



ALGEBRAIC COMBINATORICS


Dylan C. Rupel

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Volume 2, issue 6 (2019), p. 1239-1273.

[<http://alco.centre-mersenne.org/item/ALCO_2019__2_6_1239_0>](http://alco.centre-mersenne.org/item/ALCO_2019__2_6_1239_0)

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ABSTRACT We study polynomial generalizations of the Kontsevich automorphisms acting on the skew-field of formal rational expressions in two non-commuting variables. Our main result is the Laurentness and pseudo-positivity of iterations of these automorphisms. The resulting expressions are described combinatorially using a generalization (studied in [10]) of the combinatorics of compatible pairs in a maximal Dyck path developed by Lee, Li, and Zelevinsky in [8].

By specializing to quasi-commuting variables we obtain pseudo-positive expressions for rank 2 quantum generalized cluster variables. In the case that all internal exchange coefficients are zero, this quantum specialization provides a positive combinatorial construction of counting polynomials for Grassmannians of submodules in exceptional representations of valued quivers with two vertices.

Let \mathbb{k} be any field of characteristic zero. Write $\mathbb{K} = \mathbb{k}\langle X, Y \rangle$ for the skew-field of rational functions in non-commuting variables X and Y . Intuitively, writing $\pi : \mathbb{k}\langle X, Y \rangle \rightarrow \mathbb{k}\langle x, y \rangle$ for the commutative specialization, we may formally invert any element $W \in \mathbb{K}$ for which $\pi(W) \neq 0$; this idea has been made precise in [16] by considering iterated localizations of the free algebra $\mathbb{k}\langle X, Y \rangle$.

For any nonzero polynomial $P \in \mathbb{k}[z]$, consider the following \mathbb{k} -linear automorphism of \mathbb{K} :

$$F_P : \begin{cases} X \mapsto XYX^{-1} \\ Y \mapsto P(Y)X^{-1}. \end{cases}$$

We remark for later use that the element $Q := XYX^{-1}Y^{-1}$ is fixed by F_P for any nonzero polynomial P . Also note that F_P^{-1} is given by $X \mapsto P(X)Y^{-1}$ and $Y \mapsto YXY^{-1}$.

Fix nonzero monic polynomials $P_1, P_2 \in \mathbb{k}[z]$ such that $P_1(0) = 1 = P_2(0)$, say

$$P_i(z) = p_{i,0} + p_{i,1}z + \cdots + p_{i,d_i-1}z^{d_i-1} + p_{i,d_i}z^{d_i}$$

with $p_{i,0} = p_{i,d_i} = 1$ for $i = 1, 2$. Set $\mathbb{A}_+ = \mathbb{Z}_{\geq 0}[p_{1,i}, p_{2,j} : 0 \leq i \leq d_1, 0 \leq j \leq d_2]$ and call this the *pseudo-positive semiring* associated to P_1 and P_2 .

We will write $\bar{P}_1(z) := z^{d_1}P_1(z^{-1})$ and $\bar{P}_2(z) := z^{d_2}P_2(z^{-1})$ for the polynomials obtained from P_1 and P_2 by reversing the order of the coefficients. Note that these are

Manuscript received 28th July 2017, revised 24th February 2019, accepted 22nd April 2019.

KEYWORDS. non-commutative cluster, Kontsevich automorphism, maximal Dyck path, quiver Grassmannian.

again polynomials of the same form. For notational convenience, for $k \in \mathbb{Z}$ we define

$$(1) \quad P_k = p_{k,0} + p_{k,1}z + \cdots + p_{k,d_k-1}z^{d_k-1} + p_{k,d_k}z^{d_k} = \begin{cases} \bar{P}_2 & \text{if } k \equiv 0 \pmod{4}; \\ P_1 & \text{if } k \equiv 1 \pmod{4}; \\ P_2 & \text{if } k \equiv 2 \pmod{4}; \\ \bar{P}_1 & \text{if } k \equiv 3 \pmod{4}. \end{cases}$$

Here we use the notation $d_k := d_1$ if k is odd and $d_k := d_2$ if k is even.

Set $X_0 := X$ and $Y_0 := Y$.

MAIN THEOREM. For any $m \geq 0$, the elements $X_m, Y_m \in \mathbb{K}$ given by

$$(2) \quad X_m := F_{P_1}F_{P_2} \cdots F_{P_m}(X) \quad \text{and} \quad Y_m := F_{P_1}F_{P_2} \cdots F_{P_m}(Y)$$

are contained in the semiring $\mathbb{A}_+\langle X^{\pm 1}, Y^{\pm 1} \rangle \subset \mathbb{K}$ of pseudo-positive non-commutative Laurent polynomials.

REMARK. When P_1 and P_2 are monic and of the same degree but $P_1(0) = P_2(0) \neq 1$, this result also holds and can be deduced from the Main Theorem by passing to an appropriate algebraic extension of \mathbb{K} , then rescaling all variables. The same is true when the coefficients $p_{1,0}, p_{1,d_1}, p_{2,0}, p_{2,d_2} \neq 0$ are arbitrary but satisfy a balancing condition which we leave as an exercise for the reader to work out. In the absence of such a balancing condition the definitions of the polynomials P_k should be adjusted according to the exchange polynomial mutation rules developed by Chekhov and Shapiro [4]. Also, since $F_P(X) = QY$ for any polynomial P , we have $X_{m+1} = QY_m$ for $m \geq 0$; in particular, the claim for the X_m follows from the claim for the Y_m .

REMARK. If $d_1d_2 \leq 3$, the Main Theorem can be observed quite explicitly by computing X_1, X_2, \dots, X_m by hand for

$$m = \begin{cases} 4 & \text{if } d_1d_2 = 0; \\ 5 & \text{if } d_1d_2 = 1; \\ 6 & \text{if } d_1d_2 = 2; \\ 8 & \text{if } d_1d_2 = 3; \end{cases}$$

and observing in each case that these are given by pseudo-positive non-commutative Laurent polynomials with $X_m = QXQ^{-1}$. The combinatorics below can be adapted to these cases, however in everything that follows we assume $d_1d_2 \geq 4$ as such cases may be treated more uniformly.

For the following example, observe that the Y_m for $m \geq 2$ may alternatively be computed via the following non-commutative analogue of generalized cluster exchange relations:

$$(3) \quad Y_m Q Y_{m-2} = 1 + p_{m,1}Y_{m-1} + \cdots + p_{m,d_m-1}Y_{m-1}^{d_m-1} + Y_{m-1}^{d_m}.$$

EXAMPLE. Let $P_1 = 1 + p_{1,1}z + p_{1,2}z^2 + z^3$ and $P_2 = 1 + p_{2,1}z + z^2$. Then the first few non-commutative generalized cluster variables Y_m are given by:

$$Y_1 = (1 + p_{1,1}Y + p_{1,2}Y^2 + Y^3)X^{-1}, \quad Y_2 = (1 + p_{2,1}Y_1 + Y_1^2)Y^{-1}Q^{-1},$$

$$Y_3 = (1 + p_{1,2}Y_2 + p_{1,1}Y_2^2 + Y_2^3)Y_1^{-1}Q^{-1}.$$

While Y_2 is manifestly an element of $\mathbb{A}_+\langle X^{\pm 1}, Y^{\pm 1} \rangle$, a highly nontrivial cancellation must occur in the expansion of Y_3 in order for it to be a pseudo-positive non-commutative Laurent polynomial. Such cancellations indeed occur and we obtain the

expansion

$$\begin{aligned}
 Y_3 = & \left(XY^{-3} + p_{1,2}(p_{2,1} + Y_1)Y^{-1} + p_{1,1}(p_{2,1} + Y_1)Y^{-2} \right. \\
 & + p_{1,1}(1 + p_{2,1}Y_1 + Y_1^2)XY^{-1}X^{-1}(p_{2,1} + Y_1)Y^{-1} \\
 & + (p_{2,1} + Y_1)Y^{-3} + (1 + p_{2,1}Y_1 + Y_1^2)XY^{-1}X^{-1}(p_{2,1} + Y_1)Y^{-2} \\
 & \left. + (1 + p_{2,1}Y_1 + Y_1^2)XY^{-1}X^{-1}(1 + p_{2,1}Y_1 + Y_1^2)XY^{-1}X^{-1}(p_{2,1} + Y_1)Y^{-1} \right) Q^{-1}.
 \end{aligned}$$

The automorphisms F_{P_k} are generalizations of automorphisms of \mathbb{K} introduced by Kontsevich [7] which are recovered in the *binomial case* when $p_{1,i} = 0 = p_{2,j}$ for $1 \leq i \leq d_1 - 1$ and $1 \leq j \leq d_2 - 1$. In this binomial case, Kontsevich conjectured the Laurentness and positivity of the *non-commutative cluster variables* X_m and Y_m . This terminology is justified by specializing to commutative variables through which we recover the initial cluster mutations in the rank two cluster algebra [6] associated to the exchange matrix $\begin{bmatrix} 0 & d_2 \\ -d_1 & 0 \end{bmatrix}$ after composing with the transposition of initial cluster variables. In the binomial case, Laurentness was established by Usnich [15] when $d_1 = d_2 = 2$, and by Berenstein and Retakh [2] for arbitrary polynomial degrees. Positivity in the binomial case was proven by Di Francesco and Kedem [5] when $d_1 d_2 = 4$, by Lee and Schiffler [9] for $d_1 = d_2$, and by the author [12] for arbitrary polynomial degrees.

The Laurentness of X_m and Y_m was established by Usnich [14] in the special case where $P_k = P_1$ for all $k \in \mathbb{Z}$. We will prove the Main Theorem by providing a combinatorial construction of the elements Y_m , called *non-commutative generalized cluster variables*. This combinatorics was studied by the author [10] to construct greedy bases for (commutative) rank two generalized cluster algebras by building upon the notion of compatible pairs in a maximal Dyck path developed by Lee, Li, and Zelevinsky [8] for constructing greedy bases of rank two cluster algebras.

For $\mathbf{a} = (a_1, a_2) \in \mathbb{Z}_{\geq 0}^2$, let $D := D_{\mathbf{a}}$ denote the lattice path in the rectangle $[0, a_1] \times [0, a_2]$ which begins at $(0, 0)$ takes unit length East and North steps to end at (a_1, a_2) and is maximal among all such *Dyck paths* that never pass above the main diagonal of the rectangle $[0, a_1] \times [0, a_2]$. In other words, no lattice point of D lies strictly above the main diagonal and any lattice point which lies strictly above D also lies strictly above the main diagonal. Label the edges of D as $E = \{1, \dots, a_1 + a_2\}$, where this bijection of ordered sets respects the natural order on edges from $(0, 0)$ to (a_1, a_2) . There is a partition $E = H \sqcup V$, where H (resp. V) denotes the set of horizontal (resp. vertical) edges of D .

For edges $e, e' \in E$, we write ee' for the subpath of D beginning with e traveling North-East and ending with e' . By convention, this path will be empty if e is to the North-East of e' , while the path ee contains the single edge e . Let $\bar{e}e'$ (resp. $e\bar{e}'$) denote the path obtained from ee' by removing the edge e (resp. e'). Write $(ee')_H$ (resp. $(ee')_V$) for the set of horizontal (resp. vertical) edges in the path ee' . We abbreviate $|ee'|_H := |(ee')_H|$ and $|ee'|_V := |(ee')_V|$.

REMARK 1.1. In [8] and [10], the definition for subpaths ee' of D includes a “wrap-around” condition whereby ee' is non-empty for $e' < e$, however following [10, Remark 2.21] such a condition will not be necessary in our situation and all relevant results quoted from [10] will be modified accordingly.

DEFINITION 1.2. A grading $\omega : E \rightarrow \mathbb{Z}_{\geq 0}$ (on the edges) of D is called compatible if: for every $h \in H$ and $v \in V$, there exists an edge e along the path hv so that at least

one of the following holds:

$$\begin{aligned}
 \text{(HGC)} \quad & e \neq v \quad \text{and} \quad |he|_V = \sum_{h' \in (he)_H} \omega(h'); \\
 \text{(VGC)} \quad & e \neq h \quad \text{and} \quad |ev|_H = \sum_{v' \in (ev)_V} \omega(v').
 \end{aligned}$$

Recall that $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ denote the degrees of the exchange polynomials P_1 and P_2 respectively. We say that a grading ω of D is (d_1, d_2) -bounded if $\omega(h) \leq d_1$ for all $h \in H$ and $\omega(v) \leq d_2$ for all $v \in V$. For the remainder of the paper we will restrict to such bounded gradings ω , though we continue to write $\omega : E \rightarrow \mathbb{Z}_{\geq 0}$ throughout. This notion of compatible gradings was introduced in [10] building upon the notion of compatible subsets of E developed in [8] which can be recovered when $\omega(h) \in \{0, d_1\}$ for $h \in H$ and $\omega(v) \in \{0, d_2\}$ for $v \in V$.

For a (d_1, d_2) -bounded grading ω , we associate the non-commutative monomial $\text{wt}_\omega(e)$ to each edge $e \in E$ as follows:

$$(4) \quad \text{wt}_\omega(e) = \begin{cases} p_{1, \omega(e)} Y^{\omega(e)} X^{-1} & \text{if } e \in H; \\ p_{2, d_2 - \omega(e)} X^{\omega(e) + 1} Y^{-1} X^{-1} & \text{if } e \in V. \end{cases}$$

Thus we may associate a non-commutative Laurent monomial to each (d_1, d_2) -bounded grading ω by taking the product of the associated non-commutative weights in the natural order along the path D :

$$(5) \quad Y_D(\omega) := \text{wt}_\omega(1) \text{wt}_\omega(2) \cdots \text{wt}_\omega(a_1 + a_2).$$

Define $Y_D := \sum_{\omega} Y_D(\omega)$, where the sum ranges over all (d_1, d_2) -bounded compatible gradings ω of D .

We will mainly be interested in the maximal Dyck paths $D_m := D_{\mathbf{a}_m}$ for integer vectors $\mathbf{a}_m \in \mathbb{Z}^2$, $m \geq 1$, defined recursively by

$$(6) \quad \mathbf{a}_0 = (0, -1), \quad \mathbf{a}_1 = (1, 0), \quad \mathbf{a}_{m-1} + \mathbf{a}_{m+1} = \begin{cases} d_2 \mathbf{a}_m & \text{if } m \text{ is odd;} \\ d_1 \mathbf{a}_m & \text{if } m \text{ is even.} \end{cases}$$

These vectors are precisely the *almost positive roots* in the root system associated to the Cartan matrix $\begin{bmatrix} 2 & -d_2 \\ -d_1 & 2 \end{bmatrix}$ which describe the denominator vectors of cluster variables. The Main Theorem is an immediate consequence of the following combinatorial construction of the non-commutative generalized cluster variables Y_m .

THEOREM 1.3. *For $m \geq 1$, we have $Y_{D_m} = Y_m$.*

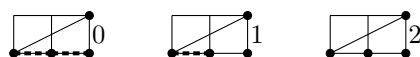
EXAMPLE 1.4. We continue the example from above with $P_1 = 1 + p_{1,1}z + p_{1,2}z^2 + z^3$ and $P_2 = 1 + p_{2,1}z + z^2$.

For $m = 1$, we get $\mathbf{a}_1 = (1, 0)$ so that $D_1 = \bullet \dashrightarrow \bullet$. This maximal Dyck path consists of a single horizontal edge which may be assigned any of the weights $0, 1, 2, 3$, a situation which we denote by the dashed edge $\bullet \dashrightarrow \bullet$. Summing the monomial contributions coming from (4) for each choice of weight, we get

$$Y_{D_1} = X^{-1} + p_{1,1} Y X^{-1} + p_{1,2} Y^2 X^{-1} + Y^3 X^{-1} = Y_1$$

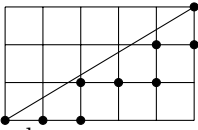
and this same equality holds for any dashed edge below.

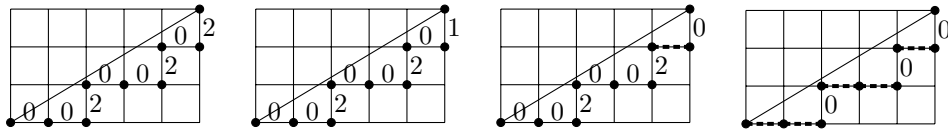
For $m = 2$, we get $\mathbf{a}_2 = (2, 1)$ so that $D_2 = \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$. In this case, the compatible weightings of the edges in D_2 are given by



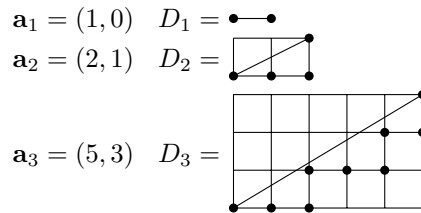
where we again use a dashed line to indicate that a horizontal edge may be assigned any of the weights 0, 1, 2, 3 without affecting compatibility. Summing the monomial contributions coming from (4) for each choice of weight, we get

$$Y_{D_2} = Y_1^2 X Y^{-1} X^{-1} + Y_1 X^{-1} p_{2,1} X^2 Y^{-1} X^{-1} + X^{-1} X^{-1} X^3 Y^{-1} X^{-1} = Y_2.$$

For $m = 3$, we get $\mathbf{a}_3 = (5, 3)$ so that $D_3 =$ . In this case, the compatible weightings of the edges in D_3 are given by



To describe the non-commutative generalized cluster variables Y_1, Y_2, Y_3 above as pseudo-positive non-commutative Laurent polynomials we consider the following roots and corresponding maximal Dyck paths:



Our proof of Theorem 1.3 requires a careful understanding of the recursive structure of the maximal Dyck paths D_m which we will establish in the next section. In Section 3, we further develop the combinatorics of compatible gradings of D_m introduced in [10]. The main aim there is to understand gradings which behave nicely with respect to the recursive structure developed in Section 2. These results produce nicely factorizable summands of Y_{D_m} , facilitating an inductive proof of Theorem 1.3. Section 4 puts these combinatorial results together to establish Theorem 1.3. We finish with Section 5 discussing the specialization from non-commutative variables to quasi-commuting variables. A main goal of this section is proving Corollary 5.7 which gives a positive combinatorial construction of counting polynomials for Grassmannians of subrepresentations in rigid indecomposable valued quiver representations.

NOTATION. We adopt the following notational conventions throughout the paper.

- For integers $a < b$, set $[a, b] = \{a, a + 1, \dots, b\}$.
- Given any quantity α defined using the tuple (d_1, d_2) or the pair of polynomials (P_1, P_2) , let α' denote the same quantity defined using the tuple $(d'_1, d'_2) = (d_0, d_1)$ or the polynomials $(P'_1, P'_2) = (P_0, P_1)$. In particular, $p'_{1,j} = p_{2,d_2-j}$ and $p'_{2,j} = p_{1,j}$ when equation (4) is applied to a (d'_1, d'_2) -bounded grading ω' on $D_{\mathbf{a}'_m}$.
- Equations that will be referenced globally will be assigned numbers, those that are referenced only locally (i.e. within a single proof) will be assigned symbols (e.g. \dagger or \ddagger). In particular, symbols labeling equations will be reused but this should not lead to any confusion.

2. MAXIMAL DYCK PATHS

In this section we study the recursive structure present in the maximal Dyck paths D_m . To accomplish this, we note that the vectors \mathbf{a}_m can be written more explicitly in terms of two-parameter Chebyshev polynomials $u_{m,k}$ ($m, k \in \mathbb{Z}$) defined recursively by:

$$(7) \quad u_{0,k} = 0, \quad u_{1,k} = 1, \quad u_{m+1,k+1} = d_k u_{m,k} - u_{m-1,k-1},$$

where d_k denotes the degree of the polynomial P_k in equation (1). Then, for $m \geq 1$, we have $\mathbf{a}_m = (u_{m,1}, u_{m-1,2})$. Write $\mathbf{a}'_m = (u'_{m,1}, u'_{m-1,2}) = (u_{m,2}, u_{m-1,1})$ and set $D'_m = D_{\mathbf{a}'_m}$ for $m \geq 1$.

REMARK 2.1. To see the equivalence with equation (6), one must use the identities $u_{m,k} = u_{m,k+1}$ for m odd and $d_k u_{m,k} = d_{k+1} u_{m,k+1}$ for m even.

We record the next simple observation for future use.

