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Semi-infinite Young tableaux and standard monomial theory for semi-infinite Lakshmibai–Seshadri paths

Motohiro Ishii

Abstract
We introduce semi-infinite Young tableaux, and show that these tableaux give a combinatorial model for the crystal basis of a level-zero extremal weight module over the quantized universal enveloping algebra of untwisted affine type $A$. The definition and characterization of these tableaux are based on standard monomial theory for semi-infinite Lakshmibai–Seshadri paths and a tableau criterion for the semi-infinite Bruhat order on affine Weyl groups of type $A$, which are also proved in this paper.

1. Introduction
The aim of this paper is to introduce semi-infinite Young tableaux (see Definition 4.2 (2)). These tableaux give a new combinatorial model for the crystal basis of a level-zero extremal weight module (see § 2.3) over the quantized universal enveloping algebra of untwisted affine type $A$. In order to accomplish our purpose, we investigate

(i) a characterization of the image of the strict embedding $\Phi_{\lambda,q=0}^{LT}$ (see § 3.1) of crystals in terms of the semi-infinite Bruhat order (see Theorem 3.4), and

(ii) a tableau criterion for the semi-infinite Bruhat order on affine Weyl groups of type $A$ in Grassmannian cases (see Theorem 4.7).

Note that the image of $\Phi_{\lambda,q=0}^{LT}$ is an isomorphic image of the crystal basis of the extremal weight module of extremal weight $\lambda$ (of level-zero) into the tensor product of crystal bases of extremal weight modules associated with level-zero fundamental weights.

Various generalizations and variations of (semi-standard) Young tableaux are concerned in many areas such as algebraic combinatorics, representation theory, algebraic geometry, and so forth. In particular, Littelmann ([21, 22]) introduced the Lakshmibai–Seshadri paths for all symmetrizable Kac–Moody root data, which can be thought of as a type-free generalization of Young tableaux. Soon after, Joseph ([7]) and Kashiwara ([13]) independently proved that, for a dominant integral weight $\Lambda$ of a symmetrizable Kac–Moody Lie algebra $g$, the set of Lakshmibai–Seshadri paths of shape $\Lambda$ equipped with Littelmann’s root operators is isomorphic, as a $g$-crystal, to the crystal basis of the integrable (irreducible) highest weight module of highest weight $\Lambda$.
over the quantized universal enveloping algebra associated with \( \mathfrak{g} \). In view of Kashiwara’s crystal (basis) theory, further generalizations and variants of Littelmann’s path model are investigated; e.g. generalized Lakshmibai–Seshadri paths for Borcherds–Kac–Moody root data ([5, 8]), quantum Lakshmibai–Seshadri paths and semi-infinite Lakshmibai–Seshadri paths for untwisted affine root data ([6, 20]). Among these general theories, it should be emphasized that the original Young tableaux have especially nice combinatorial structures (see for instance [4, 17]) due to the fact that every fundamental representation of finite type \( A \) is minuscule; namely, the Weyl group acts transitively on the crystal basis of any fundamental representation in the case of finite type \( A \). Similarly, in the case of untwisted affine type \( A \), every extremal weight module associated with a level-zero fundamental weight is minuscule (see Proposition 4.3 (2)); in this case, the affine Weyl group acts transitively on the crystal basis. Therefore, it is natural to try to find a tableau model for crystal bases of level-zero extremal weight modules in the case of untwisted affine type \( A \).

Let us give an explanation of our strategy to introduce semi-infinite Young tableaux.

For this purpose, we first briefly sketch a standard monomial theoretic characterization of (ordinary) Young tableaux in terms of crystal basis theory as follows: let \( U_q(\mathfrak{sl}_n(\mathbb{C})) \) be the quantized universal enveloping algebra of type \( A_{n-1} \) (see [9, § 4.8 TABLE Fin]). Let \( \varpi_i, 1 \leq i \leq n-1 \), be the \( i \)-th fundamental weight; we identify a dominant integral weight \( \lambda = \sum_{i=1}^{n-1} m_i \varpi_i, m_i \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq n-1 \), with the Young diagram such that the number of the columns of length \( i \) is \( m_i \) for \( 1 \leq i \leq n-1 \) (see Remark 4.1). For a dominant integral weight \( \lambda \), let \( L(\lambda) \) be the irreducible finite-dimensional highest weight \( U_q(\mathfrak{sl}_n(\mathbb{C})) \)-module of highest weight \( \lambda \), and let \( B(\lambda) \) be the crystal basis of \( L(\lambda) \). It follows that \( B(\varpi_i) \) is parametrized by the set \( \text{CST}(\varpi_i) \) of column-strict tableaux of shape \( \varpi_i \) with entries in \( \{1, 2, \ldots, n\} \) (see for instance [15, Proposition 3.3.1 (i)]). We have an injective homomorphism

\[
L(\lambda) \longrightarrow \bigotimes_{i=1}^{n-1} L(\varpi_i)^{\otimes m_i}
\]

of \( U_q(\mathfrak{sl}_n(\mathbb{C})) \)-modules sending a highest weight vector to the tensor product of highest weight vectors. Further, this homomorphism induces a strict embedding

\[
B(\lambda) \longrightarrow \bigotimes_{i=1}^{n-1} B(\varpi_i)^{\otimes m_i} \cong \prod_{i=1}^{n-1} \text{CST}(\varpi_i)^{m_i} \cong \text{CST}(\lambda)
\]

of \( U_q(\mathfrak{sl}_n(\mathbb{C})) \)-crystals, where \( \text{CST}(\lambda) \) denotes the set of column-strict tableaux of shape \( \lambda \) with entries in \( \{1, 2, \ldots, n\} \). Since the symmetric group of degree \( n \) acts transitively on \( \text{CST}(\varpi_i) \), each element in \( \text{CST}(\lambda) \) is labeled by a tuple of \( N := \sum_{i=1}^{n-1} m_i \) cosets in the symmetric group; the symmetric group of degree \( n \) will be viewed as the Weyl group of type \( A_{n-1} \). Let \( \mathbb{T} \in \text{CST}(\lambda) \), and assume that \( \mathbb{T} \) corresponds to the tuple \( (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_N) \) of \( N \) cosets in the symmetric group. Then \( \mathbb{T} \) is in the image of the strict embedding (2) if and only if there exist coset representatives \( w_\nu \in \bar{w}_\nu, 1 \leq \nu \leq N \), such that \( w_1 \succeq w_2 \succeq \cdots \succeq w_N \) in the Bruhat order \( \succeq \) on the cosets in the symmetric group (see [23, Theorem 10.1]). In consequence, a column-strict tableau satisfying this condition is just a Young tableau, and vise versa ([3, Theorem 2.6.3 (Tableau Criterion)]; see also [15, Theorem 3.4.2 (i)]). This gives an isomorphism of \( U_q(\mathfrak{sl}_n(\mathbb{C})) \)-crystals between \( B(\lambda) \) and the set of Young tableaux of shape \( \lambda \). For an explicit description of the \( U_q(\mathfrak{sl}_n(\mathbb{C})) \)-crystal structure on the set of Young tableaux, see [15, Theorem 3.4.2 (ii)].

In the case of untwisted affine type \( A \), our basic idea is to use the semi-infinite Bruhat order on the affine Weyl group in place of the Bruhat order on the symmetric
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group. Let $U$ be the quantized universal enveloping algebra of (arbitrary) untwisted affine type, and let $I_{af} = \{0\} \sqcup I$ be the index set for the simple roots. By abuse of notation, we use the same symbol $\varpi_i$, $i \in I$, for the $i$-th level-zero fundamental weight of $U$. For $\lambda = \sum_{i \in I} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$, $i \in I$, let $V(\lambda)$ be the extremal weight $U$-module of extremal weight $\lambda$, and let $B(\lambda)$ be the crystal basis of $V(\lambda)$ (see for instance [14, § 3.1]). Similarly to (1), we have a canonical homomorphism

$$\Phi_{\lambda} : V(\lambda) \to \bigotimes_{i \in I} V(\varpi_i)^{\otimes m_i}$$

of $U$-modules sending an extremal weight vector to the tensor product of extremal weight vectors. The main difficulty in carrying out the argument similar to the above is that the associated map $\Phi_{\lambda|q=0}$ at $q = 0$ of $\Phi_{\lambda}$ does not necessarily induce a morphism $B(\lambda) \to \bigotimes_{i \in I} B(\varpi_i)^{\otimes m_i}$ of $U$-crystals. In fact, each $\Phi_{\lambda|q=0}(b)$, $b \in B(\lambda)$, is a linear combination of crystal basis elements whose terms are in one-to-one correspondence with the terms of a product of some Schur polynomials (see [2, § 4.2]). To overcome this difficulty, by taking the “leading term” $\Phi^{LT}_{\lambda|q=0}(b)$ of $\Phi_{\lambda|q=0}(b)$ (see Remark 3.2), we introduce a strict embedding

$$\Phi^{LT}_{\lambda|q=0} : B(\lambda) \to \bigotimes_{i \in I} B(\varpi_i)^{\otimes m_i}$$

of $U$-crystals, which will be viewed as a counterpart of (2) in this paper (see (38) and Lemma 3.1). In the case that $U$ is of untwisted affine type $A$, it follows that $B(\varpi_i)$ is parametrized by $\text{CST}(\varpi_i) \times \mathbb{Z}$, and the affine Weyl group acts transitively on $\text{CST}(\varpi_i) \times \mathbb{Z}$ (see Proposition 4.3). Consequently, $\bigotimes_{i \in I} B(\varpi_i)^{\otimes m_i}$ is parametrized by $\text{CST}(\lambda) \times \mathbb{Z}^N$, where $N := \sum_{i \in I} m_i$, and each element in $\text{CST}(\lambda) \times \mathbb{Z}^N$ is labeled by a tuple of $N$ cosets in the affine Weyl group. Let $T \in \text{CST}(\lambda) \times \mathbb{Z}^N$, and assume that $T$ corresponds to the tuple $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N)$ of cosets in the affine Weyl group. Then $T$ is in the image of the strict embedding (4) if and only if there exist coset representatives $x_{\nu} \in \bar{x}_\nu$, $1 \leq \nu \leq N$, such that $x_1 \preceq x_2 \preceq \cdots \preceq x_N$ in the semi-infinite Bruhat order $\preceq$ on the affine Weyl group (see Theorem 3.4). Such decreasing sequences are explicitly described in terms of tableaux (see Definition 4.2 (1) and Theorem 4.7). We are thus led to the definition of semi-infinite Young tableaux (see Definition 4.2 (2)).

Let us give an example of a semi-infinite Young tableau (by using the notation in § 4.1). Let $n = 7$ and

$$\lambda = \varpi_1 + 3\varpi_2 + 2\varpi_4$$

We claim that

$$T = \begin{pmatrix} 4 & 5 & 8 & 9 & 10 \end{pmatrix}, \begin{pmatrix} -3; 5, 4, -1; 8, 7 \end{pmatrix} \in \text{CST}(\lambda) \times \mathbb{Z}^6$$

is a semi-infinite Young tableau of shape $\lambda$. Indeed, $T$ is a semi-infinite Young tableau if (and only if) its rectangle components

$$\begin{pmatrix} 4, & -3 \end{pmatrix}, \begin{pmatrix} 5, 4, & -1 \end{pmatrix}, \begin{pmatrix} 8, 7 \end{pmatrix}$$

are semi-infinite Young tableaux. Also, a tableau

$$T_1 T_2 \cdots T_m, \begin{pmatrix} c_1, c_2, \ldots, c_m \end{pmatrix} \in \text{CST}(m \varpi_i) \times \mathbb{Z}^m$$
of rectangle shape is a semi-infinite Young tableau if (and only if)
(i) $c_1 \geq c_2 \geq \cdots \geq c_m$, and
(ii) $(T_{\nu}(u + c_\nu - c_{\nu+1}) \geq T_{\nu+1}(u)$ if $1 \leq u \leq i - c_\nu + c_{\nu+1}$) for every $1 \leq \nu < m$,
where $T_{\nu}(s)$ denotes the $s$-th entry (from top) of the $\nu$-th column $T_{\nu}$.

This paper is organized as follows. In § 2, we set up notation and terminology on untwisted affine root data and crystals. Also, we have compiled some basic facts on extremal weight modules over quantized universal enveloping algebras of untwisted affine types, the semi-infinite Bruhat order on affine Weyl groups, and semi-infinite Lakshmibai–Seshadri paths. In § 3, we introduce a strict embedding $\Phi_{\lambda=q=0}^L$ of crystals. Then we state and prove a characterization of the image of $\Phi_{\lambda=q=0}^L$, which can be thought of as standard monomial theory for semi-infinite Lakshmibai–Seshadri paths (see Theorem 3.4). In § 4, we will restrict our attention to the case of untwisted affine type $A$. We introduce semi-infinite Young tableaux. We prove that the set $\mathcal{Y}^\infty(\lambda)$ of semi-infinite Young tableaux of shape $\lambda$ equals the image of $\Phi_{\lambda=q=0}^L$ (see Theorem 4.5), by showing a tableau criterion for the semi-infinite Bruhat order (see Theorem 4.7). Consequently, this proves that $\mathcal{Y}^\infty(\lambda)$ is isomorphic, as a $U$-crystal, to the crystal basis $B(\lambda)$ (see Corollary 4.6). We give an explicit description of the crystal structure on $\mathcal{Y}^\infty(\lambda)$ (see Proposition 4.17).

