ALGEBRAIC COMBINATORICS

Lucia Morotti

Composition factors of 2-parts spin representations of symmetric groups

Volume 3, issue 6 (2020), p. 1283-1291.

<http://alco.centre-mersenne.org/item/ALCO_2020__3_6_1283_0>

Algebraic Combinatorics
Volume 3, issue 6 (2020), p. 1283–1291
https://doi.org/10.5802/alco.137

Composition factors of 2-parts spin representations of symmetric groups

Lucia Morotti

Abstract Given an odd prime p, we identify possible composition factors of the reduction modulo p of spin irreducible representations of the covering groups of symmetric groups indexed by partitions with 2 parts and find some decomposition numbers.

1. Introduction

Let $\widetilde{\mathsf{S}}_n$ be a double cover of the symmetric group S_n. There exists $z \in \widetilde{\mathsf{S}}_n$ central of order 2 such that the sequence

$$1 \to \langle z \rangle \to \widetilde{\mathsf{S}}_n \to \mathsf{S}_n \to 1$$

is exact. Let F be a field of characteristic $p \neq 2$ and V be an irreducible representation of $F\widetilde{\mathsf{S}}_n$. Then z acts as ± 1 on V. If z acts as 1 then V may also be viewed as a representation of S_n, if on the other hand z acts as -1 we say that V is a spin representation of S_n. In this paper we will only consider spin representations in characteristic $\neq 2$.

It is well known that in characteristic 0 (pairs of) irreducible spin representations of the symmetric groups are labeled by partitions with distinct parts, that is strict partitions. Parametrization of the spin irreducible modules in positive characteristic have been conjectured by Leclerc and Thibon in [14] using crystal combinatorics of type $A_{p-1}^{(2)}$. The irreducible modules have been constructed using branching rules for Hecke–Clifford superalgebras in [6], and using algebraic supergroups of type Q in [7], in particular proving the Leclerc–Thibon conjecture. In [13, Theorem B], the two constructions of irreducible modules have been proved to be equivalent.

If $p = 0$ let $\mathscr{RP}_p(n)$ denote the set of strict partitions of n. If $p \geqslant 3$ let $\mathscr{RP}_p(n)$ denote the set of p-strict p-restricted partitions of n, that is partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n such that, for any $r \geqslant 1$, $\lambda_r - \lambda_{r+1} \leqslant p - \delta_{p|\lambda_r}$ and that λ_r is divisible by p if $\lambda_r = \lambda_{r+1}$. Further in either of the previous cases, for any partition λ let $h_{p'}(\lambda)$ be the number of parts of λ which are not divisible by p. In [6, 7] the spin irreducible

Manuscript received 7th February 2020, revised and accepted 17th June 2020.

Keywords. Symmetric groups, spin representations, decomposition numbers.

Acknowledgements. The author was supported by the DFG grants MO 3377/1-1 and MO 3377/1-2.

ISSN: 2589-5486 http://algebraic-combinatorics.org/

modules $D(\lambda,\varepsilon)$ labeled by $\lambda \in \mathscr{RP}_p(n)$ are constructed (where $\varepsilon = 0$ or $\pm$), and it is proved that

$$\{D(\lambda,0)|\lambda \in \mathscr{RP}_p(n) \text{ and } n-h_{p'} \text{ is even}\}$$
$$\cup\,\{D(\lambda,+),D(\lambda,-)|\lambda \in \mathscr{RP}_p(n) \text{ and } n-h_{p'} \text{ is odd}\}$$

is a complete set of non-isomorphic irreducible spin representations of S_n. For $\lambda \in \mathscr{RP}_p(n)$ we define the supermodule $D(\lambda)$ to be either $D(\lambda,0)$ or $D(\lambda,+)\oplus D(\lambda,-)$ (depending on the parity of $n-h_{p'}(\lambda)$). If $p=0$ we will write $S(\lambda,\varepsilon)$ for $D(\lambda,\varepsilon)$ and $S(\lambda)$ for $D(\lambda)$.

Not much is known about decomposition matrices of spin representations of symmetric groups. Known results include the basic and second basic spin cases, see [17], some partial results for $p=3$ and 5, see [1, 3, 4] which use ideas from [15], leading modules appearing in $S(\lambda)$, see [7, 8, 9] and the weight 1 case, see [16]. For blocks of weight 2 it is expected, due to [14, Conjecture 6.2], that results similar to those in [10] hold.

In characteristic 2, which will not be considered here, no spin representation of S_n exists. Some results about decomposition matrices in this case, in particular about leading terms, can be found in [2, 5].

