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Abstract We introduce semi-infinite Young tableaux, and show that these tableaux give a combinatorial model for the crystal basis of a level-zero extremal weight module over the quantized universal enveloping algebra of untwisted affine type A. The definition and characterization of these tableaux are based on standard monomial theory for semi-infinite Lakshmibai–Seshadri paths and a tableau criterion for the semi-infinite Bruhat order on affine Weyl groups of type A, which are also proved in this paper.

1. Introduction

The aim of this paper is to introduce semi-infinite Young tableaux (see Definition 4.2 (2)). These tableaux give a new combinatorial model for the crystal basis of a level-zero extremal weight module (see § 2.3) over the quantized universal enveloping algebra of untwisted affine type A. In order to accomplish our purpose, we investigate

(i) a characterization of the image of the strict embedding $\Phi^{\mathrm{LT}}_{\lambda|q=0}$ (see § 3.1) of crystals in terms of the semi-infinite Bruhat order (see Theorem 3.4), and
(ii) a tableau criterion for the semi-infinite Bruhat order on affine Weyl groups of type A in Grassmannian cases (see Theorem 4.7).

Note that the image of $\Phi^{\mathrm{LT}}_{\lambda|q=0}$ is an isomorphic image of the crystal basis of the extremal weight module of extremal weight λ (of level-zero) into the tensor product of crystal bases of extremal weight modules associated with level-zero fundamental weights.

Various generalizations and variations of (semi-standard) Young tableaux are concerned in many areas such as algebraic combinatorics, representation theory, algebraic geometry, and so forth. In particular, Littelmann ([21, 22]) introduced the Lakshmibai–Seshadri paths for all symmetrizable Kac–Moody root data, which can be thought of as a type-free generalization of Young tableaux. Soon after, Joseph ([7]) and Kashiwara ([13]) independently proved that, for a dominant integral weight Λ of a symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$, the set of Lakshmibai–Seshadri paths of shape Λ equipped with Littelmann's root operators is isomorphic, as a $\mathfrak{g}$-crystal, to the crystal basis of the integrable (irreducible) highest weight module of highest weight Λ

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over the quantized universal enveloping algebra associated with $\mathfrak{g}$. In view of Kashiwara's crystal (basis) theory, further generalizations and variants of Littelmann's path model are investigated; e.g. generalized Lakshmibai–Seshadri paths for Borcherds–Kac–Moody root data ([5, 8]), quantum Lakshmibai–Seshadri paths and semi-infinite Lakshmibai–Seshadri paths for untwisted affine root data ([6, 20]). Among these general theories, it should be emphasized that the original Young tableaux have especially nice combinatorial structures (see for instance [4, 17]) due to the fact that every fundamental representation of finite type A is minuscule; namely, the Weyl group acts transitively on the crystal basis of any fundamental representation in the case of finite type A. Similarly, in the case of untwisted affine type A, every extremal weight module associated with a level-zero fundamental weight is minuscule (see Proposition 4.3 (2)); in this case, the affine Weyl group acts transitively on the crystal basis. Therefore, it is natural to try to find a tableau model for crystal bases of level-zero extremal weight modules in the case of untwisted affine type A.

Let us give an explanation of our strategy to introduce semi-infinite Young tableaux.

For this purpose, we first briefly sketch a standard monomial theoretic characterization of (ordinary) Young tableaux in terms of crystal basis theory as follows: let $U_q(\mathfrak{sl}_n(\mathbb{C}))$ be the quantized universal enveloping algebra of type A_{n-1} (see [9, § 4.8 TABLE Fin]). Let ϖ_i, $1 \leqslant i \leqslant n-1$, be the i-th fundamental weight; we identify a dominant integral weight $\lambda = \sum_{i=1}^{n-1} m_i \varpi_i$, $m_i \in \mathbb{Z}_{\geqslant 0}$, $1 \leqslant i \leqslant n-1$, with the Young diagram such that the number of the columns of length i is m_i for $1 \leqslant i \leqslant n-1$ (see Remark 4.1). For a dominant integral weight λ, let $L(\lambda)$ be the irreducible finite-dimensional highest weight $U_q(\mathfrak{sl}_n(\mathbb{C}))$-module of highest weight λ, and let $B(\lambda)$ be the crystal basis of $L(\lambda)$. It follows that $B(\varpi_i)$ is parametrized by the set $\mathrm{CST}(\varpi_i)$ of column-strict tableaux of shape ϖ_i with entries in $\{1, 2, \dots, n\}$ (see for instance [15, Proposition 3.3.1 (i)]). We have an injective homomorphism

$$L(\lambda) \longrightarrow \bigotimes_{i=1}^{n-1} L(\varpi_i)^{\otimes m_i} \tag{1}$$

of $U_q(\mathfrak{sl}_n(\mathbb{C}))$-modules sending a highest weight vector to the tensor product of highest weight vectors. Further, this homomorphism induces a strict embedding

$$B(\lambda) \longrightarrow \bigotimes_{i=1}^{n-1} B(\varpi_i)^{\otimes m_i} \cong \prod_{i=1}^{n-1} \mathrm{CST}(\varpi_i)^{m_i} \cong \mathrm{CST}(\lambda) \tag{2}$$

of $U_q(\mathfrak{sl}_n(\mathbb{C}))$-crystals, where $\mathrm{CST}(\lambda)$ denotes the set of column-strict tableaux of shape λ with entries in $\{1, 2, \dots, n\}$. Since the symmetric group of degree n acts transitively on $\mathrm{CST}(\varpi_i)$, each element in $\mathrm{CST}(\lambda)$ is labeled by a tuple of $N := \sum_{i=1}^{n-1} m_i$ cosets in the symmetric group; the symmetric group of degree n will be viewed as the Weyl group of type A_{n-1}. Let $\mathbb{T} \in \mathrm{CST}(\lambda)$, and assume that $\mathbb{T}$ corresponds to the tuple $(\bar{w}_1, \bar{w}_2, \dots, \bar{w}_N)$ of N cosets in the symmetric group. Then $\mathbb{T}$ is in the image of the strict embedding (2) if and only if there exist coset representatives $w_\nu \in \bar{w}_\nu$, $1 \leqslant \nu \leqslant N$, such that $w_1 \succeq w_2 \succeq \cdots \succeq w_N$ in the Bruhat order $\succeq$ on the symmetric group (see [23, Theorem 10.1]). In consequence, a column-strict tableau satisfying this condition is just a Young tableau, and vise versa ([3, Theorem 2.6.3 (Tableau Criterion)]; see also [15, Theorem 3.4.2 (i)]). This gives an isomorphism of $U_q(\mathfrak{sl}_n(\mathbb{C}))$-crystals between $B(\lambda)$ and the set of Young tableaux of shape λ. For an explicit description of the $U_q(\mathfrak{sl}_n(\mathbb{C}))$-crystal structure on the set of Young tableaux, see [15, Theorem 3.4.2 (ii)].

In the case of untwisted affine type A, our basic idea is to use the semi-infinite Bruhat order on the affine Weyl group in place of the Bruhat order on the symmetric group. Let $\mathbf{U}$ be the quantized universal enveloping algebra of (arbitrary) untwisted affine type, and let $I_{\mathrm{af}} = \{0\} \sqcup I$ be the index set for the simple roots. By abuse of notation, we use the same symbol ϖ_i, $i \in I$, for the i-th level-zero fundamental weight of $\mathbf{U}$. For $\lambda = \sum_{i \in I} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geqslant 0}$, $i \in I$, let $V(\lambda)$ be the extremal weight $\mathbf{U}$-module of extremal weight λ, and let $\mathcal{B}(\lambda)$ be the crystal basis of $V(\lambda)$ (see for instance [14, § 3.1]). Similarly to (1), we have a canonical homomorphism

$$\Phi_\lambda : V(\lambda) \longrightarrow \bigotimes_{i \in I} V(\varpi_i)^{\otimes m_i} \tag{3}$$

of $\mathbf{U}$-modules sending an extremal weight vector to the tensor product of extremal weight vectors. The main difficulty in carrying out the argument similar to the above is that the associated map $\Phi_{\lambda|q=0}$ at $q = 0$ of Φ_λ does not necessarily induce a morphism $\mathcal{B}(\lambda) \to \bigotimes_{i \in I} \mathcal{B}(\varpi_i)^{\otimes m_i}$ of $\mathbf{U}$-crystals. In fact, each $\Phi_{\lambda|q=0}(b)$, $b \in \mathcal{B}(\lambda)$, is a linear combination of crystal basis elements whose terms are in one-to-one correspondence with the terms of a product of some Schur polynomials (see [2, § 4.2]). To overcome this difficulty, by taking the "leading term" $\Phi^{\mathrm{LT}}_{\lambda|q=0}(b)$ of $\Phi_{\lambda|q=0}(b)$ (see Remark 3.2), we introduce a strict embedding

$$\Phi^{\mathrm{LT}}_{\lambda|q=0} : \mathcal{B}(\lambda) \longrightarrow \bigotimes_{i \in I} \mathcal{B}(\varpi_i)^{\otimes m_i} \tag{4}$$

of $\mathbf{U}$-crystals, which will be viewed as a counterpart of (2) in this paper (see (38) and Lemma 3.1). In the case that $\mathbf{U}$ is of untwisted affine type A, it follows that $\mathcal{B}(\varpi_i)$ is parametrized by $\mathrm{CST}(\varpi_i) \times \mathbb{Z}$, and the affine Weyl group acts transitively on $\mathrm{CST}(\varpi_i) \times \mathbb{Z}$ (see Proposition 4.3). Consequently, $\bigotimes_{i \in I} \mathcal{B}(\varpi_i)^{\otimes m_i}$ is parametrized by $\mathrm{CST}(\lambda) \times \mathbb{Z}^N$, where $N := \sum_{i \in I} m_i$, and each element in $\mathrm{CST}(\lambda) \times \mathbb{Z}^N$ is labeled by a tuple of N cosets in the affine Weyl group. Let $\mathbb{T} \in \mathrm{CST}(\lambda) \times \mathbb{Z}^N$, and assume that $\mathbb{T}$ corresponds to the tuple $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N)$ of cosets in the affine Weyl group. Then $\mathbb{T}$ is in the image of the strict embedding (4) if and only if there exist coset representatives $x_\nu \in \bar{x}_\nu$, $1 \leqslant \nu \leqslant N$, such that $x_1 \succeq x_2 \succeq \cdots \succeq x_N$ in the semi-infinite Bruhat order $\succeq$ on the affine Weyl group (see Theorem 3.4). Such decreasing sequences are explicitly described in terms of tableaux (see Definition 4.2 (1) and Theorem 4.7). We are thus led to the definition of semi-infinite Young tableaux (see Definition 4.2 (2)).

Let us give an example of a semi-infinite Young tableau (by using the notation in § 4.1). Let $n = 7$ and

$$\lambda = \varpi_1 + 3\varpi_2 + 2\varpi_4 = \begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ \\ \hline \end{array}\!\!\begin{array}{r} \\ \begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\ \hline \end{array} \\ \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \end{array}\end{array} . \tag{5}$$

We claim that

$$\mathbb{T} = \left(\begin{array}{rrrrrr} \boxed{5} & \boxed{2} & \boxed{3} & \boxed{1} & \boxed{1} & \boxed{1} \\ & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{3} & \boxed{4} \\ & & & & \boxed{4} & \boxed{5} \\ & & & & \boxed{6} & \boxed{7} \end{array},\ (-3; 5, 4, -1; 8, 7) \right) \in \mathrm{CST}(\lambda) \times \mathbb{Z}^6 \tag{6}$$

is a semi-infinite Young tableau of shape λ. Indeed, $\mathbb{T}$ is a semi-infinite Young tableau if (and only if) its rectangle components

$$\left(\boxed{5},\ -3\right), \qquad \left(\begin{array}{|c|c|c|}\hline 2 & 3 & 1 \\ \hline 3 & 4 & 5 \\ \hline \end{array},\ (5, 4, -1) \right), \qquad \left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 4 \\ \hline 4 & 5 \\ \hline 6 & 7 \\ \hline \end{array},\ (8, 7) \right) \tag{7}$$

are semi-infinite Young tableaux. Also, a tableau

$$(\mathsf{T}_1 \mathsf{T}_2 \cdots \mathsf{T}_m, (c_1, c_2, \ldots, c_m)) \in \mathrm{CST}(m\varpi_i) \times \mathbb{Z}^m \tag{8}$$

of rectangle shape is a semi-infinite Young tableau if (and only if)

(i) $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_m$, and

(ii) $\big(\mathsf{T}_\nu(u + c_\nu - c_{\nu+1}) \geqslant \mathsf{T}_{\nu+1}(u)$ if $1 \leqslant u \leqslant i - c_\nu + c_{\nu+1}\big)$ for every $1 \leqslant \nu < m$,

where $\mathsf{T}_\nu(s)$ denotes the s-th entry (from top) of the ν-th column T_ν.

This paper is organized as follows. In § 2, we set up notation and terminology on untwisted affine root data and crystals. Also, we have compiled some basic facts on extremal weight modules over quantized universal enveloping algebras of untwisted affine types, the semi-infinite Bruhat order on affine Weyl groups, and semi-infinite Lakshmibai–Seshadri paths. In § 3, we introduce a strict embedding $\Phi^{\mathrm{LT}}_{\lambda|q=0}$ of crystals. Then we state and prove a characterization of the image of $\Phi^{\mathrm{LT}}_{\lambda|q=0}$, which can be thought of as standard monomial theory for semi-infinite Lakshmibai–Seshadri paths (see Theorem 3.4). In § 4, we will restrict our attention to the case of untwisted affine type A. We introduce semi-infinite Young tableaux. We prove that the set $\mathbb{Y}^{\frac{\infty}{2}}(\lambda)$ of semi-infinite Young tableaux of shape λ equals the image of $\Phi^{\mathrm{LT}}_{\lambda|q=0}$ (see Theorem 4.5), by showing a tableau criterion for the semi-infinite Bruhat order (see Theorem 4.7). Consequently, this proves that $\mathbb{Y}^{\frac{\infty}{2}}(\lambda)$ is isomorphic, as a $\mathbf{U}$-crystal, to the crystal basis $\mathcal{B}(\lambda)$ (see Corollary 4.6). We give an explicit description of the crystal structure on $\mathbb{Y}^{\frac{\infty}{2}}(\lambda)$ (see Proposition 4.17).

2. Preliminaries

2.1. Untwisted affine root data. Let $\mathfrak{g}_{\mathrm{af}}$ be an untwisted affine Lie algebra over $\mathbb{C}$ with a Cartan subalgebra $\mathfrak{h}_{\mathrm{af}}$. Let $\{\alpha_i\}_{i\in I_{\mathrm{af}}} \subset \mathfrak{h}^*_{\mathrm{af}} = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathrm{af}}, \mathbb{C})$ and $\{h_i\}_{i\in I_{\mathrm{af}}} \subset \mathfrak{h}_{\mathrm{af}}$ be the sets of simple roots and simple coroots, respectively. Here I_{af} denotes the vertex set of the (affine) Dynkin diagram of $\mathfrak{g}_{\mathrm{af}}$. Let $\langle\cdot,\cdot\rangle : \mathfrak{h}_{\mathrm{af}} \times \mathfrak{h}^*_{\mathrm{af}} \to \mathbb{C}$ be the canonical pairing. We take and fix an integral weight lattice $P_{\mathrm{af}} \subset \mathfrak{h}^*_{\mathrm{af}}$ satisfying the conditions that $\alpha_i \in P_{\mathrm{af}}$ and $h_i \in \mathrm{Hom}_{\mathbb{Z}}(P_{\mathrm{af}}, \mathbb{Z})$ for all $i \in I_{\mathrm{af}}$, and for each $i \in I_{\mathrm{af}}$ there exists $\Lambda_i \in P_{\mathrm{af}}$ such that $\langle h_j, \Lambda_i\rangle = \delta_{ij}$ for all $j \in I_{\mathrm{af}}$. Let $\delta = \sum_{i\in I_{\mathrm{af}}} a_i\alpha_i \in \mathfrak{h}^*_{\mathrm{af}}$ and $c = \sum_{i\in I_{\mathrm{af}}} a_i^\vee h_i \in \mathfrak{h}_{\mathrm{af}}$ be the null root and the canonical central element, respectively. We take and fix $0 \in I_{\mathrm{af}}$ such that $a_0 = a_0^\vee = 1$. Set $I = I_{\mathrm{af}} \setminus \{0\}$; note that the subset I of I_{af} corresponds to the vertex set of the Dynkin diagram of a complex finite-dimensional simple Lie subalgebra $\mathfrak{g}$ of $\mathfrak{g}_{\mathrm{af}}$. For each $i \in I_{\mathrm{af}}$, define $\varpi_i = \Lambda_i - \langle c, \Lambda_i\rangle\Lambda_0$ and call it the i-th level-zero fundamental weight; note that $\varpi_0 = 0$, $\langle c, \varpi_i\rangle = 0$ for all $i \in I_{\mathrm{af}}$, and $\langle h_i, \varpi_j\rangle = \delta_{ij}$ for all $i, j \in I$.

