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Parity of transversals of Latin squares

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ABSTRACT We introduce a notion of parity for transversals, and use it to show that in Latin squares of order $2 \pmod 4$, the number of transversals is a multiple of 4. We also demonstrate a number of relationships (mostly congruences modulo 4) involving E_1, \dots, E_n , where E_i is the number of diagonals of a given Latin square that contain exactly i different symbols.

Let $A(i | j)$ denote the matrix obtained by deleting row i and column j from a parent matrix A . Define t_{ij} to be the number of transversals in $L(i | j)$, for some fixed Latin square L . We show that $t_{ab} \equiv t_{cd} \pmod 2$ for all a, b, c, d and L . Also, if L has odd order then the number of transversals of L equals $t_{ab} \pmod 2$. We conjecture that $t_{ac} + t_{bc} + t_{ad} + t_{bd} \equiv 0 \pmod 4$ for all a, b, c, d .

In the course of our investigations we prove several results that could be of interest in other contexts. For example, we show that the number of perfect matchings in a k -regular bipartite graph on $2n$ vertices is divisible by 4 when n is odd and $k \equiv 0 \pmod 4$. We also show that

$$\text{per } A(a | c) + \text{per } A(b | c) + \text{per } A(a | d) + \text{per } A(b | d) \equiv 0 \pmod 4$$

for all a, b, c, d , when A is an integer matrix of odd order with all row and columns sums equal to $k \equiv 2 \pmod 4$.

1. INTRODUCTION

A *Latin square* is an $n \times n$ matrix consisting of n distinct symbols where each symbol appears exactly once in each row and each column. Our Latin squares will have their rows and columns indexed by $[n] = \{1, 2, \dots, n\}$ and will also have their symbols chosen from $[n]$. Latin squares can then be thought of as a set of *entries* $\{(r, c, s)\} \subset [n]^3$ where each distinct pair of entries agree in at most one coordinate. The three coordinates of an entry are its row index, column index and symbol. A *diagonal* of a Latin square is a selection of n entries, with exactly one entry from each row and each column. The *weight* of a diagonal is the number of distinct symbols on that diagonal. A diagonal of weight n is called a *transversal*. Historically, transversals in Latin squares were first used as the building blocks of mutually orthogonal Latin squares (MOLS). They have since garnered a lot of interest on their own (see [26] for a survey). A *partial transversal of length k* is a selection of k entries so that no two entries share the same row, column or symbol. A partial transversal of length $k < n$

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is not the same thing as a diagonal of weight k . While these objects are related, the distinction is important when counting them.

Over half a century ago, Ryser [22] put forward the following famous conjecture.

CONJECTURE 1.1 (Ryser's Conjecture). *Every Latin square of odd order has a transversal.*

This conjecture has been shown to be true for $n \leq 9$ by computation [18]. In 1990, Balasubramanian [6] showed that the number of transversals in a Latin square of even order is itself even. He claimed that this was a partial proof of a stronger form of Conjecture 1.1, namely that the number of transversals in a Latin square of order n should agree with $n \pmod 2$. Despite [6] attributing this conjecture to [22] it is nowhere to be found in the latter work. It is possible that Ryser made the conjecture, but we have been unable to find evidence of this. It is also worth remarking that many Latin squares of odd order have an even number of transversals, so the stronger form of Conjecture 1.1 is false. However, it does raise the intriguing possibility of proving existence of objects (in this case, transversals) by studying congruences satisfied by the number of those objects. We achieve this on a very modest scale (cf. Example 3.14), but mostly use it as motivation to unearth what we consider to be interesting patterns in numbers of transversals and related quantities.

Akbari and Alipour [2] developed the ideas from Balasubramanian's result to show that the number of diagonals with weight $n - 1$ is even in every Latin square. We outline and expand on these ideas in § 2. We then show that the number of transversals in a Latin square of order $n \equiv 2 \pmod 4$ is necessarily a multiple of 4. We show this by exploiting a notion of parity for transversals. Parity of permutations is, of course, a very well-known concept. To our knowledge it had not previously been usefully applied to transversals. However, applying it to the permutations that define the rows, columns and symbols of a single Latin square or set of MOLS has previously revealed many insights [1, 3, 4, 5, 9, 10, 11, 12, 13, 14, 15, 17, 23, 25]. In particular, each Latin square has a row-parity π_r , which is the \mathbb{Z}_2 sum of the parities of the permutations that define the rows, and a column parity π_c which is defined similarly for columns (there is also a symbol parity π_s , but it is a function of π_r and π_c , see [10, 14, 25]). We will demonstrate several new ways to partition Latin squares of certain orders into two types (independently of their π_r and π_c).

In § 3, we consider counts of transversals in (not necessarily square) submatrices of Latin squares. A *Latin array* is a matrix of symbols in which no symbol is repeated within any row, or within any column. A transversal of an $m \times n$ Latin array is a selection of $\min(m, n)$ entries in which no pair of entries share their row, column or symbol. Transversals in Latin arrays are naturally encountered in attempts to find transversals of Latin squares by induction. They have been the subject of a recent burst of activity [7, 8, 16, 19] on the question posed in [2] of how many symbols in a Latin array are enough to make a transversal unavoidable. We take a different tack, considering congruences satisfied by the number of transversals in Latin arrays formed by removing one row and/or one column from a Latin square.

Transversals are diagonals with the maximum possible number of symbols. In § 4 we count diagonals according to how many symbols they contain and demonstrate several relationships between the resulting numbers. In doing so we extend on results obtained in [2, 6].

One of the key tools in our results is a matrix function known as the permanent. Let $M_n(\mathbb{Z})$ denote the $n \times n$ integer matrices. The *permanent* of a matrix $A = [a_{ij}]$

in $M_n(\mathbb{Z})$ is defined by

$$(1) \quad \text{per } A = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^n a_{i\sigma(i)}$$

where the sum is over all permutations in the symmetric group \mathcal{S}_n on $[n]$. At several points, we use Ryser's formula [21] to compute the permanent of a matrix. It states that for $A = [a_{ij}] \in M_n(\mathbb{Z})$,

$$(2) \quad \text{per } A = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i=1}^n \sum_{j \in S} a_{ij}.$$

We also make frequent use of the fact that (1) agrees, modulo 2, with the definition of the determinant. In a determinant, some diagonals are given a negative sign, but $-1 \equiv 1 \pmod{2}$ so $\text{per } A \equiv \det A \pmod{2}$. As a simple example of how this observation can be used, we have:

LEMMA 1.2. *If $A \in M_n(\mathbb{Z})$ is such that all row sums are even, then $\det A$ and $\text{per } A$ are both even.*

Proof. Since the sum of the entries in each row is even, the columns of A are linearly dependent over \mathbb{Z}_2 . Thus, $\det A \equiv 0 \pmod{2}$, from which the claim follows. \square

Throughout the paper, J is an all-ones matrix of the appropriate order and Λ_n^k is the set of all $(0, 1)$ -matrices of order n which contain exactly k ones in each row and each column. We also need notation for the *conjugates* of a Latin square L . For each permutation abc in \mathcal{S}_3 (written in image notation) there is an abc -conjugate of L . It is the Latin square obtained by applying the permutation abc to the three coordinates in the entries of L . For example, the 213-conjugate of L is the usual matrix transpose of L .