LEMMA 2.2. *For positive integers d_1, d_2 and any integers m, k , we have $u_{m,k+1} u_{m-2,k} < u_{m-1,k+1} u_{m-1,k}$.*

Proof. We work by induction on m , the case $m = 2$ is the trivial inequality $0 < 1$. For $m \geq 3$, we have

$$\begin{aligned} u_{m,k+1} u_{m-2,k} &= d_k u_{m-1,k} u_{m-2,k} - u_{m-2,k-1} u_{m-2,k} \\ &< d_k u_{m-1,k} u_{m-2,k} - u_{m-3,k-1} u_{m-1,k} \\ &= u_{m-1,k+1} u_{m-1,k}, \end{aligned}$$

where the inequality uses induction. The case $m \leq 1$ can be handled similarly. \square

In order to establish a recursive structure for D_m we will show that the maximal Dyck paths D_m and D'_m are intimately related.

LEMMA 2.3. *For $m \geq 1$, the following hold.*

- (a) *The maximal Dyck path D'_{m+1} can be obtained from D_m via replacing each horizontal edge, together with the ℓ vertical edges which immediately follow it, by $d_1 - \ell$ horizontal edges followed by a vertical edge.*
- (b) *The maximal Dyck path D_{m+1} can be obtained from D'_m via replacing each horizontal edge, together with the ℓ vertical edges which immediately follow it, by $d_2 - \ell$ horizontal edges followed by a vertical edge.*

Proof. We prove (a) as (b) will immediately follow by interchanging the roles of d_1 and d_2 . Let D' denote the lattice path obtained from D_m as in (a). It follows from the definition that D' will contain $d_1 u_{m,1} - u_{m-1,2} = u_{m+1,2}$ horizontal edges and $u_{m,1}$ vertical edges. We need to show that D' does not cross above the main diagonal and that it is maximal with this property.

Write $v'_1, \dots, v'_{u_{m,1}}$ for the vertical edges of D' and for $1 \leq r \leq u_{m,1}$ suppose v'_r is immediately preceded by exactly ℓ_r horizontal edges of the same height. Suppose there exists t so that v'_t passes above the main diagonal, this is equivalent to the inequality $\frac{t}{\sum_{r=1}^t \ell_r} > \frac{u_{m,1}}{u_{m+1,2}}$. Using the equality $u_{m+1,2} = d_1 u_{m,1} - u_{m-1,2}$, this may be rewritten as

$$(†) \quad \frac{d_1 t - \sum_{r=1}^t \ell_r}{t} > \frac{u_{m-1,2}}{u_{m,1}}.$$

But by construction of D' , we see that $d_1 - \ell_r$ is the number of vertical edges immediately following the r -th horizontal edge of D_m . Thus, by rewriting the numerator as $d_1 t - \sum_{r=1}^t \ell_r = \sum_{r=1}^t (d_1 - \ell_r)$ in the inequality (\dagger), we see that the subpath of D_m containing the first t horizontal edges and the vertical edges immediately following these horizontal edges will cross above the main diagonal of the rectangle $[0, u_{m,1}] \times [0, u_{m-1,2}]$, a contradiction. Thus D' is a Dyck path, i.e. it does not pass above the main diagonal.

To see that D' is maximal, suppose there exists a lattice point (s, t) strictly above D' which does not lie above the main diagonal. Without loss of generality, we may take $s = \sum_{r=1}^t \ell_r - 1$ and get the inequality $\frac{t}{\sum_{r=1}^t \ell_r - 1} \leq \frac{u_{m,1}}{u_{m+1,2}}$. Using the equality $u_{m+1,2} = d_1 u_{m,1} - u_{m-1,2}$, this may be rewritten as

$$(\dagger) \quad \frac{d_1 t - \sum_{r=1}^t \ell_r + 1}{t} \leq \frac{u_{m-1,2}}{u_{m,1}}.$$

Now considering the same initial segment of D_m as above, we see that the point $(t, d_1 t - \sum_{r=1}^t \ell_r + 1)$ lies strictly above D_m , but the inequality (\dagger) implies this point does not lie above the main diagonal of the rectangle $[0, u_{m,1}] \times [0, u_{m-1,2}]$, contradicting the maximality of D_m . Thus we may conclude that $D' = D'_{m+1}$ is the maximal Dyck path in the rectangle $[0, u_{m+1,2}] \times [0, u_{m,1}]$. \square

With this we obtain the recursive structure of D_m . In what follows we always assume $d_1 d_2 \geq 4$ and take $\delta_m = \begin{cases} 1 & \text{if } d_{m-1} = 1 \text{ and } m \neq 3; \\ 0 & \text{if } d_{m-1} \neq 1 \text{ or } m = 3. \end{cases}$

COROLLARY 2.4. *The maximal Dyck paths D_m , $m \geq 1$, admit the following recursive structure:*

- (a) D_1 consists of a single horizontal edge;
- (b) D_2 consists of d_2 horizontal edges followed by a single vertical edge;
- (c) for $m \geq 3$, the Dyck path D_m can be constructed by concatenating $d_m - 1 - \delta_m$ copies of D_{m-1} followed by a copy of D_{m-1} with its first $D_{m-2-\delta_m}$ removed.

For the remainder of the paper we will understand the notation $D_{m-1} \setminus D_{m-2-\delta_m}$ to mean the terminal subpath of D_{m-1} obtained by removing its first copy of $D_{m-2-\delta_m}$ as in Corollary 2.4.

REMARK 2.5. The roles of d_1 and d_2 must be interchanged when applying Corollary 2.4 to D'_m .

Proof. Parts (a) and (b) are immediate from the definitions of D_1 and D_2 . Part (c) with $m = 3$ follows from Lemma 2.3 and part (b) since D'_2 consists of d_1 horizontal edges followed by a vertical edge.

We establish part (c) by induction on $m \geq 4$. Notice that by Remark 2.5 the claimed recursive structures of D_m and D'_{m-1} are the same for $m \geq 5$, thus we obtain the result for D_m if we know the result for D'_{m-1} by applying the construction from Lemma 2.3. Hence it suffices to establish the claimed recursive structure for D_4 .

If $d_3 \neq 1$, then $\delta_4 = 0$ and the structure of D_4 is immediately deduced from Lemma 2.3 and part (c) for D'_3 . If $d_3 = 1$, then D'_2 consists of a single horizontal edge followed by a single vertical edge and D'_3 consists of $d_2 - 1 = d_4 - 1$ copies of D'_2 followed by a vertical edge. Applying Lemma 2.3 to D'_3 shows that D_4 consists of

$d_4 - 2$ copies of D_3 followed by a copy of D_3 with its first horizontal edge (i.e. its first D_1) removed. This establishes part (c) for D_4 and completes the proof. \square

COROLLARY 2.6. *For $m \geq 2$, if the last edge of D_m is omitted, the resulting lattice path identifies with an initial subpath of the maximal Dyck path C_m obtained by concatenating d_m copies of D_{m-1} .*

Proof. We work by induction on m , the case $m = 2$ following immediately from Corollary 2.4(b).

Assume $m \geq 3$. By part (c) of Corollary 2.4, in comparing D_m to C_m the first $d_m - 1 - \delta_m$ copies of D_{m-1} inside D_m may be ignored and the problem reduces to comparing the final $D_{m-1} \setminus D_{m-2-\delta_m}$ subpath of D_m with the maximal Dyck path D_{m-1} . For $m = 3$, removing the final edge of $D_{m-1} \setminus D_{m-2-\delta_m}$ produces $d_2 - 1$ consecutive horizontal edges which clearly identifies with an initial subpath of D_{m-1} . Assume $m \geq 4$. There are two cases to consider.

- If $d_{m-1} \neq 1$, D_{m-1} consists of $d_{m-1} - 1 - \delta_{m-1}$ copies of D_{m-2} followed by a copy of $D_{m-2} \setminus D_{m-3-\delta_{m-1}}$. It follows that comparing $D_{m-1} \setminus D_{m-2}$ with D_{m-1} reduces to comparing $D_{m-2} \setminus D_{m-3-\delta_{m-1}}$ with D_{m-2} . But by induction, we know that we obtain an initial subpath of D_{m-2} by removing the last edge of $D_{m-2} \setminus D_{m-3-\delta_{m-1}}$.
- When $d_{m-1} = 1$, the maximal Dyck path D_{m-1} is just $D_{m-2} \setminus D_{m-3}$. But D_{m-2} consists of $d_m - 2$ copies of D_{m-3} followed by a copy of $D_{m-3} \setminus D_{m-5}$ and so D_{m-1} consists of $d_m - 3$ copies of D_{m-3} followed by a copy of $D_{m-3} \setminus D_{m-5}$. Hence comparing $D_{m-1} \setminus D_{m-3}$ with D_{m-1} reduces to comparing $D_{m-3} \setminus D_{m-5}$ with D_{m-3} , but by induction we know that removing the last edge of $D_{m-3} \setminus D_{m-5}$ produces an initial subpath of D_{m-3} .

The two items above show that we get an initial subpath of D_{m-1} by removing the last edge of $D_{m-1} \setminus D_{m-2-\delta_m}$ and thus removing the last edge of D_m produces an initial subpath of C_m . \square

Let E_m denote the edges of D_m , where $E_m = H_m \sqcup V_m$ for horizontal edges $H_m = \{h_1, \dots, h_{u_{m,1}}\}$ and vertical edges $V_m = \{v_1, \dots, v_{u_{m-1,2}}\}$, both labeled in the natural order along D_m . We may describe the structure of D_m as follows.

LEMMA 2.7 ([10, Lemma 3.2]). *For $m \geq 2$, the following hold.*

- (a) *There are exactly $\text{ht}(h_i) := \lfloor (i-1)u_{m-1,2}/u_{m,1} \rfloor$ vertical edges of D_m preceding the horizontal edge h_i , call this number the height of h_i ;*
- (b) *There are exactly $\text{dp}(v_t) := \lceil tu_{m,1}/u_{m-1,2} \rceil$ horizontal edges of D_m preceding the vertical edge v_t , call this number the depth of v_t .*

In the natural labeling of edges, Lemma 2.7 gives $h_i = i + \lfloor (i-1)u_{m-1,2}/u_{m,1} \rfloor$ for $1 \leq i \leq u_{m,1}$, $m \geq 1$ and $v_t = t + \lceil tu_{m,1}/u_{m-1,2} \rceil$ for $1 \leq t \leq u_{m-1,2}$, $m \geq 2$. In particular, we see that $u_{m-1,2} < u_{m,1}$ implies D_m contains no consecutive vertical edges, while $u_{m,1} < u_{m-1,2}$ implies D_m contains no consecutive horizontal edges.

For the next result, recall that we work under the assumption $d_1 d_2 \geq 4$.

COROLLARY 2.8. *For $m \geq 2$ the following hold.*

- (a) *D_m contains at most $1 + \delta_1$ vertical edges of any given depth.*
- (b) *D_m contains no consecutive horizontal edges if and only if $d_2 = 1$.*

Proof. For D_2 , both claims are immediate from Corollary 2.4(b). There are two possibilities for D_3 . If $d_2 = 1$, the result for D_2 together with Corollary 2.4(c) shows D_3 contains no consecutive horizontal edges and that the vertical edges of D_3 all have different depths except v_{d_1-1} and v_{d_1} , which both have depth $d_1 - 1$.

For $d_2 > 1$, the result for D_2 together with Corollary 2.4(c) shows all vertical edges of D_3 have different depths. To see that $d_2 > 1$ implies there are consecutive horizontal edges in D_3 we need to consider two case. When $d_1 > 1$, D_3 begins with a copy of D_2 by Corollary 2.4(c) and thus contains consecutive horizontal edges. When $d_1 = 1$, we must have $d_2 \geq 4$. But then D_3 is just $D_2 \setminus D_1$ and, since $d_2 \geq 4$, it contains consecutive horizontal edges.

For $m \geq 4$, both claims follow by induction using Corollary 2.4(c). □

Analogous statements hold for D'_m , with horizontal edges $H'_m = \{h'_1, \dots, h'_{u_{m,2}}\}$ and vertical edges $V'_m = \{v'_1, \dots, v'_{u_{m-1,1}}\}$, by interchanging the roles of d_1 and d_2 .

The proof of Theorem 1.3 will go by induction. Towards this aim we introduce notation, following Corollary 2.4, which captures the recursive structure in the edges of D_m , $m \geq 3$.

DEFINITION 2.9. For $m \geq 3$ and $1 \leq r \leq d_m - 1 - \delta_m$, define the following subsets of H_m and V_m :

$$H_{m,r} = \{h_{(r-1)u_{m-1,1}+1}, h_{(r-1)u_{m-1,1}+2}, \dots, h_{ru_{m-1,1}}\};$$

$$V_{m,r} = \{v_{(r-1)u_{m-2,2}+1}, v_{(r-1)u_{m-2,2}+2}, \dots, v_{ru_{m-2,2}}\};$$

we identify these, for each r , with the horizontal and vertical edges of D_{m-1} . Also set

$$H_{m,d_m-\delta_m} = \{h_{(d_m-1-\delta_m)u_{m-1,1}+1}, \dots, h_{u_{m,1}-1}, h_{u_{m,1}}\};$$

$$V_{m,d_m-\delta_m} = \{v_{(d_m-1-\delta_m)u_{m-2,2}+1}, \dots, v_{u_{m-1,2}-1}, v_{u_{m-1,2}}\};$$

we identify these subsets with the horizontal and vertical edges of $D_{m-1} \setminus D_{m-2-\delta_m}$.

As a notational convenience, for $1 \leq r \leq d_m - 1 - \delta_m$ and $1 \leq i \leq u_{m-1,1}$ we write $h_{i,r} := h_{(r-1)u_{m-1,1}+i}$ and similarly $v_{t,r} := v_{(r-1)u_{m-2,2}+t}$ for $1 \leq t \leq u_{m-2,2}$. For $u_{m-2-\delta_m,1} + 1 \leq i \leq u_{m-1,1}$, set $h_{i,d_m-\delta_m} := h_{(d_m-1-\delta_m)u_{m-1,1}+i-u_{m-2-\delta_m,1}}$ and set $v_{t,d_m-\delta_m} := v_{(d_m-1-\delta_m)u_{m-2,2}+t-u_{m-3-\delta_m,2}}$ for $u_{m-3-\delta_m,2} + 1 \leq t \leq u_{m-2,2}$.

3. COMBINATORICS OF COMPATIBLE PAIRS

Let $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ be a (d_1, d_2) -bounded grading of D_m , $m \geq 1$. It will be convenient to write ω_H and ω_V for the restrictions of ω to H_m and to V_m respectively. In the absence of a total grading ω , we refer to the maps $\omega_H : H_m \rightarrow [0, d_1]$ and $\omega_V : V_m \rightarrow [0, d_2]$ respectively as *horizontal gradings* and *vertical gradings* of D_m . We will often consider ω to be the pair (ω_H, ω_V) and refer to ω_H and ω_V as being *compatible* if Definition 1.2 is satisfied for ω . Since the first condition (HGC) of Definition 1.2 only involves ω_H , we refer to it as the *horizontal grading condition*. Similarly, we refer to the second condition (VGC) as the *vertical grading condition*.

Write $\text{supp}(\omega) := \{e \in E_m : \omega(e) \neq 0\}$ and call this the *support* of ω . Set $\text{supp}(\omega_H) = \text{supp}(\omega) \cap H$ and $\text{supp}(\omega_V) = \text{supp}(\omega) \cap V$. Define $|\omega|_H := \sum_{h \in H_m} \omega_H(h)$ and $|\omega|_V := \sum_{v \in V_m} \omega_V(v)$.

3.1. SHADOW STATISTICS. To begin we introduce notation to gain a more delicate grasp of the compatibility conditions (HGC) and (VGC) from Definition 1.2. For a horizontal grading $\omega_H : H_m \rightarrow [0, d_1]$ and any subpath $ee' \subset D_m$, define the *horizontal shadow statistic*

$$f_{\omega_H}(ee') := -|ee'|_V + \sum_{h \in (ee')_H} \omega_H(h).$$

We also define the *vertical shadow statistic*

$$f_{\omega_V}(ee') := -|ee'|_H + \sum_{v \in (ee')_V} \omega_V(v)$$

for each vertical grading $\omega_V : V_m \rightarrow [0, d_2]$. It immediately follows from the definitions that the shadow statistics satisfy the following additivity property with respect to concatenation of paths:

$$(8) \quad f_{\omega_H}(e_1e_3) = f_{\omega_H}(e_1e_2) + f_{\omega_H}(\bar{e}_2e_3) \quad \text{and} \quad f_{\omega_V}(e_1e_3) = f_{\omega_V}(e_1e_2) + f_{\omega_V}(\bar{e}_2e_3)$$

for edges $e_i \in E_m$ with $e_2 \in e_1e_3$.

The shadow statistics give the following alternative check for compatibility.

LEMMA 3.1. *Let $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ be a compatible grading of D_m . For $h \in H_m$ and $v \in V_m$, the following hold:*

- (a) *if $f_{\omega_H}(hv) < 0$, then the horizontal grading condition (HGC) is satisfied for the path hv ;*
- (b) *if $f_{\omega_V}(hv) < 0$, then the vertical grading condition (VGC) is satisfied for the path hv .*

Proof. We prove (a), the proof of (b) is similar.

There is nothing to show when $\omega_H(h) = 0$, so assume $h \in \text{supp}(\omega_H)$. Then $f_{\omega_H}(hh) > 0$ and as e ranges from h to v the value of $f_{\omega_H}(he)$ either increases, stays the same, or decreases by 1 with each step. Since $f_{\omega_H}(hv) < 0$, we see that $f_{\omega_H}(he)$ must eventually take the value 0 with $e \neq v$, i.e. the horizontal grading condition is satisfied for the path hv . \square

Apart from their relationship to the compatibility conditions (HGC) and (VGC), the shadow statistics f_{ω_H} and f_{ω_V} encode the following important information. For each subpath $ee' \subset D_m$, we obtain a factor $Y_{ee'}(\omega_H, \omega_V)$ of the monomial $Y_{D_m}(\omega_H, \omega_V)$ appearing in equation (5) by only multiplying the weights of edges along the path ee' .

LEMMA 3.2. *The quantities $f_{\omega_H}(ee')$ and $f_{\omega_V}(ee')$ record the total Y -degree and the total X -degree respectively of the monomial $Y_{ee'}(\omega_H, \omega_V)$.*

Proof. A horizontal edge $h \in (ee')_H$ contributes a factor of $p_{1, \omega_H(h)} Y^{\omega_H(h)} X^{-1}$ to $Y_{ee'}(\omega_H, \omega_V)$ while a vertical edge $v \in (ee')_V$ contributes a factor of $p_{2, d_2 - \omega_V(v)} X^{\omega_V(v) + 1} Y^{-1} X^{-1}$. The result now follows by comparing the total Y - and X -degrees of $Y_{ee'}(\omega_H, \omega_V)$ with the definitions of $f_{\omega_H}(ee')$ and $f_{\omega_V}(ee')$ respectively. \square

For a horizontal grading $\omega_H : H_m \rightarrow [0, d_1]$, define the *local shadow path* $D(h; \omega_H)$ of a horizontal edge $h \in H_m$ to be the shortest nonempty subpath $he \subset D_m$ such that $f_{\omega_H}(he) = 0$, if there is no such subpath we set $D(h; \omega_H) = hv_{u_{m-1,2}}$. Write $D_H(h; \omega_H) := D(h; \omega_H) \cap H_m$ and $D_V(h; \omega_H) := D(h; \omega_H) \cap V_m$ for the *local horizontal shadow* and *local vertical shadow* of h with respect to ω_H . The local shadow path $D(v; \omega_V)$ is defined similarly for a vertical edge $v \in V_m$ and a vertical grading $\omega_V : V_m \rightarrow [0, d_2]$, where $D(v; \omega_V) = h_1v$ if there is no edge $e \leq v$ for which $f_{\omega_V}(ev) = 0$. The local shadows $D_H(v; \omega_V)$, $D_V(v; \omega_V)$ are defined as above.