2. Preliminaries

2.1. Untwisted affine root data. Let $\mathfrak{g}_{af}$ be an untwisted affine Lie algebra over $\mathbb{C}$ with a Cartan subalgebra $\mathfrak{h}_{af}$. Let $\{\alpha_i\}_{i \in I_{af}} \subset \mathfrak{h}_{af}^\ast = \text{Hom}_\mathbb{C}(\mathfrak{h}_{af}, \mathbb{C})$ and $\{h_i\}_{i \in I_{af}} \subset \mathfrak{h}_{af}$ be the sets of simple roots and simple coroots, respectively. Here $I_{af}$ denotes the vertex set of the (affine) Dynkin diagram of $\mathfrak{g}_{af}$. Let \langle \cdot, \cdot \rangle : \mathfrak{h}_{af} \times \mathfrak{h}_{af}^\ast \to \mathbb{C}$ be the canonical pairing. We take and fix an integral weight lattice vertex set of the (affine) Dynkin diagram of $\mathfrak{h}_{af}$ with a Cartan subalgebra $\mathfrak{h}_{af}$. We know from [9, § 6.5] that $\mathfrak{h}_{af}^\ast \mathfrak{h}_{af}$ and $\mathfrak{h}_{af}$ are semi-infinite Young tableau if $\lambda = (\alpha_i, \alpha_i)$ for all $i \in I_{af}$ and for each $i \in I_{af}$ there exists $A_i \in P_{af}$ such that $\langle h_j, \alpha_i \rangle = \delta_{ij}$ for all $j \in I_{af}$. Let $\delta = \sum_{i \in I_{af}} a_i \alpha_i \in \mathfrak{h}_{af}^\ast$ and $c = \sum_{i \in I_{af}} a_i h_i$ be the null root and the canonical central element, respectively. We take and fix $0 \in I_{af}$ such that $a_0 = a_0' = 1$. Set $I_{af} = I_{af} \setminus \{0\}$; note that the subset $\{0\}$ of $I_{af}$ corresponds to the vertex set of the Dynkin diagram of a complex finite-dimensional simple Lie subalgebra $\mathfrak{g}$ of $\mathfrak{g}_{af}$. For each $i \in I_{af}$, define $\varpi_i = A_i = \langle \epsilon, \alpha_i \rangle A_0$ and call it the $i$-th level-zero fundamental weight; note that $\varpi_0 = 0$, $(c, \varpi_i) = 0$ for all $i \in I_{af}$, and $(h_i, \varpi_j) = \delta_{ij}$ for all $i, j \in I$. Let $W_{af} = (r_i \mid i \in I_{af})$ be the (affine) Weyl group of $\mathfrak{g}_{af}$, where $r_i$ denotes the simple reflection with respect to $\alpha_i$. The subgroup $W = (r_i \mid i \in I) \subset W_{af}$ is the (finite) Weyl group of $\mathfrak{g}$. Let $\ell : W_{af} \to \mathbb{Z}_{\geq 0}$ be the length function. Let $c \in W_{af}$ be the unit element. Let $(\epsilon, \cdot)$ be a $W_{af}$-invariant non-degenerate symmetric bilinear form on $\mathfrak{h}_{af}^\ast$ such that $(\delta, \lambda) = \langle c, \lambda \rangle$ for all $\lambda \in \mathfrak{h}_{af}^\ast$. Set $\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle} \in \mathfrak{h}_{af}^\ast$ for $i \in I_{af}$. The action of $W_{af}$ on $\mathfrak{h}_{af}^\ast$ is given by $r_i(\lambda) = \lambda - (\alpha_i^\vee, \lambda) \alpha_i = \lambda - \langle h_i, \lambda \rangle \alpha_i$ for $i \in I_{af}$ and $\lambda \in \mathfrak{h}_{af}^\ast$. Set

$$Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i, \quad Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee, \quad P^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i.$$  

We know from [9, § 6.5] that $Q^\vee \subset Q$. For $\xi \in Q^\vee$, we denote by $t_\xi \in W_{af}$ the translation by $\xi$ (see [9, §6.5]). We know from [9, Proposition 6.5] that $\{t_\xi \mid \xi \in Q^\vee\}$ forms an abelian normal subgroup of $W_{af}$, for which $t_\xi t_\zeta = t_{\xi + \zeta}$, $\xi, \zeta \in Q^\vee$, and $W_{af} = W \rtimes \{t_\xi \mid \xi \in Q^\vee\}$. For $w \in W$ and $\xi \in Q^\vee$, we have

$$wt_\xi \lambda = w(\lambda - (\xi, \lambda) \delta) \quad \text{if} \quad \lambda \in \mathfrak{h}_{af}^\ast \text{ satisfies } \langle c, \lambda \rangle = 0.$$  

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Let $\Delta$ be the root system of $g$. Set $\Delta^+ = \Delta \cap \sum_{i \in J} \mathbb{Z}_{\geq 0}\alpha_i$. For a subset $J \subset I$, set

$$Q_J = \bigoplus_{j \in J} \mathbb{Z}\alpha_j, \quad Q_J^\vee = \bigoplus_{j \in J} \mathbb{Z}\alpha_j^\vee, \quad \Delta_J = \Delta \cap Q_J, \quad \Delta_J^+ = \Delta^+ \cap Q_J.$$

Denote by $\Delta_{af}$ the set of real roots of $g_{af}$, and by $\Delta_{af}^+$ the set of positive real roots of $g_{af}$; we know from [9, Proposition 6.3] that

$$\Delta_{af} = \{\alpha + n\delta | \alpha \in \Delta, n \in \mathbb{Z}\}, \quad \Delta_{af}^+ = \Delta^+ \cup \{\alpha + n\delta | \alpha \in \Delta, n \in \mathbb{Z}_{>0}\}. $$

For $\beta \in \Delta_{af}$, let $\beta^\vee = \frac{2\beta}{(\beta, \beta)} \in h_{af}^*$, and let $r_{\beta} \in W_{af}$ be the reflection with respect to $\beta$; if $\beta = \alpha + n\delta$, $\alpha \in \Delta$ and $n \in \mathbb{Z}$, then $r_{\beta} = r_{n\alpha} r_{\alpha}^\vee$.

Let $d$ be the smallest positive integer such that $(\alpha_1, \alpha_1)/2 \in (1/d)\mathbb{Z}$ for all $i \in I_{af}$. Let $q$ be an indeterminate, and set $q_s = q^{1/d}$. Let $U$ be the quantized universal enveloping algebra over $\mathbb{Q}(q_s)$ associated with $g_{af}$. Let $U'$ be the $\mathbb{Q}(q_s)$-subalgebra of $U$ corresponding to the derived subalgebra $[g_{af}, g_{af}]$ of $g_{af}$ (see for instance [2, §2.2]).

2.2. Crystals. In this subsection, we set up notation and terminology on crystals. For a fuller treatment, we refer the reader to [1, 11, 12, 14].

A set $B$ together with the maps $\omega : B \to P_{af}$ (resp. $\omega : B \to P_{af}/(P_{af} \cap \mathbb{C}0)$), $e_i, f_i : B \to B \cup \{0\}$, $i \in I_{af}$, and $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{\infty\}$ is called a $U$-crystal (resp. $U'$-crystal) if the following conditions are satisfied:

1. $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \omega(b) \rangle$ for all $i \in I_{af}$,
2. $\omega(e_i b) = \omega(b) + \alpha_i$ if $e_i b \in B$,
3. $\omega(f_i b) = \omega(b) - \alpha_i$ if $f_i b \in B$,
4. $\varepsilon_i(e_i b) = \varepsilon_i(b) - 1$ and $\varphi_i(e_i b) = \varphi_i(b) + 1$ if $e_i b \in B$,
5. $\varepsilon_i(f_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(f_i b) = \varphi_i(b) - 1$ if $f_i b \in B$,
6. $f_i b = b'$ if and only if $b = e_i b'$ for $b, b' \in B$ and $i \in I_{af}$,
7. if $\varphi_i(b) = -\infty$, then $e_i b = f_i b = 0$.

The maps $e_i, f_i : B \to B \cup \{0\}$, $i \in I_{af}$, are called the Kashiwara operators. For a subset $B'$ of a crystal $B$, we say that $B'$ is stable under the Kashiwara operators if $e_i B', f_i B' \subset B' \cup \{0\}$ for all $i \in I_{af}$.

Let $B_1$ and $B_2$ be $U$-crystals or $U'$-crystals. A morphism $\Psi : B_1 \to B_2$ is, by definition, a map $B_1 \cup \{0\} \to B_2 \cup \{0\}$ such that

1. $\Psi(0) = 0$,
2. if $b \in B_1$ and $\Psi(b) \in B_2$, then $\omega(\Psi(b)) = \omega(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\Psi(b)) = \varphi_i(b)$ for all $i \in I_{af}$,
3. if $b, b' \in B_1$, $\Psi(b), \Psi(b') \in B_2$ and $f_i b = b'$, then $f_i \Psi(b) = \Psi(b')$ for all $i \in I_{af}$.

A morphism $\Psi : B_1 \to B_2$ is called strict if $\Psi(f_i b) = f_i \Psi(b)$ and $\Psi(e_i b) = e_i \Psi(b)$ for all $b \in B_1$ and $i \in I_{af}$. A morphism $\Psi : B_1 \to B_2$ is called a strict embedding if it is a strict morphism and the associated map $B_1 \cup \{0\} \to B_2 \cup \{0\}$ is injective. A morphism $\Psi : B_1 \to B_2$ is called an isomorphism if the associated map $B_1 \cup \{0\} \to B_2 \cup \{0\}$ is bijective.

The tensor product $B_1 \otimes B_2$ of crystals $B_1$ and $B_2$ is defined to be the set $B_1 \times B_2$ whose tensor product is defined as follows:

1. $\omega(b_1 \otimes b_2) = \omega(b_1) + \omega(b_2)$,
2. $\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \omega(b_1) \rangle\}$,
3. $\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1), \varphi_i(b_2) + \langle h_i, \omega(b_1) \rangle\}$,
4. $\varepsilon_i(b_1 \otimes b_2) = \begin{cases} (\varepsilon_i(b_1) \otimes b_2) & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_1), \\ b_1 \otimes (\varepsilon_i(b_2)) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_1). \end{cases}$
Here, we write has a crystal basis (14) \( \tilde{e} \) operators. We know from [2, § 4.2] that the map \( L \) of weight \( b \) for each \( b \) (13) \( \lambda \) extremal element if we can find elements \( b \in B \), \( x \in W_{af} \), such that

\[
S_x b = \begin{cases} 
    f_i^{(b_i, \text{wt}(b))} b & \text{if } \langle h_i, \text{wt}(b) \rangle \geq 0, \\
    e_i^{(-\langle h_i, \text{wt}(b) \rangle)} b & \text{if } \langle h_i, \text{wt}(b) \rangle \leq 0,
\end{cases}
\]

for each \( b \in B \) and \( i \in I_{af} \). Note that \( \text{wt}(S_x b) = x \text{wt}(b) \) holds for all \( x \in W_{af} \) and \( b \in B \). An element \( b \in B \) of weight \( \lambda \in P_{af} \) (resp. \( \lambda \in P_{af}/(P_{af} \cap \mathbb{C} \delta) \)) is called an extremal element if we can find elements \( b_x \in B, x \in W_{af} \), such that

1. \( b_e = b \),
2. \( \langle h_i, x \lambda \rangle \geq 0 \), then \( e_i b_x = 0 \) and \( f_i^{(h_i, x \lambda)} b_x = b_{r,x} \),
3. \( \langle h_i, x \lambda \rangle \leq 0 \), then \( f_i b_x = 0 \) and \( e_i^{(-h_i, x \lambda)} b_x = b_{r,x} \).

Then \( b_x = S_x b \) holds for all \( x \in W_{af} \).

2.3. Extremal Weight Modules and Their Crystal Bases. In this subsection, following [2, 11, 14], we review some of the standard facts on extremal weight modules and their crystal bases.

For \( \lambda \in P^+ \), let \( V(\lambda) \) be the extremal weight \( U \)-module generated by an extremal weight vector \( u_\lambda \) of extremal weight \( \lambda \), and let \( (\mathcal{L}(\lambda), B(\lambda)) \) be the crystal basis of \( V(\lambda) \) ([11, Proposition 8.2.2]; see also [14, § 3.2]). Note that \( B(\lambda) \) is a regular \( U \)-crystal in the sense of [14, § 2.2] (see § 2.2). Let \( B_0(\lambda) \) be the connected component of the crystal graph of \( B(\lambda) \) containing \( u_\lambda \mod q_\delta \mathcal{L}(\lambda) \).

Let \( z_i, i \in I \), be the \( U' \)-linear automorphism of \( V(\varpi_i) \) of weight \( \delta \) introduced in [14, § 5.2]; \( z_i \) sends a (unique) global basis element of weight \( \varpi_i \) to a (unique) global basis element of weight \( \varpi_i + \delta \). Then \( z_i \) induces a \( \mathcal{L}(\varpi_i)/q_\delta \mathcal{L}(\varpi_i) \) and an automorphism of \( B(\varpi_i) \) as a \( U' \)-crystal; by abuse of notation, we use the same letter \( z_i \) for the automorphism of \( B(\varpi_i) \). The \( U' \)-module \( \tilde{W}(\varpi_i) = V(\varpi_i)/(z_i - 1)V(\varpi_i) \) is called a level-zero fundamental representation. We know from [14, Theorem 5.17] that \( W(\varpi_i) \) is a finite-dimensional irreducible \( U' \)-module and has a (simple) crystal basis.

For \( \lambda = \sum_{i \in I} m_i \varpi_i \in P^+ \), with \( m_i \in \mathbb{Z}_{\geq 0}, i \in I \), let \( \tilde{V}(\lambda) = \bigotimes_{i \in I} V(\varpi_i)^{\otimes m_i} \), and \( \tilde{u}_\lambda = \bigotimes_{i \in I} u_{\varpi_i}^{\otimes m_i} \in \tilde{V}(\lambda) \). For each \( i \in I \) and \( 1 \leq \nu \leq m_i \), let \( z_{i, \nu} \) be the \( U' \)-linear automorphism of \( \tilde{V}(\lambda) \) obtained by the action of \( z_i \) on the \( \nu \)-th factor \( V(\varpi_i) \) of \( V(\varpi_i)^{\otimes m_i} \) in \( \tilde{V}(\lambda) \). The \( U \)-submodule

\[
\tilde{V}(\lambda) = U \left[ z_{i, \nu}, z_{i, \nu}^{-1} \mid i \in I, 1 \leq \nu \leq m_i \right] \tilde{u}_\lambda \subset \tilde{V}(\lambda)
\]

has a crystal basis \( (\tilde{\mathcal{L}}(\lambda), \tilde{B}(\lambda) = \bigotimes_{i \in I} B(\varpi_i)^{\otimes m_i}) \) such that \( \tilde{\mathcal{L}}(\lambda) \subset \bigotimes_{i \in I} \mathcal{L}(\varpi_i)^{\otimes m_i} \) ([14, Theorem 8.5]). Let \( B_0(\lambda) \) be the connected component of the crystal graph of \( B(\lambda) \) containing \( \tilde{u}_\lambda \mod q_\delta \tilde{\mathcal{L}}(\lambda) \). Since \( \tilde{V}(\lambda) \) contains an extremal weight vector \( \tilde{u}_\lambda \) of weight \( \lambda \), we have a \( U \)-linear homomorphism \( \Phi_\lambda : V(\lambda) \rightarrow \tilde{V}(\lambda) \) sending \( u_\lambda \) to \( \tilde{u}_\lambda \).

We know from [2, § 4.2] that the map \( \Phi_\lambda \) is injective, commutes with the Kashiwara operators \( e_i, f_i, i \in I_{af}, \) and induces an injective \( \mathbb{Q} \)-linear map \( \Phi_{\lambda|q=0} : \mathcal{L}(\lambda)/q_\delta \mathcal{L}(\lambda) \rightarrow \tilde{\mathcal{L}}(\lambda)/q_\delta \tilde{\mathcal{L}}(\lambda) \); note that \( \Phi_{\lambda|q=0}(B(\lambda)) \not\subset \tilde{B}(\lambda) \), in general (see Theorem 2.1 (2)).
Semi-infinite Bruhat order on affine Weyl groups.