2. TRANSVERSALS OF LATIN SQUARES OF EVEN ORDER

In this section, we layout the ideas used first by Balasubramanian [6], then again by Akbari and Alipour [2] to count the number of transversals in even ordered Latin squares modulo 2. For consistency with [2], we will define $E_m = E_m(L)$ to be the number of diagonals in L that contain exactly m distinct symbols. In particular, $E_n(L)$ is the number of transversals in L if L has order n . The key idea is to count transversals using inclusion-exclusion.

DEFINITION 2.1. *Let $f(x_1, \dots, x_n)$ be an arbitrary polynomial from \mathbb{R}^n to \mathbb{R} . Then $\langle r \rangle f$ denotes the sum of the values of f at the $\binom{n}{r}$ vectors in \mathbb{R}^n which have r coordinates equal to 1 and $n - r$ coordinates equal to 0.*

The following result is a slight generalisation of both [6, Lemma 2] and [2, Theorem 2.1].

LEMMA 2.2. *Let $f(x_1, \dots, x_n)$ be an arbitrary polynomial from \mathbb{R}^n to \mathbb{R} . Then the sum of the coefficients of monomials in f containing exactly m distinct variables is*

$$\sum_{r=0}^m (-1)^{m-r} \binom{n-r}{n-m} \langle r \rangle f.$$

Proof. Suppose that μ is a monic monomial containing t distinct variables. Then

$$\begin{aligned} \sum_{r=0}^m (-1)^{m-r} \binom{n-r}{n-m} \langle r \rangle \mu &= \sum_{r=t}^m (-1)^{m-r} \binom{n-r}{n-m} \binom{n-t}{r-t} \\ &= \binom{n-t}{n-m} \sum_{r=t}^m (-1)^{m-r} \binom{m-t}{r-t} \\ &= \begin{cases} 1 & \text{if } t = m, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The result follows. □

For any transversal, $\{(r_i, c_i, s_i)\}$, we define three corresponding permutations of the index set $[n]$ by $\sigma_r(r_i) = c_i$, $\sigma_c(c_i) = s_i$ and $\sigma_s(s_i) = r_i$. The following result is immediate.

LEMMA 2.3. *Let $T = \{(r_i, c_i, s_i)\}$ be a transversal of a Latin square L with corresponding permutations σ_r, σ_c and σ_s . Then $\sigma_r \circ \sigma_c \circ \sigma_s$ is the identity permutation.*

Proof. For any given symbol s_i , we have $s_i \xrightarrow{\sigma_s} r_i \xrightarrow{\sigma_r} c_i \xrightarrow{\sigma_c} s_i$. □

As an immediate corollary of Lemma 2.3, we have that $\varepsilon(\sigma_r) + \varepsilon(\sigma_c) + \varepsilon(\sigma_s) = 0$, where $\varepsilon : \mathcal{S}_n \rightarrow \mathbb{Z}_2$ is the standard parity homomorphism on the symmetric group \mathcal{S}_n . Thus, we can classify transversals into four types: T^{000} , T^{011} , T^{101} or T^{110} where the superscript records the parities of σ_r , σ_c and σ_s , respectively. We will use these parities to aid in counting transversals.

DEFINITION 2.4. *Let L be a Latin square of order n . The parity of a transversal is the parity of the permutation σ_r defined above. We define $E_n^\pm(L)$ to be the number of transversals in L with $\varepsilon(\sigma_r) = 0$ (even transversals) minus the number of transversals in L with $\varepsilon(\sigma_r) = 1$ (odd transversals).*

Though the symbols in our Latin square L are normally from $[n]$, we sometimes need to utilise the corresponding matrix $L[X]$, where each symbol i is replaced with a variable x_i .

THEOREM 2.5. *Let L be a Latin square of order n . Then*

$$(3) \quad E_n(L) = \sum_{r=0}^n (-1)^{n-r} \langle r \rangle \text{per } L[X],$$

and

$$(4) \quad E_n^\pm(L) = \sum_{r=0}^n (-1)^{n-r} \langle r \rangle \det L[X].$$

Proof. We use the $m = n$ case of Lemma 2.2. The terms in $\text{per } L[X]$ which include n distinct variables correspond to the transversals in L . Similar terms in $\det L[X]$ correspond to the transversals of L up to sign. Any transversal which has even parity increases the sum in (4) by 1 and any odd transversal decreases the sum by 1. □

The following lemma is adapted from [20, Lemma 1].

LEMMA 2.6. *Let $A = [a_{ij}]$ be a $(0, 1)$ -matrix of even order. Define $A^* = [b_{ij}]$ by*

$$b_{ij} = \begin{cases} a_{ij} & \text{if the } i^{\text{th}} \text{ row has an even number of ones,} \\ 1 - a_{ij} & \text{otherwise.} \end{cases}$$

Then $\det A + \det A^$ is even.*

Proof. Let δ be a row vector of ones. By permuting rows if necessary, we may assume that the first k rows of A have odd sum and the remaining rows have even sum (permuting rows may alter the sign of the determinant, but this does not matter modulo 2). Thus, we have

$$\begin{aligned} \pm \det A^* &= \det \begin{pmatrix} \delta - A_1 \\ \delta - A_2 \\ \vdots \\ \delta - A_k \\ A_{k+1} \\ \vdots \\ A_n \end{pmatrix} = \det \begin{pmatrix} \delta - A_1 \\ A_1 - A_2 \\ \vdots \\ A_1 - A_k \\ A_{k+1} \\ \vdots \\ A_n \end{pmatrix} = \det \begin{pmatrix} -A_1 \\ A_1 - A_2 \\ \vdots \\ A_1 - A_k \\ A_{k+1} \\ \vdots \\ A_n \end{pmatrix} + \det \begin{pmatrix} \delta \\ A_1 - A_2 \\ \vdots \\ A_1 - A_k \\ A_{k+1} \\ \vdots \\ A_n \end{pmatrix} \\ &= (-1)^k \det A + \det \begin{pmatrix} \delta \\ A_1 - A_2 \\ \vdots \\ A_1 - A_k \\ A_{k+1} \\ \vdots \\ A_n \end{pmatrix} \equiv \det A \pmod{2}, \end{aligned}$$

by Lemma 1.2. The result follows. □

Consider the special case of Lemma 2.6 where the row sums of A are all the same. If all of the row sums are even, then the result shows nothing interesting. However, when each row sum is odd, Lemma 2.6 tells us that $\det A + \det(J - A) \equiv 0 \pmod{2}$. Balasubramanian [6] used this result and (3) to show the following theorem (actually, [6] showed a generalisation of this result, which we discuss later). We give a full proof here, as we will use a similar technique for several of our new results.

