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The Cayley isomorphism property for
\( \mathbb{Z}_p^3 \times \mathbb{Z}_q \)

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Abstract For every pair of distinct primes \( p, q \), where \( q > 2 \) we prove that \( \mathbb{Z}_p^3 \times \mathbb{Z}_q \) is a CI-group with respect to binary relational structures.

1. Introduction

Let \( H \) be a finite group and \( S \) a subset of \( G \). The Cayley digraph \( \text{Cay}(H, S) \) is defined by having the vertex set \( H \) and \( g \) is adjacent to \( h \) if and only if \( gh^{-1} \in S \). The set \( S \) is called the connection set of the Cayley digraph \( \text{Cay}(H, S) \). An undirected Cayley digraph will be referred to as a Cayley graph. Recall that a Cayley digraph \( \text{Cay}(H, S) \) is undirected if and only if \( S = S^{-1} \), where \( S^{-1} = \{ s^{-1} \mid s \in S \} \). Every right multiplication via elements of \( H \) is an automorphism of \( \text{Cay}(H, S) \), so the automorphism group of every Cayley graph over \( H \) contains a regular subgroup denoted by \( \hat{H} \) isomorphic to \( H \). Moreover, this property characterises the Cayley graphs of \( H \).

By a binary Cayley structure (or a colored Cayley digraph) over \( H \) we mean an ordered tuple \( (\text{Cay}(H, S_1), \ldots, \text{Cay}(H, S_r)) \) of Cayley digraphs, where \( S_i \cap S_j = \emptyset \) if \( i \neq j \), which we will abbreviate as \( \text{Cay}(H, (S_1, \ldots, S_r)) \). An isomorphism between two tuples \( \text{Cay}(H, (S_1, \ldots, S_r)) \) and \( \text{Cay}(H, (T_1, \ldots, T_r)) \) is a permutation \( f \in \text{Sym}(H) \) satisfying \( \text{Cay}(H, S_i)^f = \text{Cay}(H, T_i), i = 1, \ldots, r \). With this definition, the automorphism group of the binary Cayley structure \( \text{Cay}(H, (S_1, \ldots, S_r)) \) coincides with \( \bigcap_{i=1}^r \text{Aut}(\text{Cay}(H, S_i)) \).

It is clear that every automorphism \( \mu \) of the group \( H \) induces an isomorphism between \( \text{Cay}(H, (S_1, \ldots, S_r)) \) and \( \text{Cay}(H, (S_1^\mu, \ldots, S_r^\mu)) \). Such an isomorphism is called a Cayley isomorphism. A colored Cayley digraph \( \text{Cay}(G, \Theta) \), where \( \Theta \in (2^H)^r \) has the CI-property (or is a colored CI-digraph) if, for each \( \Sigma \in \mathcal{P}(2^H)^r \) the colored Cayley digraph \( \text{Cay}(H, \Sigma) \) is isomorphic to \( \text{Cay}(G, \Theta) \) if and only if they are Cayley isomorphic, i.e. there is an automorphism \( \mu \) of \( H \) such that \( \Theta^\mu = \Sigma \). In this case we say that \( H \) has the CI-property for binary relational structures, or, it is a CI\((2)\)-group. Note that the notion of CI\((2)\)-groups was defined in a slightly different way in [12] but the two definitions are equivalent. Furthermore, a group \( H \) is called a DCI-group if every Cayley digraph of \( H \) is a CI-digraph and it is called a CI-group if every undirected Cayley digraph of \( H \) is a CI-graph.

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Investigation of the isomorphism problem of Cayley graphs started with Ádám’s conjecture [1]. Using our terminology, it was conjectured that every cyclic group is a DCI-group. This conjecture was first disproved by Elspas and Turner [8] for directed Cayley graphs of \( \mathbb{Z}_8 \) and for undirected Cayley graphs of \( \mathbb{Z}_{16} \).

Analyzing the spectrum of circulant graphs Elspas and Turner [8], and independently Đoković [5] proved that every cyclic group of order \( p \) is a CI-group if \( p \) is a prime. Also, a lot of research was devoted to the investigation of circulant graphs. One important result for our investigation is that \( \mathbb{Z}_{pq} \) is a DCI-group for every pair of primes \( p < q \). This result was first proved by Alspach and Parsons [2] and independently by Pöschel and Klin [13] using the theory of Schur rings, and also by Godsil [11]. Finally, Muzychuk [18, 19] proved that a cyclic group \( \mathbb{Z}_n \) is a DCI-group if and only if \( n = k \) or \( n = 2k \), where \( k \) is square-free. Furthermore, \( \mathbb{Z}_n \) is a CI-group if and only if \( n \) is as above or \( n = 8, 9, 18 \).

It is easy to see that every subgroup of a (D)CI-group is also a (D)CI-group so it is natural to investigate \( p \)-groups which are the Sylow \( p \)-subgroups of a finite group. Babai and Frankl [4] proved that if \( H \) is a \( p \)-group, which is a CI-group, then \( H \) can only be an elementary abelian \( p \)-group, the quaternion group of order 8 or one of a few cyclic groups \( \mathbb{Z}_4, \mathbb{Z}_8, \mathbb{Z}_9 \) or \( \mathbb{Z}_{27} \). The known results about cyclic groups show that \( \mathbb{Z}_{27} \) is not a CI-group and \( \mathbb{Z}_9, \mathbb{Z}_8 \) are not DCI-groups. Babai and Frankl also asked whether every elementary abelian \( p \)-group is a (D)CI-group.

The cyclic group of order \( p \), which is a CI-group, can also be considered as an elementary abelian \( p \)-group of rank 1. Currently, the best general result is due to Feng and Kovács [10] who proved that \( \mathbb{Z}_p^r \) is a CI-group for every prime \( p \). The proof using elementary tools for \( \mathbb{Z}_4^p \) is due to Morris [17]. It was shown by Somlai [22] that \( \mathbb{Z}_p^r \) is not a DCI-group if \( r \geq 2p + 3 \).

