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Abstract Our main theorem in [1] contained a mistake regarding the equivalence of two conditions on a graph, which we correct here. Fortunately our main result is not impacted with an additional assumption called the zero-three condition.

The following is a corrected version of the statement of our main theorem, Theorem 1.2. We say a graph of gonality 3 *satisfies the zero-three condition* if there exists a divisor D such that $\deg(D) = 3$, $r(D) = 1$, and for any three distinct vertices a, b, c with $D \sim (a) + (b) + (c)$, we have that a, b, c either share 0 edges or share 3 edges.

Theorem 1.2. *If G is a 3-edge-connected combinatorial graph, then the following are equivalent:*

(1) *G has (divisorial) gonality 3.*
(2) *There exists a non-degenerate harmonic morphism $\varphi : G \to T$, where $\deg(\varphi) = 3$ and T is a tree.*

Moreover, if G is simple and 3-vertex-connected, and also satisfies the zero-three condition, these statements imply the following condition:

(3) *There exists a cyclic automorphism $\sigma : G \to G$ of order 3 that does not fix any edge of G satisfying the property that G/σ is a tree.*

Condition (3) implies conditions (1) and (2), whether or not G satisfies the zero-three condition.

Similarly, we have the following corrected version of Theorem 4.1.

Theorem 4.1. *If G is a simple, 3-vertex-connected combinatorial graph satisfying the zero-three condition, then the following are equivalent:*

(1) *G has gonality 3.*
(2) *There exists a non-degenerate harmonic morphism $\varphi : G \to T$, where $\deg(\varphi) = 3$ and T is a tree.*
(3) *There exists a cyclic automorphism $\sigma : G \to G$ of order 3 that does not fix any edge of G, such that G/σ is a tree.*

The equivalence of (1) and (2) and the implication (3) implies (1) and (2) still hold without the zero-three condition assumption.

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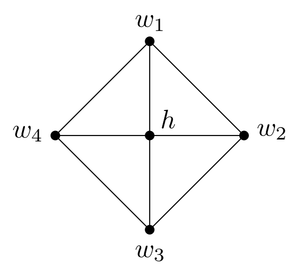

Figure 1. The wheel graph W_5

To see that the zero-three condition is necessary, consider the graph in Figure 1, which is the wheel graph W_5 on 5 vertices. It has gonality 3: the divisor $(w_1)+(h)+(w_3)$ has positive rank; and since the graph has K_4 as a minor, the treewidth of the graph, and thus its gonality, is at least 3. It is also 3-vertex-connected. However, we claim that it does not satisfy the zero-three condition. Let D be any effective rank 1 divisor of degree 3 on W_5. By Lemma 4.2, D must have support size 1 or 3, and it is equivalent to some divisor with support size 3. If W_5 satisfies the zero-three condition, then since any three vertices have at least one edge in common, there must be a rank 1 divisor $(a) + (b) + (c)$ on W_5 where a, b, c are distinct and form a K_3 in the graph. It follows that $(w_i) + (w_{i+1}) + (h)$ has rank 1 for some i, where addition is done modulo 4. However, this divisor does not have rank 1: starting Dhar's burning algorithm from w_{i+2} burns the whole graph. Thus W_5 must not satisfy the zero-three condition. Finally, the automorphism group of W_5 is the same as the automorphism group of the cycle C_4, namely the dihedral group of order 8. This group does not have any elements of order 3. Thus, W_5 is an example of a simple, 3-vertex-connected graph of gonality 3, not satisfying the zero-three condition, which does not have any automorphisms of order 3.

The gap in the original argument, which did not assume the zero-three condition, came when proving the map σ was an automorphism in Proposition 4.3. It was true that σ was well-defined and preserved adjacency relations between different classes of points under $\sim_D$, but it was omitted to prove that we could send the three points in the same class under $\sim_D$ to one another in a nontrivial way; this can be done precisely when there are zero or three edges between them. Thus, the proof needs to assume the zero-three condition, and we would add the following to the beginning of its proof: "First consider the action of σ on a triple v_1, v_2, v_3 where $D \sim (v_1) + (v_2) + (v_3)$ and v_1, v_2, v_3 are all distinct. We are assuming our graph satisfies the zero-three condition, so the map σ mapping v_1 to v_2 to v_3 to v_1 preserves the connectivity of v_1, v_2, and v_3, since either all share edges or none share edges."

The only one of our other results or examples relying on the incorrect statements of Theorems 1.2 and 4.1 was our consideration of the Frucht graph, in Figure 8. It is no longer obvious that it cannot have gonality 3: we only know that if it does have gonality 3, then it does not satisfy the zero-three property. However, we have still computationally verified that the Frucht graph has gonality 4, as claimed.

References

[1] Ivan Aidun, Frances Dean, Ralph Morrison, Teresa Yu, and Julie Yuan, *Graphs of gonality three*, Algebr. Comb. **2** (2019), no. 6, 1197–1217.

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