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Canonical decomposition of a difference of convex sets

Ana M. Botero

Abstract Let N be a lattice of rank n and let $M = N^{\vee}$ be its dual lattice. In this article we show that given two closed, bounded, full-dimensional convex sets $K_1 \subseteq K_2 \subseteq M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, there is a canonical convex decomposition of the difference $K_2 \setminus \operatorname{int}(K_1)$ and we interpret the volume of the pieces geometrically in terms of intersection numbers of toric b-divisors.

1. Introduction

Convex sets have been widely and successfully used to explore the geometry of algebraic varieties using convex geometrical methods. A well known class of examples comes from the theory of toric varieties, where the combinatorics of a lattice polytope encrypts most of the geometric properties of the corresponding projective toric variety (see [3] and [4]). More generally, Okounkov bodies (in the literature often called Newton–Okounkov bodies) are convex sets which one can attach to algebraic varieties together with some extra geometric data, e.g. a complete flag of subvarieties. These convex sets turn out to encode also important geometric information of the varieties (see [12, 13] and also [8, 9, 10] and [11] and the references therein).

More recently, generalizing the toric situation, in [1], convex sets are associated to so called toric b-divisors, which can be thought of as a limit of toric divisors keeping track of birational information. Their degree is defined as a limit. Here it is shown that under some positivity assumptions toric b-divisors are integrable and that their degree is given as the volume of a convex set. Moreover, it is shown that the dimension of the space of global sections of a nef toric b-divisor is equal to the number of lattice points in this convex set and a Hilbert–Samuel type formula for its asymptotic growth is given. This generalizes classical results for classical toric divisors on toric varieties. Finally, a relation between convex bodies associated to b-divisors and Okounkov bodies is established. We remark that the main motivation for studying toric b-divisors is to be able to do arithmetic intersection theory on mixed Shimura varieties of non-compact type. Indeed, it turns out that toric b-divisors locally encode the singularities of the invariant metric on an automorphic line bundle over a toroidal compactification of a mixed Shimura variety of non-compact type. This note is part of an overall program to

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develop an arithmetic intersection theory on mixed Shimura varieties of non-compact type via convex geometric methods whose starting point is [1].

In general, of particular interest is to compute the volume of a convex set. Aside from this being intrinsically a question of great interest, it has applications not only in the above mentioned geometric settings, but also in other mathematical fields such as in convex optimization.

For the rest of this introduction let us fix a lattice N of rank n, its dual lattice $M = N^\vee$, and two compact, full-dimensional convex sets $K_1 \subseteq K_2 \subseteq M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. The aim of this article is to show that there is a canonical convex decomposition of the difference $K_2 \setminus \mathrm{int}(K_1)$, where $\mathrm{int}(K_1)$ denotes the interior of K_1, and to interpret geometrically the volume of the pieces in terms of intersection numbers of toric b-divisors.

The outline is as follows. In Section 2 we recall the Legendre–Fenchel duality for convex sets. Most of the definitions and statements which we will state regarding this duality can be found in [14]. We also refer to [2, Chapter 2].

In Section 3 we give the canonical convex decomposition of the difference $K_2 \setminus \mathrm{int}(K_1)$. We start by defining what it means for two faces F_1 and F_2 of K_1 and K_2, respectively, to be related, denoted by $F_1 \sim F_2$. Using this relationship, we are able to show the following main result of this section, which is Theorem 3.9 in the text.

Theorem 1.1. *Let notations be as above. Then we have that*

$$\Upsilon\left(K_2 \setminus \mathrm{int}(K_1)\right) := \left\{ \mathrm{convhull}\left(F_1, F_2\right) \,\middle|\, F_1 \overset{\text{exposed}}{\leqslant} K_1,\, F_2 \overset{\text{exposed}}{\leqslant} K_2 \ \textit{and}\ F_1 \sim F_2 \right\}$$

is a convex decomposition of the difference $K_2 \setminus \mathrm{int}(K_1)$.

In the polyhedral case, the above canonical decomposition gives a polyhedral subdivision of the complement of two polytopes, one contained in the other. This subdivision appears in the literature (e.g. in [5]) although it is constructed using the so called pushing method. We haven't found in the literature the method we used in Theorem 3.9 nor have we found such a canonical decomposition in the non-polyhedral case.

In Section 4, we start by recalling the definition of toric b-divisors from [1] and the definition of the mixed degree. We then recall the definition of the surface area measure (and a mixed version thereof) associated to a convex set (and to a collection of convex sets) for which our main reference is the survey of Schneider [15]. Finally, in Corollary 4.9 we relate this measure to the intersection theory of toric b-divisors.

In Section 5, we give a geometric interpretation of the above canonical decomposition in terms of intersection numbers of toric b-divisors in the case that K_2 is polyhedral. The main result of this section is the following, which is the first part of Theorem 5.2 in the text.

Theorem 1.2. *Let notations be as above and assume that K_2 is a polytope. Then the functions ϕ_1 and ϕ_2 correspond respectively to a nef toric b-divisor $\boldsymbol{D}_1$ and to a true nef toric divisor D_2 on the toric variety determined by the normal fan of K_2. Moreover, we can express the difference of degrees $D_2^n - \boldsymbol{D}_1^n$ as a finite sum of correction terms*

$$D_2^n - \boldsymbol{D}_1^n = \sum_{F \leqslant K_2} c_F,$$

where the correction terms c_F are given by

$$c_F = \sum_{i=0}^{n-1} (n-1)! \int_{\mathrm{relint}(\sigma_F)\cap \mathbb{S}^{n-1}} \left(\phi_1(u) - \phi_2(u)\right) S(\underbrace{K_1, \dots, K_1}_{i\text{-times}}, \underbrace{K_2, \dots, K_2}_{(n-1-i)\text{-times}}, u),$$

where $S(\cdot)$ is the mixed surface area measure associated to a collection of convex sets.

2. Legendre–Fenchel duality

Throughout this article, $N \simeq \mathbb{Z}^n$ will denote a lattice of rank n and $M = N^\vee$ its dual lattice. We write $N_\mathbb{R}$ and $M_\mathbb{R}$ for the corresponding n-dimensional real vector spaces $N \otimes_\mathbb{Z} \mathbb{R}$ and $M \otimes_\mathbb{Z} \mathbb{R}$ respectively.

Recall that a non-empty subset $K \subseteq M_\mathbb{R}$ is *convex* if for each pair of points $m_1, m_2 \in K$, the line segment

$$[m_1, m_2] = \left\{ tm_1 + (1-t)m_2 \,\middle|\, 0 \leqslant t \leqslant 1 \right\}$$

is contained in K. Examples of convex sets are cones and polyhedra. (See [14] for a detailed introduction to convex geometry). Throughout this article, convex sets are assumed to be non-empty. Also, by "cone" we actually mean "rational cone" and by "polytope" we mean "rational polytope".

Definition 2.1. *Let K be a convex set in $M_\mathbb{R}$. A convex subset $F \subseteq K$ is called a* face *of K if, for every closed line segment $[m_1, m_2] \subseteq K$ such that* relint $([m_1, m_2]) \cap F \neq \varnothing$, *the inclusion $[m_1, m_2] \subseteq F$ holds. A non-empty subset $F \subseteq K$ is called an* exposed face *of K if there exists a $v \in N_\mathbb{R}$ such that*

$$F = \left\{ m \in K \,\middle|\, \langle v, m \rangle = \min_{m' \in K} \langle v, m' \rangle \right\}.$$

Remark 2.2. Every exposed face is a face. However, not every face is exposed, as can be seen in the figure 1. Here, the star is a non-exposed face. However, in the special case of polytopes, we do have that all the faces are exposed.