For \( \lambda = \sum_{i \in I} m_i \omega_i \in P^+ \), with \( m_i \in \mathbb{Z}_{\geq 0}, \ i \in I, \) set

\[
(15) \quad \text{Par}(\lambda) = \left\{ \rho = (\rho^{(i)})_{i \in I} \bigg| \rho^{(i)} \text{ is a partition of length less than } m_i \text{ for each } i \in I \right\};
\]
we understand that a partition of length less than 1 is an empty partition \( \varnothing \). For a partition \( \rho = (\rho_1 \geq \rho_2 \geq \cdots \geq \rho_l > 0) \), set \( |\rho| = \sum_{i=1}^l \rho_i \). For \( \rho = (\rho^{(i)})_{i \in I} \in \text{Par}(\lambda) \), set

\[
(16) \quad \text{wt}(\rho) = -\sum_{i \in I} |\rho^{(i)}| \delta,
\]

where \( \{ \delta \} \) is a basis of \( \mathbb{C} \).

Let \( \rho = (\rho^{(i)})_{i \in I} \in \text{Par}(\lambda) \). Let \( S^-_\rho \) be the (PBW-type) basis element of weight \( \text{wt}(\rho) \) of the negative imaginary part of \( U \) constructed in [2, the element \( S^-_c \) in § 3.1; see also Remark 4.1]. Define

\[
(17) \quad s_\rho(z^{-1}) = \prod_{i \in I} s_{\rho^{(i)}}(z_{i,1}^{-1}, z_{i,2}^{-1}, \ldots, z_{i,m_i}^{-1}),
\]

where the right-hand side is a product of Schur polynomials in the variables \( z_{i,\nu}^{-1}, \ i \in I, \ 1 \leq \nu \leq m_i \).

**Theorem 2.1** ([2, § 4.2]; see also [14, § 13]). Let \( \lambda \in P^+ \).

1. We have

\[
(18) \quad B(\lambda) = \left\{ g_1 g_2 \cdots g_l S^-_\rho u_{\lambda} \mod q_\lambda \mathcal{L}(\lambda) \right\},
\]

where \( g_k \in \{ e_i, f_i \mid i \in I' \}, \ 1 \leq k \leq l, \ l \in \mathbb{Z}_{\geq 0}, \ \rho \in \text{Par}(\lambda) \setminus \{ 0 \} \).

2. We have

\[
(19) \quad \Phi_{\lambda \mid q=0} \left( g_1 g_2 \cdots g_l S^-_\rho u_{\lambda} \mod q_\lambda \mathcal{L}(\lambda) \right) = g_1 g_2 \cdots g_l s_\rho(z^{-1}) u_{\lambda} \mod q_\lambda \mathcal{L}(\lambda)
\]

for \( g_1 g_2 \cdots g_l S^-_\rho u_{\lambda} \mod q_\lambda \mathcal{L}(\lambda) \in B(\lambda) \). In particular,

\[
(20) \quad \Phi_{\lambda \mid q=0}(B(\lambda)) = \left\{ s_\rho(z^{-1}) b \mod q_\lambda \mathcal{L}(\lambda) \bigg| \rho \in \text{Par}(\lambda), \ b \in \mathcal{B}_0(\lambda) \right\},
\]

and the map \( \Phi_{\lambda \mid q=0} \) induces an isomorphism of \( U \)-crystals from \( \mathcal{B}_0(\lambda) \) to \( \mathcal{B}_0(\lambda) \).

3. Let \( \mathcal{B}_\rho(\lambda) \) be the connected component of \( B(\lambda) \) containing \( S^-_\rho u_{\lambda} \mod q_\lambda \mathcal{L}(\lambda) \). Then we have \( B(\lambda) = \bigcup_{\rho \in \text{Par}(\lambda)} \mathcal{B}_\rho(\lambda) \). Moreover, for each \( \rho \in \text{Par}(\lambda) \), there exists an isomorphism of \( U^- \)-crystals from \( \mathcal{B}_0(\lambda) \) to \( \mathcal{B}_\rho(\lambda) \) sending \( u_{\lambda} \mod q_\lambda \mathcal{L}(\lambda) \) to \( S^-_\rho u_{\lambda} \mod q_\lambda \mathcal{L}(\lambda) \).

2.4. Semi-infinite Bruhat order on affine Weyl groups. In this subsection, we recall some basic facts on the semi-infinite Bruhat order on affine Weyl groups (see [6, 18, 25] for more details).

We take and fix \( J \subset I \). Let \( W_J = \langle r_j \mid j \in J \rangle \), and let \( W^J \) be the set of minimal coset representatives for \( W/W_J \) (see [3, Corollary 2.4.5 (ii)]). For \( w \in W \), we denote
by $w \in W^J$ the minimal coset representative for the coset $wW_J \in W/W_J$. Define
\begin{equation}
(\Delta^+)_{af} = \{\alpha + n\delta \mid \alpha \in \Delta_J, n \in \mathbb{Z}\} \subset \Delta_{af},
\end{equation}
\begin{equation}
(\Delta^+)_{af} = (\Delta_J)_{af} \cap \Delta^+ = \Delta_J^+ \cup \{\alpha + n\delta \mid \alpha \in \Delta_J, n \in \mathbb{Z}_{>0}\},
\end{equation}
\begin{equation}
(W^J)_{af} = W_J \times \{t_\xi \mid \xi \in Q_J^+\} = (r_\beta \mid \beta \in (\Delta_J^+)_{af}),
\end{equation}
\begin{equation}
(W^J)_{af} = \{x \in W_{af} \mid x\beta \in \Delta^+_{af} \text{ for all } \beta \in (\Delta_J^+)_{af}\};
\end{equation}
\begin{equation}
\text{note that } (W^e)_{af} = \{e\} \text{ and } (W^o)_{af} = W_{af}.
\end{equation}

**Lemma 2.2** ([25]; see also [18, Lemma 10.5]). For each $x \in W_{af}$, there exist a unique $x_1 \in (W^J)_{af}$ and a unique $x_2 \in (W^J)_{af}$ such that $x = x_1x_2$. In particular, $(W^J)_{af}$ is a complete system of coset representatives for $W_{af}/(W^J)_{af}$.

Define a map $\Pi^J : W_{af} \to (W^J)_{af}$ by $\Pi^J(x) = x_1$ if $x = x_1x_2$ with $x_1 \in (W^J)_{af}$ and $x_2 \in (W^J)_{af}$.

**Lemma 2.3** ([25]; see also [18, Proposition 10.8]).

1. $\Pi^J(w) = |w|$ for $w \in W$.
2. $\Pi^J(x\xi) = \Pi^J(x)\Pi^J(\xi)$ for $x \in W_{af}$ and $\xi \in Q^\vee$.

For simplicity of notation, we let $T_\xi = T^J_\xi$ stand for $\Pi^J(t_\xi) \in (W^J)_{af}$ for $\xi \in Q^\vee$. The next lemma follows immediately from (23) and Lemmas 2.2–2.3.

**Lemma 2.4.**

1. $(W^J)_{af} = \{wT_\xi \mid w \in W^J, \xi \in Q^\vee\}$.
2. Let $\xi, \zeta \in Q^\vee$. If $\xi \equiv \zeta \mod Q^+_J$, then $T^J_\xi = T^J_\zeta$.

Set $\rho_J = (1/2) \sum_{\alpha \in \Delta^+_J} \alpha$; we abbreviate $\rho_J$ to $\rho$ if $J = I$. For $x = wt_\xi \in W_{af}$ with $w \in W$ and $\xi \in Q^\vee$, define
\begin{equation}
\ell(\overrightarrow{\chi}) = \ell(w) + 2(\xi, \rho).
\end{equation}

Define the (parabolic) semi-infinite Bruhat graph SiB$^J$ to be the $\Delta^+_{af}$-colored directed graph with vertex set $(W^J)_{af}$ and edges of the form $x \xrightarrow{\beta} r_\beta x$ for $x \in (W^J)_{af}$ and $\beta \in \Delta^+_{af}$, where $r_\beta x \in (W^J)_{af}$ and $\ell(\overrightarrow{x\beta}) = \ell(\overrightarrow{x}) + 1$.

The semi-infinite Bruhat order is a partial order $\preceq$ on $(W^J)_{af}$ defined as follows: for $x, y \in (W^J)_{af}$, we write $x \preceq y$ if there exists a directed path from $x$ to $y$ in SiB$^J$.

**Proposition 2.5** ([6, Proposition A.1.2]). Let $w \in W^J$, $\xi \in Q^\vee$ and $\beta \in \Delta_J^+$. Write $\beta = w\gamma + x\delta$ with $\gamma \in \Delta$ and $\chi \in \mathbb{Z}_{>0}$. Then $r_\beta wT_\xi \in (W^J)_{af}$ and there exists an edge $wT_\xi \xrightarrow{\beta} r_\beta wT_\xi$ in SiB$^J$ if and only if $\gamma \in \Delta^+ \setminus \Delta^+_J$ and one of the following conditions holds:

- (B) $\chi = 0$ and $\ell(ww_\gamma) = \ell(w) + 1$; in this case, we have $r_\beta wT_\xi = wT_\gamma T_\xi$ and $ww_\gamma \in W_J$.
- (Q) $\chi = 1$ and $\ell([ww_\gamma]) = \ell(w) + 1 - 2(\gamma, \rho - \rho_J)$; in this case, we have $r_\beta wT_\xi = [ww_\gamma]T_\xi T_\gamma$.

**Remark 2.6.**

1. If $wT_\xi \xrightarrow{\beta} \Pi^J(r_\beta wT_\xi)$ in SiB$^J$, then $r_\beta wT_\xi = \Pi^J(r_\beta wT_\xi) \in (W^J)_{af}$ (see [6, Appendix A]).
2. The condition (B) (resp. (Q)) for $w \in W^J$ and $\gamma \in \Delta^+ \setminus \Delta^+_J$ in Proposition 2.5 corresponds to the existence of the Bruhat edge (resp. quantum edge) $w \xrightarrow{\gamma} [ww_\gamma]$ in the (parabolic) quantum Bruhat graph for $W^J$ (see [19, §4]).
2.5. Semi-infinite Lakshmibai–Seshadri paths. In this subsection, we give a brief exposition of the \( \mathbf{U} \)-crystal of semi-infinite Lakshmibai–Seshadri paths; see [6] for more details.

Let \( \lambda \in P^+ \) and set \( J_\lambda = \{ j \in I \mid \langle h_j, \lambda \rangle = 0 \} \). For a rational number \( 0 < a \leq 1 \), define \( \text{SiB}(\lambda; a) \) to be the subgraph of \( \text{SiB}^J \) with the same vertex set but having only the edges of the form

\[
x \xrightarrow{\beta} y \quad \text{with} \quad a(\beta^\vee, x\lambda) \in \mathbb{Z};
\]

note that \( \text{SiB}(\lambda; 1) = \text{SiB}^J \). A semi-infinite Lakshmibai–Seshadri path of shape \( \lambda \) is, by definition, a pair \( (x; a) \) of a decreasing sequence \( x : x_1 \geq x_2 \geq \cdots \geq x_t \) of elements in \( (W^J_{\lambda})_{af} \) and an increasing sequence \( a : 0 = a_0 < a_1 < \cdots < a_t = 1 \) of rational numbers such that there exists a directed path from \( x_{u+1} \) to \( x_u \) in \( \text{SiB}(\lambda; a_u) \) for each \( u = 1, 2, \ldots, t - 1 \). Let \( \mathbb{B} \tilde{\mathcal{P}}(\lambda) \) denote the set of semi-infinite Lakshmibai–Seshadri paths of shape \( \lambda \).

Following [6, § 3.1], we equip the set \( \mathbb{B} \tilde{\mathcal{P}}(\lambda) \) with a \( \mathbf{U} \)-crystal structure. For \( \eta = (x_1, \ldots, x_t; a_0, a_1, \ldots, a_t) \in \mathbb{B} \tilde{\mathcal{P}}(\lambda) \), define the map \( \tilde{\eta} : [0, 1] \to \mathbb{R} \otimes \mathbb{Z} P_{af} \) by

\[
\tilde{\eta}(t) = \sum_{p=1}^{u-1} (a_p - a_{p-1})x_p\lambda + (t - a_{u-1})x_u\lambda \quad \text{for} \quad t \in [a_{u-1}, a_u), \quad 1 \leq u \leq l.
\]

Define \( \text{wt} : \mathbb{B} \tilde{\mathcal{P}}(\lambda) \to P_{af} \) by \( \text{wt}(\eta) = \tilde{\eta}(1) \in P_{af} \). Set

\[
h^\eta(t) = (h_i, \tilde{\eta}(t)) \quad \text{for} \quad t \in [0, 1], \quad m_{\lambda}^\eta = \min\{h_i^\eta(t) \mid t \in [0, 1]\}.
\]

We define \( e_i, \eta, i \in I_{af} \), as follows: if \( m_{\lambda}^\eta = 0 \), then we set \( e_i\eta = 0 \). If \( m_{\lambda}^\eta \leq -1 \), then we set

\[
\begin{align*}
t_0 &= \min\{t \in [0, 1] \mid h_i^\eta(t) = m_{\lambda}^\eta + 1\}, \\
t_1 &= \max\{t \in [0, 1] \mid h_i^\eta(t) = m_{\lambda}^\eta\}.
\end{align*}
\]

Let \( 1 \leq p \leq q \leq l \) be such that \( a_{p-1} \leq t_0 < a_p \) and \( t_1 = a_p \). Then we define

\[
e_i\eta = (x_1, \ldots, x_p, r_i x_p, \ldots, r_i x_q, x_{q+1}, \ldots, x_t);
\]

if \( t_0 = a_{p-1} \), then we drop \( x_p \) and \( a_{p-1} \), and if \( r_j x_q = x_{q+1} \), then we drop \( x_{q+1} \) and \( a_q = t_1 \).

Next, we define \( f_i, \eta, i \in I_{af} \), as follows: if \( m_{\lambda}^\eta = h_i^\eta(1) \), then we set \( f_i\eta = 0 \). If \( h_i^\eta(1) - m_{\lambda}^\eta \geq 1 \), then we set

\[
\begin{align*}
t_0 &= \max\{t \in [0, 1] \mid h_i^\eta(t) = m_{\lambda}^\eta\}, \\
t_1 &= \min\{t \in [0, 1] \mid h_i^\eta(t) = m_{\lambda}^\eta + 1\}.
\end{align*}
\]

Let \( 1 \leq p \leq q \leq l - 1 \) be such that \( t_0 = a_p \) and \( a_q \leq t_1 \leq a_{q+1} \). Then we define

\[
f_i\eta = (x_1, \ldots, x_p, r_i x_{p+1}, \ldots, r_i x_{q+1}, x_{q+2}, \ldots, x_t);
\]

if \( t_1 = a_{q+1} \), then we drop \( x_{q+1} \) and \( a_{q+1} \), and if \( x_p = r_i x_{p+1} \), then we drop \( x_{p+1} \) and \( a_p = t_0 \).