By definition we have $f_{\omega_H}(D(h; \omega_H)) = 0$ whenever the final edge of $D(h; \omega_H)$ is not $v_{u_{m-1,2}}$. More importantly, writing $D(h; \omega_H) = he$, Lemma 3.1 together with equation (8) imply that $f_{\omega_H}(he') > 0$ and $f_{\omega_H}(e'e) < 0$ for any proper subpaths $he', e'e \subset D(h; \omega_H)$. Thus we see for $h \in \text{supp}(\omega_H)$ and $v \in D_V(h; \omega_H)$ that the condition (HGC) is not satisfied for the path hv , however for any ω_V compatible with ω_H the condition (VGC) is satisfied for h and v . In particular, when ω_V is compatible with ω_H , $D(v; \omega_V)$ is a proper subpath of $D(h; \omega_H)$ for any $v \in D_V(h; \omega_H)$.

Similar statements hold using the vertical shadow statistic f_{ω_V} .

3.2. RECURSIONS. We introduce in this section a recursive construction of gradings analogous to the recursive operations on Dyck paths from Lemma 2.3.

The *shadow* of a horizontal grading $\omega_H : H_m \rightarrow [0, d_1]$ is the collection of vertical edges in the local vertical shadows of all horizontal edges, i.e. $\text{sh}(\omega_H) = \bigcup_{h \in H_m} D_V(h; \omega_H)$. The *remote shadow* of a horizontal grading $\omega_H : H_m \rightarrow [0, d_1]$ is the subset $\text{rsh}(\omega_H) \subset \text{sh}(\omega_H)$ obtained by excluding for each d the (up to) $\omega_H(h_d)$ vertical edges of depth d immediately following h_d . The shadow and remote shadow of a vertical grading $\omega_V : V_m \rightarrow [0, d_2]$ are defined similarly.

REMARK 3.3. The remote shadow $\text{rsh}(\omega_H) \subset \text{sh}(\omega_H)$ of a horizontal grading ω_H can be described as the subset consisting of those vertical edges $v \in \text{sh}(\omega_H)$ for which there exists a vertical grading ω_V compatible with ω_H such that $\omega_V(v) > 0$. In particular, any vertical grading ω_V compatible with ω_H must satisfy $\omega_V(v) = 0$ for $v \in \text{sh}(\omega_H) \setminus \text{rsh}(\omega_H)$.

In order to give a relationship between gradings of D_m and gradings of D'_{m+1} , we need to partition the remote shadows according to which local shadow contains a given edge.

DEFINITION 3.4. Let $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ be a grading of D_m .

- (a) For $1 \leq j \leq d \leq u_{m,1}$, denote by $\text{rsh}(\omega_H)_{j;d}$ the set of $v \in \text{rsh}(\omega_H)$ of depth d such that $v \in D_V(h_j; \omega_H)$ and h_j is the first horizontal edge before v with this property. Define the local remote shadow of the edge h_j as $\text{rsh}(h_j; \omega_H) :=$

$$\coprod_{d \in [j+1, u_{m,1}]} \text{rsh}(\omega_H)_{j;d}.$$

- (b) For $0 \leq \ell < t \leq u_{m-1,2}$, denote by $\text{rsh}(\omega_V)_{t;\ell}$ the set of $h \in \text{rsh}(\omega_V)$ of height ℓ such that $h \in D_H(v_t; \omega_V)$ and v_t is the first vertical edge after h with this property. Define the local remote shadow of the edge v_t as $\text{rsh}(v_t; \omega_V) :=$

$$\coprod_{\ell \in [0, t-2]} \text{rsh}(\omega_V)_{t;\ell}.$$

REMARK 3.5. By the definition of the remote shadows, it is impossible to have $d = j$ or $\ell = t - 1$ in Definition 3.4.

Lemma 2.3 establishes a canonical order preserving bijection between the vertical edges V'_{m+1} of D'_{m+1} and the horizontal edges H_m of D_m which we write as $\varphi = \varphi_m : V'_{m+1} \rightarrow H_m$, $\varphi(v'_i) = h_i$ for $1 \leq i \leq u_{m,1}$. Thus we obtain a bijection from d_1 -bounded horizontal gradings of D_m to d_1 -bounded vertical gradings of D'_{m+1} taking a horizontal grading $\omega_H : H_m \rightarrow [0, d_1]$ to the vertical grading $\varphi^* \omega_H : V'_{m+1} \rightarrow [0, d_1]$ given by $\varphi^* \omega_H(v'_i) = d_1 - \omega_H(h_i)$.

REMARK 3.6. We will abuse notation slightly and also write φ_m^* for the bijection between horizontal gradings of D'_m and vertical gradings of D_{m+1} where the roles of d_1 and d_2 need to be interchanged in the definitions above, however this abuse should not lead to any confusion.

The next result shows that the remote shadows for ω_H and $\varphi^* \omega_H$ are intimately related.

LEMMA 3.7 ([10, Corollary 4.18]). Let $\omega_H : H_m \rightarrow [0, d_1]$ be a horizontal grading of D_m . For $1 \leq j < d \leq u_{m,1}$, we have $|\text{rsh}(\omega_H)_{j;d}| = |\text{rsh}(\varphi^* \omega_H)_{d;j-1}|$.

Thus for $1 \leq j < d \leq u_{m,1}$ we may define a bijection $\theta_{j;d} : \text{rsh}(\omega_H)_{j;d} \rightarrow \text{rsh}(\varphi^* \omega_H)_{d;j-1}$ which preserves the natural order determined by distance from h_j and from v'_d respectively. More explicitly, as the vertical edges of $\text{rsh}(\omega_H)_{j;d}$ are read from bottom to top the corresponding horizontal edges of $\text{rsh}(\varphi^* \omega_H)_{d;j-1}$ are read from right to left.

For a horizontal grading $\omega_H : H_m \rightarrow [0, d_1]$, write $\mathcal{G}(\omega_H)$ for the collection of all (d_1, d_2) -bounded gradings $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ such that the restriction $\omega|_{H_m}$ is precisely ω_H and denote by $\mathcal{C}(\omega_H) \subset \mathcal{G}(\omega_H)$ the subset of compatible gradings. Let $\mathcal{G}_{\text{rsh}}(\omega_H) \subset \mathcal{G}(\omega_H)$ denote those gradings ω for which $\omega(v) = 0$ whenever $v \in V_m \setminus \text{rsh}(\omega_H)$ and write $\mathcal{C}_{\text{rsh}}(\omega_H) := \mathcal{G}_{\text{rsh}}(\omega_H) \cap \mathcal{C}(\omega_H)$. Define analogous collections of gradings associated to a vertical grading $\omega_V : V_m \rightarrow [0, d_2]$.

Define a map $\Omega = \Omega_m : \mathcal{G}_{\text{rsh}}(\omega_H) \rightarrow \mathcal{G}_{\text{rsh}}(\varphi^*\omega_H)$ as follows:

$$\Omega(\omega_V)(h') = \begin{cases} 0 & \text{if } h' \in H'_{m+1} \setminus \text{rsh}(\varphi^*\omega_H); \\ \omega_V(v) & \text{if } h' = \theta_{j;d}(v) \text{ for } v \in \text{rsh}(\omega_H)_{j;d}. \end{cases}$$

Note that Ω admits an obvious inverse map.

REMARK 3.8. Given a grading $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ of D_m where $\omega_V \notin \mathcal{G}_{\text{rsh}}(\omega_H)$, the map Ω may still be applied to ω_V to produce a horizontal grading in $\mathcal{G}_{\text{rsh}}(\varphi^*\omega_H)$. This observation will be used without mention in the statements of Lemma 3.9 and Proposition 3.16 as well as in the proof of Corollary 3.23.

The following result shows that we have some control over the shadow statistics under this operation. It is also the essential ingredient for understanding the piecewise compatible gradings introduced in the next section.

LEMMA 3.9 ([10, Lemma 4.19]). *Let $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ be a grading on D_m . Suppose $h' = \theta_{j;d}(v)$ for a vertical edge $v \in \text{rsh}(\omega_H)_{j;d} \cap \text{supp}(\omega_V)$. Then $f_{\Omega(\omega_V)}(h'v'_d) = f_{\omega_V}(h_jv)$.*

This crucial result also shows that Ω restricts to a map $\mathcal{C}_{\text{rsh}}(\omega_H) \rightarrow \mathcal{C}_{\text{rsh}}(\varphi^*\omega_H)$, i.e. that the pair $(\Omega(\omega_V), \varphi^*\omega_H)$ gives a compatible grading of D'_{m+1} exactly when $\omega_V \in \mathcal{C}_{\text{rsh}}(\omega_H)$.

PROPOSITION 3.10 ([10, Lemma 4.20]). *Let $\omega_H : H_m \rightarrow [0, d_1]$ be a horizontal grading of D_m . For a vertical grading $\omega_V \in \mathcal{G}_{\text{rsh}}(\omega_H)$, we have $\omega_V \in \mathcal{C}_{\text{rsh}}(\omega_H)$ if and only if $\Omega(\omega_V) \in \mathcal{C}_{\text{rsh}}(\varphi^*\omega_H)$.*

3.3. PIECEWISE COMPATIBILITY. Our goal in this section is to understand which gradings on D_m , $m \geq 3$, are obtained by gluing together compatible gradings on the D_{m-1} subpaths of D_m found in Corollary 2.4(c).

DEFINITION 3.11. *Fix $m \geq 3$. Consider (d_1, d_2) -bounded compatible gradings $\omega_r = (\omega_{H,r}, \omega_{V,r})$ of D_{m-1} for $1 \leq r \leq d_m - \delta_m$. We assume*

$$(9) \quad \omega_{V, d_m - \delta_m}(v) = 0$$

for v in the first $D_{m-2-\delta_m}$ subpath of D_{m-1} and

$$(10) \quad \omega_{H, d_m - \delta_m}(h) = \ell$$

for h in the first $D_{m-2-\delta_m}$ subpath of D_{m-1} if h is immediately followed by exactly ℓ vertical edges inside $D_{m-2-\delta_m}$.

Define a grading $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ of D_m by

$$\omega(e) = \begin{cases} \omega_{H,r}(e) & \text{if } e \in H_{m,r}; \\ \omega_{V,r}(e) & \text{if } e \in V_{m,r}; \end{cases}$$

where we identify subsets of edges in D_m with edges of D_{m-1} as in Definition 2.9. We will refer to any grading on D_m obtained in this way as piecewise compatible.

REMARK 3.12. Every compatible grading of D_m , $m \geq 3$, is piecewise compatible. Given any grading ω of D_m and $1 \leq r \leq d_m - \delta_m$, we will denote by $\omega_r = (\omega_{H,r}, \omega_{V,r})$ the grading of D_{m-1} obtained by restricting ω to the r -th copy of D_{m-1} inside D_m , where $\omega_{d_m - \delta_m} = (\omega_{H, d_m - \delta_m}, \omega_{V, d_m - \delta_m})$ denotes the grading on D_{m-1} satisfying the conditions (9) and (10) of Definition 3.11.

REMARK 3.13. When considering piecewise compatible gradings $\omega : E'_m \rightarrow \mathbb{Z}_{\geq 0}$ of D'_m we will instead make the following assumptions on the gradings $\omega_{H',d'_m-\delta'_m}$ and $\omega_{V',d'_m-\delta'_m}$ of D'_{m-1} :

$$\omega_{H',d'_m-\delta'_m}(h') = 0$$

for h' in the first $D'_{m-2-\delta'_m}$ subpath of D'_{m-1} and

$$\omega_{V',d'_m-\delta'_m}(v') = d$$

for v' in the first $D'_{m-2-\delta'_m}$ subpath of D'_{m-1} if v' is immediately preceded by exactly d horizontal edges inside $D'_{m-2-\delta'_m}$.

The next result shows that only the final edge of D_m needs to be considered in order to verify (global) compatibility of a piecewise compatible grading.

LEMMA 3.14. *Let $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ be a piecewise compatible grading on D_m , $m \geq 3$. Then one of the compatibility conditions (HGC) or (VGC) is satisfied for every $h \in H_m$ and every $v \in V_m \setminus \{v_{u_{m-1,2}}\}$. In particular, a piecewise compatible grading on D_m is compatible if and only if one of the compatibility conditions (HGC) or (VGC) holds for all paths $hv_{u_{m-1,2}}$ with $h \in \bigsqcup_{r=1}^{d_m-1-\delta_m} H_{m,r}$.*

Proof. Following [10, Remark 2.22], we have a principle of non-interaction between adjacent D_{m-1} subpaths of D_m . More precisely, one of the compatibility conditions (HGC) or (VGC) will always be satisfied for paths hv with $h \in H_{m,r}$ and $v \in V_{m,s}$ for $1 \leq r < s \leq d_m - 1 - \delta_m$. Since each pair $(\omega_{H,r}, \omega_{V,r})$ is compatible, it only remains to verify a compatibility condition for $h \in \bigsqcup_{r=1}^{d_m-1-\delta_m} H_{m,r}$ and $v \in V_{m,d_m-\delta_m}$.

By Corollary 2.6, we may again apply [10, Remark 2.22] to see that one of the compatibility conditions will always be satisfied for all $v \in V_{m,d_m-\delta_m}$ with $v \neq v_{u_{m-1,2}}$. Thus a compatibility conditions only needs to be verified for paths $hv_{u_{m-1,2}}$ with $h \in \bigsqcup_{r=1}^{d_m-1-\delta_m} H_{m,r}$ to verify compatibility of ω . □

COROLLARY 3.15. *When $d_m = 1$, every piecewise compatible grading of D_m , $m \geq 3$, is compatible.*

Proof. When $d_m = 1$, the set of horizontal edges $\bigsqcup_{r=1}^{d_m-1-\delta_m} H_{m,r}$ is empty and thus the compatibility condition of Lemma 3.14 is trivially satisfied. □

Next we observe that piecewise compatible gradings are well-behaved under the operations φ^* and Ω introduced in Section 3.2.

PROPOSITION 3.16. *Let $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ be a (d_1, d_2) -bounded grading on D_m for $m \geq 3$. Then ω is piecewise compatible if and only if $(\Omega(\omega_V), \varphi^*\omega_H)$ is piecewise compatible.*

Proof. We prove the forward implication, the other direction can be obtained by reversing the argument.

Assume ω is piecewise compatible and, for $1 \leq r \leq d_m - \delta_m$, consider $h' \in H'_{m+1,r}$ and $v'_t \in V'_{m+1,r}$ with $h' < v'_t$. If $h' \notin D(v'_t; \varphi^*\omega_H)$, then the vertical grading condition (VGC) is satisfied for the path $h'v'_t$. So we assume $h' \in D(v'_t; \varphi^*\omega_H)$ and need to show that the horizontal grading condition (HGC) is satisfied for the path $h'v'_t$.

For $\Omega(\omega_V)(h') = 0$, there is nothing to check so assume $h' \in \text{supp}(\Omega(\omega_V))$. Set $j - 1 = \text{ht}(h')$ so that $h' \in \text{rsh}(\varphi^*\omega_H)_{d;j-1}$ with $j < d \leq t$. Then $\Omega(\omega_V)(h') = \omega_V(v)$, where $v \in \text{rsh}(\omega_H)_{j;d} \cap \text{supp}(\omega_V)$ with $h' = \theta_{j;d}(v)$.

Note that $h_j \in H_{m,r}$ and $v \in V_{m,r}$, thus by piecewise compatibility the vertical grading condition (VGC) is satisfied for the path h_jv . That is, there exists $e \in \bar{h}_jv$ so that $f_{\omega_V}(ev) = 0$. By piecewise compatibility, each vertical edge in $h_j\bar{e}$ also satisfies the vertical grading condition with h_j . It follows that $f_{\omega_V}(h_jv) < 0$.

By Lemma 3.9, we thus have $f_{\Omega(\omega_V)}(h'v'_d) < 0$ and so the horizontal grading condition is satisfied for the path $h'v'_d$ by Lemma 3.1. Since $h'v'_d$ is an initial subpath of $h'v'_t$, the horizontal grading condition is also satisfied for the path $h'v'_t$. Since h' and v'_t were arbitrary, we see that $(\Omega(\omega_V), \varphi^*\omega_H)$ is piecewise compatible. \square

We aim now to understand precisely when compatibility fails for a piecewise compatible grading. The definition below provides the necessary conditions for a piecewise compatible grading ω constructed as in Definition 3.11 to be incompatible.

DEFINITION 3.17. *Let $\omega_H : H_m \rightarrow [0, d_1]$ be a horizontal grading on D_m , $m \geq 3$. We say a horizontal edge $h \in H_m$ is blocking for ω_H if the following hold:*

- $h \in H_m \setminus H_{m,d_m-\delta_m}$;
- $D(h; \omega_H) = hv_{u_{m-1,2}}$;
- h is the maximal (i.e. furthest to the right) horizontal edge with these properties.

We call ω_H left-justified at a blocking edge $h_i \in H_m$ if there exists $k \geq i$ so that $\omega_H(h_j) > 0$ for $i \leq j \leq k$ and $\omega_H(h_j) = 0$ for $j > k$. Such a horizontal grading is strongly left-justified at h_i if in addition the following hold:

- $\omega_H(h_j) = d_1$ for $i \leq j < k$;
- $f_{\omega_H}(h_iv_{u_{m-1,2}}) = 0$.

Let $\omega_V : V_m \rightarrow [0, d_2]$ be a vertical grading on D_m , $m \geq 3$. For a horizontal edge $h_i \in H_m$, ω_V is called right-justified with respect to h_i if there is a vertical edge $v_s \in h_iv_{u_{m-1,2}}$ so that $\omega_V(v_t) > 0$ for $s \leq t \leq u_{m-1,2}$ and $\omega_V(v_t) = 0$ for all vertical edges $v_t \in (h_i\bar{v}_s)_V$. Such a vertical grading is strongly right-justified with respect to h_i if in addition the following hold:

- $\omega_V(v_t) = d_2$ for $s < t \leq u_{m-1,2}$;
- $D(v_{u_{m-1,2}}; \omega_V) = h_iv_{u_{m-1,2}}$ with $f_{\omega_V}(h_iv_{u_{m-1,2}}) = 0$.

PROPOSITION 3.18. *Let $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ be a piecewise compatible grading of D_m , $m \geq 3$, for which ω_H admits the blocking edge $h_i \in H_m$ and ω_V is strongly right-justified with respect to h_i . Then ω_H is left-justified at h_i and $\text{supp}(\omega_H) \cap h_iv_{u_{m-1,2}} = \text{rsh}(\omega_V) \cap h_iv_{u_{m-1,2}}$.*

Proof. Since ω_V is strongly right-justified with respect to h_i , we have $D(v_{u_{m-1,2}}; \omega_V) = h_iv_{u_{m-1,2}}$ and thus $\text{supp}(\omega_H) \cap h_iv_{u_{m-1,2}} \subset \text{rsh}(\omega_V) \cap h_iv_{u_{m-1,2}}$. To see equality of these sets we show that they must have the same cardinality.

As ω_V is strongly right-justified with respect to h_i , we have $\omega_V(v) = d_2$ for each vertical edge $v \in \bar{v}_sv_{u_{m-1,2}}$, where $s = u_{m-1,2} - \lfloor \frac{u_{m,1}-i+1}{d_2} \rfloor$. This implies $\omega_H(h) = 0$ for each horizontal edge $h \in \bar{v}_sv_{u_{m-1,2}}$. Otherwise both grading conditions would fail for the path hv , where v is the first vertical edge after h , a contradiction with piecewise compatibility of ω .

Then observe that the vertical edge v_s has depth $\left\lfloor \frac{(u_{m-1,2} - \lfloor \frac{u_{m,1}-i+1}{d_2} \rfloor)u_{m,1}}{u_{m-1,2}} \right\rfloor$ by Lemma 2.7. Using once more that ω_V is strongly right-justified with respect to h_i ,

we must also have $\omega_H(h) = 0$ for each of the $u_{m,1} - i + 1 - d_2 \lfloor \frac{u_{m,1}-i+1}{d_2} \rfloor$ horizontal edges h immediately preceding v_s in order for piecewise compatibility to hold.