Let $W_{\mathrm{af}} = \langle r_i \mid i \in I_{\mathrm{af}}\rangle$ be the (affine) Weyl group of $\mathfrak{g}_{\mathrm{af}}$, where r_i denotes the simple reflection with respect to α_i. The subgroup $W = \langle r_i \mid i \in I\rangle \subset W_{\mathrm{af}}$ is the (finite) Weyl group of $\mathfrak{g}$. Let $\ell : W_{\mathrm{af}} \to \mathbb{Z}_{\geqslant 0}$ be the length function. Let $e \in W_{\mathrm{af}}$ be the unit element. Let $(\cdot,\cdot)$ be a W_{af}-invariant non-degenerate symmetric bilinear form on $\mathfrak{h}^*_{\mathrm{af}}$ such that $(\delta, \lambda) = \langle c, \lambda\rangle$ for all $\lambda \in \mathfrak{h}^*_{\mathrm{af}}$. Set $\alpha_i^\vee = \dfrac{2\alpha_i}{(\alpha_i, \alpha_i)} \in \mathfrak{h}^*_{\mathrm{af}}$ for $i \in I_{\mathrm{af}}$. The action of W_{af} on $\mathfrak{h}^*_{\mathrm{af}}$ is given by $r_i(\lambda) = \lambda - (\alpha_i^\vee, \lambda)\alpha_i = \lambda - \langle h_i, \lambda\rangle\alpha_i$ for $i \in I_{\mathrm{af}}$ and $\lambda \in \mathfrak{h}^*_{\mathrm{af}}$. Set

$$Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i, \qquad Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee, \qquad P^+ = \sum_{i\in I} \mathbb{Z}_{\geqslant 0}\varpi_i. \tag{9}$$

We know from [9, § 6.5] that $Q^\vee \subset Q$. For $\xi \in Q^\vee$, we denote by $t_\xi \in W_{\mathrm{af}}$ the translation by ξ (see [9, §6.5]). We know from [9, Proposition 6.5] that $\{t_\xi \mid \xi \in Q^\vee\}$ forms an abelian normal subgroup of W_{af}, for which $t_\xi t_\zeta = t_{\xi+\zeta}$, $\xi, \zeta \in Q^\vee$, and $W_{\mathrm{af}} = W \ltimes \{t_\xi \mid \xi \in Q^\vee\}$. For $w \in W$ and $\xi \in Q^\vee$, we have

$$wt_\xi\lambda = w\lambda - (\xi, \lambda)\delta \quad \text{if } \lambda \in \mathfrak{h}^*_{\mathrm{af}} \text{ satisfies } \langle c, \lambda\rangle = 0. \tag{10}$$

Let Δ be the root system of $\mathfrak{g}$. Set $\Delta^+ = \Delta \cap \sum_{i\in I} \mathbb{Z}_{\geqslant 0}\alpha_i$. For a subset $J \subset I$, set

$$(11) \qquad Q_J = \bigoplus_{j\in J} \mathbb{Z}\alpha_j, \qquad Q_J^\vee = \bigoplus_{j\in J} \mathbb{Z}\alpha_j^\vee, \qquad \Delta_J = \Delta \cap Q_J, \qquad \Delta_J^+ = \Delta^+ \cap Q_J.$$

Denote by Δ_{af} the set of real roots of $\mathfrak{g}_{\mathrm{af}}$, and by Δ_{af}^+ the set of positive real roots of $\mathfrak{g}_{\mathrm{af}}$; we know from [9, Proposition 6.3] that

$$(12) \qquad \Delta_{\mathrm{af}} = \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\}, \qquad \Delta_{\mathrm{af}}^+ = \Delta^+ \sqcup \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}_{>0}\}.$$

For $\beta \in \Delta_{\mathrm{af}}$, let $\beta^\vee = \dfrac{2\beta}{(\beta,\beta)} \in \mathfrak{h}_{\mathrm{af}}^*$, and let $r_\beta \in W_{\mathrm{af}}$ be the reflection with respect to β; if $\beta = \alpha + n\delta$, $\alpha \in \Delta$ and $n \in \mathbb{Z}$, then $r_\beta = r_\alpha t_{n\alpha^\vee}$.

Let d be the smallest positive integer such that $(\alpha_i, \alpha_i)/2 \in (1/d)\mathbb{Z}$ for all $i \in I_{\mathrm{af}}$. Let q be an indeterminate, and set $q_s = q^{1/d}$. Let $\mathbf{U}$ be the quantized universal enveloping algebra over $\mathbb{Q}(q_s)$ associated with $\mathfrak{g}_{\mathrm{af}}$. Let $\mathbf{U}'$ be the $\mathbb{Q}(q_s)$-subalgebra of $\mathbf{U}$ corresponding to the derived subalgebra $[\mathfrak{g}_{\mathrm{af}}, \mathfrak{g}_{\mathrm{af}}]$ of $\mathfrak{g}_{\mathrm{af}}$ (see for instance [2, § 2.2]).

2.2. Crystals. In this subsection, we set up notation and terminology on crystals. For a fuller treatment, we refer the reader to [1, 11, 12, 14].

A set $\mathcal{B}$ together with the maps $\mathrm{wt} : \mathcal{B} \to P_{\mathrm{af}}$ (resp. $\mathrm{wt} : \mathcal{B} \to P_{\mathrm{af}}/(P_{\mathrm{af}} \cap \mathbb{C}\delta)$), $e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{\mathbf{0}\}$, $i \in I_{\mathrm{af}}$, and $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \sqcup \{-\infty\}$ is called a $\mathbf{U}$-crystal (resp. $\mathbf{U}'$-crystal) if the following conditions are satisfied:

- (C1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$ for all $i \in I_{\mathrm{af}}$,
- (C2) $\mathrm{wt}(e_i b) = \mathrm{wt}(b) + \alpha_i$ if $e_i b \in \mathcal{B}$,
- (C3) $\mathrm{wt}(f_i b) = \mathrm{wt}(b) - \alpha_i$ if $f_i b \in \mathcal{B}$,
- (C4) $\varepsilon_i(e_i b) = \varepsilon_i(b) - 1$ and $\varphi_i(e_i b) = \varphi_i(b) + 1$ if $e_i b \in \mathcal{B}$,
- (C5) $\varepsilon_i(f_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(f_i b) = \varphi_i(b) - 1$ if $f_i b \in \mathcal{B}$,
- (C6) $f_i b = b'$ if and only if $b = e_i b'$ for $b, b' \in \mathcal{B}$ and $i \in I_{\mathrm{af}}$,
- (C7) if $\varphi_i(b) = -\infty$, then $e_i b = f_i b = \mathbf{0}$.

The maps $e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{\mathbf{0}\}$, $i \in I_{\mathrm{af}}$, are called the Kashiwara operators. For a subset $\mathcal{B}'$ of a crystal $\mathcal{B}$, we say that $\mathcal{B}'$ is stable under the Kashiwara operators if $e_i\mathcal{B}', f_i\mathcal{B}' \subset \mathcal{B}' \sqcup \{\mathbf{0}\}$ for all $i \in I_{\mathrm{af}}$.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be $\mathbf{U}$-crystals or $\mathbf{U}'$-crystals. A morphism $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ is, by definition, a map $\mathcal{B}_1 \sqcup \{\mathbf{0}\} \to \mathcal{B}_2 \sqcup \{\mathbf{0}\}$ such that

- (M1) $\Psi(\mathbf{0}) = \mathbf{0}$,
- (M2) if $b \in \mathcal{B}_1$ and $\Psi(b) \in \mathcal{B}_2$, then $\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\Psi(b)) = \varphi_i(b)$ for all $i \in I_{\mathrm{af}}$,
- (M3) if $b, b' \in \mathcal{B}_1$, $\Psi(b), \Psi(b') \in \mathcal{B}_2$ and $f_i b = b'$, then $f_i\Psi(b) = \Psi(b')$ for all $i \in I_{\mathrm{af}}$.

A morphism $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ is called strict if $\Psi(f_i b) = f_i\Psi(b)$ and $\Psi(e_i b) = e_i\Psi(b)$ for all $b \in \mathcal{B}_1$ and $i \in I_{\mathrm{af}}$. A morphism $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ is called a strict embedding if it is a strict morphism and the associated map $\mathcal{B}_1 \sqcup \{\mathbf{0}\} \to \mathcal{B}_2 \sqcup \{\mathbf{0}\}$ is injective. A morphism $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ is called an isomorphism if the associated map $\mathcal{B}_1 \sqcup \{\mathbf{0}\} \to \mathcal{B}_2 \sqcup \{\mathbf{0}\}$ is bijective.

The tensor product $\mathcal{B}_1 \otimes \mathcal{B}_2$ of crystals $\mathcal{B}_1$ and $\mathcal{B}_2$ is defined to be the set $\mathcal{B}_1 \times \mathcal{B}_2$ whose crystal structure is defined as follows:

- (T1) $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$,
- (T2) $\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle\}$,
- (T3) $\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \mathrm{wt}(b_2)\rangle\}$,
- (T4) $e_i(b_1 \otimes b_2) = \begin{cases} (e_i b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2), \\ b_1 \otimes (e_i b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$

(T5) $f_i(b_1 \otimes b_2) = \begin{cases} (f_i b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes (f_i b_2) & \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2). \end{cases}$

Here, we write $b_1 \otimes b_2$ for $(b_1, b_2) \in \mathcal{B}_1 \times \mathcal{B}_2$, and we understand that $b_1 \otimes \mathbf{0} = \mathbf{0} \otimes b_2 = \mathbf{0}$.

Let $\mathcal{B}$ be a regular $\mathbf{U}$-crystal (resp. regular $\mathbf{U}'$-crystal) in the sense of [14, § 2.2]. By [11, § 7], we have a W_{af}-action $S : W_{\mathrm{af}} \to \mathrm{Bij}(\mathcal{B})$, $x \mapsto S_x$, on (the underlying set) $\mathcal{B}$ given by

$$S_{r_i} b = \begin{cases} f_i^{\langle h_i, \mathrm{wt}(b) \rangle} b & \text{if } \langle h_i, \mathrm{wt}(b) \rangle \geqslant 0, \\ e_i^{-\langle h_i, \mathrm{wt}(b) \rangle} b & \text{if } \langle h_i, \mathrm{wt}(b) \rangle \leqslant 0, \end{cases} \tag{13}$$

for each $b \in \mathcal{B}$ and $i \in I_{\mathrm{af}}$. Note that $\mathrm{wt}(S_x b) = x \, \mathrm{wt}(b)$ holds for all $x \in W_{\mathrm{af}}$ and $b \in \mathcal{B}$. An element $b \in \mathcal{B}$ of weight $\lambda \in P_{\mathrm{af}}$ (resp. $\lambda \in P_{\mathrm{af}}/(P_{\mathrm{af}} \cap \mathbb{C}\delta)$) is called an extremal element if we can find elements $b_x \in \mathcal{B}$, $x \in W_{\mathrm{af}}$, such that

(E1) $b_e = b$,
(E2) if $\langle h_i, x\lambda \rangle \geqslant 0$, then $e_i b_x = \mathbf{0}$ and $f_i^{\langle h_i, x\lambda \rangle} b_x = b_{r_i x}$,
(E3) if $\langle h_i, x\lambda \rangle \leqslant 0$, then $f_i b_x = \mathbf{0}$ and $e_i^{-\langle h_i, x\lambda \rangle} b_x = b_{r_i x}$.

Then $b_x = S_x b$ holds for all $x \in W_{\mathrm{af}}$.

2.3. Extremal weight modules and their crystal bases.

In this subsection, following [2, 11, 14], we review some of the standard facts on extremal weight modules and their crystal bases.

For $\lambda \in P^+$, let $V(\lambda)$ be the extremal weight $\mathbf{U}$-module generated by an extremal weight vector u_λ of extremal weight λ, and let $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ be the crystal basis of $V(\lambda)$ ([11, Proposition 8.2.2]; see also [14, § 3.2]). Note that $\mathcal{B}(\lambda)$ is a regular $\mathbf{U}$-crystal in the sense of [14, § 2.2] (see § 2.2). Let $\mathcal{B}_0(\lambda)$ be the connected component of the crystal graph of $\mathcal{B}(\lambda)$ containing $u_\lambda \mod q_s \mathcal{L}(\lambda)$.

Let z_i, $i \in I$, be the $\mathbf{U}'$-linear automorphism of $V(\varpi_i)$ of weight δ introduced in [14, § 5.2]; z_i sends a (unique) global basis element of weight ϖ_i to a (unique) global basis element of weight $\varpi_i + \delta$. Then z_i induces a $\mathbb{Q}$-linear automorphism of $\mathcal{L}(\varpi_i)/q_s\mathcal{L}(\varpi_i)$ and an automorphism of $\mathcal{B}(\varpi_i)$ as a $\mathbf{U}'$-crystal; by abuse of notation, we use the same letter z_i for the automorphism of $\mathcal{B}(\varpi_i)$. The $\mathbf{U}'$-module $W(\varpi_i) = V(\varpi_i)/(z_i - 1)V(\varpi_i)$ is called a level-zero fundamental representation. We know from [14, Theorem 5.17] that $W(\varpi_i)$ is a finite-dimensional irreducible $\mathbf{U}'$-module and has a (simple) crystal basis.

For $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$, with $m_i \in \mathbb{Z}_{\geqslant 0}$, $i \in I$, let $\tilde{V}(\lambda) = \bigotimes_{i \in I} V(\varpi_i)^{\otimes m_i}$ and $\tilde{u}_\lambda = \bigotimes_{i \in I} u_{\varpi_i}^{\otimes m_i} \in \tilde{V}(\lambda)$. For each $i \in I$ and $1 \leqslant \nu \leqslant m_i$, let $z_{i,\nu}$ be the $\mathbf{U}'$-linear automorphism of $\tilde{V}(\lambda)$ obtained by the action of z_i on the ν-th factor $V(\varpi_i)$ of $V(\varpi_i)^{\otimes m_i}$ in $\tilde{V}(\lambda)$. The $\mathbf{U}$-submodule

$$\breve{V}(\lambda) = \mathbf{U}\left[z_{i,\nu}, z_{i,\nu}^{-1} \,\middle|\, i \in I,\, 1 \leqslant \nu \leqslant m_i\right] \tilde{u}_\lambda \subset \tilde{V}(\lambda) \tag{14}$$

has a crystal basis $(\breve{\mathcal{L}}(\lambda), \breve{\mathcal{B}}(\lambda) = \bigotimes_{i \in I} \mathcal{B}(\varpi_i)^{\otimes m_i})$ such that $\breve{\mathcal{L}}(\lambda) \subset \bigotimes_{i \in I} \mathcal{L}(\varpi_i)^{\otimes m_i}$ ([14, Theorem 8.5]). Let $\breve{\mathcal{B}}_0(\lambda)$ be the connected component of the crystal graph of $\breve{\mathcal{B}}(\lambda)$ containing $\tilde{u}_\lambda \mod q_s \breve{\mathcal{L}}(\lambda)$. Since $\breve{V}(\lambda)$ contains an extremal weight vector $\tilde{u}_\lambda$ of weight λ, we have a $\mathbf{U}$-linear homomorphism $\Phi_\lambda : V(\lambda) \to \breve{V}(\lambda)$ sending u_λ to $\tilde{u}_\lambda$. We know from [2, § 4.2] that the map Φ_λ is injective, commutes with the Kashiwara operators e_i, f_i, $i \in I_{\mathrm{af}}$, and induces an injective $\mathbb{Q}$-linear map $\Phi_{\lambda|q=0} : \mathcal{L}(\lambda)/q_s\mathcal{L}(\lambda) \to \breve{\mathcal{L}}(\lambda)/q_s\breve{\mathcal{L}}(\lambda)$; note that $\Phi_{\lambda|q=0}(\mathcal{B}(\lambda)) \not\subset \breve{\mathcal{B}}(\lambda)$, in general (see Theorem 2.1 (2)).

