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Abstract In this paper we introduce the notion of an idempotent system. This linear algebraic object is motivated by the structure of an association scheme. We focus on a family of idempotent systems, said to be symmetric. A symmetric idempotent system is an abstraction of the primary module for the subconstituent algebra of a symmetric association scheme. We describe the symmetric idempotent systems in detail. We also consider a class of symmetric idempotent systems, said to be P-polynomial and Q-polynomial. In the topic of orthogonal polynomials there is an object called a Leonard system. We show that a Leonard system is essentially the same thing as a symmetric idempotent system that is P-polynomial and Q-polynomial.

1. Introduction

In this paper we introduce the notion of an idempotent system. This linear algebraic object is motivated by the structure of an association scheme. We focus on a family of idempotent systems, said to be symmetric. As we will see, a symmetric idempotent system is an abstraction of the primary module for the subconstituent algebra of a symmetric association scheme. Before we go into more detail, we recall the notion of a symmetric association scheme. A symmetric association scheme is a sequence $(X, \{R_i\}_{i=0}^d)$, where X is a finite nonempty set, and $\{R_i\}_{i=0}^d$ is a sequence of nonempty subsets of $X \times X$ such that

(i) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$ (disjoint union);
(ii) $R_0 = \{(x, x) \mid x \in X\}$;
(iii) $(x, y) \in R_i$ implies $(y, x) \in R_i$;
(iv) there exist integers p^h_{ij} $(0 \leqslant h, i, j \leqslant d)$ such that for any $(x, y) \in R_h$ the number of $z \in X$ with $(x, z) \in R_i$ and $(z, y) \in R_j$ is equal to p^h_{ij}.

The integers p^h_{ij} are called the intersection numbers. By (iii) they satisfy $p^h_{ij} = p^h_{ji}$ for $0 \leqslant h, i, j \leqslant d$. The concept of a symmetric association scheme first arose in design theory [2–4, 13] and group theory [18]. A systematic study began with [7, 9]. A comprehensive treatment is given in [1, 5].

Let $(X, \{R_i\}_{i=0}^d)$ denote a symmetric association scheme. As we study this object, the following concepts and notation will be useful. Let $\mathbb{R}$ denote the real number field. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices with rows and columns indexed by X, and all entries in $\mathbb{R}$. Let I (resp. J) denote the identity matrix (resp. all 1's matrix) in $\mathrm{Mat}_X(\mathbb{R})$. Let $\mathbb{V}$ denote the vector space over $\mathbb{R}$ consisting of the column

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vectors with coordinates indexed by X, and all entries in $\mathbb{R}$. The algebra $\mathrm{Mat}_X(\mathbb{R})$ acts on $\mathbb{V}$ by left multiplication. We define a bilinear form $\langle\ ,\ \rangle : \mathbb{V}\times\mathbb{V}\to\mathbb{R}$ such that $\langle u,v\rangle = \sum_{y\in X} u_y v_y$ for $u,v\in\mathbb{V}$. We have $\langle Bu,v\rangle = \langle u, B^{\mathrm{t}}v\rangle$ for $B\in\mathrm{Mat}_X(\mathbb{R})$ and $u,v\in\mathbb{V}$. Here B^{t} denotes the transpose of B. For $y\in X$ define $\widehat{y}\in\mathbb{V}$ that has y-entry 1 and all other entries 0. Note that $\{\widehat{y}\mid y\in X\}$ form an orthonormal basis of $\mathbb{V}$.

We now recall the Bose–Mesner algebra. For $0\leqslant i\leqslant d$ define $A_i\in\mathrm{Mat}_X(\mathbb{R})$ that has (y,z)-entry 1 if $(y,z)\in R_i$ and 0 if $(y,z)\notin R_i$ $(y,z\in X)$. The matrix A_i is symmetric. We have

$$A_0 = I, \qquad\qquad A_iA_j = \sum_{h=0}^{d} p^h_{ij}A_h \qquad (0\leqslant i,j\leqslant d).$$

The $\{A_i\}_{i=0}^d$ form a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{R})$. We call M the Bose–Mesner algebra of the scheme. Each matrix in M is symmetric. By [5, Section 2.2] there exists a basis $\{E_i\}_{i=0}^d$ for M such that

$$E_0 = |X|^{-1}J, \qquad I = \sum_{i=0}^{d} E_i, \qquad E_iE_j = \delta_{i,j}E_i \quad (0\leqslant i,j\leqslant d).$$

We have

$$\mathbb{V} = \sum_{i=0}^{d} E_i\mathbb{V} \qquad\qquad \text{(orthogonal direct sum)}.$$

For $0\leqslant i\leqslant d$, $E_i\mathbb{V}$ is the i^{th} common eigenspace for M, and E_i is the orthogonal projection from $\mathbb{V}$ onto $E_i\mathbb{V}$. There exist real numbers $p_i(j)$, $q_i(j)$ $(0\leqslant i,j\leqslant d)$ such that

$$A_i = \sum_{j=0}^{d} p_i(j)E_j, \qquad\qquad E_i = |X|^{-1}\sum_{j=0}^{d} q_i(j)A_j$$

for $0\leqslant i\leqslant d$.

We now recall the Krein parameters. Note that $A_i\circ A_j = \delta_{i,j}A_i$ $(0\leqslant i,j\leqslant d)$, where $\circ$ denotes entry-wise multiplication. Therefore M is closed under $\circ$. Consequently there exist real numbers q^h_{ij} $(0\leqslant h,i,j\leqslant d)$ such that

$$E_i\circ E_j = |X|^{-1}\sum_{h=0}^{d} q^h_{ij}E_h \qquad\qquad (0\leqslant i,j\leqslant d).$$

By [1, Theorem 3.8], $q^h_{ij}\geqslant 0$ for $0\leqslant h,i,j\leqslant d$. The q^h_{ij} are called the Krein parameters of the scheme.

We now recall the dual Bose–Mesner algebra. For the rest of this section fix $x\in X$. For $B\in M$ let B^ρ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{R})$ that has (y,y)-entry $B_{x,y}$ for $y\in X$. Roughly speaking, B^ρ is obtained by turning column x of B at a 45 degree angle. For $0\leqslant i\leqslant d$ define $E^*_i = A^\rho_i$. For $y\in X$ the (y,y)-entry of E^*_i is 1 if $(x,y)\in R_i$ and 0 if $(x,y)\notin R_i$. Note that E^*_0 has (x,x)-entry 1 and all other entries 0. The matrices $\{E^*_i\}_{i=0}^d$ satisfy

$$I = \sum_{i=0}^{d} E^*_i, \qquad\qquad E^*_iE^*_j = \delta_{i,j}E^*_i \qquad (0\leqslant i,j\leqslant d).$$

Therefore $\{E^*_i\}_{i=0}^d$ form a basis for a commutative subalgebra M^* of $\mathrm{Mat}_X(\mathbb{R})$. We call M^* the dual Bose–Mesner algebra with respect to x. We have

$$\mathbb{V} = \sum_{i=0}^{d} E^*_i\mathbb{V} \qquad\qquad \text{(orthogonal direct sum)}.$$

For $0 \leqslant i \leqslant d$, $E_i^*\mathbb{V}$ has basis $\{\widehat{y} \mid y \in X,\ (x,y) \in R_i\}$. Moreover $E_i^*\mathbb{V}$ is the i^{th} common eigenspace for M^*, and E_i^* is the orthogonal projection from $\mathbb{V}$ onto $E_i^*\mathbb{V}$.

The map $\rho : M \to M^*$, $B \mapsto B^\rho$ is $\mathbb{R}$-linear and bijective. For $0 \leqslant i \leqslant d$ define $A_i^* = |X|E_i^\rho$. The $\{A_i^*\}_{i=0}^d$ form a basis of M^*, and

$$A_0^* = I, \qquad\qquad A_i^* A_j^* = \sum_{h=0}^{d} q_{ij}^h A_h^* \qquad (0 \leqslant i, j \leqslant d).$$

For $0 \leqslant i \leqslant d$,

$$A_i^* = \sum_{j=0}^{d} q_i(j) E_j^*, \qquad\qquad E_i^* = |X|^{-1} \sum_{j=0}^{d} p_i(j) A_j^*.$$

We now recall the subconstituent algebra T and the primary T-module. Let T denote the subalgebra of $\operatorname{Mat}_X(\mathbb{R})$ generated by M and M^*. We call T the subconstituent algebra (or Terwilliger algebra) with respect to x. The algebra T is closed under the transpose map. By [15, Lemma 3.4] the algebra T is semisimple. Moreover by [15, Lemma 3.4] the T-module $\mathbb{V}$ decomposes into an orthogonal direct sum of irreducible T-modules. Among these modules there is a distinguished one, said to be primary. We now describe the primary T-module. Let $\mathbf{1}$ denote the vector in $\mathbb{V}$ that has all entries 1. So $\mathbf{1} = \sum_{y \in X} \widehat{y}$. For $0 \leqslant i \leqslant d$,

$$A_i \widehat{x} = E_i^* \mathbf{1}, \qquad\qquad |X|^{-1} A_i^* \mathbf{1} = E_i \widehat{x}.$$

Therefore $M\widehat{x} = M^*\mathbf{1}$; denote this common vector space by V. By construction V is a T-module with dimension $d+1$. By [15, Lemma 3.6] the T-module V is irreducible. The T-module V is said to be primary. For $0 \leqslant i \leqslant d$ define

$$\mathbf{1}_i = A_i \widehat{x} = E_i^* \mathbf{1}.$$

The vector $\mathbf{1}_i$ is a basis of $E_i^* V$. Moreover $\{\mathbf{1}_i\}_{i=0}^d$ is a basis of V. This basis is orthogonal and $||\mathbf{1}_i||^2 = k_i$ where $k_i = \operatorname{rank}(E_i^*)$ $(0 \leqslant i \leqslant d)$. The basis $\{\mathbf{1}_i\}_{i=0}^d$ diagonalizes M^*. For $0 \leqslant i, j \leqslant d$,

$$E_i^* \mathbf{1}_j = \delta_{i,j} \mathbf{1}_j, \qquad\qquad A_i \mathbf{1}_j = \sum_{h=0}^{d} p_{ij}^h \mathbf{1}_h.$$

For $0 \leqslant i \leqslant d$ define

$$\mathbf{1}_i^* = |X|^{-1} A_i^* \mathbf{1} = E_i \widehat{x}.$$

The vector $\mathbf{1}_i^*$ is a basis of $E_i V$. Moreover $\{\mathbf{1}_i^*\}_{i=0}^d$ is a basis of V. This basis is orthogonal and $||\mathbf{1}_i^*||^2 = k_i^*$ where $k_i^* = \operatorname{rank}(E_i)$ $(0 \leqslant i \leqslant d)$. The basis $\{\mathbf{1}_i^*\}_{i=0}^d$ diagonalizes M. For $0 \leqslant i, j \leqslant d$,

$$E_i \mathbf{1}_j^* = \delta_{i,j} \mathbf{1}_j^*, \qquad\qquad A_i^* \mathbf{1}_j^* = \sum_{h=0}^{d} q_{ij}^h \mathbf{1}_h^*.$$

The bases $\{\mathbf{1}_i\}_{i=0}^d$ and $\{\mathbf{1}_i^*\}_{i=0}^d$ are related by

$$\mathbf{1}_i = \sum_{j=0}^{d} p_i(j) \mathbf{1}_j^*, \qquad\qquad \mathbf{1}_i^* = |X|^{-1} \sum_{j=0}^{d} q_i(j) \mathbf{1}_j$$

for $0 \leqslant i \leqslant d$. The following bases for V are of interest:

(i) $\{\mathbf{1}_i\}_{i=0}^d$, (ii) $\{k_i^{-1}\mathbf{1}_i\}_{i=0}^d$, (iii) $\{\mathbf{1}_i^*\}_{i=0}^d$, (iv) $\{|X|(k_i^*)^{-1}\mathbf{1}_i^*\}_{i=0}^d$.

The bases (i), (ii) are dual with respect to $\langle\ ,\ \rangle$. Moreover the bases (iii), (iv) are dual with respect to $\langle\ ,\ \rangle$.

The algebras M and M^* are related as follows. For $0 \leqslant h, i, j \leqslant d$,

$$E_i^* A_j E_h^* = 0 \qquad \text{if and only if} \qquad p_{ij}^h = 0,$$
$$E_i A_j^* E_h = 0 \qquad \text{if and only if} \qquad q_{ij}^h = 0.$$

For $0 \leqslant i \leqslant d$,

$$A_i E_0^* E_0 = E_i^* E_0, \qquad\qquad A_i^* E_0 E_0^* = E_i E_0^*.$$

For $0 \leqslant i \leqslant d$,

$$E_0 E_i^* \neq 0, \qquad E_0^* E_i \neq 0, \qquad E_i^* E_0 \neq 0, \qquad E_i E_0^* \neq 0.$$

We summarize the above description with four statements about V:

(i) the $\{E_i\}_{i=0}^d$ act on V as a system of mutually orthogonal rank 1 idempotents;
(ii) the $\{E_i^*\}_{i=0}^d$ act on V as a system of mutually orthogonal rank 1 idempotents;
(iii) $E_0 E_i^* E_0$ is nonzero on V for $0 \leqslant i \leqslant d$;
(iv) $E_0^* E_i E_0^*$ is nonzero on V for $0 \leqslant i \leqslant d$.

The above statements (i)–(iv) have the following significance. We will show that (i)–(iv) together with the symmetry of the matrices $\{E_i\}_{i=0}^d$, $\{E_i^*\}_{i=0}^d$ are sufficient to recover the T-module V at an algebraic level.