For \( \eta \in \mathbb{B} \tilde{\mathcal{P}}(\lambda) \) and \( i \in I_{af} \), define

\[
\begin{align*}
\varepsilon_i(\eta) &= -m_{\lambda}^\eta, \\
\varphi_i(\eta) &= h_i^\eta(1) - m_{\lambda}^\eta.
\end{align*}
\]

Now we assume that \( \lambda = \sum_{i \in I} m_i \omega_i \), with \( m_i \in \mathbb{Z}_{\geq 0} \), \( i \in I \). Set \( J_\lambda' = I \setminus J_\lambda = \{ i \in I \mid m_i > 0 \} \). Following [6, Equation (7.2.2)], we define an element
Proposition 37: Let \( \eta_\rho \in \mathbb{B}^\lambda(\lambda) \) of weight \( \lambda + \text{wt}(\rho) \) for each \( \rho = (\rho(i))_{i \in I} \in \text{Par}(\lambda) \), with \( \rho(i) = \rho_1(i) \geq \rho_2(i) \geq \cdots \geq \rho_{m_i - 1}(i) \) (see (15)). Let \( s \) be the least common multiple of \( \{m_i | i \in J^g_0 \} \). Let \( c_i(\xi) \in \mathbb{Z} \) denote the coefficient of \( \alpha_i^\vee \) in \( \xi \in Q^\vee \). For \( \xi, \zeta \in Q^\vee \), write \( \xi \succeq \zeta \) if \( \xi - \zeta \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^\vee \), and write \( \xi \succ \zeta \) if \( \xi \succeq \zeta \) and \( \xi \neq \zeta \). Let \( \zeta_1, \ldots, \zeta_s \in Q^\vee \) be such that

\[
\begin{align*}
(i) \quad & c_i(\zeta_i) = \rho_i(i) \quad \text{if} \quad i \in J^g_0 \quad \text{and} \quad \frac{s(m - 1)}{m_i} < t \leq \frac{s_m}{m_i} \quad \text{and} \\
(ii) \quad & c_j(\zeta_i) = 0 \quad \text{for all} \quad j \in J_\lambda \quad \text{and} \quad 1 \leq t \leq s;
\end{align*}
\]

note that \( \zeta_1 \succeq \cdots \succeq \zeta_s \) and \( \zeta_s = 0 \). Assume that

\[
\zeta_1 = \cdots = \zeta_{s_1} \succ \cdots \succ \zeta_{s_2} \succ \cdots \succ \zeta_{s_{k-1}} = \cdots = \zeta_{s_k},
\]

where \( 1 \leq s_1 < \cdots < s_{k-1} < s_k = s \). Set

\[
\eta_\rho = \left( T_{\zeta_1}, \ldots, T_{\zeta_{s_k-1}}, e; 0, \frac{s_1}{s}, \ldots, \frac{s_{k-1}}{s}, 1 \right).
\]

Theorem 2.7 ([6, Theorems 3.1.5 and 3.2.1]). Let \( \lambda \in P^+ \).

1. The set \( \mathbb{B}^\lambda(\lambda) \) equipped with the maps \( \text{wt}, e_i, f_i, i \in I_{af} \), and \( \varepsilon_i, \varphi_i, i \in I_{af} \), defined above, is a \( U \)-crystal.

2. There exists a unique isomorphism of \( U \)-crystals from \( B(\lambda) \) to \( \mathbb{B}^\lambda(\lambda) \) sending \( S_{\rho}u_\lambda \mod q_i L(\lambda) \) to \( \eta_\rho \) for every \( \rho \in \text{Par}(\lambda) \).

For \( \lambda_1, \ldots, \lambda_N \in P^+ \), let \( \mathbb{B}^\lambda(\lambda_1) \cdots \cdots \mathbb{B}^\lambda(\lambda_N) \) be the set of symbols \( \eta_1 \cdots \cdots \eta_N \), with \( \eta_i \in \mathbb{B}^\lambda(\lambda_i) \), \( 1 \leq \nu \leq N \). We define a \( U \)-crystal structure on \( \mathbb{B}^\lambda(\lambda_1) \cdots \cdots \mathbb{B}^\lambda(\lambda_N) \) in a way similar to the above. For \( \eta = \eta_1 \cdots \cdots \eta_N \in \mathbb{B}^\lambda(\lambda_1) \cdots \cdots \mathbb{B}^\lambda(\lambda_N) \), define \( \tilde{\eta} : [0, 1] \to \mathbb{R} \otimes \mathbb{Z} P_{af} \) by

\[
\tilde{\eta}(t) = \sum_{\nu=1}^{\mu-1} \tilde{\eta}_\nu(1) + \tilde{\eta}_\nu(Nt - \mu + 1) \quad \text{for} \quad \frac{\mu-1}{N} \leq t \leq \frac{\mu}{N}, \quad 1 \leq \mu \leq N,
\]

where each \( \tilde{\eta}_\nu : [0, 1] \to \mathbb{R} \otimes \mathbb{Z} P_{af}, 1 \leq \nu \leq N, \) is defined by (27). Define \( \text{wt}(\eta) = \tilde{\eta}(1) \).

By the same way as in (28), we define \( h_i^\nu(t) \) and \( m_i^\nu \) for \( \eta = \eta_1 \cdots \cdots \eta_N \) by using (36).

We define \( e_i \eta \) (resp. \( f_i \eta \)) \( \in \mathbb{B}^\lambda(\lambda_1) \cdots \cdots \mathbb{B}^\lambda(\lambda_N) \cup \{0\} \) as follows: if \( m_i^\nu = 0 \) (resp. \( m_i^\nu = h_i^\nu(1) \)), then we set \( e_i \eta = 0 \) (resp. \( f_i \eta = 0 \)). Assume that \( m_i^\nu \neq -1 \) (resp. \( h_i^\nu(1) - m_i^\nu \geq 1 \)), and let \( 0 \leq t_0 < t_1 \leq 1 \) be as in (29) (resp. (31)). We see that there exists \( 1 \leq \nu \leq N \) such that \( \frac{\nu-1}{N} \leq t_0 < t_1 \leq \frac{\nu}{N} \). Then we set \( e_i \eta = \eta_1 \cdots \cdots \eta_{i-1} * e_i \eta_{i} * \eta_{i+1} \cdots \cdots \eta_N \) (resp. \( f_i \eta = \eta_1 \cdots \cdots \eta_{i-1} * f_i \eta_{i} * \eta_{i+1} \cdots \cdots \eta_N \)). We define the functions \( \varepsilon_i, \varphi_i \) as in (33). The proof of the next proposition is straightforward.

Proposition 2.8. Let \( \lambda_1, \ldots, \lambda_N \in P^+ \). The map \( \mathbb{B}^\lambda(\lambda_1) \otimes \cdots \otimes \mathbb{B}^\lambda(\lambda_N) \to \mathbb{B}^\lambda(\lambda_1) \cdots \cdots \mathbb{B}^\lambda(\lambda_N), \eta_1 \otimes \cdots \otimes \eta_N \mapsto \eta_1 \cdots \cdots \eta_N, \) is an isomorphism of \( U \)-crystals.

3. Standard monomial theory for semi-infinite Lakshmibai–Seshadri paths

3.1. Strict embedding \( \Phi^U_{\lambda|q=0} \). Let \( \lambda = \sum_{i \in I} m_i \omega_i \in P^+ \), with \( m_i \in \mathbb{Z}_{\geq 0}, i \in I \).

Recall the automorphisms \( z_{i, \nu}, i \in I, 1 \leq \nu \leq m_i, \) of the \( U \)-crystal \( B(\lambda) \) (see § 2.3). For \( \rho = (\rho(i))_{i \in I} \in \text{Par}(\lambda) \), with \( \rho(i) = (\rho_1(i) \geq \rho_2(i) \geq \cdots \geq \rho_{m_i - 1}(i) \geq 0), i \in I \), define the automorphism \( z^{-\rho} \) of the \( U \)-crystal \( B(\lambda) \) by

\[
z^{-\rho} = \prod_{i \in I} z_{i, 1}^{-\rho_1(i)} z_{i, 2}^{-\rho_2(i)} \cdots z_{i, m_i-1}^{-\rho_{m_i-1}(i)}.
\]
Define the map \( \Phi_{\lambda|q=0}^{LT} : B(\lambda) \rightarrow \mathcal{B}(\lambda) \) by
\[
\tag{38} g_1g_2 \cdots g_l S_{\rho} u_{\lambda} \mod q_s L(\lambda) \hookrightarrow g_1g_2 \cdots g_l z^{-\rho} \hat{u}_{\lambda} \mod q_s \hat{L}(\lambda),
\]
where \( g_k \in \{e_i, f_i \mid i \in I_{at} \} \), \( 1 \leq k \leq l \), \( l \in \mathbb{Z}_{\geq 0} \), and \( \rho \in \text{Par}(\lambda) \) (cf. (19)). Set \( \mathcal{B}_\rho(\lambda) = z^{-\rho}\left( \mathcal{B}_0(\lambda) \right) \subset \mathcal{B}(\lambda) \); note that \( \mathcal{B}_\rho(\lambda) \) is a connected component of \( \mathcal{B}(\lambda) \), and is isomorphic to \( \mathcal{B}_0(\lambda) \) as a \( U' \)-crystal.

**Lemma 3.1.** The map \( \Phi_{\lambda|q=0}^{LT} \) is well-defined, and is a strict embedding of \( U \)-crystals.

**Proof.** It suffices to show that the map \( \Phi_{\lambda|q=0}^{LT} \) induces an isomorphism of \( U \)-crystals from \( B_\rho(\lambda) \) to \( \mathcal{B}_\rho(\lambda) \) for every \( \rho \in \text{Par}(\lambda) \). We know that the maps \( B_\rho(\lambda) \rightarrow B_0(\lambda) \) in Theorem 2.1 (3), \( \Phi_{\lambda|q=0} : B_0(\lambda) \rightarrow \mathcal{B}_0(\lambda) \) in Theorem 2.1 (2), and \( z^{-\rho} : B_0(\lambda) \rightarrow \mathcal{B}_\rho(\lambda) \) are isomorphisms of \( U' \)-crystals. We check at once that the composition of these maps is describe by (38), which proves that \( \Phi_{\lambda|q=0}^{LT} \) is well-defined and induces an isomorphism of \( U' \)-crystals from \( B_\rho(\lambda) \) to \( \mathcal{B}_\rho(\lambda) \). Since \( wt(S^- u_{\lambda}) = \lambda + wt(\rho) = wt(z^{-\rho} \hat{u}_{\lambda}) \), \( \Phi_{\lambda|q=0}^{LT} \) is a morphism of \( U \)-crystals. \( \square \)

**Remark 3.2.** If we think of the Schur polynomials as the generating functions of the weights of Young tableaux (see [4, Page 3]), then the term \( z^{-\rho} \) in \( s_\rho(z^{-1}) \) (see (17)) corresponds to the tuple of the Littlewood–Richardson tableaux of shapes \( \rho(i), i \in I \) (see [4, § 5.2]), and the coefficient of \( z^{-\rho} \) in \( s_\rho(z^{-1}) \) is 1.

**3.2. Characterization of the image of \( \Phi_{\lambda|q=0}^{LT} \).** In this subsection, we give a characterization of the image of the map \( \Phi_{\lambda|q=0}^{LT} \) in terms of semi-infinite Bruhat order via semi-infinite Lakshmibai–Seshadri paths.

Recall the notation \( J_\lambda = \{i \in I \mid \langle h_i, \lambda \rangle = 0 \} \), \( \lambda \in P^+ \).

**Definition 3.3.** Let \( \lambda_\nu \in P^+ \) and \( \eta^{(\nu)} = (x_1^{(\nu)}, \ldots, x_{l_\nu}^{(\nu)}; a^{(\nu)}) \in \mathbb{B} \tilde{\tau}(\lambda_\nu) \), \( 1 \leq \nu \leq N \). We say that there exists a defining chain for \( \bigotimes_{\nu=1}^N \eta^{(\nu)} \in \bigotimes_{\nu=1}^N \mathbb{B} \tilde{\tau}(\lambda_\nu) \) if there exists \( \bar{x}^{(\nu)} \in W_{at} \), \( 1 \leq s \leq l_\nu, 1 \leq \nu \leq N \), such that
\begin{align*}
(\text{DC1}) & \quad \Pi_{p}^{\nu} (\bar{x}^{(\nu)}) = x_s^{(\nu)} \text{ for all } 1 \leq s \leq l_\nu, 1 \leq \nu \leq N, \\
(\text{DC2}) & \quad \bar{x}_p^{(\nu)} \geq \bar{x}_q^{(\nu)} \text{ for all } 1 \leq p < q \leq l_\nu, 1 \leq \nu \leq N, \text{ and} \\
(\text{DC3}) & \quad \bar{x}_1^{(\nu)} \geq \bar{x}_1^{(\nu+1)} \text{ for all } 1 \leq \nu < N,
\end{align*}
where \( \geq \) denotes the semi-infinite Bruhat order on \( W_{at} \) defined by using SiB\(^R \) (see § 2.4). The tuple \( (\bar{x}_s^{(\nu)})_{1 \leq s \leq l_\nu, 1 \leq \nu \leq N} \) above is called a defining chain for \( \bigotimes_{\nu=1}^N \eta^{(\nu)} \).

Let \( \lambda = \sum_{i \in I} m_i \omega_i \in P^+ \). Recall the notation \( J_\lambda^c = I \setminus J_\lambda \). Set \( \mathbb{B} \tilde{\tau}(\lambda) = \bigotimes_{i \in J_\lambda^c} \mathbb{B} \tilde{\tau}(\omega_i)^{\otimes m_i} \). We know from Theorem 2.7 that there exists an isomorphism
\[
\Psi_\lambda : \mathcal{B}(\lambda) \rightarrow \mathbb{B} \tilde{\tau}(\lambda)
\]
of \( U \)-crystals defined as the tensor product of the isomorphisms \( B(\omega_i) \rightarrow \mathbb{B} \tilde{\tau}(\omega_i) \) sending \( u_{\omega_i} \) to \((e;0,1), i \in I \). Write
\[
\Psi_\lambda(b) = \bigotimes_{i \in J_\lambda^c} \Psi_\lambda^{(i)}(b) \in \mathbb{B} \tilde{\tau}(\lambda), \text{ where } \Psi_\lambda^{(i)}(b) \in \mathbb{B} \tilde{\tau}(\omega_i)^{\otimes m_i}, \text{ } i \in J_\lambda^c.
\]
Set
\[
\mathbb{B} \tilde{\tau}(\lambda) = \left\{ \eta \in \mathbb{B} \tilde{\tau}(\lambda) \left| \text{there exists a defining chain for } \eta \right. \right\}.
\]
Theorem 3.4. Let \( \lambda \in P^+ \). For \( b \in \mathcal{B}(\lambda) \), the following conditions are equivalent:

1. \( b \in \Phi_{\lambda\eta=0}^{|}(B(\lambda)) \).
2. \( \Psi_{\lambda}(b) \in \mathcal{S}^\circ \).
3. \( \Psi_{\lambda}^{(i)}(b) \in \mathcal{S}^\circ \)(\( m_i, \xi_i \)) for every \( i \in J^\circ \).