For $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$, with $m_i \in \mathbb{Z}_{\geqslant 0}$, $i \in I$, set

(15)
$$\mathrm{Par}(\lambda) = \left\{ \boldsymbol{\rho} = \left(\rho^{(i)}\right)_{i \in I} \;\middle|\; \rho^{(i)} \text{ is a partition of length less than } m_i \text{ for each } i \in I \right\};$$

we understand that a partition of length less than 1 is an empty partition $\varnothing$. For a partition $\rho = (\rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_l > 0)$, set $|\rho| = \sum_{\nu=1}^{l} \rho_\nu$. For $\boldsymbol{\rho} = \left(\rho^{(i)}\right)_{i \in I} \in \mathrm{Par}(\lambda)$, set

$$\mathrm{wt}(\boldsymbol{\rho}) = -\sum_{i \in I} \left|\rho^{(i)}\right| \delta. \tag{16}$$

Let $\boldsymbol{\rho} = \left(\rho^{(i)}\right)_{i \in I} \in \mathrm{Par}(\lambda)$. Let $S_{\boldsymbol{\rho}}^-$ be the (PBW-type) basis element of weight $\mathrm{wt}(\boldsymbol{\rho})$ of the negative imaginary part of $\mathbf{U}$ constructed in [2, the element $S_{\mathbf{c}_0}^-$ in § 3.1; see also Remark 4.1]. Define

$$s_{\boldsymbol{\rho}}(z^{-1}) = \prod_{i \in I} s_{\rho^{(i)}}(z_{i,1}^{-1}, z_{i,2}^{-1}, \ldots, z_{i,m_i}^{-1}), \tag{17}$$

where the right-hand side is a product of Schur polynomials in the variables $z_{i,\nu}^{-1}$, $i \in I$, $1 \leqslant \nu \leqslant m_i$.

Theorem 2.1 ([2, § 4.2]; see also [14, § 13]). *Let* $\lambda \in P^+$.

(1) *We have*

$$\begin{aligned}\mathcal{B}(\lambda) = \Big\{ & g_1 g_2 \cdots g_l S_{\boldsymbol{\rho}}^- u_\lambda \mod q_s \mathcal{L}(\lambda) \\ & \Big|\ g_k \in \{e_i, f_i \mid i \in I_{\mathrm{af}}\},\ 1 \leqslant k \leqslant l,\ l \in \mathbb{Z}_{\geqslant 0},\ \boldsymbol{\rho} \in \mathrm{Par}(\lambda) \Big\} \setminus \{0\}.\end{aligned} \tag{18}$$

(2) *We have*

$$\begin{aligned}\Phi_{\lambda|q=0}\left(g_1 g_2 \cdots g_l S_{\boldsymbol{\rho}}^- u_\lambda \mod q_s \mathcal{L}(\lambda)\right) &= g_1 g_2 \cdots g_l s_{\boldsymbol{\rho}}(z^{-1}) \tilde{u}_\lambda \mod q_s \check{\mathcal{L}}(\lambda) \\ &= s_{\boldsymbol{\rho}}(z^{-1}) g_1 g_2 \cdots g_l \tilde{u}_\lambda \mod q_s \check{\mathcal{L}}(\lambda)\end{aligned} \tag{19}$$

for $g_1 g_2 \cdots g_l S_{\boldsymbol{\rho}}^- u_\lambda \mod q_s \mathcal{L}(\lambda) \in \mathcal{B}(\lambda)$. *In particular,*

$$\Phi_{\lambda|q=0}(\mathcal{B}(\lambda)) = \left\{ s_{\boldsymbol{\rho}}(z^{-1}) b \mod q_s \check{\mathcal{L}}(\lambda) \;\middle|\; \boldsymbol{\rho} \in \mathrm{Par}(\lambda),\ b \in \check{\mathcal{B}}_0(\lambda) \right\}, \tag{20}$$

and the map $\Phi_{\lambda|q=0}$ *induces an isomorphism of* $\mathbf{U}$*-crystals from* $\mathcal{B}_0(\lambda)$ *to* $\check{\mathcal{B}}_0(\lambda)$.

(3) *Let* $\mathcal{B}_{\boldsymbol{\rho}}(\lambda)$ *be the connected component of* $\mathcal{B}(\lambda)$ *containing* $S_{\boldsymbol{\rho}}^- u_\lambda \mod q_s \mathcal{L}(\lambda)$. *Then we have* $\mathcal{B}(\lambda) = \bigsqcup_{\boldsymbol{\rho} \in \mathrm{Par}(\lambda)} \mathcal{B}_{\boldsymbol{\rho}}(\lambda)$. *Moreover, for each* $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$, *there exists an isomorphism of* $\mathbf{U}'$*-crystals from* $\mathcal{B}_0(\lambda)$ *to* $\mathcal{B}_{\boldsymbol{\rho}}(\lambda)$ *sending* $u_\lambda \mod q_s \mathcal{L}(\lambda)$ *to* $S_{\boldsymbol{\rho}}^- u_\lambda \mod q_s \mathcal{L}(\lambda)$.

2.4. Semi-infinite Bruhat order on affine Weyl groups. In this subsection, we recall some basic facts on the semi-infinite Bruhat order on affine Weyl groups (see [6, 18, 25] for more details).

We take and fix $J \subset I$. Let $W_J = \langle r_j \mid j \in J \rangle$, and let W^J be the set of minimal coset representatives for W/W_J (see [3, Corollary 2.4.5 (i)]). For $w \in W$, we denote by $\lfloor w \rfloor \in W^J$ the minimal coset representative for the coset $wW_J \in W/W_J$. Define

$$(\Delta_J)_{\mathrm{af}} = \{\alpha + n\delta \mid \alpha \in \Delta_J, n \in \mathbb{Z}\} \subset \Delta_{\mathrm{af}}, \tag{21}$$

$$(\Delta_J)_{\mathrm{af}}^+ = (\Delta_J)_{\mathrm{af}} \cap \Delta_{\mathrm{af}}^+ = \Delta_J^+ \sqcup \{\alpha + n\delta \mid \alpha \in \Delta_J, n \in \mathbb{Z}_{>0}\}, \tag{22}$$

$$(W_J)_{\mathrm{af}} = W_J \ltimes \{t_\xi \mid \xi \in Q_J^\vee\} = \langle r_\beta \mid \beta \in (\Delta_J)_{\mathrm{af}}^+ \rangle, \tag{23}$$

$$(W^J)_{\mathrm{af}} = \{x \in W_{\mathrm{af}} \mid x\beta \in \Delta_{\mathrm{af}}^+ \text{ for all } \beta \in (\Delta_J)_{\mathrm{af}}^+\}; \tag{24}$$

note that $(W_\varnothing)_{\mathrm{af}} = \{e\}$ and $(W^\varnothing)_{\mathrm{af}} = W_{\mathrm{af}}$.

Lemma 2.2 ([25]; see also [18, Lemma 10.5]). *For each $x \in W_{\mathrm{af}}$, there exist a unique $x_1 \in (W^J)_{\mathrm{af}}$ and a unique $x_2 \in (W_J)_{\mathrm{af}}$ such that $x = x_1 x_2$. In particular, $(W^J)_{\mathrm{af}}$ is a complete system of coset representatives for $W_{\mathrm{af}}/(W_J)_{\mathrm{af}}$.*

Define a map $\Pi^J : W_{\mathrm{af}} \to (W^J)_{\mathrm{af}}$ by $\Pi^J(x) = x_1$ if $x = x_1 x_2$ with $x_1 \in (W^J)_{\mathrm{af}}$ and $x_2 \in (W_J)_{\mathrm{af}}$.

Lemma 2.3 ([25]; see also [18, Proposition 10.8]).

1. $\Pi^J(w) = \lfloor w \rfloor$ *for* $w \in W$.
2. $\Pi^J(xt_\xi) = \Pi^J(x)\Pi^J(t_\xi)$ *for* $x \in W_{\mathrm{af}}$ *and* $\xi \in Q^\vee$.

For simplicity of notation, we let $T_\xi = T_\xi^J$ stand for $\Pi^J(t_\xi) \in (W^J)_{\mathrm{af}}$ for $\xi \in Q^\vee$. The next lemma follows immediately from (23) and Lemmas 2.2–2.3.

Lemma 2.4.

1. $(W^J)_{\mathrm{af}} = \{wT_\xi \mid w \in W^J,\ \xi \in Q^\vee\}$.
2. *Let* $\xi, \zeta \in Q^\vee$. *If* $\xi \equiv \zeta \mod Q_J^\vee$, *then* $T_\xi^J = T_\zeta^J$.

Set $\rho_J = (1/2)\sum_{\alpha \in \Delta_J^+} \alpha$; we abbreviate ρ_J to ρ if $J = I$. For $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$, define

$$\ell^{\frac{\infty}{2}}(x) = \ell(w) + 2(\xi, \rho). \tag{25}$$

Define the (parabolic) semi-infinite Bruhat graph SiB^J to be the Δ_{af}^+-colored directed graph with vertex set $(W^J)_{\mathrm{af}}$ and edges of the form $x \xrightarrow{\beta} r_\beta x$ for $x \in (W^J)_{\mathrm{af}}$ and $\beta \in \Delta_{\mathrm{af}}^+$, where $r_\beta x \in (W^J)_{\mathrm{af}}$ and $\ell^{\frac{\infty}{2}}(r_\beta x) = \ell^{\frac{\infty}{2}}(x) + 1$.

The semi-infinite Bruhat order is a partial order $\preceq$ on $(W^J)_{\mathrm{af}}$ defined as follows: for $x, y \in (W^J)_{\mathrm{af}}$, we write $x \preceq y$ if there exists a directed path from x to y in SiB^J.

Proposition 2.5 ([6, Proposition A.1.2]). *Let $w \in W^J$, $\xi \in Q^\vee$ and $\beta \in \Delta_{\mathrm{af}}^+$. Write $\beta = w\gamma + \chi\delta$ with $\gamma \in \Delta$ and $\chi \in \mathbb{Z}_{\geqslant 0}$. Then $r_\beta wT_\xi \in (W^J)_{\mathrm{af}}$ and there exists an edge $wT_\xi \xrightarrow{\beta} r_\beta wT_\xi$ in SiB^J if and only if $\gamma \in \Delta^+ \setminus \Delta_J^+$ and one of the following conditions holds:*

- (B) $\chi = 0$ *and* $\ell(wr_\gamma) = \ell(w) + 1$; *in this case, we have* $r_\beta wT_\xi = wr_\gamma T_\xi$ *and* $wr_\gamma \in W^J$.
- (Q) $\chi = 1$ *and* $\ell(\lfloor wr_\gamma \rfloor) = \ell(w) + 1 - 2(\gamma^\vee, \rho - \rho_J)$; *in this case, we have* $r_\beta wT_\xi = \lfloor wr_\gamma \rfloor T_{\xi+\gamma^\vee}$.

Remark 2.6.

1. If $wT_\xi \xrightarrow{\beta} \Pi^J(r_\beta wT_\xi)$ in SiB^J, then $r_\beta wT_\xi = \Pi^J(r_\beta wT_\xi) \in (W^J)_{\mathrm{af}}$ (see [6, Appendix A]).
2. The condition (B) (resp. (Q)) for $w \in W^J$ and $\gamma \in \Delta^+ \setminus \Delta_J^+$ in Proposition 2.5 corresponds to the existence of the Bruhat edge (resp. quantum edge) $w \xrightarrow{\gamma} \lfloor wr_\gamma \rfloor$ in the (parabolic) quantum Bruhat graph for W^J (see [19, § 4]).

2.5. Semi-infinite Lakshmibai–Seshadri paths. In this subsection, we give a brief exposition of the **U**-crystal of semi-infinite Lakshmibai–Seshadri paths; see [6] for more details.

Let $\lambda \in P^+$ and set $J_\lambda = \{j \in I \mid \langle h_j, \lambda \rangle = 0\}$. For a rational number $0 < a \leqslant 1$, define $\mathrm{SiB}(\lambda; a)$ to be the subgraph of SiB^{J_λ} with the same vertex set but having only the edges of the form

$$x \xrightarrow{\beta} y \quad \text{with} \quad a(\beta^\vee, x\lambda) \in \mathbb{Z}; \tag{26}$$

note that $\mathrm{SiB}(\lambda; 1) = \mathrm{SiB}^{J_\lambda}$. A semi-infinite Lakshmibai–Seshadri path of shape λ is, by definition, a pair $(\boldsymbol{x}; \boldsymbol{a})$ of a decreasing sequence $\boldsymbol{x} : x_1 \succeq x_2 \succeq \cdots \succeq x_l$ of elements in $(W^{J_\lambda})_{\mathrm{af}}$ and an increasing sequence $\boldsymbol{a} : 0 = a_0 < a_1 < \cdots < a_l = 1$ of rational numbers such that there exists a directed path from x_{u+1} to x_u in $\mathrm{SiB}(\lambda; a_u)$ for each $u = 1, 2, \ldots, l-1$. Let $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ denote the set of semi-infinite Lakshmibai–Seshadri paths of shape λ.

Following [6, § 3.1], we equip the set $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ with a **U**-crystal structure. For $\eta = (x_1, \ldots, x_l; a_0, a_1, \ldots, a_l) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, define the map $\bar{\eta} : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{af}}$ by

$$\bar{\eta}(t) = \sum_{p=1}^{u-1} (a_p - a_{p-1}) x_p \lambda + (t - a_{u-1}) x_u \lambda \quad \text{for } t \in [a_{u-1}, a_u],\ 1 \leqslant u \leqslant l. \tag{27}$$

Define $\mathrm{wt} : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to P_{\mathrm{af}}$ by $\mathrm{wt}(\eta) = \bar{\eta}(1) \in P_{\mathrm{af}}$. Set

$$h_i^\eta(t) = \langle h_i, \bar{\eta}(t) \rangle \quad \text{for } t \in [0,1], \qquad m_i^\eta = \min\{h_i^\eta(t) \mid t \in [0,1]\}. \tag{28}$$

We define $e_i \eta$, $i \in I_{\mathrm{af}}$, as follows: if $m_i^\eta = 0$, then we set $e_i \eta = \mathbf{0}$. If $m_i^\eta \leqslant -1$, then we set

$$\begin{cases} t_1 = \min\{t \in [0,1] \mid h_i^\eta(t) = m_i^\eta\}, \\ t_0 = \max\{t \in [0, t_1] \mid h_i^\eta(t) = m_i^\eta + 1\}. \end{cases} \tag{29}$$

Let $1 \leqslant p \leqslant q \leqslant l$ be such that $a_{p-1} \leqslant t_0 < a_p$ and $t_1 = a_q$. Then we define

$$\begin{aligned} e_i \eta = (x_1, \ldots, x_p, r_i x_p, \ldots, r_i x_q, x_{q+1}, \ldots, x_l; \\ a_0, \ldots, a_{p-1}, t_0, a_p, \ldots, a_q = t_1, \ldots, a_l); \end{aligned} \tag{30}$$

if $t_0 = a_{p-1}$, then we drop x_p and a_{p-1}, and if $r_j x_q = x_{q+1}$, then we drop x_{q+1} and $a_q = t_1$.

Next, we define $f_i \eta$, $i \in I_{\mathrm{af}}$, as follows: if $m_i^\eta = h_i^\eta(1)$, then we set $f_i \eta = \mathbf{0}$. If $h_i^\eta(1) - m_i^\eta \geqslant 1$, then we set

$$\begin{cases} t_0 = \max\{t \in [0,1] \mid h_i^\eta(t) = m_i^\eta\}, \\ t_1 = \min\{t \in [t_0, 1] \mid h_i^\eta(t) = m_i^\eta + 1\}. \end{cases} \tag{31}$$

Let $1 \leqslant p \leqslant q \leqslant l-1$ be such that $t_0 = a_p$ and $a_q < t_1 \leqslant a_{q+1}$. Then we define

$$\begin{aligned} f_i \eta = (x_1, \ldots, x_p, r_i x_{p+1}, \ldots, r_i x_{q+1}, x_{q+1}, \ldots, x_l; \\ a_0, \ldots, a_p = t_0, \ldots, a_q, t_1, a_{q+1}, \ldots, a_l); \end{aligned} \tag{32}$$

if $t_1 = a_{q+1}$, then we drop x_{q+1} and a_{q+1}, and if $x_p = r_i x_{p+1}$, then we drop x_p and $a_p = t_0$.

