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Abstract In a recent article, Duval, Goeckner, Klivans and Martin disproved the longstanding conjecture by Stanley, that every Cohen–Macaulay simplicial complex is partitionable. We construct counterexamples to this conjecture that are even *balanced*, i.e. their underlying graph has a minimal coloring. This answers a question by Duval et al. in the negative.

1. Introduction

Cohen–Macaulay simplicial complexes are among the best studied classes of simplicial complexes in topological combinatorics and combinatorial commutative algebra and they have been proven to be extremely useful for various problems in these areas. The most prominent such example is probably provided by Stanley's proof of the Upper Bound Conjecture for spheres, which also marks the birth of Cohen–Macaulay complexes [11]. Though the original definition by Stanley is algebraic, Reisner [9] – using results of Hochster [7] – could show that Cohen–Macaulayness is a purely topological property. In particular, all triangulations of balls and spheres are known to be Cohen–Macaulay. One of the longstanding conjectures concerning this class of simplicial complexes is the so-called *Partitionability Conjecture* by Stanley [13, p. 14] (for all Cohen–Macaulay simplicial complexes) and Garsia [4] (for all for order complexes of Cohen–Macaulay posets), stating that every Cohen–Macaulay simplicial complex is partitionable. An affirmative answer to this conjecture would also have provided a combinatorial interpretation of the h-vectors of Cohen–Macaulay complexes. However, in 2016 Duval, Goeckner, Klivans and Martin [3] provided an infinite family of non-partitionable Cohen–Macaulay simplicial complexes – together with a general construction method for such counterexamples – and thereby disproved Stanley's conjecture. Even though we now know that the Partitionability Conjecture is false in full generality, one could still hope for a more restricted version to be true. In particular, Duval et al. suggested the following question, which is the main focus of this article [3, Question 4.2].

Question 1.1. *Is every balanced Cohen–Macaulay simplicial complex partitionable?*

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We recall that a $(d-1)$-dimensional simplicial complex is called *balanced* if its underlying graph is d-colorable (in the graph-theoretic sense). Balanced simplicial complexes were introduced by Stanley [13] and they comprise Coxeter complexes, Tits buildings and also barycentric subdivisions of regular CW complexes. Hence, a counterexample to the Partitionability Conjecture for order complexes of Cohen–Macaulay posets [4], would also answer Question 1.1 in the negative.

In this article, we construct an infinite family of balanced non-partitionable Cohen–Macaulay complexes. For this aim we start with the counterexample from [3] and *remove* the obvious obstructions to balancedness via edge subdivisions. A priori the subdivided complex might be partitionable, but we prove in Theorem 3.5 that for carefully chosen subdivisions non-partitionability is preserved. Since our counterexample is not only Cohen–Macaulay but constructible, it is the first example of a balanced constructible non-partitionable simplicial complex [5, § 4].

2. Background on simplicial complexes

We recall basics on (relative) simplicial complexes, including some of their combinatorial and algebraic properties. We refer to [2] and [14] for more details.

Given a finite set V, an (abstract) *simplicial complex* Δ on vertex set V is a collection of subsets of V that is closed under inclusion. We write $V(\Delta)$ for the vertex set of a simplicial complex Δ. Throughout this article, all simplicial complexes are assumed to be finite. Elements of Δ are called *faces* of Δ and inclusion-maximal faces are called *facets* of Δ. The *dimension* of a face $F \in \Delta$ is its cardinality minus one, and the *dimension* of Δ is defined as $\dim \Delta \coloneqq \max\{\dim F \ : \ F \in \Delta\}$. 0-dimensional and 1-dimensional faces are called *vertices* and *edges*, respectively. A simplicial complex Δ is *pure* if all its facets have the same dimension. The *link* of a face $F \in \Delta$ is the subcomplex

$$\mathrm{lk}_\Delta(F) = \{G \in \Delta \ : \ G \cap F = \varnothing, \ G \cup F \in \Delta\}.$$