In particular, \( \mathcal{S}^\circ \)(\( \lambda \)) = \( \bigotimes_{i \in J^\circ} \mathcal{S}^\circ \)(\( m_i, \xi_i \)) (cf. [2, Remark 4.17]); see also [14, Conjecture 13.1 (iii)]. \( \mathcal{S}^\circ \)(\( \lambda \)) is an isomorphism of \( \mathcal{U} \)-crystals.

Remark 3.5.

1. Our argument in the proof of Theorem 3.4 in \S \S 3.3 does not imply [14, Conjecture 13.1 (iii)] since [2, Remark 4.17] is used in the proof of Theorem 2.7 (Theorems 3.1.5 and 3.2.1).

2. Similar result to Theorem 3.4 is obtained in [16, Theorem 3.1], where they proved that, for any \( \lambda, \mu \in P^+ \), the subset \( \mathcal{S}^\circ \)(\( \lambda + \mu \)) of elements in \( \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \) having a defining chain is stable under the Kashiwara operators, and is isomorphic to \( \mathcal{B}(\lambda + \mu) \) as a \( \mathcal{U} \)-crystal. But the proof is slightly different from ours. The main task in the proof of [16, Theorem 3.1] is to construct an isomorphism of \( \mathcal{U} \)-crystals between \( \mathcal{S}^\circ \)(\( \lambda + \mu \)) and \( \mathcal{B}(\lambda + \mu) \). This is achieved by giving an explicit parametrization of the connected components of \( \mathcal{S}^\circ \)(\( \lambda + \mu \)). In contrast, this argument starts from a specific choice of a map (see (38)), and aims to give an explicit description of the image of this map; in fact, there are infinitely many strict embeddings of \( \mathcal{U} \)-crystals from \( \mathcal{B}(\lambda) \) to \( \mathcal{S}(\lambda) \), in general.

3.3. Proof of Theorem 3.4. This subsection is devoted to the proof of Theorem 3.4.

We see from [14, Theorem 5.17] that, for each \( i \in I \), there exists a strict surjective morphism of \( \mathcal{U} \)-crystals from \( \mathcal{B}(\xi_i) \) to the crystal basis of a finite-dimensional \( \mathcal{U} \)-module \( W(\xi_i) \) (see \S 2.3). Hence, the next lemma follows from [1, Lemmas 1.5–1.6].

Lemma 3.6. Let \( i_1, \ldots, i_N \in I \).

1. Any connected component of \( \bigotimes_{\nu=1}^{N} \mathcal{B}(\xi_{i_{\nu}}) \) contains an extremal element.
2. If \( b = \bigotimes_{\nu=1}^{N} b_{\nu} \in \bigotimes_{\nu=1}^{N} \mathcal{B}(\xi_{i_{\nu}}) \) is an extremal element, then \( S_{\nu} b = \bigotimes_{\nu=1}^{N} S_{\nu} b_{\nu} \) for all \( x \in W_{\mathcal{af}} \).
3. \( \bigotimes_{\nu=1}^{N} b_{\nu} \in \bigotimes_{\nu=1}^{N} \mathcal{B}(\xi_{i_{\nu}}) \) is an extremal element if and only if there exist \( w \in W \) and \( \xi_1, \ldots, \xi_N \in Q^\nu \) such that \( b_{\nu} = S_{w_{\xi_{i_{\nu}}}, u_{\xi_{i_{\nu}}}} \) for \( 1 \leq \nu \leq N \).

By Theorem 2.7, the \( W_{\mathcal{af}} \)-action on \( \mathcal{B}(\lambda) \) (see (13)) induces a \( W_{\mathcal{af}} \)-action on \( \mathcal{B}_{\mathcal{BG}}^\circ (\lambda) \).

The next lemma follows from Theorem 2.7 (2) and Lemmas 2.3–2.4 and 3.6.

Lemma 3.7. Let \( i_1, \ldots, i_N \in I \).

1. Any connected component of \( \bigotimes_{\nu=1}^{N} \mathcal{B}_{\mathcal{BG}}^\circ (\xi_{i_{\nu}}) \) contains an extremal element.
2. If \( \eta = \bigotimes_{\nu=1}^{N} \eta_{\nu} \in \bigotimes_{\nu=1}^{N} \mathcal{B}_{\mathcal{BG}}^\circ (\xi_{i_{\nu}}) \) is an extremal element, then \( S_{\nu} \eta = \bigotimes_{\nu=1}^{N} S_{\nu} \eta_{\nu} \) for all \( x \in W_{\mathcal{af}} \).
3. \( \bigotimes_{\nu=1}^{N} \eta_{\nu} \in \bigotimes_{\nu=1}^{N} \mathcal{B}_{\mathcal{BG}}^\circ (\xi_{i_{\nu}}) \) is an extremal element if and only if there exist \( w \in W \) and \( \sigma_\nu \in Z, 1 \leq \nu \leq N, \) such that \( \eta_{\nu} = S_w \left( T_{\nu, \alpha_{i_{\nu}}}^{\lambda, \rho_{\nu}}(0, 1) \right) \in \mathcal{B}_{\mathcal{BG}}^\circ (\xi_{i_{\nu}}) \) for \( 1 \leq \nu \leq N \).

We also denote by \( z_i \) the automorphism, as a \( \mathcal{U} \)-crystal, of \( \mathcal{B}_{\mathcal{BG}}^\circ (\xi_{i_{\nu}}) \) corresponding to the automorphism \( z_i \) of \( \mathcal{B}(\xi_{i_{\nu}}) \). Recall that \( c_i(\xi) \) denotes the coefficient of \( \alpha_{i_{\nu}}^\lambda \) in \( \xi \in Q^\nu \).
Lemma 3.8. Let $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$. 

(1) For each $i \in I$, we have $z_i^k(e; 0, 1) = \left( T_{-\kappa \lambda_i}^{J - \{i\}}; 0, 1 \right)$ in $\mathbb{B}^+_{\varpi_i}$ for all $k \in \mathbb{Z}$.

In particular, for every $\rho = (\rho^{(i)})_{i \in I} \in \mathrm{Par}(\lambda)$, with $\rho^{(i)} = (\rho^{(i)}_1 \geq \cdots \geq \rho^{(i)}_{m_i - 1} \geq 0)$, $i \in I$, we have
\begin{equation}
\Psi_{\lambda} \left( z^p \bar{u}_\lambda \right) = \left( T^{J - \{i\}}_{\rho^{(i)}_1 \alpha_i}; 0, 1 \right) \otimes \cdots \otimes \left( T^{J - \{i\}}_{\rho^{(i)}_{m_i - 1} \alpha_i}; 0, 1 \right) \otimes (e; 0, 1)
\end{equation}
in $\mathbb{B}^+_{\varpi_i} \otimes m_i$ for each $i \in I$.

(2) Any connected component of $\mathbb{B}^+_{\varpi} \left( \lambda \right)$ contains an extremal element of the form
\begin{equation}
\bigotimes_{i \in J_x \nu = 1} \left( T^{J - \{i\}}_{\rho^{(i)}_1 \alpha_i \nu}; 0, 1 \right),
\end{equation}
where $\rho^{(i)}_\nu \in \mathbb{Z}$, $1 \leq \nu \leq m_i$, $i \in J_x$, and $\rho^{(i)}_0 = 0$ for all $i \in J_x$.

Proof. (1): Since $z_i^k(e; 0, 1)$ is an extremal element of weight $\varpi_i + k \delta$, there exists $x \in (W^{J - \{i\}})_\mathrm{af}$ such that $z_i^k(e; 0, 1) = S_x(e; 0, 1) = (x; 0, 1)$ by [14, Proposition 5.4(i)]. If we write $x = w T^{J - \{i\}}_\nu$, $w \in \mathcal{W}$, $\xi \in Q^\nu$, then $\mathrm{wt}(x; 0, 1) = w \varpi_i - (\xi, \varpi_i) \delta = w \varpi_i - c_i(\xi) \delta$, which implies that $w = e$ and $c_i(\xi) = -k$. By Lemma 2.4 (2), we have $T^{J - \{i\}}_\nu = T_{-\kappa \lambda_i}^{J - \{i\}}$, which proves that $z_i^k(e; 0, 1) = \left( T_{-\kappa \lambda_i}^{J - \{i\}} \right)$. 

(2): By Lemma 3.7, any connected component $C$ of $\mathbb{B}^+_{\varpi} \left( \lambda \right)$ contains an extremal element of the form $\eta = \bigotimes_{i \in J_x \nu = 1} \left( T^{J - \{i\}}_{\sigma^{(i)}_1 \alpha_i \nu}; 0, 1 \right)$, with $\sigma^{(i)}_\nu \in \mathbb{Z}$, $1 \leq \nu \leq m_i$, $i \in J_x$. Set $\rho^{(i)}_\nu = \sigma^{(i)}_\nu - \sigma^{(i)}_0$ and $\xi = -\sum_{i \in J_x} \sigma^{(i)}_0 \alpha_i^\nu$. By Lemmas 2.4 (2) and 3.7 (2),
\begin{equation}
C \supseteq S_{\bar{u}} \eta = \bigotimes_{i \in J_x \nu = 1} \left( T^{J - \{i\}}_{\sigma^{(i)}_1 \alpha_i \nu}; 0, 1 \right) = \bigotimes_{i \in J_x \nu = 1} \left( T^{J - \{i\}}_{\rho^{(i)}_1 \alpha_i \nu}; 0, 1 \right),
\end{equation}
which is the desired conclusion. \hfill \Box

For $J \subset I$, set $J^c = I \setminus J$. Let $[\cdot, \cdot] : Q^I = Q^J \oplus Q^{J^c} \to Q^J$ be the projection. Recall that we write $\xi \geq \zeta$ for $\xi, \zeta \in Q^J$ if $\xi - \zeta \in \sum_{i \in I} \mathbb{Z} \alpha_i^\nu$.

Lemma 3.9.

(1) For $\xi_1, \xi_2 \in Q^J$, $T^{J_1}_{\xi_1} \geq T^{J_2}_{\xi_2}$ in $(W^J)_\mathrm{af}$ if and only if $[\xi_1]_{J^c} \geq [\xi_2]_{J^c}$ in $Q^J$.

(2) For $x, y \in W_{\mathrm{af}}$ and $\xi \in Q^J$, $x \geq y$ in $W_{\mathrm{af}}$ if and only if $xt_\xi \geq yt_\xi$ in $W_{\mathrm{af}}$. In particular, we have $w t_\xi \geq t_\xi$ in $W_{\mathrm{af}}$ for all $w \in W$.

(3) For any $y \in W_{\mathrm{af}}$ and $\xi \in Q^J$, there exists $\vartheta \in Q^J$ such that $\vartheta \geq \xi$ in $Q^J$ and $t_\vartheta \geq y$ in $W_{\mathrm{af}}$.

(4) Let $J, K \subset I$ be such that $J \cap K^c = \emptyset$. Then, for any $x, y \in W_{\mathrm{af}}$, there exist $\vartheta_1 \in Q^J$ and $\vartheta_2 \in Q^K$ such that $\Pi^J(x t_{\vartheta_1}) = \Pi^K(x)$, $\Pi^K(y t_{\vartheta_2}) = \Pi^K(y)$ and $x t_{\vartheta_1} \geq y t_{\vartheta_2}$ in $W_{\mathrm{af}}$.

(5) Let $J \subset I$ and $x, y \in W_{\mathrm{af}}$. If $x \geq y$ in $W_{\mathrm{af}}$, then $\Pi^J(x) \geq \Pi^J(y)$ in $(W^J)_\mathrm{af}$.

Proof. (1): This is a special case of [6, Lemma 6.2.1].

(2): The assertion follows immediately from the formula $t_{(\xi)} = t_{(\xi)} + 2(\xi, \rho)$.

(3): Let $y = v t_{\xi}$, $v \in W$ and $\xi \in Q^J$, and let $v = r_1 r_2 \cdots r_i$, $i_1, i_2, \ldots, i_l \in I_{af}$, be a reduced expression. If we set $w_k = r_1 r_2 \cdots r_k$ and $\gamma_k = w_k \alpha_i$ for $k = 1, 2, \ldots, l$, then
\begin{equation}
y = w t_\xi \rightarrow_{\xi + \alpha_i}^\delta \left( w_{i-1} t_{\xi + \alpha_i} \rightarrow_{\xi + \alpha_i + \gamma_1}^\delta \cdots \rightarrow_{\xi + \alpha_i + \gamma_{i_l}}^\delta t_{\xi + \alpha_i + \gamma_1 + \cdots + \gamma_{i_l}} \right) \in \mathrm{SiB}^{\varpi},
\end{equation}
which proves that \( \vartheta' = \zeta + \alpha \gamma_1 + \alpha \gamma_2 + \cdots + \alpha \gamma_t \in Q^\vee \) satisfies \( t_{\vartheta'} \succeq y \). We see that 
\[
\vartheta = \sum_{i \in I} \max \{ c_i(1), c_i(2') \} \alpha_i^\vee \in Q^\vee \text{ satisfies } t_{\vartheta} \succeq y, \text{ and hence } t_{\vartheta'} \succeq t_{\vartheta} \text{ by (1)}.
\]

(4): Assume that \( x = wt_x \) with \( w \in W \) and \( \xi \in Q^\vee \). By (3), there exists \( \vartheta \in Q^\vee \) such that \( \vartheta \succeq \xi \) and \( t_{\vartheta} \succeq y \). If we set \( \vartheta_1 = [\vartheta - \xi]_J \) and \( \vartheta_2 = [\xi - \vartheta]_K \), then \( \Pi^J(xt_{\vartheta_1}) = \Pi^I(x) \) and \( \Pi^K(yt_{\vartheta_2}) = \Pi^K(y) \). Moreover, we have \( \xi + \vartheta_1 \succeq \vartheta + \vartheta_2 \), because \( \vartheta \succeq \xi \), \( \xi + \vartheta_1 = [\xi]_J + [\vartheta]_J + [\vartheta]_{J,K} \) since \( J = K^c \cup (J \cap K) \), and \( \vartheta + \vartheta_2 = [\xi]_K + [\vartheta]_{K^c} + [\vartheta]_{J,K^c} \). Then (1)–(2) shows that
\[
x t_{\vartheta_1} = wt_{\xi+\vartheta_1} \succeq t_{\xi+\vartheta_1} = t_{\vartheta_2} t_{\vartheta_2} \succeq yt_{\vartheta_2}.
\]

(5): By induction on \( \ell(\vartheta)(x) - \ell(\vartheta)(y) \), the assertion follows from [6, Lemma 6.1.1 for \( K = \emptyset \)].

\[\square\]

\textbf{Proof of Theorem 3.4.} We first prove that (2) and (3) are equivalent. Clearly, (2) implies (3). We prove that (3) implies (2). The proof is by induction on \( \# J_\lambda \). If \( \# J_\lambda = 1 \), then (2) and (3) are equivalent. Assume that \( \# J_\lambda > 1 \), \( b \in \check{B}(\lambda) \) satisfies (3), and \( \Psi_\lambda(b) = \Psi_\lambda(i, b) \otimes_{\check{J}_\lambda} \xi_{\lambda}(b) \). By (3), there exists a defining chain \( (x_1, \ldots, x_N) \) for \( \Psi_\lambda(i, b) \). By induction hypothesis, there exists a defining chain \( (y_1, \ldots, y_M) \) for \( \bigotimes_{\check{J}_\lambda} \xi_{\lambda}(b) \). Applying Lemma 3.9 (4) to \( x = x_N, y = y_1, J = I \setminus \{ i \} \), and \( K = J_\lambda \cup \{ i \} \) to obtain \( \vartheta_1 \in Q^\vee_{I \setminus \{ i \}} \) and \( \vartheta_2 \in Q^\vee_{J_\lambda \cup \{ i \}} \) such that \( x_N t_{\vartheta_1} \succeq y_1 t_{\vartheta_2} \). By Lemma 3.9 (2), we conclude that \( (x_1 t_{\vartheta_1}, \ldots, x_N t_{\vartheta_1}, y_1 t_{\vartheta_2}, \ldots, y_M t_{\vartheta_2}) \) is a defining chain for \( \Psi_\lambda(b) \).

