



ALGEBRAIC COMBINATORICS

Ada Chan

Complex Hadamard matrices, instantaneous uniform mixing and cubes

Volume 3, issue 3 (2020), p. 757-774.

[<http://alco.centre-mersenne.org/item/ALCO_2020__3_3_757_0>](http://alco.centre-mersenne.org/item/ALCO_2020__3_3_757_0)

© The journal and the authors, 2020.
Some rights reserved.

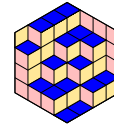


This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>

Access to articles published by the journal *Algebraic Combinatorics* on
the website <http://alco.centre-mersenne.org/> implies agreement with the
Terms of Use (<http://alco.centre-mersenne.org/legal/>).



Algebraic Combinatorics is member of the
Centre Mersenne for Open Scientific Publishing
www.centre-mersenne.org



Complex Hadamard matrices, instantaneous uniform mixing and cubes

Ada Chan

ABSTRACT We study the continuous-time quantum walks on graphs in the adjacency algebra of the n -cube and its related distance regular graphs.

For $k \geq 2$, we find graphs in the adjacency algebra of $(2^{k+2} - 8)$ -cube that admit instantaneous uniform mixing at time $\pi/2^k$ and graphs that have perfect state transfer at time $\pi/2^k$.

We characterize the folded n -cubes, the halved n -cubes and the folded halved n -cubes whose adjacency algebra contains a complex Hadamard matrix. We obtain the same conditions for the characterization of these graphs admitting instantaneous uniform mixing.

1. INTRODUCTION

The continuous-time quantum walk on a graph X is given by the transition operator

$$e^{-itA} = \sum_{k \geq 0} \frac{(-it)^k}{k!} A^k,$$

where A is the adjacency matrix of X . For example, if X is the complete graph on two vertices, K_2 , then

$$\begin{aligned} e^{-itA} &= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots\right) I - i \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right) A \\ &= \begin{pmatrix} \cos t & -i \sin t \\ -i \sin t & \cos t \end{pmatrix}. \end{aligned}$$

Being the quantum analogue of the random walks on graphs, there is a lot of research interest on quantum walks for the development of quantum algorithms. Moreover, quantum walks are proved to be universal for quantum computations [7]. In this paper, we focus on the continuous-time quantum walks introduced by Farhi and Gutmann in [10]. Please see [12] and [13] for surveys on quantum walks.

Since A is real and symmetric, the operator e^{-itA} is unitary. We say the continuous-time quantum walk on X is *instantaneous uniform mixing at time τ* if

$$|(e^{-i\tau A})_{a,b}| = \frac{1}{\sqrt{|V(X)|}}, \quad \text{for all vertices } a \text{ and } b.$$

This condition is equivalent to $\sqrt{|V(X)|} e^{-i\tau A}$ being a complex Hadamard matrix. Thus if X admits instantaneous uniform mixing then its adjacency algebra contains a

Manuscript received 22nd June 2019, revised and accepted 9th February 2020.

KEYWORDS. Association schemes, Hamming schemes, complex Hadamard matrix, continuous-time quantum walks, instantaneous uniform mixing, perfect state transfer.

complex Hadamard matrix. In K_2 , the continuous-time quantum walk is instantaneous uniform mixing at time $\pi/4$.

In [14], Moore and Russell discovered that the continuous-time quantum walk on the n -cube is instantaneous uniform mixing at time $\pi/4$ which is faster than its classical analogue. Ahmadi et al. [1] showed that the complete graph K_q admits instantaneous uniform mixing if and only if $q \in \{2, 3, 4\}$. Best et al. [2] proved that instantaneous uniform mixing occurs in graphs X and Y at time τ if and only if instantaneous uniform mixing occurs in their Cartesian product at the same time. They concluded that the Hamming graph $H(n, q)$, which is the Cartesian product of n copies of K_q , has instantaneous uniform mixing if and only if $q \in \{2, 3, 4\}$. In the same paper, they also proved that a folded n -cube admits instantaneous uniform mixing if and only if n is odd.