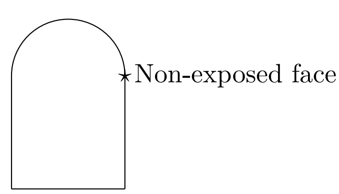

Figure 1. Example of a non-exposed face

Definition 2.3. *Let Υ be a non-empty collection of convex subsets of $M_\mathbb{R}$. Υ is called a* convex subdivision *if the following conditions hold:*

1. *Every face of an element of Υ is also in Υ.*
2. *Every two elements of Υ are either disjoint or they intersect in a common face.*

If only (2) is satisfied, then we call Υ a convex decomposition. *Let Υ be a convex subdivision or decomposition in $M_\mathbb{R}$. The* support *of Υ is the set $|\Upsilon| := \bigcup_{C \in \Upsilon} C$. We say Υ is* complete *if its support is the whole of $M_\mathbb{R}$. For a given subset $E \subseteq M_\mathbb{R}$, if $|\Upsilon| = E$, we say Υ is a convex subdivision or decomposition of E.*

Example 2.4. The set of all faces of a convex set K is a convex subdivision of K. The set of all exposed faces of a convex set K is a convex decomposition of K.

Recall that a function $f\colon N_\mathbb{R} \to \underline{\mathbb{R}}$ $(:= \mathbb{R} \cup \{-\infty\})$ is said to be *concave* if for all $x, y \in N_\mathbb{R}$, the following inequality

$$f(tx + (1-t)y) \geqslant tf(x) + (1-t)f(y)$$

is satisfied for $0 \leqslant t \leqslant 1$ and for all $x, y \in N_\mathbb{R}$. A concave function is said to be *closed* if it is upper semicontinuous. We now define some important classes of concave functions which arise from convex sets. We refer to [2, Section 2] for more details.

Definition 2.5. *The* support function *of a (not necessarily bounded) convex set K is the function*

$$\phi_K \colon N_{\mathbb{R}} \longrightarrow \underline{\mathbb{R}}\ (= \mathbb{R} \cup \{-\infty\})$$

given by the assignment

$$v \longmapsto \inf_{m \in K} \langle m, v \rangle .$$

Support functions of convex sets are concave, closed and conical, i.e. they satisfy $\phi_K(\lambda x) = \lambda \phi_K(x)$ for any non-negative real number λ.

Definition 2.6. *The* Legendre–Fenchel dual *of a concave function $f \colon N_{\mathbb{R}} \to \underline{\mathbb{R}}$ is the function*

$$f^{\vee} \colon M_{\mathbb{R}} \longrightarrow \underline{\mathbb{R}},$$

defined by

$$m \longmapsto \inf_{v \in N_{\mathbb{R}}} \left(\langle m, v \rangle - f(v) \right).$$

The stability set *of f, which is denoted by K_f, is defined to be the effective domain* $\operatorname{dom}(f^{\vee})$ *of the Legendre–Fenchel dual $f^{\vee}$, i.e. we have*

$$K_f = \operatorname{dom}(f^{\vee}) = \left\{ m \in M \,\middle|\, f^{\vee}(m) \neq \varnothing \right\}.$$

The Legendre–Fenchel dual of a concave function can be shown to be concave and closed.

Definition 2.7. *The* indicator function *of a closed convex set $K \subseteq M_{\mathbb{R}}$ is the function*

$$\iota_K \colon M_{\mathbb{R}} \longrightarrow \underline{\mathbb{R}}$$

defined by

$$\iota_K(m) = \begin{cases} 0 & \text{if } m \in K, \\ -\infty & \text{if } m \notin K. \end{cases}$$

The indicator function of a closed convex set can be shown to be concave and closed.

The following useful remark can be found in [2, Section 2.1].

Remark 2.8. Let $K \subseteq M_{\mathbb{R}}$ be a closed convex set and let $\iota_K \colon M_{\mathbb{R}} \to \underline{\mathbb{R}}$ be its indicator function. Then we have that $\phi_K = \iota_K^{\vee}$ and $\phi_K^{\vee} = \iota_K$. Hence, the Legendre–Fenchel duality gives a bijective correspondence between indicator functions of closed convex sets in $M_{\mathbb{R}}$ and concave, closed, conical functions on $N_{\mathbb{R}}$.

Definition 2.9. *Let f be a concave function on $N_{\mathbb{R}}$. The* sup-differential $\partial f(u)$ *of f at $u \in N_{\mathbb{R}}$ is defined by*

$$\partial f(u) \coloneqq \left\{ m \in M_{\mathbb{R}} \,\middle|\, \langle m, u - v \rangle \geqslant f(u) - f(v),\ \forall\, v \in N_{\mathbb{R}} \right\},$$

if $f(u) \neq -\infty$, and $\varnothing$ if $f(u) = -\infty$.

This is a generalization to the non-smooth setting of the *gradient* of a smooth function at a point. Note that in general, the sup-differential may contain more than one point. The following definition is taken from [2, Section 2.2].

Definition 2.10. *We say that f is* sup-differentiable *at a point $u \in N_{\mathbb{R}}$ if $\partial f(u) \neq \varnothing$. The* effective domain *of ∂f is the set of points where f is sup-differentiable. We denote it by* $\operatorname{dom}(\partial f)$. *For a subset $V \subseteq N_{\mathbb{R}}$, the set $\partial f(V)$ is defined by*

$$\partial f(V) \coloneqq \bigcup_{u \in V} \partial f(u).$$

In particular, the image *of ∂f is defined as* $\operatorname{Im}(\partial f) = \partial f(N_{\mathbb{R}})$.

The following propositions can be found in [2, Section 2].

Proposition 2.11. *The sup-differential $\partial f(u)$ is a closed, convex set for all $u \in \operatorname{dom}(\partial f)$. It is bounded if and only if $u \in \operatorname{relint}(\operatorname{dom}(f))$. Moreover, the effective domain of ∂f is close to being convex, in the sense that*

$$\operatorname{relint}(\operatorname{dom}(f)) \subseteq \operatorname{dom}(\partial f) \subseteq \operatorname{dom}(f).$$

In particular, if $\operatorname{dom}(f) = N_{\mathbb{R}}$, we have $\operatorname{dom}(\partial f) = N_{\mathbb{R}}$.

Proposition 2.12. *If f is closed, then we have that $\operatorname{Im}(\partial f) = \operatorname{dom}(\partial f^{\vee})$. Moreover, the image of the sup-differential is close to being convex, in the sense that*

$$\operatorname{relint}(K_f) \subseteq \operatorname{Im}(\partial f) \subseteq K_f.$$

Definition 2.13. *Let f be a closed, concave function on $N_{\mathbb{R}}$. We denote by $\Upsilon(f)$ the collection of all sets of the form*

$$C_m \coloneqq \partial f^{\vee}(m) \subseteq \mathcal{P}(N_{\mathbb{R}}),$$

for $m \in \operatorname{dom}(f^{\vee}) \subseteq M_{\mathbb{R}}$.

The following is [2, Proposition 2.2.8].

Proposition 2.14. *Let f be a closed, concave function on $N_{\mathbb{R}}$. Then $\Upsilon(f)$ is a convex decomposition of $\operatorname{dom}(\partial f)$. In particular, if $\operatorname{dom}(f) = N_{\mathbb{R}}$, then $\Upsilon(f)$ is complete.*

Definition 2.15. *Let f be a closed, concave function on $N_{\mathbb{R}}$. The* Legendre–Fenchel *correspondence of f*

$$\mathcal{L}f \colon \Upsilon(f) \longrightarrow \Upsilon(f^{\vee})$$

is given by the assignment

$$C \longmapsto \bigcap_{u \in C} \partial f(u) \quad \left(= \partial f(u_0), \text{ for any } u_0 \in \operatorname{relint}(C)\right).$$

Definition 2.16. *Let V, V^* be subsets of $N_{\mathbb{R}}$ and of $M_{\mathbb{R}}$, respectively. Moreover, let Υ, Υ^* be convex decompositions of V and V^*, respectively. We say that Υ and Υ^* are* dual *convex decompositions if there exists a bijective map*

$$\Upsilon \longrightarrow \Upsilon^*$$

given by the assignment

$$C \longmapsto C^*$$

and satisfying the following properties:

1. *For every C, D in Υ we have that $C \subseteq D$ if and only if $C^* \supseteq D^*$.*
2. *For every C in Υ, the sets C and C^* are contained in orthogonal affine spaces of $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$, respectively.*

The following theorem is taken from [2, Theorem 2.2.12].