We next prove that (1) and (2) are equivalent. The proof is completed by showing that

1. \( \check{S}(\lambda) \) is stable under the Kashiwara operators, and
2. each connected component of \( \check{S}(\lambda) \) contains \( \Psi_\lambda(z^{-\rho}u) \) for some \( \rho \in Par(\lambda) \),

because \( \Psi_\lambda \circ \varphi_{LT,\lambda|q=0} : B(\lambda) \to \check{S}(\lambda) \) is a strict embedding of \( U \)-crystals, and \( \Psi_\lambda(z^{-\rho}u) \in \check{S}(\lambda) \) for every \( \rho \in Par(\lambda) \); indeed, by Lemmas 3.8 (1) and 3.9 (1), we have a defining chain \( (t_{\rho^{(i)}(1)} \alpha_1^{-\gamma}, \ldots, t_{\rho^{(i)}(s)} \alpha_s^{-\gamma}) \) for \( \Psi_\lambda(i, \rho^{(i)}) \) for every \( \rho^{(i)} \in I \) in \( \varphi_{LT,\lambda|q=0} \), with \( \rho^{(i)} = (\rho^{(i)}_1 \geq \cdots \geq \rho^{(i)}_{m_i-1} > 0) \), \( i \in I \), and hence there exists a defining chain for \( \Psi_\lambda(z^{-\rho}u) \) by the implication (3) \( \Rightarrow \) (2).

We prove (i) only for the action of \( e_j, j \in I_{af} \); the proof for \( f_j \) is similar. Let \( \eta = \bigotimes_{i \in I_{\text{gr}}} \eta^{(i)} \in \check{S}(\lambda) \), with \( \eta^{(i)} \in \check{S}(m_i, \omega_i) \), \( i \in I_{\text{gr}} \). By tensor product rule and the implication (3) \( \Rightarrow \) (2), we only need to show that, for each \( i \in I_{af} \), if \( e_j \eta^{(i)} \neq 0 \), then \( e_j \eta^{(i)} \in \check{S}(m_i, \omega_i) \). Write \( \eta^{(i)} = \bigotimes_{\nu=1}^{m_i} \eta^{(i)}_\nu \), with \( \eta^{(i)}_\nu \in B(\lambda; \omega_i) \), \( 1 \leq \nu \leq m_i \), and let \( (x^{[\nu]}_{\nu=1} \in W(\lambda; \omega_i), 1 \leq s \leq l_\nu, 1 \leq \nu \leq m_i \) be a defining chain for \( \eta^{(i)} \); by Lemma 3.9 (5), we may assume that \( x^{[\nu]}_{\nu=1} \) is a defining chain for \( \eta^{(i)} \); by Lemma 3.9 (5), we may assume that \( x^{[\nu]}_{\nu=1} \in W(\lambda; \omega_i), 1 \leq s \leq l_\nu, 1 \leq \nu \leq m_i \). By tensor product rule, \( e_j \eta^{(i)} = \bigotimes_{\nu=1}^{m_i} \otimes e_j \eta^{(i)}_\nu \otimes \otimes \eta^{(i)}_\nu \) for some \( 1 \leq \nu \leq m_i \). Let \( 1 \leq p < q \leq l_\nu \) be as in (30) for \( \eta^{(i)}_\nu \). It follows from Proposition 2.8 and [6, Lemma 4.1.6] that the tuple
\[
\left( x^{[\nu]}_{[1]} \otimes \cdots \otimes x^{[\nu]}_{[1]} \right)_{\nu=1}^{m_i}
\]

is a defining chain for \( e_j \eta^{(i)} \), and hence \( e_j \eta^{(i)} \in \check{S}(m_i, \omega_i) \).

Finally, we prove (ii). Let \( C \) be an arbitrary connected component of \( \check{S}(\lambda) \); we see from (i) that \( C \) is a connected component of \( \check{S}(\lambda) \). By Lemma 3.8 (2), \( C \) contains an element of the form \( \eta = \bigotimes_{i \in I_{\text{e}}} \bigotimes_{\nu=1}^{m_i} \left( T_{\rho^{(i)}(\nu)}(\alpha^{\gamma}_i), 0, 1 \right) \), with \( \rho^{(i)} = 0, i \in I_{af} \). Since
there exists a defining chain for $\eta$, we see from Lemma 3.9 (1) that $\rho^{(i)} = (\rho_1^{(i)} \geq \rho_2^{(i)} \cdots \geq \rho_{m_i}^{(i)})$ is a partition of length less than $m_i$ for each $i \in I$; here, we set $\rho^{(i)} = \emptyset$ if $i \in J$. Hence $\rho = (\rho^{(i)})_{i \in I} \in \text{Par}(\lambda)$. We have $\Psi_\lambda(z^{-\rho} \tilde{\alpha}_\lambda) = \eta \in C$ by Lemma 3.8 (1), which proves (ii). \hfill \Box

4. Semi-infinite Young Tableaux

Throughout this section, we will make the following assumptions: $\mathfrak{g}_{af}$ is of type $A^{(1)}_{n-1}$ (see [9, § 4.8 TABLE Aff1]), and $I = \{1, 2, \ldots, n-1\}$ satisfies

\begin{equation}
(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i - j \equiv \pm 1 \mod n, \\ 0 & \text{otherwise} \end{cases}
\end{equation}

for all $i, j \in I_{af} = \{0\} \cup I$. In this case, $\alpha_i^\vee = \alpha_i$, $i \in I_{af}$, and hence $Q^\vee = Q$. We sometimes think of $W$ as the permutation group of $\{1, 2, \ldots, n\}$, namely the symmetric group of degree $n$, where $r_i$, $i \in I$, acts as the transposition $(i \ i+1)$. Observe that this action extends to the $W_{af}$-action, where $r_0$ acts as the transposition $(n \ 1)$; note that each $t_\xi$, $\xi \in Q$, acts as the identity.

4.1. Semi-infinite Young Tableaux and Isomorphism Theorem. We identify each element $\lambda = \sum_{i \in I} m_i \omega_i \in P^+$ with the Young diagram such that the number of the columns of length $i$ is $m_i$ for each $i \in I$. A column-strict tableau of shape $\lambda \in P^+$ with entries in $\{1, 2, \ldots, n\}$, by definition, an assignment of a number in $\{1, 2, \ldots, n\}$ to each box of the Young diagram $\lambda$ such that the entries are strictly increasing from top to bottom in each column. Let $\text{CST}(\lambda)$ be the set of column-strict tableaux of shape $\lambda$ with entries in $\{1, 2, \ldots, n\}$. For a tuple $(T_1, T_2, \ldots, T_M)$ of column-strict tableaux of one-column shapes, let $\prod_{\nu=1}^M T_\nu = T_1 T_2 \cdots T_M$ denote the column-strict tableau whose $\nu$-th column is $T_\nu$. For $T \in \text{CST}(\omega_i)$, let $T(s) \in \{1, 2, \ldots, n\}$, $1 \leq s \leq i$, denote the $s$-th entry (from top) of $T$.

Remark 4.1. In this paper, we consider a Young diagram as a collection of boxes, arranged in right-justified rows, with a weakly decreasing number of boxes in each row from top to bottom. For example, the Young diagram $\lambda = 5\omega_1 + 3\omega_2 + 4\omega_3 + 2\omega_4 + \omega_6$ is as follows:

\begin{equation}
(49)
\end{equation}

Definition 4.2.

1. Define the partial order $\preceq$ on $\text{CST}(\omega_i) \times Z$ as follows: for $(T, c), (T', c') \in \text{CST}(\omega_i) \times Z$, set $(T, c) \preceq (T', c')$ if $c \leq c'$ and $(T(u) \leq T'(u + c' - c))$ if $1 \leq u \leq i - c' + c$.

2. Let $\lambda = \sum_{i \in I} m_i \omega_i \in P^+$, $m_i \in \mathbb{Z}_{\geq 0}$, $i \in I$, and $N = \sum_{i \in I} m_i$. Let

\begin{equation}
(51)
\end{equation}

where $T_\nu^{(i)} \in \text{CST}(\omega_i)$ and $c_\nu^{(i)} \in \mathbb{Z}$ for $1 \leq \nu \leq m_i$, $i \in I$. We call $T$ a semi-infinite Young tableau of shape $\lambda$ if

\begin{equation}
(52)
\end{equation}

in $\text{CST}(\omega_i) \times Z$ for every $i \in I$. 

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Let $\mathcal{Y}(\lambda)$ be the set of semi-infinite Young tableaux of shape $\lambda$; note that $\mathcal{Y}(\lambda_i) = \text{CST}(\lambda_i) \times \mathbb{Z}$.

In § 4.3, we define a $U$-crystal structure on $\mathcal{Y}(\lambda_i)$, and prove the next proposition.

**Proposition 4.3.** Let $i \in I$.

1. There exists a unique isomorphism $\Upsilon_i : \mathcal{B}(\lambda_i) \to \mathcal{Y}(\lambda_i)$ of $U$-crystals.
2. We have $\mathcal{B}(\lambda_i) = \{ u_x := S_x u_{\lambda_i} \mid x \in (W^T \setminus \{i\})_{af} \}$, and the map $(W^T \setminus \{i\})_{af} \to \mathcal{B}(\lambda_i), \ x \mapsto u_x$, is bijective. In particular, $V(\lambda_i)$ is a minuscule representation of $U$.

**Remark 4.4.** It follows from Theorem 2.7 (2) and Proposition 4.3 that $\mathcal{B}(\lambda_i) = \{ \{x;0,1\} \mid x \in (W^T \setminus \{i\})_{af} \}$, and the map $\mathcal{B}(\lambda_i) \to \mathcal{B}(\lambda_i), \ u_x \mapsto (x;0,1), \ x \in (W^T \setminus \{i\})_{af}$, equals the isomorphism in Theorem 2.7 (2).

Let $\lambda = \sum_{i \in I} m_i \omega_i \in P^+$ and $N = \sum_{i \in I} m_i$. We have a bijection from $\bigotimes_{i=1}^{n-1} \mathcal{Y}(\lambda_i)^{m_i}$ to $\text{CST}(\lambda) \times \mathbb{Z}^N$ defined by

\[
\bigotimes_{i=1}^{n-1} \mathcal{Y}(\lambda_i)^{m_i} \rightarrow \prod_{i=1}^{n-1} \prod_{\nu=1}^{m_i} \left( T_i^{(0)}, c_i^{(0)} \right),
\]

where $(T_i^{(0)}, c_i^{(0)}) \in \mathcal{Y}(\lambda_i), \ 1 \leq \nu \leq m_i, \ i \in I$. Define a $U$-crystal structure on $\text{CST}(\lambda) \times \mathbb{Z}^N$ to be such that the map (53) is an isomorphism of $U$-crystals. From now on we assume that $\hat{B}(\lambda) = \bigotimes_{i=1}^{n-1} \mathcal{B}(\lambda_i)^{m_i} = \mathcal{B}(\lambda_1)^{m_1} \otimes \cdots \otimes \mathcal{B}(\lambda_{n-1})^{m_{n-1}}$.

**Theorem 4.5.** Let $\lambda = \sum_{i \in I} m_i \omega_i \in P^+$ and $N = \sum_{i \in I} m_i$. Then, $\mathcal{Y}(\lambda)$ equals the image of the composition of the maps

\[
\hat{B}(\lambda) \xrightarrow{\phi_{\lambda_1}^{m_1}} \mathcal{B}(\lambda) \xrightarrow{\otimes_{i=1}^{n-1} \mathcal{Y}(\lambda_i)^{m_i}} \mathcal{Y}(\lambda_i)^{m_i} \xrightarrow{(53)} \text{CST}(\lambda) \times \mathbb{Z}^N.
\]

Since the map (54) is a strict embedding of $U$-crystals, we have the following.

**Corollary 4.6.** Let $\lambda = \sum_{i \in I} m_i \omega_i \in P^+$ and $N = \sum_{i \in I} m_i$. Then, $\mathcal{Y}(\lambda)$ is stable under the Kashiwara operators on $\bigotimes_{i=1}^{n-1} \mathcal{Y}(\lambda_i)^{m_i} \cong \text{CST}(\lambda) \times \mathbb{Z}^N$, and is isomorphic, as a $U$-crystal, to the crystal basis $\hat{B}(\lambda)$.

**Theorem 4.7.** follows from Theorem 3.4, Definition 4.2, Remark 4.4, and the following table criterion for the semi-infinite Bruhat order.

**Theorem 4.7.** Let $i \in I$ and $x, y \in (W^T \setminus \{i\})_{af}$. Write $T_i(u_x) = (T, c)$ and $T_i(u_y) = (T', c')$. The following conditions are equivalent:

1. $x \leq y$ in $(W^T \setminus \{i\})_{af}$.
2. $c' \leq c$ and $T(u) \leq T(u + c' - c)$ if $1 \leq u \leq i - c' + c$.

Theorem 4.7 will be proved in § 4.4.

**4.2. Explicit description of $(W^T)_{af}$.** In this subsection, following [19, § 3], we give an explicit description of $(W^T)_{af}$ for later use.