Severe restrictions on the structure of DCI-groups were given by Li and Praeger and then a more precise list of candidates for DCI-groups was given by Li, Lu and Pálfy [16]. A new family of CI-groups was found by Kovács and Muzychuk [14], that is, \( \mathbb{Z}_p^2 \times \mathbb{Z}_q \) is a DCI-group for every prime \( p \) and \( q \). One example of DCI-groups connected to the question treated in this paper is \( \mathbb{Z}_3^3 \times \mathbb{Z}_p \), see [6]. It was also conjectured in [14], that the direct product of DCI-groups of coprime order is a DCI-group\(^{(1)}\). Note that the conjecture is not true for CI-groups as it was shown recently by Dobson [7]. Dobson also proved that the product of relatively prime order elementary abelian DCI-groups is a DCI-group by posing a serious assumption on the prime divisors of the order of the group [6].

In this paper we prove the following result which supports this conjecture.

**Theorem 1.1.** For every pair of primes \( p, q \), where \( q > 2 \) the group \( \mathbb{Z}_p^3 \times \mathbb{Z}_q \) is a DCI-group.

In fact we prove here a more general fact: the above group is a CI\(^{(2)}\)-group. Our paper is organized as follows. In Section 2 we introduce the basic notation from Schur rings theory that is needed in this paper. In Section 3 we prove general results about Schur rings over abelian groups of special order. Finally, Section 4 contains the proof of Theorem 1.1.

## 2. Schur rings

This section is devoted to presenting a standard approach for dealing with the CI-problem via Schur rings so the results collected here are not new. The result below is a direct consequence of Babai’s lemma [3].

\(^{(1)}\)The cited paper deals in fact with DCI-groups while it talks about CI-groups.
Lemma 2.1. A colored Cayley graph $\text{Cay}(H, G) \in \mathcal{P}(H)^c$ has the CI-property if and only if any $H$-regular subgroup\(^{(2)}\) of the full automorphism group $\text{Aut}(\text{Cay}(H, G))$ is conjugate to $H$ inside $\text{Aut}(\text{Cay}(H, G))$.

According to this result, in order to prove the CI-property for binary Cayley structures, it is sufficient to go through the whole set of automorphism groups of all colored Cayley graph over $H$. This could be done using the method of Schur rings. Let $G := \text{Aut}(\text{Cay}(H, G)), G = (S_1, \ldots, S_r)$ denote the full automorphism group of a colored digraph $\text{Cay}(H, G)$. Its intersection with $\text{Aut}(H)$ will be denoted as $\text{Aut}_H(\text{Cay}(H, G))$. Let us order the orbits of $G_e$ in an arbitrary way, say $O_{11}, \ldots, O_{1r}$. Since $\text{Aut}(\text{Cay}(H, (S_1, \ldots, S_r))) = \text{Aut}(\text{Cay}(H, (O_{11}, \ldots, O_{1r})))$, we have to analyze only those colored Cayley graphs which correspond to overgroups $G \leq \text{Sym}(H)$ of $H$. It turns out that these colored Cayley graphs are closely related to Schur rings.

2.1. Schur rings over finite groups. We start with the basic definitions [23]. Given a group $G$, we denote its group algebra over the rationals as $\mathbb{Q}[G]$. If $S \subseteq H$, then by $\hat{S}$ we denote the element $\sum_{s \in S} s \in \mathbb{Q}[H]$. Following [23] we call elements of this type simple quantities.

A subalgebra $\mathfrak{A}$ of the group ring $\mathbb{Q}[H]$ is called a Schur ring, an $S$-ring for short, if it satisfies the following conditions.

1. There exists a partition $T = \{T_0, T_1, \ldots, T_l\}$ of $H$ such that $\mathfrak{A}$ is generated as a vector space by the elements of the following form: $T = \sum_{t \in T} t$.
2. $T_0 = \{e\}$.
3. For each $0 \leq i \leq l$ the subset $T_i^{(-1)} := \{t^{-1} | t \in T_i\}$ belongs to $T$.

The elements of the partition $T$ are called basic sets of $\mathfrak{A}$ and $T_i$’s are called basic quantities. In what follows the notation $\text{Bsets}(\mathfrak{A})$ will stand for $T$ and any partition satisfying the above conditions will be referred to as a Schur partition. We say that a Schur ring is non-trivial if $H \setminus \{e\}$ is the union of at least two basic sets.

One of the most natural examples of Schur rings are the transitivity modules. Let $\hat{H} \leq \text{Sym}(H)$ be the right regular representation of a finite group $H$ and $G \leq \text{Sym}(H)$ its subgroup, i.e. $\hat{H} \leq G$. Then the orbits of the stabilizer $G_e$ are the basic sets of a Schur ring over $H$ [21]. Such a Schur ring will be called the transitivity module of $H$ induced by $G$ and denoted by $V(H, G_e)$. If $G = \hat{H}M$ for some $M \leq \text{Aut}(H)$, then the Schur ring $V(H, G_e)$ is called cyclotomic. In this case, the basic sets of $V(H, G_e)$ coincide with the orbits of $M$.

Every Schur partition (equivalently every $S$-ring) $T = \{T_0, \ldots, T_d\}$ gives rise to an association scheme $\text{Cay}(H, T)$ whose basic graphs are the Cayley graphs $\text{Cay}(H, T), T \in T$. Two Schur partitions (Schur rings) $\mathfrak{A} \subseteq \mathbb{Q}[H], \mathfrak{B} \subseteq \mathbb{Q}[F]$ are called (combinatorially) isomorphic if the corresponding association schemes are isomorphic, i.e. there exists a bijection $f : H \to F$ which maps the basic Cayley graphs $\text{Cay}(H, T), T \in T$ bijectively onto the set $\{\text{Cay}(F, S)\}_{S \in \text{Bsets}(\mathfrak{B})}$. The bijection $f$ is called a combinatorial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. The isomorphism $f$ is called normalized if $f(e_H) = e_F$. If $f$ is a normalized isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, then $\text{Bsets}(\mathfrak{A})/f = \text{Bsets}(\mathfrak{B})$.