Theorem 2.17. *Let f be a closed, concave function. Then $\mathcal{L}f$ gives a duality between $\Upsilon(f)$ and $\Upsilon(f^{\vee})$ with inverse given by $(\mathcal{L}f)^{-1} = \mathcal{L}f^{\vee}$.*

We make the following remark which can be found in [7, Proposition 2.1.5].

Remark 2.18. Consider a full-dimensional, closed convex set $K \subseteq M_{\mathbb{R}}$. Let ϕ_K be the corresponding closed, concave, conical support function and let $C \in \Upsilon(\phi_K)$. Then, for any $u \in \operatorname{relint}(C)$ we have that $\partial \phi_K(u) \in \Upsilon(\phi_K^{\vee})$ is an exposed face of $K_{\phi_K} = K$. Conversely, every exposed face F of K can be obtained as $\partial \phi_K(u)$ for some $u \in N_{\mathbb{R}}$. Explicitly, consider $m \in \operatorname{relint}(F)$. Then we may take any $u \in \operatorname{relint}\left(\partial h_K^{\vee}(m)\right) = \operatorname{relint}\left(\partial \iota_K(m)\right)$. In particular, if K is bounded, we get a duality between the set of exposed faces of K and a complete convex decomposition of $N_{\mathbb{R}}$.

Example 2.19. Let notations be as in Remark 2.18 and assume that $K = P$ is a polytope. Then the Legendre–Fenchel duality gives back the classical duality between the faces of a polytope and the cones of its normal fan Σ_P.

If K is not polyhedral, our convex decompositions will not be finite, as can be seen in Figure 2. Here we have

$$\phi_K(a,b) = \begin{cases} \frac{ab}{a+b}, & \text{if } a,b \in \mathbb{R}_{\geqslant 0}, \\ \min\{0,a,b\}, & \text{otherwise.} \end{cases}$$

Note that here the convex decomposition of $N_{\mathbb{R}} \simeq \mathbb{R}^2$ gives us a foliation of the positive quadrant by rays.

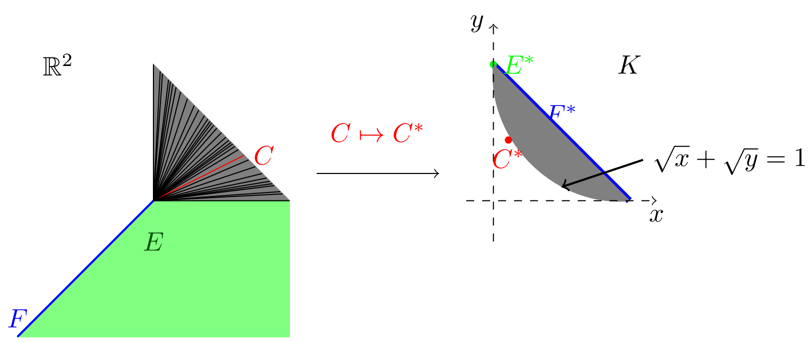

Figure 2. Legendre–Fenchel correspondence in the non-polyhedral case

3. Canonical decomposition of a difference of convex sets

Let $K_1 \subseteq K_2$ be two full-dimensional, closed and bounded convex sets in $M_{\mathbb{R}}$ with corresponding support functions $\phi_{K_1}, \phi_{K_2} \colon N_{\mathbb{R}} \to \mathbb{R}$. The aim of this section is to give a canonical decomposition of the difference $K_2 \smallsetminus \operatorname{int}(K_1)$.

Definition 3.1. *We define two complete convex decompositions Σ_{K_1} and Σ_{K_2} of $N_{\mathbb{R}}$ by setting*

$$\Sigma_{K_i} \coloneqq \Upsilon(\phi_{K_i})$$

for $i = 1, 2$.

Note that the elements in Σ_{K_i} for $i = 1, 2$ are cones. This follows from the fact that the convex set C_m corresponding to an $m \in \operatorname{relint}(K_i)$ is $\{0\}$. Hence, we will call Σ_{K_i} a fan, eventhough it may not be finite nor rational.

It follows from Remark 2.18 that the Legendre–Fenchel duality gives an order-reversing, bijective correspondence between cones in Σ_{K_i} and the set of exposed faces of K_i for $i = 1, 2$. For $F \leqslant K_i$ an exposed face, we will denote by σ_F the cone in Σ_{K_i} given by this correspondence.

The following is a key definition for giving the canonical decomposition of the difference $K_2 \smallsetminus \operatorname{int}(K_1)$.

Definition 3.2. *Let $F_1 \leqslant K_1$ and $F_2 \leqslant K_2$ be exposed faces. We say that F_1* is related to *F_2 (and denote it by $F_1 \sim F_2$) if and only if*

$$\operatorname{relint}(\sigma_{F_1}) \cap \operatorname{relint}(\sigma_{F_2}) \neq \varnothing$$

is satisfied.

In the case where both K_1 and K_2 are polytopes, we make the following definition. This will be useful for the polytopal case in Theorem 5.2.

Definition 3.3. *Assume that K_1 and K_2 are polytopes. Let $\Sigma \subseteq N_{\mathbb{R}}$ be a complete fan in $N_{\mathbb{R}}$. We say that Σ is a* difference fan *for K_1 and K_2, and denote it by $\Sigma = \Sigma_{K_2 \smallsetminus \mathrm{int}(K_1)}$, if the following two conditions are satisfied:*

(1) *Σ is a smooth refinement of both Σ_{K_1} and Σ_{K_2}.*
(2) *Let $F_1 \leqslant K_1$ and $F_2 \leqslant K_2$ be exposed faces. If $F_1 \sim F_2$, then there exists a $\tau \in \Sigma(1)$ such that $\tau \in relint(\sigma_{F_1}) \cap relint(\sigma_{F_2})$.*

Remark 3.4. Note that given any two full-dimensional polytopes $K_1 \subseteq K_2$, we can always find a difference fan $\Sigma_{K_1 \smallsetminus \mathrm{int}(K_2)}$.

Now, before giving the canonical decomposition of the difference $K_2 \smallsetminus \mathrm{int}(K_1)$, we need some auxiliary results.

Let I be the incidence set

$$I \coloneqq \bigcup_{F_1,F_2} F_1 \times F_2,$$

where the union is taken over all proper, exposed faces $F_i \leqslant K_i$, for $i = 1,2$, such that $F_1 \sim F_2$.

Note that by Definition 3.2 of being related, we have that

$$\mathrm{I} = \left\{(x,y) \in \partial K_1 \times \partial K_2 \,\middle|\, \sigma_{F_x} \cap \sigma_{F_y} \neq \varnothing\right\},$$

where σ_{F_x} and σ_{F_y} denote the smallest exposed faces of K_1 and K_2 respectively containing x and y respectively.

Definition 3.5. *We define the function*

$$H\colon\ \mathrm{I} \times [0,1] \longrightarrow M_{\mathbb{R}}$$

by

$$\big((x,y),t\big) \longmapsto tx + (1-t)y.$$

The following proposition follows from the definitions.

Proposition 3.6. *With the notations given above, we have that*

$$\bigcup_{F_1,F_2} \mathrm{convhull}\,(F_1,F_2) = \mathrm{Im}(H),$$

where the union on the LHS is taken over all proper, exposed faces $F_i \leqslant K_i$, for $i = 1,2$, such that $F_1 \sim F_2$.

Now, choose any identification $N_{\mathbb{R}} \simeq \mathbb{R}^n$. We may then consider the unit sphere $\mathbb{S}^{n-1} \subseteq N_{\mathbb{R}}$. Moreover, for $i = 1,2$, we define the incidence sets

$$\mathrm{I}\big(K_1,K_2,\mathbb{S}^{n-1}\big) \coloneqq \left\{(x,y,z) \in \partial K_1 \times \partial K_2 \times \mathbb{S}^{n-1} \,\middle|\, z \in \sigma_{F_x} \cap \sigma_{F_y}\right\} \subseteq \partial K_1 \times \partial K_2 \times \mathbb{S}^{n-1}$$
$$\mathrm{I}\left(K_i,\mathbb{S}^{n-1}\right) \coloneqq \left\{(w,z) \in \partial K_i \times \mathbb{S}^{n-1} \,\middle|\, z \in \sigma_{F_w}\right\} \subseteq \partial K_i \times \mathbb{S}^{n-1},$$

where the sets $\sigma_{F_x}, \sigma_{F_y}$ and σ_{F_w} denote, as above, the smallest exposed faces of K_1, K_2 and K_i respectively containing x, y and z respectively.