For $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ and $i \in I_{\mathrm{af}}$, define

$$\begin{cases} \varepsilon_i(\eta) = -m_i^\eta, \\ \varphi_i(\eta) = h_i^\eta(1) - m_i^\eta. \end{cases} \tag{33}$$

Now we assume that $\lambda = \sum_{i \in I} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geqslant 0}$, $i \in I$. Set $J_\lambda^{\mathrm{c}} = I \setminus J_\lambda = \{i \in I \mid m_i > 0\}$. Following [6, Equation (7.2.2)], we define an element

$\eta_{\boldsymbol{\rho}} \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ of weight $\lambda + \mathrm{wt}(\boldsymbol{\rho})$ for each $\boldsymbol{\rho} = \left(\rho^{(i)}\right)_{i\in I} \in \mathrm{Par}(\lambda)$, with $\rho^{(i)} = \left(\rho^{(i)}_1 \geqslant \rho^{(i)}_2 \geqslant \cdots \geqslant \rho^{(i)}_{m_i-1}\right)$ (see (15)). Let s be the least common multiple of $\{m_i \mid i \in J^c_\lambda\}$. Let $c_i(\xi) \in \mathbb{Z}$ denote the coefficient of $\alpha^\vee_i$ in $\xi \in Q^\vee$. For $\xi, \zeta \in Q^\vee$, write $\xi \succeq \zeta$ if $\xi - \zeta \in \sum_{i\in I} \mathbb{Z}_{\geqslant 0}\alpha^\vee_i$, and write $\xi \succ \zeta$ if $\xi \succeq \zeta$ and $\xi \neq \zeta$. Let $\zeta_1, \dots, \zeta_s \in Q^\vee$ be such that

(i) $c_i(\zeta_t) = \rho^{(i)}_u$ if $i \in J^c_\lambda$ and $\dfrac{s(u-1)}{m_i} < t \leqslant \dfrac{su}{m_i}$, and

(ii) $c_j(\zeta_t) = 0$ for all $j \in J_\lambda$ and $1 \leqslant t \leqslant s$;

note that $\zeta_1 \succeq \cdots \succeq \zeta_s$ and $\zeta_s = 0$. Assume that

$$\zeta_1 = \cdots = \zeta_{s_1} \succ \zeta_{s_1+1} = \cdots = \zeta_{s_2} \succ \cdots\cdots \succ \zeta_{s_{k-1}+1} = \cdots = \zeta_{s_k}, \tag{34}$$

where $1 \leqslant s_1 < \cdots < s_{k-1} < s_k = s$. Set

$$\eta_{\boldsymbol{\rho}} = \left(T_{\zeta_{s_1}}, \dots, T_{\zeta_{s_{k-1}}}, e;\, 0, \frac{s_1}{s}, \dots, \frac{s_{k-1}}{s}, 1\right). \tag{35}$$

Theorem 2.7 ([6, Theorems 3.1.5 and 3.2.1]). *Let $\lambda \in P^+$.*

1. *The set $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ equipped with the maps* wt, e_i, f_i, $i \in I_{\mathrm{af}}$, *and* ε_i, φ_i, $i \in I_{\mathrm{af}}$, *defined above, is a* $\mathbf{U}$*-crystal.*
2. *There exists a unique isomorphism of* $\mathbf{U}$*-crystals from $\mathcal{B}(\lambda)$ to $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ sending $S^-_{\boldsymbol{\rho}} u_\lambda \mod q_s\mathcal{L}(\lambda)$ to $\eta_{\boldsymbol{\rho}}$ for every $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$.*

For $\lambda_1, \dots, \lambda_N \in P^+$, let $\mathbb{B}^{\frac{\infty}{2}}(\lambda_1) * \cdots * \mathbb{B}^{\frac{\infty}{2}}(\lambda_N)$ be the set of symbols $\eta_1 * \cdots * \eta_N$, with $\eta_\nu \in \mathbb{B}^{\frac{\infty}{2}}(\lambda_\nu)$, $1 \leqslant \nu \leqslant N$. We define a $\mathbf{U}$-crystal structure on $\mathbb{B}^{\frac{\infty}{2}}(\lambda_1) * \cdots * \mathbb{B}^{\frac{\infty}{2}}(\lambda_N)$ in a way similar to the above. For $\eta = \eta_1 * \cdots * \eta_N \in \mathbb{B}^{\frac{\infty}{2}}(\lambda_1) * \cdots * \mathbb{B}^{\frac{\infty}{2}}(\lambda_N)$, define $\bar{\eta} : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{af}}$ by

$$\bar{\eta}(t) = \sum_{\nu=1}^{\mu-1} \bar{\eta}_\nu(1) + \bar{\eta}_\mu(Nt - \mu + 1) \quad \text{for } \frac{\mu-1}{N} \leqslant t \leqslant \frac{\mu}{N},\ 1 \leqslant \mu \leqslant N, \tag{36}$$

where each $\bar{\eta}_\nu : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{af}}$, $1 \leqslant \nu \leqslant N$, is defined by (27). Define $\mathrm{wt}(\eta) = \bar{\eta}(1)$. By the same way as in (28), we define $h^\eta_i(t)$ and m^η_i for $\eta = \eta_1 * \cdots * \eta_N$ by using (36). We define $e_i\eta$ (resp. $f_i\eta$) $\in \mathbb{B}^{\frac{\infty}{2}}(\lambda_1) * \cdots * \mathbb{B}^{\frac{\infty}{2}}(\lambda_N) \sqcup \{\mathbf{0}\}$ as follows: if $m^\eta_i = 0$ (resp. $m^\eta_i = h^\eta_i(1)$), then we set $e_i\eta = \mathbf{0}$ (resp. $f_i\eta = \mathbf{0}$). Assume that $m^\eta_i \leqslant -1$ (resp. $h^\eta_i(1) - m^\eta_i \geqslant 1$), and let $0 \leqslant t_0 < t_1 \leqslant 1$ be as in (29) (resp. (31)). We see that there exists $1 \leqslant \nu \leqslant N$ such that $\dfrac{\nu-1}{N} \leqslant t_0 < t_1 \leqslant \dfrac{\nu}{N}$. Then we set $e_i\eta = \eta_1 * \cdots * \eta_{\nu-1} * e_i\eta_\nu * \eta_{\nu+1} * \cdots * \eta_N$ (resp. $f_i\eta = \eta_1 * \cdots * \eta_{\nu-1} * f_i\eta_\nu * \eta_{\nu+1} * \cdots * \eta_N$). We define the functions ε_i, φ_i as in (33). The proof of the next proposition is straightforward.

Proposition 2.8. *Let $\lambda_1, \dots, \lambda_N \in P^+$. The map $\mathbb{B}^{\frac{\infty}{2}}(\lambda_1) \otimes \cdots \otimes \mathbb{B}^{\frac{\infty}{2}}(\lambda_N) \to \mathbb{B}^{\frac{\infty}{2}}(\lambda_1) * \cdots * \mathbb{B}^{\frac{\infty}{2}}(\lambda_N)$, $\eta_1 \otimes \cdots \otimes \eta_N \mapsto \eta_1 * \cdots * \eta_N$, is an isomorphism of* $\mathbf{U}$*-crystals.*

3. Standard monomial theory for semi-infinite Lakshmibai–Seshadri paths

3.1. Strict embedding $\Phi^{\mathrm{LT}}_{\lambda|q=0}$.

Let $\lambda = \sum_{i\in I} m_i\varpi_i \in P^+$, with $m_i \in \mathbb{Z}_{\geqslant 0}$, $i \in I$. Recall the automorphisms $z_{i,\nu}$, $i \in I$, $1 \leqslant \nu \leqslant m_i$, of the $\mathbf{U}'$-crystal $\breve{\mathcal{B}}(\lambda)$ (see § 2.3). For $\boldsymbol{\rho} = (\rho^{(i)})_{i\in I} \in \mathrm{Par}(\lambda)$, with $\rho^{(i)} = (\rho^{(i)}_1 \geqslant \rho^{(i)}_2 \geqslant \cdots \geqslant \rho^{(i)}_{m_i-1} \geqslant 0)$, $i \in I$, define the automorphism $z^{-\boldsymbol{\rho}}$ of the $\mathbf{U}'$-crystal $\breve{\mathcal{B}}(\lambda)$ by

$$z^{-\boldsymbol{\rho}} = \prod_{i\in I} z_{i,1}^{-\rho^{(i)}_1} z_{i,2}^{-\rho^{(i)}_2} \cdots z_{i,m_i-1}^{-\rho^{(i)}_{m_i-1}}. \tag{37}$$

Define the map $\Phi^{\mathrm{LT}}_{\lambda|q=0} : \mathcal{B}(\lambda) \to \breve{\mathcal{B}}(\lambda)$ by

$$g_1 g_2 \cdots g_l S^-_{\boldsymbol{\rho}} u_\lambda \mod q_s \mathcal{L}(\lambda) \longmapsto g_1 g_2 \cdots g_l z^{-\boldsymbol{\rho}} \tilde{u}_\lambda \mod q_s \breve{\mathcal{L}}(\lambda), \tag{38}$$

where $g_k \in \{e_i, f_i \mid i \in I_{\mathrm{af}}\}$, $1 \leqslant k \leqslant l$, $l \in \mathbb{Z}_{\geqslant 0}$, and $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$ (cf. (19)). Set $\breve{\mathcal{B}}_{\boldsymbol{\rho}}(\lambda) = z^{-\boldsymbol{\rho}}\left(\breve{\mathcal{B}}_0(\lambda)\right) \subset \breve{\mathcal{B}}(\lambda)$; note that $\breve{\mathcal{B}}_{\boldsymbol{\rho}}(\lambda)$ is a connected component of $\breve{\mathcal{B}}(\lambda)$, and is isomorphic to $\breve{\mathcal{B}}_0(\lambda)$ as a $\mathbf{U}'$-crystal.

Lemma 3.1. *The map $\Phi^{\mathrm{LT}}_{\lambda|q=0}$ is well-defined, and is a strict embedding of $\mathbf{U}$-crystals.*

Proof. It suffices to show that the map $\Phi^{\mathrm{LT}}_{\lambda|q=0}$ induces an isomorphism of $\mathbf{U}$-crystals from $\mathcal{B}_{\boldsymbol{\rho}}(\lambda)$ to $\breve{\mathcal{B}}_{\boldsymbol{\rho}}(\lambda)$ for every $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$. We know that the maps $\mathcal{B}_{\boldsymbol{\rho}}(\lambda) \to \mathcal{B}_0(\lambda)$ in Theorem 2.1 (3), $\Phi_{\lambda|q=0} : \mathcal{B}_0(\lambda) \to \breve{\mathcal{B}}_0(\lambda)$ in Theorem 2.1 (2), and $z^{-\boldsymbol{\rho}} : \breve{\mathcal{B}}_0(\lambda) \to \breve{\mathcal{B}}_{\boldsymbol{\rho}}(\lambda)$ are isomorphisms of $\mathbf{U}'$-crystals. We check at once that the composition of these maps is describe by (38), which proves that $\Phi^{\mathrm{LT}}_{\lambda|q=0}$ is well-defined and induces an isomorphism of $\mathbf{U}'$-crystals from $\mathcal{B}_{\boldsymbol{\rho}}(\lambda)$ to $\breve{\mathcal{B}}_{\boldsymbol{\rho}}(\lambda)$. Since $\mathrm{wt}(S^-_{\boldsymbol{\rho}} u_\lambda) = \lambda + \mathrm{wt}(\boldsymbol{\rho}) = \mathrm{wt}(z^{-\boldsymbol{\rho}} \tilde{u}_\lambda)$, $\Phi^{\mathrm{LT}}_{\lambda|q=0}$ is a morphism of $\mathbf{U}$-crystals. ☐

Remark 3.2. If we think of the Schur polynomials as the generating functions of the weights of Young tableaux (see [4, Page 3]), then the term $z^{-\boldsymbol{\rho}}$ in $s_{\boldsymbol{\rho}}(z^{-1})$ (see (17)) corresponds to the tuple of the Littlewood–Richardson tableaux of shapes $\rho^{(i)}$, $i \in I$ (see [4, § 5.2]), and the coefficient of $z^{-\boldsymbol{\rho}}$ in $s_{\boldsymbol{\rho}}(z^{-1})$ is 1.

3.2. Characterization of the image of $\Phi^{\mathrm{LT}}_{\lambda|q=0}$.

In this subsection, we give a characterization of the image of the map $\Phi^{\mathrm{LT}}_{\lambda|q=0}$ in terms of semi-infinite Bruhat order via semi-infinite Lakshmibai–Seshadri paths.

Recall the notation $J_\lambda = \{i \in I \mid \langle h_i, \lambda\rangle = 0\}$, $\lambda \in P^+$.

Definition 3.3. *Let $\lambda_\nu \in P^+$ and $\eta^{(\nu)} = \left(x_1^{(\nu)}, \ldots, x_{l_\nu}^{(\nu)}; \boldsymbol{a}^{(\nu)}\right) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda_\nu)$, $1 \leqslant \nu \leqslant N$. We say that there exists a defining chain for $\bigotimes_{\nu=1}^N \eta^{(\nu)} \in \bigotimes_{\nu=1}^N \mathbb{B}^{\frac{\infty}{2}}(\lambda_\nu)$ if there exists $\tilde{x}_s^{(\nu)} \in W_{\mathrm{af}}$, $1 \leqslant s \leqslant l_\nu$, $1 \leqslant \nu \leqslant N$, such that*

- (DC1) $\Pi^{J_{\lambda_\nu}}(\tilde{x}_s^{(\nu)}) = x_s^{(\nu)}$ *for all* $1 \leqslant s \leqslant l_\nu$, $1 \leqslant \nu \leqslant N$,
- (DC2) $\tilde{x}_p^{(\nu)} \succeq \tilde{x}_q^{(\nu)}$ *for all* $1 \leqslant p \leqslant q \leqslant l_\nu$, $1 \leqslant \nu \leqslant N$, *and*
- (DC3) $\tilde{x}_{l_\nu}^{(\nu)} \succeq \tilde{x}_1^{(\nu+1)}$ *for all* $1 \leqslant \nu < N$,

where $\succeq$ denotes the semi-infinite Bruhat order on W_{af} defined by using $\mathrm{SiB}^{\varnothing}$ *(see § 2.4). The tuple $(\tilde{x}_s^{(\nu)})_{1\leqslant s\leqslant l_\nu, 1\leqslant \nu\leqslant N}$ above is called a defining chain for $\bigotimes_{\nu=1}^N \eta^{(\nu)}$.*

Let $\lambda = \sum_{i\in I} m_i \varpi_i \in P^+$. Recall the notation $J^c_\lambda = I \setminus J_\lambda$. Set $\breve{\mathbb{B}}^{\frac{\infty}{2}}(\lambda) = \bigotimes_{i \in J^c_\lambda} \mathbb{B}^{\frac{\infty}{2}}(\varpi_i)^{\otimes m_i}$. We know from Theorem 2.7 that there exists an isomorphism

$$\Psi_\lambda : \breve{\mathcal{B}}(\lambda) \to \breve{\mathbb{B}}^{\frac{\infty}{2}}(\lambda) \tag{39}$$

of $\mathbf{U}$-crystals defined as the tensor product of the isomorphisms $\mathcal{B}(\varpi_i) \to \mathbb{B}^{\frac{\infty}{2}}(\varpi_i)$ sending u_{ϖ_i} to $(e; 0, 1)$, $i \in I$. Write

$$\Psi_\lambda(b) = \bigotimes_{i\in J^c_\lambda} \Psi^{(i)}_\lambda(b) \in \breve{\mathbb{B}}^{\frac{\infty}{2}}(\lambda), \text{ where } \Psi^{(i)}_\lambda(b) \in \mathbb{B}^{\frac{\infty}{2}}(\varpi_i)^{\otimes m_i},\ i \in J^c_\lambda. \tag{40}$$

Set

$$\breve{\mathbb{S}}^{\frac{\infty}{2}}(\lambda) = \left\{\eta \in \breve{\mathbb{B}}^{\frac{\infty}{2}}(\lambda) \,\middle|\, \text{there exists a defining chain for } \eta\right\}. \tag{41}$$

Theorem 3.4. *Let $\lambda \in P^+$. For $b \in \breve{\mathcal{B}}(\lambda)$, the following conditions are equivalent:*

(1) $b \in \Phi^{\mathrm{LT}}_{\lambda|q=0}(\mathcal{B}(\lambda))$.

(2) $\Psi_\lambda(b) \in \breve{\mathbb{S}}^{\frac{\infty}{2}}(\lambda)$.

(3) $\Psi^{(i)}_\lambda(b) \in \breve{\mathbb{S}}^{\frac{\infty}{2}}(m_i\varpi_i)$ *for every* $i \in J^c_\lambda$.

In particular, $\breve{\mathbb{S}}^{\frac{\infty}{2}}(\lambda) = \bigotimes_{i \in J^c_\lambda} \breve{\mathbb{S}}^{\frac{\infty}{2}}(m_i\varpi_i)$ *(cf.* [2, Remark 4.17]*; see also* [14, Conjecture 13.1 (iii)]*),* $\breve{\mathbb{S}}^{\frac{\infty}{2}}(\lambda)$ *is stable under the Kashiwara operators, and the map* $\Psi_\lambda \circ \Phi^{\mathrm{LT}}_{\lambda|q=0} : \mathcal{B}(\lambda) \to \breve{\mathbb{S}}^{\frac{\infty}{2}}(\lambda)$ *is an isomorphism of* **U***-crystals.*

Remark 3.5.

(1) Our argument in the proof of Theorem 3.4 in § 3.3 does not imply [14, Conjecture 13.1 (iii)] since [2, Remark 4.17] is used in the proof of Theorem 2.7 ([6, Theorems 3.1.5 and 3.2.1]).