In this paper, we give a necessary condition for the Bose–Mesner algebra of a symmetric association scheme to contain a complex Hadamard matrix. Applying this condition, we generalize the result of Best et al. to show that the adjacency algebra of $H(n, q)$ contains the adjacency matrix of a graph that admits instantaneous uniform mixing if and only if $q \in \{2, 3, 4\}$. We characterize the halved n -cubes and the folded halved n -cubes that have instantaneous uniform mixing. We obtain the same characterization for the folded n -cubes, the halved n -cubes and the folded halved n -cubes to have a complex Hadamard matrix in their adjacency algebras.

A *cubelike graph* is a Cayley graph of the elementary abelian group $\mathbb{Z}_2^d$. The graphs appear in this paper are distance regular cubelike graphs. For $k \geq 2$, we find graphs in the adjacency algebra of $H(2^{k+2} - 8, 2)$ that admit instantaneous uniform mixing at time $\pi/2^k$. Hence, for all $\tau > 0$, there exists graphs that admit instantaneous uniform mixing at time less than τ .

In a graph X , perfect state transfer occurs from vertex u to vertex w at time τ if

$$|(e^{-i\tau A(X)})_{u,w}| = 1.$$

In the n -cube, perfect state transfer occurs between antipodal vertices at time $\pi/4$ [8].

Given a graph X , we use $A(X)$ to denote its adjacency matrix, and X_r to denote the graph on the vertex set $V(X)$ in which two vertices are adjacent if they are at distance r in X . We use I_v and J_v to denote the $v \times v$ identity matrix and the $v \times v$ matrix of all ones, respectively. We drop the subscript if the order of the matrices is clear.

2. A NECESSARY CONDITION

The graphs we study in this paper are distance regular. The adjacency algebra of a distance regular graph is the Bose–Mesner algebra of a symmetric association scheme. In this section, we give a necessary condition for a Bose–Mesner algebra to contain a complex Hadamard matrix. This condition is also necessary for a Bose–Mesner algebra to contain the adjacency matrix of a graph that admits instantaneous uniform mixing.

A *symmetric association scheme* of order v with d classes is a set

$$\mathcal{A} = \{A_0, A_1, \dots, A_d\}$$

of $v \times v$ symmetric 01-matrices satisfying

- (1) $A_0 = I$.
- (2) $\sum_{j=0}^d A_j = J$.
- (3) $A_j A_k = A_k A_j$, for $j, k = 0, \dots, d$.
- (4) $A_j A_k \in \text{span } \mathcal{A}$, for $j, k = 0, \dots, d$.

For example, if X is a distance regular graph with diameter d and X_j is the j -th distance graph of X , for $j = 1, \dots, d$, then the set $\{I, A(X_1), A(X_2), \dots, A(X_d)\}$ is a symmetric association scheme.

The Bose–Mesner algebra of an association scheme $\mathcal{A}$ is the span of $\mathcal{A}$ over $\mathbb{C}$. It is known [3] that the Bose–Mesner algebra contains another basis $\{E_0, E_1, \dots, E_d\}$ satisfying

- (a) $E_j E_k = \delta_{j,k} E_j$, for $j, k = 0, \dots, d$, and
- (b) $\sum_{j=0}^d E_j = I$.

Now there exist complex numbers $p_r(s)$'s such that

$$(1) \quad A_r = \sum_{s=0}^d p_r(s) E_s, \quad \text{for } r = 0, \dots, d.$$

It follows from Condition (a) that

$$A_r E_s = p_r(s) E_s, \quad \text{for } r, s = 0, \dots, d.$$

We call the $p_r(s)$'s the eigenvalues of the association schemes. Since the matrices in $\mathcal{A}$ are symmetric, the $p_r(s)$'s are real.

A $v \times v$ matrix W is type II if, for $a, b = 1, \dots, v$,

$$(2) \quad \sum_{c=1}^v \frac{W_{ac}}{W_{bc}} = \begin{cases} v & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

A complex Hadamard matrix is a type II matrix whose entries have absolute value one.