Consider the maps in the following diagram.

$$
\begin{array}{ccccc}
 & & \mathrm{I}\left(K_1, K_2, \mathbb{S}^{n-1}\right) \subseteq \partial K_1 \times \partial K_2 \times \mathbb{S}^{n-1} & & \\
 & \swarrow^{p_0} & \downarrow p_1 & \searrow^{p_2} & \\
\mathrm{I}\left(K_1, \mathbb{S}^{n-1}\right) & & \mathrm{I}\left(K_2, \mathbb{S}^{n-1}\right) & & \mathrm{I} \subseteq \partial K_1 \times \partial K_2 \\
p_3 \downarrow & \searrow^{p_4} & \downarrow p_5 & \searrow^{p_6} & \\
\partial K_1 & & \mathbb{S}^{n-1} & & \partial K_2
\end{array}
$$

We have the following propositions.

Proposition 3.7. *All the sets in the above diagram are closed, hence they are compact and all the maps $p_0, \ldots, p_6$ are proper.*

Proof. Let $i \in \{1,2\}$. We show that $\mathrm{I}\left(K_i, \mathbb{S}^{n-1}\right) \subseteq \partial K_i \times \mathbb{S}^{n-1}$ is closed. Consider the concave support function $\phi_{K_i} \colon N_{\mathbb{R}} \to \mathbb{R}$. Then the set

$$\tilde{\mathrm{I}} \coloneqq \left\{(x,y) \in N_{\mathbb{R}} \times M_{\mathbb{R}} \,\middle|\, \phi_{K_i} + \phi^{\vee}_{K_i}(y) = \langle x, y\rangle\right\},$$

where $\phi^{\vee}_{K_i}$ denotes the Legendre–Fenchel dual (see Section 2), is closed and

$$\mathrm{I}\left(K_i, \mathbb{S}^{n-1}\right) = \tilde{\mathrm{I}} \cap \left(\mathbb{S}^{n-1} \times M_{\mathbb{R}}\right),$$

hence, it is closed. Now, we have that

$$\mathrm{I}\left(K_1, K_2, \mathbb{S}^{n-1}\right) = p_0^{-1}\left(\mathrm{I}\left(K_1, \mathbb{S}^{n-1}\right)\right) \cap p_1^{-1}\left(\mathrm{I}\left(K_2, \mathbb{S}^{n-1}\right)\right),$$

hence it is closed. Moreover, we have that

$$\mathrm{I} = p_2\left(\mathrm{I}\left(K_1, K_2, \mathbb{S}^{n-1}\right)\right)$$

and since $\mathrm{I}\left(K_1, K_2, \mathbb{S}^{n-1}\right)$ is compact, we get that I is compact as well. This concludes the proof of the proposition. □

Proposition 3.8. *For all $p_0, \ldots, p_6$, the corresponding maps in homology*

$$(\pi_i)_* \colon \mathrm{H}_*(\quad, \mathbb{Z}) \longrightarrow \mathrm{H}_*(\quad, \mathbb{Z})$$

are isomorphisms.

Proof. One can think of the dual maps in cohomology. Note that for $i = 1, \ldots, 6$, the fibers of the p_i's are contractible. Using this together with the properness of the p_i's we apply the Leray spectral sequence in cohomology (see e.g. [6, III Section 5]) and deduce the result. Indeed consider for example the map $p_5 \colon I(K_2, \mathbb{S}^{n-1}) \to \mathbb{S}^{n-1}$. Consider the constatnt sheaf $\mathbb{Z}$ and let F be any fiber of p_5. Then the Leray spectral sequence $E^{p,q}_{\infty}$ already converges at the second page

$$E_2^{pq} = H^p\left(\mathbb{S}^{n-1}, \mathcal{H}^q(F,\mathbb{Z})\right) = \begin{cases} H^p(\mathbb{S}^{n-1}, \mathbb{Z}), & \text{if } q = 0, \\ 0, & \text{otherwise.}\end{cases}$$

Hence we get that

$$H^n(I(K_2, \mathbb{S}^{n-1}), \mathbb{Z}) = \oplus_p E_{\infty}^{p,n-p} = \oplus_p E_2^{p,n-p} = E_2^{n,0} = H^n(\mathbb{S}^{n-1}, \mathbb{Z}).$$

We can do the same with the other maps p_i. □

The next theorem gives the canonical decomposition of the difference $K_2 \smallsetminus \mathrm{int}(K_1)$. It is the main result of this section.

THEOREM 3.9. *Let $K_1 \subseteq K_2$ be two n-dimensional closed and bounded convex sets in $M_{\mathbb{R}}$. Then we have that*

$$\begin{aligned}&\Upsilon\left(K_2 \smallsetminus \operatorname{int}(K_1)\right)\\ &\quad:= \left\{\operatorname{convhull}\left(F_1, F_2\right) \,\middle|\, F_1 \overset{\text{exposed,proper}}{\leqslant} K_1,\ F_2 \overset{\text{exposed,proper}}{\leqslant} K_2 \text{ and } F_1 \sim F_2\right\}\end{aligned}$$

is a convex decomposition of the difference $K_2 \smallsetminus \operatorname{int}(K_1)$.

Proof. Let $F_1 \leqslant K_1$ and $F_2 \leqslant K_2$ be proper, exposed faces such that $F_1 \sim F_2$. We may fix a $v \in N_{\mathbb{R}}$ such that

$$F_1 = \left\{m \in K_1 \,\middle|\, \langle v, m\rangle = \min_{m' \in K_1} \langle v, m'\rangle\right\} \quad \text{and} \quad F_2 = \left\{m \in K_2 \,\middle|\, \langle v, m\rangle = \min_{m' \in K_2} \langle v, m'\rangle\right\}.$$

Note that related faces live in parallel hyperplanes.

Now, let us show that $\operatorname{convhull}(F_1, F_2) \subseteq K_2 \smallsetminus \operatorname{int}(K_1)$. Let $m \in \operatorname{convhull}(F_1, F_2)$. The fact that $m \in K_2$ is clear. Now, let λ_1, λ_2 be non-negative real numbers satisfying $\lambda_1 + \lambda_2 = 1$ and such that

$$m = \lambda_1 m_1 + \lambda_2 m_2$$

for $m_1 \in F_1$ and $m_2 \in F_2$. Since $m_1 \in F_1$, $m_2 \in F_2$ and $K_1 \subseteq K_2$, we have

$$\langle v, m_1\rangle = \min_{m' \in K_1} \langle v, m'\rangle \geqslant \min_{m' \in K_2} \langle v, m'\rangle = \langle v, m_2\rangle.$$

Hence, we obtain

$$\langle v, m\rangle = \lambda_1\langle v, m_1\rangle + \lambda_2\langle v, m_2\rangle \leqslant \lambda_1\langle v, m_1\rangle + \lambda_2\langle v, m_1\rangle = \langle v, m_1\rangle,$$

which implies that

$$\operatorname{convhull}(F_1, F_2) \cap K_1 = F_1, \tag{1}$$

in particular $\operatorname{convhull}(F_1, F_2) \subseteq K_2 \smallsetminus \operatorname{int}(K_1)$.

Now we show the other inclusion

$$K_2 \smallsetminus \operatorname{int}(K_1) \subseteq \bigcup_{F_1, F_2} \operatorname{convhull}(F_1, F_2),$$

where the union on the RHS is taken over all proper, exposed faces $F_i \leqslant K_i$, for $i = 1, 2$, such that $F_1 \sim F_2$.

By Proposition 3.8 we have that $\mathrm{H}_{n-1}(\mathrm{I}, \mathbb{Z}) = \mathbb{Z}$. Let γ be a chain representing a generator of this group. Recall the map H from Definition 3.5. Note that the chains $\gamma \times \{0\}$ and $\gamma \times \{1\}$ are homologous, i.e. they give the same homology class in $\mathrm{I} \times [0, 1]$, hence their images $H_*(\gamma \times \{0\})$ and $H_*(\gamma \times \{1\})$ are homologous in $\operatorname{Im}(H)$.