The above discussion has shown that $\omega_H(h_j) = 0$ whenever j is larger than the following quantity:

$$\begin{aligned} & \left\lceil \frac{\left(u_{m-1,2} - \lfloor \frac{u_{m,1}-i+1}{d_2} \rfloor\right) u_{m,1}}{u_{m-1,2}} \right\rceil - \left(u_{m,1} - i + 1 - d_2 \left\lfloor \frac{u_{m,1} - i + 1}{d_2} \right\rfloor\right) \\ &= i - 1 + \left\lceil \frac{-\lfloor \frac{u_{m,1}-i+1}{d_2} \rfloor u_{m,1}}{u_{m-1,2}} \right\rceil + d_2 \left\lfloor \frac{u_{m,1} - i + 1}{d_2} \right\rfloor \\ &= i - 1 + \left\lceil \frac{\lfloor \frac{u_{m,1}-i+1}{d_2} \rfloor u_{m-2,1}}{u_{m-1,2}} \right\rceil, \end{aligned}$$

where both equalities follow from the identity $\lceil n+x \rceil = n + \lceil x \rceil$ which holds for all real numbers x and all integers n . This discussion also shows that $h_i v_{u_{m-1,2}} \cap (\text{sh}(\omega_V) \setminus \text{rsh}(\omega_V)) = (h_{i+d} v_{u_{m-1,2}})_H$, where $d = \left\lceil \frac{\lfloor \frac{u_{m,1}-i+1}{d_2} \rfloor u_{m-2,1}}{u_{m-1,2}} \right\rceil$. Since $D(v_{u_{m-1,2}}; \omega_V) = h_i v_{u_{m-1,2}}$, it follows that

$$|\text{rsh}(\omega_V) \cap h_i v_{u_{m-1,2}}| = \left\lceil \frac{\lfloor \frac{u_{m,1}-i+1}{d_2} \rfloor u_{m-2,1}}{u_{m-1,2}} \right\rceil.$$

Now observe the inequality

$$\left\lceil \frac{\lfloor \frac{u_{m,1}-i+1}{d_2} \rfloor u_{m-2,1}}{u_{m-1,2}} \right\rceil \leq \left\lceil \frac{(u_{m,1} - i + 1)u_{m-2,1}}{d_2 u_{m-1,2}} \right\rceil = \left\lceil \frac{(u_{m,1} - i + 1)u_{m-2,2}}{d_1 u_{m-1,1}} \right\rceil,$$

where the equality can be deduced from the identities in Remark 2.1. By Lemma 2.2, the last expression above is not larger than

$$(11) \quad \left\lceil \frac{(u_{m,1} - i + 1)u_{m-1,2}}{d_1 u_{m,1}} \right\rceil = \left\lceil \frac{u_{m-1,2} - \lfloor \frac{(i-1)u_{m-1,2}}{u_{m,1}} \rfloor}{d_1} \right\rceil,$$

where the equality follows from right to left using the identities $-\lfloor x \rfloor = \lceil -x \rceil$, $\lceil n + x \rceil = n + \lceil x \rceil$, and $\left\lceil \frac{\lfloor x \rfloor}{n} \right\rceil = \left\lceil \frac{x}{n} \right\rceil$ which hold for all real numbers x and all positive integers n .

But h_i is blocking and ω_H is d_1 -bounded so that

$$|\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}}| \geq \left\lceil \frac{|h_i v_{u_{m-1,2}}|_V}{d_1} \right\rceil = \left\lceil \frac{u_{m-1,2} - \lfloor \frac{(i-1)u_{m-1,2}}{u_{m,1}} \rfloor}{d_1} \right\rceil.$$

Combining this observation with the inequalities leading up to equation (11), we see that

$$\begin{aligned} |\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}}| &\geq \left\lceil \frac{u_{m-1,2} - \left\lfloor \frac{(i-1)u_{m-1,2}}{u_{m,1}} \right\rfloor}{d_1} \right\rceil \\ &\geq \left\lceil \frac{\left\lfloor \frac{u_{m,1}-i+1}{d_2} \right\rfloor u_{m-2,1}}{u_{m-1,2}} \right\rceil = |\text{rsh}(\omega_V) \cap h_i v_{u_{m-1,2}}|. \end{aligned}$$

But either inequality being strict is impossible since $\text{supp}(\omega_H)$ does not intersect $\text{sh}(\omega_V) \setminus \text{rsh}(\omega_V)$. Thus ω_H must be left-justified at h_i with

$$(12) \quad |\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}}| = \left\lceil \frac{u_{m-1,2} - \left\lfloor \frac{(i-1)u_{m-1,2}}{u_{m,1}} \right\rfloor}{d_1} \right\rceil \\ = \left\lceil \frac{\left\lfloor \frac{u_{m,1}-i+1}{d_2} \right\rfloor u_{m-2,1}}{u_{m-1,2}} \right\rceil = |\text{rsh}(\omega_V) \cap h_i v_{u_{m-1,2}}|.$$

In particular, $\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}} = \text{rsh}(\omega_V) \cap h_i v_{u_{m-1,2}}$. □

REMARK 3.19. The middle equality of equation (12) does not hold for all i , this equality is a consequence of the hypotheses and thus provides a necessary condition for the existence of a piecewise compatible grading as in Proposition 3.18.

The next result will show that this condition is also sufficient and that such gradings are the only piecewise compatible gradings which are not compatible.

THEOREM 3.20. *Let $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ be a piecewise compatible grading of D_m , $m \geq 3$.*

- (a) *If ω_H does not admit a blocking edge, then ω is compatible.*
- (b) *Suppose ω_H admits a blocking edge h_i but ω is not compatible. Then the following hold:*
 - (i) *$D(v_{u_{m-1,2}}; \omega_V) = h_i v_{u_{m-1,2}}$ with $f_{\omega_V}(h_i v_{u_{m-1,2}}) = 0$;*
 - (ii) *ω_H is left-justified at h_i and ω_V is strongly right-justified with respect to h_i .*

If in addition $m \geq 4$, the following also hold:

- (iii) *$f_{\omega_H}(h_i v_{u_{m-1,2}}) = 0$;*
- (iv) *ω_H must be strongly left-justified at h_i ;*
- (v) *$\text{ht}(h_{i+1}) = \text{ht}(h_i) + \delta_1$ when $|\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}}| > 1$.*

REMARK 3.21. When $d_m = 1$, the hypotheses of Theorem 3.20 cannot apply by Corollary 3.15.

Proof. If ω_H does not admit a blocking edge, any horizontal edge $h \in H_m$ has a local shadow path of the form $D(h; \omega_H) = he$ with $e < v_{u_{m-1,2}}$, i.e. the horizontal grading condition is satisfied for h and $v_{u_{m-1,2}}$. By Lemma 3.14, this implies ω is compatible. This establishes (a).

From now on we assume ω_H admits a blocking edge h_i and $d_m \neq 1$. There are two possible cases. First consider the case $\text{ht}(h_i) \geq u_{m-1,2} - d_1$. Since by definition h_i is not contained in the final $D_{m-1} \setminus D_{m-2-\delta_m}$ subpath of D_m , we must have $d_1 \geq 2$ and so this case can occur only if one of the following holds:

- $m = 3$ with $d_1 \neq 1$;
- $m = 4$ with $d_1, d_2 \neq 1$;
- $m = 5$ with $d_2 = 1$.

When $m = 4$ or $m = 5$ above, we must have $h_i = h_{u_{m-1,1}, d_{m-1} - \delta_m}$ (see Definition 2.9 for notation).

Let $\ell = \text{ht}(h_i)$. Then, since $d_1 \neq 1$ in all cases above, each vertical edge in $\bar{v}_{\ell+1}\bar{v}_{u_{m-1,2}}$ if $m = 3$ (resp. in $h_{i+1}\bar{v}_{u_{m-1,2}}$ if $m = 4$ or $m = 5$) is immediately preceded by exactly d_2 horizontal edges inside $h_i v_{u_{m-1,2}}$ while $v_{u_{m-1,2}}$ is immediately preceded by exactly $d_2 - 1$ horizontal edges. In particular, we see that the horizontal grading condition fails for the path $h_i v_{u_{m-1,2}}$ exactly when:

- $\omega_V(v_{\ell+1}) = \text{dp}(v_{\ell+1}) - i$ and $\omega_V(v) = d_2$ for $v \in (\bar{v}_{\ell+1}v_{u_{m-1,2}})_V$ for $m = 3$;
- $\omega_V(v) = d_2$ for all $v \in (h_i v_{u_{m-1,2}})_V$ for $m = 4$ or $m = 5$.

In either case we have $D(v_{u_{m-1,2}}; \omega_V) = h_i v_{u_{m-1,2}}$ with $f_{\omega_V}(h_i v_{u_{m-1,2}}) = 0$. Note that in each of the cases above, ω_H is left-justified at h_i with $k = i$ in Definition 3.17 and ω_V is strongly right-justified with respect to h_i . This establishes the claims in the first part of (b) for these cases. Observe that our assumptions when $m = 4$ or $m = 5$ imply $f_{\omega_H}(h_i v_{u_{m-1,2}}) = 0$ and that ω_H is strongly left-justified at h_i . Since $\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}} = \{h_i\}$, this establishes the second part of (b) in these cases.

Now assume $m \geq 4$ and $\text{ht}(h_i) < u_{m-1,2} - d_1$. Then there must exist $j > i$ so that $D(h_j; \omega_H) = h_j v_{u_{m-1,2} - \ell}$ with $1 \leq \ell \leq \omega_H(h_i) - \delta_1$ (the extra δ_1 must be included here since $d_2 = 1$ implies all horizontal edges of D_m have different heights, in other words $d_2 = 1$ implies h_i is immediately followed by a vertical edge). Assume that j is chosen so that ℓ is minimal, in particular when $d_1 = 1$ we must have $\ell = 1$.

By Lemma 3.14, the vertical grading condition must be satisfied for the paths $h_i v$ with $v \in (h_i \bar{v}_{u_{m-1,2}})_V$. For each such v , we have $D(v; \omega_V) = h_{j(v)} v$ for some $j(v) > i$, in particular $f_{\omega_V}(h_{j(v)} v) = 0$. Since h_i is blocking, it cannot be contained in the shadow of any of these vertical edges. Moreover, when $d_1 = 1$, the edge h_j will also not be contained in the shadow of any of these vertical edges. Thus we see that there are at least $1 + \delta_2$ horizontal edges of the path $h_i v_{u_{m-1,2} - 1}$ lying outside the shadows of its vertical edges and applying equation (8) shows $f_{\omega_V}(h_i v_{u_{m-1,2} - 1}) \leq -(1 + \delta_2)$. But by Corollary 2.4 there are $d_2 - 1 - \delta_2$ horizontal edges immediately preceding $v_{u_{m-1,2}}$ and, since $\omega_V(v_{u_{m-1,2}}) \leq d_2$, we must have $f_{\omega_V}(h_i v_{u_{m-1,2}}) \leq 0$. We conclude that one of the following holds:

- $D(v_{u_{m-1,2}}; \omega_V)$ is a proper subpath of $h_i v_{u_{m-1,2}}$ by Lemma 3.1 and thus ω is compatible;
- $D(v_{u_{m-1,2}}; \omega_V) = h_i v_{u_{m-1,2}}$ with $f_{\omega_V}(h_i v_{u_{m-1,2}}) = 0$ and both compatibility conditions fail for the path $h_i v_{u_{m-1,2}}$.

This establishes claim (i) of (b) in this case. When $d_1 = 1$, we must have $f_{\omega_H}(h_i v_{u_{m-1,2}}) = 0$ for otherwise h_i could not be blocking. This gives claim (iii) of (b) when $d_1 = 1$. To complete the proof of (iii) for $d_1 > 1$ and $m \geq 4$, we observe that h_i being a blocking edge implies $f_{\omega_H}(h_i v_{u_{m-1,2}}) \geq 0$. Our aim then is to show that $f_{\omega_H}(h_i v_{u_{m-1,2}}) > 0$ implies the second situation above is impossible.

Indeed, $f_{\omega_H}(h_i v_{u_{m-1,2}}) > 0$ can only occur if we take $\ell \leq \omega_H(h_i) - 1 - \delta_1$ above. But, assuming $d_1 > 1$ and $m \geq 4$, there are d_2 horizontal edges of D_m immediately preceding each of the vertical edges

$$v_{u_{m-1,2} - d_1 + 2 + \delta_1}, v_{u_{m-1,2} - d_1 + 3 + \delta_1}, \dots, v_{u_{m-1,2} - 1},$$

and $d_2 - 1$ horizontal edges immediately preceding $v_{u_{m-1,2}}$ (by Corollary 2.4, the terminal subpath of D_m containing all these edges identifies with the terminal subpath $D_3 \setminus D_2$ inside D_3). It follows that $D(v_{u_{m-1,2}}; \omega_V)$ must be a subpath of $h_j v_{u_{m-1,2}}$ and so the vertical grading condition is satisfied for the path $h_i v_{u_{m-1,2}}$. In particular, ω is compatible by Lemma 3.14, this completes the proof of (iii).

The arguments above also establish the following when $m \geq 4$, $d_m \neq 1$, and $\text{ht}(h_i) < u_{m-1,2} - d_1$:

- if $\omega_H(h_i) < d_1$ or h_i is immediately followed by $1 + \delta_1$ vertical edges, then either $d_1 = 1$ and h_i cannot possibly be blocking or there must exist a horizontal edge h_j as in the previous paragraph and compatibility again holds, this gives (v) once we have established (iv), i.e. once we know that ω_H is strongly left-justified at h_i ;
- if $\omega_V(v_{u_{m-1,2-t}}) < d_2$ for any $0 \leq t \leq d_1 - \delta_1$, then the piecewise compatible grading ω must be compatible.

We prove (iii) and (iv) by induction on $m \geq 3$, $d_m \neq 1$. The base case $m = 3$ of (ii) was established in the first part of the proof. Suppose $m \geq 4$ and ω is not compatible. By Proposition 3.16 the grading $((\varphi_{m-1}^*)^{-1}\omega_V, \Omega_{m-1}^{-1}(\omega_H)) =: (\omega_{H'}, \omega_{V'})$ of D'_{m-1} is piecewise compatible, but not compatible by Proposition 3.10. By part (a), there must be a blocking edge h'_j for $\omega_{H'}$. Applying (ii) to the grading $(\omega_{H'}, \omega_{V'})$ we see that $\omega_{H'}$ is left-justified at h'_j and $\omega_{V'}$ is strongly right-justified with respect to this blocking edge.

When $m = 4$, we have $\text{supp}(\omega_{H'}) \cap h'_j v'_{u'_{m-2,2}} = \{h'_j\}$ and from the definition of φ_{m-1}^* we see that ω_V is strongly right-justified with respect to h_i . This requires the extra observation above that we had to take $k = i$ in the definition of left-justification for the case $m = 3$. For $m \geq 5$, claim (iv) applied to the grading $(\omega_{H'}, \omega_{V'})$ shows that $\omega_{H'}$ is strongly left-justified at h'_j and again the definition of φ_{m-1}^* shows that ω_V is strongly right-justified with respect to h_i . By Proposition 3.18, we see that ω_H must be left-justified at h_i .

It remains to argue that ω_H is strongly left-justified at h_i , but this is immediate from Lemma 3.7 and the definition of the maps θ . Indeed, since $\omega_{H'}$ is strongly left-justified at its blocking edge h'_j , the remote shadows of the horizontal edges in $h'_j v'_{u'_{m-2,1}}$ are linearly ordered in the opposite order to the horizontal edges in $\text{supp}(\omega_{H'}) \cap h'_j v'_{u'_{m-2,1}}$. Since ω_V is strongly right-justified with respect to h_i , analogous statements can be made about the remote shadows of the vertical edges in $h_i v_{u_{m-1,2}}$. But the maps θ are compatible with these orderings and so $\omega_{V'}$ being strongly right-justified with respect to h'_j forces $\omega_H = \Omega_{m-1}(\omega_{V'})$ to be strongly left-justified at h_i . This completes the proof of (ii) and (iv). \square

The next result severely restricts which horizontal edges can be blocking.

COROLLARY 3.22. *Let $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ be a piecewise compatible grading of D_m , $m \geq 5$, which is not compatible. Write $h_i \in H_m$ for the blocking edge of ω_H . Then either $i = 1$ or h_i is immediately preceded by a vertical edge.*

Proof. By Proposition 3.16 and Proposition 3.10, the grading $(\omega_{H'}, \omega_{V'}) := ((\varphi_{m-1}^*)^{-1}\omega_V, \Omega_{m-1}^{-1}\omega_H)$ of D'_{m-1} is piecewise compatible but not compatible. Let $h'_j \in H'_{m-1}$ denote the blocking edge of $\omega_{H'}$. Then since $m \geq 5$, we have $|\text{rsh}(h'_j; \omega_{H'})| = d_2 - \ell$, where ℓ is the number of vertical edges immediately following h'_j . By Lemma 3.7, this implies there are $d_2 - \ell$ horizontal edges of height $j - 1$ in the remote shadow of ω_V . But there are exactly $d_2 - \ell$ horizontal edges of height $j - 1$ inside D_m by Lemma 2.3. Since $D(v_{u_{m-1,2}}; \omega_V) = h_i v_{u_{m-1,2}}$, the edge h_i lies furthest to the left among all horizontal edges in $\text{rsh}(\omega_V) \cap h_i v_{u_{m-1,2}}$, this gives the result. \square

We also obtain the following analogue of Proposition 3.18.

COROLLARY 3.23. *Let $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ be a piecewise compatible grading of D_m , $m \geq 3$, which is not compatible. If $h_i \in H_m$ denotes the blocking edge for ω_H , then $\text{supp}(\omega_V) \cap h_i v_{u_{m-1,2}} = \text{rsh}(\omega_H) \cap h_i v_{u_{m-1,2}}$.*

Proof. Since ω is not compatible, the grading $(\Omega_m(\omega_V), \varphi_m^* \omega_H) =: (\omega_{H'}, \omega_{V'})$ of D'_{m+1} is not compatible by Proposition 3.10, but is piecewise compatible by Proposition 3.16.

By Theorem 3.20 the grading $(\omega_{H'}, \omega_{V'})$ satisfies the hypotheses of Proposition 3.18 and so $\text{supp}(\omega_{H'}) \cap h'_j v'_{u'_{m,2}} = \text{rsh}(\omega_{V'}) \cap h'_j v'_{u'_{m,2}}$, where h'_j denotes the blocking edge of $\omega_{H'}$.

By piecewise compatibility, we must have $\text{supp}(\omega_V) \cap h_i v_{u_{m-1,2}} \subset \text{rsh}(\omega_H) \cap h_i v_{u_{m-1,2}}$ since every vertical edge in $\text{supp}(\omega_V) \cap h_i v_{u_{m-1,2}}$ is contained in the shadow of ω_H . If there exists $v \in \text{rsh}(\omega_H) \cap h_i v_{u_{m-1,2}}$ with $\omega_V(v) = 0$, by Lemma 3.7 there will be a horizontal edge $h' \in \text{rsh}(\omega_{V'}) \cap h'_j v'_{u'_{m,2}}$ with $\omega_{H'}(h') = 0$, a contradiction. Therefore we must have $\text{supp}(\omega_V) \cap h_i v_{u_{m-1,2}} = \text{rsh}(\omega_H) \cap h_i v_{u_{m-1,2}}$. \square

As a final consequence we show that the piecewise compatible gradings which are not compatible satisfy a certain upper bound property with respect to compatible gradings.

COROLLARY 3.24. *Suppose $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ is a piecewise compatible grading of D_m , $m \geq 3$, which is not compatible. Write h_i for the blocking edge of ω_H . Then the following hold:*

- (a) *for any vertical grading $\chi_V \in \mathcal{C}(\omega_H)$ and any edge $v \in (h_i v_{u_{m-1,2}})_V$, we have $\chi_V(v) \leq \omega_V(v)$;*
- (b) *for any horizontal grading $\chi_H \in \mathcal{C}(\omega_V)$ and any edge $h \in (h_i v_{u_{m-1,2}})_H$, we have $\chi_H(h) \leq \omega_H(h)$.*

Proof. We begin by making a few basic observations which allow to deduce part (b) for D_m from part (a) for D'_{m-1} .

Consider a horizontal grading $\chi_H \in \mathcal{C}(\omega_V)$ and suppose $\omega_H(h) < \chi_H(h)$ for some $h \in (h_i v_{u_{m-1,2}})_H$. This implies $\omega_H(h) < d_1$ since we only consider d_1 -bounded horizontal gradings. By Theorem 3.20, we have $D(v_{u_{m-1,2}}; \omega_V) = h_i v_{u_{m-1,2}}$ and so every edge of $h_i v_{u_{m-1,2}}$ is in the shadow of ω_V . Thus we have

$$\text{supp}(\chi_H) \cap h_i v_{u_{m-1,2}} \subset \text{rsh}(\omega_V) \cap h_i v_{u_{m-1,2}} = \text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}},$$

where the equality comes from Proposition 3.18. By Theorem 3.20, ω_H is strongly left-justified at h_i and so the only edge $h \in \text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}}$ which could satisfy $\omega_H(h) < d_1$ is $h = h_{i-1+d}$, where $d = |\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}}|$.