A subcomplex $\Gamma \subseteq \Delta$ is *induced* if for any $F \subseteq V(\Gamma)$ with $F \in \Delta$, it holds that $F \in \Gamma$. We write $\Gamma = \Delta_{V(\Gamma)}$ in this case. A *relative simplicial complex* is a pair (Δ, Γ) of simplicial complexes, where $\Gamma \subseteq \Delta$ is a subcomplex of Δ. Elements of $\Delta \smallsetminus \Gamma$ are called *faces* of (Δ, Γ) and the *dimension* of (Δ, Γ) is the maximal dimension of a face in $\Delta \smallsetminus \Gamma$. Other notions from arbitrary simplicial complexes carry over to relative simplicial complexes in exactly the same way. Let (Δ, Γ) be a relative simplicial complex, let $\Omega = \Delta \smallsetminus \Gamma$ and $\overline{\Omega} = \{F \ : \ F \subseteq G \text{ for some } G \in \Omega\}$ be the so-called *combinatorial closure* of Ω. $(\overline{\Omega}, \overline{\Omega} \smallsetminus \Omega)$ is called the *minimal representation* of (Δ, Γ) for the obvious reason. We will make use of this representation of relative simplicial complexes in the construction of our counterexample.

The *f-vector* of a $(d-1)$-dimensional (relative) simplicial complex Δ is $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{d-1}(\Delta))$, where $f_i(\Delta)$ denotes the number of i-dimensional faces of Δ. The *h-vector* $h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$ of Δ is defined by the relation

$$\sum_{i=0}^{d} f_{i-1}(\Delta)(t-1)^{d-i} = \sum_{i=0}^{d} h_i(\Delta) t^{d-i}.$$

We turn to several combinatorial and algebraic properties of simplicial complexes that will be of importance for this article.

A $(d-1)$-dimensional simplicial complex Δ is called *balanced* if its underlying graph is d-colorable, that is, there exists a map $\kappa : \ V(\Delta) \to [d] = \{1, \ldots, d\}$ such that $\kappa(v) \neq \kappa(w)$ if $\{v, w\} \in \Delta$. Balanced simplicial complexes were originally introduced by Stanley [13] and prominent examples of such simplicial complexes are provided by barycentric subdivisions, Coxeter complexes and Tits buildings.

Definition 2.1 ([1, 8]). *A pure (relative) simplicial complex* Δ *with facets* $F_1, \dots, F_n$ *is called* partitionable *if there exists a partitioning of* Δ *into pairwise disjoint Boolean intervals*

$$\Delta = \dot{\bigcup_{i=1}^{n}} [R_i, F_i],$$

where $[R_i, F_i] = \{G \in \Delta \ : \ R_i \subseteq G \subseteq F_i\}$.

It was shown by Stanley [14, Proposition III.2.3] that the h-vector of a partitionable simplicial complex Δ has the following combinatorial interpretation:

$$h_i(\Delta) = \#\{1 \leqslant j \leqslant n \ : \ \#R_j = i\}. \qquad (1)$$

In particular, all h-vector entries are non-negative in this case.

We define three classes of simplicial complexes that share the same set of h-vectors [12, Theorem 6]: shellable, constructible and Cohen–Macaulay complexes.

A pure simplicial complex Δ is *shellable* if there exists an ordering $F_1, \dots, F_n$ of the facets of Δ such that for each $1 \leqslant i \leqslant n$ there exists a unique minimal element R_i in

$$\{G \subseteq F_i \ : \ G \nsubseteq F_j \text{ for } 1 \leqslant j \leqslant i-1\}.$$

Since, obviously, $\dot{\bigcup}_{i=1}^{n} [R_i, F_i]$ is a partitioning of Δ, the h-vector of a shellable simplicial complex can be computed using (1).

A $(d-1)$-dimensional simplicial complex Δ is *constructible* if Δ is a simplex or $\Delta = \Delta_1 \cup \Delta_2$, where Δ_1 and Δ_2 are constructible $(d-1)$-dimensional simplicial complexes and $\Delta_1 \cap \Delta_2$ is constructible of dimension $d-2$.

In the following, let $\mathbb{F}$ be an arbitrary field. A $(d-1)$-dimensional simplicial complex Δ is *Cohen–Macaulay* over $\mathbb{F}$ if and only if, for every face $F \in \Delta$ (including the empty face), $\tilde{\beta}_i(\mathrm{lk}_\Delta(F); \mathbb{F}) = 0$ for all $i \neq d-1-\#F$ (see [14, Corollary II.4.2]). Here, we use $\tilde{\beta}_i(\Gamma; \mathbb{F}) := \dim_{\mathbb{F}} \widetilde{H}_i(\Gamma; \mathbb{F})$ to denote the dimension of the i^{th} reduced homology group of a simplicial complex Γ over $\mathbb{F}$.