PROPOSITION 2.1. Let $\mathcal{A} = \{A_0, A_1, \dots, A_d\}$ be a symmetric association scheme. Let $t_0, \dots, t_d \in \mathbb{C} \setminus \{0\}$. The matrix $W = \sum_{j=0}^d t_j A_j$ is type II if and only if

$$\left[\sum_{h=0}^d p_h(s) t_h \right] \left[\sum_{j=0}^d p_j(s) \frac{1}{t_j} \right] = v, \quad \text{for } s = 0, 1, \dots, d.$$

Proof. The matrix W is type II if and only if

$$\left[\sum_{h=0}^d t_h A_h \right] \left[\sum_{j=0}^d \frac{1}{t_j} A_j \right] = vI.$$

It follows from Equation (1) and Condition (b) that

$$\left[\sum_{h=0}^d \sum_{l=0}^d t_h p_h(l) E_l \right] \left[\sum_{j=0}^d \sum_{k=0}^d \frac{1}{t_j} p_j(k) E_k \right] = v \sum_{r=0}^d E_r.$$

By Condition (a), the left-hand side becomes

$$\sum_{r=0}^d \left[\sum_{h=0}^d t_h p_h(r) \right] \left[\sum_{j=0}^d \frac{1}{t_j} p_j(r) \right] E_r,$$

multiplying E_s to both sides yields the equations of this proposition. $\square$

Finding type II matrices in the Bose–Mesner algebra of a symmetric association scheme amounts to solving the system of equations in Proposition 2.1, which is not easy as d gets large. When we limit the scope of the search to complex Hadamard matrices, we get the following necessary condition which can be checked efficiently.

PROPOSITION 2.2. *If the Bose–Mesner algebra of $\mathcal{A}$ contains a complex Hadamard matrix, then*

$$v \leq \left[\sum_{r=0}^d |p_r(s)| \right]^2, \quad \text{for } s = 0, 1, \dots, d.$$

Proof. Suppose $W = \sum_{j=0}^d t_j A_j$ is a complex Hadamard matrix. By Proposition 2.1, for $s = 0, \dots, d$,

$$v = \sum_{r=0}^d p_r(s)^2 + \sum_{0 \leq h < j \leq d} \left(\frac{t_h}{t_j} + \frac{t_j}{t_h} \right) p_h(s) p_j(s).$$

Since $|\frac{t_h}{t_j}| = 1$, we have $|\frac{t_h}{t_j} + \frac{t_j}{t_h}| \leq 2$ and

$$v \leq \sum_{r=0}^d |p_r(s)|^2 + \sum_{0 \leq h < j \leq d} 2|p_h(s) p_j(s)| = \left[\sum_{r=0}^d |p_r(s)| \right]^2. \quad \square$$

Suppose $A(X)$ belongs to the Bose–Mesner algebra of $\mathcal{A}$. If instantaneous uniform mixing occurs in X at time τ then $\sqrt{v} e^{-i\tau A(X)}$ is a complex Hadamard matrix and the eigenvalues of $\mathcal{A}$ satisfy the inequalities in Proposition 2.2. For example, the association scheme $\{I_q, J_q - I_q\}$ has eigenvalues $p_0(1) = 1$ and $p_1(1) = -1$. By Proposition 2.2, if the adjacency algebra of K_q contains a complex Hadamard matrix then $q \leq 4$. Hence instantaneous uniform mixing does not occur in K_q , for $q \geq 5$.

PROPOSITION 2.3. *Let X be a graph whose adjacency matrix belongs to the Bose–Mesner algebra of $\mathcal{A}$. Let $\theta_0, \dots, \theta_d$ be the eigenvalues of $A(X)$ satisfying*

$$A(X) = \sum_{s=0}^d \theta_s E_s.$$

The continuous-time quantum walk of X is instantaneous uniform mixing at time τ if and only if there exist scalars $t_0, \dots, t_d$ such that

$$|t_0| = \dots = |t_d| = 1$$

and

$$\sqrt{v} e^{-i\tau\theta_s} = \sum_{j=0}^d p_j(s) t_j, \quad \text{for } s = 0, \dots, d.$$

Proof. It follows from Condition (a) that $A(X)^k = \sum_{s=0}^d \theta_s^k E_s$, for $k \geq 0$. Therefore,

$$(3) \quad \sqrt{v} e^{-i\tau A(X)} = \sqrt{v} \sum_{s=0}^d e^{-i\tau\theta_s} E_s$$

belongs to $\text{span } \mathcal{A}$, and there exists $t_0, \dots, t_d$ such that

$$\sqrt{v} e^{-i\tau A(X)} = \sum_{j=0}^d t_j A_j.$$

By Equation (1), we get

$$\sqrt{v} e^{-i\tau\theta_s} = \sum_{j=0}^d p_j(s) t_j, \quad \text{for } s = 0, \dots, d.$$