For $m = 3$, we have $\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}} = \{h_i\}$. Since the horizontal grading condition (HGC) of ω_H is not satisfied for the path $h_i v_{u_{m-1,2}}$, the inequality $\omega_H(h_i) < \chi_H(h_i)$ implies the horizontal grading condition of χ_H is also not satisfied for the path $h_i v_{u_{m-1,2}}$. In particular, (χ_H, ω_V) is not compatible, a contradiction.

For $m \geq 4$, consider the compatible grading $(\omega_{H'}, \chi_{V'}) := ((\varphi_{m-1}^*)^{-1} \omega_V, \Omega_{m-1}^{-1} \chi_H)$ of D'_{m-1} (see Proposition 3.10) and the piecewise compatible grading $(\omega_{H'}, \omega_{V'}) := ((\varphi_{m-1}^*)^{-1} \omega_V, \Omega_{m-1}^{-1} \omega_H)$ of D'_{m-1} (see Proposition 3.16). By the definition of Ω , we have

$$\chi_{V'}(\theta^{-1} h_{i-1+d}) = \chi_H(h_{i-1+d}) > \omega_H(h_{i-1+d}) = \omega_{H'}(\theta^{-1} h_{i-1+d}).$$

This contradicts part (a) applied to the grading $(\omega_{H'}, \omega_{V'})$ of D'_{m-1} and so there can be no grading χ_H as above. Thus part (b) holds for m once we have established part (a) for $m - 1$, $m \geq 4$.

To continue we suppose there exists a vertical grading $\chi_V \in \mathcal{C}(\omega_H)$ such that $\chi_V(v) > \omega_V(v)$ for some $v \in (h_i v_{u_{m-1,2}})_V$. As above, this implies $0 < \omega_V(v) < d_2$ and thus $v = v_{u_{m-1,2}-t+1}$, where $t = |\text{supp}(\omega_V) \cap h_i v_{u_{m-1,2}}|$. In particular, we must have $d_2 \geq 2$ and by Corollary 2.8 the Dyck path D_m has no consecutive vertical edges.

Note that, by Proposition 3.18, there are only two possibilities for the height of the edge h_{i-1+d} . Either $\text{ht}(h_{i-1+d}) = u_{m-1,2} - t$ so that $v_{u_{m-1,2}-t+1} \in \text{rsh}(h_{i-1+d}; \omega_H)$ or $\text{ht}(h_{i-1+d}) = u_{m-1,2} - t - 1$ with h_{i-1+d} immediately followed by a single vertical edge. In the latter case, $\omega_H(h_{i-1+d}) > 1$ also implies $v_{u_{m-1,2}-t+1} \in \text{rsh}(h_{i-1+d}; \omega_H)$.

If $v_{u_{m-1,2}-t+1} \in \text{rsh}(h_{i-1+d}; \omega_H)$, the horizontal grading condition (HGC) is not satisfied for the path $h_{i-1+d}v_{u_{m-1,2}-t+1}$ and we have $D(v_{u_{m-1,2}-t+1}; \omega_V) = h_{i+d}v_{u_{m-1,2}-t+1}$ by Proposition 3.18. But then for χ_V as above, the vertical grading condition (VGC) is not satisfied for the path $h_{i-1+d}v_{u_{m-1,2}-t+1}$. In particular, this implies (ω_H, χ_V) is not compatible, a contradiction.

Thus we must have $\text{ht}(h_{i-1+d}) = u_{m-1,2} - t - 1$ with h_{i-1+d} immediately followed by exactly one vertical edge and $\omega_H(h_{i-1+d}) = 1$. Then, since ω_H is strongly left-justified at h_i , we have $v_{u_{m-1,2}-t+1} \in \text{rsh}(h_{i-2+d}; \omega_H)$ and so the horizontal grading condition (HGC) is not satisfied for the path $h_{i-2+d}v_{u_{m-1,2}-t+1}$. If $h_{i-1+d} \in \text{rsh}(v_{u_{m-1,2}-t+1}; \omega_V)$, we must have $D(v_{u_{m-1,2}-t+1}; \omega_V) = h_{i-1+d}v_{u_{m-1,2}-t+1}$. But then for χ_V as above, the vertical grading condition (VGC) is not satisfied for the path $h_{i-2+d}v_{u_{m-1,2}-t+1}$. In particular, this implies (ω_H, χ_V) is not compatible, a contradiction.

Thus the horizontal edge h_{i-1+d} must lie beyond the shadow of $v_{u_{m-1,2}-t+1}$. By Proposition 3.18, there can be no horizontal edges of height $u_{m-1,2} - t$ in the remote shadow of ω_V and so we must have $\omega_V(v_{u_{m-1,2}-t+1}) = \ell$, where $\ell < d_2$ is the number of horizontal edges immediately preceding $v_{u_{m-1,2}-t+1}$. For $m = 3$, this can only occur for $t = 1$, but $v_{u_{m-1,2}}$ is immediately preceded by $d_2 - 1$ horizontal edges inside D_3 and thus ω is compatible, a contradiction.

So we must have $m \geq 4$. Consider the piecewise compatible grading $(\omega_{H'}, \omega_{V'}) := ((\varphi_{m-1}^*)^{-1}\omega_V, \Omega_{m-1}^{-1}\omega_H)$ of D'_{m-1} (see Proposition 3.16). Since ω_V is strongly right-justified and $\omega_V(v_{u_{m-1,2}-t+1}) < d_2$, the last horizontal edge in $\text{supp}(\omega_{H'})$ must be $h'_{u_{m-1,2}-t+1}$ with $\omega_{H'}(h'_{u_{m-1,2}-t+1}) = d_2 - \ell$, this being exactly the number of vertical edges immediately following $h'_{u_{m-1,2}-t+1}$ by Lemma 2.3. Moreover, by Lemma 3.7, the first vertical edge v' in $\text{rsh}(\omega_{H'})$ lies in the remote shadow of $h'_{u_{m-1,2}-t}$ and $\omega_{V'}(v') = 1$. By Corollary 3.23, v' cannot be immediately preceded by a vertical edge. But then there exists $\chi_{V'}$ with $\chi_{V'}(v') = 2$ compatible with $\omega_{H'}$, a contradiction with part (b) for D'_{m-1} .

This contradiction shows there can be no vertical χ_V as above and thus proves (a). □

4. PROOF OF MAIN THEOREM

We begin this section with a general statement about non-commutative weights associated to certain gradings of an arbitrary (i.e. not necessarily maximal) Dyck path, here we make no boundedness assumptions on the gradings.

PROPOSITION 4.1. *Let D be any Dyck path with edges $E = H \sqcup V$, where $H = \{h_1, \dots, h_{a_1}\}$ with $a_1 \geq 1$ and $V = \{v_1, \dots, v_{a_2}\}$ denote the sets of horizontal and vertical edges of D . Write $E = \{1, 2, \dots, a_1 + a_2\}$ for the edges of D taken in the natural order. Let $\omega : E \rightarrow \mathbb{Z}_{\geq 0}$ be any grading of D . Given $q_{i,j} \in \mathbb{k}$ for $i \in \{1, 2\}$ and $j \in \mathbb{Z}_{\geq 0}$, define non-commutative weights*

$$(13) \quad \text{wt}_\omega(e) = \begin{cases} q_{1,\omega(e)} Y^{\omega(e)} X^{-1} & \text{if } e \in H; \\ q_{2,\omega(e)} X^{\omega(e)+1} Y^{-1} X^{-1} & \text{if } e \in V; \end{cases}$$

and let $Y_D(\omega) = \text{wt}_\omega(1) \text{wt}_\omega(2) \cdots \text{wt}_\omega(a_1 + a_2)$. Assume ω is compatible and satisfies the following:

- (1) the local shadow path $D(h_1; \omega_H) = D$ with $f_{\omega_H}(D) = 0$;
- (2) for any other vertical grading $\chi_V \in \mathcal{C}(\omega_H)$ and any vertical edge $v_t \in V$ so that $\chi_V(v_s) = \omega_V(v_s)$ for $s < t$, we have $\chi_V(v_t) \leq \omega_V(v_t)$.

Then $Y_D(\omega) = pX^{-1}$, where $p = \prod_{i=1}^{a_1} q_{1,\omega(h_i)} \cdot \prod_{t=1}^{a_2} q_{2,\omega(v_t)}$.

Proof. We first note that the coefficient p is immediate from the definition of the non-commutative edge weights in equation (13). Thus we assume all $q_{i,j} = 1$ for the remainder of the proof.

We work by induction on a_2 . For $a_2 = 0$, assumption 1 implies $a_1 = 1$ and $\omega_H(h_1) = 0$. The claim follows in this case directly from the definition of the non-commutative edge weights in equation (13).

Suppose $a_2 \geq 1$ and consider $h_i \in \text{supp}(\omega_H)$ with i maximal. Let $v_r \in V$ denote the next vertical edge after h_i , i.e. the path $h_i v_r$ consists of several consecutive horizontal edges, say d of them, followed by a single vertical edge. By assumption 1, we have $r < a_2$. By assumption 2, we have $\omega_V(v_r) = d - 1$ so that

$$(\dagger) \quad Y_{h_i v_r}(\omega) = (Y^{\omega_H(h_i)} X^{-1})(X^{-1})^{d-1} (X^d Y^{-1} X^{-1}) = Y^{\omega_H(h_i)-1} X^{-1}.$$

Let \tilde{D} be the Dyck path obtained from D by replacing the path $h_i v_r$ by a single horizontal edge. Write $\tilde{E} = \tilde{H} \sqcup \tilde{V}$ for the edges of \tilde{D} , where $\tilde{H} = \{\tilde{h}_1, \dots, \tilde{h}_{a_1-d+1}\}$ and $\tilde{V} = \{\tilde{v}_1, \dots, \tilde{v}_{a_2-1}\}$ denote the horizontal and vertical edges of \tilde{D} . Define a grading $\tilde{\omega} : \tilde{E} \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\tilde{\omega}_H(\tilde{h}_j) = \begin{cases} \omega_H(h_j) & \text{if } j < i; \\ \omega_H(h_i) - 1 & \text{if } j = i; \\ 0 & \text{if } j > i; \end{cases} \quad \tilde{\omega}_V(\tilde{v}_s) = \begin{cases} \omega_V(v_s) & \text{if } s < r; \\ \omega_V(v_{s+1}) & \text{if } s \geq r. \end{cases}$$

It is not hard to see that $\tilde{\omega}$ satisfies assumptions 1 and 2, thus by induction we have $Y_{\tilde{D}}(\tilde{\omega}) = X^{-1}$. By (\dagger) , we have $Y_D(\omega) = Y_{\tilde{D}}(\tilde{\omega})$ and so $Y_D(\omega) = X^{-1}$ as desired. \square

Now we turn to the proof of Theorem 1.3 and return to our standard boundedness assumptions on gradings.

LEMMA 4.2. *Let $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ be a piecewise compatible grading of D_m , $m \geq 3$, which is not compatible. Denote by $h_i \in H_m$ the blocking edge of ω_H . Set*

$$d = |\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}}| \quad \text{and} \quad t = |\text{supp}(\omega_V) \cap h_i v_{u_{m-1,2}}|.$$

Then for any $h \in (\bar{h}_i h_{i-1+d})_H$, we have $Y_{D(h;\omega_H)}(\omega) = pX^{-1}$, where $p = p_{1,\omega_H(h_{i-1+d})} p_{2,d_2-\omega_V(v_{u_{m-1,2}-t+1})}$.

Proof. Since h_i is blocking, no local shadow path $D(h;\omega_H)$ for $h \in (\bar{h}_i h_{i-1+d})_H$ contains $v_{u_{m-1,2}}$. Thus Lemma 3.14 shows $\omega|_{D(h;\omega_H)}$ is compatible. By definition of local shadow paths, $\omega|_{D(h;\omega_H)}$ satisfies condition 1 of Proposition 4.1. Condition 2 follows directly from Corollary 3.24. The conclusion immediately follows since the only edges in $D(h;\omega_H)$ for $h \in (\bar{h}_i h_{i-1+d})_H$ whose non-commutative weights have nontrivial coefficients are h_{i-1+d} and $v_{u_{m-1,2}-t+1}$. \square

This leads to the following result which is key to our induction argument.

COROLLARY 4.3. *Let $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ be a piecewise compatible grading of D_m , $m \geq 3$, which is not compatible. Write h_i for the blocking edge of ω_H and assume $f_{\omega_H}(h_i v_{u_{m-1,2}}) = 0$. Set*

$$d = |\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}}| \quad \text{and} \quad t = |\text{supp}(\omega_V) \cap h_i v_{u_{m-1,2}}|.$$

Then $Y_{h_i v_{u_{m-1,2}}}(\omega) = pYXY^{-1}X^{-1}$, where $p = p_{1,\omega_H(h_{i-1+d})} p_{2,d_2-\omega_V(v_{u_{m-1,2}-t+1})}$.

Proof. We distinguish two cases as in the proof of Theorem 3.20. First consider the case $\text{ht}(h_i) \geq u_{m-1,2} - d_1$ in which one of the following holds: $m = 3$ with $d_1 \neq 1$, $m = 4$ with $d_1, d_2 \neq 1$, or $m = 5$ with $d_2 = 1$. In each of these cases $\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}} = \{h_i\}$ and by assumption $\omega_H(h_i) = u_{m-1,2} - \text{ht}(h_i)$. We use the description of ω from the proof of Theorem 3.20 in each case.

For $m = 3$ with $d_1 \neq 1$, set $\delta = 1$ if h_i is immediately followed by a vertical edge and $\delta = 0$ otherwise. Then we have

$$\begin{aligned} Y_{h_i v_{u_{2,2}}}(\omega) &= (p_{1, \omega_H(h_i)} Y^{\omega_H(h_i)} X^{-1}) (XY^{-1} X^{-1})^\delta (X^{-1})^{\omega_V(v_{u_{m-1,2-t+1}})} \\ &\quad \times (p_{2, d_2 - \omega_V(v_{u_{m-1,2-t+1}})} X^{\omega_V(v_{u_{m-1,2-t+1}}) + 1} Y^{-1} X^{-1}) \\ &\quad \times \left[(X^{-1})^{d_2} (X^{d_2+1} Y^{-1} X^{-1}) \right]^{\omega_H(h_i) - 2 - \delta} (X^{-1})^{d_2 - 1} (X^{d_2+1} Y^{-1} X^{-1}) \\ &= p (Y^{\omega_H(h_i) - 1 - \delta} X^{-1}) \left[XY^{-\omega_H(h_i) + 2 + \delta} X^{-1} \right] (X^2 Y^{-1} X^{-1}) \\ &= p Y X Y^{-1} X^{-1}. \end{aligned}$$

For $m = 4$ with $d_1, d_2 \neq 1$ or $m = 5$ with $d_2 = 1$, we have $p = 1$ and so

$$\begin{aligned} Y_{h_i v_{u_{m-1,2}}}(\omega) &= (Y^{d_1} X^{-1}) (XY^{-1} X^{-1})^{1 + \delta_1} \left[(X^{-1})^{d_2} (X^{d_2+1} Y^{-1} X^{-1}) \right]^{d_1 - 2 - \delta_1} \\ &\quad \times (X^{-1})^{d_2 - 1} (X^{d_2+1} Y^{-1} X^{-1}) \\ &= (Y^{d_1 - 1 - \delta_1} X^{-1}) \left[XY^{-d_1 + 2 + \delta_1} X^{-1} \right] (X^2 Y^{-1} X^{-1}) \\ &= Y X Y^{-1} X^{-1}. \end{aligned}$$

Now suppose $\text{ht}(h_i) < u_{m-1,2} - d_1$ so that $|\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}}| > 1$ and, by Theorem 3.20(iii), $\text{ht}(h_{i+1}) = \text{ht}(h_i) + \delta_1$. By Lemma 4.2, we have $Y_{D(h_{i+1}; \omega_H)}(\omega) = p X^{-1}$. As in the proof of Theorem 3.20, we have $\omega_H(h_i) = d_1$ and $\omega_V(v_{u_{m-1,2-\ell}}) = d_2$ for $0 \leq \ell \leq d_1 - \delta_1$. Combining these observations, we get

$$\begin{aligned} Y_{h_i v_{u_{m-1,2}}}(\omega) &= (Y^{d_1} X^{-1}) (XY^{-1} X^{-1})^{\delta_1} [p X^{-1}] \left[(X^{-1})^{d_2 - 1} (X^{d_2+1} Y^{-1} X^{-1}) \right] \\ &\quad \times \left[(X^{-1})^{d_2} (X^{d_2+1} Y^{-1} X^{-1}) \right]^{d_1 - 2 - \delta_1} (X^{-1})^{d_2 - 1} (X^{d_2+1} Y^{-1} X^{-1}) \\ &= p (Y^{d_1 - 1 - \delta_1} X^{-1}) \left[XY^{-d_1 + 2 + \delta_1} X^{-1} \right] (X^2 Y^{-1} X^{-1}) \\ &= p Y X Y^{-1} X^{-1}. \end{aligned} \quad \square$$

For $m \geq 1$, we consider summands of Y_{D_m} given as follows:

$$(14) \quad Y_{D_m} = \sum_{\omega_H: H_m \rightarrow [0, d_1]} Y_{D_m}(\omega_H), \quad Y_{D_m}(\omega_H) := \sum_{\omega_V \in \mathcal{C}(\omega_H)} Y_{D_m}(\omega_H, \omega_V).$$

Our goal will be to understand the action of F_{P_0} on each of these summands. The first step is given by the following factorization results which allow for an induction argument.

LEMMA 4.4. *Let $\omega_H : H_m \rightarrow [0, d_1]$ be a horizontal grading of D_m , $m \geq 3$. Write*

$$Y_{D_m}^{pc}(\omega_H) = \sum_{\omega: E_m \rightarrow \mathbb{Z}_{\geq 0}} Y_{D_m}(\omega),$$

where the sum ranges over piecewise compatible gradings ω of D_m for which $\omega|_{H_m} = \omega_H$. Then there is the following factorization:

$$(15) \quad Y_{D_m}^{pc}(\omega_H) = Y_{D_{m-1}}(\omega_{H,1}) Y_{D_{m-1}}(\omega_{H,2}) \cdots Y_{D_{m-1}}(\omega_{H,d_m-1-\delta_m}) p X^{|H_{m-2-\delta_m}|} Y_{D_{m-1}}(\omega_{H,d_m-\delta_m}),$$

where

$$p = \begin{cases} p_{1,1}^{|V_{m-2-\delta_m}| - 2|H_{m-2-\delta_m}|} p_{1,2}^{|H_{m-2-\delta_m}| - |V_{m-2-\delta_m}|} & \text{if } d_2 = 1 \text{ and } m > 3; \\ p_{1,1}^{-|V_{m-2-\delta_m}|} & \text{if } d_2 > 1 \text{ or } m = 3. \end{cases}$$

Proof. By the assumptions on the horizontal grading $\omega_{H,d_m-\delta_m}$ of D_{m-1} from Definition 3.11, each term contributing to $Y_{D_{m-1}}(\omega_{H,d_m-\delta_m})$ begins with the monomial $p^{-1}X^{-|H_{m-2-\delta_m}|}$ associated to the initial $D_{m-2-\delta_m}$ subpath of D_{m-1} . To see the coefficient p^{-1} , we observe the following:

- when $d_2 > 1$, there are $|V_{m-2-\delta_m}|$ horizontal edges of $D_{m-2-\delta_m}$ which are immediately followed by a single vertical edge and all other horizontal edges are not immediately followed by any vertical edges;
- when $d_2 = 1$, each horizontal edge is immediately followed by a vertical edge and so there are $|V_{m-2-\delta_m}| - |H_{m-2-\delta_m}|$ horizontal edges of $D_{m-2-\delta_m}$ which are immediately followed by exactly two vertical edges (see Corollary 2.8) and the remaining $2|H_{m-2-\delta_m}| - |V_{m-2-\delta_m}|$ horizontal edges are immediately followed by a single vertical edge.