We now turn our attention to idempotent systems. An idempotent system is defined as follows. Let $\mathbb{F}$ denote a field. Let d denote a nonnegative integer, and let V denote a vector space over $\mathbb{F}$ with dimension $d+1$. Let $\mathrm{End}(V)$ denote the $\mathbb{F}$-algebra consisting of the $\mathbb{F}$-linear maps from V to V. An idempotent system on V is a sequence $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ such that

(i) $\{E_i\}_{i=0}^d$ is a system of mutually orthogonal rank 1 idempotents in $\mathrm{End}(V)$;
(ii) $\{E_i^*\}_{i=0}^d$ is a system of mutually orthogonal rank 1 idempotents in $\mathrm{End}(V)$;
(iii) $E_0 E_i^* E_0 \neq 0 \;\; (0 \leqslant i \leqslant d)$;
(iv) $E_0^* E_i E_0^* \neq 0 \;\; (0 \leqslant i \leqslant d)$.

The above idempotent system Φ is said to be symmetric whenever there exists an antiautomorphism $\dagger$ of $\mathrm{End}(V)$ that fixes each of E_i, E_i^* for $0 \leqslant i \leqslant d$. The map $\dagger$ corresponds to the transpose map.

Let $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote a symmetric idempotent system on V. Using Φ we will define some elements $\{A_i\}_{i=0}^d$, $\{A_i^*\}_{i=0}^d$ in $\mathrm{End}(V)$ and some scalars

$$\nu, \quad k_i, \quad k_i^*, \quad p_{ij}^h, \quad q_{ij}^h, \quad p_i(j), \quad q_i(j) \tag{1}$$

in $\mathbb{F}$. The scalar ν corresponds to $|X|$. We will endow V with a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$. We will define four orthogonal bases of V that correspond to the four earlier bases of interest. We will show that the resulting construction matches the primary T-module at an algebraic level.

Our definitions are summarized as follows. Note that $\{E_i\}_{i=0}^d$ form a basis for a commutative subalgebra $\mathcal{M}$ of $\mathrm{End}(V)$. We show that for $0 \leqslant i \leqslant d$ there exists a unique $A_i \in \mathcal{M}$ such that

$$A_i E_0^* E_0 = E_i^* E_0.$$

We show that $\{A_i\}_{i=0}^d$ is a basis for the vector space $\mathcal{M}$. Similarly, the $\{E_i^*\}_{i=0}^d$ form a basis for a commutative subalgebra $\mathcal{M}^*$ of $\mathrm{End}(V)$. We show that for $0 \leqslant i \leqslant d$ there exists a unique $A_i^* \in \mathcal{M}^*$ such that

$$A_i^* E_0 E_0^* = E_i E_0^*.$$

We show that $\{A_i^*\}_{i=0}^d$ is a basis for the vector space $\mathcal{M}^*$.

Concerning the scalars (1), we show that $\mathrm{tr}(E_0 E_0^*) \neq 0$. The scalar ν is defined by

$$\nu = \mathrm{tr}(E_0 E_0^*)^{-1}.$$

The scalars k_i, k_i^* are defined by

$$k_i = \nu \ \mathrm{tr}(E_0 E_i^*), \qquad k_i^* = \nu \ \mathrm{tr}(E_0^* E_i) \qquad (0 \leqslant i \leqslant d).$$

We show that $\sum_{i=0}^d k_i = \nu = \sum_{i=0}^d k_i^*$, and each of k_i, k_i^* is nonzero for $0 \leqslant i \leqslant d$. The scalars p^h_{ij}, q^h_{ij} are defined by

$$A_i A_j = \sum_{h=0}^d p^h_{ij} A_h, \qquad A_i^* A_j^* = \sum_{h=0}^d q^h_{ij} A_h^* \qquad (0 \leqslant i, j \leqslant d).$$

The scalars $p_i(j)$, $q_i(j)$ are defined by

$$A_i = \sum_{j=0}^d p_i(j) E_j, \qquad A_i^* = \sum_{j=0}^d q_i(j) E_j^* \qquad (0 \leqslant i \leqslant d).$$

We define a bilinear form $\langle \ , \ \rangle$ on V as follows. By linear algebra, there exists a nondegenerate bilinear form $\langle \ , \ \rangle$ on V such that $\langle Bu, v\rangle = \langle u, B^\dagger v\rangle$ for all $B \in \mathrm{End}(V)$ and $u, v \in V$. The bilinear form $\langle \ , \ \rangle$ is unique up to multiplication by a nonzero scalar in $\mathbb{F}$. The bilinear form $\langle \ , \ \rangle$ is symmetric.

Fix nonzero ξ, ζ in E_0V and nonzero ξ^*, ζ^* in E_0^*V. We show that each of the following (i)–(iv) is an orthogonal basis for V:

(i) $\{E_i^*\xi\}_{i=0}^d$, (ii) $\{k_i^{-1} E_i^*\zeta\}_{i=0}^d$, (iii) $\{E_i \xi^*\}_{i=0}^d$, (iv) $\{(k_i^*)^{-1} E_i \zeta^*\}_{i=0}^d$.

The bases (i), (ii) are dual if and only if $\langle \xi, \zeta\rangle = \nu$, and the bases (iii), (iv) are dual if and only if $\langle \xi^*, \zeta^*\rangle = \nu$.

We just summarized our definitions. In the main body of the paper, we show that the resulting defined objects are related in a manner that matches the primary T-module. To describe this relationship, we use some equations involving the $\{E_i\}_{i=0}^d$, $\{E_i^*\}_{i=0}^d$, $\{A_i\}_{i=0}^d$, $\{A_i^*\}_{i=0}^d$ called the reduction rules.

Near the end of the paper we introduce the P-polynomial and Q-polynomial properties for symmetric idempotent systems. We show that a symmetric idempotent system that is P-polynomial and Q-polynomial is essentially the same thing as a Leonard system in the sense of [16, Definition 4.1].

The paper is organized as follows. In Section 2 we recall some basic results from linear algebra. In Section 3 we introduce the concept of an idempotent system. In Section 4 we introduce the scalar ν and discuss some related topics. In Section 5 we introduce the symmetric idempotent systems. In Sections 6, 7 we introduce a certain linear bijection $\rho : \mathcal{M} \to \mathcal{M}^*$ and use it to define the elements A_i, A_i^*. In Sections 8, 9 we introduce the scalars k_i, k_i^* and obtain some reduction rules involving these scalars. In Sections 10, 11 we introduce the scalars p^h_{ij}, q^h_{ij} and obtain some reduction rules involving these scalars. In Sections 12, 13 we introduce the scalars $p_i(j)$, $q_i(j)$ and obtain some reduction rules involving these scalars. In Section 14 we put some of our earlier results in matrix form. In Sections 15–17 we introduce the four bases of interest and discuss their properties. In Section 18 we obtain the transition matrices between these four bases, and the inner products between these four bases. We also obtain the matrices representing A_i, A_i^*, E_i, E_i^* with respect to these four bases. In Section 19 we introduce the P-polynomial and Q-polynomial properties. In Section 20 we recall the notion of a Leonard pair and a Leonard system. In Section 21 we show that a Leonard system is essentially the same thing as a symmetric idempotent system that is P-polynomial and Q-polynomial.

The reader might wonder how the concept of a symmetric idempotent system is related to the concept of a character algebra [10]. Roughly speaking, a symmetric idempotent system is obtained by gluing together a character algebra and its dual; we will discuss this in a future paper.

2. Preliminaries

In this section we fix some notation and recall some basic concepts. Throughout this paper $\mathbb{F}$ denotes a field. By a *scalar* we mean an element of $\mathbb{F}$. All algebras and vector spaces discussed in this paper are over $\mathbb{F}$. All algebras discussed in this paper are associative and have a multiplicative identity. For an algebra $\mathcal{A}$, by an *automorphism* of $\mathcal{A}$ we mean an algebra isomorphism $\mathcal{A} \to \mathcal{A}$, and by an *antiautomorphism* of $\mathcal{A}$ we mean an $\mathbb{F}$-linear bijection $\tau : \mathcal{A} \to \mathcal{A}$ such that $(YZ)^\tau = Z^\tau Y^\tau$ for $Y, Z \in \mathcal{A}$. For the rest of this paper, fix an integer $d \geqslant 0$ and let V denote a vector space with dimension $d+1$. Let $\mathrm{End}(V)$ denote the algebra consisting of the $\mathbb{F}$-linear maps from V to V. Let $\mathrm{Mat}_{d+1}(\mathbb{F})$ denote the algebra consisting of the $d+1$ by $d+1$ matrices that have all entries in $\mathbb{F}$. We index the rows and columns by $0, 1, \ldots, d$. The identity of $\mathrm{End}(V)$ or $\mathrm{Mat}_{d+1}(\mathbb{F})$ is denoted by I. For $A \in \mathrm{End}(V)$, the dimension of AV is called the *rank of* A. A matrix $M \in \mathrm{Mat}_{d+1}(\mathbb{F})$ is said to be *tridiagonal* whenever the (i,j)-entry $M_{i,j} = 0$ if $|i-j| > 1$ $(0 \leqslant i, j \leqslant d)$. Assume for the moment that M is tridiagonal. Then M is said to be *irreducible* whenever $M_{i,j} \neq 0$ if $|i-j| = 1$ $(0 \leqslant i, j \leqslant d)$. We recall how each basis $\{v_i\}_{i=0}^d$ of V gives an algebra isomorphism $\mathrm{End}(V) \to \mathrm{Mat}_{d+1}(\mathbb{F})$. For $A \in \mathrm{End}(V)$ and $M \in \mathrm{Mat}_{d+1}(\mathbb{F})$, we say that M *represents* A *with respect to* $\{v_i\}_{i=0}^d$ whenever $Av_j = \sum_{i=0}^d M_{i,j} v_i$ for $0 \leqslant j \leqslant d$. The isomorphism sends A to the unique matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$ that represents A with respect to $\{v_i\}_{i=0}^d$. Next we recall the transition matrix between two bases of V. Let $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ denote bases of V. By the *transition matrix from* $\{u_i\}_{i=0}^d$ *to* $\{v_i\}_{i=0}^d$ we mean the matrix $T \in \mathrm{Mat}_{d+1}(\mathbb{F})$ such that $v_j = \sum_{i=0}^d T_{i,j} u_i$ for $0 \leqslant j \leqslant d$. Let T denote the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. Then T is invertible and T^{-1} is the transition matrix from$\{v_i\}_{i=0}^d$ to $\{u_i\}_{i=0}^d$. Let T' denote the transition matrix from $\{v_i\}_{i=0}^d$ to a basis $\{w_i\}_{i=0}^d$ of V. Then the transition matrix from $\{u_i\}_{i=0}^d$ to $\{w_i\}_{i=0}^d$ is TT'. For $A \in \mathrm{End}(V)$ let M denote the matrix representing A with respect to $\{u_i\}_{i=0}^d$. Then $T^{-1}MT$ represents A with respect to $\{v_i\}_{i=0}^d$. Let $A \in \mathrm{End}(V)$. A subspace $W \subseteq V$ is called an *eigenspace* of A whenever $W \neq 0$ and there exists a scalar θ such that $W = \{v \in V \mid Av = \theta v\}$; in this case θ is the *eigenvalue* of A associated with W. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A. We say that A is *multiplicity-free* whenever A is diagonalizable and its eigenspaces all have dimension one.

Definition 2.1. *By a* decomposition of V *we mean a sequence* $\{V_i\}_{i=0}^d$ *consisting of one-dimensional subspaces of* V *such that* $V = \sum_{i=0}^d V_i$ *(direct sum).*

Definition 2.2 ([6, Section 6A]). *By a* system of mutually orthogonal rank 1 idempotents *in* $\mathrm{End}(V)$ *we mean a sequence* $\{E_i\}_{i=0}^d$ *of elements in* $\mathrm{End}(V)$ *such that*

$$E_i E_j = \delta_{i,j} E_i \qquad (0 \leqslant i, j \leqslant d),$$

$$\mathrm{rank}(E_i) = 1 \qquad (0 \leqslant i \leqslant d).$$

Lemma 2.3. *The following hold.*

(i) *Let* $\{V_i\}_{i=0}^d$ *denote a decomposition of* V. *For* $0 \leqslant i \leqslant d$ *define* $E_i \in \mathrm{End}(V)$ *such that* $(E_i - I)V_i = 0$ *and* $E_i V_j = 0$ *if* $j \neq i$ $(0 \leqslant j \leqslant d)$. *Then* $\{E_i\}_{i=0}^d$ *is a system of mutually orthogonal rank* 1 *idempotents in* $\mathrm{End}(V)$.

(ii) *Let* $\{E_i\}_{i=0}^d$ *denote a system of mutually orthogonal rank* 1 *idempotents in* $\mathrm{End}(V)$. *Then* $\{E_i V\}_{i=0}^d$ *is a decomposition of* V.

Definition 2.4. *Let A denote a multiplicity-free element in* $\operatorname{End}(V)$*, and let* $\{V_i\}_{i=0}^d$ *denote an ordering of the eigenspaces of A. Then $\{V_i\}_{i=0}^d$ is a decomposition of V. Let $\{E_i\}_{i=0}^d$ denote the corresponding system of mutually orthogonal rank 1 idempotents from Lemma* 2.3(i)*. We call $\{E_i\}_{i=0}^d$ the* primitive idempotents of A.

For the rest of this section, let $\{E_i\}_{i=0}^d$ denote a system of mutually orthogonal rank 1 idempotents in $\operatorname{End}(V)$. The next two lemmas are routinely verified.

Lemma 2.5. *The following hold:*

(i) $\operatorname{tr}(E_i) = 1$ $(0 \leqslant i \leqslant d)$*, where* tr *means trace.*
(ii) $I = \sum_{i=0}^d E_i$*;*
(iii) $\{E_i\}_{i=0}^d$ *form a basis for a commutative subalgebra of* $\operatorname{End}(V)$*.*

Lemma 2.6. *For* $\mathcal{A} = \operatorname{End}(V)$*,*

(i) *the sum* $\mathcal{A} = \sum_{i=0}^d \sum_{j=0}^d E_i \mathcal{A} E_j$ *is direct;*
(ii) $\dim E_i \mathcal{A} E_j = 1$ *for* $0 \leqslant i, j \leqslant d$*.*

3. Idempotent systems

Recall the vector space V with dimension $d+1$. In this section we introduce the notion of an idempotent system on V.