We denote by $\text{Iso}(\mathfrak{A}, \mathfrak{B})$ the set of all combinatorial isomorphisms between $\mathfrak{A}, \mathfrak{B}$ and by $\text{Iso}_e(\mathfrak{A}, \mathfrak{B})$ its subset consisting of the normalized ones. It is easy to see that $\text{Iso}(\mathfrak{A}, \mathfrak{B}) = H \text{Iso}_e(\mathfrak{A}, \mathfrak{B}) = \text{Iso}_e(\mathfrak{A}, \mathfrak{B})F$.

Note that $\text{Iso}(\mathfrak{A}, \mathfrak{B})$ is empty if and only if $\mathfrak{A}, \mathfrak{B}$ are not combinatorially isomorphic.

\(^{(2)}\)An $H$-regular subgroup is any regular subgroup of the symmetric group isomorphic to $H$.
\(^{(3)}\)The notation $T^{(-1)}$ is a particular case of a more general one $T^{(m)}$ introduced later.
In what follows we write \( \text{Iso}(\mathfrak{A}, \star) \) for the union of \( \text{Iso}(\mathfrak{A}, \mathfrak{B}) \), where the second argument runs among all \( \mathfrak{S} \)-rings over the group \( H \). As before, \( \text{Iso}(\mathfrak{A}, \star) = \hat{H}\text{Iso}(\mathfrak{A}, \star) = \text{Iso}(\mathfrak{A}, \star) \hat{H} \).

Two \( \mathfrak{S} \)-rings \( \mathfrak{A} \subseteq \mathbb{Q}[H] \) and \( \mathfrak{B} \subseteq \mathbb{Q}[F] \) are \textit{Cayley isomorphic} if there exists a group isomorphism \( \varphi : H \rightarrow F \) such that \( \varphi(\mathfrak{A}) = \mathfrak{B} \). Note that Cayley isomorphic \( \mathfrak{S} \)-rings are always combinatorially isomorphic but not vice versa.

As an application of Babai’s lemma [3] we have the following statement [12].

\begin{proposition}
Let \( \Gamma := \text{Cay}(H, \Sigma) \) be a colored Cayley graph over \( H \) and \( G := \text{Aut}(\Gamma) \). The following are equivalent
\begin{enumerate}
  \item \( \Gamma \) has the CI-property;
  \item any \( H \)-regular subgroup of \( G \) is conjugate to \( \hat{H} \) in \( G \);
  \item the transitivity module \( V(H, \text{Aut}(\Gamma)_c) \) is a CI-S-ring.
\end{enumerate}
\end{proposition}

This implies the following result.

\begin{theorem}
A group \( H \) has a CI-property for binary relational structures (CI\textsuperscript{2}-group, for short) if and only if every transitivity module over \( H \) is a CI-S-ring.
\end{theorem}

Thus one has to check all transitivity modules over the group \( H \). To reduce the number of checks we use the following partial order on the set \( \text{Sup}(\hat{H}) \) consisting of all overgroups of \( \hat{H} \).

Given two overgroups \( X, Y \in \text{Sup}(\hat{H}) \), we write \( X \preceq_{\hat{H}} Y \) if any \( H \)-regular subgroup of \( Y \) may be conjugated into \( X \) by an element of \( Y \), i.e.

\[ \forall g \in \text{Sym}(H) : \hat{H}^g \leq Y \Rightarrow \exists y \in Y : (\hat{H}^g)^y \leq X. \]

One can easily check that \( \preceq_{\hat{H}} \) is a partial order on the set of all overgroups of \( \hat{H} \). Note that any two \( H \)-regular subgroups of \( X \in \text{Sup}(\hat{H}) \) are conjugate inside \( X \) if and only if \( \hat{H} \preceq_{\hat{H}} X \).

The statement below allows us to consider transitivity modules of \( \preceq_{\hat{H}} \)-minimal groups only.

\begin{proposition}
Let \( G_1 \leq G_2 \) be two overgroups of \( \hat{H} \) and \( \mathfrak{A}_i := \text{V}(H, (G_i)_c) \) their transitivity modules. Then \( \mathfrak{A}_1 \supseteq \mathfrak{A}_2 \). If \( G_1 \preceq_{\hat{H}} \text{Aut}(\mathfrak{A}_2) \) and \( \mathfrak{A}_1 \) is CI, then \( \mathfrak{A}_2 \) is also a CI-S-ring.
\end{proposition}

\textit{Proof.} First we note that the inclusion \( \mathfrak{A}_1 \supseteq \mathfrak{A}_2 \) is obvious.

To show the CI-property of \( \mathfrak{A}_2 \) we have to verify that \( \text{Iso}(\mathfrak{A}_2, \star) \subseteq \text{Aut}(\mathfrak{A}_2)\text{Aut}(\hat{H}) \) (the converse inclusion is obvious). Pick an arbitrary \( f \in \text{Iso}(\mathfrak{A}_2, \star) \). Then \( \mathfrak{A}_2^f = \mathfrak{B} \) for some \( \mathfrak{S} \)-ring \( \mathfrak{B} \) over \( H \). Then \( \hat{H} \subseteq \text{Aut}(\mathfrak{B}) = \text{Aut}(\mathfrak{A}_2)^f \) implying \( \hat{H}^f \leq \text{Aut}(\mathfrak{A}_2) \). It follows from the assumption that there exists \( g \in \text{Aut}(\mathfrak{A}_2) \) such that \( (\hat{H}^f)^g \leq G_1 \).