In this paper we will describe modules which are composition factors of reduction modulo p of some supermodules $S((\lambda_1,\lambda_2))$. For irreducible representations of symmetric groups an exact description of the composition factors, and their multiplicities, of the reduction modulo p of the modules $S^{(\lambda_1,\lambda_2)}$ had been obtained by James in [11]. Although we cannot in general obtain similar exact formulas for spin representations (or even tell if the possible composition factor have non-zero multiplicity), we can still describe the composition factors of $S((\lambda_1,\lambda_2))$ which are not composition factors of $S((\mu_1,\mu_2))$ with $\mu_1+\mu_2=\lambda_1+\lambda_2$ and $\mu_1>\lambda_1$ and compute their multiplicity in $S((\lambda_1,\lambda_2))$.

Before stating results about decomposition matrices, we need to define certain particular partitions. Let $n=bp+c$ with $0\leqslant c<p$. If $n=0$ define $\beta^n := ()$. If $n>0$ define $\beta^n := (p^b,c)$ if $c>0$ or $\beta^n := (p^{b-1},p-1,1)$ if $c=0$. Then $\beta^n \in \mathscr{RP}_p(n)$. Then β^n is the partition labeling the basic spin modules in characteristic p, see [12, Lemma 22.3.3]. Further let $\ell=\ell_p := (p-1)/2$. For $p\geqslant 5$ we define partitions μ^k for $1\leqslant k\leqslant \ell$ as follows:

c	k	μ^k
0	1	$(p^{b-2},p-1,p-2,2,1)$
0	$2\leqslant k\leqslant \ell$	$(p^{b-1},p-k,k)$
1	1	$(p^{b-1},p-2,2,1)$
1	$2\leqslant k\leqslant \ell$	$(p^{b-1},p+1-k,k)$
$2\leqslant c\leqslant p-2$	$1\leqslant k\leqslant \lceil c/2\rceil-1$	$(p^b,\lfloor c/2\rfloor+k,\lceil c/2\rceil-k)$
$2\leqslant c\leqslant p-2$	$\lceil c/2\rceil$	$(p^{b-1},p-1,c,1)$
$2\leqslant c\leqslant p-2$	$\lceil c/2\rceil+1\leqslant k\leqslant \ell$	$(p^{b-1},p+\lceil c/2\rceil-k,\lfloor c/2\rfloor+k)$
$p-1$	$1\leqslant k\leqslant \ell-1$	$(p^b,\ell+k,\ell-k)$
$p-1$	ℓ	$(p^{b-1},p-1,p-2,2)$

Table 1.

It can be checked that $\mu^k \in \mathscr{RP}_p(n)$ if $n \geqslant p \geqslant 5$ and $1 + \delta_{n=p} \leqslant k \leqslant \ell$.

If $n < p$ then $F\tilde{\mathsf{S}}_n$ is semisimple, so this case does not need to be considered. Due to partitions μ^k not being defined for $p = 3$, as well as μ^1 not being a partition when $n = p$, we will present results considering the cases $n = p$, $n > p = 3$ and $n > p \geqslant 5$ separately.

Theorem 1.1. *Let $n = p \geqslant 3$. Define $D_0 := D(\beta^p) = D((p-1, 1))$ and for $1 \leqslant j \leqslant \ell - 1$ define $D_j := D(\mu^{j+1}) = D((p-j-1, j+1))$. Further define $D_{-1}, D_\ell := 0$. Then $[S((p-j, j))] = [D_j] + [D_{j-1}]$ for $0 \leqslant j \leqslant \ell$.*

Theorem 1.2. *Let $n > p = 3$ and $m := \lfloor (n-1)/2 \rfloor - 1 - \delta_{n \equiv 3 \bmod 6}$. For $0 \leqslant j \leqslant m$ define $D_j := D(\beta^{n-j} + \beta^j)$. If $\lambda = (\lambda_1, \lambda_2) \in \mathscr{RP}_0(n)$ then any composition factor of the reduction modulo 3 of $S(\lambda)$ is of the form D_j with $0 \leqslant j \leqslant \min\{\lambda_2, m\}$. Further if $\lambda_2 \leqslant m$ then $[S(\lambda) : D_{\lambda_2}] = 2^a$ with $a = 1$ if at least one of the following holds:*

- *$\lambda_1 > \lambda_2 > 0$ are both divisible by 3,*
- *$\lambda_1 > \lambda_2 > 0$, one of them is divisible by 3 and n is odd,*
- *$\lambda_2 = 0$ and n is even and divisible by 3,*

or $a = 0$ else.

Theorem 1.3. *Let $n > p \geqslant 5$, $m := \lfloor (n-1)/2 \rfloor - \delta_{n \equiv p \bmod 2p}$ and $0 \leqslant c \leqslant p-1$ with $n \equiv c \bmod p$. For $1 \leqslant k \leqslant \ell$ let μ^k be as in Table I. Define*

$$D_j := \begin{cases} D(\beta^{n-j} + \beta^j), & \text{if } 0 \leqslant j \leqslant m - \ell, \\ D(\mu^{m+1-j}), & \text{if } m - \ell < j \leqslant m \text{ and } n + c \text{ is even}, \\ D(\mu^{\ell-m+j}), & \text{if } m - \ell < j \leqslant m \text{ and } n + c \text{ is odd}. \end{cases}$$