(2) Similar result to Theorem 3.4 is obtained in [16, Theorem 3.1], where they proved that, for any $\lambda, \mu \in P^+$, the subset $\mathbb{S}^{\frac{\infty}{2}}(\lambda+\mu)$ of elements in $\mathbb{B}^{\frac{\infty}{2}}(\lambda) \otimes \mathbb{B}^{\frac{\infty}{2}}(\mu)$ having a defining chain is stable under the Kashiwara operators, and is isomorphic to $\mathbb{B}^{\frac{\infty}{2}}(\lambda+\mu)$ as a **U**-crystal. But the proof is slightly different from ours. The main task in the proof of [16, Theorem 3.1] is to construct an isomorphism of **U**-crystals between $\mathbb{S}^{\frac{\infty}{2}}(\lambda+\mu)$ and $\mathbb{B}^{\frac{\infty}{2}}(\lambda+\mu)$. This is achieved by giving an explicit parametrization of the connected components of $\mathbb{S}^{\frac{\infty}{2}}(\lambda+\mu)$. In contrast of this, our argument starts from a specific choice of a map (see (38)), and aims to give an explicit description of the image of this map; in fact, there are infinitely many strict embeddings of **U**-crystals from $\mathcal{B}(\lambda)$ to $\breve{\mathcal{B}}(\lambda)$, in general.

3.3. Proof of Theorem 3.4. This subsection is devoted to the proof of Theorem 3.4.

We see from [14, Theorem 5.17] that, for each $i \in I$, there exists a strict surjective morphism of $\mathbf{U}'$-crystals from $\mathcal{B}(\varpi_i)$ to the crystal basis of a finite-dimensional $\mathbf{U}'$-module $W(\varpi_i)$ (see § 2.3). Hence, the next lemma follows from [1, Lemmas 1.5–1.6].

Lemma 3.6. *Let $i_1, \dots, i_N \in I$.*

(1) *Any connected component of* $\bigotimes_{\nu=1}^N \mathcal{B}(\varpi_{i_\nu})$ *contains an extremal element.*

(2) *If* $b = \bigotimes_{\nu=1}^N b^{(\nu)} \in \bigotimes_{\nu=1}^N \mathcal{B}(\varpi_{i_\nu})$ *is an extremal element, then* $S_x b = \bigotimes_{\nu=1}^N S_x b^{(\nu)}$ *for all* $x \in W_{\mathrm{af}}$.

(3) $\bigotimes_{\nu=1}^N b^{(\nu)} \in \bigotimes_{\nu=1}^N \mathcal{B}(\varpi_{i_\nu})$ *is an extremal element if and only if there exist* $w \in W$ *and* $\xi_1, \dots, \xi_N \in Q^\vee$ *such that* $b^{(\nu)} = S_{wt_{\xi_\nu}} u_{\varpi_{i_\nu}}$ *for* $1 \leqslant \nu \leqslant N$.

By Theorem 2.7, the W_{af}-action on $\mathcal{B}(\lambda)$ (see (13)) induces a W_{af}-action on $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$. The next lemma follows from Theorem 2.7 (2) and Lemmas 2.3–2.4 and 3.6.

Lemma 3.7. *Let $i_1, \dots, i_N \in I$.*

(1) *Any connected component of* $\bigotimes_{\nu=1}^N \mathbb{B}^{\frac{\infty}{2}}(\varpi_{i_\nu})$ *contains an extremal element.*

(2) *If* $\eta = \bigotimes_{\nu=1}^N \eta^{(\nu)} \in \bigotimes_{\nu=1}^N \mathbb{B}^{\frac{\infty}{2}}(\varpi_{i_\nu})$ *is an extremal element, then* $S_x\eta = \bigotimes_{\nu=1}^N S_x\eta^{(\nu)}$ *for all* $x \in W_{\mathrm{af}}$.

(3) $\bigotimes_{\nu=1}^N \eta^{(\nu)} \in \bigotimes_{\nu=1}^N \mathbb{B}^{\frac{\infty}{2}}(\varpi_{i_\nu})$ *is an extremal element if and only if there exist* $w \in W$ *and* $\sigma_\nu \in \mathbb{Z}$, $1 \leqslant \nu \leqslant N$, *such that* $\eta^{(\nu)} = S_w\left(T^{I\setminus\{i_\nu\}}_{\sigma_\nu\alpha^\vee_{i_\nu}}; 0, 1\right) \in \mathbb{B}^{\frac{\infty}{2}}(\varpi_{i_\nu})$ *for* $1 \leqslant \nu \leqslant N$.

We also denote by z_i the automorphism, as a $\mathbf{U}'$-crystal, of $\mathbb{B}^{\frac{\infty}{2}}(\varpi_i)$ corresponding to the automorphism z_i of $\mathcal{B}(\varpi_i)$. Recall that $c_i(\xi)$ denotes the coefficient of $\alpha^\vee_i$ in $\xi \in Q^\vee$.

Lemma 3.8. *Let* $\lambda = \sum_{i\in I} m_i\varpi_i \in P^+$.

(1) *For each* $i \in I$, *we have* $z_i^k(e;0,1) = \left(T^{I\setminus\{i\}}_{-k\alpha_i^\vee};0,1\right)$ *in* $\mathbb{B}^{\frac{\infty}{2}}(\varpi_i)$ *for all* $k \in \mathbb{Z}$. *In particular, for every* $\boldsymbol{\rho} = (\rho^{(i)})_{i\in I} \in \mathrm{Par}(\lambda)$, *with* $\rho^{(i)} = (\rho^{(i)}_1 \geqslant \cdots \geqslant \rho^{(i)}_{m_i-1} \geqslant 0)$, $i \in I$, *we have*

$$\Psi^{(i)}_\lambda(z^{-\boldsymbol{\rho}}\tilde{u}_\lambda) = \left(T^{I\setminus\{i\}}_{\rho^{(i)}_1\alpha_i^\vee};0,1\right) \otimes \cdots \otimes \left(T^{I\setminus\{i\}}_{\rho^{(i)}_{m_i-1}\alpha_i^\vee};0,1\right) \otimes (e;0,1) \tag{42}$$

in $\mathbb{B}^{\frac{\infty}{2}}(\varpi_i)^{\otimes m_i}$ *for each* $i \in I$.

(2) *Any connected component of* $\breve{\mathbb{B}}^{\frac{\infty}{2}}(\lambda)$ *contains an extremal element of the form*

$$\bigotimes_{i\in J_\lambda^c} \bigotimes_{\nu=1}^{m_i} \left(T^{I\setminus\{i\}}_{\rho^{(i)}_\nu\alpha_i^\vee};0,1\right), \tag{43}$$

where $\rho^{(i)}_\nu \in \mathbb{Z}$, $1 \leqslant \nu \leqslant m_i$, $i \in J_\lambda^c$, *and* $\rho^{(i)}_{m_i} = 0$ *for all* $i \in J_\lambda^c$.

Proof. (1): Since $z_i^k(e;0,1)$ is an extremal element of weight $\varpi_i + k\delta$, there exists $x \in (W^{I\setminus\{i\}})_{\mathrm{af}}$ such that $z_i^k(e;0,1) = S_x(e;0,1) = (x;0,1)$ by [14, Proposition 5.4 (i)]. If we write $x = wT^{I\setminus\{i\}}_\xi$, $w \in W$, $\xi \in Q^\vee$, then $\mathrm{wt}(x;0,1) = w\varpi_i - (\xi,\varpi_i)\delta = w\varpi_i - c_i(\xi)\delta$, which implies that $w = e$ and $c_i(\xi) = -k$. By Lemma 2.4 (2), we have $T^{I\setminus\{i\}}_\xi = T^{I\setminus\{i\}}_{-k\alpha_i^\vee}$, which proves that $z_i^k(e;0,1) = \left(T^{I\setminus\{i\}}_{-k\alpha_i^\vee};0,1\right)$.

(2): By Lemma 3.7, any connected component C of $\breve{\mathbb{B}}^{\frac{\infty}{2}}(\lambda)$ contains an extremal element of the form $\eta = \bigotimes_{i\in J_\lambda^c}\bigotimes_{\nu=1}^{m_i}\left(T^{I\setminus\{i\}}_{\sigma^{(i)}_\nu\alpha_i^\vee};0,1\right)$, with $\sigma^{(i)}_\nu \in \mathbb{Z}$, $1 \leqslant \nu \leqslant m_i$, $i \in J_\lambda^c$. Set $\rho^{(i)}_\nu = \sigma^{(i)}_\nu - \sigma^{(i)}_{m_i}$ and $\xi = -\sum_{i\in J_\lambda^c}\sigma^{(i)}_{m_i}\alpha_i^\vee$. By Lemmas 2.4 (2) and 3.7 (2),

$$C \ni S_{t_\xi}\eta = \bigotimes_{i\in J_\lambda^c}\bigotimes_{\nu=1}^{m_i} S_{t_\xi}\left(T^{I\setminus\{i\}}_{\sigma^{(i)}_\nu\alpha_i^\vee};0,1\right) = \bigotimes_{i\in J_\lambda^c}\bigotimes_{\nu=1}^{m_i}\left(T^{I\setminus\{i\}}_{\rho^{(i)}_\nu\alpha_i^\vee};0,1\right), \tag{44}$$

which is the desired conclusion. □

For $J \subset I$, set $J^c = I \setminus J$. Let $[\,\cdot\,]_J : Q^\vee = Q^\vee_J \oplus Q^\vee_{J^c} \to Q^\vee_J$ be the projection. Recall that we write $\xi \succeq \zeta$ for $\xi,\zeta \in Q^\vee$ if $\xi - \zeta \in \sum_{i\in I}\mathbb{Z}_{\geqslant 0}\alpha_i^\vee$.

Lemma 3.9.

(1) *For* $\xi_1,\xi_2 \in Q^\vee$, $T^J_{\xi_1} \succeq T^J_{\xi_2}$ *in* $(W^J)_{\mathrm{af}}$ *if and only if* $[\xi_1]_{J^c} \succeq [\xi_2]_{J^c}$ *in* $Q^\vee$.
(2) *For* $x,y \in W_{\mathrm{af}}$ *and* $\xi \in Q^\vee$, $x \succeq y$ *in* W_{af} *if and only if* $xt_\xi \succeq yt_\xi$ *in* W_{af}. *In particular, we have* $wt_\xi \succeq t_\xi$ *in* W_{af} *for all* $w \in W$.
(3) *For any* $y \in W_{\mathrm{af}}$ *and* $\xi \in Q^\vee$, *there exists* $\vartheta \in Q^\vee$ *such that* $\vartheta \succeq \xi$ *in* $Q^\vee$ *and* $t_\vartheta \succeq y$ *in* W_{af}.
(4) *Let* $J,K \subset I$ *be such that* $J^c \cap K^c = \varnothing$. *Then, for any* $x,y \in W_{\mathrm{af}}$, *there exist* $\vartheta_1 \in Q^\vee_J$ *and* $\vartheta_2 \in Q^\vee_K$ *such that* $\Pi^J(xt_{\vartheta_1}) = \Pi^J(x)$, $\Pi^K(yt_{\vartheta_2}) = \Pi^K(y)$ *and* $xt_{\vartheta_1} \succeq yt_{\vartheta_2}$ *in* W_{af}.
(5) *Let* $J \subset I$ *and* $x,y \in W_{\mathrm{af}}$. *If* $x \succeq y$ *in* W_{af}, *then* $\Pi^J(x) \succeq \Pi^J(y)$ *in* $(W^J)_{\mathrm{af}}$.

Proof. (1): This is a special case of [6, Lemma 6.2.1].

(2): The assertion follows immediately from the formula $\ell^{\frac{\infty}{2}}(xt_\xi) = \ell^{\frac{\infty}{2}}(x) + 2(\xi,\rho)$.

(3): Let $y = vt_\zeta$, $v \in W$ and $\zeta \in Q^\vee$, and let $v = r_{i_1}r_{i_2}\cdots r_{i_l}$, $i_1,i_2,\ldots,i_l \in I_{\mathrm{af}}$, be a reduced expression. If we set $w_k = r_{i_1}r_{i_2}\cdots r_{i_k}$ and $\gamma_k = w_k\alpha_{i_k}$ for $k = 1,2,\ldots,l$, then

$$y = w_lt_\zeta \xrightarrow{\delta+\gamma_l} w_{l-1}t_{\zeta+\alpha^\vee_{i_l}} \xrightarrow{\delta+\gamma_{l-1}} \cdots \xrightarrow{\delta+\gamma_1} t_{\zeta+\alpha^\vee_{i_1}+\alpha^\vee_{i_2}+\cdots+\alpha^\vee_{i_l}} \quad \text{in } \mathrm{SiB}^{\varnothing}, \tag{45}$$

which proves that $\vartheta' = \zeta + \alpha_{i_1}^\vee + \alpha_{i_2}^\vee + \cdots + \alpha_{i_l}^\vee \in Q^\vee$ satisfies $t_{\vartheta'} \succeq y$. We see that $\vartheta = \sum_{i\in I} \max\{c_i(\xi), c_i(\vartheta')\}\alpha_i^\vee$ satisfies $\vartheta \succeq \xi$, $\vartheta \succeq \vartheta'$, and hence $t_\vartheta \succeq t_{\vartheta'}$ by (1).

(4): Assume that $x = wt_\xi$ with $w \in W$ and $\xi \in Q^\vee$. By (3), there exists $\vartheta \in Q^\vee$ such that $\vartheta \succeq \xi$ and $t_\vartheta \succeq y$. If we set $\vartheta_1 = [\vartheta - \xi]_J$ and $\vartheta_2 = [\xi - \vartheta]_K$, then $\Pi^J(xt_{\vartheta_1}) = \Pi^J(x)$ and $\Pi^K(yt_{\vartheta_2}) = \Pi^K(y)$. Moreover, we have $\xi + \vartheta_1 \succeq \vartheta + \vartheta_2$, because $\vartheta \succeq \xi$, $\xi + \vartheta_1 = [\xi]_{J^c} + [\vartheta]_{K^c} + [\vartheta]_{J\cap K}$ since $J = K^c \sqcup (J \cap K)$, and $\vartheta + \vartheta_2 = [\xi]_{J^c} + [\vartheta]_{K^c} + [\xi]_{J\cap K}$ since $K = J^c \sqcup (J\cap K)$. Then (1)–(2) shows that

$$xt_{\vartheta_1} = wt_{\xi+\vartheta_1} \succeq t_{\xi+\vartheta_1} \succeq t_{\vartheta+\vartheta_2} = t_\vartheta t_{\vartheta_2} \succeq yt_{\vartheta_2}. \tag{46}$$

(5): By induction on $\ell^{\frac{\infty}{2}}(x) - \ell^{\frac{\infty}{2}}(y)$, the assertion follows from [6, Lemma 6.1.1 for $K = \varnothing$]. □

Proof of Theorem 3.4. We first prove that (2) and (3) are equivalent. Clearly, (2) implies (3). We prove that (3) implies (2). The proof is by induction on $\#J_\lambda^c$. If $\#J_\lambda^c = 1$, then (2) and (3) are equivalent. Assume that $\#J_\lambda^c > 1$, $b \in \breve{\mathcal{B}}(\lambda)$ satisfies (3), and $\Psi_\lambda(b) = \Psi_\lambda^{(i)}(b) \otimes \bigotimes_{j\in J_\lambda^c \setminus \{i\}} \Psi_\lambda^{(j)}(b)$. By (3), there exists a defining chain $(x_1, \ldots, x_N)$ for $\Psi_\lambda^{(i)}(b)$. By induction hypothesis, there exists a defining chain $(y_1, \ldots, y_M)$ for $\bigotimes_{j\in J_\lambda^c\setminus\{i\}} \Psi_\lambda^{(j)}(b)$. Applying Lemma 3.9 (4) to $x = x_N$, $y = y_1$, $J = I \setminus \{i\}$, and $K = J_\lambda \cup \{i\}$ to obtain $\vartheta_1 \in Q_{I\setminus\{i\}}^\vee$ and $\vartheta_2 \in Q_{J_\lambda\cup\{i\}}^\vee$ such that $x_N t_{\vartheta_1} \succeq y_1 t_{\vartheta_2}$. By Lemma 3.9 (2), we conclude that $(x_1 t_{\vartheta_1}, \ldots, x_N t_{\vartheta_1}, y_1 t_{\vartheta_2}, \ldots, y_M t_{\vartheta_2})$ is a defining chain for $\Psi_\lambda(b)$.

We next prove that (1) and (2) are equivalent. The proof is completed by showing that

(i) $\breve{\mathbb{S}}^{\frac{\infty}{2}}(\lambda)$ is stable under the Kashiwara operators, and
(ii) each connected component of $\breve{\mathbb{S}}^{\frac{\infty}{2}}(\lambda)$ contains $\Psi_\lambda(z^{-\boldsymbol{\rho}}\tilde{u}_\lambda)$ for some $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$,

because $\Psi_\lambda \circ \Phi_{\lambda|q=0}^{\mathrm{LT}} : \mathcal{B}(\lambda) \to \breve{\mathbb{B}}^{\frac{\infty}{2}}(\lambda)$ is a strict embedding of $\mathbf{U}$-crystals, and $\Psi_\lambda(z^{-\boldsymbol{\rho}}\tilde{u}_\lambda) \in \breve{\mathbb{S}}^{\frac{\infty}{2}}(\lambda)$ for every $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$; indeed, by Lemmas 3.8 (1) and 3.9 (1), we have a defining chain $\left(t_{\rho_1^{(i)}\alpha_i^\vee}, \ldots, t_{\rho_{m_i-1}^{(i)}\alpha_i^\vee}, e\right)$ for $\Psi_\lambda^{(i)}(z^{-\boldsymbol{\rho}}\tilde{u}_\lambda)$ for every $\boldsymbol{\rho} = (\rho^{(i)})_{i\in I} \in \mathrm{Par}(\lambda)$, with $\rho^{(i)} = (\rho_1^{(i)} \geqslant \cdots \geqslant \rho_{m_i-1}^{(i)} \geqslant 0)$, $i \in I$, and hence there exists a defining chain for $\Psi_\lambda(z^{-\boldsymbol{\rho}}\tilde{u}_\lambda)$ by the implication (3) ⇒ (2).