THEOREM 2.7. *If L is a Latin square of even order n then L has an even number of transversals.*

Proof. Note that $E_n(L) \equiv E_n^\pm(L) \pmod{2}$. We pair up complementary terms in (4). In other words, each term of the sum $\langle r \rangle \det L[X]$ is paired with the unique term in $\langle n - r \rangle \det L[X]$ for which the indexing zero-one vectors sum to the all-ones vector. For each of these pairs of terms, we have one of two situations. If r is even, then $n - r$ is also even and so both determinants are even, by Lemma 1.2. Alternatively, if r is odd, then each row sum in $L[X]$ is odd, so $\det L[X] + \det(J - L[X]) \equiv 0 \pmod{2}$ by Lemma 2.6. Thus, each of the 2^{n-1} pairs contributes a multiple of two to the summation in (4). The result follows. □

To proceed, we need a few linear algebraic results.

LEMMA 2.8. *If $A \in \Lambda_n^k$ where both n and k are even, then $\det A \equiv 0 \pmod{4}$.*

Proof. Since $A \in \Lambda_n^k$, we have that $\det A$ is a multiple of $k \cdot \gcd(n, k)$, by [20, Theorem 2]. The desired result follows, since k and $\gcd(n, k)$ are both even. □

LEMMA 2.9. *Let $n \equiv 2 \pmod{4}$ and $k \equiv 1 \pmod{2}$. If $A \in \Lambda_n^k$, then*

$$\det A + \det(J - A) \equiv 0 \pmod{4}.$$

Proof. Since $A \in \Lambda_n^k$, we have that

$$(5) \quad k \det(J - A) = (-1)^{n-1} (n - k) \det A,$$

by [20, Lemma 1]. The result follows by noting that $n - k \equiv k \pmod 4$ and that k has a multiplicative inverse modulo 4. \square

We now have the framework to start counting transversals. The proof of our next result is very similar to that of Theorem 2.7.

THEOREM 2.10. *If L is a Latin square of order $n \equiv 2 \pmod 4$ then $E_n^\pm(L) \equiv 0 \pmod 4$.*

Proof. Again, we pair up complementary terms in (4). That is, each term of the sum $\langle r \rangle \det L[X]$ is paired with the unique term in $\langle n - r \rangle \det L[X]$ for which the indexing zero-one vectors sum to the all-ones vector. For each of these pairs of terms, we have one of two situations. If r is even, then $n - r$ is also even and we use Lemma 2.8 twice to show that both terms are a multiple of four. Alternatively, if r is odd, then we use Lemma 2.9 to show that the terms sum to a multiple of four. The result follows. \square

We now use Theorem 2.10 to show our first main result, which strengthens Theorem 2.7 for Latin squares of singly-even order.

THEOREM 2.11. *If L is a Latin square of order $n \equiv 2 \pmod 4$ then $E_n(L) \equiv 0 \pmod 4$.*

Proof. Let T be the set of transversals of L . We define w, x, y and z to be the number of transversals of type $T^{000}, T^{011}, T^{101}$ and T^{110} , respectively. By definition, we have

$$(6) \quad w + x + y + z = E_n(L)$$

and

$$(7) \quad w + x - y - z = E_n^\pm(L).$$

Let L' be the 312-conjugate of L . There is a natural bijection between T and the set of transversals of L' . Each transversal in L' must be of the form $\{(c_i, s_i, r_i)\}$, where $\{(r_i, c_i, s_i)\} \in T$. The parity of each transversal in L' depends on σ_c for the corresponding transversal in T , so we have

$$(8) \quad w - x + y - z = E_n^\pm(L').$$

Similarly, if L'' is the 231-conjugate of L , then each transversal of L'' has parity matching that of σ_s for the corresponding transversal in T . So we have

$$(9) \quad w - x - y + z = E_n^\pm(L'').$$

The sum of (6), (7), (8) and (9) gives us

$$(10) \quad 4w = E_n(L) + E_n^\pm(L) + E_n^\pm(L') + E_n^\pm(L'').$$

Theorem 2.10 applied to L, L' and L'' tells us that $E_n(L) \equiv 0 \pmod 4$. \square

Based on computation of small squares, it seems that Theorem 2.7 and Theorem 2.11 are the only general modular restrictions on the number of transversals of a Latin square. By considering sets of Latin squares that are connected by turning intercalates (that is, replacing a subsquare $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ with $\begin{bmatrix} b & a \\ a & b \end{bmatrix}$), we were able to find Latin squares that satisfy every other congruence with small modulus. For example, suppose that $9 \leq n \leq 11$ and $0 \leq k < m \leq 32$. Except where it would violate Theorem 2.7 or

Theorem 2.11, there is some subset of intercalates in these Latin squares

1	2	3	4	5	6	7	8	9
2	1	4	3	6	5	9	7	8
3	6	1	8	7	9	5	2	4
4	3	5	6	9	7	8	1	2
5	4	2	9	8	1	6	3	7
6	9	7	5	3	8	2	4	1
7	8	9	1	2	3	4	5	6
8	5	6	7	4	2	1	9	3
9	7	8	2	1	4	3	6	5

1	2	3	4	5	6	7	8	9	10
2	1	4	3	6	5	8	7	10	9
3	6	5	7	2	8	10	9	4	1
4	5	6	8	7	9	2	10	1	3
5	8	7	9	1	10	4	3	2	6
6	4	8	10	9	7	1	2	3	5
7	3	10	5	8	1	9	4	6	2
8	7	9	6	10	2	3	1	5	4
9	10	1	2	3	4	5	6	7	8
10	9	2	1	4	3	6	5	8	7

1	2	3	4	5	6	7	8	9	10	11
2	1	4	3	6	5	8	7	11	9	10
3	8	1	6	7	10	11	9	4	5	2
4	11	2	8	9	7	5	10	1	3	6
5	3	6	10	8	9	1	2	7	11	4
6	4	7	9	10	11	2	3	8	1	5
7	5	8	11	4	2	10	1	3	6	9
8	7	9	5	11	1	6	4	10	2	3
9	10	11	1	2	3	4	5	6	7	8
10	6	5	7	3	8	9	11	2	4	1
11	9	10	2	1	4	3	6	5	8	7

that can be turned to give a Latin square of order n with $k \bmod m$ transversals. We also found examples for $12 \leq n \leq 16$ with the same property, but we do not display them here for the sake of space. For $n \leq 7$, there are some sporadic values of $k, m \leq 16$ where no Latin square of order n contains $k \bmod m$ transversals. For $n = 8$, there is no Latin square that contains $22 \bmod 63$ transversals, while there exists a Latin square that contains $k \bmod m$ transversals for all other $0 \leq k < m \leq 64$ that satisfy Theorem 2.7. However, we believe that the restrictions for $n \leq 8$ are not interesting; they are simply a result of there being comparatively few Latin squares of these orders.

The proof of Theorem 2.11 leads us to the following interesting property.