Definition 3.1. *By an* idempotent system *on V we mean a sequence*

$$(\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$$

such that

(i) $\{E_i\}_{i=0}^d$ *is a system of mutually orthogonal rank* 1 *idempotents in* $\operatorname{End}(V)$*;*
(ii) $\{E_i^*\}_{i=0}^d$ *is a system of mutually orthogonal rank* 1 *idempotents in* $\operatorname{End}(V)$*;*
(iii) $E_0 E_i^* E_0 \neq 0 \quad (0 \leqslant i \leqslant d)$*;*
(iv) $E_0^* E_i E_0^* \neq 0 \quad (0 \leqslant i \leqslant d)$*.*

Let $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote an idempotent system on V. Define

$$\Phi^* = \left(\{E_i^*\}_{i=0}^d; \{E_i\}_{i=0}^d\right).$$

Then Φ^* is an idempotent system on V, called the *dual* of Φ. We have $(\Phi^*)^* = \Phi$. For an object f attached to Φ, the corresponding object attached to Φ^* is denoted by f^*.

Let $\Phi' = (\{E_i'\}_{i=0}^d; \{E_i^{*\prime}\}_{i=0}^d)$ denote an idempotent system on a vector space V'. By an *isomorphism of idempotent systems from Φ to Φ'* we mean an algebra isomorphism $\operatorname{End}(V) \to \operatorname{End}(V')$ that sends $E_i \mapsto E_i'$, $E_i^* \mapsto E_i^{*\prime}$ for $0 \leqslant i \leqslant d$. We say that Φ and Φ' are *isomorphic* whenever there exists an isomorphism of idempotent systems from Φ to Φ'. By the Skolem–Noether theorem (see [14, Corollary 7.125]), a map $\sigma : \operatorname{End}(V) \to \operatorname{End}(V')$ is an algebra isomorphism if and only if there exists an $\mathbb{F}$-linear bijection $S : V \to V'$ such that $A^\sigma = SAS^{-1}$ for all $A \in \operatorname{End}(V)$.

Definition 3.2. *Let $\mathcal{M}$ denote the subalgebra of* $\operatorname{End}(V)$ *generated by $\{E_i\}_{i=0}^d$. Note that $\mathcal{M}$ is commutative, and $\{E_i\}_{i=0}^d$ form a basis of the vector space $\mathcal{M}$.*

4. The scalars m_i, ν

Let $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote an idempotent system on V. In this section we use Φ to introduce some scalars $\{m_i\}_{i=0}^d$, ν.

Definition 4.1. *For* $0 \leqslant i \leqslant d$ *define*

$$m_i = \operatorname{tr}(E_0^* E_i). \tag{2}$$

Lemma 4.2. *For $0 \leqslant i \leqslant d$ the following hold:*

(i) $E^*_0 E_i E^*_0 = m_i E^*_0$;
(ii) $E_0 E^*_i E_0 = m^*_i E_0$.

Proof. (i) Abbreviate $\mathcal{A} = \mathrm{End}(V)$. By Lemma 2.6(ii), E^*_0 is a basis for the vector space $E^*_0 \mathcal{A} E^*_0$. So there exists a scalar α_i such that $E^*_0 E_i E^*_0 = \alpha_i E^*_0$. In this equation, take the trace of each side and simplify the result using Lemma 2.5(i) and $\mathrm{tr}(MN) = \mathrm{tr}(NM)$ to obtain $\alpha_i = m_i$. The result follows.

(ii) Apply (i) to Φ^*. □

Lemma 4.3. *For $0 \leqslant i \leqslant d$ the following hold:*

(i) $E_i E^*_0 E_i = m_i E_i$;
(ii) $E^*_i E_0 E^*_i = m^*_i E^*_i$.

Proof. Similar to the proof of Lemma 4.2. □

Lemma 4.4. *The following hold:*

(i) $m_i \neq 0 \quad (0 \leqslant i \leqslant d)$;
(ii) $\sum_{i=0}^d m_i = 1$.

Proof. (i) Use Definition 3.1(iv) and Lemma 4.2(i).

(i) By Lemma 2.5(ii), $\sum_{i=0}^d E_i = I$. In this equation, multiply each side on the left by E^*_0 to get $\sum_{i=0}^d E^*_0 E_i = E^*_0$. In this equation, take the trace of each side, and evaluate the result using Lemma 2.5(i) and Definition 4.1. □

Definition 4.5. *Setting $i = 0$ in (2) we find that $m_0 = m^*_0$; let ν denote the multiplicative inverse of this common value. We emphasize $\nu = \nu^*$ and*

$$\mathrm{tr}(E_0 E^*_0) = \nu^{-1}. \tag{3}$$

Lemma 4.6. *We have*

$$\nu E_0 E^*_0 E_0 = E_0, \qquad\qquad \nu E^*_0 E_0 E^*_0 = E^*_0. \tag{4}$$

Proof. To get the equation on the left in (4), set $i = 0$ in Lemma 4.2(ii) and use Definition 4.5. Applying this to Φ^* we get the equation on the right in (4). □

Lemma 4.7. *Each of the following is a basis of the vector space* $\mathrm{End}(V)$*:*

(i) $\{E_i E^*_0 E_j \,|\, 0 \leqslant i, j \leqslant d\}$;
(ii) $\{E^*_i E_0 E^*_j \,|\, 0 \leqslant i, j \leqslant d\}$.

Proof. (i) In view of Lemma 2.6, it suffices to show that $E_i E^*_0 E_j \neq 0$ for $0 \leqslant i, j \leqslant d$. Let i, j be given, and suppose $E_i E^*_0 E_j = 0$. Using Lemmas 4.2(i) and 4.4(i),

$$0 = E^*_0 (E_i E^*_0 E_j) E^*_0 = m_i m_j E^*_0 \neq 0$$

for a contradiction. The result follows.

(ii) Apply (i) to Φ^*. □

Lemma 4.8. *Each of the following is a generating set for the algebra* $\mathrm{End}(V)$*:*

(i) E^*_0 *and* $\mathcal{M}$;
(ii) E_0 *and* $\mathcal{M}^*$;
(iii) $\mathcal{M}$ *and* $\mathcal{M}^*$.

Proof. (i) By Definition 3.2 and Lemma 4.7(i).

(ii) Apply (i) to Φ^*.

(iii) By (i) above and Definition 3.2. □

5. Symmetric idempotent systems

We continue to discuss an idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ on V.

Definition 5.1. *We say that Φ is* symmetric *whenever there exists an antiautomorphism $\dagger$ of $\mathrm{End}(V)$ that fixes each of E_i, E_i^* for $0 \leqslant i \leqslant d$.*

Recall the algebra $\mathcal{M}$ from Definition 3.2.

Lemma 5.2. *Assume that Φ is symmetric, and let $\dagger$ denote an antiautomorphism of $\mathrm{End}(V)$ from Definition 5.1. Then the following hold:*

(i) *$\dagger$ is unique;*
(ii) *$(A^\dagger)^\dagger = A$ for $A \in \mathrm{End}(V)$;*
(iii) *$\dagger$ fixes every element in $\mathcal{M}$ and every element in $\mathcal{M}^*$.*

Proof. (iii) By Definitions 3.2 and 5.1.

(ii) The composition $\dagger \circ \dagger$ is an automorphism of $\mathrm{End}(V)$ that fixes everything in $\mathcal{M}$ and everything in $\mathcal{M}^*$. This automorphism is the identity in view of Lemma 4.8(iii).

(i) Let μ denote an antiautomorphism of $\mathrm{End}(V)$ that fixes each of E_i, E_i^* for $0 \leqslant i \leqslant d$. We show $\mu = \dagger$. The composition $\dagger \circ \mu$ is an automorphism of $\mathrm{End}(V)$ that fixes everything in $\mathcal{M}$ and everything in $\mathcal{M}^*$. So this automorphism is the identity. We have $\dagger = \dagger^{-1}$ by (ii) above, so $\mu = \dagger$. $\square$

6. The map ρ

Let $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote a symmetric idempotent system on V. Recall the algebra $\mathcal{M}$ from Definition 3.2. In this section we introduce a certain map $\rho : \mathcal{M} \to \mathcal{M}^*$ that will play an essential role in our theory. As we will see, ρ is an isomorphism of vector spaces but not algebras.

Lemma 6.1. *For $\mathcal{A} = \mathrm{End}(V)$,*

(i) *the elements $\{E_i E_0^*\}_{i=0}^d$ form a basis of $\mathcal{A}E_0^*$;*
(ii) *the elements $\{E_i^* E_0\}_{i=0}^d$ form a basis of $\mathcal{A}E_0$.*

Proof. (i) By Lemmas 2.5(ii) and 2.6(i) the sum $\mathcal{A}E_0^* = \sum_{i=0}^d E_i^* \mathcal{A} E_0^*$ is direct. Each summand has dimension one by Lemma 2.6(ii), so $\mathcal{A}E_0^*$ has dimension $d+1$. The elements $\{E_i E_0^*\}_{i=0}^d$ are contained in $\mathcal{A}E_0^*$. We show that these elements are linearly independent. For scalars $\{\alpha_i\}_{i=0}^d$ suppose $0 = \sum_{i=0}^d \alpha_i E_i E_0^*$. For $0 \leqslant r \leqslant d$, multiply each side of this equation on the left by E_r to obtain $0 = \alpha_r E_r E_0^*$. We have $E_r E_0^* \neq 0$ by Definition 3.1(iv), so $\alpha_r = 0$. We have shown that $\{E_i E_0^*\}_{i=0}^d$ are linearly independent, and hence a basis of $\mathcal{A}E_0^*$.

(ii) Apply (i) to Φ^*. $\square$

Lemma 6.2. *For $\mathcal{A} = \mathrm{End}(V)$,*

(i) *the map $\mathcal{M} \to \mathcal{A}E_0^*$, $Y \mapsto YE_0^*$ is an $\mathbb{F}$-linear bijection;*
(ii) *the map $\mathcal{M}^* \to \mathcal{A}E_0$, $Y \mapsto YE_0$ is an $\mathbb{F}$-linear bijection.*

Proof. (i) Clearly the map is $\mathbb{F}$-linear. By Lemma 6.1(i), the map sends the basis $\{E_i\}_{i=0}^d$ of $\mathcal{M}$ to the basis $\{E_i E_0^*\}_{i=0}^d$ of $\mathcal{A}E_0^*$. So the map is bijective.

(ii) Apply (ii) to Φ^*. $\square$

Lemma 6.3. *There exists a unique $\mathbb{F}$-linear map $\rho : \mathcal{M} \to \mathcal{M}^*$ such that for $Y \in \mathcal{M}$,*

$$YE_0^* E_0 = Y^\rho E_0. \tag{5}$$

Proof. Abbreviate $\mathcal{A} = \mathrm{End}(V)$. Concerning existence, consider the $\mathbb{F}$-linear map $g : \mathcal{M} \to \mathcal{A}E_0$, $Y \mapsto YE_0^*E_0$. Let μ denote the map in Lemma 6.2(ii). The composition

$$\rho : \mathcal{M} \xrightarrow{g} \mathcal{A}E_0 \xrightarrow{\mu^{-1}} \mathcal{M}^*$$

satisfies (5). We have shown that ρ exists. The map ρ is unique by Lemma 6.2(ii). $\square$

Lemma 6.4. *The maps ρ and $\nu\rho^*$ are inverses. In particular, the maps ρ, ρ^* are bijective.*

Proof. Pick $Y \in \mathcal{M}$. Using Lemma 4.6 and applying (5) to both Φ and Φ^*,

$$(Y^\rho)^{\rho^*}E_0^* = Y^\rho E_0 E_0^* = YE_0^*E_0E_0^* = \nu^{-1}YE_0^*.$$

By this and Lemma 6.2(i) we get $(Y^\rho)^{\rho^*} = \nu^{-1}Y$. Applying this to Φ^*, $(Z^{\rho^*})^\rho = \nu^{-1}Z$ for $Z \in \mathcal{M}^*$. Thus the maps ρ and $\nu\rho^*$ are inverses. $\square$

Lemma 6.5. *The map ρ sends $I \mapsto E_0^*$ and $E_0 \mapsto \nu^{-1}I$.*

Proof. Using Lemma 6.3, $E_0^*E_0 = IE_0^*E_0 = I^\rho E_0$. This forces $E_0^* = I^\rho$ by Lemma 6.2(ii). Using Lemmas 4.6 and 6.3, $E_0^\rho E_0 = E_0E_0^*E_0 = \nu^{-1}E_0$. This forces $E_0^\rho = \nu^{-1}I$ by Lemma 6.2(ii). $\square$

7. The elements A_i

We continue to discuss a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ on V.

Definition 7.1. *For $0 \leqslant i \leqslant d$ define*

$$A_i = \nu(E_i^*)^{\rho^*}. \tag{6}$$

Lemma 7.2. *For $0 \leqslant i \leqslant d$, ρ sends $A_i \mapsto E_i^*$ and $E_i \mapsto \nu^{-1}A_i^*$.*

Proof. By Lemma 6.4 and Definition 7.1, $A_i^\rho = \nu((E_i^*)^{\rho^*})^\rho = E_i^*$. Applying (6) to Φ^*, $E_i^\rho = \nu^{-1}A_i^*$. $\square$

Lemma 7.3. *The antiautomorphism $\dagger$ from Definition 5.1 fixes each of A_i, A_i^* for $0 \leqslant i \leqslant d$.*

Proof. By Lemma 5.2(iii) and since $A_i \in \mathcal{M}$, $A_i^* \in \mathcal{M}^*$ for $0 \leqslant i \leqslant d$. $\square$

Lemma 7.4. *For $0 \leqslant i \leqslant d$ the following hold:*
- (i) $A_iE_0^*E_0 = E_i^*E_0$;
- (ii) $A_i^*E_0E_0^* = E_iE_0^*$;
- (iii) $E_0E_0^*A_i = E_0E_i^*$;
- (iv) $E_0^*E_0A_i^* = E_0^*E_i$.