If $\lambda = (\lambda_1, \lambda_2) \in \mathscr{RP}_0(n)$ then any composition factor of the reduction modulo p of $S(\lambda)$ is of the form D_j with $0 \leqslant j \leqslant \min\{\lambda_2, m\}$. Further if $\lambda_2 \leqslant m$ then $[S(\lambda) : D_{\lambda_2}] = 2^a$ with $a = 1$ if at least one of the following holds:

- *$\lambda_1 > \lambda_2 > 0$ are both divisible by p,*
- *$\lambda_1 > \lambda_2 > 0$, one of them is divisible by p and n is odd,*
- *$\lambda_2 = 0$ and n is even and divisible by p,*
- *$n \equiv 0 \bmod 2p$ and $\lambda = (n/2 + 1, n/2 - 1)$,*

or $a = 0$ else.

In particular by Theorems 1.1, 1.2 and 1.3 we have that part of the decomposition matrix of the supermodules of $\widetilde{\mathsf{S}}_n$ is given by

$$\begin{array}{c|cccc} & D_0 & \cdots & D_{\overline{m}-b} & \text{other modules} \\ \hline S((n)) & a_0 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ S((n-\overline{m}+b, \overline{m}-b)) & * & & a_{\overline{m}-b} & 0 \\ S((n-\overline{m}+b-1, \overline{m}-b+1)) & * & \cdots & * & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ S((n-\overline{m}, \overline{m})) & * & \cdots & * & 0 \end{array}$$

with $\overline{m} = \lfloor (n-1)/2 \rfloor$, $0 \leqslant b \leqslant 2$ and $a_j \in \{1, 2\}$. Theorems 1.1 and 1.2 can be easily obtained from known results and will be proved in Sections 4 and 5. Theorem 1.3 will be proved in Section 6 after having studied certain projective modules in Section 3.

Maximal composition factors $D((n-j, j)^R)$ for the reduction modulo p of $S((n-j, j))$ (with respect to ordering according to the lexicographic order) can be obtained from [7, 8, 9] (see Lemma 2.1). It can be checked that the modules D_j for $j > \lfloor (n-1)/2\rfloor - \ell$ are not of the form $D((n-j,j)^R)$. So Theorem 1.3 gives new information about composition factors (and their multiplicities) of the reduction modulo p of the modules $S((n-j,j))$. It also further restricts the possible composition factors, since for example $D((p+1,p^a,2,1))$ cannot be a composition factor of $S((n-1,1))$ for $n=(a+1)p+4$ and $p\geqslant 5$ by Theorem 1.3, but it cannot be excluded as a composition factor using Lemma 2.1.

2. Basic lemmas

For $\lambda\in\mathscr{RP}_0(n)$ let $\lambda^R\in\mathscr{RP}_p(n)$ be the regularization of λ as defined in [9, Section 2]. In particular if $\lambda=(\lambda_1,\ldots,\lambda_h)$ with $\lambda_r-\lambda_{r+1}\geqslant p+\delta_{p|\lambda_r}$ for $1\leqslant r<h$ then $\lambda^R=\beta^{\lambda_1}+\cdots+\beta^{\lambda_h}$, while if $\lambda\in\mathscr{RP}_p(n)$ then $\lambda^R=\lambda$. The next lemma can be obtained combining [7, Theorem 10.8], [8, Theorem 10.4] and [9, Theorem 4.4]. For $\lambda\in\mathscr{RP}_p(n)$ let $a_p(\lambda):=0$ if $n-h_{p'}(\lambda)$ is even or $a_p(\lambda):=1$ if $n-h_{p'}(\lambda)$ is odd.

Lemma 2.1. *Let $p\geqslant 3$ and $\lambda\in\mathscr{RP}_0(n)$. Then*

$$[S(\lambda):D(\lambda^R)]=2^{(h(\lambda)-h_{p'}(\lambda)+a_0(\lambda)-a_p(\lambda^R))/2}$$

and if $\nu\in\mathscr{RP}_p(n)$ and $D(\nu)$ is a composition factor of the reduction modulo p of $S(\lambda)$ then $\nu\trianglelefteq\lambda^R$.

Normal nodes play an important role when considering branching (see for example [6, §9-a] for the definition of normal nodes). The next results will be used in the proof of Theorem 1.3.

Lemma 2.2. [13, Theorem 8.4.5] *Let $p\geqslant 3$ and $\lambda\in\mathscr{RP}_p(n)$. If A is a normal node of λ and $\lambda\setminus A\in\mathscr{RP}_p(n-1)$ then $D(\lambda\setminus A)$ is a composition factor of $D(\lambda){\downarrow_{\widetilde{\mathsf{S}}_{n-1}}}$.*

Lemma 2.3. [6, Theorem 9.13] *Let $p\geqslant 3$ and $\lambda\in\mathscr{RP}_p(n)$. If λ has a unique normal node A and A has residue 0 then $D(\lambda){\downarrow_{\tilde{\mathsf{S}}_{n-1}}}\cong D(\lambda\setminus A)$.*

3. Projective modules

In this section we will construct certain projective modules whose structure will play a major role in the proof of Theorem 1.3. This section uses ideas from [15] and extends results from [1, 3, 4] to characteristic $p\geqslant 5$ for two-parts partitions.