The following implications between the just described classes of simplicial complexes are well-known:

$$\begin{array}{ccccc} \text{shellable} & \Rightarrow & \text{constructible} & \Rightarrow & \text{Cohen–Macaulay} \\ \Downarrow & & & & \\ \text{partitionable.} & & & & \end{array}$$

Examples of constructible 3-balls that are not shellable show that the left implication is strict (see e.g. [10, 15]). As any triangulation of the dunce hat is Cohen–Macaulay but not constructible [6, § 2], also the right implication is strict. Finally Björner exhibited a partitionable simplicial complex that is not Cohen–Macaulay and hence neither constructible nor shellable [14, p. 85].

Example 2.2. The construction in [3] starts with a particular subcomplex Q of Ziegler's famous example of a non-shellable 3-ball on 10 vertices, labeled $0, \dots, 9$ [15]. More precisely, the subcomplex Q is the combinatorial closure of the following set of facets

$$\begin{aligned} \mathcal{F}(Q) = \{&\{1,2,4,9\}, \{1,2,6,9\}, \{1,5,6,9\}, \{1,5,8,9\}, \{1,4,8,9\}, \\ &\{1,4,5,8\}, \{1,4,5,7\}, \{4,5,7,8\}, \{1,2,5,6\}, \{0,1,2,5\}, \\ &\{0,2,5,6\}, \{0,1,2,3\}, \{1,2,3,4\}, \{1,3,4,7\}\}. \end{aligned}$$

Let $A = Q_{\{0,2,3,4,6,7,8\}}$ be the induced subcomplex of Q on vertex set $\{0,2,3,4,6,7,8\}$, i.e. A is the combinatorial closure of

$$\{\{0,2,6\}, \{0,2,3\}, \{2,3,4\}, \{3,4,7\}, \{4,7,8\}\}.$$

The complexes A and Q, which will be the starting point of the construction in the next section, are depicted in Figure 1. Note that Q is not balanced and that $\tau = (0\,7)(2\,4)(6\,8)$ is an automorphism of Q.

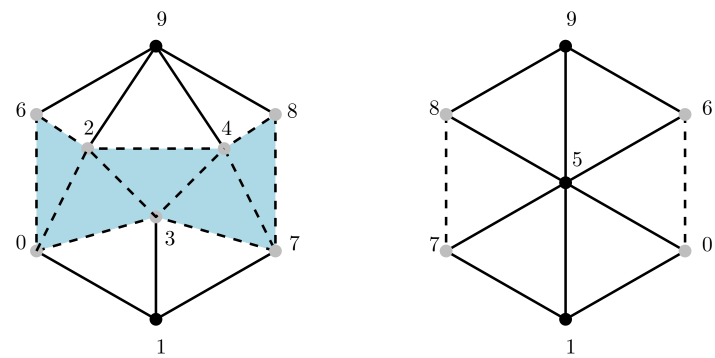

Figure 1. Front (*left*) and back (*right*) view of Q. The blue and dashed faces belong to A.

We end this section with the following main tool from [3], which is also crucial for the balanced construction.

Theorem 2.3 ([3, Theorem 3.1]). *Let $X = (Q, A)$ be a relative simplicial complex such that:*

(1) *Q and A are Cohen–Macaulay;*
(2) *A is an induced subcomplex of Q of codimension at most 1;*
(3) *X is not partitionable.*

For a positive integer N let C_N denote the simplicial complex obtained by identifying N disjoint copies of Q along A. If N is larger than the total number of faces of A, then C_N is Cohen–Macaulay and not partitionable.

3. The balanced construction

Our counterexample makes use of the following basic observation.