We prove (i) only for the action of e_j, $j \in I_{\mathrm{af}}$; the proof for f_j is similar. Let $\eta = \bigotimes_{i\in J_\lambda^c} \eta^{(i)} \in \breve{\mathbb{S}}^{\frac{\infty}{2}}(\lambda)$, with $\eta^{(i)} \in \breve{\mathbb{S}}^{\frac{\infty}{2}}(m_i\varpi_i)$, $i \in J_\lambda^c$. By tensor product rule and the implication (3) ⇒ (2), we only need to show that, for each $i \in J_\lambda^c$, if $e_j\eta^{(i)} \neq \mathbf{0}$, then $e_j\eta^{(i)} \in \breve{\mathbb{S}}^{\frac{\infty}{2}}(m_i\varpi_i)$. Write $\eta^{(i)} = \bigotimes_{\nu=1}^{m_i} \eta_\nu^{(i)}$, with $\eta_\nu^{(i)} \in \mathbb{B}^{\frac{\infty}{2}}(\varpi_i)$, $1 \leqslant \nu \leqslant m_i$, and let $(x_s^{[\nu]})_{1\leqslant s\leqslant l_\nu, 1\leqslant \nu\leqslant m_i}$ be a defining chain for $\eta^{(i)}$; by Lemma 3.9 (5), we may assume that $x_s^{[\nu]} \in (W^{I\setminus\{i\}})_{\mathrm{af}}$, $1 \leqslant s \leqslant l_\nu$, $1 \leqslant \nu \leqslant m_i$. By tensor product rule, $e_j\eta^{(i)} = \eta_1^{(i)} \otimes \cdots \otimes e_j\eta_\nu^{(i)} \otimes \cdots \otimes \eta_{m_i}^{(i)}$ for some $1 \leqslant \nu \leqslant m_i$. Let $1 \leqslant p < q \leqslant l_\nu$ be as in (30) for $\eta_\nu^{(i)}$. It follows from Proposition 2.8 and [6, Lemma 4.1.6] that the tuple

$$\left(x_1^{[1]}, \ldots, x_{l_{\nu-1}}^{[\nu-1]}, x_1^{[\nu]}, \ldots, x_p^{[\nu]}, r_i x_p^{[\nu]}, \ldots, r_i x_q^{[\nu]}, x_{q+1}^{[\nu]}, \ldots, x_{l_\nu}^{[\nu]}, x_1^{[\nu+1]}, \ldots, x_{l_{m_i}}^{[m_i]}\right) \tag{47}$$

is a defining chain for $e_j\eta^{(i)}$, and hence $e_j\eta^{(i)} \in \breve{\mathbb{S}}^{\frac{\infty}{2}}(m_i\varpi_i)$.

Finally, we prove (ii). Let C be an arbitrary connected component of $\breve{\mathbb{S}}^{\frac{\infty}{2}}(\lambda)$; we see from (i) that C is a connected component of $\breve{\mathbb{B}}^{\frac{\infty}{2}}(\lambda)$. By Lemma 3.8 (2), C contains an element of the form $\eta = \bigotimes_{i\in J_\lambda^c} \bigotimes_{\nu=1}^{m_i} \left(T_{\rho_\nu^{(i)}\alpha_i^\vee}^{I\setminus\{i\}}; 0, 1\right)$, with $\rho_{m_i}^{(i)} = 0$, $i \in J_\lambda^c$. Since there exists a defining chain for η, we see from Lemma 3.9 (1) that $\rho^{(i)} = (\rho_1^{(i)} \geqslant \rho_2^{(i)} \geqslant \cdots \geqslant \rho_{m_i-1}^{(i)})$ is a partition of length less than m_i for each $i \in I$; here, we set $\rho^{(i)} = \varnothing$ if $i \in J_\lambda$. Hence $\boldsymbol{\rho} = (\rho^{(i)})_{i\in I} \in \mathrm{Par}(\lambda)$. We have $\Psi_\lambda(z^{-\boldsymbol{\rho}}\tilde{u}_\lambda) = \eta \in C$ by Lemma 3.8 (1), which proves (ii). □

4. Semi-infinite Young tableaux

Throughout this section, we will make the following assumptions: $\mathfrak{g}_{\mathrm{af}}$ is of type $A_{n-1}^{(1)}$ (see [9, § 4.8 TABLE Aff1]), and $I = \{1, 2, \dots, n-1\}$ satisfies

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i - j \equiv \pm 1 \mod n, \\ 0 & \text{otherwise} \end{cases} \tag{48}$$

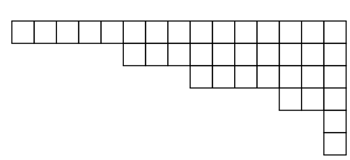
for all $i, j \in I_{\mathrm{af}} = \{0\} \sqcup I$. In this case, $\alpha_i^\vee = \alpha_i$, $i \in I_{\mathrm{af}}$, and hence $Q^\vee = Q$. We sometimes think of W as the permutation group of $\{1, 2, \dots, n\}$, namely the symmetric group of degree n, where r_i, $i \in I$, acts as the transposition $(i\ i+1)$. Observe that this action extends to the W_{af}-action, where r_0 acts as the transposition $(n\ 1)$; note that each t_ξ, $\xi \in Q$, acts as the identity.

4.1. Semi-infinite Young tableaux and isomorphism theorem.

We identify each element $\lambda = \sum_{i\in I} m_i\varpi_i \in P^+$ with the Young diagram such that the number of the columns of length i is m_i for each $i \in I$. A column-strict tableau of shape $\lambda \in P^+$ with entries in $\{1, 2, \dots, n\}$ is, by definition, an assignment of a number in $\{1, 2, \dots, n\}$ to each box of the Young diagram λ such that the entries are strictly increasing from top to bottom in each column. Let $\mathrm{CST}(\lambda)$ be the set of column-strict tableaux of shape λ with entries in $\{1, 2, \dots, n\}$. For a tuple $(\mathsf{T}_1, \mathsf{T}_2, \dots, \mathsf{T}_M)$ of column-strict tableaux of one-column shapes, let $\prod_{\nu=1}^M \mathsf{T}_\nu = \mathsf{T}_1\mathsf{T}_2\cdots\mathsf{T}_M$ denote the column-strict tableau whose ν-th column is T_ν. For $\mathsf{T} \in \mathrm{CST}(\varpi_i)$, let $\mathsf{T}(s) \in \{1, 2, \dots, n\}$, $1 \leqslant s \leqslant i$, denote the s-th entry (from top) of T.

Remark 4.1. In this paper, we consider a Young diagram as a collection of boxes, arranged in right-justified rows, with a weakly decreasing number of boxes in each row from top to bottom. For example, the Young diagram $\lambda = 5\varpi_1 + 3\varpi_2 + 4\varpi_3 + 2\varpi_4 + \varpi_6$ is as follows:

(49)



Definition 4.2.

(1) *Define the partial order $\preceq$ on $\mathrm{CST}(\varpi_i) \times \mathbb{Z}$ as follows: for $(\mathsf{T}, c), (\mathsf{T}', c') \in \mathrm{CST}(\varpi_i) \times \mathbb{Z}$, set $(\mathsf{T}, c) \preceq (\mathsf{T}', c')$ if*

$$\big(c \leqslant c'\big) \quad \text{and} \quad \big(\mathsf{T}(u) \leqslant \mathsf{T}'(u + c' - c) \text{ if } 1 \leqslant u \leqslant i - c' + c\big). \tag{50}$$

(2) *Let $\lambda = \sum_{i\in I} m_i\varpi_i \in P^+$, $m_i \in \mathbb{Z}_{\geqslant 0}$, $i \in I$, and $N = \sum_{i\in I} m_i$. Let*

$$\mathbb{T} = \left(\prod_{i=1}^{n-1} \prod_{\nu=1}^{m_i} \mathsf{T}_\nu^{(i)}, (c_\nu^{(i)})_{1\leqslant\nu\leqslant m_i, 1\leqslant i\leqslant n-1} \right) \in \mathrm{CST}(\lambda) \times \mathbb{Z}^N, \tag{51}$$

where $\mathsf{T}_\nu^{(i)} \in \mathrm{CST}(\varpi_i)$ and $c_\nu^{(i)} \in \mathbb{Z}$ for $1 \leqslant \nu \leqslant m_i$, $i \in I$. We call $\mathbb{T}$ a semi-infinite Young tableau of shape λ if

$$(\mathsf{T}_1^{(i)}, c_1^{(i)}) \succeq (\mathsf{T}_2^{(i)}, c_2^{(i)}) \succeq \cdots \succeq (\mathsf{T}_{m_i}^{(i)}, c_{m_i}^{(i)}) \text{ in } \mathrm{CST}(\varpi_i) \times \mathbb{Z} \text{ for every } i \in I. \tag{52}$$

Let $\mathbb{Y}^{\frac{\infty}{2}}(\lambda)$ be the set of semi-infinite Young tableaux of shape λ; note that $\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i) = \mathrm{CST}(\varpi_i) \times \mathbb{Z}$.

In § 4.3, we define a $\mathbf{U}$-crystal structure on $\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$, and prove the next proposition.

Proposition 4.3. *Let $i \in I$.*

1. *There exists a unique isomorphism $\Upsilon_i : \mathcal{B}(\varpi_i) \to \mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$ of $\mathbf{U}$-crystals.*
2. *We have $\mathcal{B}(\varpi_i) = \{u_x := S_x u_{\varpi_i} \mid x \in (W^{I\setminus\{i\}})_{\mathrm{af}}\}$, and the map $(W^{I\setminus\{i\}})_{\mathrm{af}} \to \mathcal{B}(\varpi_i)$, $x \mapsto u_x$, is bijective. In particular, $V(\varpi_i)$ is a minuscule representation of $\mathbf{U}$.*

Remark 4.4. It follows from Theorem 2.7 (2) and Proposition 4.3 that $\mathbb{B}^{\frac{\infty}{2}}(\varpi_i) = \{(x;0,1) \mid x \in (W^{I\setminus\{i\}})_{\mathrm{af}}\}$, and the map $\mathcal{B}(\varpi_i) \to \mathbb{B}^{\frac{\infty}{2}}(\varpi_i)$, $u_x \mapsto (x;0,1)$, $x \in (W^{I\setminus\{i\}})_{\mathrm{af}}$, equals the isomorphism in Theorem 2.7 (2).

Let $\lambda = \sum_{i\in I} m_i\varpi_i \in P^+$ and $N = \sum_{i\in I} m_i$. We have a bijection from $\bigotimes_{i=1}^{n-1} \mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)^{\otimes m_i}$ to $\mathrm{CST}(\lambda) \times \mathbb{Z}^N$ defined by

$$\bigotimes_{i=1}^{n-1}\bigotimes_{\nu=1}^{m_i}\left(\mathsf{T}_\nu^{(i)}, c_\nu^{(i)}\right) \longmapsto \left(\prod_{i=1}^{n-1}\prod_{\nu=1}^{m_i}\mathsf{T}_\nu^{(i)}, (c_\nu^{(i)})_{1\leqslant\nu\leqslant m_i, 1\leqslant i\leqslant n-1}\right), \tag{53}$$

where $(\mathsf{T}_\nu^{(i)}, c_\nu^{(i)}) \in \mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$, $1 \leqslant \nu \leqslant m_i$, $i \in I$. Define a $\mathbf{U}$-crystal structure on $\mathrm{CST}(\lambda) \times \mathbb{Z}^N$ to be such that the map (53) is an isomorphism of $\mathbf{U}$-crystals. From now on we assume that $\breve{\mathcal{B}}(\lambda) = \bigotimes_{i=1}^{n-1} \mathcal{B}(\varpi_i)^{\otimes m_i} = \mathcal{B}(\varpi_1)^{\otimes m_1} \otimes \cdots \otimes \mathcal{B}(\varpi_{n-1})^{\otimes m_{n-1}}$.

Theorem 4.5. *Let $\lambda = \sum_{i\in I} m_i\varpi_i \in P^+$ and $N = \sum_{i\in I} m_i$. Then, $\mathbb{Y}^{\frac{\infty}{2}}(\lambda)$ equals the image of the composition of the maps*

$$\mathcal{B}(\lambda) \xrightarrow[\S\ 3.1]{\Phi^{\mathrm{LT}}_{\lambda|q=0}} \breve{\mathcal{B}}(\lambda) = \bigotimes_{i=1}^{n-1}\mathcal{B}(\varpi_i)^{\otimes m_i} \xrightarrow[\text{Proposition } 4.3\ (1)]{\bigotimes_{i=1}^{n-1}\Upsilon_i^{\otimes m_i}} \bigotimes_{i=1}^{n-1}\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)^{\otimes m_i} \xrightarrow[(53)]{\cong} \mathrm{CST}(\lambda)\times\mathbb{Z}^N. \tag{54}$$

Since the map (54) is a strict embedding of $\mathbf{U}$-crystals, we have the following.

Corollary 4.6. *Let $\lambda = \sum_{i\in I} m_i\varpi_i \in P^+$ and $N = \sum_{i\in I} m_i$. Then, $\mathbb{Y}^{\frac{\infty}{2}}(\lambda)$ is stable under the Kashiwara operators on $\bigotimes_{i=1}^{n-1}\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)^{\otimes m_i} \cong \mathrm{CST}(\lambda)\times\mathbb{Z}^N$, and is isomorphic, as a $\mathbf{U}$-crystal, to the crystal basis $\mathcal{B}(\lambda)$.*

Theorem 4.5 follows from Theorem 3.4, Definition 4.2, Remark 4.4, and the following tableau criterion for the semi-infinite Bruhat order.

Theorem 4.7. *Let $i \in I$ and $x, y \in (W^{I\setminus\{i\}})_{\mathrm{af}}$. Write $\Upsilon_i(u_x) = (\mathsf{T}, c)$ and $\Upsilon_i(u_y) = (\mathsf{T}', c')$. The following conditions are equivalent:*

1. *$x \preceq y$ in $(W^{I\setminus\{i\}})_{\mathrm{af}}$.*
2. *$c \leqslant c'$ and $\mathsf{T}(u) \leqslant \mathsf{T}'(u + c' - c)$ if $1 \leqslant u \leqslant i - c' + c$.*

Theorem 4.7 will be proved in § 4.4.

4.2. Explicit description of $(W^J)_{\mathrm{af}}$. In this subsection, following [19, § 3], we give an explicit description of $(W^J)_{\mathrm{af}}$ for later use.

We take and fix $J = \bigsqcup_{m=1}^k I_m \subset I$, where $I_1, I_2, \ldots, I_k$ are the sets of vertices of the connected components of the Dynkin diagram of Δ_J; note that $\Delta_J = \bigsqcup_{m=1}^k \Delta_{I_m}$ and each Δ_{I_m}, $1 \leqslant m \leqslant k$, is of finite type A. Set $(I_m)_{\mathrm{af}} = \{0\} \sqcup I_m \subset I_{\mathrm{af}}$, $1 \leqslant m \leqslant k$. For $1 \leqslant s \leqslant t \leqslant n-1$, set $\alpha_{s,t} = \sum_{i=s}^t \alpha_i$; note that $\alpha_s = \alpha_{s,s}$. It follows that

$$\Delta = \{\pm\alpha_{s,t} \mid 1 \leqslant s \leqslant t \leqslant n-1\}. \tag{55}$$

Set

$$Q^J = \{\xi \in Q \mid (\xi, \alpha) \in \{-1, 0\} \text{ for all } \alpha \in \Delta_J^+\}. \tag{56}$$

Lemma 4.8 ([19, Equation (3.6)]). *For each $\xi \in Q$ there exist a unique $\phi_J(\xi) \in Q_J$ and a unique $(j_1, j_2, \ldots, j_k) \in \prod_{m=1}^k (I_m)_{\mathrm{af}}$ such that*

$$\xi + \phi_J(\xi) + \sum_{m=1}^{k} \varpi_{j_m} \in \bigoplus_{i \in I \setminus J} \mathbb{Z}\varpi_i \oplus \mathbb{C}\delta. \tag{57}$$

In particular, $\xi + \phi_J(\xi) \in Q^J$ for any $\xi \in Q$, and hence Q^J is a complete system of coset representatives for Q/Q_J.