The content of a node (r,s) is given by $\min\{c-1,p-c\}$, where $1\leqslant c\leqslant p$ and $r\equiv c \mod p$. So nodes on any row have residues

$$0,1,\ldots,\ell-1,\ell,\ell-1,\ldots,1,0,0,1,\ldots,\ell-1,\ell,\ell-1,\ldots,1,0,\ldots.$$

The content of a partition is the multiset of the contents of its nodes. It is known that two partitions have the same content if and only if they have the same p-bar core. Further two irreducible modules are contained in the same block if and only if they are labeled by partitions with the same p-bar core (unless possibly if they are labeled by a p-bar core, in which case the weight is 0). In particular the content of a block is well defined. For $0\leqslant i\leqslant\ell$ and a $\widetilde{\mathsf{S}}_n$-module M contained in the block(s) with content I, let $\mathrm{Ind}_i\, M$ be the block component(s) of $M{\uparrow^{\widetilde{\mathsf{S}}_{n+1}}}$ corresponding to the block(s) with content $I\cup\{i\}$.

Lemma 3.1. *Let $p\geqslant 5$ and $m:=\lfloor (n-1)/2\rfloor-\delta_{n\equiv p \mod 2p}$. For $0\leqslant j\leqslant m$ there exist projective modules $[P_j:S(\mu)]=0$ if $\mu\in\mathscr{RP}_0(n)$ with $\mu_1>n-j$ and with $[P_j:S((n-j,j))]=2^a$, where $a=1$ if at least one of the following holds:*

- *$n-j>j>0$ are both divisible by p,*
- *$n-j>j>0$, one of them is divisible by p and n is even,*

- $j=0$ *and* n *is odd and divisible by* p,
- $n\equiv 0 \mod 2p$ *and* $(n-j,j)=(n/2+1,n/2-1)$,

or $a=0$ *else.*

Proof. For $I=(i_1,i_2,\ldots)$ let $P_I:=\cdots\operatorname{Ind}_{i_2}\operatorname{Ind}_{i_1}D_0(())$. Then P_I is projective. For $0\leqslant j\leqslant m$ we will now construct a residue sequence I_j and define a positive integer k_j. We will then show that there exist a projective modules P_j with $[P_j]=[P_{I_j}]/k_j$ for which the lemma holds.

Case 1: $j=0$. Let I_0 be the residues of (n) (taken starting from the node $(1,1)$ until the node $(1,n)$) and $k_0:=2^{\lfloor (n-1)/2\rfloor -a}$.

Case 2: $1\leqslant j\leqslant m$. Write $(n-j,j):=(bp+c,dp+e)$ with $2\leqslant c\leqslant p+1$ and $1\leqslant e\leqslant p$. The case $b=d$ and $(c,e)=(\ell+1,\ell)$ is excluded by assumption. Let I_j be obtained as concatenation of the following tuples: $(0,1,0)$, then d times the tuples $(2,3,\ldots,\ell)$, $(1,2,\ldots,\ell-1)$, $(\ell-1,\ell)$, $(\ell-2,\ell-3,\ldots,0)$, (0), $(\ell-1,\ell-2,\ldots,1)$, $(1,0,0)$ and then I_j' where I_j' is the concatenation of the following:

Case 2.1: $b>d$, $1\leqslant e\leqslant \ell$. $(2,3,\ldots,\ell)$, $(1,2,\ldots,e-1)$ followed by the residues of $(n-j)$, starting from that of the node $(1,dp+(p+3)/2)$. Further let $k_j:=2^{\lfloor (n-2)/2\rfloor+2d+\delta_{e=\ell}-a}$.

Case 2.2: $b>d$, $\ell+1\leqslant e\leqslant p-1$. $(2,3,\ldots,\ell)$, $(1,2,\ldots,\ell-1)$, $(\ell-1,\ell)$, $(\ell-2,\ell-3,\ldots,0)$, (0), $(\ell-1,\ell-2,\ldots,p-e)$ followed by the residues of $(n-j)$, starting from that of the node $(1,(d+1)p+2)$. Further let $k_j:=2^{\lfloor (n-2)/2\rfloor+2d+1+\delta_{e=p-1}-a}$.

Case 2.3: $b>d$, $e=p$. $(2,3,\ldots,\ell)$, $(1,2,\ldots,\ell-1)$,$(\ell-1,\ell)$, $(\ell-2,\ell-3,\ldots,0)$, (0), $(\ell-1,\ell-2,\ldots,1)$, $(1,0)$ followed by the residues of $(n-j)$, starting from that of the node $(1,(d+1)p+3)$. Further let $k_j:=2^{\lfloor (n-2)/2\rfloor+2d+2-a}$.