We show that $K_2 \smallsetminus \operatorname{int}(K_1) \subseteq \operatorname{Im}(H)$. Indeed, suppose this is not the case. Then there exists an $x \in K_2 \smallsetminus \operatorname{int}(K_1)$ such that $x \notin \operatorname{Im}(H)$. Note that since ∂K_1 and ∂K_2 are clearly in the image of H then such an x must belong to $\operatorname{int}(K_2 \smallsetminus \operatorname{int}(K_1))$. But this in turn implies that $H_*(\gamma \times \{0\})$ is not homologous to $H_*(\gamma \times \{1\})$ (see figure 3) and this gives a contradiction. Hence we have that $K_2 \smallsetminus \operatorname{int}(K_1) \subseteq \operatorname{Im}(H)$. Using Proposition 3.6 we finally obtain

$$K_2 \smallsetminus \operatorname{int}(K_1) \subseteq \bigcup_{F_1, F_2} \operatorname{convhull}(F_1, F_2),$$

as we wanted to show.

It remains to show that this is indeed a convex decomposition. To this end, let $F_1' \leqslant K_1$ and $F_2' \leqslant K_2$ be any other proper, exposed faces satisfying $F_1' \sim F_2'$. We have to show that

$$\operatorname{convhull}(F_1, F_2) \cap \operatorname{convhull}(F_1', F_2') \tag{2}$$

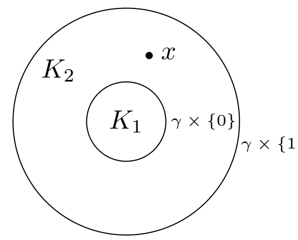

Figure 3.

is either empty or a face of both. If the intersection is empty, then we are done. Hence, assume that $\operatorname{convhull}(F_1, F_2) \cap \operatorname{convhull}(F_1', F_2') \neq \varnothing$. Using (1) , we can show that if $\operatorname{convhull}(F_1, F_2) \subseteq \operatorname{convhull}(F_1', F_2')$ then F_1 is a face of F_1' and F_2 is a face of F_2'. Hence, in this case, and similarly in the case that $\operatorname{convhull}(F_1', F_2') \subseteq \operatorname{convhull}(F_1, F_2)$, the statement is clear. Hence, assume that

$$\operatorname{convhull}(F_1, F_2) \smallsetminus \operatorname{int}(\operatorname{convhull}(F_1', F_2')) \neq \varnothing$$

and

$$\operatorname{convhull}(F_1', F_2') \smallsetminus \operatorname{int}(\operatorname{convhull}(F_1, F_2)) \neq \varnothing.$$

Let A be a hyperplane separating F_1 and F_1', i.e. A is a hyperplane such that F_1 lies entirely in one of the affine half spaces defined by this hyperplane and F_2 in the other. This exists by definition of an exposed face together with the fact that $F_1 \cap F_2$ is an exposed face. Then, since F_1 is parallel to F_2 and F_1' is parallel to F_2', we can choose A to be a separating hyperplane of F_2 and F_2' as well.

The existence of this separating hyperplane implies that

$$\operatorname{convhull}(F_1, F_2) \cap \operatorname{convhull}(F_1', F_2') = \operatorname{convhull}(F_1 \cap F_1', F_2 \cap F_2')$$

which proves that the intersection in (2) is a face of both. This concludes the proof of the proposition. □

Remark 3.10. As was mentioned in the introduction, in the polyhedral case, the above canonical decomposition is a polyhedral subdivision of the complement of two polytopes, one contained in the other. This subdivision appears in the literature (e.g. in [5]) although it is constructed using the so called pushing method. We haven't found in the literature the method we used in Theorem 3.9 nor have we found such a canonical decomposition in the non-polyhedral case.

Again, let $K_1 \subseteq K_2$ be full-dimensional, closed and bounded convex sets in $M_{\mathbb{R}}$.

Definition 3.11. *Let $F \leqslant K_2$ be an exposed face. To F we associate the* correction set

$$K_F := \bigcup_{\tau \in \operatorname{relint}(\sigma_F)} \operatorname{convhull}(F, F_{1,\tau}),$$

where for $\tau \in \operatorname{relint}(\sigma_F)$, the face $F_{1,\tau}$ is the unique exposed face of K_1 such that $\tau \in \operatorname{relint}\left(\sigma_{F_{1,\tau}}\right)$. The associated correction term *c_F is defined as $n!$ times the volume of K_F, i.e.*

$$c_F := n! \operatorname{vol}(K_F).$$

Remark 3.12. Note that by Theorem 3.9 we have that

$$K_2 \smallsetminus \operatorname{int}(K_1) = \bigcup_{F \overset{\text{exposed}}{\leqslant} K_2} K_F$$

is a convex decomposition and hence

$$n! \operatorname{vol}(K_2 \smallsetminus \operatorname{int}(K_1)) = \sum_{F \overset{\text{exposed}}{\leqslant} K_2} c_F.$$

Let's look at a simple 2-dimensional polyhedral example.

Example 3.13. Consider the simplex K_1 contained in the square K_2 as in the Figure 4. Here, the different colors show the correction sets associated to the faces of K_2. Figure 5 shows the dual picture with the fans Σ_{K_2}, $\Sigma_{K_2 \smallsetminus \operatorname{int}(K_1)}$, Σ_{K_1}.

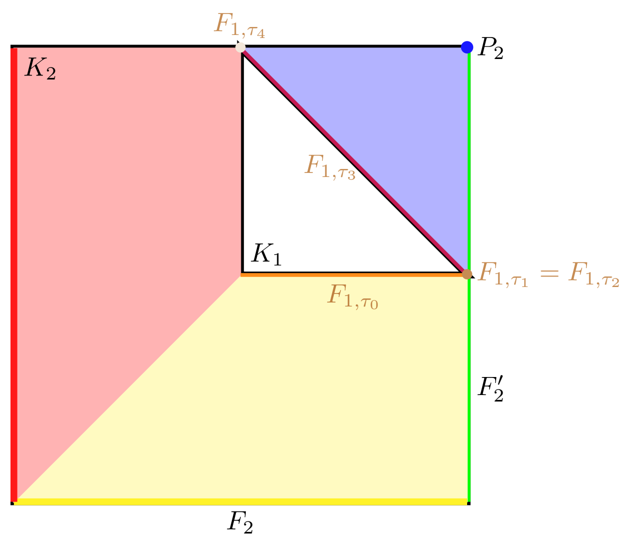

Figure 4. Canonical decomposition of the complement of the simplex contained in the square

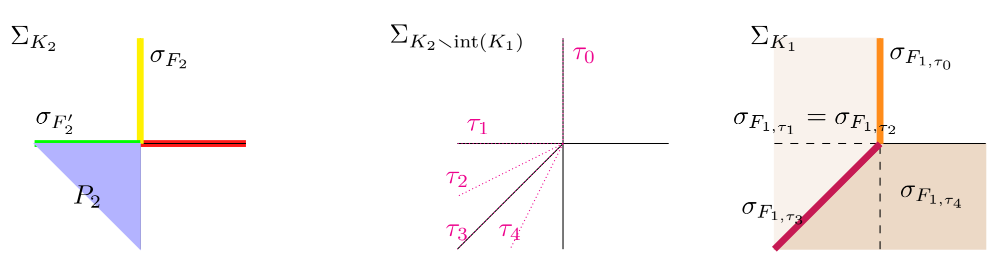

Figure 5. Difference conical subdivision of the simplex contained in the square

4. Toric b-divisors and surface area measures

The goal of this section is to recall the main definitions and facts regarding toric b-divisors (see [1]) and to relate the intersection theory of toric b-divisors with the so called surface area measure (and a mixed version thereof) associated to a convex set (and to a collection of convex sets) (see [15]).