Lemma 3.1. *Let $d-1 \geqslant 2$ and let Δ be a pure balanced $(d-1)$-dimensional simplicial complex. Then all vertex links of Δ are balanced.*

Proof. The claim follows immediately from the facts that Δ is balanced, $\dim \mathrm{lk}_\Delta(\{v\}) = d-2$ for any vertex $v \in V(\Delta)$ and that $\kappa(v) \neq \kappa(w)$ for all $v \in V(\Delta)$, $w \in \mathrm{lk}_\Delta(\{v\})$ and any proper coloring $\kappa : V(\Delta) \to \{1, \ldots, d\}$. □

Note that the converse of the previous lemma is not necessarily true. In the following, we call a vertex v (or its vertex link $\mathrm{lk}_\Delta(\{v\})$) *critical* if $\mathrm{lk}_\Delta(\{v\})$ is not balanced and *uncritical* otherwise.

Before proceeding to our construction, we describe its underlying idea. If we look at the simplicial complex Q from Example 2.2, we easily see that vertices $0, 3, 7$ are uncritical, whereas all the other vertices are critical. Hence, by the previous lemma, those are obvious obstructions that prevent Q from being balanced. The idea now is to perform some (possibly few) edge subdivisions that make the critical vertex links balanced without affecting balancedness of other uncritical vertex links and without altering the symmetry of the simplicial complex Q. Luckily, – though this is not guaranteed by Lemma 3.1 – it will turn out, that the simplicial complex obtained in this way, is already balanced. We now make this idea more precise. We perform the following subdivision steps:

Step 1. We first subdivide the edge $\{2,4\}$ by introducing a new vertex 10. In this way, the link of the former critical vertex 9 becomes the cone over a 6-gon and as such is balanced. Figure 2 (left and middle) shows the link of the vertex 9 before and after the subdivision. Moreover, the permutation $\tau = (0\,7)(2\,4)(6\,8)$ is still an automorphism of the subdivided complex (see Example 2.2).

Step 2. In the next step, we subdivide the edge $\{5,9\}$ by adding a vertex 11. It is easy to check that the vertices 6 and 8 are now uncritical and that τ is still an automorphism of this new complex. This step is depicted in Figure 2 (middle and right). On the right the vertices are properly colored. Namely, 1 is colored red, 2, 4 and 11 are colored yellow and 6, 8 and 10 are colored green.

Step 3. Subdividing the edges $\{0,6\}$ and $\{7,8\}$, the vertices 2 and 4, respectively become uncritical. Moreover, also 5 has an uncritical link now. Labeling the new vertex on the edge $\{0,6\}$ with 12 and the one on $\{7,8\}$ with 13, we also see that the permutation $\tau' = (0\,7)(2\,4)(6\,8)(12\,13)$ is an automorphism of the subdivided complex.

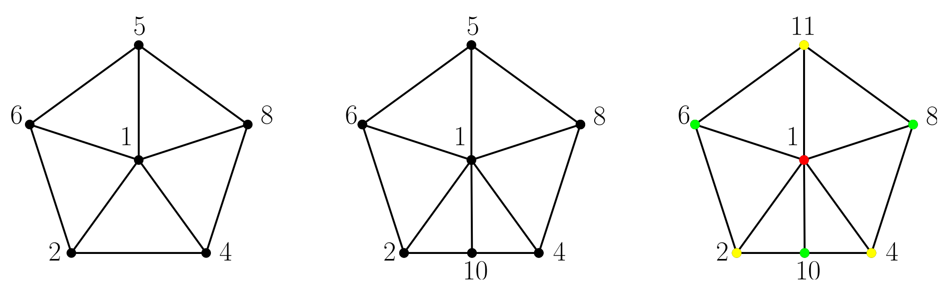

Figure 2. The link of 9 before (*left*) and after the edge subdivision of $\{2,4\}$ (*middle*) and $\{5,9\}$ (*right*).

We call the simplicial complex obtained from the just described edge subdivisions Q^*. The top row of Figure 3 depicts the front and back view of Q^*. It is easy to check that − though we did not treat the critical vertex 1 separately − the simplicial complex Q^* has only uncritical vertices. We even have the following:

Lemma 3.2. *The simplicial complex Q^* constructed above is balanced.*

Proof. The bottom row of Figure 3 shows the 3-dimensional simplicial complex Q^* together with the proper 4-coloring obtained by partitioning the vertices as

$$\kappa^{-1}(\{1\}) = \{0,6,7,8,10\}, \qquad \kappa^{-1}(\{2\}) = \{1,12,13\},$$
$$\kappa^{-1}(\{3\}) = \{2,4,11\}, \qquad \kappa^{-1}(\{4\}) = \{3,5,9\}. \qquad \square$$

Another reasonable approach to construct a balanced counterexample to the partitionability conjecture could have been to start with a balanced non-shellable ball and to try to apply the technique from [3]. However, all known examples of balanced non-shellable balls are relatively big and it is hard to decide which subcomplex to choose. We also want to remark that applying the same edge subdivisions as above directly to Ziegler's non-shellable ball does not produce a balanced ball.