Proof. (i) Use Lemmas 6.3, 7.2.

(ii) Apply (i) to Φ^*.

(iii), (iv) For the equations in (i) and (ii), apply $\dagger$ to each side and use Lemma 7.3. $\square$

Lemma 7.5. *We have $A_0 = I$.*

Proof. By Lemma 6.5, $I^\rho = E_0^*$. In this equation, apply ρ^* to each side and evaluate the result using Lemma 6.4 and Definition 7.1. $\square$

Lemma 7.6. *We have $\sum_{i=0}^d A_i = \nu E_0$.*

Proof. In the equation $\sum_{i=0}^d E_i^* = I$, apply ρ^* to each side and evaluate the result using Definition 7.1 along with Lemma 6.5 applied to Φ^*. $\square$

Lemma 7.7. *The elements* $\{A_i\}_{i=0}^d$ *form a basis of the vector space* $\mathcal{M}$.

Proof. By Lemmas 6.4, 7.2 and since $\{E_i^*\}_{i=0}^d$ form a basis of the vector space $\mathcal{M}^*$. $\square$

8. The scalars k_i

We continue to discuss a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ on V. In this section we use Φ to define some scalars k_i that will play a role in our theory.

Definition 8.1. *For* $0 \leqslant i \leqslant d$ *let* k_i *denote the eigenvalue of* A_i *corresponding to* E_0.

Lemma 8.2. *For* $0 \leqslant i \leqslant d$ *the following hold:*

(i) $A_i E_0 = E_0 A_i = k_i E_0$;
(ii) $A_i^* E_0^* = E_0^* A_i^* = k_i^* E_0^*$.

Proof. (i) By Definition 8.1.
(ii) Apply (i) to Φ^*. $\square$

Lemma 8.3. *For* $0 \leqslant i \leqslant d$ *the following hold:*

(i) $k_i = \nu m_i^*$;
(ii) $k_i^* = \nu m_i$.

Proof. (i) By Lemma 7.4(i), $E_0 A_i E_0^* E_0 = E_0 E_i^* E_0$. In this equation, evaluate the left-hand side using Lemmas 4.6, 8.2(i), and evaluate the right-hand side using Lemma 4.2(ii). This gives $k_i \nu^{-1} E_0 = m_i^* E_0$. The result follows.
(ii) Apply (i) to Φ^*. $\square$

Lemma 8.4. *The following hold:*

(i) $k_i \neq 0 \qquad (0 \leqslant i \leqslant d)$;
(ii) $\nu = \sum_{i=0}^d k_i$;
(iii) $k_0 = 1$.

Proof. (i) Apply Lemma 4.4(i) to Φ^* and use Lemma 8.3(i).
(ii) Apply Lemma 4.4(ii) to Φ^* and use Lemma 8.3(i).
(iii) By Definition 4.5 and Lemma 8.3(i). $\square$

9. Some reduction rules

We continue to discuss a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ on V. In this section we obtain some reduction rules for Φ. Recall the antiautomorphism $\dagger$ of $\operatorname{End}(V)$ from Definition 5.1.

Lemma 9.1. *For* $0 \leqslant i \leqslant d$ *the following hold:*

(i) $E_i E_0^* E_0 = \nu^{-1} A_i^* E_0$;
(ii) $E_i^* E_0 E_0^* = \nu^{-1} A_i E_0^*$;
(iii) $E_0 E_0^* E_i = \nu^{-1} E_0 A_i^*$;
(iv) $E_0^* E_0 E_i^* = \nu^{-1} E_0^* A_i$.

Proof. (i) Set $Y = E_i$ in (5) and use Lemma 7.2.
(ii) Apply (i) to Φ^*.
(iii), (iv) For the equations in (i) and (ii), apply $\dagger$ to each side. $\square$

Lemma 9.2. *For* $0 \leqslant i, j \leqslant d$ *the following hold:*

(i) $E_j^* A_i E_0^* = \delta_{i,j} A_i E_0^*$;
(ii) $E_j A_i^* E_0 = \delta_{i,j} A_i^* E_0$;
(iii) $E_0^* A_i E_j^* = \delta_{i,j} E_0^* A_i$;
(iv) $E_0 A_i^* E_j = \delta_{i,j} E_0 A_i^*$.

Proof. (i) For the equation in Lemma 9.1(ii), multiply each side on the left by E^*_j to get $\delta_{i,j} E^*_i E_0 E^*_0 = \nu^{-1} E^*_j A_i E^*_0$. In this equation, evaluate the left-hand side using Lemma 9.1(ii).

(ii) Apply (i) to Φ^*.

(iii), (iv) For the equations in (i) and (ii), apply † to each side. □

Lemma 9.3. *For $0 \leqslant i, j \leqslant d$ the following hold:*

(i) $E_0 E^*_j A_i E^*_0 = \delta_{i,j} k_i E_0 E^*_0$;
(ii) $E^*_0 E_j A^*_i E_0 = \delta_{i,j} k^*_i E^*_0 E_0$;
(iii) $E^*_0 A_i E^*_j E_0 = \delta_{i,j} k_i E^*_0 E_0$;
(iv) $E_0 A^*_i E_j E^*_0 = \delta_{i,j} k^*_i E_0 E^*_0$.

Proof. (i) Using Lemmas 9.2(i), 8.2(i) in order,

$$E_0 E^*_j A_i E_0 = \delta_{i,j} E_0 A_i E^*_0 = \delta_{i,j} k_i E_0 E^*_0.$$

(ii) Apply (i) to Φ^*.

(iii), (iv) For the equations in (i) and (ii), apply † to each side. □

Lemma 9.4. *For $0 \leqslant i, j \leqslant d$ the following hold:*

(i) $A_i E^*_0 A_j = \nu E^*_i E_0 E^*_j$;
(ii) $A^*_i E_0 A^*_j = \nu E_i E^*_0 E_j$.

Proof. (i) Using Lemmas 9.1(iv), 7.4(i) in order,

$$A_i E^*_0 A_j = \nu A_i E^*_0 E_0 E^*_j = \nu E^*_i E_0 E^*_j.$$

(ii) Apply (i) to Φ^*. □

Lemma 9.5. *For $0 \leqslant i, j \leqslant d$ the following hold:*

(i) $E_i E^*_0 A_j = A^*_i E_0 E^*_j$;
(ii) $E^*_i E_0 A^*_j = A_i E^*_0 E_j$;
(iii) $A_j E^*_0 E_i = E^*_j E_0 A^*_i$;
(iv) $A^*_j E_0 E^*_i = E_j E^*_0 A_i$.

Proof. (i) Using Lemmas 9.1(iv), 9.1(i) in order,

$$E_i E^*_0 A_j = \nu E_i E^*_0 E_0 E^*_j = A^*_i E_0 E^*_j.$$

(ii) Apply (i) to Φ^*.

(iii), (iv) For the equations in (i) and (ii), apply † to each side. □

10. The scalars p^h_{ij}, q^h_{ij}

We continue to discuss a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d)$ on V.

Lemma 10.1. *There exist scalars p^h_{ij} $(0 \leqslant h, i, j \leqslant d)$ such that*

$$A_i A_j = \sum_{h=0}^{d} p^h_{ij} A_h \qquad (0 \leqslant i, j \leqslant d). \tag{7}$$

Proof. By Lemma 7.7. □

Definition 10.2. *Referring to Lemma* 10.1*, the scalars p^h_{ij} are called the* intersection numbers *of* Φ.

Definition 10.3. *For $0 \leqslant h, i, j \leqslant d$ define $q^h_{ij} = (p^h_{ij})^*$. We call these scalars the* Krein parameters *of* Φ.

Lemma 10.4. *For $0 \leqslant i, j \leqslant d$,*

$$A_i^* A_j^* = \sum_{h=0}^{d} q_{ij}^h A_h^*. \tag{8}$$

Proof. Apply Lemma 10.1 to Φ^* and use Definition 10.3. □

Lemma 10.5. *For $0 \leqslant h, i, j \leqslant d$ the following hold:*

(i) $p_{ij}^h = p_{ji}^h$;
(ii) $q_{ij}^h = q_{ji}^h$.

Proof. (i) By (7) and since the algebra $\mathcal{M}$ is commutative.
(ii) Apply (i) to Φ^*. □

Lemma 10.6. *For $0 \leqslant h, i \leqslant d$ the following hold:*

(i) $p_{i0}^h = \delta_{h,i}$;
(ii) $p_{0i}^h = \delta_{h,i}$;
(iii) $q_{i0}^h = \delta_{h,i}$;
(iv) $q_{0i}^h = \delta_{h,i}$.

Proof. (i) In (7) set $j = 0$ and use Lemmas 7.5, 7.7.
(ii) By (i) and Lemma 10.5(i).
(iii), (iv) Apply (i), (ii) to Φ^*. □

Lemma 10.7. *For $0 \leqslant h, i, j, t \leqslant d$ the following hold:*

(i) $\sum_{r=0}^{d} p_{hr}^t p_{ij}^r = \sum_{s=0}^{d} p_{hi}^s p_{sj}^t$;
(ii) $\sum_{r=0}^{d} q_{hr}^t q_{ij}^r = \sum_{s=0}^{d} q_{hi}^s q_{sj}^t$.

Proof. (i) Expand $A_h(A_i A_j) = (A_h A_i) A_j$ in two ways using (7), and compare the coefficients using Lemma 7.7.
(ii) Apply (i) to Φ^*. □

Lemma 10.8. *For $0 \leqslant h, i \leqslant d$ the following hold:*

(i) $k_i = \sum_{j=0}^{d} p_{ij}^h$;
(ii) $k_i^* = \sum_{j=0}^{d} q_{ij}^h$.

Proof. (i) Using Lemmas 7.6 and 8.2(i),

$$A_i \sum_{j=0}^{d} A_j = k_i \sum_{h=0}^{d} A_h.$$

By (7),

$$A_i \sum_{j=0}^{d} A_j = \sum_{h=0}^{d} \sum_{j=0}^{d} p_{ij}^h A_h.$$

Compare the above two equations using Lemma 7.7.
(ii) Apply (i) to Φ^*. □

Lemma 10.9. *For $0 \leqslant i, j \leqslant d$ the following hold:*

(i) $p_{ij}^0 = \delta_{i,j} k_i$;
(ii) $q_{ij}^0 = \delta_{i,j} k_i^*$.

Proof. (i) For the equation (7), multiply each side on the left by $E_0 E_0^*$ and on the right by $E_0^* E_0$. Evaluate the result using Lemma 7.4(i),(iii) along with Lemmas 4.2, 8.3.
(ii) Apply (i) to Φ^*. □

Lemma 10.10. *For $0 \leqslant i, j \leqslant d$ the following hold:*

(i) $k_i k_j = \sum_{h=0}^{d} p^h_{ij} k_h$;
(ii) $k^*_i k^*_j = \sum_{h=0}^{d} q^h_{ij} k^*_h$.

Proof. (i) In (7), multiply each side by E_0, and simplify the result using Lemma 8.2(i).
(ii) Apply (i) to Φ^*. □

Lemma 10.11. *For $0 \leqslant h, i, j \leqslant d$ the following hold:*

(i) $k_h p^h_{ij} = k_i p^i_{jh} = k_j p^j_{hi}$;
(ii) $k^*_h q^h_{ij} = k^*_i q^i_{jh} = k^*_j q^j_{hi}$.

Proof. (i) In view of Lemma 10.5(i), it suffices to show that $k_h p^h_{ij} = k_j p^j_{hi}$. To obtain this equation, set $t = 0$ in Lemma 10.7(i), and evaluate the result using Lemma 10.9(i).
(ii) Apply (i) to Φ^*. □

11. Reduction rules involving p^h_{ij}, q^h_{ij}

We continue to discuss a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d)$ on V. In this section we give some reduction rules for Φ that involve the intersection numbers and Krein parameters.

Lemma 11.1. *For $0 \leqslant i, j \leqslant d$ the following hold:*

(i) $A_j E^*_i E_0 = \sum_{h=0}^{d} p^h_{ij} E^*_h E_0$;
(ii) $A^*_j E_i E^*_0 = \sum_{h=0}^{d} q^h_{ij} E_h E^*_0$;
(iii) $E_0 E^*_i A_j = \sum_{h=0}^{d} p^h_{ij} E_0 E^*_h$;
(iv) $E^*_0 E_i A^*_j = \sum_{h=0}^{d} q^h_{ij} E^*_0 E_h$.

Proof. (i) Using Lemmas 7.4(i), 10.1, 7.4(i) in order,

$$A_j E^*_i E_0 = A_j A_i E^*_0 E_0 = \sum_{h=0}^{d} p^h_{ij} A_h E^*_0 E_0 = \sum_{h=0}^{d} p^h_{ij} E^*_h E_0.$$

(ii) Apply (i) to Φ^*.
(iii), (iv) For the equations in (i) and (ii), apply † to each side. □

Lemma 11.2. *For $0 \leqslant h, i, j \leqslant d$ the following hold:*

(i) $E^*_h A_j E^*_i E_0 = p^h_{ij} E^*_h E_0$;
(ii) $E_h A^*_j E_i E^*_0 = q^h_{ij} E_h E^*_0$;
(iii) $E_0 E^*_i A_j E^*_h = p^h_{ij} E_0 E^*_h$;
(iv) $E^*_0 E_i A^*_j E_h = q^h_{ij} E^*_0 E_h$.