Combining this with \( G_1 \leq \text{Aut}(\mathfrak{A}_1) \) we conclude that \( \hat{H}^{-1}g^{-1} \leq \text{Aut}(\mathfrak{A}_1) \). Since \( \mathfrak{A}_1 \) is a CI-S-ring, there exists \( g_1 \in \text{Aut}(\mathfrak{A}_1) \) such that \( \hat{H}g_1 = \hat{H}^{-1}g \). This implies \( g_1^{-1}g^{-1} \in \hat{H}\text{Aut}(H) \), or, equivalently, \( g_1g^{-1}f \in \hat{H}\text{Aut}(H) \). It follows from \( \mathfrak{A}_1 \supseteq \mathfrak{A}_2 \) that \( \text{Aut}(\mathfrak{A}_1) \subseteq \text{Aut}(\mathfrak{A}_2) \). Therefore \( g_1g^{-1} \in \text{Aut}(\mathfrak{A}_2) \), and, consequently, \( f \in \text{Aut}(\mathfrak{A}_2)\text{Aut}(\hat{H}) \), as required. \[ \square \]
Sylow’s theorem shows that if \( H \) is a \( p \)-group, then any \( \leq H \)-minimal overgroup of \( H \) is a \( p \)-group. In this case we are left to investigate transitivity modules whose basic sets have a \( p \)-power cardinality. These Schur rings are called \( p \)-Schur rings.

2.2. Structural properties of Schur rings. As before, \( H \) is a finite group and \( Q[H] \) is its group algebra. For an element of the group algebra \( U = \sum_{g \in H} a_g g \) let \( U^m = \sum_{g \in H} a_g g^m \). We extend this notation to an arbitrary subset \( T \) of \( H \) by \( T^m = \{ t^m \mid t \in T \} \).

The two lemmas below are taken from [23].

**Lemma 2.5.** Let \( \mathfrak{A} \) be an \( \mathfrak{S} \)-ring over an abelian group \( H \). If \( \gcd(m, |H|) = 1 \), then \( T^m \in \mathfrak{A} \) for every \( T \in \mathfrak{A} \).

A similar statement holds if \( m \) divides \( |H| \).

**Lemma 2.6.** Let \( T \) be a simple quantity and \( m \) a prime divisor of \( |G| \) and let \( T^m = \sum_{h \in H} a_h h \). Then for any integer \( i \) the simple quantity \( \sum_{h \in H; a_h \equiv i \pmod{m}} h \) belongs to \( \mathfrak{A} \).

A subgroup \( L \leq H \) is called an \( \mathfrak{A} \)-subgroup if \( L \in \mathfrak{A} \). We say that \( \mathfrak{A} \) is primitive if the only \( \mathfrak{A} \)-subgroups are \( \{ e \} \) and \( H \). A Schur ring \( \mathfrak{A} \) is called imprimitive if \( L \in \mathfrak{A} \) for some non-trivial and proper subgroup \( L \leq H \).

If \( T \) is an \( \mathfrak{A} \)-set, then we may define its radical \( \text{Rad}(T) = \{ g \in T \mid Tg = gT = T \} \).

It is well known that the radical of an \( \mathfrak{A} \)-set \( T \) is an \( \mathfrak{A} \)-subgroup [23].

It is a simple observation that a trivial \( \mathfrak{S} \)-ring is always primitive. The converse is not true (e.g. [23, Theorem 25.7]). The result below proved by Wielandt ([23, Theorem 25.4]) provides a sufficient condition for the converse implication.

**Theorem 2.7.** A primitive \( \mathfrak{S} \)-ring over an abelian group \( H \) of a composite order is trivial if \( H \) has a cyclic Sylow subgroup.

For an \( \mathfrak{A} \)-subgroup \( U \) one can define \( \mathfrak{A} U \) as the restriction of \( \mathfrak{A} \) to \( U \) spanned by the basic sets of \( \mathfrak{A} \) contained in \( U \). For a pair of \( \mathfrak{A} \)-subgroups \( L \leq U \) we define \( \mathfrak{A}_{U/L} \) as a subring of \( Q[U/L] \) spanned by \( \{ X^\pi \mid X \subset U \}, X \in \text{Bsets}(\mathfrak{A}) \) \( \pi \) denotes the canonical epimorphism from \( U \) to \( U/L \) [9].

We say that the Schur ring \( \mathfrak{A} \) is a generalized wreath product if there exists \( \mathfrak{A} \)-subgroups \( L \leq U \) such that \( L \) is a normal subgroup in \( H \) and every basic set outside of \( U \) is the union of \( L \)-cosets. Such a wreath product is called trivial if \( L = \{ e \} \) or \( U = H \). In the case of \( L = U \) we obtain the usual wreath product of Schur rings.

Let \( K \) and \( L \) be two \( \mathfrak{A} \)-subgroups. We say that \( \mathfrak{A} \) is the star product of \( \mathfrak{A}_K \) and \( \mathfrak{A}_L \) (or \( \mathfrak{A} \) admits a star decomposition) if the following conditions hold:

1. \( K \cap L \leq L \)
2. each basic set \( T \in \mathfrak{A} \) with \( T \subseteq (L \setminus K) \) is the union of \( K \cap L \)-cosets
3. for each basic set \( T \subseteq H \setminus (K \cup L) \) there exists \( R, S \in \text{Bsets}(\mathfrak{A}) \), where \( R \subseteq K \), \( S \subseteq L \) such that \( T = RS \).

Note that in order to verify (3) it is enough to find \( \mathfrak{A} \)-sets \( \mathfrak{R}' \) and \( \mathfrak{S}' \) with \( T = \mathfrak{R}' \mathfrak{S}' \).

In this case we write \( \mathfrak{A} = \mathfrak{A}_K \ast \mathfrak{A}_L \). A star-decomposition is called trivial if \( K = \{ e \} \) or \( H \). In the case of \( L = H \) a star decomposition coincides with the wreath product of \( \mathfrak{A}_K \) and \( \mathfrak{A}/K \).

The theorems below provide us sufficient conditions for these products to have the CI-property. Although the first statement was originally proved for elementary abelian groups only [12], the proof works for a more general class of groups, namely: the abelian groups with elementary abelian Sylow subgroups. In what follows we refer to these groups as \( \mathcal{E} \)-groups.
Theorem 2.8 ([14, Theorem 3.2]). Let \( H \) be an \( \mathcal{E} \)-group and let \( G \leq \text{Sym}(H) \) be an overgroup of \( H \). Assume that the transitivity module \( \mathfrak{A} := V(H,G_e) \) admits a non-trivial star-decomposition \( \mathfrak{A}_K \rtimes \mathfrak{A}_L \). If \( \mathfrak{A}_K \) and \( \mathfrak{A}_{L/K\cap L} \) are CI-S-rings, then \( \mathfrak{A} \) is a CI-S-ring.