The following simple remark will be useful later.

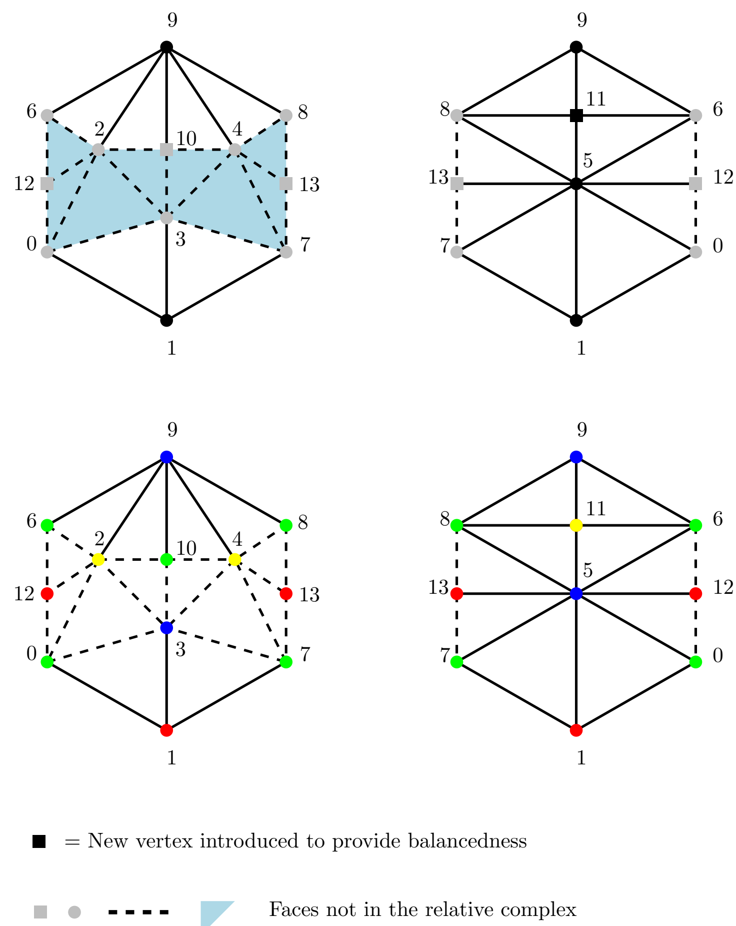

Figure 3. Front and back view of Δ.

Remark 3.3. If Δ is a balanced simplicial complex, then any simplicial complex built from Δ by taking a certain number of copies of Δ and identifying them along a fixed subcomplex is balanced.

We define A^* of Q^* to be the induced subcomplex $Q^*_{\{0,2,3,4,6,7,8,10,12,13\}}$. Note that $\dim A^* = 2$ and that A^* can be obtained from A in the same way we constructed Q^* from Q; namely, by subdividing the edges $\{2,4\}$, $\{0,6\}$ and $\{7,8\}$. (We do not subdivide $\{5,9\}$ since it is not present in A.) As edge subdivisions preserve shellability, we get the following lemma:

Lemma 3.4. *The simplicial complexes Q^* and A^* are shellable, hence constructible and Cohen–Macaulay.*

Our final goal is to apply Theorem 2.3 to the relative simplicial complex (Q^*, A^*). The only ingredient missing to be able to do so is to verify that condition (3) of Theorem 2.3 is fulfilled. Indeed, we have the following statement:

Theorem 3.5. *The relative simplicial complex $X := (Q^*, A^*)$ is not partitionable.*

Proof. The proof uses similar ideas as the ones employed in the proof of [3, Theorem 3.3], even though there are some more cases to consider due to the new vertices introduced with the subdivision. Assume by contradiction that X is partitionable. We will show that, in this case, the vertex 5 has to be contained in at least two intervals of any partitioning, which gives a contradiction.