Using the notation of Definition 2.9, for any grading $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ there is the factorization

$$Y_{D_m}(\omega) = Y_{h_{1,1}v_{u_{m-2,2},1}}(\omega) \cdots Y_{h_{1,d_m-1-\delta_m}v_{u_{m-2,2},d_m-1-\delta_m}}(\omega) Y_{h_{u_{m-2-\delta_m,1}+1,d_m-\delta_m}v_{u_{m-2,2},d_m-\delta_m}}(\omega).$$

The result then immediately follows from the definition of piecewise compatible gradings in Definition 3.11. \square

Using Remark 3.13 instead of Definition 3.11, we obtain a similar factorization for piecewise compatible gradings of D'_{m+1} . Below we use the notation $Y'_{D'_m}(\omega_{V'}) := \sum_{\omega_{H'} \in \mathcal{C}(\omega_{V'})} Y'_{D'_m}(\omega_{H'}, \omega_{V'})$ for a vertical grading $\omega_{V'} : V'_m \rightarrow [0, d_1]$. Note that $d'_{m+1} = d_m$, $d'_m = d_{m-1}$, and so $\delta'_{m+1} = \delta_m$ when $m \geq 3 + \delta'_{m+1}$.

LEMMA 4.5. *Let $\omega_{V'} : V'_{m+1} \rightarrow [0, d_1]$ be a vertical grading of D'_{m+1} for $m \geq 3 + \delta'_{m+1}$. Write*

$$Y'^{pc}_{D'_{m+1}}(\omega_{V'}) = \sum_{\omega' : E'_{m+1} \rightarrow \mathbb{Z}_{\geq 0}} Y'_{D'_{m+1}}(\omega'),$$

where the sum ranges over piecewise compatible gradings ω' of D'_{m+1} for which $\omega'|_{V'_{m+1}} = \omega_{V'}$. Then there is the following factorization:

$$(16) \quad Y'^{pc}_{D'_{m+1}}(\omega_{V'}) = Y'_{D'_m}(\omega_{V',1})Y'_{D'_m}(\omega_{V',2}) \cdots Y'_{D'_m}(\omega_{V',d_m-1-\delta_m})pXY'^{|V'_{m-1-\delta_m}|}X^{-1}Y'_{D'_m}(\omega_{V',d_m-\delta_m}),$$

where

$$p = \begin{cases} p_{1,1}^{|H_{m-2-\delta_m}|-2|H_{m-2-\delta_m}|} p_{1,2}^{|H_{m-2-\delta_m}|-|V_{m-2-\delta_m}|} & \text{if } d_2 = 1; \\ p_{1,1}^{-|V_{m-2-\delta_m}|} & \text{if } d_2 > 1. \end{cases}$$

Proof. By the assumptions on the vertical grading $\omega_{V',d_m-\delta_m}$ from Remark 3.13, each term contributing to $Y'_{D'_m}(\omega_{V',d_m-\delta_m})$ begins with the monomial $p^{-1}XY^{-|V'_{m-1-\delta_m}|}X^{-1}$ associated to the initial $D'_{m-1-\delta_m}$ subpath of D'_m . The coefficient p^{-1} here can be seen as follows. Applying Lemma 2.3(a), we see that the structure of $D'_{m-1-\delta_m}$ is determined by the structure of $D_{m-2-\delta_m}$ observed in the last part of the previous proof. More precisely, we have the following:

- when $d_2 > 1$, there are $|V_{m-2-\delta_m}|$ vertical edges of $D'_{m-1-\delta_m}$ which are immediately preceded by $d_1 - 1$ horizontal edges and all other vertical edges are immediately followed by d_1 horizontal edges;

- when $d_2 = 1$, there are $|V_{m-2-\delta_m}| - |H_{m-2-\delta_m}|$ vertical edges of $D'_{m-1-\delta_m}$ which are immediately preceded by $d_1 - 2$ horizontal edges and the remaining $2|H_{m-2-\delta_m}| - |V_{m-2-\delta_m}|$ vertical edges are immediately preceded by $d_1 - 1$ horizontal edges.

Then observe that in the computation of $Y'_{D'_m}(\omega_{V',d_m-\delta_m})$ the coefficients are given by $p'_{2,d_1-k} = p_{1,k}$ for $k = 1, 2$. □

The analogous factorization in the special case where $m = 3$ and $\delta'_4 = 1$ is handled in the following result which is proven exactly as Lemma 4.4.

LEMMA 4.6. *Suppose $d_2 = 1$. Let $\omega_{V'} : V'_4 \rightarrow [0, d_1]$ be a vertical grading of D'_4 . Write*

$$Y'_{D'_4}{}^{pc}(\omega_{V'}) = \sum_{\omega': E'_4 \rightarrow \mathbb{Z}_{\geq 0}} Y'_{D'_4}(\omega'),$$

where the sum ranges over piecewise compatible gradings ω' of D'_4 for which $\omega'|_{V'_4} = \omega_{V'}$. Then there is the following factorization:

$$(17) \quad Y'_{D'_4}{}^{pc}(\omega_{V'}) = Y'_{D'_3}(\omega_{V',1})Y'_{D'_3}(\omega_{V',2}) \cdots Y'_{D'_3}(\omega_{V',d_1-2})XY'_{D'_3}(\omega_{V',d_1-1}).$$

The factorizations above concerned sums over piecewise compatible gradings. Our goal is to understand sums over compatible gradings, however it will be easier to first focus on piecewise compatible gradings which are not compatible.

LEMMA 4.7. *Let $\omega_H : H_m \rightarrow [0, d_1]$ be a horizontal grading of D_m , $m \geq 3$, for which there exists a vertical grading $\omega_V^* : V_m \rightarrow [0, d_2]$ of D_m so that (ω_H, ω_V^*) is piecewise compatible but not compatible. Write*

$$Y_{D_m}{}^{nc}(\omega_H) = \sum_{\omega: E_m \rightarrow \mathbb{Z}_{\geq 0}} Y_{D_m}(\omega),$$

where the sum ranges over piecewise compatible gradings ω of D_m which are not compatible and satisfy $\omega|_{H_m} = \omega_H$. Let $h_i \in H_m$ denote the blocking edge of ω_H and set

$$d = |\text{supp}(\omega_H) \cap h_i v_{u_{m-1},2}| \quad \text{and} \quad t = |\text{supp}(\omega_V^*) \cap h_i v_{u_{m-1},2}|.$$

Let $s = \left\lceil \frac{i}{u_{m-1,1}} \right\rceil$ denote the index so that $h_i \in H_{m,s}$. Define a horizontal grading $\chi_H : H_{m-1} \rightarrow [0, d_1]$ of D_{m-1} with $\chi_H(h) = \omega_{H,s}(h)$ for $h \in (h_{1,s}\bar{h}_i)_H$ and $\chi_H(h) = \ell$ for $h \in (h_i v_{u_{m-2},2,s})_H$ if h is immediately followed by exactly ℓ vertical edges in this copy of D_{m-1} . Then there is the following factorization:

$$(18) \quad Y_{D_m}{}^{nc}(\omega_H) = Y_{D_{m-1}}(\omega_{H,1}) \cdots Y_{D_{m-1}}(\omega_{H,s-1})Y_{D_{m-1}}(\chi_H)p_1 X^{|h_i v_{u_{m-2},2,s}|_H} p_2 Y X Y^{-1} X^{-1},$$

where $p_2 = p_{1,\omega_H(h_{i-1+d})}p_{2,d_2-\omega_V^*(v_{u_{m-1},2-t+1})}$ and

$$p_1 = \begin{cases} |h_i v_{u_{m-2},2,s}|_V - 2|h_i v_{u_{m-2},2,s}|_H & |h_i v_{u_{m-2},2,s}|_H - |h_i v_{u_{m-2},2,s}|_V \\ p_{1,1} & p_{1,2} \\ -|h_i v_{u_{m-2},2,s}|_V & \end{cases} \quad \begin{array}{l} \text{if } d_2 = 1; \\ \\ \text{if } d_2 > 1. \end{array}$$

Proof. Using the notation of Definition 2.9, for any grading $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ there is the factorization

$$Y_{D_m}(\omega) = Y_{h_{1,1}v_{u_{m-2},2,1}}(\omega) \cdots Y_{h_{1,s-1}v_{u_{m-2},2,s-1}}(\omega)Y_{h_{1,s}\bar{h}_i}(\omega)Y_{h_i v_{u_{m-1},2}}(\omega).$$

By definition of χ_H , every term of $Y_{D_{m-1}}(\chi_H)$ ends with the monomial

$$p_1^{-1} X^{-|h_i v_{u_{m-2},2,s}|_H}.$$

Theorem 3.20 shows that any piecewise compatible grading $\omega : E_m \rightarrow \mathbb{Z}_{\geq 0}$ of D_m which is not compatible agrees with (ω_H, ω_V^*) on the path $h_i v_{u_{m-1,2}}$. The result then immediately follows from Corollary 4.3 and the definition of piecewise compatible gradings in Definition 3.11. \square

The next result gives an analogous factorization for sums over piecewise compatible gradings of D'_{m+1} which are not compatible.

LEMMA 4.8. *Let $\omega_{V'} : V'_{m+1} \rightarrow [0, d_1]$ be a vertical grading of D'_{m+1} , $m \geq 3$, for which there exists a horizontal grading $\omega_{H'}^* : H'_{m+1} \rightarrow [0, d_2]$ of D'_{m+1} so that $(\omega_{H'}^*, \omega_{V'})$ is piecewise compatible but not compatible. Write*

$$Y'^{nc}_{D'_{m+1}}(\omega_{V'}) = \sum_{\omega : E'_{m+1} \rightarrow \mathbb{Z}_{\geq 0}} Y'_{D'_{m+1}}(\omega),$$

where the sum ranges over piecewise compatible gradings ω of D'_{m+1} which are not compatible and satisfy $\omega|_{V'_{m+1}} = \omega_{V'}$. Let $h'_j \in H'_{m+1}$ denote the blocking edge of $\omega_{H'}^*$, where $\text{ht}(h'_j) = i - 1$, and set

$$d = |\text{supp}(\omega_{V'}) \cap h'_j v'_{u'_{m,2}}| \quad \text{and} \quad t = |\text{supp}(\omega_{H'}^*) \cap h'_j v'_{u'_{m,2}}|.$$

Let $s = \left\lceil \frac{i}{u_{m-1,1}} \right\rceil$ denote the index so that $h'_j \in H'_{m+1,s}$. Define a horizontal grading $\chi_{V'} : V'_m \rightarrow [0, d_1]$ of D'_m with $\chi_{V'}(v') = \omega_{V',s}(v')$ for $v' \in (h'_{1,s} \bar{h}'_j)_V$ and $\chi_{V'}(v') = \ell$ for $v' \in (h'_j v'_{u'_{m-1,2},s})_V$ if v' is immediately preceded by exactly ℓ horizontal edges in this copy of D'_m . Then there is the following factorization:

$$(19) \quad Y'^{nc}_{D'_{m+1}}(\omega_{V'}) = Y'_{D'_m}(\omega_{V',1}) \cdots Y'_{D'_m}(\omega_{V',s-1}) Y'_{D'_m}(\chi_{V'}) p_1 X Y'^{|h'_j v'_{u'_{m-1,2},s}|_V} X^{-1} p_2 Y X Y^{-1} X^{-1},$$

where $p_2 = p_{1,d_1 - \omega_V(v'_{u'_{m,2}-d+1})} p_{2,d_2 - \omega_{H'}^*(h'_{j-1+t})}$ and

$$p_1 = \begin{cases} p_{1,1} & \text{if } d_2 = 1; \\ -|h_i v_{u_{m-2,2},s}|_V & \text{if } d_2 > 1; \end{cases}$$

with $h_i v_{u_{m-2,2},s}$ being the subpath in the s -th copy of D_{m-1} inside D_m .

Proof. By definition of $\chi_{V'}$, every term of $Y'_{D'_m}(\chi_{V'})$ ends with the monomial $p_1^{-1} X Y'^{-|h'_j v'_{u'_{m-1,2},s}|_V} X^{-1}$. To see the coefficient p_1^{-1} , note that by Lemma 2.3 the structure of D'_m is determined by the structure of D_{m-1} .

Theorem 3.20 shows that any piecewise compatible grading $\omega : E'_{m+1} \rightarrow \mathbb{Z}_{\geq 0}$ of D'_{m+1} which is not compatible agrees with $(\omega_{H'}^*, \omega_{V'})$ on the path $h'_j v'_{u'_{m,2}}$. The result then immediately follows from Corollary 4.3 and the definition of piecewise compatible gradings in Definition 3.11. \square

4.1. PROOF OF MAIN THEOREM. We work by induction on $m \geq 1$. From the definition of the non-commutative weights in equation (4), we immediately see

$$Y_{D_1} = P_1(Y) X^{-1} = F_{P_1}(Y) = Y_1$$

and

$$\begin{aligned} Y_{D_2} &= \sum_{\ell=0}^{d_2} (P_1(Y) X^{-1})^{d_2-\ell} (X^{-1})^\ell (p_{2,d_2-\ell} X^{\ell+1} Y^{-1} X^{-1}) \\ &= P_2(P_1(Y) X^{-1}) X Y^{-1} X^{-1} = F_{P_1} F_{P_2}(Y) = Y_2. \end{aligned}$$

Write $Y_{D_2} = \sum_{\omega_H: H_2 \rightarrow [0, d_1]} Y_{D_2}(\omega_H)$. We will show that $F_{P_0}(Y_{D_2}(\omega_H)) = Y'_{D'_3}(\varphi_2^* \omega_H)$ for each horizontal grading $\omega_H : H_2 \rightarrow [0, d_1]$. Fix a horizontal grading $\omega_H : H_2 \rightarrow [0, d_1]$. If $\text{supp}(\omega_H) = \emptyset$, then

$$Y_{D_2}(\omega_H) = (X^{-1})^{d_2} P_0(X) X Y^{-1} X^{-1}$$

and so

$$\begin{aligned} F_{P_0}(Y_{D_2}(\omega_H)) &= (X Y^{-1} X^{-1})^{d_2} (X P_0(Y) X^{-1}) (X Y X^{-1}) (P_0(Y) X^{-1})^{-1} (X Y^{-1} X^{-1}) \\ &= (X Y^{-1} X^{-1})^{d_2} X Y X Y^{-1} X^{-1} \\ &= (X Y^{-1} X^{-1})^{d_2-1} X^2 Y^{-1} X^{-1} \\ &= Y'_{D'_3}(\varphi_2^* \omega_H). \end{aligned}$$

Suppose $\text{supp}(\omega_H) \neq \emptyset$. Let h_i be the last horizontal edge in $\text{supp}(\omega_H)$. Then (ω_H, ω_V) will be compatible if and only if $\omega_V(v_1) \leq d_2 - i$. This gives

$$\begin{aligned} &Y_{D_2}(\omega_H) \\ &= (p_{1, \omega(h_1)} Y^{\omega(h_1)} X^{-1}) \cdots (p_{1, \omega(h_i)} Y^{\omega(h_i)} X^{-1}) (X^{-1})^{d_2-i} \sum_{\ell=0}^{d_2-i} p_{2, d_2-\ell} X^{\ell+1} Y^{-1} X^{-1}. \end{aligned}$$

Applying F_{P_0} gives

$$\begin{aligned} &F_{P_0}(Y_{D_2}(\omega_H)) \\ &= (p_{1, \omega(h_1)} (P_0(Y) X^{-1})^{\omega(h_1)} X Y^{-1} X^{-1}) \cdots (p_{1, \omega(h_i)} (P_0(Y) X^{-1})^{\omega(h_i)} X Y^{-1} X^{-1}) \\ &\quad \times [X Y^{-1} X^{-1}]^{d_2-i} \sum_{\ell=0}^{d_2-i} p_{2, d_2-\ell} (X Y X^{-1})^{\ell+1} (P_0(Y) X^{-1})^{-1} (X Y^{-1} X^{-1}) \\ &= (P_0(Y) X^{-1})^{\omega(h_1)} (X^{-1})^{d_1-\omega(h_1)} (p_{1, \omega(h_1)} X^{d_1-\omega(h_1)+1} Y^{-1} X^{-1}) \cdots (P_0(Y) X^{-1})^{\omega(h_i)-1} \\ &\quad \times \left(\sum_{\ell=0}^{d_2-i} p_{2, d_2-\ell} Y^\ell X^{-1} \right) (X^{-1})^{d_1-\omega(h_i)} (p_{1, \omega(h_i)} X^{d_1-\omega(h_i)+1} Y^{-1} X^{-1}) \\ &\quad [(X^{-1})^{d_1} (X^{d_1+1} Y^{-1} X^{-1})]^{d_2-i-1} (X^{-1})^{d_1-1} (X^{d_1+1} Y^{-1} X^{-1}) \\ &= Y'_{D'_3}(\varphi_2^* \omega_H). \end{aligned}$$

Suppose $m \geq 3$ and let $\omega_H : H_m \rightarrow [0, d_1]$ be a horizontal grading of D_m . Following Theorem 3.20, there are two cases to consider.

- (a) Suppose that (ω_H, ω_V) is compatible for every piecewise compatible grading (ω_H, ω_V) of D_m . Then Lemma 4.4 shows there is the factorization

$$\begin{aligned} (\dagger) \quad &Y_{D_m}(\omega_H) \\ &= Y_{D_{m-1}}(\omega_{H,1}) Y_{D_{m-1}}(\omega_{H,2}) \cdots Y_{D_{m-1}}(\omega_{H, d_m-1-\delta_m}) p X^{|H_{m-2-\delta_m}|} Y_{D_{m-1}}(\omega_{H, d_m-\delta_m}), \end{aligned}$$

where

$$p = \begin{cases} p_{1,1}^{|V_{m-2-\delta_m}|-2|H_{m-2-\delta_m}|} p_{1,2}^{|H_{m-2-\delta_m}|-|V_{m-2-\delta_m}|} & \text{if } d_2 = 1; \\ p_{1,1}^{-|V_{m-2-\delta_m}|} & \text{if } d_2 > 1. \end{cases}$$

If $m \geq 3 + \delta'_{m+1}$, we may apply Lemma 4.5 to conclude by induction that $F_{P_0}(Y_{D_m}(\omega_H)) = Y'_{D'_{m+1}}(\varphi_m^* \omega_H)$.

It remains to consider the case $m = 3$ with $\delta'_4 = 1$, i.e. $d_2 = 1$. In this case D_2 consists of a single horizontal edge followed by a single vertical edge and, by Corollary 2.4(c), D_3 consists of $d_1 - 1$ copies of D_2 followed

by a single vertical edge, in particular D_3 ends with two consecutive vertical edges. The factorization (†) still holds in this case and by induction we have $F_{P_0}(Y_{D_2}(\omega_{H,r})) = Y'_{D'_3}(\varphi_2^* \omega_{H,r})$ for $1 \leq r \leq d_1$. In particular, by Lemma 4.6 to see that $F_{P_0}(Y_{D_3}(\omega_H)) = Y'_{D'_4}(\varphi_3^* \omega_H)$ it suffices to compare $F_{P_0}(Y_{D_2}(\omega_{H,d_1-1})pXY_{D_2}(\omega_{H,d_1}))$ with $XY'_{D'_3}(\omega_{V',d_1-1})$, where we write $\omega_{V'} = \varphi_3^* \omega_H$.

There are two cases to consider. If $\omega(h_{d_1-1}) = 0$, we have $Y_{D_2}(\omega_{H,d_1-1}) = (X^{-1})P_0(X)XY^{-1}X^{-1}$ and so

$$\begin{aligned} F_{P_0}(Y_{D_2}(\omega_{H,d_1-1})) &= (XY^{-1}X^{-1})(XP_0(Y)X^{-1})(XYX^{-1})(P_0(Y)X^{-1})^{-1}XY^{-1}X^{-1} \\ &= X^2Y^{-1}X^{-1}. \end{aligned}$$

The same calculation shows $F_{P_0}(Y_{D_2}(\omega_{H,d_1})) = X^2Y^{-1}X^{-1}$ by the assumptions on ω_{H,d_1} in Definition 3.11. But then, since $p = 1$ in this case, we have

$$\begin{aligned} F_{P_0}(Y_{D_2}(\omega_{H,d_1-1})pXY_{D_2}(\omega_{H,d_1})) &= (X^2Y^{-1}X^{-1})(XYX^{-1})(X^2Y^{-1}X^{-1}) \\ &= X^3Y^{-1}X^{-1} \\ &= X(X^{-1})^{d_1-1}(X^{d_1+1}Y^{-1}X^{-1}), \end{aligned}$$

which is exactly $XY'_{D'_3}(\omega_{V',d_1-1})$.