We fix a complete, smooth fan $\Sigma \subseteq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and we denote by X_Σ the corresponding n-dimensional, complete, smooth toric variety with dense open torus $\mathbb{T}$. We refer to [3] and to [4] for a more detailed introduction to toric geometry. The set $R(\Sigma)$ consists of all smooth sudivisions of Σ. This is a directed set with partial order given by $\Sigma'' \geqslant \Sigma'$ in $R(\Sigma)$ if and only if Σ'' is a smooth subdivision of Σ'. The *toric Riemann–Zariski space* of X_Σ is defined as the inverse limit

$$\mathfrak{X}_\Sigma \coloneqq \varprojlim_{\Sigma' \in R(\Sigma)} X_{\Sigma'},$$

with maps given by the toric proper birational morphisms $\pi_{\Sigma''} \colon X_{\Sigma''} \to X_{\Sigma'}$ induced whenever $\Sigma'' \geqslant \Sigma'$. The group of *toric Weil b-divisors* on X_Σ consists of elements in the inverse limit

$$\mathrm{We}(\mathfrak{X}_\Sigma)_{\mathbb{R}} \coloneqq \varprojlim_{\Sigma' \in R(\Sigma)} \mathbb{T}\text{-}\mathrm{Div}(X_{\Sigma'})_{\mathbb{R}},$$

where $\mathbb{T}\text{-}\mathrm{Div}(X_{\Sigma'})_{\mathbb{R}}$ denotes the set of toric Weil $\mathbb{R}$-divisors of $X_{\Sigma'}$, with maps given by the push-forward map of toric Weil $\mathbb{R}$-divisors. We will denote b-divisors with bold $\boldsymbol{D}$ to distinguish them from classical divisors D. We can think of a toric b-divisor as a net of toric Weil $\mathbb{R}$-divisors $(D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$, being compatible under push-forward.

A toric b-divisor $\boldsymbol{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$ is said to be *nef*, if $D_{\Sigma'} \in \mathbb{T}\text{-}\mathrm{Div}(X_{\Sigma'})_{\mathbb{R}}$ is nef for all Σ' in a cofinal subset of $R(\Sigma)$. It follows from basic toric geometry that there is a bijective correspondence between the set of nef toric b-divisors and the set of $\mathbb{R}$-valued, conical, concave functions on $N_{\mathbb{Q}}$.

The *mixed degree* $\boldsymbol{D}_1 \cdots \boldsymbol{D}_n$ of a collection of toric b-divisors is defined as the limit (in the sense of nets)

$$\boldsymbol{D}_1 \cdots \boldsymbol{D}_n \coloneqq \lim_{\Sigma' \in R(\Sigma)} D_{1_{\Sigma'}} \cdots D_{n_{\Sigma'}}$$

of top intersection numbers of toric divisors, provided this limit exists and is finite. In particular, if $\boldsymbol{D} = \boldsymbol{D}_1 = \ldots = \boldsymbol{D}_n$, then the limit (in the sense of nets)

$$\boldsymbol{D}^n \coloneqq \lim_{\Sigma' \in R(\Sigma)} D^n_{\Sigma'},$$

provided this limit exists and is finite, is called the *degree* of the toric b-divisor $\boldsymbol{D}$. A toric b-divisor whose degree exists, is said to be *integrable*.

It turns out that we can compute (mixed) degrees of nef toric b-divisors using the so called (mixed) surface area measure. We give some definitions in order to state our result.

Definition 4.1. *Consider the vector space $\mathbb{R}^n$ equipped with the standard euclidean metric. For $0 \leqslant k \leqslant n$, we let $\mathcal{H}^k$ be the k-dimensional* Hausdorff measure *on $\mathbb{R}^n$. In particular, if ω is a Borel subset of a k-dimensional euclidean space E^k or a k-dimensional sphere $\mathbb{S}^k$ in $\mathbb{R}^n$, then $\mathcal{H}^k(\omega)$ coincides with the k-dimensional* Lebesgue *measure of ω computed in E^k or with the k-dimensional* spherical Lebesgue *measure of ω computed in $\mathbb{S}^k$, respectively.*

Let $K \subseteq \mathbb{R}^n$ be a closed, bounded, full-dimensional convex set with corresponding support function $\phi_K \colon \mathbb{R}^n \to \mathbb{R}$. Moreover, let

$$g_K \colon \mathbb{S}^{n-1} \longrightarrow \mathcal{P}(\partial K), \tag{3}$$

where $\mathcal{P}(\partial K)$ denotes the power set of the boundary ∂K of K, be the map given in the following way. In the case that ϕ_K is of class C^2, g_K sends $u \in \mathbb{S}^{n-1}$ to the gradient $\nabla \phi_K(u)$. In general, the inverse g^{-1} is what in the literature is called the *Gauss* map, which assigns the outer unit normal vector $v_K(x)$ to an $x \in \partial_* K$, where

$\partial_* K$ consists of all points in the boundary ∂K of K having a unique outer normal vector. In other words, we have that

$$g_K(u) = \left\{ m \in \mathbb{R}^n \,\middle|\, \langle m, u\rangle = \phi_K(u) \text{ and } \langle m, v\rangle \geqslant \phi_K(v),\, \forall v \in \mathbb{R}^n \right\},$$

for every $u \in \mathbb{S}^{n-1}$.

Remark 4.2. This map is related to the Legendre–Fenchel duality of Definition 2.15 in the following way. Let $u \in \mathbb{S}^{n-1} \subseteq \mathbb{N}_{\mathbb{R}}$. Let $C \in \Upsilon(\phi_K)$ be the smallest convex set containing u. Recall that $\Upsilon(\phi_K)$ is a complete conical subdivision of $N_{\mathbb{R}}$. Then we have that

$$\mathcal{L}f(C) = g_K(u).$$

Definition 4.3. *The* surface area measure $S_{n-1}(K, \cdot)$ *associated to* K *is the finite Borel measure on the unit sphere* $\mathbb{S}^{n-1}$ *defined by*

$$S_{n-1}(K, \omega) = \mathcal{H}^{n-1}\left(g_K(\omega)\right)$$

for every Borel subset ω *of* $\mathbb{S}^{n-1}$.

In particular, for a polytope P *with unitary normal vectors* $u_1, \dots, u_r$ *at its facets* $F_1, \dots, F_r$*, respectively, the surface area measure of a Borel subset* $\omega \subseteq \mathbb{S}^{n-1}$ *is given by*

$$S_{n-1}(P, \omega) = \sum_{u_i \in \omega} \mathrm{vol}_{n-1}(F_i),$$

where vol_k *denotes the* k*-dimensional volume operator. In other words, we have that*

$$S_{n-1}(P, \cdot) = \sum_{i=1}^{r} \mathrm{vol}_{n-1}(F_i)\, \delta_{u_i},$$

where δ_{u_i} *denotes the Dirac delta measure supported on* $u_i \subset \mathbb{S}^{n-1}$ *for all* $i = 1, \dots, r$.

Example 4.4. Let P be a polytope and let ϕ_P be its corresponding piecewise linear, concave support function. We get the formula

$$n \,\mathrm{vol}_n(P) = \sum_{i=1}^{r} \phi_P(u_i)\, \mathrm{vol}_{n-1}(F_i) = \int_{\mathbb{S}^{n-1}} \phi_P(u) S_{n-1}(P, u)$$

for the volume of P. This formula can be generalized to any full dimensional, closed and bounded convex set K. Indeed, the volume of K can be shown to be given by

$$\mathrm{vol}_n(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \phi_K(u) S_{n-1}(K, u). \tag{4}$$

As is described in [15, Section 5], one can generalize the surface area measure associated to a single convex set to a collection of $n-1$ (not necessarily distinct) convex sets. This is the so called *mixed surface area measure*. We denote by $\mathcal{K}_n$ the set of closed, bounded, full-dimensional convex sets in $\mathbb{R}^n$. The next theorem follows from [15, Theorem 5.1.7].