Case 2.4: $b=d$, $1\leqslant e\leqslant \ell-1$, $e<c\leqslant \ell+1$. $(2,3,\ldots,c-1)$, $(1,2,\ldots,e-1)$. Further let $k_j:=2^{\lfloor (n-2)/2\rfloor+2d}$.

Case 2.5: $b=d$, $1\leqslant e\leqslant \ell$, $\ell+2\leqslant c\leqslant p+1$. $(2,3,\ldots,\ell)$, $(1,2,\ldots,e-1)$ followed by the residues of $(n-j)$, starting from that of the node $(1,dp+(p+3)/2)$. Further let $k_j:=2^{\lfloor (n-2)/2\rfloor+2d+\delta_{e=\ell}-a}$.

Case 2.6: $b=d$, $\ell+1\leqslant e\leqslant p-1$, $e<c\leqslant p$. $(2,3,\ldots,\ell)$, $(1,2,\ldots,\ell-1)$, $(\ell-1,\ell)$, $(\ell-2,\ell-3,\ldots,p-c)$, $(\ell-1,\ell-2,\ldots,p-e)$. Further let $k_j:=2^{\lfloor (n-2)/2\rfloor+2d+1-a}$.

Case 2.7: $b=d$, $\ell+1\leqslant e\leqslant p-2$, $c=p+1$. $(2,3,\ldots,\ell)$, $(1,2,\ldots,\ell-1)$, $(\ell-1,\ell)$, $(\ell-2,\ell-3,\ldots,0)$, (0), $(\ell-1,\ell-2,\ldots,p-e)$. Further let $k_j:=2^{\lfloor (n-2)/2\rfloor+2d+1}$.

Case 2.8: $b=d$, $p-1\leqslant e\leqslant p$, $c=p+1$. $(2,3,\ldots,\ell)$, $(1,2,\ldots,\ell-1)$, $(\ell-1,\ell)$, $(\ell-2,\ell-3,\ldots,0)$, $(\ell-1,\ell-2,\ldots,p-e)$, (0). Further let $k_j:=2^{\lfloor (n-2)/2\rfloor+2d+1-a}$.

In any of the above cases, if $I_j=(i_1,i_2,\ldots)$ and $i_r=i_{r+1}$ for some r then $i_r\in\{0,1,\ell-1\}$ and, if it exists, $i_{r+2}\neq i_r$. Let x_j to be the number of successive tuples $(1,1)$ or $(\ell-1,\ell-1)$ which appear in I_j and y_j to be the number of single 0. If $[P_{I_j}]=\sum_{\mu\in\mathscr{RP}_0(n)}c_{j,\mu}[S(\mu)]$ then $c_{j,\mu}$ is divisible by $2^{\lfloor (n-h(\mu))/2\rfloor+x_j+\max\{0,h(\mu)-y_j\}}$, since to obtain μ from $()$ adding a single node at each step we have to switch $\lfloor (n-h(\mu))/2\rfloor$ times from partitions with $a_0(\nu)=1$ to partitions with $a_0(\nu)=0$ and whenever adding two nodes of either the same residues $\neq 0$ (so on different rows and far enough) or both of residue 0, one on a new row and the other on a different row their order can be exchanged (and there exists at least $x_j+\max\{0,h(\mu)-y_j\}$ such pairs). It can then be checked that $[P_{I_j}:S(\mu)]$ is divisible by k_j for each $\mu\in\mathscr{RP}_0(n)$.

Further it can be computed that $[P_{I_j} : S((n-j,j))] = 2^a k_j$ and that $[P_{I_j} : S(\mu)] = 0$ if $\mu_1 > n-j$.

The lemma then follows by taking P_j with $[P_j] = [P_{I_j}]/k_j$. $\square$

Remark 3.2. The sequences of residues I_j given above roughly correspond to adding nodes according to the following sequence (as long as nodes are contained in the partition $(n-j,j)$ and possibly with minor modifications at the end) given by

$$y_1, y_2, x_{1,1}, \ldots, x_{1,8}, x_{2,1}, \ldots, x_{2,8}, \ldots,$$

where the subsets of nodes y_i and $x_{1,i}$ are as follows

1	2	$3 \cdots \ell-1$	ℓ	$\ell+1$	$\ell+2$	$\ell+3 \cdots p-2$	$p-1$	p	$p+1$	$p+2$
y_1	y_1	$x_{1,1}$		$x_{1,1}$	$x_{1,3}$	$x_{1,5}$			$x_{1,5}$	$x_{1,7}$
y_2	$x_{1,2}$	$x_{1,2}$	$x_{1,4}$	$x_{1,4}$	$x_{1,6}$	$x_{1,6}$	$x_{1,8}$		$x_{1,8}$	

and the subsets $x_{c,i}$ are obtained by shifting $x_{1,i}$ to the right by $(c-1)p$ columns.