When $h_{d_1-1} \in \text{supp}(\omega_H)$, we have $Y_{D_2}(\omega_{H,d_1-1}) = p_{1,\omega(h_{d_1-1})}Y^{\omega(h_{d_1-1})-1}X^{-1}$ so that

$$\begin{aligned} F_{P_0}(Y_{D_2}(\omega_{H,d_1-1})) &= (P_0(Y)X^{-1})^{\omega(h_{d_1-1})-1}(X^{-1})^{d_1-\omega(h_{d_1-1})}p_{1,\omega(h_{d_1-1})}X^{d_1-\omega(h_{d_1-1})+1}Y^{-1}X^{-1}. \end{aligned}$$

We saw above that $F_{P_0}(Y_{D_2}(\omega_{H,d_1})) = X^2Y^{-1}X^{-1}$ and so

$$\begin{aligned} F_{P_0}(Y_{D_2}(\omega_{H,d_1-1})pXY_{D_2}(\omega_{H,d_1})) &= (P_0(Y)X^{-1})^{\omega(h_{d_1-1})-1}(X^{-1})^{d_1-\omega(h_{d_1-1})-1}p_{1,\omega(h_{d_1-1})}X^{d_1-\omega(h_{d_1-1})+1}Y^{-1}X^{-1}, \end{aligned}$$

which is exactly $XY'_{D'_3}(\omega_{V',d_1-1})$. The claim then follows by induction from Lemma 4.6.

- (b) Suppose there exists a vertical grading $\omega_V^* : V_m \rightarrow [0, d_2]$ of D_m so that (ω_H, ω_V^*) is piecewise compatible, but not compatible. By Theorem 3.20, there must exist a blocking edge h_i for ω_H . Set

$$d = |\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}}| \quad \text{and} \quad t = |\text{supp}(\omega_V^*) \cap h_i v_{u_{m-1,2}}|.$$

By Proposition 3.10 and Proposition 3.16, $(\Omega_m \omega_V^*, \varphi_m^* \omega_H)$ is a piecewise compatible grading of D'_{m+1} which is not compatible. Let $D(v'_{u'_{m,2}}; \varphi_m^* \omega_H) = h'_j v'_{u'_{m,2}}$ and observe that $\text{ht}(h'_j) = i - 1$ by definition of Ω_m . By Proposition 3.18, Lemma 3.7, and Corollary 3.23, we have

$$\begin{aligned} |\text{supp}(\Omega_m \omega_V^*) \cap h'_j v'_{u'_{m,2}}| &= |\text{rsh}(\varphi_m^* \omega_H) \cap h'_j v'_{u'_{m,2}}| = |\text{rsh}(\omega_H) \cap h_i v_{u_{m-1,2}}| \\ &= |\text{supp}(\omega_V) \cap h_i v_{u_{m-1,2}}|. \end{aligned}$$

Moreover, we have

$$|\text{supp}(\varphi_m^* \omega_H) \cap h'_j v'_{u'_{m,2}}| = u_{m,1} - i + 1 - |\text{supp}(\omega_H) \cap h_i v_{u_{m-1,2}}| + \delta,$$

where $\delta = 0$ if $\omega_H(h_{i-1+d}) = d_1$ and $\delta = 1$ otherwise. It then follows from the definitions of φ_m^* and Ω_m that the coefficients p_2 agree in Lemma 4.7 and Lemma 4.8.

Using the notation of Lemma 4.4 and Lemma 4.7, we have

$$Y_{D_m}(\omega_H) = Y_{D_m}^{pc}(\omega_H) - Y_{D_m}^{nc}(\omega_H).$$

By induction we have $F_{P_0}(Y_{D_{m-1}}(\omega_{H,r})) = Y'_{D'_m}(\varphi_{m-1}^* \omega_{H,r})$ for $1 \leq r \leq d_m - \delta_m$ and $F_{P_0}(Y_{D_{m-1}}(\chi_H)) = Y'_{D'_m}(\chi_{V'})$. It follows that

$$F_{P_0}(Y_{D_m}^{pc}(\omega_H)) = Y'^{pc}_{D'_{m+1}}(\varphi_m^* \omega_H) \quad \text{and} \quad F_{P_0}(Y_{D_m}^{nc}(\omega_H)) = Y'^{nc}_{D'_{m+1}}(\varphi_m^* \omega_H).$$

Since $Y'_{D'_{m+1}}(\varphi_m^* \omega_H) = Y'^{pc}_{D'_{m+1}}(\varphi_m^* \omega_H) - Y'^{nc}_{D'_{m+1}}(\varphi_m^* \omega_H)$, the result follows.

REMARK 4.9. Our proof of the Main Theorem developed a combinatorial model for the analogue (2) of initial cluster mutations. It would be interesting and highly non-trivial to understand the direct combinatorial interpretation for the non-commutative exchange relations (3).

5. SPECIALIZATIONS

In this section we consider the specialization to quantum generalized cluster variables. Assume $v \in \mathbb{k}$ is transcendental over \mathbb{Q} . Define the quantum torus algebra $\mathcal{T} := \mathcal{T}_v = \mathbb{k}\langle Z_1, Z_2 : Z_1 Z_2 = v^2 Z_2 Z_1 \rangle$ and let \mathcal{F} denote the skew-field of fractions of \mathcal{T} . It will be convenient to consider elements $Z^{\mathbf{a}} := v^{-a_1 a_2} Z_1^{a_1} Z_2^{a_2}$ for $\mathbf{a} = (a_1, a_2) \in \mathbb{Z}^2$, these form a \mathbb{k} -basis of \mathcal{T} .

Recall the notation (1) for the polynomials $P_k, k \in \mathbb{Z}$. Consider *quantum generalized cluster variables* $Z_k^{(\alpha)} \in \mathcal{F}, \alpha, k \in \mathbb{Z}$, defined recursively by

$$(20) \quad Z_1^{(\alpha)} = Z_1, \quad Z_2^{(\alpha)} = Z_2, \quad Z_{k-1}^{(\alpha)} Z_{k+1}^{(\alpha)} = P_{\alpha+k}(v Z_k^{(\alpha)}).$$

Observe that equation (20) immediately implies $Z_k^{(\alpha)} Z_{k+1}^{(\alpha)} = v^2 Z_{k+1}^{(\alpha)} Z_k^{(\alpha)}$ for all $\alpha, k \in \mathbb{Z}$.

For a fixed $\alpha \in \mathbb{Z}$, the *quantum generalized cluster algebra* $\mathcal{A}_v^{(\alpha)}(P_1, P_2) \subset \mathcal{F}$ is the \mathbb{k} -subalgebra generated by the $Z_k^{(\alpha)}, k \in \mathbb{Z}$. Although they are defined as elements of \mathcal{F} , the quantum generalized cluster variables actually live in \mathcal{T} . We give a direct proof here, however the combinatorial construction below provides an alternate proof. See [1] for a proof of this result in the special case when $P_1 = \overline{P}_1$ and $P_2 = \overline{P}_2$.

THEOREM 5.1. *Each quantum generalized cluster variable $Z_k^{(\alpha)}$ is an element of $\mathcal{T} \subset \mathcal{F}$.*

Proof. Consider the monomial $v^{d_{\alpha+k}} Z_{k-1}^{(\alpha)} (Z_{k+2}^{(\alpha)})^{d_{\alpha+k}}$. Expanding $Z_{k-1}^{(\alpha)}$ in terms of $Z_k^{(\alpha)}$ and $Z_{k+1}^{(\alpha)}$ using equation (20) gives $v^{d_{\alpha+k}} Z_{k-1}^{(\alpha)} (Z_{k+2}^{(\alpha)})^{d_{\alpha+k}}$ as

$$\begin{aligned} & v^{d_{\alpha+k}} P_{\alpha+k}(v Z_k^{(\alpha)}) (Z_{k+1}^{(\alpha)})^{-1} (Z_{k+2}^{(\alpha)})^{d_{\alpha+k}} \\ &= v^{-d_{\alpha+k}} P_{\alpha+k}(v Z_k^{(\alpha)}) (Z_{k+2}^{(\alpha)})^{d_{\alpha+k}} (Z_{k+1}^{(\alpha)})^{-1} \\ &= \sum_{i=0}^{d_{\alpha+k}} p_{\alpha+k,i} v^{-d_{\alpha+k}+i} [(Z_k^{(\alpha)})^i (Z_{k+2}^{(\alpha)})^i - 1] (Z_{k+2}^{(\alpha)})^{d_{\alpha+k}-i} (Z_{k+1}^{(\alpha)})^{-1} \\ &\quad + P_{\alpha+k+2}(v^{-1} Z_{k+2}^{(\alpha)}) (Z_{k+1}^{(\alpha)})^{-1} \\ &= \sum_{i=0}^{d_{\alpha+k}} p_{\alpha+k,i} v^{d_{\alpha+k}-i} [(Z_k^{(\alpha)})^i (Z_{k+2}^{(\alpha)})^i - 1] (Z_{k+1}^{(\alpha)})^{-1} (Z_{k+2}^{(\alpha)})^{d_{\alpha+k}-i} \\ &\quad + (Z_{k+1}^{(\alpha)})^{-1} P_{\alpha+k+2}(v Z_{k+2}^{(\alpha)}). \end{aligned}$$

But for $0 \leq i \leq d_{\alpha_k}$, the term $(Z_k^{(\alpha)})^i (Z_{k+2}^{(\alpha)})^i - 1$ above is a polynomial in $Z_{k+1}^{(\alpha)}$ with no constant term and so $[(Z_k^{(\alpha)})^i (Z_{k+2}^{(\alpha)})^i - 1] (Z_{k+1}^{(\alpha)})^{-1}$ is a polynomial in $Z_{k+1}^{(\alpha)}$. Thus we may solve for $Z_{k+3}^{(\alpha)} = (Z_{k+1}^{(\alpha)})^{-1} P_{\alpha+k+2}(vZ_{k+2}^{(\alpha)})$ above and see that this generalized cluster variable can be written as a polynomial in $\mathbb{k}[Z_{k-1}^{(\alpha)}, Z_k^{(\alpha)}, Z_{k+1}^{(\alpha)}, Z_{k+2}^{(\alpha)}]$. A similar calculation shows $Z_{k-2}^{(\alpha)} \in \mathbb{k}[Z_{k-1}^{(\alpha)}, Z_k^{(\alpha)}, Z_{k+1}^{(\alpha)}, Z_{k+2}^{(\alpha)}]$. Then by induction we see $Z_k^{(\alpha)} \in \mathbb{k}[Z_0^{(\alpha)}, Z_1^{(\alpha)}, Z_2^{(\alpha)}, Z_3^{(\alpha)}] \subset \mathcal{T}$ for all $\alpha, k \in \mathbb{Z}$. \square

REMARK 5.2. The proof above actually shows more. We see from this proof that

$$\mathcal{A}_v^{(\alpha)}(P_1, P_2) = \mathbb{k}[Z_{k-1}^{(\alpha)}, Z_k^{(\alpha)}, Z_{k+1}^{(\alpha)}, Z_{k+2}^{(\alpha)}]$$

for each $\alpha, k \in \mathbb{Z}$.

Define the quantum specialization $\pi_v : \mathbb{k} \rightarrow \mathcal{F}$ by

$$(21) \quad \pi_v(X) = vZ_1, \quad \pi_v(Y) = v^{-1}Z_2.$$

Note that for $Q = XYX^{-1}Y^{-1}$ we have $\pi_v(Q) = v^2$. For $\alpha \in \mathbb{Z}$, set $X_0^{(\alpha)} = X$ and for $m \geq 1$ define elements $X_m^{(\alpha)}, X_{-m}^{(\alpha)} \in \mathbb{k}$ by

$$X_m^{(\alpha)} = F_{P_{\alpha+1}} F_{P_{\alpha+2}} \cdots F_{P_{\alpha+m}}(X) \quad \text{and} \quad X_{-m}^{(\alpha)} = F_{P_{\alpha}}^{-1} F_{P_{\alpha-1}}^{-1} \cdots F_{P_{\alpha-m+1}}^{-1}(X)$$

and observe that Theorem 1.3 provides a combinatorial construction of each $X_m^{(\alpha)}$. The following specialization result will provide a combinatorial construction of the quantum generalized cluster variables $Z_m^{(\alpha)}$.

THEOREM 5.3. For $m, \alpha \in \mathbb{Z}$, we have $\pi_v(X_m^{(\alpha+1)}) = vZ_{m+1}^{(\alpha)}$.

Proof. We work by induction on m . Since $X_0^{(\alpha)} = X$ and $X_1^{(\alpha)} = QY$ for all $\alpha \in \mathbb{Z}$, the cases $m = 0, 1$ follow immediately from equation (21).

For any nonzero polynomial $P \in \mathbb{k}[z]$, define a \mathbb{k} -linear automorphism $\mu_P : \mathcal{F} \rightarrow \mathcal{F}$ given by $\mu_P(Z_1) = Z_2$ and $\mu_P(Z_2) = Z_1^{-1}P(vZ_2)$. These satisfy the functional identities $\pi_v \circ F_P = \mu_P \circ \pi_v$. Note that $\mu_P^{-1}(Z_1) = P(vZ_1)Z_2^{-1}$ and $\mu_P^{-1}(Z_2) = Z_1$ so that $\pi_v \circ F_P^{-1} = \mu_P^{-1} \circ \pi_v$.

Moreover, observe that $\mu_{P_{\alpha+2}}(Z_2) = Z_3^{(\alpha)}$ and $\mu_{P_{\alpha+1}}^{-1}(Z_1) = Z_0^{(\alpha)}$ for $\alpha \in \mathbb{Z}$. By the symmetry of the exchange relations (20), these imply $\mu_{P_{\alpha+2}}(Z_m^{(\alpha+1)}) = Z_{m+1}^{(\alpha)}$ and $\mu_{P_{\alpha+1}}^{-1}(Z_{m+2}^{(\alpha-1)}) = Z_{m+1}^{(\alpha)}$ for any $\alpha, m \in \mathbb{Z}$. Indeed, by induction on $m \geq 3$ we have

$$\begin{aligned} Z_{m+1}^{(\alpha)} &= \left(Z_{m-1}^{(\alpha)}\right)^{-1} P_{\alpha+m} \left(vZ_m^{(\alpha)}\right) = \mu_{P_{\alpha+2}} \left(\left(Z_{m-2}^{(\alpha+1)}\right)^{-1} P_{\alpha+m} \left(vZ_{m-1}^{(\alpha+1)}\right) \right) \\ &= \mu_{P_{\alpha+2}} \left(Z_m^{(\alpha+1)} \right). \end{aligned}$$

Similarly, by induction on $m \leq -2$ we have

$$\begin{aligned} Z_{m+1}^{(\alpha)} &= \left(Z_{m-1}^{(\alpha)}\right)^{-1} P_{\alpha+m} \left(vZ_m^{(\alpha)}\right) = \mu_{P_{\alpha+1}}^{-1} \left(\left(Z_m^{(\alpha-1)}\right)^{-1} P_{\alpha+m} \left(vZ_{m+1}^{(\alpha-1)}\right) \right) \\ &= \mu_{P_{\alpha+1}}^{-1} \left(Z_{m+2}^{(\alpha-1)} \right). \end{aligned}$$

Thus, by induction on $m \geq 1$ we see

$$\begin{aligned} \pi_v \left(X_m^{(\alpha+1)} \right) &= \pi_v \left(F_{P_{\alpha+2}} \left(X_{m-1}^{(\alpha+2)} \right) \right) = \mu_{P_{\alpha+2}} \left(\pi_v \left(X_{m-1}^{(\alpha+2)} \right) \right) = \mu_{P_{\alpha+2}} \left(vZ_m^{(\alpha+1)} \right) \\ &= vZ_{m+1}^{(\alpha)} \end{aligned}$$

and by induction on $m \leq -1$ we see

$$\begin{aligned} \pi_v \left(X_m^{(\alpha+1)} \right) &= \pi_v \left(F_{P_{\alpha+1}}^{-1} \left(X_{m+1}^{(\alpha)} \right) \right) = \mu_{P_{\alpha+1}}^{-1} \left(\pi_v \left(X_{m+1}^{(\alpha)} \right) \right) = \mu_{P_{\alpha+1}}^{-1} \left(v Z_{m+2}^{(\alpha-1)} \right) \\ &= v Z_{m+1}^{(\alpha)}. \end{aligned} \quad \square$$

Applying the quantum specialization π_v to Theorem 1.3, Theorem 5.3 gives the following combinatorial construction of the quantum generalized cluster variables $Z_m^{(\alpha)}$. For notational convenience, we restrict to the quantum generalized cluster variables $Z_m := Z_m^{(3)}$.

COROLLARY 5.4.

(1) For $m \geq 3$, the quantum generalized cluster variable Z_m is computed as follows:

$$(22) \quad Z_m = \sum_{\omega: E_{m-2} \rightarrow \mathbb{Z}_{\geq 0}} p_{\omega} v^{1-u_{m-2,1}-u_{m-3,2}+\gamma_{\omega}+\beta_{\omega}} Z^{(-u_{m-2,1}+|\omega_V|, -u_{m-3,2}+|\omega_H|)},$$

where

- the sum ranges over (d_1, d_2) -bounded compatible gradings ω of D_{m-2} ;
- $p_{\omega} = \prod_{i=1}^{u_{m-2,1}} p_{1, \omega_H(h_i)} \prod_{t=1}^{u_{m-3,2}} p_{2, d_2 - \omega_V(v_t)}$;
- $\gamma_{\omega} = \sum_{e < e' \in E_{m-2}} \gamma_{\omega}(e, e')$ for

$$(23) \quad \gamma_{\omega}(e, e') = \begin{cases} 0 & \text{if } e \in H_{m-2} \setminus \text{supp}(\omega_H) \text{ or } e' \in V_{m-2} \setminus \text{supp}(\omega_V); \\ -2\omega(e)\omega(e') & \text{if } e \in \text{supp}(\omega_H) \text{ and } e' \in \text{supp}(\omega_V); \\ 2\omega(e) & \text{if } e \in \text{supp}(\omega_H) \text{ and } e' \in H_{m-2}; \\ 2\omega(e') & \text{if } e \in V_{m-2} \text{ and } e' \in \text{supp}(\omega_V); \\ -2 & \text{if } e \in V_{m-2} \text{ and } e' \in H_{m-2}; \end{cases}$$

- $\beta_{\omega} = \sum_{e < e' \in E_{m-2}} \beta_{\omega}(e, e')$ for

$$(24) \quad \beta_{\omega}(e, e') = \begin{cases} \omega(e)\omega(e') + 1 & \text{if } e \in H_{m-2} \text{ and } e' \in V_{m-2} \text{ or } e \in V_{m-2} \text{ and } e' \in H_{m-2}; \\ -(\omega(e) + \omega(e')) & \text{if } e, e' \in H_{m-2} \text{ or } e, e' \in V_{m-2}. \end{cases}$$

(2) For $m \leq 0$, the quantum generalized cluster variable Z_m is computed as follows:

$$(25) \quad Z_m = \sum_{\omega: E'_{-m+1} \rightarrow \mathbb{Z}_{\geq 0}} p'_{\omega} v^{-1+u'_{-m+1,1}+u'_{-m,2}+\gamma'_{\omega}+\beta'_{\omega}} Z^{(-u'_{-m,2}+|\omega_{H'}|, -u'_{-m+1,1}+|\omega_{V'}|)},$$

where

- the sum ranges over (d_2, d_1) -bounded compatible gradings ω of D'_{-m+1} ;
- $p'_{\omega} = \prod_{i=1}^{u'_{-m+1,1}} p_{2, d_2 - \omega_{H'}(h'_i)} \prod_{t=1}^{u'_{-m,2}} p_{1, \omega_{V'}(v'_t)}$;
- $\gamma'_{\omega} = \sum_{e < e' \in E'_{-m+1}} \gamma'_{\omega}(e, e')$ for

$$\gamma'_{\omega}(e, e') = \begin{cases} 0 & \text{if } e \in V'_{-m+1} \setminus \text{supp}(\omega_{V'}) \text{ or } e' \in H'_{-m+1} \setminus \text{supp}(\omega_{H'}); \\ -2\omega(e)\omega(e') & \text{if } e \in \text{supp}(\omega_{V'}) \text{ and } e' \in \text{supp}(\omega_{H'}); \\ 2\omega(e) & \text{if } e \in \text{supp}(\omega_{V'}) \text{ and } e' \in V'_{-m+1}; \\ 2\omega(e') & \text{if } e \in H'_{-m+1} \text{ and } e' \in \text{supp}(\omega_{H'}); \\ -2 & \text{if } e \in H'_{-m+1} \text{ and } e' \in V'_{-m+1}; \end{cases}$$

• $\beta'_\omega = \sum_{e < e' \in E'_{-m+1}} \beta'_\omega(e, e')$ for

$$\beta'_\omega(e, e') = \begin{cases} \omega(e)\omega(e') + 1 & \text{if } e \in H'_{-m+1} \text{ and } e' \in V'_{-m+1} \text{ or } e \in V'_{-m+1} \text{ and } e' \in H'_{-m+1}; \\ -(\omega(e) + \omega(e')) & \text{if } e, e' \in H'_{-m+1} \text{ or } e, e' \in V'_{-m+1}. \end{cases}$$

Proof. We prove part 1, the proof of part 2 is essentially the same where the roles of X and Y are interchanged in equation (4).