Proof. (i) Using Lemma 11.1(i),

$$E^*_h A_j E^*_i E_0 = \sum_{s=0}^{d} p^s_{ij} E^*_h E^*_s E_0 = p^h_{ij} E^*_h E_0.$$

(ii) Apply (i) to Φ^*.
(iii), (iv) For the equations in (i) and (ii), apply † to each side. □

Lemma 11.3. *For $0 \leqslant h, i, j \leqslant d$ the following hold:*

(i) $E_i A^*_j E_h = m_i^{-1} q^h_{ij} E_i E^*_0 E_h$;
(ii) $E^*_i A_j E^*_h = (m^*_i)^{-1} p^h_{ij} E^*_i E_0 E^*_h$.

Proof. (i) In Lemma 11.2(iv), multiply each side on the left by E_i. Simplify the result using Lemma 4.3(i).
(ii) Apply (i) to Φ^*. $\square$

LEMMA 11.4. *For $0 \leqslant h, i, j \leqslant d$ the following hold:*
(i) $E_i^* A_j E_h^* = 0$ *if and only if* $p_{ij}^h = 0$*;*
(ii) $E_i A_j^* E_h = 0$ *if and only if* $q_{ij}^h = 0$*.*

Proof. By Lemmas 4.7 and 11.3. $\square$

LEMMA 11.5. *For $0 \leqslant h, i, j \leqslant d$ the following hold:*
(i) $p_{ij}^h = (m_h^*)^{-1} \operatorname{tr}(E_0 E_i^* A_j E_h^*)$*;*
(ii) $q_{ij}^h = m_h^{-1} \operatorname{tr}(E_0^* E_i A_j^* E_h)$*.*

Proof. (i) In Lemma 11.2(iii), take the trace of each side, and simplify the result using Definition 4.1.
(ii) Apply (i) to Φ^*. $\square$

12. The scalars $p_i(j)$, $q_i(j)$

We continue to discuss a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ on V. In this section we use Φ to define some scalars $p_i(j)$, $q_i(j)$ that will play a role in our theory. Recall the algebra $\mathcal{M}$ from Definition 3.2.

LEMMA 12.1. *There exist scalars $p_i(j)$ $(0 \leqslant i, j \leqslant d)$ such that*

$$A_i = \sum_{j=0}^{d} p_i(j) E_j \qquad (0 \leqslant i \leqslant d). \tag{9}$$

Proof. By Definition 3.2 the elements $\{E_i\}_{i=0}^d$ form a basis of $\mathcal{M}$. By Definition 7.1, $A_i \in \mathcal{M}$ for $0 \leqslant i \leqslant d$. The result follows. $\square$

DEFINITION 12.2. *For $0 \leqslant i, j \leqslant d$ define $q_i(j) = (p_i(j))^*$.*

LEMMA 12.3. *For $0 \leqslant i, j \leqslant d$,*

$$A_i^* = \sum_{j=0}^{d} q_i(j) E_j^* \qquad (0 \leqslant i \leqslant d). \tag{10}$$

Proof. Apply Lemma 12.1 to Φ^* and use Definition 12.2. $\square$

LEMMA 12.4. *For $0 \leqslant i, j \leqslant d$ the following hold:*
(i) $A_i E_j = E_j A_i = p_i(j) E_j$*;*
(ii) $A_i^* E_j^* = E_j^* A_i^* = q_i(j) E_j^*$*.*

In other words, $p_i(j)$ (resp. $q_i(j)$) is the eigenvalue of A_i (resp. A_i^*) associated with $E_j V$ (resp. $E_j^* V$).

Proof. (i) Use (9).
(ii) Apply (i) to Φ^*. $\square$

LEMMA 12.5. *For $0 \leqslant i \leqslant d$ the following hold:*
(i) $E_i^* = \nu^{-1} \sum_{j=0}^d p_i(j) A_j^*$*;*
(ii) $E_i = \nu^{-1} \sum_{j=0}^d q_i(j) A_j$*.*

Proof. (i) In (9), apply ρ to each side and use Lemma 7.2.
(ii) Apply (i) to Φ^*. $\square$

Lemma 12.6. *For $0 \leqslant i, j \leqslant d$ the following hold:*

(i) $\sum_{h=0}^{d} p_i(h) q_h(j) = \delta_{i,j}\nu$;
(ii) $\sum_{h=0}^{d} q_i(h) p_h(j) = \delta_{i,j}\nu$.

Proof. (i) By (9), $A_i = \sum_{h=0}^{d} p_i(h) E_h$. In this equation, eliminate E_h using Lemma 12.5(ii), and compare the coefficients of each side.
(ii) Apply (i) to Φ^*. □

Lemma 12.7. *For $0 \leqslant j \leqslant d$ the following hold:*

(i) $p_0(j) = 1$;
(ii) $q_0(j) = 1$.

Proof. (i) Set $i = 0$ in (9) and recall that $A_0 = I$.
(ii) Apply (i) to Φ^*. □

Lemma 12.8. *For $0 \leqslant i \leqslant d$ the following hold:*

(i) $p_i(0) = k_i$;
(ii) $q_i(0) = k_i^*$.

Proof. (i) Set $j = 0$ in Lemma 12.4(i) and compare the result with Lemma 8.2(i).
(ii) Apply (i) to Φ^*. □

Lemma 12.9. *For $0 \leqslant j \leqslant d$ the following hold:*

(i) $\sum_{h=0}^{d} p_h(j) = \delta_{0,j}\nu$;
(ii) $\sum_{h=0}^{d} q_h(j) = \delta_{0,j}\nu$.

Proof. (i) Set $i = 0$ in Lemma 12.6(ii), and evaluate the result using Lemma 12.7(ii).
(ii) Apply (i) to Φ^*. □

Lemma 12.10. *For $0 \leqslant i \leqslant d$ the following hold:*

(i) $\sum_{h=0}^{d} m_h p_i(h) = \delta_{i,0}$;
(ii) $\sum_{h=0}^{d} m_h^* q_i(h) = \delta_{i,0}$.

Proof. (i) Set $j = 0$ in Lemma 12.6(i), and evaluate the result using Lemmas 8.3(ii), 12.8(ii).
(ii) Apply (i) to Φ^*. □

Lemma 12.11. *For $0 \leqslant i, j, r \leqslant d$ the following hold:*

(i) $p_i(r) p_j(r) = \sum_{h=0}^{d} p_{ij}^h p_h(r)$;
(ii) $q_i(r) q_j(r) = \sum_{h=0}^{d} q_{ij}^h q_h(r)$.

Proof. (i) In (7), multiply each side by E_r, and simplify the result using Lemma 12.4(i).
(ii) Apply (i) to Φ^*. □

Lemma 12.12. *For $0 \leqslant h, i, j \leqslant d$ the following hold:*

(i) $p_{ij}^h = \nu^{-1} \sum_{r=0}^{d} p_i(r) p_j(r) q_r(h)$;
(ii) $q_{ij}^h = \nu^{-1} \sum_{r=0}^{d} q_i(r) q_j(r) p_r(h)$.

Proof. (i) Expand the sum $\sum_{r=0}^{d} p_i(r) p_j(r) q_r(h)$ using Lemma 12.11(i), and simplify the result using Lemma 12.6(i).
(ii) Apply (i) to Φ^*. □

13. Reduction rules involving $p_i(j)$, $q_i(j)$

We continue to discuss a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ on V.

Lemma 13.1. *For $0 \leqslant i, j \leqslant d$ the following hold:*

(i) $E_0 A_i^* A_j = p_j(i) E_0 A_i^*$;
(ii) $E_0^* A_i A_j^* = q_j(i) E_0^* A_i$;
(iii) $A_j A_i^* E_0 = p_j(i) A_i^* E_0$;
(iv) $A_j^* A_i E_0^* = q_j(i) A_i E_0^*$.

Proof. (i) Using Lemmas 12.1 and 9.2(iv) in order,

$$E_0 A_i^* A_j = \sum_{h=0}^{d} p_j(h) E_0 A_i^* E_h = \sum_{h=0}^{d} p_j(h) \delta_{i,h} E_0 A_i^* = p_j(i) E_0 A_i^*.$$

(ii) Apply (i) to Φ^*.
(iii), (iv) For the equations in (i) and (ii), apply † to each side. □

Lemma 13.2. *For $0 \leqslant i, j \leqslant d$ the following hold:*

(i) $E_0 E_i^* E_j = \nu^{-1} p_i(j) E_0 A_j^*$;
(ii) $E_0^* E_i E_j^* = \nu^{-1} q_i(j) E_0^* A_j$;
(iii) $E_j E_i^* E_0 = \nu^{-1} p_i(j) A_j^* E_0$;
(iv) $E_j^* E_i E_0^* = \nu^{-1} q_i(j) A_j E_0^*$.

Proof. (i) Using Lemmas 12.5(i) and 9.2(iv) in order,

$$E_0 E_i^* E_j = E_0 \left(\nu^{-1} \sum_{h=0}^{d} p_i(h) A_h^* \right) E_j = \nu^{-1} \sum_{h=0}^{d} p_i(h) \delta_{h,j} E_0 A_h^* = \nu^{-1} p_i(j) E_0 A_j^*.$$

(ii) Apply (i) to Φ^*.
(iii), (iv) For the equations in (i) and (ii), apply † to each side. □

Lemma 13.3. *For $0 \leqslant i, j \leqslant d$ the following hold:*

(i) $E_0 A_i^* A_j E_0^* = p_j(i) k_i^* E_0 E_0^*$;
(ii) $E_0^* A_i A_j^* E_0 = q_j(i) k_i E_0^* E_0$;
(iii) $E_0^* A_i A_j^* E_0 = p_i(j) k_j^* E_0^* E_0$;
(iv) $E_0 A_i^* A_j E_0^* = q_i(j) k_j E_0 E_0^*$.

Proof. (i) Using Lemmas 13.1(i), 12.4(ii), 12.8(ii) in order,

$$E_0 A_i^* A_j E_0^* = p_j(i) E_0 A_i^* E_0^* = p_j(i) q_i(0) E_0 E_0^* = p_j(i) k_i^* E_0 E_0^*.$$

(ii) Apply (i) to Φ^*.
(iii), (iv) For the equations in (i) and (ii), apply † to each side. □

Lemma 13.4. *For $0 \leqslant i, j \leqslant d$ the following hold:*

(i) $E_0 E_i^* E_j E_0^* = p_i(j) m_j E_0 E_0^*$;
(ii) $E_0^* E_i E_j^* E_0 = q_i(j) m_j^* E_0^* E_0$.

Proof. (i) Using Lemmas 13.2(i), 12.4(ii), 12.8(ii) in order,

$$E_0 E_i^* E_j E_0^* = \nu^{-1} p_i(j) E_0 A_j^* E_0^* = \nu^{-1} p_i(j) q_j(0) E_0 E_0^* = \nu^{-1} p_i(j) k_j^* E_0 E_0^*.$$

Now use Lemma 8.3(ii).
(ii) Apply (i) to Φ^*. □

Lemma 13.5. *For $0 \leqslant i, j \leqslant d$,*

$$\frac{p_i(j)}{k_i} = \frac{q_j(i)}{k^*_j}. \tag{11}$$

Proof. By Lemma 13.3(ii),(iii), $p_i(j) k^*_j E^*_0 E_0 = q_j(i) k_i E^*_0 E_0$. The result follows since $E^*_0 E_0 \neq 0$ by Definition 3.1(iii). □

Lemma 13.6. *For $0 \leqslant i, j \leqslant d$ the following hold:*

(i) $p_i(j) = \nu m_j^{-1} \operatorname{tr}(E_0 E^*_i E_j E^*_0)$;
(ii) $p_i(j) = \nu m_j^{-1} \operatorname{tr}(E^*_0 E_j E^*_i E_0)$;
(iii) $q_i(j) = \nu (m^*_j)^{-1} \operatorname{tr}(E^*_0 E_i E^*_j E_0)$;
(iv) $q_i(j) = \nu (m^*_j)^{-1} \operatorname{tr}(E_0 E^*_j E_i E^*_0)$.

Proof. (i) Using Lemma 13.4(i) and Definition 4.5,

$$\operatorname{tr}(E_0 E^*_i E_j E^*_0) = p_i(j) m_j \operatorname{tr}(E_0 E^*_0) = \nu^{-1} p_i(j) m_j.$$

(iii) Apply (i) to Φ^*.
(ii) In (iii), exchange i, j, and use Lemmas 8.3, 13.5.
(iv) Apply (ii) to Φ^*. □

14. Some matrices

We continue to discuss a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d)$ on V. In the previous sections we used Φ to define several kinds of scalars, and we described how these scalars are related. In this section we express these relationships in matrix form.

Definition 14.1. *Let K (resp. K^*) denote the diagonal matrix in $\operatorname{Mat}_{d+1}(\mathbb{F})$ that has (i,i)-entry k_i (resp. k^*_i) for $0 \leqslant i \leqslant d$. Let P (resp. Q) denote the matrix in $\operatorname{Mat}_{d+1}(\mathbb{F})$ that has (i,j)-entry $p_j(i)$ (resp. $q_j(i)$) for $0 \leqslant i, j \leqslant d$.*

Lemma 14.2. *The following hold:*

(i) $PQ = QP = \nu I$;
(ii) $P^{\mathsf{t}} K^* = KQ$;
(iii) $K^* P = Q^{\mathsf{t}} K$.