We take and fix $J = \bigsqcup_{m=1}^{k} I_m \subset I$, where $I_1, I_2, \ldots, I_k$ are the sets of vertices of the connected components of the Dynkin diagram of $\Delta_j$; note that $\Delta_J = \bigsqcup_{m=1}^{k} \Delta_{m}$ and each $\Delta_m, 1 \leq m \leq k$, is of finite type $A$. Set $(I_m)_{af} = \{0\} \sqcup I_m \subset I_{af}, 1 \leq m \leq k$. For $1 \leq s \leq t \leq n - 1$, set $\alpha_{s,t} = \sum_{s=1}^{t} \alpha_i$; note that $\alpha_s = \alpha_{s, s}$. It follows that

\[
\Delta = \{ \pm \alpha_{s,t} \mid 1 \leq s \leq t \leq n - 1 \}.
\]
Set
\[ Q^J = \{ \xi \in Q \mid (\xi, \alpha) \in \{-1, 0\} \text{ for all } \alpha \in \Delta_J^+ \}. \]

Lemma 4.8 ([19, Equation (3.6)]). For each \( \xi \in Q \) there exist a unique \( \phi_J(\xi) \in Q_J \) and a unique \( (j_1, j_2, \ldots, j_k) \in \prod_{m=1}^k (I_m)_{af} \) such that
\[ \xi + \phi_J(\xi) + \sum_{m=1}^k \varepsilon_{j_m} \in \bigoplus_{i \in I^J} \mathbb{Z} \varpi_i \oplus \mathbb{C} \delta. \]

In particular, \( \xi + \phi_J(\xi) \in Q^J \) for any \( \xi \in Q \), and hence \( Q^J \) is a complete system of coset representatives for \( Q/Q_J \).

For a subset \( K \subset I \), let \( w_K^0 \) be the longest element of \( W_K \). For \( j_m \in (I_m)_{af} \), set
\[ v^I_{j_m} = w_0^{I_m} w_0^{I_m \setminus \{j_m\}} \in W_{I_m} \subset W_J; \]
\[ v^I_{j_m} = e. \]
\[ z_{j_m} = z_{j_m}^J = v^{I_m \setminus \{j_m\}} v^{I_m \setminus \{j_m\}} \ldots v^{I_m \setminus \{j_m\}} \in W_J, \]
where \( (j_1, j_2, \ldots, j_k) \in \prod_{m=1}^k (I_m)_{af} \), satisfying (57) for \( \xi \), is determined uniquely by Lemma 4.8; note that \( z_\xi = z_{j_m} \) if \( \xi \equiv j_m \mod Q_J \).

Lemma 4.9 ([19, Lemma 3.7]). We have \( T_\xi = \Pi^J(t_\xi) = z_{\xi \varepsilon_{j_1} + \phi_J(\xi)} \) for every \( \xi \in Q \). Therefore, by Lemma 2.3, \( \Pi^J(w_\xi) = [w][z_{\xi \varepsilon_{j_1} + \phi_J(\xi)}] \) for every \( w \in W \) and \( \xi \in Q \), and we have a bijection \( W^J \times Q^J \rightarrow (W^J)_{af}; (w, \xi) \mapsto wT_\xi. \) In particular,
\[ (W^J)_{af} = \{ wT_\xi = wz_{j_1} \mid w \in W_J, \xi \in Q^J \}. \]

4.3. Crystal structure on \( \mathbb{C}^{\infty}(\varpi_I) \). In this subsection, we define a \( U \)-crystal structure on \( \mathbb{C}^{\infty}(\varpi_I) \), and give a proof of Proposition 4.3.
We know from [3, Lemma 2.4.7] that
\[ W^{I \setminus \{i\}} = \{ w \in W \mid w(1) < w(2) < \cdots < w(i), \text{ and } w(i + 1) < w(i + 2) < \cdots < w(n) \}. \]

For \( w \in W^{I \setminus \{i\}} \), set
\[ T_w = \begin{bmatrix} w(1) \\ w(2) \\ \vdots \\ w(i) \end{bmatrix} \in \text{CST}(\varpi_I). \]

By (61), we have \( \text{CST}(\varpi_I) = \{ T_w \mid w \in W^{I \setminus \{i\}} \} \) and the map \( W^{I \setminus \{i\}} \rightarrow \text{CST}(\varpi_I) \), \( w \mapsto T_w \), is bijective. Let \( c_i(\xi) \) be the coefficient of \( \alpha_i (= \alpha_i^\vee) \) in \( \xi \in Q \). It follows from Lemma 4.8 that \( Q^{I \setminus \{i\}} = \{ c \alpha_i + \phi_{I \setminus \{i\}}(c \alpha_i) \mid c \in \mathbb{Z} \} \), and the maps \( \mathbb{Z} \rightarrow Q^{I \setminus \{i\}} \), \( c \mapsto c \alpha_i + \phi_{I \setminus \{i\}}(c \alpha_i) \), and \( Q^{I \setminus \{i\}} \rightarrow \mathbb{Z}, \xi \mapsto c_i(\xi) \), are inverses of each other. We have thus proved that the map
\[ \mathcal{Y}_i : (W^{I \setminus \{i\}})_{af} \rightarrow \mathbb{C}^{\infty}(\varpi_I), \quad wT_\xi \mapsto \mathcal{Y}_i(wT_\xi) = (T_w, c_i(\xi)), \]
is bijective, where \( w \in W^{I \setminus \{i\}} \) and \( \xi \in Q^{I \setminus \{i\}} \) (see Lemma 4.9).
Following [26, § 3.7] (see also [10, § 4.1]), we equip the set CST(ϖ_i) with a U’-crystal structure as follows: let T ∈ CST(ϖ_i). For k ∈ {1, 2, . . . , n}, write k ∈ T if T(s) = k for some 1 ≤ s ≤ i.

(i) Define wt(T_w) = wϖ_i (mod Cδ) for w ∈ W(ω_i^−(i)).

(ii) For j ∈ I, if T(s) = j + 1 and j ∈ T, then we define e_jT ∈ CST(ϖ_i) to be such that (e_jT)(s) = j and (e_jT)(u) = T(u) for 1 ≤ u < i, u ≠ s.

(iii) If 1 ∈ T and u /∈ T, then we define e_0T ∈ CST(ϖ_i) to be such that (e_0T)(i) = n and (e_0T)(u) = T(u + 1) for 1 ≤ u ≤ i − 1.

(iv) Otherwise, we set e_jT = 0 for j ∈ I_{af}.

(v) For j ∈ I, if T(s) = j and j + 1 /∈ T, then we define f_jT ∈ CST(ϖ_i) to be such that (f_jT)(s) = j + 1 and (f_jT)(u) = T(u) for 1 ≤ u < i, u ≠ s.

(vi) If j /∈ T and n ∈ T, then we define f_0T ∈ CST(ϖ_i) to be such that (f_0T)(1) = 1 and (f_0T)(u) = T(u − 1) for 2 ≤ u ≤ i.

(vii) Otherwise, we set f_jT = 0 for j ∈ I_{af}.

(viii) Define

\[ \varepsilon_j(T) = \begin{cases} 1 & \text{if } e_jT \neq 0, \\ 0 & \text{if } e_jT = 0, \end{cases} \quad \varphi_j(T) = \begin{cases} 1 & \text{if } f_jT \neq 0, \\ 0 & \text{if } f_jT = 0. \end{cases} \]

**Remark 4.10.** The U’-crystal CST(ϖ_i) defined above is isomorphic to the crystal basis of the U’-module W(ϖ_i) (see § 2.3). Indeed, we see from [14, Theorem 5.17 (ix)] (see also [24, Remark 3.3]) that W(ϖ_i) is isomorphic to a Kirillov–Reshetikhin module, whose crystal basis is a perfect crystal of level 1 in the sense of [10, Definition 1.1.1]. It follows that the U’-crystal CST(ϖ_i) and the crystal basis of W(ϖ_i) satisfy the conditions in [10, Proposition 1.2.1 for l = 1], and hence they must be isomorphic to each other.

The set Y_{U}^2(ϖ_i) = CST(ϖ_i) × Z can be identified with the affinization of the U’-crystal CST(ϖ_i) in the sense of [14, § 4.2]. We have thus obtained a U-crystal structure on Y_{U}^2(ϖ_i) as follows: for w ∈ W(ω_i^−(i)), c ∈ Z, T ∈ CST(ϖ_i), and j ∈ I_{af},

\[
\begin{align*}
\text{wt}(T_w, c) &= wϖ_i - cδ_i, \\
\varepsilon_j(T, c) &= (e_jT, c - δ_j, 0), \\
f_j(T, c) &= (f_jT, c + δ_j, 0), \\
\varphi_j(T, c) &= (φ_j(T), φ_j(T, c) = φ_j(T));
\end{align*}
\]

we understand that (0, c) = 0. By (10), (63), and (65), we have wt(Y_i(x)) = xϖ_i for all x ∈ (W(ω_i^−(i))_{af}.

**Proof of Proposition 4.3.** (1): Since Y_{U}^2(ϖ_i) is isomorphic to the affinization of the crystal basis of W(ϖ_i) (see Remark 4.10), we see from [14, Proposition 5.4 (ii) and Theorem 5.17 (vii)] that B(ϖ_i) is isomorphic to a U-crystal, to Y_{U}^2(ϖ_i). Note that (T_w, 0) ∈ Y_{U}^2(ϖ_i) (and hence u_{ϖ_i} ∈ B(ϖ_i)) is a unique element of weight ϖ_i (indeed, by (65), wt(T_w, c) = ϖ_i holds if and only if w = e and c = 0). This and the connectedness of B(ϖ_i) (see [14, Proposition 5.4 (ii)]) prove the uniqueness of the isomorphism between B(ϖ_i) and Y_{U}^2(ϖ_i).

(2): By (1) and (64)–(65), we see that \( ε_j(b), φ_j(b) ∈ \{0, 1\} \) for all \( j ∈ I_{af} \) and \( b ∈ B(ϖ_i) \). Hence \( ε_j(b) = S_a b \) (resp. \( f_j(b) = S_a b \)) if \( ε_j(b) ≠ 0 \) (resp. \( f_j(b) ≠ 0 \)) for \( j ∈ I_{af} \) and \( b ∈ B(ϖ_i) \). Since B(ϖ_i) is connected ([14, Proposition 5.4 (ii)]), this proves that the action of W_{af} is transitive. We have \( \{ x ∈ W_{af} \mid S_a u_ϖ_i = u_ϖ_i \} = (W(ω_i^−(i))_{af} \) (see (23)) by [6, Proposition 5.1.1], and hence \( B(ω_i) = \{ S_a u_ϖ_i \mid x ∈ (W(ω_i^−(i))_{af} \) by Lemma 2.2.

□
4.4. Tableau criterion for semi-infinite Bruhat order. This subsection is devoted to the proof of Theorem 4.7.

We take and fix $i \in I$. It is easily seen from (55) that

$$\Delta^+ \setminus \Delta^+_{I \setminus \{i\}} = \{\alpha_{s,t} \mid 1 \leq s \leq t \leq n-1\}. \tag{66}$$

We have

$$T_i(u_s) = Y_i(x) \text{ for all } x \in (W^{I \setminus \{i\}})_{\text{af}} \tag{67}$$

(see Proposition 4.3 (1) and (63)) because both elements are of weight $xw\iota_i$, and there is only one element of weight $xw\iota_i$ in $\mathbb{Y}^\bar{\iota}(w\iota_i)$.

**Proposition 4.11.** Let $w \in W^{I \setminus \{i\}}$, $\xi \in Q^{I \setminus \{i\}}$, $\beta = w\gamma + \chi\delta \in \Delta^+_w$, $\gamma = \alpha_{s,t} \in \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$, $1 \leq s \leq i \leq t \leq n-1$, and $\chi \in \mathbb{Z}_{\geq 0}$. Write $Y_i(wT\xi) = (T, c)$ and $Y_i(\Pi^{I \setminus \{i\}}(r_{\beta}wT\xi)) = (T', c')$. Then $r_{\beta}wT\xi \in (W^{I \setminus \{i\}})_{\text{af}}$ and there exists an edge $wT_{\xi} \xrightarrow{\beta} r_{\beta}wT_{\xi}$ in $\text{SiB}^{I \setminus \{i\}}$ if and only if one of the following conditions holds:

- (B) $c' = c$, $T'(s) = T(s) + 1$, and $T'(u) = T(u)$ for $1 \leq u \leq i$, $u \neq s$.
- (Q) $c' = c + 1$, $T'(1) = 1$, $T'(u) = T(u - 1)$ for $2 \leq u \leq i$, and $T(i) = n$.

**Remark 4.12.** Under the assumptions of Proposition 4.11, the following holds:

1. (B) is equivalent to $w(s) \in I$ and $(T', c') = f_{w(s)}(T, c)$ in $\mathbb{Y}^\bar{\iota}(w\iota_i)$; note that $T = T_w$ and $T' = T_{w\iota_i}$ in this case.
2. (Q) is equivalent to $(T', c') = f_0(T, c)$ in $\mathbb{Y}^\bar{\iota}(w\iota_i)$; note that $T = T_w$ and $T' = T_{[w\iota_i]}$ in this case.

Proposition 4.11 is established by combining Proposition 2.5 and Lemmas 4.13–4.15 below.

**Lemma 4.13.** Under the assumptions of Proposition 4.11, we have the following:

1. $\chi = 0$ is equivalent to $c' = c$.
2. $\chi = 1$ is equivalent to $c' = c + 1$.

**Proof.** It suffices to prove that $c' = c + \chi$. We have $r_{\beta}wT\xi = r_{w\gamma}t_{w\gamma}wz_t\xi = wr_\gamma z_t\xi$, and hence $\Pi^{I \setminus \{i\}}(r_{\beta}wT\xi) = [w_\gamma T\xi]_{z_t^{-1}\gamma}$ by Lemma 2.3. This gives $c' = c_\gamma(1 + \chi z_{t}^{-1}\gamma) = c_\gamma(1 + \chi z_{t}^{-1}\gamma) = c + \chi c_\gamma(z_t^{-1}\gamma)$. Since $z_t^{-1} \in W_{I \setminus \{i\}}$, it follows that $z_t^{-1}\gamma \in \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$. Therefore $c_\gamma(z_t^{-1}\gamma) = 1$ by (66). \hfill $\square$

**Lemma 4.14** ([3, Proposition 2.4.8]). Let $w \in W^{I \setminus \{i\}}$ and $\gamma = \alpha_{s,t} \in \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$, $1 \leq s \leq i \leq t \leq n-1$. The following conditions are equivalent:

1. $\ell(w_\gamma) = \ell(w) + 1$.
2. $w_\gamma \in W^{I \setminus \{i\}}$, $w_\gamma(s) = w(s) + 1$, and $w_\gamma(u) = w(u)$ for $1 \leq u \leq i$, $u \neq s$.

**Lemma 4.15.** Let $w \in W^{I \setminus \{i\}}$ and $\gamma \in \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$. The following conditions are equivalent:

1. $\ell([w_\gamma]) = \ell(w) + 1 - 2(\gamma, \rho_{I \setminus \{i\}})$.
2. $[w_\gamma](1) = 1$, $[w_\gamma](u) = w(u - 1)$ for $2 \leq u \leq i$, and $w(i) = n$.