COROLLARY 2.12. *Let L be a Latin square of order $n \equiv 2 \pmod 4$. The numbers of transversals in L of types T^{000} , T^{110} , T^{101} and T^{110} are all equal modulo 2.*

Proof. Define w, x, y and z as in Theorem 2.11. Adding (6) to (7) we find that $2w + 2x = E_n(L) + E_n^\pm(L) \equiv 0 \pmod 4$ (by Theorem 2.10 and Theorem 2.11) which gives us that $w \equiv x \pmod 2$. Similarly, (6) + (8) and (6) + (9) tell us that $w \equiv y \pmod 2$ and $w \equiv z \pmod 2$, respectively. \square

It is important to remark that Theorem 2.11 is less general in one respect than Balasubramanian's Theorem [6]. Balasubramanian proved that the number of transversals in any row-Latin square of even order is even (a *row-Latin square* of order n is an $n \times n$ matrix in which each row is a permutation of $[n]$). Theorem 2.11 does not generalise to row-Latin squares. Below we give two row-Latin squares whose number of

transversals is not a multiple of 4. The row-Latin square of order 2 has 2 transversals and the row-Latin square of order 6 has 6 transversals.

1	2
1	2

1	3	6	2	5	4
2	1	5	6	4	3
3	2	4	1	5	6
4	2	1	5	6	3
5	2	3	6	1	4
6	5	2	3	4	1

Computational evidence suggests the following generalisation of Theorem 2.11 and Corollary 2.12.

CONJECTURE 2.13. *Let L be a Latin square of even order n . Let w, x, y and z be the number of transversals in L of types $T^{000}, T^{011}, T^{101}$ and T^{110} , respectively. Then*

- (a) $E_n(L) \equiv E_n^\pm(L) \pmod{4}$ and
- (b) $w \equiv x \equiv y \equiv z \pmod{2}$.

Conjecture 2.13 is true for $n \equiv 2 \pmod{4}$ since $E_n(L) \equiv E_n^\pm(L) \equiv 0 \pmod{4}$ (by Theorem 2.10 and Theorem 2.11) and $w \equiv x \equiv y \equiv z \pmod{2}$ (by Corollary 2.12). Note that there are many Latin squares of odd order for which Conjecture 2.13 is not true.

LEMMA 2.14. *The conditions (a) and (b) in Conjecture 2.13 are equivalent for Latin squares of even order.*

Proof. By Theorem 2.7, we know that $E_n(L) \equiv 0 \pmod{2}$ when n is even, showing condition (a) is equivalent to $E_n(L) + E_n^\pm(L) \equiv 0 \pmod{4}$. We may use the same idea as the proof of Corollary 2.12 to show the result. □

3. TRANSVERSALS OF DEPLETED LATIN SQUARES

In this section we give a number of results around the common theme of transversals of *depleted Latin squares*, that is, matrices formed by removing a row and/or a column of a Latin square. This depleted Latin square is a Latin array.

Given an $n \times n$ matrix A , we use $\nu(A)$ to denote the \mathbb{Z}_2 -nullity of A and we use $A(i|j)$ to denote the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A . We start by considering the permanent of this submatrix, which is analogous to the consideration of minors when computing the determinant.

THEOREM 3.1. *Let $A \in M_n(\mathbb{Z})$ for $n > 1$. Then*

- $\text{per } A(i|j) \equiv 0 \pmod{2}$ for all i, j if and only if $\nu(A) \geq 2$.
- $\text{per } A(i|j) \equiv 1 \pmod{2}$ for all i, j if and only if $\nu(A) = 1$ and all row and column totals of A are even.

Proof. It suffices to show analogous properties for determinants since the determinant and permanent agree modulo 2. All calculations in this proof will be working over \mathbb{Z}_2 , and all minors will be of order $n-1$.

If $\nu(A) \geq 2$, then for all i, j we know that $\nu(A(i|j)) \geq 1$ so $\det A(i|j) \equiv 0 \pmod{2}$.

If $\nu(A) = 0$, then A has an inverse so the adjugate $\text{adj}(A)$ has full rank and hence is not a multiple of J (given that $n > 1$). Hence not all minors of A are equal.

So suppose that $\nu(A) = 1$, and hence $\det A = 0$. Since $\nu(A) = 1$ there is at least one minor of A that equals 1.

If there is any row or column of A with odd sum, then expanding the determinant in that row/column shows that A has at least one minor which is zero, and hence not all minors are equal.

It remains to treat the case where each row and column sum of A is even. It suffices to show $\det A(1|1) \equiv \det A(2|1) \pmod 2$. But

$$\det A(1|1) + \det A(2|1) = \det \begin{pmatrix} a_{12} + a_{22} & a_{13} + a_{23} & \cdots & a_{1n} + a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ a_{42} & a_{43} & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \equiv 0 \pmod 2,$$

since all column sums are even (cf. Lemma 1.2). □

Since $\nu(A) \geq 1$ whenever all row totals are even, we have:

COROLLARY 3.2. *Let $A \in M_n(\mathbb{Z})$ be such that all row and column sums are even. Then $\text{per } A(a|c) \equiv \text{per } A(b|d) \pmod 2$ for all a, b, c, d .*

The previous result gave a congruence mod 2. Our next results involve congruences mod 4.

THEOREM 3.3. *Suppose that n is odd and $k \equiv 2 \pmod 4$. If $A \in \Lambda_n^k$, then*

$$\text{per } A(a|c) + \text{per } A(b|c) + \text{per } A(a|d) + \text{per } A(b|d) \equiv 0 \pmod 4$$

for any a, b, c, d .

Proof. If $a = b$ (or symmetrically, $c = d$), then $2(\text{per } A(a|c) + \text{per } A(a|d)) \equiv 0 \pmod 4$, by Corollary 3.2. Hence, it suffices to consider the case when $a = c = 1$ and $b = d = 2$. Define

$$B = \begin{pmatrix} a_{11} + a_{21} + a_{12} + a_{22} & a_{13} + a_{23} & \cdots & a_{1n} + a_{2n} \\ a_{31} + a_{32} & a_{33} & \cdots & a_{3n} \\ a_{41} + a_{42} & a_{43} & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

Note that B has order $n - 1$ and that its first row and column each sum to $2k \equiv 0 \pmod 4$, while its other rows and columns each sum to k . Also, by multilinearity of the permanent,

$$\begin{aligned} \text{per } B &= \text{per} \begin{pmatrix} a_{12} + a_{22} & a_{13} + a_{23} & \cdots & a_{1n} + a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ a_{42} & a_{43} & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \\ &\quad + \text{per} \begin{pmatrix} a_{11} + a_{21} & a_{13} + a_{23} & \cdots & a_{1n} + a_{2n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ a_{41} & a_{43} & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \\ &= \text{per } A(1|1) + \text{per } A(2|1) + \text{per } A(1|2) + \text{per } A(2|2). \end{aligned}$$

Next, apply (2) to calculate per B :

$$\text{per } B = \sum_{S \subseteq [n-1]} (-1)^{n-1-|S|} \prod_{i=1}^{n-1} \sum_{j \in S} b_{ij}.$$