Proof. (i) By Lemma 12.6.
(ii) By Lemma 13.5.
(iii) In (ii), take the transpose of each side. □

Definition 14.3. *Note by Lemma 14.2 that $K^{-1} P^{\mathsf{t}} = Q (K^*)^{-1}$ and $(K^*)^{-1} Q^{\mathsf{t}} = P K^{-1}$; we define*

$$U = K^{-1} P^{\mathsf{t}} = Q (K^*)^{-1}, \qquad\qquad U^* = (K^*)^{-1} Q^{\mathsf{t}} = P K^{-1}. \tag{12}$$

Lemma 14.4. *The following hold:*

(i) $P = U^* K$;
(ii) $P^{\mathsf{t}} = KU$;
(iii) $Q = UK^*$;
(iv) $Q^{\mathsf{t}} = K^* U^*$.

Proof. Immediate from Definition 14.3. □

LEMMA 14.5. *We have $U_{i,0} = 1$ and $U^*_{i,0} = 1$ for $0 \leqslant i \leqslant d$. Moreover $U_{0,j} = 1$ and $U^*_{0,j} = 1$ for $0 \leqslant j \leqslant d$.*

Proof. Use Lemmas 8.4(iii), 12.7, 12.8. □

LEMMA 14.6. *The following hold:*

(i) $U^{\mathrm{t}} = U^*$;
(ii) $UK^*U^*K = \nu I$;
(iii) $U^*KUK^* = \nu I$.

Proof. (i) By Definition 14.3.
(ii), (iii) By Lemma 14.4(i),(iii) and Lemma 14.2(i). □

DEFINITION 14.7. *For $0 \leqslant i \leqslant d$ let B_i and B^*_i denote the matrices in $\mathrm{Mat}_{d+1}(\mathbb{F})$ that have entries*

$$(B_i)_{h,j} = p^h_{ij}, \qquad (B^*_i)_{h,j} = q^h_{ij} \qquad (0 \leqslant h, j \leqslant d).$$

We call B_i (resp. B^*_i) the i^{th} intersection matrix *(resp. i^{th}* dual intersection matrix*) of Φ.*

DEFINITION 14.8. *For $0 \leqslant i \leqslant d$ let H_i and H^*_i denote the diagonal matrices in $\mathrm{Mat}_{d+1}(\mathbb{F})$ that have diagonal entries*

$$(H_i)_{j,j} = p_i(j), \qquad (H^*_i)_{j,j} = q_i(j) \qquad (0 \leqslant j \leqslant d).$$

LEMMA 14.9. *For $0 \leqslant r \leqslant d$,*

$$H_rP = PB_r, \qquad H^*_rQ = QB^*_r, \tag{13}$$

$$QH_r = B_rQ, \qquad PH^*_r = B^*_rP, \tag{14}$$

$$KB_r = (B_r)^{\mathrm{t}}K, \qquad K^*B^*_r = (B^*_r)^{\mathrm{t}}K^*, \tag{15}$$

$$UH_r = B_rU, \qquad U^*H^*_r = B^*_rU^*. \tag{16}$$

Proof. To get the equation on the left in (13), compare the entries of each side using Lemma 12.11(i). In the equation on the left in (13), multiply each side on the left and on the right by Q and simplify the result using Lemma 14.2(i). This gives the equation on the left in (14). To obtain the equation on the left in (15), compare the entries of each side using Lemma 10.11(i). The equation on the left in (16) follows from $QH_r = B_rQ$ and Lemma 14.4(iii) together with the fact that H_r, K^* commute since they are both diagonal. To get the equations on the right in (13)–(16), apply the equations on the left in (13)–(16) to Φ^*. □

LEMMA 14.10. *For $0 \leqslant i, j \leqslant d$ the following hold:*

(i) $B_iB_j = \sum_{h=0}^{d} p^h_{ij} B_h$;
(ii) $B^*_iB^*_j = \sum_{h=0}^{d} q^h_{ij} B^*_h$;
(iii) $H_iH_j = \sum_{h=0}^{d} p^h_{ij} H_h$;
(iv) $H^*_iH^*_j = \sum_{h=0}^{d} q^h_{ij} H^*_h$.

Proof. (i), (ii) By Lemma 10.7.
(iii), (iv) By Lemma 12.11. □

15. THE Φ-STANDARD BASIS

We continue to discuss a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d)$ on V. In this section we introduce the notion of a Φ-standard basis.

Lemma 15.1. *For* $0 \leqslant i \leqslant d$*,* $E_i^* V = E_i^* E_0 V$*.*

Proof. The vector space $E_i^* V$ has dimension 1 and contains $E_i^* E_0 V$. By Definition 3.1(iii), $E_i^* E_0 V \neq 0$. The result follows. □

Lemma 15.2. *Let* ξ *denote a nonzero vector in* $E_0 V$*. Then for* $0 \leqslant i \leqslant d$ *the vector* $E_i^* \xi$ *is nonzero and hence a basis of* $E_i^* V$*. Moreover the vectors* $\{E_i^* \xi\}_{i=0}^d$ *form a basis of* V*.*

Proof. Let the integer i be given. We show $E_i^* \xi \neq 0$. The vector space $E_0 V$ has dimension 1 and ξ is a nonzero vector in $E_0 V$, so ξ spans $E_0 V$. Therefore $E_i^* \xi$ spans $E_i^* E_0 V$. The vector space $E_i^* E_0 V$ has dimension 1 by Lemma 15.1 so $E_i^* \xi$ is nonzero. The remaining assertions are clear. □

Definition 15.3. *By a* Φ-standard basis *of* V *we mean a sequence* $\{E_i^* \xi\}_{i=0}^d$*, where* ξ *is a nonzero vector in* $E_0 V$*.*

We give a characterization of a Φ-standard basis.

Lemma 15.4. *Let* $\{u_i\}_{i=0}^d$ *denote a sequence of vectors in* V*, not all* 0*. Then this sequence is a* Φ*-standard basis if and only if both* (i)*,* (ii) *hold below:*

(i) $u_i \in E_i^* V$ *for* $0 \leqslant i \leqslant d$*;*
(ii) $\sum_{i=0}^d u_i \in E_0 V$*.*

Proof. To prove the lemma in one direction, assume that $\{u_i\}_{i=0}^d$ is a Φ-standard basis of V. By Definition 15.3 there exists a nonzero $\xi \in E_0 V$ such that $u_i = E_i^* \xi$ for $0 \leqslant i \leqslant d$. By construction $u_i \in E_i^* V$ for $0 \leqslant i \leqslant d$, so (i) holds. Recall $I = \sum_{i=0}^d E_i^*$. In this equation we apply each side to ξ, to find that $\xi = \sum_{i=0}^d u_i$, and (ii) follows. We have now proved the lemma in one direction. To prove the lemma in the other direction, assume that $\{u_i\}_{i=0}^d$ satisfy (i) and (ii). Define $\xi = \sum_{i=0}^d u_i$ and observe $\xi \in E_0 V$. Using (i) we find that $E_i^* u_j = \delta_{i,j} u_i$ for $0 \leqslant i, j \leqslant d$. It follows $u_i = E_i^* \xi$ for $0 \leqslant i \leqslant d$. Observe $\xi \neq 0$ since at least one of $\{u_i\}_{i=0}^d$ is nonzero. Now $\{u_i\}_{i=0}^d$ is a Φ-standard basis of V by Definition 15.3. □

16. Bilinear forms

In this section we recall some basic facts concerning bilinear forms on V. See [14, Section 8.5] for more information. By a *bilinear form on* V we mean a map $\langle\ ,\ \rangle : V \times V \to \mathbb{F}$ that satisfies the following four conditions for $u, v, w \in V$ and $\alpha \in \mathbb{F}$: (i) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$; (ii) $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$; (iii) $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$; (iv) $\langle u, \alpha v\rangle = \alpha\langle u, v\rangle$. Let $\langle\ ,\ \rangle$ denote a bilinear form on V. We abbreviate $||v||^2 = \langle v, v\rangle$ for $v \in V$. The following are equivalent: (i) there exists a nonzero $u \in V$ such that $\langle u, v\rangle = 0$ for all $v \in V$; (ii) there exists a nonzero $v \in V$ such that $\langle u, v\rangle = 0$ for all $u \in V$. The form $\langle\ ,\ \rangle$ is said to be *degenerate* whenever (i), (ii) hold and *nondegenerate* otherwise.

We recall from [8, Theorem 1.1] or [11, Ch. 1, Theorem. 4.2] how bilinear forms on V are related to antiautomorphisms of $\mathrm{End}(V)$. Let γ denote an antiautomorphism of $\mathrm{End}(V)$. Then there exists a nonzero bilinear form $\langle\ ,\ \rangle$ on V such that $\big\langle Au, v\big\rangle = \big\langle u, A^\gamma v\big\rangle$ for $u, v \in V$ and $A \in \mathrm{End}(V)$. The form is unique up to multiplication by a nonzero scalar. The form is nondegenerate. We refer to this form as a *bilinear form on* V *associated with* γ.

For the rest of this section let $\langle\ ,\ \rangle$ denote a nondegenerate bilinear form on V.

Definition 16.1. *For bases* $\{u_i\}_{i=0}^d$ *and* $\{v_i\}_{i=0}^d$ *of* V*, the* inner product matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$ *is the matrix in* $\mathrm{Mat}_{d+1}(\mathbb{F})$ *that has* (i,j)*-entry* $\langle u_i, v_j\rangle$ *for* $0 \leqslant i, j \leqslant d$*.*

Referring to Definition 16.1, the inner product matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$ is invertible.

Definition 16.2. *The form* $\langle\ ,\ \rangle$ *is said to be* symmetric *whenever* $\langle u, v\rangle = \langle v, u\rangle$ *for* $u, v \in V$.

Definition 16.3. *Assume that* $\langle\ ,\ \rangle$ *is symmetric. Then two bases* $\{u_i\}_{i=0}^d$, $\{v_i\}_{i=0}^d$ *of* V *are said to be* dual with respect to $\langle\ ,\ \rangle$ *whenever* $\langle u_i, v_j\rangle = \delta_{i,j}$ *for* $0 \leqslant i, j \leqslant d$.

Lemma 16.4. *Assume that* $\langle\ ,\ \rangle$ *is symmetric. Then each basis of* V *has a unique dual with respect to* $\langle\ ,\ \rangle$.

Lemma 16.5. *Assume that* $\langle\ ,\ \rangle$ *is symmetric. Let* $\{u_i\}_{i=0}^d$ *and* $\{v_i\}_{i=0}^d$ *denote bases of* V. *Then the following are the same:*

(i) *the inner product matrix from* $\{u_i\}_{i=0}^d$ *to* $\{v_i\}_{i=0}^d$;
(ii) *the inner product matrix from* $\{u_i\}_{i=0}^d$ *to* $\{u_i\}_{i=0}^d$, *times the transition matrix from* $\{u_i\}_{i=0}^d$ *to* $\{v_i\}_{i=0}^d$.

Proof. Routine linear algebra. □

17. The dual Φ-standard basis

We return our attention to a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ on V. In this section we introduce the notion of a dual Φ-standard basis of V. Recall the antiautomorphism † of $\mathrm{End}(V)$ from Definition 5.1. For the rest of the paper $\langle\ ,\ \rangle$ denotes a bilinear form on V associated with †. By the construction, for $A \in \mathrm{End}(V)$ we have

$$\langle Au, v\rangle = \langle u, A^\dagger v\rangle \qquad (u, v \in V). \tag{17}$$

Recall the algebra $\mathcal{M}$ from Definition 3.2.

Lemma 17.1. *For* $A \in \mathcal{M} \cup \mathcal{M}^*$,

$$\langle Au, v\rangle = \langle u, Av\rangle \qquad (u, v \in V). \tag{18}$$

Proof. By Definition 5.1 and (17). □

Lemma 17.2. *For* $\xi \in E_0V$,

$$\langle E_i^*\xi, E_j^*\xi\rangle = \delta_{i,j}\nu^{-1}k_i||\xi||^2 \qquad (0 \leqslant i, j \leqslant d). \tag{19}$$

Proof. Using (18) and $E_0\xi = \xi$,

$$\langle E_i^*\xi, E_j^*\xi\rangle = \langle E_i^*E_0\xi, E_j^*E_0\xi\rangle = \langle \xi, E_0E_i^*E_j^*E_0\xi\rangle = \delta_{i,j}\langle \xi, E_0E_i^*E_0\xi\rangle.$$

By this and Lemmas 4.2(ii), 8.3(i) we get the result. □

Lemma 17.3. *The bilinear form* $\langle\ ,\ \rangle$ *is symmetric.*

Proof. Consider a Φ-standard basis $\{E_i^*\xi\}_{i=0}^d$ of V, where $0 \neq \xi \in E_0V$. By Lemma 17.2, $\langle E_i^*\xi, E_j^*\xi\rangle = \langle E_j^*\xi, E_i^*\xi\rangle$ for $0 \leqslant i, j \leqslant d$. Therefore $\langle u, v\rangle = \langle v, u\rangle$ for u, $v \in V$. □

Lemma 17.4. *The following hold for $0 \neq \xi \in E_0V$ and $0 \neq \xi^* \in E_0^*V$:*

(i) *each of $||\xi||^2$, $||\xi^*||^2$, $\langle \xi, \xi^* \rangle$ is nonzero;*
(ii) $E_0^*\xi = \frac{\langle \xi, \xi^* \rangle}{||\xi^*||^2}\, \xi^*$;
(iii) $E_0\xi^* = \frac{\langle \xi, \xi^* \rangle}{||\xi||^2}\, \xi$;
(iv) $||\xi||^2 ||\xi^*||^2 = \nu \langle \xi, \xi^* \rangle^2$.