Note that the above theorem implies that if \( \mathfrak{A} \) admits a usual wreath product decomposition, then \( \mathfrak{A} \) is a CI-S-ring. In the case of a generalized wreath product we have the following result.

Theorem 2.9 ([15]). Let \( H \) be an \( \mathcal{E} \)-group and let \( G \leq \text{Sym}(H) \) be an overgroup of \( H \). Assume that \( \mathfrak{A} := V(H,G_e) \) is a non-trivial generalized wreath product with respect to \( \mathfrak{A} \)-subgroups \( \{e\} \neq L \leq U \neq H \). Assume that \( \mathfrak{A}_U \) and \( \mathfrak{A}_{U/L} \) are CI-S-rings and \( \text{Aut}_{U/L}(\mathfrak{A}_{U/L}) = \text{Aut}_U(\mathfrak{A}_U)^{U/L} \text{Aut}_H/L(\mathfrak{A}_{H/L})^{U/L} \). Then \( \mathfrak{A} \) is a CI-S-ring.

3. Schur rings over abelian group of non-powerful order

Recall that a number \( n \) is call powerful if \( p^2 \) divides \( n \) for every prime divisor \( p \) of \( n \). In this section and in what follows we assume that \( H \) is an abelian group of a non-powerful order, i.e. there exists a prime divisor \( q \) of \(|H|\) such that \(|H| = nq \) where \( n \) is coprime to \( q \). In what follows we call such \( q \) a simple prime divisor of \(|H|\). We assume that \( q > 2 \).

Let \( P \) and \( Q \) denote the unique subgroups of \( H \) of orders \( n \) and \( q \), respectively. Let \( \langle q \rangle = Q \setminus \{1\} \). Let \( t \) be the exponent of \( P \). The group \( \mathbb{Z}_{tq} \cong \mathbb{Z}_t \times \mathbb{Z}_q \) acts on \( H \) via raising to the power as \( h \mapsto h^t \), where \( t \in \mathbb{Z}_{tq} \). Denote \( M_q := \{t \in \mathbb{Z}_{tq} \mid t \equiv 1 \pmod{\ell}\} \). Clearly \( M_q \cong \mathbb{Z}_q^* \).

Every element \( h \in H \) has a unique decomposition into the product \( h = h_qh_q \) where \( h_q \in P \) and \( h_q \in Q \). Notice that two elements \( h, f \in H \) belong to the same \( Q \)-coset if and only if \( h_q = f_q \). Let \( q^* \in \mathbb{Z}_{tq}^* \) be an element satisfying \( q^*q \equiv 1 \pmod{\ell} \) and \( q^* \equiv 1 \pmod{q} \). Then \( h_q = h^{q^*} \).

Given a subset \( T \subseteq H \). We write \( T_q \) for the set \( \{h_q \mid h \in T\} \). Notice that \( T_q \) is always contained in \( P \). We always have the decomposition \( T = \bigcup_{s \in T_q} sR_s \) where \( R_s := s^{-1}T \cap Q \).

In what follows \( A \) stands for a non-trivial \( S \)-ring over \( H \). Let \( P_1 \) be the maximal \( \mathfrak{A} \)-subgroup contained in \( P \) while \( Q_1 \) is the minimal \( \mathfrak{A} \)-subgroup which contains \( Q \).

The statement below describes the structure of \( M_q \)-invariant basic sets.

Proposition 3.1. Let \( T \) be a basic set of \( \mathfrak{A} \) which is \( M_q \)-invariant. Denote \( S := T_q \). There exists a partition\(^{(4)} \) \( S = S_1 \cup S_{-1} \cup S_0 \) such that \( T = S_1 \cup S_{-1}Q^{\#} \cup S_0Q \) and \( S_1, S_{-1} \) are \( \mathfrak{A} \)-subsets (not necessarily basic). In addition the sets \( S_1, S_{-1} \) and \( S_0 \) satisfy the following conditions

\begin{enumerate}
  \item If \( S_1 \neq \emptyset \), then \( S_{-1} = S_0 = \emptyset \) and \( T \subseteq P_1 \);
  \item If \( S_1 = \emptyset \) and \( S_{-1} \neq \emptyset \), then \( T = S_{-1}(Q_1 \setminus P_1) \);
  \item If \( S_1 = S_{-1} = \emptyset \), then \( Q_1P = T \).
\end{enumerate}

Proof. Write \( T = \bigcup_{s \in \mathfrak{A}} sR_s \) where \( R_s := s^{-1}T \cap Q \). Since \( T \) is \( M_q \)-invariant, the sets \( R_s \) are \( \mathbb{Z}_q^* \)-invariant. Therefore \( R_s \in \{\{1\}, Q^{\#}, Q\} \). Now the sets

\[ 
S_1 := \{s \mid R_s = \{1\}\}, \quad S_{-1} := \{s \mid R_s = Q^{\#}\}, \quad S_0 := \{s \mid R_s = Q\} 
\]

produce the required partition. Raising the simple quantity \( T = S_1 + S_{-1}Q^{\#} + S_0Q \) to the \( q \)-th power modulo \( q \) we obtain

\[ 
T^q \equiv (S_1)^q - (S_{-1})^q \equiv (S_1^{(q)}) - (S_{-1}^{(q)}) \pmod{q} 
\]

\(^{(4)}\)Notice that some of its parts may be empty.
Now Lemma 2.6 applied to $T^q$ with $m = q$ and $i = \pm 1$ ($-1 \neq 1$, because $q > 2$) implies that $S_1, S_1^{-1}$ are $\mathfrak{A}$-subsets. Applying $q^*$ we conclude that $S_1$ and $S_{-1}$ are $\mathfrak{A}$-subsets too.