Fix a set S_0 and consider the terms corresponding to S_0 and its complement in the outer summation. We have,

$$\begin{aligned} & (-1)^{n-1-|S_0|} \prod_{i=1}^{n-1} \sum_{j \in S_0} b_{ij} + (-1)^{|S_0|} \prod_{i=1}^{n-1} \sum_{j \notin S_0} b_{ij} \\ &= (-1)^{|S_0|} \left(\prod_{i=1}^{n-1} x_i + (2k - x_1) \prod_{i=2}^{n-1} (k - x_i) \right) \\ (11) \quad & \equiv (-1)^{|S_0|} \left(2 \prod_{i=1}^{n-1} x_i - k \sum_{j=2}^{n-1} \prod_{i \neq j} x_i \right) \pmod{4}, \end{aligned}$$

where

$$x_i = \sum_{j \in S_0} b_{ij}.$$

Now $\sum_i x_i$ is even, so there is an even number of choices of i for which x_i is even. If this number is non-zero, then (11) is clearly 0 modulo 4. So we may assume that every x_i is odd. But then (11) is 0 modulo 4 again, since each term in the sum is odd and there is an odd number of summands. \square

THEOREM 3.4. *Let $n \equiv 1 \pmod{2}$ and $k \equiv 2 \pmod{4}$. If $A \in \Lambda_n^k$, then*

$$\text{per } A + 2 \text{per}(J - A) \equiv 0 \pmod{4}.$$

Proof. By inclusion-exclusion,

$$\text{per}(J - A) = \sum_{i=0}^n (-1)^i (n - i)! \tau_i(A) \equiv \tau_{n-1}(A) - \text{per } A \pmod{2},$$

where $\tau_i(A)$ is the sum of the permanents of all $i \times i$ submatrices of A . However, per A is even by Lemma 1.2, so $\text{per } A + 2 \text{per}(J - A) \equiv \text{per } A + 2\tau_{n-1}(A) \pmod{4}$. Next, define an $n \times n$ matrix $C = [c_{ij}]$ by $c_{ij} = \text{per } A(i | j)$. By Corollary 3.2 and Theorem 3.3, we know that modulo 4 each pair of rows of C either agrees in every position or differs by 2 in every position. Hence, up to row and column permutations, C has the block form

$$\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

where each entry in $C_2 \cup C_3$ differs from each entry in $C_1 \cup C_4$ by 2 mod 4. Note that some blocks may be vacuous, but one of the four blocks must have odd dimensions. Without loss of generality, we choose it to be C_1 . Now, partition A into 4 blocks

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

whose dimensions and locations match the corresponding block of C . Define n_r to be the total of a row r in block A_1 . Next, consider calculating per A mod 4 by taking an expansion along row r :

$$\text{per } A = \sum_{j=1}^n a_{rj} c_{rj} \equiv n_r c_{r1} + (k - n_r)(c_{r1} + 2) \equiv 2c_{r1} - 2n_r \pmod{4}.$$

The answer must be independent of r , which means that $n_r \pmod 2$ is constant. Analogous statements hold for row totals in each block. In particular, A_3 , which has an even number of rows, must contain an even number of ones. But then A_1 must also contain an even number of ones, given that the column totals of A are even. It follows that n_r must be even, so $\text{per } A \equiv 2c_{r_1} \pmod 4$. Now,

$$\tau_{n-1}(A) = \sum_{i,j} c_{ij} \equiv n^2 c_{r_1} \equiv c_{r_1} \pmod 2.$$

So $\text{per } A + 2\tau_{n-1}(A) \equiv 4c_{r_1} \equiv 0 \pmod 4$ and we are done. □

In our next major result, we show that a stronger form of Lemma 1.2 can be obtained under some circumstances.

THEOREM 3.5. *Let $A \in M_n(\mathbb{Z})$ where n is odd. If all row sums are multiples of 4 and all column sums are even, then $\text{per } A \equiv 0 \pmod 4$.*

Proof. We compute the permanent via (2):

$$\text{per } A = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i=1}^n \sum_{j \in S} a_{ij}.$$

Let r_i be the sum of row i and c_j be the sum of column j of A . Fix a set $S = S_0$ of odd cardinality and consider the contribution from S_0 and its complement. We have,

$$\begin{aligned} \prod_{i=1}^n \sum_{j \in S_0} a_{ij} - \prod_{i=1}^n \sum_{j \notin S_0} a_{ij} &= \prod_{i=1}^n \sum_{j \in S_0} a_{ij} - \prod_{i=1}^n \left(r_i - \sum_{j \in S_0} a_{ij} \right) \\ (12) \qquad \qquad \qquad &\equiv 2 \prod_{i=1}^n \sum_{j \in S_0} a_{ij} \pmod 4. \end{aligned}$$

Since each column of A has an even total,

$$(13) \qquad \sum_{i=1}^n \sum_{j \in S_0} a_{ij} = \sum_{j \in S_0} c_j \equiv 0 \pmod 2.$$

Since n is odd, $\sum_{j \in S_0} a_{ij}$ must be even for at least one value of i in (13). Thus, the product of the partial row sums must be even and (12) must be a multiple of 4. Summing over S_0 , the result follows. □

COROLLARY 3.6. *Let $A \in \Lambda_n^{4k}$ for integers k, n with n odd. Then $\text{per } A \equiv 0 \pmod 4$.*

It is well-known that perfect matchings in bipartite graphs can be counted using the permanent of the bi-adjacency matrix of the graph. Corollary 3.6 says that the number of perfect matchings will be a multiple of 4 in any $4k$ -regular bipartite graph with an odd number of vertices in each class of the bipartition. Indeed, Theorem 3.5 says that the same conclusion can be reached under weaker hypotheses. It suffices for all vertices in one class to have even degree, and all vertices in the other class to have their degree divisible by 4.

We define $t_{ij}(L)$ to be the number of transversals in the Latin array formed by deleting the i th row and j th column of L . When clear from context, the shorthand t_{ij} is used.

THEOREM 3.7. *Let L be a row-Latin square of order n . Then for all a, b, c ,*

$$t_{ab} \equiv t_{ac} \pmod 2.$$

Proof. Without loss of generality, we may assume that $n \geq 2$, $a = 1$, $b = 1$, $c = 2$. Let $L[X] = [x_{ij}]$ and define

$$L[X]' = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{pmatrix}.$$

Then $t_{ab} + t_{ac}$ is the number of terms in per $L[X]'$ which have exactly $n - 1$ symbols. Thus, by Lemma 2.2,

$$t_{ab} + t_{ac} = \sum_{r=0}^{n-1} (-1)^{n-1-r} (n-r) \langle r \rangle \text{ per } L[X]' \equiv \sum_{r=1}^{n-1} (n-r) \langle r \rangle \det L[X]' \pmod{2}.$$

If n is odd, then we have two subcases. If r is even, then $\langle r \rangle \det L[X]'$ is even, by Lemma 1.2. If r is odd, then $n - r$ is even, and so each term in the summation is even.