Proof. (i) Observe $||\xi||^2 \neq 0$ by Lemma 17.2 and since $\langle\ ,\ \rangle$ is nonzero. Applying this to Φ^* we get $||\xi^*||^2 \neq 0$. To see that $\langle \xi, \xi^* \rangle \neq 0$, observe that ξ^* is a basis of E_0^*V so there exists a scalar α such that $E_0^*\xi = \alpha\xi^*$. Recall $E_0^*\xi \neq 0$ by Lemma 15.2 so $\alpha \neq 0$. Using (18) and $E_0^*\xi^* = \xi^*$ we routinely find that $\langle \xi, \xi^* \rangle = \alpha||\xi^*||^2$ and it follows $\langle \xi, \xi^* \rangle \neq 0$.

(ii) In the proof of part (i) we found $E_0^*\xi = \alpha\xi^*$ where $\langle \xi, \xi^* \rangle = \alpha||\xi^*||^2$. The result follows.

(iii) Apply (ii) to Φ^*.

(iv) Using $\xi = E_0\xi$ and Lemma 4.6 one finds that $\nu^{-1}\xi = E_0E_0^*\xi$. To finish the proof, evaluate $E_0E_0^*\xi$ using (ii), (iii). □

Definition 17.5. *By a* dual Φ-standard basis *of V we mean the dual of a Φ-standard basis with respect to $\langle\ ,\ \rangle$.*

Shortly we will describe the dual Φ-standard bases. We will use the following definition.

Definition 17.6. *Note that for nonzero ξ, $\zeta \in E_0V$ the following are equivalent:*

(i) $\langle \xi, \zeta \rangle = \nu$; (ii) $\zeta = \nu\xi/||\xi||^2$; (iii) $\xi = \nu\zeta/||\zeta||^2$.

We say that ξ, ζ are partners *whenever they satisfy* (i)–(iii).

Lemma 17.7. *For nonzero ξ, ζ in E_0V the following are equivalent:*

(i) *the bases $\{E_i^*\xi\}_{i=0}^d$ and $\{k_i^{-1}E_i^*\zeta\}_{i=0}^d$ are dual with respect to $\langle\ ,\ \rangle$;*
(ii) *ξ, ζ are partners.*

Proof. The vector space E_0V has dimension 1, so there exists a scalar α such that $\zeta = \alpha\xi$. By this and Lemma 17.2,

$$\langle E_i^*\xi, k_j^{-1}E_j^*\zeta \rangle = \delta_{i,j}\alpha\nu^{-1}||\xi||^2.$$

So (i) holds if and only if $\alpha||\xi||^2 = \nu$. By this and Definition 17.6 we obtain the result. □

Lemma 17.8. *A given basis of V is dual Φ-standard if and only if it has the form $\{k_i^{-1}E_i^*\zeta\}_{i=0}^d$ for some nonzero $\zeta \in E_0V$.*

Proof. Use Lemma 17.7. □

We mention a result for later use.

Lemma 17.9. *For $0 \neq \xi \in E_0V$ and $0 \neq \xi^* \in E_0^*V$,*

$$\langle E_i^*\xi, E_j\xi^* \rangle = \nu^{-1}p_i(j)k_j^*\langle \xi, \xi^* \rangle \qquad (0 \leqslant i, j \leqslant d).$$

Proof. Using $E_0\xi = \xi$, $E_0^*\xi^* = \xi^*$ and Lemma 13.4(i),

$$\langle E_i^*\xi, E_j\xi^*\rangle = \langle \xi, E_0E_i^*E_jE_0^*\xi^*\rangle = p_i(j)m_j\langle \xi, \xi^*\rangle.$$

By this and Lemma 8.3(ii) we obtain the result. □

18. Four bases of V

We continue to discuss a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ on V. Recall the elements A_i from Definition 7.1. Recall the matrices K, K^*, U, U^* from Definitions 14.1, 14.3, and the matrices B_i, B_i^*, H_i, H_i^* from Definitions 14.7, 14.8. Recall the bilinear form $\langle\ ,\ \rangle$ from above Lemma 17.1.

Throughout this section, we fix nonzero vectors $\xi, \zeta \in E_0V$ and $\xi^*, \zeta^* \in E_0^*V$, and consider the following four bases of V.

(20)

basis type	basis
Φ-standard	$\{E_i^*\xi\}_{i=0}^d$
dual Φ-standard	$\{k_i^{-1}E_i^*\zeta\}_{i=0}^d$
Φ^*-standard	$\{E_i\xi^*\}_{i=0}^d$
dual Φ^*-standard	$\{(k_i^*)^{-1}E_i\zeta^*\}_{i=0}^d$

In this section we display the matrices that represent $\{A_r\}_{r=0}^d$, $\{A_r^*\}_{r=0}^d$, $\{E_r\}_{r=0}^d$, $\{E_r^*\}_{r=0}^d$ with respect to these bases. We display the inner product matrices between these bases. We display the transition matrices between these bases.

We introduce some notation. For $0 \leqslant i, j \leqslant d$ define $\Delta_{i,j} \in \mathrm{Mat}_{d+1}(\mathbb{F})$ that has (i,j)-entry 1 and all other entries 0.

Proposition 18.1. *In the table below we give some matrix representations. For $0 \leqslant r \leqslant d$, each entry in the table is the matrix that represents the map in the given column with respect to the basis in the given row.*

basis	A_r	A_r^*	E_r	E_r^*
$\{E_i^*\xi\}_{i=0}^d$	B_r	H_r^*	$\nu^{-1}UK^*\Delta_{r,r}U^*K$	$\Delta_{r,r}$
$\{k_i^{-1}E_i^*\zeta\}_{i=0}^d$	B_r^{t}	H_r^*	$(U^*)^{-1}\Delta_{r,r}U^*$	$\Delta_{r,r}$
$\{E_i\xi^*\}_{i=0}^d$	H_r	B_r^*	$\Delta_{r,r}$	$\nu^{-1}U^*K\Delta_{r,r}UK^*$
$\{(k_i^*)^{-1}E_i\zeta^*\}_{i=0}^d$	H_r	$(B_r^*)^{\mathrm{t}}$	$\Delta_{r,r}$	$U^{-1}\Delta_{r,r}U$

Proof. We first consider the matrices representing A_r. The matrix representing A_r with respect to $\{E_i^*\xi\}_{i=0}^d$ is obtained using Lemma 11.1(i) and Definition 14.7. The matrix representing A_r with respect to $\{k_i^{-1}E_i^*\zeta\}_{i=0}^d$ is obtained using Lemmas 10.11(i) and 11.1(i). The matrices representing A_r with respect to $\{E_i\xi^*\}_{i=0}^d$ and $\{(k_i^*)^{-1}E_i\zeta^*\}_{i=0}^d$ are obtained using Lemma 12.4(i) and Definition 14.8. Applying these results to Φ^* we obtain the matrices representing A_r^*. Next we consider the matrices representing E_r. The matrix representing E_r with respect to $\{E_i^*\xi\}_{i=0}^d$ is obtained using Lemmas 13.2(iii), 12.3, 14.4(i),(iii). Multiply this matrix on the left (resp. right) by K (resp. K^{-1}) and use Lemma 14.6(iii) to obtain the matrix representing E_r with respect to $\{k_i^{-1}E_i^*\zeta\}_{i=0}^d$. The matrices representing E_r with respect to $\{E_i\xi^*\}_{i=0}^d$ and $\{(k_i^*)^{-1}E_i\zeta^*\}_{i=0}^d$ are routinely obtained. Applying these results to Φ^* we obtain the matrices representing E_r^*. □

Proposition 18.2. *In the table below we give the inner product matrices between the bases in* (20)*. Each entry of the table is the inner product matrix from the basis in the given row to the basis in the given column.*

	$\{E^*_i\xi\}_{i=0}^d$	$\{k_i^{-1}E^*_i\zeta\}_{i=0}^d$	$\{E_i\xi^*\}_{i=0}^d$	$\{(k^*_i)^{-1}E_i\zeta^*\}_{i=0}^d$
$\{E^*_i\xi\}_{i=0}^d$	$\frac{\|\|\xi\|\|^2}{\nu}K$	$\frac{\langle\xi,\zeta\rangle}{\nu}I$	$\frac{\langle\xi,\xi^*\rangle}{\nu}KUK^*$	$\frac{\langle\xi,\zeta^*\rangle}{\nu}KU$
$\{k_i^{-1}E^*_i\zeta\}_{i=0}^d$	$\frac{\langle\zeta,\xi\rangle}{\nu}I$	$\frac{\|\|\zeta\|\|^2}{\nu}K^{-1}$	$\frac{\langle\zeta,\xi^*\rangle}{\nu}UK^*$	$\frac{\langle\zeta,\zeta^*\rangle}{\nu}U$
$\{E_i\xi^*\}_{i=0}^d$	$\frac{\langle\xi^*,\xi\rangle}{\nu}K^*U^*K$	$\frac{\langle\xi^*,\zeta\rangle}{\nu}K^*U^*$	$\frac{\|\|\xi^*\|\|^2}{\nu}K^*$	$\frac{\langle\xi^*,\zeta^*\rangle}{\nu}I$
$\{(k^*_i)^{-1}E_i\zeta^*\}_{i=0}^d$	$\frac{\langle\zeta^*,\xi\rangle}{\nu}U^*K$	$\frac{\langle\zeta^*,\zeta\rangle}{\nu}U^*$	$\frac{\langle\zeta^*,\xi^*\rangle}{\nu}I$	$\frac{\|\|\zeta^*\|\|^2}{\nu}(K^*)^{-1}$

Proof. Note that ζ (resp. ζ^*) is a nonzero scalar multiple of ξ (resp. ξ^*). Using this and Lemmas 17.2, 17.9 we represent the inner products in terms of P, Q, K, K^*. Now eliminate P, Q using Lemma 14.4 to get the result. $\square$

In the diagram below we display the inner product matrices between the four bases in (20):

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\{E^*_i\xi\}_{i=0}^d \arrow[loop above, "\frac{||\xi||^2}{\nu}K"] \arrow[rr, no head, "\frac{\langle\xi,\zeta\rangle}{\nu}I"] \arrow[d, shift right, "\frac{\langle\xi,\xi^*\rangle}{\nu}KUK^*"'] \arrow[drr, shift left, "\frac{\langle\xi,\zeta^*\rangle}{\nu}KU"] & & \{k_i^{-1}E^*_i\zeta\}_{i=0}^d \arrow[loop above, "\frac{||\zeta||^2}{\nu}K^{-1}"] \arrow[d, shift left, "\frac{\langle\zeta,\zeta^*\rangle}{\nu}U"] \arrow[dll, shift right, "\frac{\langle\zeta,\xi^*\rangle}{\nu}UK^*"'] \\
\{E_i\xi^*\}_{i=0}^d \arrow[loop below, "\frac{||\xi^*||^2}{\nu}K^*"] \arrow[rr, no head, "\frac{\langle\xi^*,\zeta^*\rangle}{\nu}I"'] \arrow[u, shift right, "\frac{\langle\xi^*,\xi\rangle}{\nu}K^*U^*K"'] \arrow[urr, shift right, "\frac{\langle\xi^*,\zeta\rangle}{\nu}K^*U^*"'] & & \{(k^*_i)^{-1}E_i\zeta^*\}_{i=0}^d \arrow[loop below, "\frac{||\zeta^*||^2}{\nu}(K^*)^{-1}"] \arrow[u, shift left, "\frac{\langle\zeta^*,\zeta\rangle}{\nu}U^*"] \arrow[ull, shift left, "\frac{\langle\zeta^*,\xi\rangle}{\nu}U^*K"]
\end{tikzcd}
$$

Inner products

$\{u_i\}_{i=0}^d \xrightarrow{M} \{v_i\}_{i=0}^d$ means $M_{ij} = \langle u_i, v_j\rangle$ $(0 \leqslant i, j \leqslant d)$

The direction arrow is left off if M is symmetric

Proposition 18.3. *In the table below we give the transition matrices between the four bases in* (20)*. Each entry of the table is the transition matrix from the basis in the given row to the basis in the given column.*

	$\{E^*_i\xi\}_{i=0}^d$	$\{k_i^{-1}E^*_i\zeta\}_{i=0}^d$	$\{E_i\xi^*\}_{i=0}^d$	$\{(k^*_i)^{-1}E_i\zeta^*\}_{i=0}^d$
$\{E^*_i\xi\}_{i=0}^d$	I	$\frac{\langle\xi,\zeta\rangle}{\|\|\xi\|\|^2}K^{-1}$	$\frac{\langle\xi,\xi^*\rangle}{\|\|\xi\|\|^2}UK^*$	$\frac{\langle\xi,\zeta^*\rangle}{\|\|\xi\|\|^2}U$
$\{k_i^{-1}E^*_i\zeta\}_{i=0}^d$	$\frac{\langle\zeta,\xi\rangle}{\|\|\zeta\|\|^2}K$	I	$\frac{\langle\zeta,\xi^*\rangle}{\|\|\zeta\|\|^2}KUK^*$	$\frac{\langle\zeta,\zeta^*\rangle}{\|\|\zeta\|\|^2}KU$
$\{E_i\xi^*\}_{i=0}^d$	$\frac{\langle\xi^*,\xi\rangle}{\|\|\xi^*\|\|^2}U^*K$	$\frac{\langle\xi^*,\zeta\rangle}{\|\|\xi^*\|\|^2}U^*$	I	$\frac{\langle\xi^*,\zeta^*\rangle}{\|\|\xi^*\|\|^2}(K^*)^{-1}$
$\{(k^*_i)^{-1}E_i\zeta^*\}_{i=0}^d$	$\frac{\langle\zeta^*,\xi\rangle}{\|\|\zeta^*\|\|^2}K^*U^*K$	$\frac{\langle\zeta^*,\zeta\rangle}{\|\|\zeta^*\|\|^2}K^*U^*$	$\frac{\langle\zeta^*,\xi^*\rangle}{\|\|\zeta^*\|\|^2}K^*$	I

Proof. Use Lemma 16.5 and Proposition 18.2. $\square$

In the diagram below we display the transition matrices between the four bases in (20).