4. Proof of Theorem 1.1

From [16, Theorem 4.4] there exist simple supermodules E_j for $0 \leqslant j \leqslant \ell-1$ which are pairwise non-isomorphic such that, if $E_{-1} = E_\ell = 0$, then $[S((p-j,j))] = [E_j]+[E_{j-1}]$ for $0 \leqslant j \leqslant \ell$. The theorem then follows from Lemma 2.1.

5. Proof of Theorem 1.2

Note that for $p = 3$ we have that $\mu \in \mathscr{RP}_p(n)$ if and only if $\mu = \beta^{\nu_1} + \cdots + \beta^{\nu_h}$ with $(\nu_1, \ldots, \nu_h)$ a partition of n with $\nu_r - \nu_{r+1} \geqslant 3 + \delta_{3|\nu_r}$ for $1 \leqslant r < h$. Further, if $(\pi_1, \ldots, \pi_k)$ is also a partition of n with $\pi_r - \pi_{r+1} \geqslant 3 + \delta_{3|\pi_r}$ for $1 \leqslant r < k$, then $\beta^{\nu_1} + \cdots + \beta^{\nu_h} \trianglelefteq \beta^{\pi_1} + \cdots + \beta^{\pi_k}$ if and only if $(\nu_1, \ldots, \nu_h) \trianglerighteq (\pi_1, \ldots, \pi_k)$.

The theorem then holds by Lemma 2.1. See also [4, Theorem 4.1] for an alternative partial proof.

6. Proof of Theorem 1.3

We will first prove that any composition factor of $S(\lambda)$ with $\lambda \in \mathscr{RP}_0(n)$ with at most 2 rows is of the form D_j for some j. This will be done by induction on n (using Theorem 1.1 if $n = p+1$). Let $\lambda \in \mathscr{RP}_0(n)$ with at most 2 rows and $D(\mu)$ be a composition factor of the reduction modulo p of $S(\lambda)$. Then any composition factor of $D(\mu)\downarrow_{\widetilde{\mathsf{S}}_{n-1}}$ is a composition factor of some $S(\nu)$ with $\nu \in \mathscr{RP}_0(n-1)$ with at most 2 rows. In particular by Lemma 2.2 there exists $\psi \in \mathscr{RP}_p(n-1)$ such that $D(\psi)$ is a composition factor of some $S(\nu)$ with $\nu \in \mathscr{RP}_0(n-1)$ with at most 2 rows such that μ is obtained from ψ by adding an addable node. By induction we then have that $D(\mu) \cong D_j$ for some $0 \leqslant j \leqslant m$ or μ is of one of the following forms:

- $(p^a, b, c, 1)$ with $a \geqslant 0$, $1 < c < b < p-1$ and $(b,c) \neq (p-2,2)$,
- $(p+1, p^a, b, c)$ with $a \geqslant 0$, $0 < c < b < p$ and $(b,c) \neq (p-1,1)$,
- $(p+1, p^a, p-1, b, 1)$ with $a \geqslant 0$ and $1 < b < p-1$,
- $(p+1, p^a, p-2, 2, 1)$ with $a \geqslant 0$,
- $(p+1, p^a, p-1, p-2, 2)$ with $a \geqslant 0$,
- $(p+1, p^a, p-1, p-2, 2, 1)$ with $a \geqslant 0$,
- $(2p^a, b, p^c, d, 1)$ with $a, c \geqslant 0$, $p < b < 2p$ and $1 < d < p-1$,
- $(2p^a, 2p-1, p+1, p^c, d, 1)$ with $a, c \geqslant 0$ and $1 < d < p-1$,
- $(2p^a, b, p^c, p-1, 2)$ with $a, c \geqslant 0$ and $p < b < 2p$,
- $(2p^a, 2p-1, p+1, p^c, p-1, 2)$ with $a, c \geqslant 0$,

- $(2p^a, b, p+1, p^c, d)$ with $a, c \geqslant 0$, $p+1 < b < 2p-1$ and $1 < d < p-1$,
- $(2p^a, 2p-1, p+2, p^c, d)$ with $a, c \geqslant 0$ and $1 < d < p-1$,
- $(2p^a, b, p+1, p^c, p-1, 1)$ with $a, c \geqslant 0$ and $p+1 < b < 2p-1$,
- $(2p^a, 2p-1, p+2, p^c, p-1, 1)$ with $a, c \geqslant 0$,
- $(2p+1, 2p^a, b, p^c, d)$ with $a, c \geqslant 0$, $p < b < 2p$ and $0 < d < p$,
- $(2p+1, 2p^a, 2p-1, p+1, p^c, d)$ with $a, c \geqslant 0$ and $0 < d < p$,
- $(2p+1, 2p^a, b, p^c, p-1, 1)$ with $a, c \geqslant 0$ and $p < b < 2p$,
- $(2p+1, 2p^a, 2p-1, p+1, p^c, p-1, 1)$ with $a, c \geqslant 0$.