First note that we have $X_{m-1} = QY_{m-2}$ so that $vZ_m = \pi_v(X_{m-1}) = v^2\pi_v(Y_{m-2})$, in particular this accounts for the 1 appearing in the exponent of v in equation (22). By Theorem 1.3, we may compute Y_{m-2} by considering compatible gradings on the maximal Dyck path D_{m-2} and thus Z_m can be computed by applying the quantum projection π_v to equation (5). Then the exponents of Z_1 and Z_2 in equation (22) are immediate from Lemma 3.2. The coefficient p_ω also follows directly from the definition of the non-commutative edge weights in equation (4), so for the remainder of the proof we assume $p_{i,j} = 1$ for all i and j .

Note that

$$(26) \quad Z^{(a_1, a_2)} Z^{(b_1, b_2)} = v^{a_1 b_2 - a_2 b_1} Z^{(a_1 + b_1, a_2 + b_2)}$$

for $a_i, b_i \in \mathbb{Z}$, $i = 1, 2$. The rest of the exponent of v in equation (22) can be seen as follows:

(a) for an edge $e \in E_{m-2}$ we have

$$\begin{aligned} \pi_v(\text{wt}_\omega(e)) &= \begin{cases} \pi_v(Y^{\omega(e)} X^{-1}) & \text{if } e \in H_{m-2} \\ \pi_v(X^{\omega(e)+1} Y^{-1} X^{-1}) & \text{if } e \in V_{m-2} \end{cases} \\ &= \begin{cases} v^{-\omega_H(e)-1} Z_2^{\omega_H(e)} Z_1^{-1} i & \text{if } e \in H_{m-2} \\ v^{\omega(e)+1} Z_1^{\omega(e)+1} Z_2^{-1} Z_1^{-1} & \text{if } e \in V_{m-2} \end{cases} \\ &= \begin{cases} v^{-1} Z^{(-1, \omega_H(e))} & \text{if } e \in H_{m-2}; \\ v^{-1} Z^{(\omega(e), -1)} & \text{if } e \in V_{m-2}; \end{cases} \end{aligned}$$

The v^{-1} in each possibility above accounts for the terms $-u_{m,1}$ and $-u_{m-1,2}$ in equation (22).

(b) for $e, e' \in E_{m-2}$, the quantity $\gamma_\omega(e, e')$ from equation (23) records the power of v which appears when commuting powers of Z_2 appearing in $\pi_v(\text{wt}_\omega(e))$ past powers of Z_1 appearing in $\pi_v(\text{wt}_\omega(e'))$;

(c) for $e, e' \in E_{m-2}$, the quantity $\beta_\omega(e, e')$ from equation (24) records the power of v so that

$$\pi_v(\text{wt}_\omega(e))\pi_v(\text{wt}_\omega(e')) = v^{\gamma_\omega(e, e') + \beta_\omega(e, e') - 2} Z^{(a_1, a_2)}$$

for appropriate $a_1, a_2 \in \mathbb{Z}$ depending on $e, e' \in E_{m-2}$ (the -2 here accounts for part (a) above).

Since we have $Z_m = v\pi_v(Y_{D_{m-2}})$, the result follows by combining the observations above. \square

Let $\mathbb{k} = \mathbb{Q}(v)$ for an indeterminate v . When $p_{i,j} = 0$ for $i = 1, 2$ and $1 \leq j \leq d_i - 1$, the expansions of the quantum generalized cluster variables as elements of \mathcal{T} have been computed [11] using the representation theory of valued quivers as follows. In this case, we drop the adjective “generalized” and refer to the $Z_k^{(\alpha)}$ simply as *quantum cluster variables*.

Let $d = \gcd(d_1, d_2)$. Consider the quiver Λ with vertices $\Lambda_0 = \{1, 2\}$ with d arrows $a_j : 2 \rightarrow 1$, $1 \leq j \leq d$. Write \mathbb{F}_q for the finite field with q elements and fix an algebraic closure $\overline{\mathbb{F}}$ of \mathbb{F}_q . Let $\mathbb{F}_{q^{d_1}}, \mathbb{F}_{q^{d_2}}, \mathbb{F}_{q^d} \subset \overline{\mathbb{F}}$ denote the extension fields of \mathbb{F}_q of degree d_1, d_2, d , respectively. Note that $\mathbb{F}_{q^{d_1}}$ and $\mathbb{F}_{q^{d_2}}$ are naturally identified as vector spaces over \mathbb{F}_{q^d} .

A valued representation $V = (V_1, V_2, V_{a_j})$ of Λ consists of $\mathbb{F}_{q^{d_i}}$ -vector spaces V_i for $i = 1, 2$ and \mathbb{F}_{q^d} -linear maps $V_{a_j} : V_2 \rightarrow V_1$ for $1 \leq j \leq d$. For representations $V = (V_1, V_2, V_{a_j})$ and $W = (W_1, W_2, W_{a_j})$, a morphism $\theta : V \rightarrow W$ consists of $\mathbb{F}_{q^{d_i}}$ -linear maps $\theta_i : V_i \rightarrow W_i$ for $i = 1, 2$ such that the following diagram commutes for $1 \leq j \leq d$:

$$\begin{array}{ccc} V_1 & \xleftarrow{V_{a_j}} & V_2 \\ \theta_1 \downarrow & & \downarrow \theta_2 \\ W_1 & \xleftarrow{W_{a_j}} & W_2 \end{array}$$

Thus the finite-dimensional valued representations of Λ form a category $\text{rep}(\Lambda)$. In fact, this category is well-known to be abelian, \mathbb{F}_q -linear, and Krull-Schmidt. Write $\mathcal{K}(\Lambda)$ for the Grothendieck group of the category $\text{rep}(\Lambda)$, then $\mathcal{K}(\Lambda) \cong \mathbb{Z}^2$ where the class $[V] = (\dim_{\mathbb{F}_{q^{d_1}}} V_1, \dim_{\mathbb{F}_{q^{d_2}}} V_2)$ of a valued representation V of Λ gives its *dimension vector*. Define a \mathbb{Z} -bilinear pairing $\langle \cdot, \cdot \rangle : \mathcal{K}(\Lambda) \times \mathcal{K}(\Lambda) \rightarrow \mathbb{Z}$ on the natural basis $\alpha_1 = (1, 0)$ and $\alpha_2 = (0, 1)$ by

$$\langle \alpha_i, \alpha_i \rangle = d_i, \quad \langle \alpha_1, \alpha_2 \rangle = 0, \quad \langle \alpha_2, \alpha_1 \rangle = -d_1 d_2.$$

For a valued representation V of Λ and a dimension vector $\mathbf{e} = (e_1, e_2) \in \mathcal{K}(\Lambda)$, write $Gr_{\mathbf{e}}(V)$ for the *Grassmannian of subrepresentations* of V with dimension vector \mathbf{e} :

$$Gr_{\mathbf{e}}(V) = \{E \subset V : [E] = \mathbf{e}\}.$$

The *quiver Grassmannian* $Gr_{\mathbf{e}}(V)$ naturally embeds as a closed subvariety in the product $Gr_{e_1}(V_1) \times Gr_{e_2}(V_2)$, in particular it is a projective variety. When V is rigid, i.e. $\text{Ext}^1(V, V) = 0$, Caldero and Reineke have shown [3] that $Gr_{\mathbf{e}}(V)$ is smooth.

Since the field \mathbb{F}_q is finite, each Grassmannian $Gr_{\mathbf{e}}(V)$ is a finite set. For V rigid, a result of [13] shows that the number of points in $Gr_{\mathbf{e}}(V)$ can be computed by evaluating a polynomial $P_{\mathbf{e}, V}(t) \in \mathbb{Z}[t]$ at $q = |\mathbb{F}_q|$. Note that since V is rigid, it is uniquely determined up to isomorphism by its dimension vector $[V] \in \mathcal{K}(\Lambda)$.

THEOREM 5.5 ([13, Corollary 1.2]). *Let V be a rigid valued representations of Λ . For each dimension vector $\mathbf{e} \in \mathcal{K}(\Lambda)$, there exists a polynomial $P_{\mathbf{e}, V}(t) \in \mathbb{Z}[t]$ depending only on the dimension vector of V so that*

$$|Gr_{\mathbf{e}}(V)| = P_{\mathbf{e}, V}(q).$$

It was conjectured in [13] that for a rigid representation V the counting polynomials $P_{\mathbf{e}, V}(t)$ have positive coefficients and are unimodal. Corollary 5.7 proves this positivity conjecture by giving a positive combinatorial construction of these counting polynomials. It remains an interesting open question to see how this combinatorics can be used to establish unimodality.

Define the *quantum cluster character* of a rigid valued representation V of Λ by

$$Z_V = \sum_{\mathbf{e} \in \mathcal{K}(\Lambda)} v^{-\langle \mathbf{e}, \mathbf{v} - \mathbf{e} \rangle} P_{\mathbf{e}, V}(v^2) Z^{(-v_1 + d_2 e_2, -v_2 + d_1(v_1 - e_1))},$$

where $[V] = \mathbf{v} = (v_1, v_2)$ and $\mathbf{e} = (e_1, e_2)$. Write P_m (resp. I_m), $m \geq 1$, for the preprojective (resp. preinjective) valued representations of Λ (definitions can be found in [11] where it is shown that $[P_m] = \mathbf{a}_m$ and $[I_m] = \mathbf{a}'_m$). Then the Laurent expansions

of the non-initial quantum cluster variables Z_m , $m \in \mathbb{Z} \setminus \{1, 2\}$, can be computed as follows.

THEOREM 5.6 ([11]). *Assume the intermediate exchange coefficients $p_{i,j} = 0$ for $i = 1, 2$ and $1 \leq j \leq d_i - 1$. Then the following hold:*

- (a) *for $m \geq 3$, the quantum cluster variable Z_m is equal to $Z_{P_{m-2}}$;*
- (b) *for $m \leq 0$, the quantum cluster variable Z_m is equal to $Z_{I_{-m+1}}$.*

Combining Corollary 5.4 with Theorem 5.6, we obtain a combinatorial construction of the counting polynomials for Grassmannians of subrepresentations in rigid valued quiver representations.

COROLLARY 5.7. *For $m \geq 1$, the counting polynomials $P_{\mathbf{e}, P_m}(t)$ and $P_{\mathbf{e}, I_m}(t)$ are given by*

$$(27) \quad P_{\mathbf{e}, P_m}(t) = \sum_{\omega: E_m \rightarrow \mathbb{Z}_{\geq 0}} t^{\bar{\gamma}_\omega},$$

where

- *the sum ranges over (d_1, d_2) -bounded compatible gradings ω of D_m such that $\omega(H_m) \subset \{0, d_1\}$, $\omega(V_m) \subset \{0, d_2\}$, $|\text{supp}(\omega_H)| = u_{m,1} - e_1$, and $|\text{supp}(\omega_V)| = e_2$;*
- $\bar{\gamma}_\omega = \sum_{e < e' \in E_m} \bar{\gamma}_\omega(e, e')$ for

$$(28) \quad \bar{\gamma}_\omega(e, e') = \begin{cases} -d_1 d_2 & \text{if } e \in \text{supp}(\omega_H) \text{ and } e' \in \text{supp}(\omega_V); \\ d_1 & \text{if } e \in \text{supp}(\omega_H) \text{ and } e' \in H_m \setminus \text{supp}(\omega_H); \\ d_2 & \text{if } e \in V_m \setminus \text{supp}(\omega_V) \text{ and } e' \in \text{supp}(\omega_V); \\ 0 & \text{otherwise;} \end{cases}$$

and

$$(29) \quad P_{\mathbf{e}, I_m}(t) = \sum_{\omega: E'_m \rightarrow \mathbb{Z}_{\geq 0}} t^{\bar{\gamma}'_\omega},$$

where

- *the sum ranges over (d_2, d_1) -bounded compatible gradings ω of D'_m such that $\omega(H'_m) \subset \{0, d_2\}$, $\omega(V'_m) \subset \{0, d_1\}$, $|\text{supp}(\omega_{H'})| = e_2$, and $|\text{supp}(\omega_{V'})| = u'_{m,1} - e_1$;*
- $\bar{\gamma}'_\omega = \sum_{e < e' \in E'_m} \bar{\gamma}'_\omega(e, e')$ for

$$\bar{\gamma}'_\omega(e, e') = \begin{cases} -d_1 d_2 & \text{if } e \in \text{supp}(\omega_{V'}) \text{ and } e' \in \text{supp}(\omega_{H'}); \\ d_1 & \text{if } e \in \text{supp}(\omega_{V'}) \text{ and } e' \in V'_m \setminus \text{supp}(\omega_{V'}); \\ d_2 & \text{if } e \in H'_m \setminus \text{supp}(\omega_{H'}) \text{ and } e' \in \text{supp}(\omega_{H'}); \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We prove equation (27), the proof of equation (29) is essentially the same.

By Corollary 5.4 and Theorem 5.6, we have

$$P_{\mathbf{e}, P_m}(v^2) = \sum_{\omega: E_m \rightarrow \mathbb{Z}_{\geq 0}} v^{-\langle \mathbf{e}, \mathbf{a}_m - \mathbf{e} \rangle + 1 - u_{m,1} - u_{m-1,2} + \gamma_\omega + \beta_\omega},$$

where the sum ranges over all (d_1, d_2) -bounded compatible gradings of D_m with $\omega(H_m) \subset \{0, d_1\}$, $\omega(V_m) \subset \{0, d_2\}$, $|\text{supp}(\omega_H)| = u_{m,1} - e_1$, and $|\text{supp}(\omega_V)| = e_2$. But observe that

$$\langle \mathbf{e}, \mathbf{a}_m - \mathbf{e} \rangle = d_1 e_1 (u_{m,1} - e_1) + d_2 e_2 (u_{m-1,2} - e_2) - d_1 d_2 e_2 (u_{m,1} - e_1)$$

and under the assumptions on ω we have

$$\beta_\omega = d_1 d_2 e_2 (u_{m,1} - e_1) + u_{m,1} u_{m-1,2} - d_1 e_1 (u_{m,1} - e_1) - 2d_1 \binom{u_{m,1} - e_1}{2} - d_2 e_2 (u_{m-1,2} - e_2) - 2d_2 \binom{e_2}{2}.$$

Canceling like terms gives

$$P_{\mathbf{e}, P_m}(v^2) = \sum_{\omega: E_m \rightarrow \mathbb{Z}_{\geq 0}} v^{(u_{m,1}-1)(u_{m-1,2}-1)-2d_1 \binom{u_{m,1}-e_1}{2}-2d_2 \binom{e_2}{2}+\gamma_\omega}.$$

When $|\text{supp}(\omega_H)| = 0$ and $|\text{supp}(\omega_V)| = 0$, we have $\gamma_\omega = -2|\{e, e' \in E_m : e < e', e \in V_m, e' \in H_m\}|$. But these assumptions imply $\mathbf{e} = (u_{m,1}, 0)$ so that $P_{\mathbf{e}, P_m}(t) = 1$ and thus

$$(u_{m,1} - 1)(u_{m-1,2} - 1) = 2|\{e, e' \in E_m : e < e', e \in V_m, e' \in H_m\}|.$$

In particular, the case $e \in V_m$ and $e' \in H_m$ can be ignored when computing γ_ω if we omit the term $(u_{m,1} - 1)(u_{m-1,2} - 1)$ from the exponent of v . Since $|\text{supp}(\omega_H)| = u_{m,1} - e_1$ and $|\text{supp}(\omega_V)| = e_2$, the cases $e, e' \in \text{supp}(\omega_H)$ and $e, e' \in \text{supp}(\omega_V)$ can also be ignored giving

$$P_{\mathbf{e}, P_m}(v^2) = \sum_{\omega: E_m \rightarrow \mathbb{Z}_{\geq 0}} v^{2\bar{\gamma}_\omega}.$$

This gives the result since v was an indeterminate. □

REMARK 5.8. The exponents in equation (27) are not manifestly positive, however equation (28) giving the exponents can be refined as follows. Consider $e \in \text{supp}(\omega_H)$ and $e' \in \text{supp}(\omega_V)$ with $e < e'$ which contributes a term $-d_1 d_2$ in equation (28). The d_2 horizontal edges preceding e' cannot be in the support of ω_H by compatibility, moreover each such horizontal edge h satisfies $e < h$. In particular, these pairs $e < h$ together contribute a term $d_1 d_2$ in equation (28). Thus the negative contribution to $\bar{\gamma}_\omega$ will always cancel and equation (27) indeed gives $P_{\mathbf{e}, P_m}(t)$ as a polynomial in t .

REFERENCES

- [1] Liqian Bai, Xueqing Chen, Ming Ding, and Fan Xu, *A quantum analogue of generalized cluster algebras*, <https://arxiv.org/abs/1610.09803>, 2016.
- [2] Arkady Berenstein and Vladimir Retakh, *A short proof of Kontsevich cluster conjecture*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 3-4, 119–122.
- [3] Philippe Caldero and Markus Reineke, *On the quiver Grassmannian in the acyclic case*, J. Pure Appl. Algebra **212** (2008), no. 11, 2369–2380.
- [4] Leonid Chekhov and Michael Shapiro, *Teichmüller spaces of Riemann surfaces with orbifold points of arbitrary order and cluster variables*, Int. Math. Res. Not. (2014), no. 10, 2746–2772.
- [5] Philippe Di Francesco and Rinat Kedem, *Discrete non-commutative integrability: Proof of a conjecture of M. Kontsevich*, Int. Math. Res. Not. (2010), no. 21, 4042–4063.
- [6] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.
- [7] Maxim Kontsevich, *Noncommutative identities*, <https://arxiv.org/abs/1109.2469>.
- [8] Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, Selecta Math. **20** (2014), no. 1, 57–82.
- [9] Kyungyong Lee and Ralf Schiffler, *Proof of a positivity conjecture of M. Kontsevich on non-commutative cluster variables*, Compos. Math. **148** (2012), no. 6, 1821–1832.
- [10] Dylan Rupel, *Greedy bases in rank 2 generalized cluster algebras*, <https://arxiv.org/abs/1309.2567>.
- [11] ———, *On a quantum analog of the Caldero–Chapoton formula*, Int. Math. Res. Not. (2011), no. 14, 3207–3236.
- [12] ———, *Proof of the Kontsevich non-commutative cluster positivity conjecture*, C. R. Math. Acad. Sci. Paris **350** (2012), no. 21-22, 929–932.
- [13] ———, *Quantum cluster characters for valued quivers*, Trans. Amer. Math. Soc. **367** (2015), no. 10, 7061–7102.

- [14] Alexandr Usnich, *Non-commutative Laurent phenomenon for two variables*, <https://arxiv.org/abs/1006.1211>.
- [15] ———, *Non-commutative cluster mutations*, Dokl. Nats. Akad. Nauk Belarusi **53** (2009), no. 4, 27–29.
- [16] ———, *Action of the Cremona group on a non-commutative ring*, Adv. Math. **228** (2011), no. 4, 1863–1893.

DYLAN C. RUPEL, Michigan State University, Department of Mathematics, 619 Red Cedar Road,
C212 Wells Hall, East Lansing, MI 48824, USA
E-mail : dylanrupel@gmail.com