If $S_1 \neq \emptyset$, then $S_1 = T$ because $T$ is basic and $S_1$ is a nonempty $\mathfrak{A}$-subset contained in $T$. Hence $S_{-1} = S_0 = \emptyset$.

Assume now that $S_1 = \emptyset$ and $S_{-1} \neq \emptyset$. Since $Q_1 \setminus P_1 = Q_1 \setminus (Q_1 \cap P_1)$ is an $\mathfrak{A}$-subset which contains $Q^\#$, we conclude that $S_{-1}(Q_1 \cap P_1)$ is an $\mathfrak{A}$-subset which intersects $T$ non-trivially (the part $S_{-1}Q^\#$ is in common). Therefore $S_{-1}(Q_1 \cap P_1) \supseteq T$.

The union $S_{-1} \cup T = (S_{-1} \cup S_0)Q$ is an $\mathfrak{A}$-subset the radical of which contains $Q$. Therefore, by the minimality of $Q_1$, we have $Q_1 \subseteq \operatorname{Rad}(S_{-1} \cup T)$. This implies $Q_1S_{-1} \cup Q_1T = S_{-1} \cup T$ so $S_{-1}Q_1 \subseteq S_{-1} \cup T$. Thus $T \subseteq S_{-1}(Q_1 \cap P_1) \subseteq S_{-1} \cup T$. If $S_{-1}(Q_1 \cap P_1) \cap S_{-1} \neq \emptyset$, then $t = s'$ for some $s, s' \in S_{-1}$ and $t \in Q_1 \cap P_1$. But in this case we would obtain $t = s's^{-1} \subseteq S_{-1}S_{-1}^{-1} \subseteq P_1$, a contradiction. Hence $S_{-1}(Q_1 \cap P_1) \cap S_{-1} = \emptyset$ implying that $T = S_{-1}(Q_1 \cap P_1)$.

If $S_1 = S_{-1} = \emptyset$, then $T = S_0Q$ so $\operatorname{Rad}(T)$ contains $Q$ by the minimality of $Q_1$ we have $Q_1 \subseteq \operatorname{Rad}(T)$ so $Q_1T = T$. □

**Corollary 3.2.** $\mathfrak{A}$ is a generalized wreath product with respect to $Q_1$ and $P_1Q_1$.

**Proof.** There is nothing to prove if $Q_1P_1 = H$. So, in what follows we assume that $Q_1P_1 \neq H$.

We have to show that $Q_1T = T$ holds for each $\mathfrak{A}$-basic set $T$ outside of $P_1Q_1$. Let $T$ be such a basic set, that is, $T \cap P_1Q_1 = \emptyset$.

If $T$ contains a $q$'-element, then $T$ is $M_q$-invariant, and therefore, $T$ fits one of the cases described in Proposition 3.1. The cases (a) and (b) contradict $T \cap P_1Q_1 = \emptyset$, since in both of them $T \subseteq P_1Q_1$. Therefore the case 3 of Proposition 3.1 occurs and $TQ_1 = T$, as required.

It remains to show that every basic $\mathfrak{A}$-set disjoint with $P_1Q_1$ contains $q'$-elements. Assume that there exists one, say $T$, which does not contain a $q'$-element. Denote $R := T_q$. Then $T$ can uniquely be written as $T = \cup_{h \in Q_1}hQ_h$, where $Q^\# \supseteq Q_h \neq \emptyset$. Then by Lemma 2.6 $T^{(q)} = R^{(q)}$ is an $\mathfrak{A}$-set, implying that $R^{(q)} \subseteq P_1$ and $R \subseteq P_1$.

Again we have $T \subseteq RQ \subseteq P_1Q_1$, contrary to the choice of $T$. □

**3.1. The structure of the section $\mathfrak{A}_{P_1Q_1}$.** In what follows we abbreviate $H_1 := P_1Q_1$ and $\mathfrak{A}_1 := \mathfrak{A}_H$. We start with the following simple statement.

**Proposition 3.3.** $P_1$ is an $\mathfrak{A}_1$-maximal subgroup.

**Proof.** Let $P_1$ denote a proper $\mathfrak{A}_1$-maximal subgroup which contains $P_1$. If $q$ divides $P_1$, then $Q_1$ is contained in $P_1$ implying $P_1Q_1 \subseteq P_1 = H_1$, a contradiction. Hence $P_1$ is a $\mathfrak{A}$-subgroup, which is an $\mathfrak{A}_1$-subgroup. Therefore, $P_1 = P_1$. □

**Proposition 3.4.** If $|H_1/P_1| \neq q$, then $\mathfrak{A}_1/P_1$ has rank two and $\mathfrak{A}_1 = (\mathfrak{A}_1)_{P_1} \ast (\mathfrak{A}_1)_{Q_1}$.

**Proof.** $P_1$ is an $\mathfrak{A}_1$-maximal subgroup, by Proposition 3.3. Thus the quotient S-ring is primitive. The Sylow $q'$-subgroup of $H_1/P_1$ is cyclic. Therefore by Wielandt’s Theorem 2.7 either the quotient S-ring has rank two or $H_1/P_1$ is of prime order. In the latter case, $|H_1/P_1| = q$, which contradicts our assumptions.

The quotient S-ring $\mathfrak{A}_1/P_1$ has rank two iff $TP_1 = H_1 \cap P_1$ holds for each basic set $T \in \mathfrak{B}ets(\mathfrak{A}_1)$ outside of $P_1$.

It follows from $|H_1/P_1| \neq q$ that $P_1 \neq (H_1)^{q'}$. Pick an arbitrary $T \in \mathfrak{B}ets(\mathfrak{A}_1)$ with $T \cap P_1 = \emptyset$. Then $TP_1 = H_1 \cap P_1 \supseteq (H_1)^{q'} \times P_1$ implying $T \cap (H_1)^{q'} \neq \emptyset$. Thus $T$ contains $q'$-elements, and, therefore, is $M_q$-invariant and Proposition 3.1 is applicable.

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The first case of the Proposition is not possible because $T \cap P_1 = \emptyset$.