For the sake of clarity we list the facets of both Q^* and A^*.

$$\begin{aligned}\mathcal{F}(Q^*) = \big\{&\{1,2,9,10\},\{1,4,9,10\},\{1,2,6,9\},\{1,5,6,11\},\{1,6,9,11\},\{1,5,8,11\},\\ &\{1,8,9,11\},\{1,4,8,9\},\{1,4,5,8\},\{1,4,5,7\},\{4,5,7,13\},\{4,5,8,13\},\\ &\{1,2,5,6\},\{0,1,2,5\},\{0,2,5,12\},\{2,5,6,12\},\{0,1,2,3\},\{1,2,3,10\},\\ &\{1,3,4,10\},\{1,3,4,7\}\big\}\\ \mathcal{F}(A^*) = \big\{&\{0,2,3\},\{4,7,13\},\{3,4,10\},\{0,2,12\},\{3,4,7\},\{2,3,10\},\{2,6,12\},\\ &\{4,8,13\}\big\}\end{aligned}$$

Given a partitioning $\mathcal{P}$ of X and a facet $F \in \mathcal{F}(Q^*)$, we denote by I_F the interval of $\mathcal{P}$ with top element F.

As $\{1,4,8,9\}$ is the only facet containing the triangle $\{4,8,9\}$, we have $\{4,8,9\} \in I_{\{1,4,8,9\}}$. If also $\{1,4,8\} \in I_{\{1,4,8,9\}}$, it follows that $I_{\{1,4,8,9\}}$ must contain $\{4,8\} = \{4,8,9\} \cap \{1,4,8\}$, which is a contradiction since $\{4,8\} \in A^*$. Therefore, $\{1,4,8\} \notin I_{\{1,4,8,9\}}$ and since $\{1,4,5,8\}$ is the only other facet of Q^* containing $\{1,4,8\}$, we conclude that $\{1,4,8\} \in I_{\{1,4,5,8\}}$. Again, as $\{4,8\} \in A^*$ and $\{4,8\} = \{1,4,8\} \cap \{4,5,8\}$, it must hold that $\{4,5,8\} \notin I_{\{1,4,5,8\}}$ and hence also $\{4,5\} \notin I_{\{1,4,5,8\}}$. The other facets of X containing $\{4,5\}$ are

$$\{4,5,8,13\},\{4,5,7,13\},\{1,4,5,7\}. \tag{2}$$

Using that τ' is an automorphism of Q^* and A^*, the same line of arguments applied to $\{2,6,9\}$ yields that the edge $\{2,5\}$ has to be contained in an interval with one of the following top elements:

$$\{2,5,6,12\},\{0,2,5,12\},\{0,1,2,5\}.$$

We now distinguish four cases:

Case 1. $\{4,5\} \in I_{\{4,5,8,13\}}$ and $\{2,5\} \in I_{\{2,5,6,12\}}$
As $\{4,5,8,13\}$ is the only facet containing $\{5,8,13\}$, we must have $\{5,8,13\} \in \{4,5,8,13\}$. As $\{5\} = \{4,5\} \cap \{5,8,13\}$ we infer that that $\{5\} \in I_{\{4,5,8,13\}}$. Similarly, using again that τ' is an automorphism of Q^* and A^*, we get that $\{5\} \in I_{\{2,5,6,12\}}$. Hence $\{5\}$ is contained in two intervals, which is a contradiction.

Case 2. $\{4,5\} \notin I_{\{4,5,8,13\}}$ and $\{2,5\} \notin I_{\{2,5,6,12\}}$
As $\{4,5\} \notin I_{\{4,5,8,13\}}$ it follows from (2) that $\{4,5\} \in I_{\{4,5,7,13\}}$ or $\{4,5\} \in I_{\{1,4,5,7\}}$. Since $\{4,5,7,13\}$ and $\{1,4,5,7\}$ are also the only facets of Q^* containing $\{4,5,7\}$ and since $\{4,5\},\{5,7\} \subseteq \{4,5,7\}$, it follows that they have to lie in the same interval, together with $\{5\} = \{4,5\} \cap \{5,7\}$. Therefore, we either have

$$\{5\} \in I_{\{1,4,5,7\}} \qquad \text{or} \qquad \{5\} \in I_{\{4,5,7,13\}}. \tag{3}$$

Applying the automorphism τ' to the above argument yields

$$\{5\} \in I_{\{0,1,2,5\}} \qquad \text{or} \qquad \{5\} \in I_{\{0,2,5,12\}}.$$

Hence, $\{5\}$ belongs to two intervals, which is a contradiction.