Theorem/Definition 4.5. *There is a nonnegative symmetric function* $\mathrm{MV}\colon (\mathcal{K}_n)^n \to \mathbb{R}$*, called the* mixed volume*, such that for every natural number* ℓ *and for every nonnegative real numbers* $\lambda_1, \dots, \lambda_\ell$*, the equation*

$$\mathrm{vol}_n(\lambda_1 K_1 + \cdots + \lambda_\ell K_\ell) = \sum_{i_1, \dots, i_n = 1}^{\ell} \lambda_{i_1} \cdots \lambda_{i_n} \frac{1}{n!} \mathrm{MV}\left(K_{i_1}, \dots, K_{i_n}\right),$$

where the sum on the left hand side is the Minkowski sum of convex sets, is satisfied for any collection of convex sets $K_1, \dots, K_\ell \in \mathcal{K}_n$.

Furthermore, there is a symmetric map S from $(\mathcal{K}_n)^{n-1}$ into the space of finite Borel measures on $\mathbb{S}^{n-1}$, called the mixed surface area measure*, such that for every natural number ℓ and for every non-negative real numbers $\lambda_1, \ldots, \lambda_\ell$, the equation*

$$S_{n-1}\left(\lambda_1 K_1 + \cdots + \lambda_\ell K_\ell, \omega\right) = \sum_{i_1,\ldots,i_{n-1}=1}^{\ell} \lambda_{i_1} \cdots \lambda_{i_{n-1}} S\left(K_{i_1}, \ldots, K_{i_{n-1}}, \omega\right)$$

is satisfied for $K_1, \ldots, K_\ell \in \mathcal{K}_n$ and for every Borel subset $\omega \subseteq \mathbb{S}^{n-1}$. Moreover, for $K_1, \ldots, K_n \in \mathcal{K}_n$, the mixed volume $\mathrm{MV}(K_1, \ldots, K_n)$ can be expressed in terms of the mixed surface area measure in the following way

$$\mathrm{MV}\left(K_1, \ldots, K_n\right) = (n-1) \int_{\mathbb{S}^{n-1}} \phi_{K_1}(u) S\left(K_2, \ldots, K_n, u\right).$$

We make the following remarks.

Remark 4.6.

(1) Setting $K = K_1 = \cdots = K_n$ we get

$$\begin{aligned}\mathrm{vol}_n(K) &= \frac{1}{n!}\mathrm{MV}(K, \ldots, K)\\ &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} \phi_K(u) S\left(K, \ldots, K, u\right)\\ &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} \phi_K(u) S_{n-1}\left(K, u\right)\end{aligned}$$

as in Equation (4).

(2) The mixed volume "V(·)" defined in [15, Theorem 5.1.7] is related to the Mixed Volume "MV(·)" from above by the formula

$$\mathrm{V}\left(K_1, \ldots, K_n\right) = \frac{1}{n!}\mathrm{MV}\left(K_1, \ldots, K_n\right)$$

for $K_1, \ldots, K_n \in \mathcal{K}_n$.

We now come back to b-divisors. Let $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ be a collection of nef toric b-divisors on a smooth and complete toric variety X_Σ of dimension n. Let $\tilde{\phi}_i \colon N_{\mathbb{Q}} \to \mathbb{R}$ be the corresponding concave functions for $i = 1, \ldots, n$.

The following theorem relates the mixed degree $\boldsymbol{D}_1 \ldots \boldsymbol{D}_n$ with the mixed volume of convex bodies. It is a combination of [1, Theorems 4.9 and 4.12].

Theorem 4.7. *With notations as above, the functions $\tilde{\phi}_i$ extend to continuous, concave functions $\phi_i \colon N_{\mathbb{R}} \to \mathbb{R}$. Moreover, the mixed degree $\boldsymbol{D}_1 \cdots \boldsymbol{D}_n$ exists, and is given by the mixed volume of the stability sets K_{ϕ_i} of the concave functions ϕ_i, i.e. we have that*

$$\boldsymbol{D}_1 \cdots \boldsymbol{D}_n = \mathrm{MV}\left(K_{\phi_1}, \ldots, K_{\phi_n}\right).$$

In particular, a nef toric b-divisor $\boldsymbol{D}$ is integrable, and its degree is given by

$$\boldsymbol{D}^n = n!\ \mathrm{vol}\left(K_\phi\right),$$

where ϕ is the corresponding concave function.

Remark 4.8. Note that here we consider toric $\mathbb{R}$-divisors instead of $\mathbb{Q}$-divisors (as is done in [1]). All the definitions and results can be extended to this case without difficulties. Moreover, we have that if $K \subseteq M_{\mathbb{R}}$ is a convex set in $\mathcal{K}_n$, then the support function ϕ_K of K corresponds to a nef toric b-divisor $\boldsymbol{D}_K$. Hence, in this case, there is a bijection between convex sets in $\mathcal{K}_n$ and nef toric b-divisors on a smooth and complete toric variety X_Σ of dimension n.

The following corollary follows from the definition of the surface area measure and Theorem 4.7.

Corollary 4.9. *We fix an identification $N_{\mathbb{R}} \simeq \mathbb{R}^n$. Let $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ be a collection of nef toric b-divisors associated to full-dimensional, closed and bounded convex sets $K_i \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ with corresponding support functions ϕ_i, for all $i = 1, \ldots, n$. Then the mixed degree $\boldsymbol{D}_1 \cdots \boldsymbol{D}_m$ is related to the mixed surface area measure $S_{n-1}(K_1, \ldots, K_{n-1}, \cdot)$ by the formula*

$$\boldsymbol{D}_1 \cdots \boldsymbol{D}_n = (n-1)! \int_{\mathbb{S}^{n-1}} \phi_1(u) S(K_2, \ldots, K_n, u).$$

Moreover, by the symmetry of the mixed surface area measure, we have integral formulae

$$\boldsymbol{D}_1 \cdots \boldsymbol{D}_n = (n-1)! \int_{\mathbb{S}^{n-1}} \phi_i(u) S(K_1, \ldots, \widehat{K}_i, \ldots, K_n, u)$$

for all $i = 1, \ldots, n$.

Remark 4.10. Assuming some smoothness conditions on the support functions of the convex sets, one can compute integrals with respect to (mixed) surface area measure measures explicitly in terms of Lebesgue measures of determinants of Hessians of smooth functions (see [15, Corollary 2.5.3] and the results in [15, Section 5.3]).

5. Volumes and intersection numbers

Let $K_1 \subseteq K_2$ be two full-dimensional, closed and bounded convex sets in $M_{\mathbb{R}}$.

The goal of this section is to relate the correction terms of Definition 3.11 with intersection numbers of toric b-divisors in the case that K_2 is a polytope.

Note that two related exposed faces $F_1 \sim F_2$ with $F_1 \leqslant K_1$ and $F_2 \leqslant K_2$ are contained in parallel hyperplane sections (defined by the v given in the proof of Theorem 3.9). The following is a key Lemma.

Lemma 5.1. *Let F_1, $F_2 \subseteq \mathbb{R}^{d+1}$ be polytopes. Here, $d = \max\{\dim(F_1), \dim(F_2)\}$. Assume that $F_1 \subseteq \{x_{d+1} = 0\}$ and that $F_2 \subseteq \{x_{d+1} = 1\}$. Then the volume of the convex hull of F_1, F_2 is given by*

$$\operatorname{vol}(\operatorname{convhull}(F_1, F_2)) = \frac{1}{d+1} \sum_{i=0}^{d} \operatorname{MV}\Bigg(\underbrace{F_1, \ldots, F_1}_{i\text{-times}}, \underbrace{F_2, \ldots, F_2}_{(d-i)\text{-times}}\Bigg).$$

Proof. We start with the following three claims:
Claim 1: Let λ be any real number between 0 and 1. Then the slice of the convex hull of F_1 and F_2 at $x_{d+1} = \lambda$ is given by

$$\operatorname{convhull}(F_1, F_2) \cap \{x_{d+1} = \lambda\} = \lambda F_1 + (1-\lambda) F_2,$$

where the sum in the right hand side is the Minkowski sum of convex sets.
Claim 2: Let λ be any real number between 0 and 1. Then it follows that the volume of the slice $\lambda F_1 + (1-\lambda)F_2 \subseteq \operatorname{convhull}(F_1, F_2)$ is given by

$$\operatorname{vol}(\lambda F_1 + (1-\lambda)F_2) = \sum_{i=0}^{d} \binom{d}{i} \lambda^i (1-\lambda)^{d-i} \operatorname{MV}\Bigg(\underbrace{F_1, \ldots, F_1}_{i\text{-times}}, \underbrace{F_2, \ldots, F_2}_{(d-i)\text{-times}}\Bigg).$$

Claim 3: Let λ be as before and let ℓ, k be two non-negative integers with $0 \leqslant k \leqslant \ell$. We define the number $\mathrm{I}(\ell, k)$ by

$$\mathrm{I}(\ell, k) \coloneqq \int_0^1 \lambda^k (1-\lambda)^{\ell-k} \, \mathrm{d}\,\lambda.$$

If $k < \ell$, then the formula

$$\mathrm{I}(\ell, k) = \left((\ell+1)\binom{\ell}{k}\right)^{-1}$$

holds true.