In the second case we obtain that $T$ is the product of two $\mathcal{A}_1$-sets $S_{-1} \subset P_1$ and $Q_1 \setminus P_1 \subset Q_1$ so $T$ fits the definition of star decomposition.

Finally, if $Q_1 \cap T = T$, then $T$ is the union of $Q_1$-cosets. Since $P_1 Q_1 = H_1$ we have that $P_1$ intersects every $Q_1$-coset. Hence $T \cap P_1 \neq \emptyset$, contradicting the choice of $T$.

Thus, we have proven that any basic set $T$ of $\mathcal{A}_1$ disjoint to $P_1$ has the form $S(Q_1 \setminus P_1)$ where $S \subseteq P_1$ is an $\mathcal{A}_1$-subset so is a union of $P_1 \cap Q_1$-cosets. This immediately implies that $Q_1 \setminus P_1$ is a basic set of $\mathcal{A}_1$ and $\mathcal{A}_1 = (\mathcal{A}_1)(P_1) \ast (\mathcal{A}_1)Q_1$. \quad \Box

Note that it follows from the Corollary 3.2 that if $H_1 = Q_1$, then $\mathcal{A}$ is a wreath product with respect to $P_1$.

$P_1$ is a maximal $\mathcal{A}_1$-subgroup by Proposition 3.3, and the order of $H_1 / P_1$ is divisible by $q$ but not divisible by $q^2$. Thus by Theorem 2.7 if $\mathcal{A}_1 / P_1$ is non-trivial, then $\mathcal{A}_1 / P_1$ is a non-trivial $S$-ring over a cyclic group of order $q$. In particular, $[H_1 : P_1] = q$. Although the structure of $S$-rings over $C_q$ is known [20] we do not need it, because for our purposes we need to settle the case when $\mathcal{A}_1 / P_1$ coincides with full group algebra.

From now on we denote the cyclic group of order $m$ by $C_m$ in order to make the notation more readable.

**Proposition 3.5.** If $\mathcal{A}_1 / P_1 \cong \mathbb{Z}[C_q]$, then $\mathcal{A}_1 = (\mathcal{A}_1)(P_1) \ast (\mathcal{A}_1)Q_1$.

**Proof.** It follows from the assumption that cosets $hP_1, h \in Q^\#$ are $\mathcal{A}_1$-subsets. Therefore $hP_1$ is partitioned into a disjoint union of basic sets yielding a partition $\Sigma_h$ of $P_1$:

$$S \in \Sigma_h \iff hS \in \text{Bsets}(\mathcal{A}_1).$$

Since $M_q$ permutes basic sets and acts transitively on $Q^\#$, the partitions $\Sigma_h$ does not depend on the choice of $h \in Q^\#$ by Lemma 2.5. So, in what follows we write just $\Sigma$ without an index.

Pick a basic set $T$ outside of $P_1$. Then $T = hS$ for some $h \in Q^\#$ and $S \in \Sigma$. Now it follows from $T^q \equiv S^{(q)} \pmod{q}$ that $S^{(q)}$ is an $\mathcal{A}_1$-subset contained in $P_1$. Applying $q^*$ to $S^{(q)}$ we conclude that $S$ is an $\mathcal{A}_1$-subset.

Since $(T \setminus T \in \text{Bsets}(\mathcal{A}_1) \wedge T \subseteq hP_1)$ is an $(\mathcal{A}_1)P_1$-invariant subspace, the linear span $\Sigma := \langle S \rangle_{S \in \Sigma}$ is an ideal of $(\mathcal{A}_1)P_1$. Let $S_e \in \Sigma$ be a class containing $e$.

We claim that $S_e$ is a $\mathcal{A}_e$-subgroup and every class of $\Sigma$ is a union of $S_e$-cosets. This will imply our claim.

Pick a basic set $T$ of $(\mathcal{A}_1)P_1$ contained in $S_e$. Then $e$ appears in the product $T^{(-1)}S_e$ with coefficient $[T]$. Therefore $S_e$ appears $[T]$ times in this product. This implies $T^{(-1)}S_e = [T]S_e$ and, consequently, $T^{(-1)}S_e = S_e$. Since this equality holds for any basic set $T$ contained in $S_e$, we conclude that $S_e^{(-1)}S_e = S_e$, hereby proving that $S_e$ is a subgroup of $P_1$.

Pick now an arbitrary $S \in \Sigma$. Then $S^{(-1)}S_e \in \Sigma$. The identity $e$ appear in the product $|S|$ times. Therefore $S_e$ appears in the product $S^{(-1)}S$ with coefficient $|S|$. Therefore $S$ is a union of $S_e$-cosets.

It is easy to see that $S_e h$ generates an $\mathcal{A}_1$-subgroup, whose order is divisible by $q$ so it contains $Q_1$. On the other hand $S_e h$ is a basic set intersecting $Q$ non-trivially so it is contained in $Q_1$. Thus $S_e = Q_1 \cap P_1$, which gives that $\mathcal{A}_1$ admits a star decomposition. \quad \Box

4. Proof of the main result

In this section we show that every transitivity module over the group $H \equiv C_p^3 \times C_q, p \neq q$ are primes, is a CI-S-ring. Since $q$ is a simple prime divisor of $|H|$, the structural
results from the previous section are applicable. We also keep the notation $P_1$ and $Q_1$ defined in Section 3.

For the rest of the section $\mathfrak{A} = V(H, G_c)$ is a transitivity module of an $\preceq_H$-minimal subgroup $G$.

In this section we prove the following.

**Theorem 4.1.** $\mathfrak{A}$ is a CI-S-ring.

Combining this result with Theorem 2.3 we obtain the main result of the paper.

**4.1. Proof of Theorem 4.1 in the Case of $P_1Q_1 \neq H$.** If $P_1Q_1 \neq H$, then by Corollary 3.2 the S-ring $\mathfrak{A}$ is a non-trivial generalized wreath product of $\mathfrak{A}_{P_1Q_1}$ and $\mathfrak{A}_{H/Q_1}$. Therefore, the results of [15] are applicable.