For the proof of Lemma 4.15, we need the following lemma. Let $(i_1 i_2 \cdots i_l) \in W$ denote the cyclic permutation $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_l \mapsto i_1$, where $l \in \mathbb{Z}_{\geq 2}$ and $i_1, i_2, \ldots, i_l \in \{1, 2, \ldots, n\}$ are all distinct.
Lemma 4.16.

(1) $Q_{I_1} \cap \left( \Delta^+ \setminus \Delta^+_{I,\{i\}} \right) = \{ \alpha_i \}.$

(2) $2(\gamma, \rho - \rho_{I,\{i\}}) = n$ for $\gamma \in \Delta^+ \setminus \Delta^+_{I,\{i\}}.$

(3) $\ell(w) = (w(1) - 1)i + \sum_{u=2}^{i} (w(u) - w(u - 1) - 1)(i - u + 1)$ for $w \in W^{I,\{i\}}.$

(4) $\ell_{\alpha_i} = (1 2 \cdots i)(n n - 1 \cdots i + 1).$

Proof. (1): It is clear that $\alpha_i \in Q_{I_1} \cap \left( \Delta^+ \setminus \Delta^+_{I,\{i\}} \right)$, $1 \leq s \leq i \leq t \leq n - 1.$ If $s < i$ (resp. $i < t$), then $\alpha_s \in \Delta^+_{I,\{i\}}$ (resp. $\alpha_t \in \Delta^+_{I,\{i\}}$) and $\langle \gamma, \alpha_i \rangle = 1$ (resp. $\langle \gamma, \alpha_i \rangle = 1$). This proves that $\gamma \notin Q_{I_1} \setminus \{i\}$ unless $s = t = i.$

(2): The assertion follows from $2(\xi, \rho - \rho_{I,\{i\}}) = 0$ for $\xi \in Q_{I_1} \setminus \{i\},$ $\gamma \equiv \alpha_i \text{ mod } Q_{I_1} \setminus \{i\}$ for $\gamma \in \Delta^+ \setminus \Delta^+_{I,\{i\}},$ and $2(\alpha_s, \rho - \rho_{I,\{i\}}) = 2(\alpha_t, \rho) - 2(\alpha_t, \rho_{I,\{i\}}) = 2 + \#(I \setminus \{i\}) = n.$

(3): This is an immediate consequence of (1) and the fact that the length of a permutation equals the number of its inversions (see [3, Proposition 1.5.2]).

(4): Let $I_1 = \{1, \ldots, i - 1\}$ and $I_2 = \{i + 1, \ldots, n - 1\}$ be connected components of $I \setminus \{i\}.$ We see that $(i - 1, i + 1) \in (I_1)_{af} \times (I_2)_{af}$ satisfies the condition in Lemma 4.8 for $\alpha_i,$ because $\alpha_i \in Q_{I_1} \setminus \{i\}$ by (1), $(\alpha_i, \alpha_{i-1}) = -1$ if $i < i,$ and $(\alpha_i, \alpha_{i+1}) = -1$ if $i < n - 1.$ Therefore

\[
\ell_{\alpha_i} = \ell_{\alpha_i} z_{\alpha_i}^{(i)} = v_{i+1}^{(i)} v_{i+1}^{(i+1)} w_{i+1}^{(i+1)} w_{i+1}^{(i)} w_{i+1}^{(i-1)} w_{i-1}^{(i)} w_{i-1}^{(i+1)} w_{i+1}^{(i+1)}.
\]

Now the assertion is shown by the fact that the longest element of the symmetric group of degree $N$ is the permutation $j \mapsto N - j + 1,$ $j \in \{1, 2, \ldots, N\}.$

Proof of Lemma 4.15. We see from [18, Proof of Theorem 10.16] that (1) is equivalent to

(3) $\ell(wr_\gamma) = \ell(w) + 1 - 2(\gamma, \rho)$ and $wr_\gamma t_\gamma \in (W^{I,\{i\}})_{af}.$

It follows immediately from Lemmas 4.9 and 4.16 (1) that (3) is equivalent to

(4) $\gamma = \alpha_i,$ $\ell(w) = \ell(wr_\gamma) + 1$ and $wr_\gamma = \left[ wr_\gamma \right] z_{\alpha_i}^{(i)}.$

Let us prove that (1) (and (4)) imply (2). By (4) and Lemma 4.16 (4), we have

\[
\left[ wr_\gamma \right] = wr_\gamma z_{\alpha_i}^{(i)} - 1 = w(i + 1) (i + 2 \cdots n - 1 n - 1) n - 1).
\]

Hence $\left[ wr_\gamma \right](1) = w(i + 1).

We first assume that $i = 1.$ Then $\left[ wr_\gamma \right](1) = w(2).$ The condition $\ell(w) = \ell(wr_\gamma) + 1$ in (4) shows, by [3, Proposition 1.5.3], that $w(1) > w(2).$ Since $w \in W^{I,\{i\}},$ it follows from (61) that $w(2) = 1$ and, in consequence, $\left[ wr_\gamma \right](1) = 1.$ Since $\left[ wr_\gamma \right] \in W^{I,\{i\}},$ this implies that $\left[ wr_\gamma \right] = e$ and hence $w = (n n - 1 \cdots 2 1)$ by (69). This gives $w(1) = n.$

We next assume that $1 < i \leq n - 1.$ By (69), $\left[ wr_\gamma \right](u) = w(u - 1)$ for $2 \leq u \leq i.$ As $\left[ wr_\gamma \right] \in W^{I,\{i\}},$ we have $1 \leq \left[ wr_\gamma \right](1) < \left[ wr_\gamma \right](2) = w(1).$ Since $w \in W^{I,\{i\}},$ we see from (61) that $w(i + 1) = 1,$ and so $\left[ wr_\gamma \right](1) = 1.$ It follows from Lemma 4.16 (3) that

\[
\ell(\left[ wr_\gamma \right]) = (w(1) - 2)(i - 1) + \sum_{u=2}^{i-1} (w(u) - w(u - 1) - 1)(i - u - 1),
\]

and

\[
\ell(w) = (w(1) - 1)i + \sum_{u=2}^{i} (w(u) - w(u - 1) - 1)(i - u + 1),
\]

which gives $\ell(\left[ wr_\gamma \right]) - \ell(w) = 1 - w(i).$ By (1), (4) and Lemma 4.16 (2), we have $\ell(\left[ wr_\gamma \right]) - \ell(w) = 1 - 2(\alpha_i, \rho - \rho_{I,\{i\}}) = 1 - n,$ and consequently $w(i) = n.$
Finally, we prove that (2) implies (1). In a way similar to the above, we have \( \ell([w_{ij}]) - \ell(w) = 1 - w(i) = 1 - n \). Lemma 4.16 (2) now shows that (1) holds. \( \square \)

**Proof of Theorem 4.7.** If \( x \preceq y \), then \( c \preceq c' \) by Proposition 4.11. Therefore, we may assume that \( d := c' - c \geq 0 \). The proof is by induction on \( d \).

If \( d = 0 \), then it is obvious from Proposition 4.11 that \( x \preceq y \) is equivalent to \( T(u) \leq T'(u) \) for all \( 1 \leq u \leq i \).

Let \( d > 0 \). We first assume that \( T(u) \leq T(u + d) \) if \( 1 \leq u \leq i - d \), and show that \( x \preceq y \). Let \( x_1, x_2 \in (W^{I \setminus \{i\}})_{ad} \) be such that \( Y_i(x_1) = (T_1, c'), Y_i(x_2) = (T_2, c' - 1) \in \mathcal{Y}_i(w_i) \), where

\[
\begin{align*}
(1) \quad T_1(1) &= 1 \\
(72) \quad T_1(u) &= \begin{cases} 
T'(u) & \text{if } T'(u) < n - i + u, \\
T'(u) - 1 & \text{if } T'(u) = n - i + u 
\end{cases}
for 2 \leq u \leq i, \\
(2) \quad T_2(u) = T_1(u + 1) & \text{for } 1 \leq u \leq i - 1, \text{ and } T_2(i) = n.
\end{align*}
\]

By Proposition 4.11, we have \( x_2 < x_1 \leq y \). If we prove that

\[
(73) \quad T(u) \leq T_2(u + d - 1) \text{ if } 1 \leq u \leq i - (d - 1),
\]
then \( x \preceq x_2 \) by induction hypothesis, and hence \( x \preceq y \). Note that

\[
(74) \quad T_2(u + d - 1) = \begin{cases} 
T'(u + d) & \text{if } 1 \leq u + d - 1 \leq i - 1 \text{ and } T'(u + d) < n - i + u + d, \\
T'(u + d) - 1 & \text{if } 1 \leq u + d - 1 \leq i - 1 \text{ and } T'(u + d) = n - i + u + d, \\
n & \text{if } u + d - 1 = i.
\end{cases}
\]

We prove (73) as follows.

1. If \( 1 \leq u + d - 1 \leq i - 1 \) and \( T'(u + d) < n - i + u + d \), then \( T(u) \leq T'(u + d) = T_2(u + d - 1) \).
2. If \( 1 \leq u + d - 1 \leq i - 1 \) and \( T'(u + d) = n - i + u + d \), then \( T_2(u + d - 1) - T(u) = T'(u + d) - 1 - T(u) \geq (n - i + u + d) - 1 - (n - i + u + d - 1) - 1 = 0 \).
3. If \( u + d - 1 = i \), then \( T(u) \leq n = T_2(u + d - 1) \).

We next assume that \( x \preceq y \), and show that \( T(u) \leq T'(u + d) \) if \( 1 \leq u \leq i - d \). We see from Proposition 4.11 that there exist \( x_3, x_4 \in (W^{I \setminus \{i\}})_{ad} \) such that

1. \( x \preceq x_4 < x_3 \preceq y \),
2. \( Y_i(x_3) = (T_3, c'), Y_i(x_4) = (T_4, c' - 1) \in \mathcal{Y}_i(w_i) \),
3. \( T_3(u) \leq T'(u) \) for \( 1 \leq u \leq i \),
4. \( T_3(1) = 1, T_3(u + 1) = T_4(u) \) for \( 1 \leq u \leq i - 1 \), and \( T_4(i) = n \).

By induction hypothesis, \( T(u) \leq T_3(u + d - 1) \) if \( 1 \leq u \leq i - (d - 1) \). We have \( T'(u + d) - T(u) \geq T_4(u + d - 1) - T_4(u + d - 1) = 0 \) if \( 1 \leq u \leq i - d \). \( \square \)

### 4.5. Explicit description of crystal structure on \( \mathcal{Y}_i(w_i) \)

**Proposition 4.17.** Let \( \lambda = \sum_{i \in I} m_i w_i \), \( m_i \in \mathbb{Z}_{\geq 0}, \) \( i \in I \), and set \( N = \sum_{i \in I} m_i \). Let \( \mathcal{T} = (T_1T_2 \cdots T_N, (c_1, c_2, \ldots, c_N)) \in \text{CST}(\lambda) \times \mathbb{Z}^N \) and \( j \in I_{af} \). Then \( \text{wt}(\mathcal{T}) \in P_{af} \), \( e_j(\mathcal{T}), f_j(\mathcal{T}) \in \mathbb{Z} \), and \( e_j T, f_j T \in \text{CST}(\lambda) \times \mathbb{Z}^N \cup \{0\} \) are computed by the following procedure:

1. \( \text{wt}(\mathcal{T}) = \sum_{\nu=1}^{N} \text{wt}(T_\nu) - \sum_{\nu=1}^{N} c_\nu \delta \).
Let \( T \in \CST(\varpi_i) \). If \( j \in I \), then define \( e^{(j)}(T) \in \{\oplus, \ominus, \bullet\} \) by

\[
e^{(j)}(T) = \begin{cases} 
\ominus & \text{if } j \in T \text{ and } j + 1 \not\in T, \\
\oplus & \text{if } j \not\in T \text{ and } j + 1 \in T, \\
\bullet & \text{otherwise}.
\end{cases}
\]

Likewise, define \( e^{(0)}(T) \in \{\oplus, \ominus, \bullet\} \) by

\[
e^{(0)}(T) = \begin{cases} 
\ominus & \text{if } n \in T \text{ and } 1 \not\in T, \\
\oplus & \text{if } n \not\in T \text{ and } 1 \in T, \\
\bullet & \text{otherwise}.
\end{cases}
\]

(iii) In \( (e^{(j)}(T_1), \ldots, e^{(j)}(T_N)) \), continue replacing a pair \((e^{(j)}(T_\nu), e^{(j)}(T_{\nu'}))\) = \((\oplus, \ominus)\) with \((\bullet, \bullet)\) if \( \nu < \nu' \) and \( e^{(j)}(T_\mu) = \bullet \) for all \( \nu < \mu < \nu' \) untill no such pair exists. Let \( e^{(j)}(T) \in \{\oplus, \ominus, \bullet\}^N \) be the resulting tuple such that no \( \oplus \) placed to the left of \( \ominus \).

(iv) \( e_j(T) \) (resp. \( \varphi_j(T) \)) equals the number of \( \ominus \) (resp. \( \oplus \)) in \( e^{(j)}(T) \).

(v) If \( \ominus \) is not in \( e^{(j)}(T) \), then \( e_j T = 0 \). If there exists \( \ominus \) in \( e^{(j)}(T) \), and the right-most \( \ominus \) is at the \( \nu \)-th place, then

\[
e_j T = (T_1 \cdots T_{\nu-1} (e_j T_\nu) T_{\nu+1} \cdots T_N, (c_1, \ldots, c_{\nu-1}, c_\nu - \delta_j, 0, c_{\nu+1}, \ldots, c_N)).
\]

(vi) If \( \ominus \) is not in \( e^{(j)}(T) \), then \( f_j T = 0 \). If there exists \( \ominus \) in \( e^{(j)}(T) \), and the left-most \( \ominus \) is at the \( \nu \)-th place, then

\[
f_j T = (T_1 \cdots T_{\nu-1} (f_j T_\nu) T_{\nu+1} \cdots T_N, (c_1, \ldots, c_{\nu-1}, c_\nu + \delta_j, 0, c_{\nu+1}, \ldots, c_N)).
\]

Proof. Let \( T \in \CST(\varpi_i) \). We check at once that the following holds:

1. \( e^{(j)}(T) = \oplus \) if and only if \( e_j(T) = 0 \), \( \varphi_j(T) = 1 \), and \( \langle h_j, \text{wt}(T) \rangle = 1 \).
2. \( e^{(j)}(T) = \ominus \) if and only if \( e_j(T) = 1 \), \( \varphi_j(T) = 0 \), and \( \langle h_j, \text{wt}(T) \rangle = -1 \).
3. \( e^{(j)}(T) = \bullet \) if and only if \( e_j(T) = \varphi_j(T) = \langle h_j, \text{wt}(T) \rangle = 0 \).

Then the assertion follows by the same method as in [15, §2.1].

REFERENCES

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