Now, Claim 1 is clear and Claim 2 is a standard result in convex geometry. We proceed to give a proof of Claim 3: integrating by parts, we get

$$\begin{aligned}
\mathrm{I}(\ell, k) &= \int_0^1 \lambda^k (1-\lambda)^{\ell-k} \,\mathrm{d}\,\lambda \\
&= \frac{\lambda^{k+1}(1-\lambda)^{\ell-k}}{k+1}\Big|_0^1 + \int_0^1 \frac{\lambda^{k+1}}{k+1}(\ell-k)(1-\lambda)^{\ell-k-1} \,\mathrm{d}\,\lambda \\
&= \frac{\ell-k}{k+1}\ \mathrm{I}(\ell, k+1).
\end{aligned}$$

Moreover the values for $k = \ell$ and for $k = 0$ are given by

$$\begin{aligned}
\mathrm{I}(\ell, \ell) &= \int_0^1 \lambda^\ell \,\mathrm{d}\,\lambda = \frac{\lambda^{\ell+1}}{\ell+1}\Big|_0^1 = \frac{1}{\ell+1}, \\
\mathrm{I}(\ell, 0) &= \int_0^1 (1-\lambda)^\ell \,\mathrm{d}\,\lambda = \frac{-(1-\lambda)^{\ell+1}}{\ell+1}\Big|_0^1 = \frac{1}{\ell+1}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\mathrm{I}(\ell, \ell-1) &= \frac{1}{\ell} \cdot \frac{1}{\ell+1}, \\
\mathrm{I}(\ell, \ell-2) &= \frac{2}{\ell-1} \cdot \frac{1}{\ell} \cdot \frac{1}{\ell+1}, \\
&\vdots \\
\mathrm{I}(\ell, k) &= \left((\ell+1)\binom{\ell}{k}\right)^{-1},
\end{aligned}$$

as we wanted to show.

Finally, note that Claim 1, Claim 2 and Claim 3 imply that

$$\begin{aligned}
\mathrm{vol}\left(\mathrm{convhull}\left(F_1, F_2\right)\right) &= \int_0^1 \mathrm{vol}\left(\lambda F_1 + (1-\lambda) F_2\right) \mathrm{d}\,\lambda \\
&= \int_0^1 \sum_{i=0}^{d} \binom{d}{i} \lambda^i (1-\lambda)^{d-i}\ \mathrm{MV}\left(\underbrace{F_1, \dots, F_1}_{i\text{-times}}, \underbrace{F_2, \dots, F_2}_{(d-i)\text{-times}}\right) \mathrm{d}\,\lambda \\
&= \sum_{i=0}^{d} \binom{d}{i}\ \mathrm{I}(d, i)\ \mathrm{MV}\left(F_1, \dots, F_1, F_2, \dots, F_2\right) \\
&= \frac{1}{d+1} \sum_{i=0}^{d} \mathrm{MV}\left(F_1, \dots, F_1, F_2, \dots, F_2\right),
\end{aligned}$$

concluding the proof of the lemma. □

Consider the convex sets $K_1 \subseteq K_2$ with corresponding support functions ϕ_1, ϕ_2. Moreover, in the case where both K_1 and K_2 are polytopes, let $\Sigma = \Sigma_{K_2 \smallsetminus \mathrm{int}(K_1)}$ be a difference fan as in Definition 3.3.

We have the following theorem.

Theorem 5.2. *Let notations be as above and assume that K_2 is a polytope. Then the functions ϕ_1 and ϕ_2 correspond respectively to a nef toric b-divisor $\boldsymbol{D}_1$ and to a true nef toric divisor D_2 on the toric variety determined by the normal fan of K_2. Moreover, we can express the difference of degrees $D_2^n - \boldsymbol{D}_1^n$ as a finite sum of correction terms*

$$D_2^n - \boldsymbol{D}_1^n = \sum_{F \leqslant K_2} c_F,$$

where the correction terms c_F are given by

$$c_F = \sum_{i=0}^{n-1} (n-1)! \int_{\mathrm{relint}(\sigma_F)\cap \mathbb{S}^{n-1}} (\phi_1(u) - \phi_2(u))\, S(\underbrace{K_1, \dots, K_1}_{i\text{-times}}, \underbrace{K_2, \dots, K_2}_{(n-1-i)\text{-times}}, u),$$

where $S(\cdot)$ is the mixed surface area measure defined in the previous section. In particular, if K_1 is also polyhedral, then Σ is a real rational, polyhedral fan and we get

$$c_F = \sum_{i=0}^{n-1} \sum_{\substack{r \in relint(\sigma_F) \\ r \in \Sigma(1)}} (\phi_1(r) - \phi_2(r))\, \boldsymbol{D}_1^{n-1-i} D_2^i\ D_r,$$

where D_r is the divisor corresponding to the ray $r \in \Sigma(1)$ and all the intersection products are done in Σ.

Proof. The first and last statement of the theorem follow from Theorem 3.9 and Definition 3.11. For the statement regarding the expression of the correction terms, using Corollary 4.9, we have

$$\begin{aligned} &\phi_2^n - \phi_1^n \\ &= (\phi_2 - \phi_1) \sum_{i=0}^{n-1} \phi_1^i \phi_2^{n-1-i} \\ &= \sum_{\substack{\text{exposed} \\ F \ \leqslant \ K_2}} \sum_{i=0}^{n-1} (n-1)! \int_{\mathrm{relint}(\sigma_F)\cap \mathbb{S}^{n-1}} (\phi_1(u) - \phi_2(u)) S(\underbrace{K_1, \dots, K_1}_{i\text{-times}}, \underbrace{K_2, \dots, K_2}_{(n-1-i)\text{-times}}, u). \end{aligned}$$

This concludes the proof of the theorem. □

Example 5.3. Consider the fan of $\mathbb{P}^2$ and the nef toric b-divisors ϕ_1 and ϕ_2 given by the concave functions $\phi_1, \phi_2 \colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\phi_1(a,b) = \begin{cases} \frac{ab}{a+b}\,, & a, b \in \mathbb{R}_{\geqslant 0}, \\ \min\{0, a, b\}\,, & \text{otherwise,} \end{cases}$$

and

$$\phi_2(a,b) = \min\{0, a, b\}$$

and consider the corresponding convex sets $K_1 \subseteq K_2$. Note that K_2 is the 2-dimensional symplex and K_1 is the convex set from Example 2.19. The only face of the simplex K_2 whose associated correction term is non-zero is the vertex F_0 in Figure 6.

On the one hand, we can calculate the difference $\phi_2^2 - \phi_1^2$ as the difference of volumes of convex sets

$$c_{F_0} = \text{ full correction term } = \phi_2^2 - \phi_1^2 = 2\ \mathrm{vol}\,(K_2) - 2\ \mathrm{vol}\,(K_1) = 1 - \frac{2}{3} = \frac{1}{3}.$$

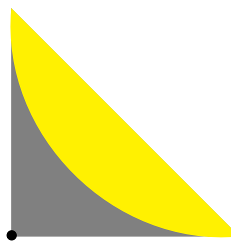

Figure 6. Convex sets $K_1 \subseteq K_2$

On the other hand, Theorem 5.2 tells us that we can compute the correction term c_{F_0} by

$$c_{F_0} = \int_0^{\pi/2} \phi_1(\theta) S_1 (K_1, \theta) = \frac{1}{3},$$

where the last equality follows from a computation using Remark 4.10.

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