Case 3. $\{4,5\} \notin I_{\{4,5,8,13\}}$ and $\{2,5\} \in I_{\{2,5,6,12\}}$

As $\{4,5\} \notin I_{\{4,5,8,13\}}$, the argument of Case 2 shows that (3) holds. We now show that $\{5\}$ has to lie in a second interval.

Note that the only two facets containing $\{5,12\}$ are $\{2,5,6,12\}$ and $\{0,2,5,12\}$. Since both of these contain $\{2,5,12\}$ and $\{5,12\} \subseteq \{2,5,12\}$, it follows that $\{5,12\}$ and $\{2,5,12\}$ have to belong to the same interval. Moreover, since $\{2,5\} \in I_{\{2,5,6,12\}}$

by assumption, we must have $\{2,5,12\} \in I_{\{2,5,6,12\}}$ and hence $\{5,12\} \in I_{\{2,5,6,12\}}$. Finally, this implies $\{5\} = \{2,5\} \cap \{5,12\} \in I_{\{2,5,6,12\}}$ and therefore again $\{5\}$ lies in two intervals, which is a contradiction.

Case 4. $\{4,5\} \in I_{\{4,5,8,13\}}$ and $\{2,5\} \notin I_{\{2,5,6,12\}}$
We reach a contradiction in this case by applying the automorphism τ' to the arguments of Case 3. This finishes the proof. □

The (relative) simplicial complexes Q^*, A^* and $X = (Q^*, A^*)$ have the following f-vectors:

$$f(Q^*) = (1, 14, 45, 52, 20), \quad f(A^*) = (1, 10, 17, 8), \quad f(X) = (0, 4, 28, 44, 20).$$

In particular, the subcomplex A^* has a total number of 36 faces. Theorem 2.3, Theorem 3.5, Lemma 3.4 and Remark 3.3 therefore imply our main result:

Theorem 3.6. *The simplicial complex C_{37} constructed from 37 disjoint copies of Q^* and identifying them along A^* is balanced, Cohen–Macaulay and not partitionable.*

Analogous to the situation in [3, Theorem 3.5] we note that a much smaller counterexample to the balanced partitionability conjecture can be found by glueing together only 3 copies of Q^*.

Theorem 3.7. *The simplicial complex C_3 obtained by taking 3 disjoint copies of Q^* and identifying them along A^* is balanced, Cohen–Macaulay and not partitionable.*

We omit the proof of the above theorem since it is verbatim the same as the one of Theorem 3.5 in [3], if one exchanges the automorphism τ by τ'.

The f-vector of the simplicial complex C_3 is $f(C_3) = (1, 22, 101, 140, 60)$.

Remark 3.8. We do not know if C_3 is the smallest balanced simplicial complex that is Cohen–Macaulay but not partitionable. However, it is easy to see, e.g. by solving a certain integer linear program, that C_2 is partitionable. Moreover, it is not possible to change Q into a balanced simplicial complex by fewer than 4 edge subdivisions. On the other hand, if one finds a counterexample to the partitionability conjecture, which is smaller than the one of [3], then it might well be the case that one can also construct a counterexample to the balanced partitionability conjecture that is smaller than C_3.

Remark 3.9. As Q^* and A^* are both constructible by Lemma 3.4, it follows by definition that C_3 is also constructible. The simplicial complex C_3 therefore is the first balanced counterexample to the conjecture that every constructible simplicial complex is partitionable [5, § 4].

Remark 3.10. The conjecture by [4] still remains open, since we do not know if any of the balanced counterexamples is the order complex of a Cohen–Macaulay poset. By solving the associated linear program we checked that the barycentric subdivision of the simplicial complex C_3 from [3] is partitionable.

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