fixed least eigenvalue are the point graphs of partial geometries, these geometric graphs warrant some consideration with respect to our results. We mentioned previously that geometric graphs always have Delsarte cliques. This is because the Hoffman bound for these graphs is equal to the size of a line in the underlying partial geometry, and thus the points on a line induce a Delsarte clique, though there may be others. It follows from this that all geometric graphs are of types B or C. Therefore, a geometric graph is type B if and only if it has a Hoffman coloring, and otherwise is type C. Recall that every color class in a Hoffman coloring is a Delsarte coclique. For geometric graphs, it is known that a Delsarte coclique corresponds to a set of points in the underlying partial geometry that meets every line exactly once, and vice versa. Such an object is called an *ovoid*. Therefore, a Hoffman coloring of a geometric graph is a partition of its partial geometry into ovoids. A partition into ovoids is called a *fan*. So we see that the point graph of a partial geometry is type B if and only if the geometry has a fan, and otherwise the graph is type C.

In light of the generalization of our main result presented in Section 6, it is interesting to ask what graphs are (α, β)-graphs for which real numbers α and β. We are presently preparing a paper addressing this question, but we will discuss some basic points here. First, we have seen that strongly regular graphs are (α, β)-graphs for $\alpha = \tau/k$ and $\beta = \bar{\theta}/\bar{k}$. As we mentioned in Section 2.2, it is possible for different parameter sets to result in the same values of both τ/k and $\bar{\theta}/\bar{k}$. This brings us to an interesting question: for fixed α and β, are there an infinite number of (α, β)-graphs?

---

(2) Copying and pasting this graph6 string may result in some errors. In particular, the ‘ and possibly the ˜ characters may need to be changed manually after pasting.

If we restrict to strongly regular graphs, it turns out the answer is no. This is because, as we show in our upcoming paper, the second largest eigenvalue of a regular (α,β)-graph is determined by α and β. Thus the least eigenvalue of its complement is determined. So for fixed α and β, the least eigenvalue of the complement of a strongly regular (α,β)-graph is fixed, and thus Neumaier's result can be applied. One can then simply check the infinite families to see that these do not provide infinitely many (α,β)-graphs.

In the positive direction, any graph which is transitive on its non-edges is an (α,β)-graph for some values of α and β. This is because the Gram matrix of any optimal strict vector coloring of a non-edge-transitive graph can be "smoothed out" on the non-edges by taking a uniform convex combination of the Gram matrix conjugated by permutation matrices representing automorphisms of the graph. This provides a large class of (α,β)-graphs that includes many graphs which are not strongly regular.

The fact that every strongly regular graph is a pseudocore has implications in the study of synchronizing groups. A permutation group Γ acting on a set S *synchronizes* a function f from S to itself if the monoid generated by Γ and f contains a transformation whose image is a single element of S. The group Γ is said to be *synchronizing* if it synchronizes every function that is not a permutation. This definition is motivated by concerns in the theory of finite automata, in particular the Černý conjecture. In [1], Cameron et. al. define *almost synchronizing* permutation groups as those which synchronize all functions which are non-uniform, i.e. whose preimages are not all the same size. They note that the automorphism group of any vertex transitive pseudocore is almost synchronizing whenever it is primitive. Therefore, our main result shows that the automorphism group of any vertex transitive strongly regular graph is almost synchronizing whenever it is primitive. In particular, they note[(3)] that this implies any primitive group with permutation rank 3 is almost synchronizing.

In [2], the *hull* of a graph was introduced by Cameron & Kazanidis in order to prove that rank 3 graphs are core-complete. The hull of a graph G has the same vertex set as G, with two vertices being adjacent in the hull if there does not exist an endomorphism of G which identifies these vertices. In particular, this means that every edge of G is an edge of its hull. Cameron & Kazanidis proved several results about the hull of a graph, showing that it is in some sense a dual notion to that of the core. It therefore may be natural to ask whether the hull of a strongly regular graph is always either the graph itself or a complete graph. This turns out to not be the case, and in fact we have found through direct computations that there are strongly regular graphs whose hulls are not even regular. In fact this happens for 14 of the 23 type B $SRG(45,12,3,3)$'s. One such example is the $SRG(45,12,3,3)$ whose graph6 string is given in Section 6.1.

As we mentioned in the introduction, we believe that the technique presented here has wider application to the study of homomorphisms of highly regular graphs. In fact, with Godsil and Rooney, we have already obtained results in this direction, though the work is still in progress. In particular, we consider graphs obtained from a construction based on Hoffman colorings of strongly regular graphs. Depending on whether the Hoffman bound for the initial strongly regular graphs is less than or at least the square root of the number of vertices, the resulting graphs will either be uniquely (vector) colorable or pseudocores.

Another avenue to consider is the generalization of homomorphism results for symmetric graphs to results for highly regular graphs, as we have done here for the

(3) Cameron et. al. received a preprint of this manuscript before it became publicly available.

Cameron and Kazanidis theorem for rank 3 graphs. There are many other such results we could look at. One possibility is the following: it is known [13] that if G is vertex transitive, then any homomorphism from G to its core has fibres (preimages of single vertices) of uniform size. Can we use our techniques to generalize this? We would also need to determine the correct combinatorial analog of vertex transitivity. Simply being regular is not strong enough. Perhaps we need to assume walk regularity, which means that the number of closed walks of length ℓ on a vertex is independent of the vertex for all ℓ. Algebraically, this is equivalent to the powers of the adjacency matrix all having constant diagonals. Or maybe the assumption should more directly concern optimal primal and dual solutions for $\bar{\vartheta}(G)$, which are what we really used in the end.