If n is even, then we use a trick similar to Theorem 2.10 by pairing up complementary terms. Our result follows from Lemma 1.2 and Lemma 2.6 when r is even and r is odd, respectively. \square

This immediately gives us a surprisingly simple result which lays the groundwork for the patterns found in the remainder of the section.

COROLLARY 3.8. *Let L be a Latin square of order n . Then for all a, b, c, d ,*

$$t_{ab} \equiv t_{cd} \pmod{2}.$$

Proof. Since L is a row-Latin square, $t_{ab} \equiv t_{ad} \pmod{2}$ by Theorem 3.7. Moreover, since the transpose of L is a row-Latin square, $t_{ad} \equiv t_{cd} \pmod{2}$. \square

This simple observation leads to several patterns relating to deleting a row and a column of a Latin square.

COROLLARY 3.9. *Let R be an $(n - 1) \times n$ row-Latin rectangle, where n is even. Then the number of transversals in R is even.*

Proof. Let L be some row-Latin square formed by adding one row to R . By definition, the number of transversals in R is

$$t_{n1}(L) + t_{n2}(L) + \cdots + t_{nn}(L).$$

Each of these terms is congruent modulo 2 (by Theorem 3.7) and n is even. \square

Each $(n - 1) \times n$ Latin rectangle R has a unique completion to a Latin square L , and each transversal of R corresponds to a so-called *near transversal* of L . Corollary 3.9 does not generalise to odd orders, as there are some rectangles that have an even number of transversals and other rectangles that have an odd number of transversals. If any row is removed from the Cayley table of a cyclic group of odd order, the resulting Latin rectangle has an odd number of transversals. This can be seen by combining two well-known features of the cyclic group tables of odd order. Firstly each near transversal extends to a (unique) transversal, and secondly there are an odd number of transversals.

We define $N_r = N_r(L)$ to be the number of diagonals of weight $n - 1$ in L where the symbol that appears in row r also appears in another row of the diagonal. The following two results follow directly from the definition of t_{ij} .

LEMMA 3.10. *Let L be a Latin square of order n . Then for any row r ,*

$$\sum_{c=1}^n t_{rc} = E_n + N_r.$$

Proof. Each transversal in the matrix formed by deleting row r and column c extends to either a transversal of L or a diagonal of weight $n - 1$ depending on which symbol is in the cell (r, c) . \square

LEMMA 3.11. *Let L be a Latin square of order n . Then*

$$\sum_{r=1}^n \sum_{c=1}^n t_{rc} = nE_n + 2E_{n-1}.$$

Proof. Across the whole summation, each transversal of L is counted n times (once for each entry in the transversal) and each diagonal with weight $n - 1$ is counted twice (once for each entry containing the duplicated symbol). \square

Our next main result has a curious feature, which we explain after proving the result.

THEOREM 3.12. *Let L be a Latin square of odd order n . Then for any r and c ,*

$$t_{rc} \equiv E_n \pmod{2}.$$

Proof. Since n is odd, Corollary 3.8 ensures that

$$t_{rc} \equiv \sum_{i=1}^n \sum_{j=1}^n t_{ij} \pmod{2}.$$

Then Lemma 3.11 gives $t_{rc} \equiv nE_n + 2E_{n-1} \equiv E_n \pmod{2}$, as desired. \square

COROLLARY 3.13. *Let L be a Latin square of order n . Then N_r is even for all rows r .*

Proof. Each term in $\sum_{c=1}^n t_{rc}$ is the same modulo 2. If n is even, this sum is even, whereas if n is odd, the sum is equivalent to t_{r1} modulo 2. In either case, the sum is equivalent to E_n modulo 2, by Theorem 2.7 and Theorem 3.12, respectively. The result now follows from Lemma 3.10. \square

An interesting feature of Theorem 3.12 lies in the fact that a transversal of L can be inferred without locating one. In each of the other previous results, the number of diagonals with specific properties is of a similar form: congruent to 0 modulo m for some m . Congruences like this cannot be used to show existence of transversals. However, Theorem 3.12 gives a slightly different approach. In particular, if $t_{rc} \equiv 1 \pmod{2}$ for any row and column, then there must exist a transversal in L even if none go through the cell (r, c) .

EXAMPLE 3.14. Consider L_5 :

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

Every transversal in L_5 goes through the shaded entry. In particular, there are no transversals including the entry in the top left corner. However, the main diagonal is the sole transversal in $L_5(1|1)$, so $t_{11} = 1$. Thus, we can use Theorem 3.12 to deduce that at least one transversal exists in L_5 without finding such a transversal.

We finish the discussion of t_{ij} with a rather curious pattern found for small orders. It is very much in the spirit of Theorem 3.3 (but does not seem to follow directly from it).

CONJECTURE 3.15. *Let L be a Latin square of order n . Then $t_{ac} + t_{bc} + t_{ad} + t_{bd} \equiv 0 \pmod 4$ for all a, b, c, d .*

In light of Corollary 3.8, Conjecture 3.15 implies a very specific structure for the matrix $[t_{ij}]$. Each pair of rows either agrees modulo 4 or differs in every column by 2 modulo 4. A similar observation holds for columns.

4. COUNTING DIAGONALS BY THEIR NUMBER OF SYMBOLS

In this section, we look at relationships between the $E_i = E_i(L)$, that is, the counts of diagonals of L according to how many symbols they contain. We will also be interested in $R_i = R_i(L)$ which we define to be shorthand for $\langle i \rangle$ per $L[X]$. Note that $R_0 = 0$. The R_i are related to the E_i by

$$(14) \quad E_m = \sum_{r=1}^m (-1)^{m-r} \binom{n-r}{n-m} R_r,$$

where n is the order of L . This relationship was given explicitly in [2] and can easily be derived from Lemma 2.2.

In several proofs we will encounter d_n , the number of *derangements* in \mathcal{S}_n . From the well-known recurrence $d_n = nd_{n-1} + (-1)^n$, we learn that

$$(15) \quad d_n \equiv 1 \pmod 4 \text{ when } n \text{ is even.}$$

The following proposition is a list of identities which are either immediate from the definition of a Latin square or are proved in [6].

LEMMA 4.1. *Let L be a Latin square of order n .*

- (a) $R_1 = n$,
- (b) $R_{n-1} = nd_n$,
- (c) $R_n = n!$,
- (d) R_{2i} is even for each integer i ,
- (e) $R_i + R_{n-i}$ is even if n is even, and
- (f) $R_{n/2}$ is even if n is even.

Balasubramanian [6] used (d) and (e) to show Theorem 2.7, while Akbari and Alipour [2] showed the following two results.

THEOREM 4.2. *If L is a Latin square of order $n \equiv 2 \pmod 4$ then E_{n-3} is even.*

THEOREM 4.3. *If L is a Latin square of order n then E_{n-1} is even.*

We start with patterns in Latin squares of odd order. We have two direct corollaries of earlier results.