Since $\mathfrak{H} := H/Q_1$ is an elementary abelian $p$-group, we may assume that the basic sets of $\mathfrak{A} := \mathfrak{A}/Q_1$ are of $p$-power length. Such a Schur ring is called a $p$-S-ring and so $\mathfrak{A}$ is a transitivity module of the quotient group $\mathfrak{G} := \mathfrak{G}/Q_1$. Since $\mathfrak{G}$ is $\preceq_H$-minimal, the group $\mathfrak{G}$ is $\preceq_{\mathfrak{P}}$-minimal.

If $|P_1Q_1/Q_1| \leq p$, then $\mathfrak{A}_{P_1Q_1/Q_1}$ is the full group ring and we are done by Proposition 4.1 of [15]. Thus we may assume that $|P_1Q_1/Q_1| = p^a$ with $a \geq 2$. Since $q$ divides $|P_1Q_1|$ and $P_1Q_1 \neq H$, we conclude that $|P_1| = p^a|Q_1| = q$. Thus $\mathfrak{A}_{P_1Q_1/Q_1} \cong \mathbb{Z}[C_p] \wr \mathbb{Z}[C_p]$ since if $\mathfrak{A}_{P_1Q_1/Q_1} \cong \mathbb{Z}[C_p]$ we may apply Proposition 4.1 of [15] and these are the only $p$-Schur rings over $\mathbb{Z}_p$. Further it follows from $|Q_1| = q$ that $\mathfrak{H} \cong C_p^3$.

The S-ring $\mathfrak{A}_{H}$ is a Schurian $p$-S-ring over the group $\mathfrak{H} \cong C_p^3$. The classification of such S-rings is well-known [12]. They are

- $\mathfrak{B}_1 = \mathbb{Z}[C_p^3]$,
- $\mathfrak{B}_2 = \mathbb{Z}[C_p^3] \wr \mathbb{Z}[C_p]$,
- $\mathfrak{B}_3 = (\mathbb{Z}[C_p] \wr \mathbb{Z}[C_p]) 
\otimes \mathbb{Z}[C_p]$,
- $\mathfrak{B}_4 = \mathbb{Z}[C_p] \wr \mathbb{Z}[C_p]$,
- $\mathfrak{B}_5 = \mathbb{Z}[C_p] \wr \mathbb{Z}[C_p] \otimes \mathbb{Z}[C_p]$,
- $\mathfrak{B}_6 = V(C_p^3, (C_p^3 \simeq (\alpha)_e)$

Here $\alpha \in \text{Aut}(C_p^3)$ is an automorphism of order $p$ which has $p$ fixed points. We can exclude the S-ring $\mathfrak{B}_6$, because in this case the group $\mathfrak{G}$ is not $\preceq_{\mathfrak{P}}$-minimal.

It follows from $\mathfrak{A}_{Q_1P_1/Q_1} \cong \mathbb{Z}[C_p] \wr \mathbb{Z}[C_p]$ that there exists an $\mathfrak{A}$-subgroup of order $p^2$ on which the induced Schur ring is isomorphic to $\mathbb{Z}[C_p] \wr \mathbb{Z}[C_p]$. This excludes $\mathfrak{A} \cong \mathfrak{B}_4$ or $\mathfrak{B}_5$.

It remains to settle the cases $\mathfrak{A} \cong \mathfrak{B}_i$, $i = 3, 4, 5$.

The inclusion $\text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}}) \leq \text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}})$ is trivial. To prove the inverse inclusion we note that each of the S-rings $\mathfrak{B}_i$, $i = 3, 4, 5$ is cyclotomic. In particular this implies that $\text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}})$ acts transitively on each basic set of $\mathfrak{A}$. Therefore $\text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}})$ is non-trivial whenever the induced S-ring $\mathfrak{A}_{\mathfrak{F}}$ is non-trivial for any $\mathfrak{A}$-subgroup $F$. This implies that $\text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{F}})$ is non-trivial. Therefore, $p \leq |\text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{F}})| \leq |\text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}})|$.

On the other hand, $\text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{F}}) = \text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}}) = \text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}})$ is contained in a Sylow $p$-subgroup of $\text{Aut}(C_p^3) \cong GL_2(p)$. Since the latter one has order $p$, we conclude that $|\text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}})| \leq p$ implying $\text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}}) = \text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}})$.

Therefore $\text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{F}}) = \text{Aut}_{\mathfrak{G}}(\mathfrak{A}_{\mathfrak{G}})$ and by Theorem 2.9 of [15] the corresponding S-ring is CI.

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4.2. PROOF OF THEOREM 4.1 IN THE CASE OF $P_1 Q_1 = H$. Note, first, that $|H/P_1|$ is divisible by $q$.

If $|H/P_1| \neq q$, then by Proposition 3.4 we have $\mathfrak{A} = \mathfrak{A}_{P_1} \ast \mathfrak{A}_{Q_1}$. Since both $P_1$ and $Q_1/(P_1 \cap Q_1)$ are $\mathcal{E}$-groups with at most three prime factors, they are CI$^{(2)}$-groups by [12] and [14]. Therefore, $\mathfrak{A}_{P_1}$ and $\mathfrak{A}_{Q_1}/(P_1 \cap Q_1)$ are CI-S-rings. By Theorem 2.8 $\mathfrak{A}$ is a CI-S-ring.

Assume now that $|H/P_1| = q$. Since $G$ is $\preceq_{H}$-minimal, its quotient $G^{H/P_1}$ is $\preceq_{H/P_1}$-minimal too. Therefore $G^{H/P_1} \cong C_q$ and $\mathfrak{A}_{H/P_1} \cong \mathbb{Z}[C_q]$. By Proposition 3.5 $\mathfrak{A} = \mathfrak{A}_{P_1} \ast \mathfrak{A}_{Q_1}$. As before, we conclude that $\mathfrak{A}$ is a CI-S-ring.

Although the case of $q = 2$ is not considered in the paper, the main result remains true also in this case.

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