It can be easily checked that in each of the above cases there exists a normal node B with $\mu \setminus B \in \mathscr{RP}_p(n-1)$ and $D(\mu \setminus B)$ not a composition factor of some $S(\nu)$ with $\nu \in \mathscr{RP}_0(n-1)$ with at most 2 rows unless μ is of one of the following forms:

- $(3, 2, 1)$ with $p \geqslant 7$,
- $(p+1, 2, 1)$,
- $(p+2, p+1, 1)$,
- $(2p+1, p+1, 1)$.

Since the p-bar cores of these partitions have 3 rows, the corresponding modules are not a composition factors of some $S(\lambda)$ with $\lambda \in \mathscr{RP}_0(n)$ with at most 2 rows.

In view of Lemma 3.1 the theorem holds, up to identification of the modules D_j and their multiplicity in $S((n-j, j))$. For $0 \leqslant j < m - \ell$ the theorem then holds by Lemma 2.1, since in this case $(n-j, j)^R = \beta^{n-j} + \beta^j$. For $m - \ell < j \leqslant m$ we can then identify the modules D_j comparing normal/removable nodes. We will now check the multiplicity. For $\mu \in \mathscr{RP}_p(n)$ with $a_p(\mu) = 0$ let $P(\mu) = P(\mu, 0)$ be the projective indecomposable module with socle $D(\mu, 0)$, while if $a_p(\mu) = 1$ let $P(\mu) = P(\mu, +) \oplus P(\mu, -)$ be the sum of the projective indecomposable modules with socles $D(\mu, \pm)$. For $\mu \in \mathscr{RP}_p(n)$ and $\nu \in \mathscr{RP}_0(n)$ we have that:

- if $a_p(\mu) = a_0(\nu) = 0$ then
$$\begin{aligned}[S(\nu) : D(\mu)] &= [S(\nu, 0) : D(\mu, 0)],\\ [P(\mu) : S(\nu)] &= [P(\mu, 0) : S(\nu, 0)],\end{aligned}$$
- if $a_p(\mu) = 0$, $a_0(\nu) = 1$ then
$$\begin{aligned}[S(\nu) : D(\mu)] &= 2[S(\nu, \pm) : D(\mu, 0)],\\ [P(\mu) : S(\nu)] &= [P(\mu, 0) : S(\nu, \pm)],\end{aligned}$$
- if $a_p(\mu) = 1$, $a_0(\nu) = 0$ then
$$\begin{aligned}[S(\nu) : D(\mu)] &= [S(\nu, 0) : D(\mu, \pm)],\\ [P(\mu) : S(\nu)] &= 2[P(\mu, \pm) : S(\nu, 0)],\end{aligned}$$
- if $a_p(\mu) = a_0(\nu) = 1$ then for $\delta = \pm$
$$\begin{aligned}[S(\nu) : D(\mu)] &= [S(\nu, \delta) : D(\mu, +)] + [S(\nu, \delta) : D(\mu, -)]\\ &= [S(\nu, +) : D(\mu, \delta)] + [S(\nu, -) : D(\mu, \delta)],\\ [P(\mu) : S(\nu)] &= [P(\mu, \delta) : S(\nu, +)] + [P(\mu, \delta) : S(\nu, -)]\\ &= [P(\mu, +) : S(\nu, \delta)] + [P(\mu, -) : S(\nu, \delta)].\end{aligned}$$

In particular $[S(\nu) : D(\mu)] = 2^{a_0(\nu) - a_p(\mu)}[P(\mu) : S(\nu)]$. The multiplicity $[S((n-j, j)) : D_j]$ is then as given in the theorem also for $m - (p-1)/2 \leqslant j \leqslant m$, unless possibly if $n \equiv 0 \bmod 2p$ and $j = n/2 - 1$ or $n \equiv p \bmod 2p$ and $j = (n-p)/2$. In either of these two cases $D_j = D((p^k, p-1, p-2, 2, 1))$ and $D_j{\downarrow_{\widetilde{\mathsf{S}}_{n-1}}} = D((p^k, p-1, p-2, 2))$ by Lemma 2.3.

If $n \equiv 0 \mod 2p$ and $j = n/2 - 1$ then

$$[S((n-j,j))\downarrow_{\widetilde{\mathsf{S}}_{n-1}}] = [S((n/2, n/2-1))] + [S((n/2+1, n/2-2))].$$

Further if $[D_r\downarrow_{\widetilde{\mathsf{S}}_{n-1}} : D((p^k, p-1, p-2, 2))] > 0$ then

$$\begin{aligned}
&[S((n-r,r))\downarrow_{\widetilde{\mathsf{S}}_{n-1}} : D((p^k, p-1, p-2, 2))] \\
&\qquad = c_r[S((n-r-1, n-r)) : D((p^k, p-1, p-2, 2))] \\
&\qquad\quad + c_{r-1}[S((n-r, r-1)) : D((p^k, p-1, p-2, 2))] \\
&\qquad > 0
\end{aligned}$$

and so $r = m = j$. Since $[S((n/2, n/2-1)) : D((p^k, p-1, p-2, 2))] = 2$ and $[S((n/2+1, n/2-2)) : D((p^k, p-1, p-2, 2))] = 0$ by induction, it follows that $[S((n-j,j)) : D_j] = 2$ in this case.