For a subset $K \subset I$, let w_0^K be the longest element of W_K. For $j_m \in (I_m)_{\mathrm{af}}$, set

$$v_{j_m}^{I_m} = w_0^{I_m} w_0^{I_m \setminus \{j_m\}} \in W_{I_m} \subset W_J; \tag{58}$$

note that $v_0^{I_m} = e$. For $\xi \in Q$, define

$$z_\xi = z_\xi^J = v_{j_1}^{I_1} v_{j_2}^{I_2} \cdots v_{j_k}^{I_k} \in W_J, \tag{59}$$

where $(j_1, j_2, \ldots, j_k) \in \prod_{m=1}^k (I_m)_{\mathrm{af}}$, satisfying (57) for ξ, is determined uniquely by Lemma 4.8; note that $z_\xi = z_\zeta$ if $\xi \equiv \zeta \mod Q_J$.

Lemma 4.9 ([19, Lemma 3.7]). *We have $T_\xi = \Pi^J(t_\xi) = z_\xi t_{\xi + \phi_J(\xi)}$ for every $\xi \in Q$. Therefore, by Lemma 2.3, $\Pi^J(wt_\xi) = \lfloor w \rfloor z_\xi t_{\xi+\phi_J(\xi)}$ for every $w \in W$ and $\xi \in Q$, and we have a bijection $W^J \times Q^J \to (W^J)_{\mathrm{af}}$, $(w, \xi) \mapsto wT_\xi$. In particular,*

$$(W^J)_{\mathrm{af}} = \{wT_\xi = wz_\xi t_\xi \mid w \in W^J,\ \xi \in Q^J\}. \tag{60}$$

4.3. Crystal structure on $\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$. In this subsection, we define a $\mathbf{U}$-crystal structure on $\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$, and give a proof of Proposition 4.3.

We know from [3, Lemma 2.4.7] that

$$W^{I \setminus \{i\}} = \{w \in W \mid w(1) < w(2) < \cdots < w(i), \text{ and } w(i+1) < w(i+2) < \cdots < w(n)\}. \tag{61}$$

For $w \in W^{I \setminus \{i\}}$, set

$$\mathsf{T}_w = \begin{array}{|c|} \hline w(1) \\ \hline w(2) \\ \hline \vdots \\ \hline w(i) \\ \hline \end{array} \in \mathrm{CST}(\varpi_i). \tag{62}$$

By (61), we have $\mathrm{CST}(\varpi_i) = \{\mathsf{T}_w \mid w \in W^{I \setminus \{i\}}\}$ and the map $W^{I \setminus \{i\}} \to \mathrm{CST}(\varpi_i)$, $w \mapsto \mathsf{T}_w$, is bijective. Let $c_i(\xi)$ be the coefficient of $\alpha_i\ (= \alpha_i^\vee)$ in $\xi \in Q$. It follows from Lemma 4.8 that $Q^{I \setminus \{i\}} = \{c\alpha_i + \phi_{I \setminus \{i\}}(c\alpha_i) \mid c \in \mathbb{Z}\}$, and the maps $\mathbb{Z} \to Q^{I \setminus \{i\}}$, $c \mapsto c\alpha_i + \phi_{I \setminus \{i\}}(c\alpha_i)$, and $Q^{I \setminus \{i\}} \to \mathbb{Z}$, $\xi \mapsto c_i(\xi)$, are inverses of each other. We have thus proved that the map

$$\mathcal{Y}_i : (W^{I \setminus \{i\}})_{\mathrm{af}} \to \mathbb{Y}^{\frac{\infty}{2}}(\varpi_i),\ wT_\xi \mapsto \mathcal{Y}_i(wT_\xi) = (\mathsf{T}_w, c_i(\xi)), \tag{63}$$

is bijective, where $w \in W^{I \setminus \{i\}}$ and $\xi \in Q^{I \setminus \{i\}}$ (see Lemma 4.9).

Following [26, § 3.7] (see also [10, § 4.1]), we equip the set $\mathrm{CST}(\varpi_i)$ with a $\mathbf{U}'$-crystal structure as follows: let $\mathsf{T} \in \mathrm{CST}(\varpi_i)$. For $k \in \{1, 2, \ldots, n\}$, write $k \in \mathsf{T}$ if $\mathsf{T}(s) = k$ for some $1 \leqslant s \leqslant i$.

(i) Define $\mathrm{wt}(\mathsf{T}_w) = w\varpi_i \pmod{\mathbb{C}\delta}$ for $w \in W^{I \setminus \{i\}}$.
(ii) For $j \in I$, if $\mathsf{T}(s) = j+1$ and $j \notin \mathsf{T}$, then we define $e_j\mathsf{T} \in \mathrm{CST}(\varpi_i)$ to be such that $(e_j\mathsf{T})(s) = j$ and $(e_j\mathsf{T})(u) = \mathsf{T}(u)$ for $1 \leqslant u \leqslant i$, $u \neq s$.
(iii) If $1 \in \mathsf{T}$ and $n \notin \mathsf{T}$, then we define $e_0\mathsf{T} \in \mathrm{CST}(\varpi_i)$ to be such that $(e_0\mathsf{T})(i) = n$ and $(e_0\mathsf{T})(u) = \mathsf{T}(u+1)$ for $1 \leqslant u \leqslant i-1$.
(iv) Otherwise, we set $e_j\mathsf{T} = \mathbf{0}$ for $j \in I_{\mathrm{af}}$.
(v) For $j \in I$, if $\mathsf{T}(s) = j$ and $j+1 \notin \mathsf{T}$, then we define $f_j\mathsf{T} \in \mathrm{CST}(\varpi_i)$ to be such that $(f_j\mathsf{T})(s) = j+1$ and $(f_j\mathsf{T})(u) = \mathsf{T}(u)$ for $1 \leqslant u \leqslant i$, $u \neq s$.
(vi) If $1 \notin \mathsf{T}$ and $n \in \mathsf{T}$, then we define $f_0\mathsf{T} \in \mathrm{CST}(\varpi_i)$ to be such that $(f_0\mathsf{T})(1) = 1$ and $(f_0\mathsf{T})(u) = \mathsf{T}(u-1)$ for $2 \leqslant u \leqslant i$.
(vii) Otherwise, we set $f_j\mathsf{T} = \mathbf{0}$ for $j \in I_{\mathrm{af}}$.
(viii) Define

$$\varepsilon_j(\mathsf{T}) = \begin{cases} 1 & \text{if } e_j\mathsf{T} \neq \mathbf{0}, \\ 0 & \text{if } e_j\mathsf{T} = \mathbf{0}, \end{cases} \qquad \varphi_j(\mathsf{T}) = \begin{cases} 1 & \text{if } f_j\mathsf{T} \neq \mathbf{0}, \\ 0 & \text{if } f_j\mathsf{T} = \mathbf{0}. \end{cases} \tag{64}$$

Remark 4.10. The $\mathbf{U}'$-crystal $\mathrm{CST}(\varpi_i)$ defined above is isomorphic to the crystal basis of the $\mathbf{U}'$-module $W(\varpi_i)$ (see § 2.3). Indeed, we see from [14, Theorem 5.17 (ix)] (see also [24, Remark 3.3]) that $W(\varpi_i)$ is isomorphic to a Kirillov–Reshetikhin module, whose crystal basis is a perfect crystal of level 1 in the sense of [10, Definition 1.1.1]. It follows that the $\mathbf{U}'$-crystal $\mathrm{CST}(\varpi_i)$ and the crystal basis of $W(\varpi_i)$ satisfy the conditions in [10, Proposition 1.2.1 for $l = 1$], and hence they must be isomorphic to each other.

The set $\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i) = \mathrm{CST}(\varpi_i) \times \mathbb{Z}$ can be identified with the affinization of the $\mathbf{U}'$-crystal $\mathrm{CST}(\varpi_i)$ in the sense of [14, § 4.2]. We have thus obtained a $\mathbf{U}$-crystal structure on $\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$ as follows: for $w \in W^{I \setminus \{i\}}$, $c \in \mathbb{Z}$, $\mathsf{T} \in \mathrm{CST}(\varpi_i)$, and $j \in I_{\mathrm{af}}$,

$$\begin{cases} \mathrm{wt}(\mathsf{T}_w, c) = w\varpi_i - c\delta, \\ e_j(\mathsf{T}, c) = (e_j\mathsf{T}, c - \delta_{j,0}),\ f_j(\mathsf{T}, c) = (f_j\mathsf{T}, c + \delta_{j,0}), \\ \varepsilon_j(\mathsf{T}, c) = \varepsilon_j(\mathsf{T}),\ \varphi_j(\mathsf{T}, c) = \varphi_j(\mathsf{T}); \end{cases} \tag{65}$$

we understand that $(\mathbf{0}, c) = \mathbf{0}$. By (10), (63), and (65), we have $\mathrm{wt}(\mathcal{Y}_i(x)) = x\varpi_i$ for all $x \in (W^{I \setminus \{i\}})_{\mathrm{af}}$.

Proof of Proposition 4.3. (1): Since $\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$ is isomorphic to the affinization of the crystal basis of $W(\varpi_i)$ (see Remark 4.10), we see from [14, Proposition 5.4 (ii) and Theorem 5.17 (vii)] that $\mathcal{B}(\varpi_i)$ is isomorphic, as a $\mathbf{U}$-crystal, to $\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$. Note that $(\mathsf{T}_e, 0) \in \mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$ (and hence $u_{\varpi_i} \in \mathcal{B}(\varpi_i)$) is a unique element of weight ϖ_i; indeed, by (65), $\mathrm{wt}(\mathsf{T}_w, c) = \varpi_i$ holds if and only if $w = e$ and $c = 0$. This and the connectedness of $\mathcal{B}(\varpi_i)$ (see [14, Proposition 5.4 (ii)]) prove the uniqueness of the isomorphism between $\mathcal{B}(\varpi_i)$ and $\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$.

(2): By (1) and (64)–(65), we see that $\varepsilon_j(b), \varphi_j(b) \in \{0, 1\}$ for all $j \in I_{\mathrm{af}}$ and $b \in \mathcal{B}(\varpi_i)$. Hence $e_jb = S_{r_j}b$ (resp. $f_jb = S_{r_j}b$) if $e_jb \neq \mathbf{0}$ (resp. $f_jb \neq \mathbf{0}$) for $j \in I_{\mathrm{af}}$ and $b \in \mathcal{B}(\varpi_i)$. Since $\mathcal{B}(\varpi_i)$ is connected ([14, Proposition 5.4 (ii)]), this proves that the action of W_{af} is transitive. We have $\{x \in W_{\mathrm{af}} \mid S_xu_{\varpi_i} = u_{\varpi_i}\} = (W_{I \setminus \{i\}})_{\mathrm{af}}$ (see (23)) by [6, Proposition 5.1.1], and hence $\mathcal{B}(\varpi_i) = \{S_xu_{\varpi_i} \mid x \in (W^{I \setminus \{i\}})_{\mathrm{af}}\}$ by Lemma 2.2. □

4.4. Tableau criterion for semi-infinite Bruhat order. This subsection is devoted to the proof of Theorem 4.7.

We take and fix $i \in I$. It is easily seen from (55) that

$$\Delta^+ \setminus \Delta^+_{I \setminus \{i\}} = \{\alpha_{s,t} \mid 1 \leqslant s \leqslant i \leqslant t \leqslant n-1\}. \tag{66}$$

We have

$$\Upsilon_i(u_x) = \mathcal{Y}_i(x) \text{ for all } x \in (W^{I \setminus \{i\}})_{\mathrm{af}} \tag{67}$$

(see Proposition 4.3 (1) and (63)) because both elements are of weight $x\varpi_i$, and there is only one element of weight $x\varpi_i$ in $\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$.

Proposition 4.11. *Let $w \in W^{I \setminus \{i\}}$, $\xi \in Q^{I \setminus \{i\}}$, $\beta = w\gamma + \chi\delta \in \Delta^+_{\mathrm{af}}$, $\gamma = \alpha_{s,t} \in \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$, $1 \leqslant s \leqslant i \leqslant t \leqslant n-1$, and $\chi \in \mathbb{Z}_{\geqslant 0}$. Write $\mathcal{Y}_i(wT_\xi) = (\mathsf{T}, c)$ and $\mathcal{Y}_i\left(\Pi^{I \setminus \{i\}}(r_\beta w T_\xi)\right) = (\mathsf{T}', c')$. Then $r_\beta w T_\xi \in (W^{I \setminus \{i\}})_{\mathrm{af}}$ and there exists an edge $wT_\xi \xrightarrow{\beta} r_\beta w T_\xi$ in $\mathrm{SiB}^{I \setminus \{i\}}$ if and only if one of the following conditions holds:*

- (B) *$c' = c$, $\mathsf{T}'(s) = \mathsf{T}(s) + 1$, and $\mathsf{T}'(u) = \mathsf{T}(u)$ for $1 \leqslant u \leqslant i$, $u \neq s$.*
- (Q) *$c' = c + 1$, $\mathsf{T}'(1) = 1$, $\mathsf{T}'(u) = \mathsf{T}(u-1)$ for $2 \leqslant u \leqslant i$, and $\mathsf{T}(i) = n$.*

Remark 4.12. Under the assumptions of Proposition 4.11, the following holds:

1. (B) is equivalent to $w(s) \in I$ and $(\mathsf{T}', c') = f_{w(s)}(\mathsf{T}, c)$ in $\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$; note that $\mathsf{T} = \mathsf{T}_w$ and $\mathsf{T}' = \mathsf{T}_{wr_\gamma}$ in this case.
2. (Q) is equivalent to $(\mathsf{T}', c') = f_0(\mathsf{T}, c)$ in $\mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$; note that $\mathsf{T} = \mathsf{T}_w$ and $\mathsf{T}' = \mathsf{T}_{\lfloor wr_\gamma \rfloor}$ in this case.

Proposition 4.11 is established by combining Proposition 2.5 and Lemmas 4.13–4.15 below.

Lemma 4.13. *Under the assumptions of Proposition 4.11, we have the following:*

1. *$\chi = 0$ is equivalent to $c' = c$.*
2. *$\chi = 1$ is equivalent to $c' = c + 1$.*

Proof. It suffices to prove that $c' = c + \chi$. We have $r_\beta w T_\xi = r_{w\gamma} t_{\chi w\gamma} w z_\xi t_\xi = w r_\gamma z_\xi t_{\xi + \chi z_\xi^{-1}\gamma}$, and hence $\Pi^{I \setminus \{i\}}(r_\beta w T_\xi) = \lfloor wr_\gamma \rfloor T_{\xi + \chi z_\xi^{-1}\gamma}$ by Lemma 2.3. This gives $c' = c_i(\xi + \chi z_\xi^{-1}\gamma) = c_i(\xi) + \chi c_i(z_\xi^{-1}\gamma) = c + \chi c_i(z_\xi^{-1}\gamma)$. Since $z_\xi^{-1} \in W_{I \setminus \{i\}}$, it follows that $z_\xi^{-1}\gamma \in \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$. Therefore $c_i(z_\xi^{-1}\gamma) = 1$ by (66). □

Lemma 4.14 ([3, Proposition 2.4.8]). *Let $w \in W^{I \setminus \{i\}}$ and $\gamma = \alpha_{s,t} \in \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$, $1 \leqslant s \leqslant i \leqslant t \leqslant n-1$. The following conditions are equivalent:*

1. *$\ell(wr_\gamma) = \ell(w) + 1$.*
2. *$wr_\gamma \in W^{I \setminus \{i\}}$, $wr_\gamma(s) = w(s) + 1$, and $wr_\gamma(u) = w(u)$ for $1 \leqslant u \leqslant i$, $u \neq s$.*

Lemma 4.15. *Let $w \in W^{I \setminus \{i\}}$ and $\gamma \in \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$. The following conditions are equivalent:*

1. *$\ell(\lfloor wr_\gamma \rfloor) = \ell(w) + 1 - 2(\gamma, \rho - \rho_{I \setminus \{i\}})$.*
2. *$\lfloor wr_\gamma \rfloor(1) = 1$, $\lfloor wr_\gamma \rfloor(u) = w(u-1)$ for $2 \leqslant u \leqslant i$, and $w(i) = n$.*

For the proof of Lemma 4.15, we need the following lemma. Let $(i_1\ i_2\ \cdots\ i_l) \in W$ denote the cyclic permutation $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_l \mapsto i_1$, where $l \in \mathbb{Z}_{\geqslant 2}$ and $i_1, i_2, \ldots, i_l \in \{1, 2, \ldots, n\}$ are all distinct.

Lemma 4.16.

(1) $Q^{I\smallsetminus\{i\}} \cap (\Delta^+ \smallsetminus \Delta^+_{I\smallsetminus\{i\}}) = \{\alpha_i\}$.

(2) $2(\gamma, \rho - \rho_{I\smallsetminus\{i\}}) = n$ *for* $\gamma \in \Delta^+ \smallsetminus \Delta^+_{I\smallsetminus\{i\}}$.