COROLLARY 4.4. *If L is a Latin square of odd order then $R_{4k} \equiv 0 \pmod 4$ for each integer k .*

Proof. Simply apply Corollary 3.6 to each matrix in the sum that defines R_{4k} . □

COROLLARY 4.5. *If L is a Latin square of odd order n and $k \equiv 2 \pmod{4}$, then*

$$R_k + 2R_{n-k} \equiv 0 \pmod{4}.$$

Proof. Apply Theorem 3.4 to each of the complementary pairs in R_k and R_{n-k} . \square

In addition, we have:

THEOREM 4.6. *If L is a Latin square of odd order n then E_i is even whenever i is even.*

Proof. By (14) we have

$$E_i \equiv \sum_{j=1}^i \binom{n-j}{n-i} R_j \pmod{2}.$$

Now, R_j is even for even j , by Lemma 4.1(d). When j is odd, $\binom{n-j}{n-i}$ is even by Lucas' Theorem, given that $n-j$ is even and $n-i$ is odd. The result follows. \square

We also have the following strengthening of Theorem 4.3 for odd orders:

THEOREM 4.7. *If L is a Latin square of odd order n then $E_{n-1} \equiv 0 \pmod{4}$.*

Proof. We compute E_{n-1} utilising (14). We pair up the complementary terms in this summation, $(n-r)R_r - rR_{n-r}$. Within each of these pairs, we assume that r is even, by replacing r by $n-r$ if necessary. We examine two cases. First, if $r \equiv 0 \pmod{4}$, then the second term vanishes modulo 4 and $(n-r)R_r \equiv 0 \pmod{4}$ by Corollary 4.4. Alternatively, if $r \equiv 2 \pmod{4}$ then R_r is even, by Lemma 4.1(d), so $(n-r)R_r \equiv R_r \pmod{4}$. Thus, $(n-r)R_r - rR_{n-r} \equiv R_r + 2R_{n-r} \pmod{4}$. We may now use Corollary 4.5. Each pair of complementary terms sums to a multiple of four, so the result follows. \square

We now shift our attention to Latin squares of even order, where the results are based on the global relationship between the different R_i values in contrast with the local nature of Corollary 4.4 and Corollary 4.5.

THEOREM 4.8. *If L is a Latin square of even order $n > 2$ then*

$$E_1 + E_3 + \dots + E_{n-1} \equiv E_2 + E_4 + \dots + E_n \equiv n \pmod{4}.$$

Proof. By (14),

$$\begin{aligned} \sum_{m=1}^{n/2} E_{2m-1} &= \sum_{m=1}^{n/2} \sum_{r=1}^{2m-1} (-1)^{2m-1-r} \binom{n-r}{n-2m+1} R_r \\ &= \sum_{r=1}^{n-1} (-1)^{n-r-1} R_r \sum_{s=1}^{\lceil (n-r)/2 \rceil} \binom{n-r}{2s-1} \\ &= \sum_{r=1}^{n-1} (-2)^{n-r-1} R_r \equiv R_{n-1} - 2R_{n-2} \equiv nd_n - 0 \equiv n \pmod{4}, \end{aligned}$$

by Lemma 4.1 and (15). If $n \geq 4$ then $\sum_{i=1}^n E_i = n! \equiv 0 \pmod{4}$, and the second congruence follows. \square

COROLLARY 4.9. *Every Latin square has an even number of diagonals that contain an even number of symbols.*

Proof. The order 2 case is trivial and Theorem 4.8 takes care of all larger even orders. The odd case is immediate from Theorem 4.6. \square

COROLLARY 4.10. *Every Latin square of order $n > 1$ has an even number of diagonals that contain an odd number of symbols.*

Proof. There are $n! \equiv 0 \pmod 2$ diagonals, so the result follows from Corollary 4.9. \square

The *even permanent* per^{ev} is defined as the sum of the products of the entries on the even diagonals of a matrix. In other words, it has the same definition as (1) except that the sum is taken over the alternating group rather than the symmetric group. Let R_i^{ev} be defined the same as R_i , but using per^{ev} in place of per . Similarly, let E_i^{ev} be the number of *even* diagonals with exactly i different symbols on them. We considered even permanents as one possible approach to Conjecture 2.13. While that effort was unsuccessful, we did manage to prove this weak analogue of Theorem 4.8:

THEOREM 4.11. *If L is a Latin square of even order $n > 2$ then*

$$E_3^{\text{ev}} + E_5^{\text{ev}} + \cdots + E_{n-1}^{\text{ev}} \equiv E_1^{\text{ev}} + E_2^{\text{ev}} + E_4^{\text{ev}} + E_6^{\text{ev}} + \cdots + E_n^{\text{ev}} \equiv 0 \pmod 2.$$

Proof. Similar to the proof of Theorem 4.8, we have that

$$\sum_{m=1}^{n/2} E_{2m-1}^{\text{ev}} \equiv R_{n-1}^{\text{ev}} \pmod 2.$$

Hence

$$(16) \quad \sum_{m=3}^{n/2} E_{2m-1}^{\text{ev}} \equiv E_1^{\text{ev}} + R_{n-1}^{\text{ev}} \equiv R_1^{\text{ev}} + R_{n-1}^{\text{ev}} \pmod 2.$$

Let P_0 and P_1 be permutation matrices corresponding to arbitrary even and odd permutations, respectively. Then $\text{per}^{\text{ev}} P_0 = 1$ and $\text{per}^{\text{ev}} P_1 = 0$. Also $\text{per}^{\text{ev}}(J - P_0) = a$ and $\text{per}^{\text{ev}}(J - P_1) = b$, where $a + b = \text{per}(J - P_0) = d_n$, and $a - b = \det(J - P_0) = 1 - n$, by (5). Thus $a = (d_n + 1 - n)/2$ and $b = (d_n - 1 + n)/2$. By (15),

$$\begin{aligned} \text{per}^{\text{ev}} P_0 + \text{per}^{\text{ev}}(J - P_0) &\equiv 1 + (d_n + 1 - n)/2 \equiv n/2 \equiv (d_n - 1 + n)/2 \\ &\equiv \text{per}^{\text{ev}} P_1 + \text{per}^{\text{ev}}(J - P_1) \end{aligned}$$

mod 2. Thus, in calculating (16) we can pair up complementary terms in R_1^{ev} and R_{n-1}^{ev} to show that $R_1^{\text{ev}} + R_{n-1}^{\text{ev}} \equiv n(n/2) \equiv 0 \pmod 2$. The result follows, since $\sum_i E_i^{\text{ev}} = n!/2 \equiv 0 \pmod 2$. \square

Note that $E_1^{\text{ev}} \equiv n - \pi_s \pmod 2$, where π_s is the symbol parity described in § 1. It is curious that the E_1^{ev} term in Theorem 4.11 appears on the side of the congruence that it does. The analogous statement for standard permanents follows by considering Theorem 4.8 modulo 2, and noting that the E_1 term can be written on either side of the congruence, since $E_1 = n$ is even.