If $n \equiv p \mod 2p$ and $j = (n-p)/2$ then, since $n > p$,

$$\begin{aligned}
[S((n-j,j))\downarrow_{\widetilde{\mathsf{S}}_{n-1}}] = {} & 2[S(((n+p)/2, (n-p)/2-1))] \\
& + 2[S(((n+p)/2-1, (n-p)/2))].
\end{aligned}$$

If $r \leqslant j$ and $[D_r\downarrow_{\widetilde{\mathsf{S}}_{n-1}} : D((p^k, p-1, p-2, 2))] > 0$ then

$$\begin{aligned}
&[S((n-r,r))\downarrow_{\widetilde{\mathsf{S}}_{n-1}} : D((p^k, p-1, p-2, 2))] \\
&\qquad = c_r[S((n-r-1, n-r)) : D((p^k, p-1, p-2, 2))] \\
&\qquad\quad + c_{r-1}[S((n-r, r-1)) : D((p^k, p-1, p-2, 2))] \\
&\qquad > 0
\end{aligned}$$

and so $r = j$. As

$$\begin{aligned}
&[S(((n+p)/2, (n-p)/2-1)) : D((p^k, p-1, p-2, 2))] = 0, \\
&[S(((n+p)/2-1, (n-p)/2)) : D((p^k, p-1, p-2, 2))] = 1,
\end{aligned}$$

it follows that $[S((n-j,j)) : D_{\lambda_2}] = 2$ also in this case.

Acknowledgements. The author thanks the referee for comments and suggestions on the paper.

References

[1] George E. Andrews, Christine Bessenrodt, and Jørn B. Olsson, *Partition identities and labels for some modular characters*, Trans. Amer. Math. Soc. **344** (1994), 597–615.

[2] Dave Benson, *Spin modules for symmetric groups*, J. London Math. Soc. (2) **38** (1988), 250–262.

[3] Christine Bessenrodt, *Representations of the covering groups of the symmetric groups and their combinatorics*, Sém. Lothar. Combin. **33** (1994), Art. B33a (29 pages).

[4] Christine Bessenrodt, Alun O. Morris, and Jørn B. Olsson, *Decomposition matrices for spin characters of symmetric groups at characteristic* 3, J. Algebra **164** (1994), no. 1, 146–172.

[5] Christine Bessenrodt and Jørn B. Olsson, *The* 2*-blocks of the covering groups of the symmetric groups*, Adv. Math. **129** (1997), no. 2, 261–300.

[6] Jonathan Brundan and Alexander Kleshchev, *Hecke–Clifford superalgebras, crystals of type $A_{2l}^{(2)}$ and modular branching rules for $\hat{S}_n$*, Represent. Theory **5** (2001), 317–403.

[7] ———, *Projective representations of symmetric groups via Sergeev duality*, Math. Z. **239** (2002), no. 1, 27–68.

[8] ———, *Modular representations of the supergroup $Q(n)$. I*, J. Algebra **260** (2003), no. 1, 64–98.

[9] ———, *James' regularization theorem for double covers of symmetric groups*, J. Algebra **306** (2006), no. 1, 128–137.

[10] Matthew Fayers, *Defect* 2 *spin blocks of symmetric groups and canonical basis coefficients*, `https://arxiv.org/abs/1905.04080`, 2019.

[11] Gordon D. James, *On the decomposition matrices of the symmetric groups. I*, J. Algebra **43** (1976), no. 1, 42–44.
[12] Alexander Kleshchev, *Linear and projective representations of symmetric groups*, Cambridge Tracts in Mathematics, vol. 163, Cambridge University Press, Cambridge, 2005.
[13] Alexander Kleshchev and Vladimir Shchigolev, *Modular branching rules for projective representations of symmetric groups and lowering operators for the supergroup $Q(n)$*, Mem. Amer. Math. Soc. **220** (2012), no. 1034, xviii+123.
[14] Bernard Leclerc and Jean-Yves Thibon, *q-deformed Fock spaces and modular representations of spin symmetric groups*, J. Phys. A **30** (1997), no. 17, 6163–6176.
[15] Alun O. Morris and A. K. Yaseen, *Decomposition matrices for spin characters of symmetric groups*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), 145–164.
[16] Jürgen Müller, *Brauer trees for the Schur cover of the symmetric group*, J. Algebra **266** (2003), no. 2, 427–445.
[17] David B. Wales, *Some projective representations of S_n*, J. Algebra **61** (1979), no. 1, 37–57.

Lucia Morotti, Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, 30167 Hannover, Germany
E-mail : morotti@math.uni-hannover.de