(3) $\ell(w) = (w(1)-1)i + \sum_{u=2}^{i}(w(u) - w(u-1) - 1)(i-u+1)$ *for* $w \in W^{I\smallsetminus\{i\}}$.

(4) $z_{\alpha_i}^{I\smallsetminus\{i\}} = (1\ 2\ \cdots\ i)(n\ n-1\ \cdots\ i+1)$.

Proof. (1): It is clear that $\alpha_i \in Q^{I\smallsetminus\{i\}} \cap (\Delta^+ \smallsetminus \Delta^+_{I\smallsetminus\{i\}})$. Let $\gamma = \alpha_{s,t} \in \Delta^+ \smallsetminus \Delta^+_{I\smallsetminus\{i\}}$, $1 \leqslant s \leqslant i \leqslant t \leqslant n-1$. If $s < i$ (resp. $i < t$), then $\alpha_s \in \Delta^+_{I\smallsetminus\{i\}}$ (resp. $\alpha_t \in \Delta^+_{I\smallsetminus\{i\}}$) and $(\gamma, \alpha_s) = 1$ (resp. $(\gamma, \alpha_t) = 1$). This proves that $\gamma \notin Q^{I\smallsetminus\{i\}}$ unless $s = t = i$.

(2): The assertion follows from $2(\xi, \rho - \rho_{I\smallsetminus\{i\}}) = 0$ for $\xi \in Q_{I\smallsetminus\{i\}}$, $\gamma \equiv \alpha_i \mod Q_{I\smallsetminus\{i\}}$ for $\gamma \in \Delta^+ \smallsetminus \Delta^+_{I\smallsetminus\{i\}}$, and $2(\alpha_i, \rho - \rho_{I\smallsetminus\{i\}}) = 2(\alpha_i, \rho) - 2(\alpha_i, \rho_{I\smallsetminus\{i\}}) = 2 + \#(I \smallsetminus \{i\}) = n$.

(3): This is an immediate consequence of (61) and the fact that the length of a permutation equals the number of its inversions (see [3, Proposition 1.5.2]).

(4): Let $I_1 = \{1, \dots, i-1\}$ and $I_2 = \{i+1, \dots, n-1\}$ be connected components of $I \smallsetminus \{i\}$. We see that $(i-1, i+1) \in (I_1)_{\mathrm{af}} \times (I_2)_{\mathrm{af}}$ satisfies the condition in Lemma 4.8 for α_i, because $\alpha_i \in Q^{I\smallsetminus\{i\}}$ by (1), $(\alpha_i, \alpha_{i-1}) = -1$ if $1 < i$, and $(\alpha_i, \alpha_{i+1}) = -1$ if $i < n-1$. Therefore

$$z_{\alpha_i}^{I\smallsetminus\{i\}} = v_{i-1}^{I_1} v_{i+1}^{I_2} = w_0^{I_1} w_0^{I_1\smallsetminus\{i-1\}} w_0^{I_2} w_0^{I_2\smallsetminus\{i+1\}}. \tag{68}$$

Now the assertion is shown by the fact that the longest element of the symmetric group of degree N is the permutation $j \mapsto N - j + 1$, $j \in \{1, 2, \dots, N\}$. □

Proof of Lemma 4.15. We see from [18, Proof of Theorem 10.16] that (1) is equivalent to

(3) $\ell(wr_\gamma) = \ell(w) + 1 - 2(\gamma, \rho)$ and $wr_\gamma t_\gamma \in (W^{I\smallsetminus\{i\}})_{\mathrm{af}}$.

It follows immediately from Lemmas 4.9 and 4.16 (1) that (3) is equivalent to

(4) $\gamma = \alpha_i$, $\ell(w) = \ell(wr_i) + 1$ and $wr_i = \lfloor wr_i \rfloor z_{\alpha_i}^{I\smallsetminus\{i\}}$.

Let us prove that (1) (and (4)) imply (2). By (4) and Lemma 4.16 (4), we have

$$\lfloor wr_i \rfloor = wr_i (z_{\alpha_i}^{I\smallsetminus\{i\}})^{-1} = w(i\ i+1)(i\ \cdots\ 2\ 1)(i+1\ \cdots\ n-1\ n). \tag{69}$$

Hence $\lfloor wr_i \rfloor(1) = w(i+1)$.

We first assume that $i = 1$. Then $\lfloor wr_1 \rfloor(1) = w(2)$. The condition $\ell(w) = \ell(wr_1) + 1$ in (4) shows, by [3, Proposition 1.5.3], that $w(1) > w(2)$. Since $w \in W^{I\smallsetminus\{1\}}$, it follows from (61) that $w(2) = 1$ and, in consequence, $\lfloor wr_1 \rfloor(1) = 1$. Since $\lfloor wr_1 \rfloor \in W^{I\smallsetminus\{1\}}$, this implies that $\lfloor wr_1 \rfloor = e$ and hence $w = (n\ n-1\ \cdots\ 2\ 1)$ by (69). This gives $w(1) = n$.

We next assume that $1 < i \leqslant n-1$. By (69), $\lfloor wr_i \rfloor(u) = w(u-1)$ for $2 \leqslant u \leqslant i$. As $\lfloor wr_i \rfloor \in W^{I\smallsetminus\{i\}}$ we have $1 \leqslant \lfloor wr_i \rfloor(1) < \lfloor wr_i \rfloor(2) = w(1)$. Since $w \in W^{I\smallsetminus\{i\}}$, we see from (61) that $w(i+1) = 1$, and so $\lfloor wr_i \rfloor(1) = 1$. It follows from Lemma 4.16 (3) that

$$\ell(\lfloor wr_i \rfloor) = (w(1) - 2)(i-1) + \sum_{u=2}^{i-1} (w(u) - w(u-1) - 1)(i-u), \tag{70}$$

$$\ell(w) = (w(1) - 1)i + \sum_{u=2}^{i} (w(u) - w(u-1) - 1)(i-u+1), \tag{71}$$

which gives $\ell(\lfloor wr_i \rfloor) - \ell(w) = 1 - w(i)$. By (1), (4) and Lemma 4.16 (2), we have $\ell(\lfloor wr_i \rfloor) - \ell(w) = 1 - 2(\alpha_i, \rho - \rho_{I\smallsetminus\{i\}}) = 1 - n$, and consequently $w(i) = n$.

Finally, we prove that (2) implies (1). In a way similar to the above, we have $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = 1 - w(i) = 1 - n$. Lemma 4.16 (2) now shows that (1) holds. □

Proof of Theorem 4.7. If $x \preceq y$, then $c \leqslant c'$ by Proposition 4.11. Therefore, we may assume that $d := c' - c \geqslant 0$. The proof is by induction on d.

If $d = 0$, then it is obvious from Proposition 4.11 that $x \preceq y$ is equivalent to $\mathsf{T}(u) \leqslant \mathsf{T}'(u)$ for all $1 \leqslant u \leqslant i$.

Let $d > 0$. We first assume that $\mathsf{T}(u) \leqslant \mathsf{T}'(u+d)$ if $1 \leqslant u \leqslant i-d$, and show that $x \preceq y$. Let $x_1, x_2 \in (W^{I \setminus \{i\}})_{\mathrm{af}}$ be such that $\mathcal{Y}_i(x_1) = (\mathsf{T}_1, c'), \mathcal{Y}_i(x_2) = (\mathsf{T}_2, c'-1) \in \mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$, where

(1) $\mathsf{T}_1(1) = 1$ and

$$\mathsf{T}_1(u) = \begin{cases} \mathsf{T}'(u) & \text{if } \mathsf{T}'(u) < n-i+u, \\ \mathsf{T}'(u) - 1 & \text{if } \mathsf{T}'(u) = n-i+u \end{cases} \tag{72}$$

for $2 \leqslant u \leqslant i$,

(2) $\mathsf{T}_2(u) = \mathsf{T}_1(u+1)$ for $1 \leqslant u \leqslant i-1$, and $\mathsf{T}_2(i) = n$.

By Proposition 4.11, we have $x_2 \prec x_1 \preceq y$. If we prove that

$$\mathsf{T}(u) \leqslant \mathsf{T}_2(u+d-1) \text{ if } 1 \leqslant u \leqslant i-(d-1), \tag{73}$$

then $x \preceq x_2$ by induction hypothesis, and hence $x \preceq y$. Note that

$$\begin{aligned} &\mathsf{T}_2(u+d-1) \\ &= \begin{cases} \mathsf{T}'(u+d) & \text{if } 1 \leqslant u+d-1 \leqslant i-1 \text{ and } \mathsf{T}'(u+d) < n-i+u+d, \\ \mathsf{T}'(u+d) - 1 & \text{if } 1 \leqslant u+d-1 \leqslant i-1 \text{ and } \mathsf{T}'(u+d) = n-i+u+d, \\ n & \text{if } u+d-1 = i. \end{cases} \end{aligned} \tag{74}$$

We prove (73) as follows.

(1) If $1 \leqslant u+d-1 \leqslant i-1$ and $\mathsf{T}'(u+d) < n-i+u+d$, then $\mathsf{T}(u) \leqslant \mathsf{T}'(u+d) = \mathsf{T}_2(u+d-1)$.

(2) If $1 \leqslant u+d-1 \leqslant i-1$ and $\mathsf{T}'(u+d) = n-i+u+d$, then $\mathsf{T}_2(u+d-1) - \mathsf{T}(u) = \mathsf{T}'(u+d) - 1 - \mathsf{T}(u) \geqslant (n-i+u+d) - 1 - (n-i+u) = d-1 \geqslant 0$.

(3) If $u+d-1 = i$, then $\mathsf{T}(u) \leqslant n = \mathsf{T}_2(u+d-1)$.

We next assume that $x \preceq y$, and show that $\mathsf{T}(u) \leqslant \mathsf{T}'(u+d)$ if $1 \leqslant u \leqslant i-d$. We see from Proposition 4.11 that there exist $x_3, x_4 \in (W^{I \setminus \{i\}})_{\mathrm{af}}$ such that

(1) $x \preceq x_4 \prec x_3 \preceq y$,

(2) $\mathcal{Y}_i(x_3) = (\mathsf{T}_3, c'), \mathcal{Y}_i(x_4) = (\mathsf{T}_4, c'-1) \in \mathbb{Y}^{\frac{\infty}{2}}(\varpi_i)$,

(3) $\mathsf{T}_3(u) \leqslant \mathsf{T}'(u)$ for $1 \leqslant u \leqslant i$,

(4) $\mathsf{T}_3(1) = 1$, $\mathsf{T}_3(u+1) = \mathsf{T}_4(u)$ for $1 \leqslant u \leqslant i-1$, and $\mathsf{T}_4(i) = n$.

By induction hypothesis, $\mathsf{T}(u) \leqslant \mathsf{T}_4(u+d-1)$ if $1 \leqslant u \leqslant i-(d-1)$. We have $\mathsf{T}'(u+d) - \mathsf{T}(u) \geqslant \mathsf{T}_3(u+d) - \mathsf{T}_4(u+d-1) = 0$ if $1 \leqslant u \leqslant i-d$. □

4.5. Explicit description of crystal structure on $\mathbb{Y}^{\frac{\infty}{2}}(\lambda)$.

Proposition 4.17. *Let $\lambda = \sum_{i \in I} m_i \varpi_i$, $m_i \in \mathbb{Z}_{\geqslant 0}$, $i \in I$, and set $N = \sum_{i \in I} m_i$. Let $\mathbb{T} = (\mathsf{T}_1 \mathsf{T}_2 \cdots \mathsf{T}_N, (c_1, c_2, \ldots, c_N)) \in \mathrm{CST}(\lambda) \times \mathbb{Z}^N$ and $j \in I_{\mathrm{af}}$. Then $\mathrm{wt}(\mathbb{T}) \in P_{\mathrm{af}}$, $\varepsilon_j(\mathbb{T}), \varphi_j(\mathbb{T}) \in \mathbb{Z}$, and $e_j\mathbb{T}, f_j\mathbb{T} \in \mathrm{CST}(\lambda) \times \mathbb{Z}^N \sqcup \{\mathbf{0}\}$ are computed by the following procedure:*

(i) $\mathrm{wt}(\mathbb{T}) = \sum_{\nu=1}^{N} \mathrm{wt}(\mathsf{T}_\nu) - \sum_{\nu=1}^{N} c_\nu \delta$.

(ii) *Let* $\mathsf{T} \in \mathrm{CST}(\varpi_i)$*. If* $j \in I$*, then define* $\epsilon^{(j)}(\mathsf{T}) \in \{\oplus, \ominus, \bullet\}$ *by*

$$\epsilon^{(j)}(\mathsf{T}) = \begin{cases} \oplus & \text{if } j \in \mathsf{T} \text{ and } j+1 \notin \mathsf{T}, \\ \ominus & \text{if } j \notin \mathsf{T} \text{ and } j+1 \in \mathsf{T}, \\ \bullet & \text{otherwise.} \end{cases} \tag{75}$$

Likewise, define $\epsilon^{(0)}(\mathsf{T}) \in \{\oplus, \ominus, \bullet\}$ *by*

$$\epsilon^{(0)}(\mathsf{T}) = \begin{cases} \oplus & \text{if } n \in \mathsf{T} \text{ and } 1 \notin \mathsf{T}, \\ \ominus & \text{if } n \notin \mathsf{T} \text{ and } 1 \in \mathsf{T}, \\ \bullet & \text{otherwise.} \end{cases} \tag{76}$$

(iii) *In* $\left(\epsilon^{(j)}(\mathsf{T}_1), \dots, \epsilon^{(j)}(\mathsf{T}_N)\right)$*, continue replacing a pair* $\left(\epsilon^{(j)}(\mathsf{T}_\nu), \epsilon^{(j)}(\mathsf{T}_{\nu'})\right) = (\oplus, \ominus)$ *with* $(\bullet, \bullet)$ *if* $\nu < \nu'$ *and* $\epsilon^{(j)}(\mathsf{T}_\mu) = \bullet$ *for all* $\nu < \mu < \nu'$ *until no such pair exists. Let* $\epsilon^{(j)}(\mathbb{T}) \in \{\oplus, \ominus, \bullet\}^N$ *be the resulting tuple such that no* $\oplus$ *placed to the left of* $\ominus$*.*

(iv) $\varepsilon_j(\mathbb{T})$ *(resp.* $\varphi_j(\mathbb{T})$*) equals the number of* $\ominus$ *(resp.* $\oplus$*) in* $\epsilon^{(j)}(\mathbb{T})$*.*

(v) *If* $\ominus$ *is not in* $\epsilon^{(j)}(\mathbb{T})$*, then* $e_j \mathbb{T} = \mathbf{0}$*. If there exists* $\ominus$ *in* $\epsilon^{(j)}(\mathbb{T})$*, and the right-most* $\ominus$ *is at the* ν*-th place, then*

$$e_j \mathbb{T} = \left(\mathsf{T}_1 \cdots \mathsf{T}_{\nu-1} \left(e_j \mathsf{T}_\nu\right) \mathsf{T}_{\nu+1} \cdots \mathsf{T}_N, \left(c_1, \dots, c_{\nu-1}, c_\nu - \delta_{j,0}, c_{\nu+1}, \dots, c_N\right)\right). \tag{77}$$

(vi) *If* $\oplus$ *is not in* $\epsilon^{(j)}(\mathbb{T})$*, then* $f_j \mathbb{T} = \mathbf{0}$*. If there exists* $\oplus$ *in* $\epsilon^{(j)}(\mathbb{T})$*, and the left-most* $\oplus$ *is at the* ν*-th place, then*

$$f_j \mathbb{T} = \left(\mathsf{T}_1 \cdots \mathsf{T}_{\nu-1} \left(f_j \mathsf{T}_\nu\right) \mathsf{T}_{\nu+1} \cdots \mathsf{T}_N, \left(c_1, \dots, c_{\nu-1}, c_\nu + \delta_{j,0}, c_{\nu+1}, \dots, c_N\right)\right). \tag{78}$$

Proof. Let $\mathsf{T} \in \mathrm{CST}(\varpi_i)$. We check at once that the following holds:

(1) $\epsilon^{(j)}(\mathsf{T}) = \oplus$ if and only if $\varepsilon_j(\mathsf{T}) = 0$, $\varphi_j(\mathsf{T}) = 1$, and $\langle h_j, \mathrm{wt}(\mathsf{T})\rangle = 1$.
(2) $\epsilon^{(j)}(\mathsf{T}) = \ominus$ if and only if $\varepsilon_j(\mathsf{T}) = 1$, $\varphi_j(\mathsf{T}) = 0$, and $\langle h_j, \mathrm{wt}(\mathsf{T})\rangle = -1$.
(3) $\epsilon^{(j)}(\mathsf{T}) = \bullet$ if and only if $\varepsilon_j(\mathsf{T}) = \varphi_j(\mathsf{T}) = \langle h_j, \mathrm{wt}(\mathsf{T})\rangle = 0$.

Then the assertion follows by the same method as in [15, § 2.1]. ☐

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