We briefly remark that two variants of $\bar{\vartheta}$ might be useful for generalizing our results to larger classes of graphs. The Schrijver [20] and Szegedy [22] theta functions (of the complement), denoted $\bar{\vartheta}^-$ and $\bar{\vartheta}^+$ respectively, satisfy $\bar{\vartheta}^-(G) \leqslant \bar{\vartheta}(G) \leqslant \bar{\vartheta}^+(G)$ for all graphs G. They also have primal/dual semidefinite programming formulations with analogous complementary slackness conditions, though these conditions are not quite as nice as for $\bar{\vartheta}$. The parameter $\bar{\vartheta}^-$ is also known as the vector chromatic number. Recall from Section 5 that every optimal vector coloring of an strongly regular graph is a strict vector coloring. This is not true in general and so $\bar{\vartheta}^-$ (and similarly $\bar{\vartheta}^+$) may be able to be used in cases where $\bar{\vartheta}$ does not get the job done.

Acknowledgements. I would like to thank Chris Godsil, Brendan Rooney, Robert Šámal, and Antonis Varvitsiotis for all I learned through our work on vector colorings. In particular, I thank Robert for first showing me a paper about unique vector colorings which initiated this collaboration. Brendan was the unfortunate soul that first checked the proof of the main result for correctness, and I thank him for that. Chris, Krystal Guo, and Gordon Royle also read early versions of this work and I thank them for their helpful input. I would also like to thank Laura Mančinska for encouraging me to keep going after I found a mistake in my original proof.

References

[1] J. Araújo, P. J. Cameron, and B. Steinberg, *Between primitive and 2-transitive: Synchronization and its friends*, EMS Surveys in Mathematical Sciences **4** (2017), no. 2, 101–184.

[2] P. J. Cameron and P. A. Kazanidis, *Cores of symmetric graphs*, Journal of the Australian Mathematical Society **85** (2008), no. 2, 145–154.

[3] P. Delsarte, *An algebraic approach to the association schemes of coding theory*, Philips Research Reports Suppl. **10** (1973).

[4] A. M. Duval, *A directed graph version of strongly regular graphs*, Journal of Combinatorial Theory, Series A **47** (1988), no. 1, 71–100.

[5] A. D. Gardiner, C. D. Godsil, A. D. Hensel, and G. F. Royle, *Second neighbourhoods of strongly regular graphs*, Discrete Mathematics **103** (1992), no. 2, 161–170.

[6] C. D. Godsil, S. A. Hobart, and W. J. Martin, *Representations of directed strongly regular graphs*, European Journal of Combinatorics **28** (2007), no. 7, 1980–1993.

[7] C. D. Godsil, D. E. Roberson, B. Rooney, R. Šámal, and A. Varvitsiotis, *Graph homomorphisms via vector colorings*, `https://arxiv.org/abs/1610.10002`, 2016.

[8] ———, *Universal completability, least eigenvalue frameworks, and vector colorings*, Discrete & Computational Geometry **58** (2017), no. 2, 265–292.

[9] ———, *Vector coloring the categorical product of graphs*, `https://arxiv.org/abs/1801.08243`, 2018.

[10] C. D. Godsil and G. F. Royle, *Cores of geometric graphs*, Annals of Combinatorics **15** (2011), no. 2, 267–276.

[11] ———, *Algebraic graph theory*, vol. 207, Springer Science & Business Media, 2013.

[12] W. H. Haemers, *Eigenvalue techniques in design and graph theory*, Ph.D. thesis, Eindhoven University of Technology (Netherlands), 1979.

[13] G. Hahn and C. Tardif, *Graph homomorphisms: structure and symmetry*, in Graph symmetry, Springer, 1997, pp. 107–166.

[14] A. J. Hoffman, *On eigenvalues and colorings of graphs*, Graph Theory and its Applications, Academic Press, New York, 1970, pp. 79–91.

[15] L.-P. Huang, J.-Q. Huang, and K. Zhao, *On endomorphisms of alternating forms graph*, Discrete Mathematics **338** (2015), no. 3, 110–121.

[16] J. Jonušas, J. D. Mitchell, M. Torpey, and W. Wilson, *Digraphs – GAP package, version 0.2*, September 2015.

[17] D. Karger, R. Motwani, and M. Sudan, *Approximate graph coloring by semidefinite programming*, J. ACM **45** (1998), no. 2, 246–265.

[18] L. Lovász, *On the Shannon capacity of a graph*, IEEE Trans. Inform. Theory **25** (1979), no. 1, 1–7.

[19] A. Neumaier, *Strongly regular graphs with smallest eigenvalue $-m$*, Archiv der Mathematik **33** (1979), no. 1, 392–400.

[20] A. Schrijver, *A comparison of the Delsarte and Lovász bounds*, IEEE Trans. Inform. Theory **25** (1979), no. 4, 425–429.

[21] E. Spence, *Personal webpage: Strongly regular graphs on at most 64 vertices*, `http://www.maths.gla.ac.uk/~es/srgraphs.php`.

[22] M. Szegedy, *A note on the ϑ number of Lovász and the generalized Delsarte bound*, Proceedings of the 35th Annual IEEE Symposium on Foundations of Computer Science, 1994.

[23] The Sage Developers, *Sage Mathematics Software (Version 6.9)*, 2015, `http://www.sagemath.org`.

David E. Roberson, Technical University of Denmark, Dept. of Applied Mathematics and Computer Science, Richard Petersens Plads, Lyngby, DK-2400, Denmark
E-mail : `dero@dtu.dk`
Url : `https://sites.google.com/site/davideroberson`