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\{E_i^*\xi\}_{i=0}^d \arrow[r, shift left, "\frac{\langle\xi,\zeta\rangle}{\|\xi\|^2}K^{-1}"] \arrow[d, shift right, "\frac{\langle\xi,\xi^*\rangle}{\|\xi\|^2}UK^*"'] \arrow[dr, shift left, "\frac{\langle\xi,\zeta^*\rangle}{\|\xi\|^2}U"] & \{k_i^{-1}E_i^*\zeta\}_{i=0}^d \arrow[l, shift left, "\frac{\langle\zeta,\xi\rangle}{\|\zeta\|^2}K"] \arrow[d, shift left, "\frac{\langle\zeta,\zeta^*\rangle}{\|\zeta\|^2}KU"] \arrow[dl, shift right, "\frac{\langle\zeta,\xi^*\rangle}{\|\zeta\|^2}KUK^*"'] \\
\{E_i\xi^*\}_{i=0}^d \arrow[u, shift right, "\frac{\langle\xi^*,\xi\rangle}{\|\xi^*\|^2}U^*K"'] \arrow[r, shift right, "\frac{\langle\xi^*,\zeta^*\rangle}{\|\xi^*\|^2}(K^*)^{-1}"'] \arrow[ur, shift right, "\frac{\langle\xi^*,\zeta\rangle}{\|\xi^*\|^2}U^*"'] & \{(k_i^*)^{-1}E_i\zeta^*\}_{i=0}^d \arrow[l, shift right, "\frac{\langle\zeta^*,\xi^*\rangle}{\|\zeta^*\|^2}K^*"'] \arrow[u, shift left, "\frac{\langle\zeta^*,\zeta\rangle}{\|\zeta^*\|^2}K^*U^*"] \arrow[ul, shift left, "\frac{\langle\zeta^*,\xi\rangle}{\|\zeta^*\|^2}K^*U^*K"]
\end{tikzcd}
$$

Transition matrices

$\{u_i\}_{i=0}^d \xrightarrow{M} \{v_i\}_{i=0}^d$ means $v_j = \sum_{i=0}^d M_{ij} u_i$ $(0 \leqslant j \leqslant d)$

19. P-polynomial and Q-polynomial idempotent systems

We continue to discuss a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ on V.

Definition 19.1. *We say that Φ is P-*polynomial *whenever p_{ij}^h is zero (resp. nonzero) if one of h, i, j is greater than (resp. equal to) the sum of the other two $(0 \leqslant h, i, j \leqslant d)$.*

For the moment, assume that $d \geqslant 1$ and Φ is P-polynomial. Then the first intersection matrix B_1 has the form

$$
B_1 = \begin{pmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & c_2 & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \quad b_{d-1} \\ \mathbf{0} & & & c_d & a_d \end{pmatrix},
$$

where

$$
c_i = p_{1,i-1}^i \quad (1 \leqslant i \leqslant d), \qquad a_i = p_{1,i}^i \quad (0 \leqslant i \leqslant d), \qquad b_i = p_{1,i+1}^i \quad (0 \leqslant i \leqslant d-1).
$$

Moreover $c_i \neq 0$ for $1 \leqslant i \leqslant d$ and $b_i \neq 0$ for $0 \leqslant i \leqslant d-1$. So B_1 is irreducible tridiagonal. Shortly we will show that this feature of B_1 characterizes the P-polynomial property.

Lemma 19.2. *Assume that $d \geqslant 1$ and Φ is P-polynomial. Then*

$$
\begin{aligned}
A_1 A_0 &= a_0 A_0 + c_1 A_1, \\
A_1 A_i &= b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \qquad\qquad (1 \leqslant i \leqslant d-1), \\
A_1 A_d &= b_{d-1} A_{d-1} + a_d A_d.
\end{aligned}
$$

Proof. By Lemma 10.1 and the comments below Definition 19.1. □

For elements A, B in any algebra, we say that B is an *affine transformation* of A whenever there exist scalars α, β such that $\alpha \neq 0$ and $B = \alpha A + \beta I$.

Proposition 19.3. *Assume that $d \geqslant 1$. Then for $A \in \mathrm{End}(V)$ the following are equivalent:*

(i) *Φ is P-polynomial and A is an affine transformation of A_1;*
(ii) *for $0 \leqslant i \leqslant d$ there exists $f_i \in \mathbb{F}[x]$ such that $\deg(f_i) = i$ and $A_i = f_i(A)$.*

Proof. (i) $\Rightarrow$ (ii) By Lemma 19.2 and since $A_0 = I$.

(ii) $\Rightarrow$ (i) The elements $\{A_i\}_{i=0}^d$ are linearly independent by Lemma 7.7, so the elements $\{A^i\}_{i=0}^d$ are linearly independent. Pick integers i, j $(0 \leqslant i, j \leqslant d)$ such that $i + j \leqslant d$. We show that

$$f_i f_j = \sum_{h=0}^{d} p^h_{ij} f_h. \tag{21}$$

Define a polynomial $g = f_i f_j - \sum_{h=0}^d p^h_{ij} f_h$. The degree of g is at most d, and $g(A) = 0$. Therefore $g = 0$. We have shown (21). In (21) we examine the degrees to find

$$i + j = \max\{h \mid 0 \leqslant h \leqslant d,\ p^h_{ij} \neq 0\}.$$

By this and Lemma 10.11(i), we find that Φ is P-polynomial. Since $A_1 = f_1(A)$ and $\deg(f_1) = 1$, A is an affine transformation of A_1. □

Proposition 19.4. *Assume that $d \geqslant 1$ and Φ is P-polynomial. Then the following hold:*

(i) *$\{A_1^i\}_{i=0}^d$ form a basis for the vector space $\mathcal{M}$, where $\mathcal{M}$ is from Definition 3.2;*
(ii) *$\{p_1(j)\}_{j=0}^d$ are mutually distinct;*
(iii) *$\{E_i V\}_{i=0}^d$ are the eigenspaces of A_1;*
(iv) *A_1 is multiplicity-free;*
(v) *$\{E_i\}_{i=0}^d$ are the primitive idempotents of A_1.*

Proof. (i) By Lemma 7.7 and Proposition 19.3(ii).

(ii) By Lemma 12.4, $p_1(j)$ is the eigenvalue of A_1 corresponding to $E_j V$ for $0 \leqslant j \leqslant d$. So the characteristic polynomial of A_1 is $\prod_{j=0}^d (x - p_1(j))$. By (i) the minimal polynomial of A_1 has degree $d+1$. By these comments, the minimal polynomial of A_1 is $\prod_{j=0}^d (x - p_1(j))$. The result follows.

(iii) By Lemma 12.4(i) and (ii) above.

(iv) By (iii) above and since $E_i V$ has dimension one for $0 \leqslant i \leqslant d$.

(v) By (iii), (iv) above. □

Proposition 19.5. *For $d \geqslant 1$ the following are equivalent:*

(i) *Φ is P-polynomial;*
(ii) *the first intersection matrix B_1 is irreducible tridiagonal.*

Proof. (i) $\Rightarrow$ (ii) We saw this above Lemma 19.2.

(ii) $\Rightarrow$ (i) Since B_1 is irreducible tridiagonal, we have the equations in Lemma 19.2. So for $0 \leqslant i \leqslant d$ there exists $f_i \in \mathbb{F}[x]$ such that $\deg(f_i) = i$ and $A_i = f_i(A_1)$. By Proposition 19.3 (with $A = A_1$) we see that Φ is P-polynomial. □

Definition 19.6. *We say that Φ is Q-*polynomial *whenever q^h_{ij} is zero (resp. nonzero) if one of h, i, j is greater than (resp. equal to) the sum of the other two $(0 \leqslant h, i, j \leqslant d)$.*

Lemma 19.7. *Φ is Q-polynomial if and only if Φ^* is P-polynomial.*

Proof. Immediate from Definitions 10.3, 19.1, 19.6. □

20. Leonard pairs and Leonard systems

In this section we recall the notion of a Leonard pair and a Leonard system.

Definition 20.1 ([16, Definition 1.1]). *By a* Leonard pair *on V we mean an ordered pair A, A^* of elements in* $\mathrm{End}(V)$ *that satisfy the following* (i), (ii).

(i) *There exists a basis of V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.*
(ii) *There exists a basis of V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.*

Let A, A^* denote a Leonard pair on V. By [16, Lemma 1.3] each of A, A^* is multiplicity-free. Let $\{E_i\}_{i=0}^d$ denote an ordering of the primitive idempotents of A. For $0 \leqslant i \leqslant d$ pick a nonzero $v_i \in E_iV$. Then $\{v_i\}_{i=0}^d$ form a basis of V. We say that the ordering $\{E_i\}_{i=0}^d$ is *standard* whenever $\{v_i\}_{i=0}^d$ satisfies Definition 20.1(ii). In this case, the ordering $\{E_{d-i}\}_{i=0}^d$ is standard and no further ordering is standard. A standard ordering of the primitive idempotents of A^* is similarly defined.

Definition 20.2 ([16, Definition 1.4]). *By a* Leonard system *on V we mean a sequence*

$$(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \tag{22}$$

of elements in $\mathrm{End}(V)$ *that satisfy the following* (i)–(iii)*:*

(i) *A, A^* is a Leonard pair on V;*
(ii) *$\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A;*
(iii) *$\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^*.*

For the rest of this section let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V. Note that $(A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$ is a Leonard system on V.

Lemma 20.3 ([17, Lemma 9.2]). *The following hold:*

(i) $E_0E_i^*E_0 \neq 0 \quad (0 \leqslant i \leqslant d)$*;*
(ii) $E_0^*E_iE_0^* \neq 0 \quad (0 \leqslant i \leqslant d)$.

Lemma 20.4 ([17, Theorem 6.1 and Lemma 6.3]). *There exists a unique antiautomorphism $\dagger$ of* $\mathrm{End}(V)$ *that fixes each of A, A^*. Moreover $\dagger$ fixes each of E_i, E_i^* for $0 \leqslant i \leqslant d$.*

Lemma 20.5 ([17, Theorem 13.4]). *There exist polynomials $\{f_i\}_{i=0}^d$ in $\mathbb{F}[x]$ such that $\deg(f_i) = i$ and $f_i(A)E_0^*E_0 = E_i^*E_0$ for $0 \leqslant i \leqslant d$.*

Lemma 20.6 ([12, Theorem 4.2]). *For elements B, B^* in* $\mathrm{End}(V)$ *the following are equivalent:*

(i) *$(B; \{E_i\}_{i=0}^d; B^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system;*
(ii) *B (resp. B^*) is an affine transformation of A (resp. A^*).*

21. Idempotent systems and Leonard systems

In this section we show that a Leonard system is essentially the same thing as a symmetric idempotent system that is P-polynomial and Q-polynomial.

Theorem 21.1. *Let $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote a sequence of elements in* $\mathrm{End}(V)$*. Then the following are equivalent:*

(i) *Φ is a symmetric idempotent system that is P-polynomial and Q-polynomial;*
(ii) *there exist A, A^* in* $\mathrm{End}(V)$ *such that $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system.*

Proof. We assume $d \geqslant 1$; otherwise the assertion is obvious.

(i) $\Rightarrow$ (ii) We show that $(A_1; \{E_i\}_{i=0}^d; A_1^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system on V, where A_1, A_1^* are from Definition 7.1. By Proposition 18.1, with respect to a Φ-standard basis of V the matrix representing A_1 is B_1 and the matrix representing A_1^* is H_1^*. By Definition 14.7 the matrix H_1^* is diagonal, and by Proposition 19.5 the matrix B_1 is irreducible tridiagonal. Thus with respect to a Φ-standard basis the matrix representing A_1 is irreducible tridiagonal and the matrix representing A_1^* is diagonal. Applying this to Φ^*, with respect to a Φ^*-standard basis the matrix representing A_1^* is irreducible tridiagonal and the matrix representing A_1 is diagonal. By these comments A_1, A_1^* is a Leonard pair on V. By Proposition 19.4(v) and the construction, $\{E_i\}_{i=0}^d$ (resp. $\{E_i^*\}_{i=0}^d$) is a standard ordering of the primitive idempotents of A_1 (resp. A_1^*). We have shown that $(A_1; \{E_i\}_{i=0}^d; A_1^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system on V.

(ii) $\Rightarrow$ (i) By Lemmas 20.3 and 20.4, Φ is a symmetric idempotent system on V. By Lemma 20.5 there exist polynomials $\{f_i\}_{i=0}^d$ in $\mathbb{F}[x]$ such that $\deg(f_i) = i$ and $f_i(A)E_0^*E_0 = E_i^*E_0$ for $0 \leqslant i \leqslant d$. By Lemmas 6.3, 6.4, 7.4(i), A_i is the unique element in $\mathcal{M}$ such that $A_iE_0^*E_0 = E_i^*E_0$ $(0 \leqslant i \leqslant d)$. By these comments $f_i(A) = A_i$ for $0 \leqslant i \leqslant d$. By this and Proposition 19.3, Φ is P-polynomial. Apply this to the Leonard system $(A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$ to find that Φ is Q-polynomial. □

Lemma 21.2. *Assume that $d \geqslant 1$ and the equivalent conditions* (i), (ii) *hold in Theorem* 21.1. *Then for A, A^* in* $\mathrm{End}(V)$ *the following are equivalent:*

(i) $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ *is a Leonard system on V;*

(ii) *A (resp. A^*) is an affine transformation of A_1 (resp. A_1^*), where A_1, A_1^* are from Definition* 7.1.

Proof. (i) $\Rightarrow$ (ii) By the proof of Theorem 21.1, $(A_1; \{E_i\}_{i=0}^d; A_1^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system on V. By this and Lemma 20.6, A (resp. A^*) is an affine transformation of A_1 (resp. A_1^*).

(ii) $\Rightarrow$ (i) By Lemma 20.6. □

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