We next show a parity relationship between consecutive pairs in the sequence E_1, \dots, E_n .

THEOREM 4.12. *If L is a Latin square of even order then $E_{2i-1} \equiv E_{2i} \pmod 2$ for each integer i .*

Proof. By (14),

$$\begin{aligned} E_{2i} + E_{2i-1} &= R_{2i} + \sum_{r=1}^{2i-1} \left[\binom{n-r}{n-2i} - \binom{n-r}{n-2i+1} \right] (-1)^r R_r \\ &= R_{2i} + \sum_{r=1}^{2i-1} \left[\binom{n-r}{n-2i} - \binom{n-r}{n-2i} \binom{2i-r}{n-2i+1} \right] (-1)^r R_r \\ &= R_{2i} + \sum_{r=1}^{2i-1} \binom{n-r}{n-2i} \left[\frac{n-4i+r+1}{n-2i+1} \right] (-1)^r R_r. \end{aligned}$$

If r is even, then R_r is even. If r is odd, then $n - 4i + r + 1$ is even, while $n - 2i + 1$ is odd, so $\binom{n-r}{n-2i}(n - 4i + r + 1)/(n - 2i + 1)$ must be even (it is an integer, since our proof shows that it is the difference of two integers). The result follows. \square

It seems that E_{2i} and E_{2i+1} are unrelated except for the case E_{n-2} and E_{n-1} when $n \equiv 0 \pmod 4$, which is covered in the following conjecture.

CONJECTURE 4.13. *Let L be a Latin square of order n . The following holds for all i, j and r . If $n \equiv 0 \pmod 4$, then*

$$E_n \equiv E_{n-1} \equiv 2E_{n-2} \equiv 2t_{ij} \equiv N_r \pmod 4,$$

$$(17) \quad R_1 + R_3 + \dots + R_{n-1} \equiv 0 \pmod 4 \quad \text{and} \quad R_2 + R_4 + \dots + R_n \equiv E_n \pmod 4.$$

If $n \equiv 2 \pmod 4$, then $E_{n-1} \equiv 2t_{ij} \equiv N_r \pmod 4$.

Using (14), Theorem 2.7 and Lemma 4.1(d), if $n \equiv 0 \pmod 4$, then $\sum R_i \equiv E_n \pmod 4$. Also, $E_{n-1} + 2E_{n-2} \equiv R_1 + R_3 + \dots + R_{n-1} \pmod 4$. Thus, $E_{n-1} \equiv 2E_{n-2} \pmod 4$ is equivalent to (17). Note, for even n , that E_n, E_{n-1} and N_r are even by Theorem 2.7, Theorem 4.3 and Corollary 3.13, respectively.

Suppose that $n \equiv 0 \pmod 4$ and that Conjecture 4.13 holds. It follows that $N_r + E_n \equiv 0 \pmod 4$, which combines with Lemma 3.10 to imply that the number of transversals in any $(n - 1) \times n$ Latin rectangle is divisible by 4, whenever n itself is divisible by 4.

Our results to this point have all been congruences mod 2 or 4. We finish by showing for any given order n that R_2 and E_2 have only two possible values mod 6. The main interest in this result is that it involves a different modulus to our other results.

LEMMA 4.14. *Let L be a Latin square of any order n . Then $R_2 \equiv E_2 \equiv 0 \pmod 2$ and*

$$R_2 \not\equiv (-1)^n(n + 1) \pmod 3 \quad \text{and} \quad E_2 \not\equiv (-1)^n(n + 1) - n(n - 1) \pmod 3.$$

Proof. By (14) we have that $E_2 = R_2 - (n - 1)R_1 = R_2 - n(n - 1)$, so it suffices to prove the claims about R_2 . By Lemma 4.1(d), we know that R_2 is even, and it follows immediately that E_2 is even as well.

For each symbol s of L , define a permutation $\theta_s : [n] \rightarrow [n]$ by $\theta_s(i) = j$ if $L_{ij} = s$. Then R_2 is the sum over symbols $s, s' \in [n]$ of $2^{c(s,s')}$, where $c(s, s')$ is the number of cycles in $(\theta_s)^{-1}\theta_{s'}$. The number of cycles in any permutation $\sigma \in \mathcal{S}_n$ is $n - \varepsilon(\sigma) \pmod 2$. Hence

$$\begin{aligned} R_2 &= \sum_{s,s'} 2^{c(s,s')} \equiv \sum_{s,s'} (-1)^{n-\varepsilon((\theta_s)^{-1}\theta_{s'})} \equiv (-1)^n \sum_{s,s'} (-1)^{\varepsilon(\theta_s)+\varepsilon(\theta_{s'})} \\ &\equiv (-1)^n \left(\binom{e}{2} + \binom{n-e}{2} - e(n-e) \right) \pmod 3, \end{aligned}$$

where $e = |\{s \in [n] : \varepsilon(\theta_s) = 0\}|$. The required result now follows by a simple case analysis concerning the value of $e \pmod 3$. \square

5. CONCLUDING REMARKS

We have shown a number of congruences satisfied by various quantities motivated by the study of transversals in Latin squares. There are many others which are direct consequences of the results we have given. For example, it is easy to use (14), Lemma 4.1(d) and Corollary 4.4 to show that $E_8 \equiv 0 \pmod 4$ when $n \equiv 3 \pmod 4$, given

that

$$\binom{n-7}{1} \equiv \binom{n-5}{3} \equiv \binom{n-3}{5} \equiv \binom{n-2}{6} \equiv \binom{n-1}{7} \equiv 0 \pmod{4} \quad \text{and}$$

$$\binom{n-6}{2} \equiv 0 \pmod{2}.$$

As mentioned in the introduction, a notion of parity has been useful in a number of different studies of Latin squares. In this paper we have introduced parity for transversals of Latin squares, and used it in the analysis of the number of transversals. In our investigation we uncovered a number of interesting patterns, some of which we have proved, and others we conjecture. Several of the conjectures classify Latin squares of a given even order into two types which seem to have different properties. These classifications do not seem to be related to each other, or to pre-existing notions of parity. In Conjecture 2.13 the value of $w \pmod{2}$, say, partitions Latin squares based on the parity of their transversals, while Conjecture 4.13 partitions Latin squares into two classes based on E_{n-1} modulo 4. However, these partitions seem to be independent of each other and of the previously studied parities π_r and π_c . By randomly generating Latin squares, we found a Latin square with each of the 16 possibilities for $(w, E_{n-1}/2, \pi_r, \pi_c) \pmod{2}$ for orders 8, 10 and 12.

Finally, we remark that we have only considered the classical 2-dimensional case in this paper. However, transversals are of interest in the context of Latin hypercubes and permanents can also be generalised to higher dimensions. All of the questions that we have investigated could also be asked in these higher dimensional contexts. A first step in this direction has been taken by Taranenko [24], who noted that Theorem 2.7 generalises to Latin hypercubes.

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