Lastly, $\sqrt{v} e^{-i\tau A(X)}$ is a complex Hadamard matrix exactly when

$$|t_0| = \dots = |t_d| = 1. \quad \square$$

For $n, q \geq 2$, the *Hamming graph* $H(n, q)$ is the Cartesian product of n copies of K_q . Equivalently, the vertex set V of the Hamming graph $H(n, q)$ is the set of words of length n over an alphabet of size q , and two words are adjacent if they differ in exactly one coordinate. The Hamming graph is a distance regular graph on q^n vertices with diameter n . For $j = 1, \dots, n$, X_j is the graph with vertex set V where two vertices are adjacent when they differ in exactly j coordinates. Let $A_0 = I$ and $A_j = A(X_j)$, for $j = 1, \dots, n$. Then $\mathcal{H}(n, q) = \{A_0, A_1, \dots, A_n\}$ is a symmetric association scheme, called the *Hamming scheme*. For more information on Hamming scheme, please see [3] and [11].

It follows from Equation (4.1) of [11] that

$$\sum_{j=0}^n x^j A_j = [I_q + x(J_q - I_q)]^{\otimes n},$$

and the eigenvalues of $\mathcal{H}(n, q)$ satisfy

$$(4) \quad \sum_{j=0}^n p_j(s) x^j = (1 + (q-1)x)^{n-s} (1-x)^s, \quad \text{for } s = 0, \dots, n.$$

Using $[x^k]g(x)$ to denote the coefficient of x^k in a polynomial $g(x)$, we have for $r, s = 0, \dots, n$,

$$\begin{aligned} p_r(s) &= [x^r] (1 + (q-1)x)^{n-s} (1-x)^s \\ &= [x^r] (1 + (q-1)x)^{n-s} ((1 + (q-1)x) - qx)^s \\ &= [x^r] \sum_h \binom{s}{h} (1 + (q-1)x)^{n-h} (-qx)^h \\ (5) \quad &= \sum_h (-q)^h (q-1)^{r-h} \binom{n-h}{r-h} \binom{s}{h}. \end{aligned}$$

We now quote the following characterization from [14] and [2].

THEOREM 2.4. *The Hamming graph $H(n, q)$ admits instantaneous uniform mixing if and only if $q \in \{2, 3, 4\}$.*

We see from Proposition 2.3 that whether a graph X admits instantaneous uniform mixing depends on only the spectrum of X and the eigenvalues of the Bose–Mesner algebra containing $A(X)$. A *Doob graph* $D(m_1, m_2)$ is a Cartesian product of m_1 copies of the Shrikhande graph and m_2 copies of K_4 . It is a distance regular graph with the same parameters as the Hamming graph $H(2m_1 + m_2, 4)$, see Section 9.2B of [3]. Since instantaneous uniform mixing occurs in $H(n, 4)$ for all $n \geq 1$, we see that the Doob graph $D(m_1, m_2)$ admits instantaneous uniform mixing for all $m_1, m_2 \geq 1$.

COROLLARY 2.5. *The Bose–Mesner algebra of $\mathcal{H}(n, q)$ contains a complex Hadamard matrix if and only if $q \in \{2, 3, 4\}$.*

Proof. It follows from Equation (4) that

$$p_r(n) = (-1)^r \binom{n}{r}.$$

By Proposition 2.2, if the Bose–Mesner algebra of $\mathcal{H}(n, q)$ contains a complex Hadamard matrix, then

$$q^n \leq \left[\sum_{r=0}^n |p_r(n)| \right]^2 = 4^n.$$

Hence $q \in \{2, 3, 4\}$.

The converse follows directly from Theorem 2.4. $\square$

We conclude that if $A(X)$ belongs to the Bose–Mesner algebra of $\mathcal{H}(n, q)$, for $q \geq 5$, then instantaneous uniform mixing does not occur in X .

3. THE CUBES

The Hamming graph $H(n, 2)$ is also called the n -cube. It is a distance regular graph on 2^n vertices with intersection numbers

$$a_j = 0, \quad b_j = (n - j) \quad \text{and} \quad c_j = j, \quad \text{for } j = 0, \dots, n.$$

It is both bipartite and antipodal, see Section 9.2 of [3] for details.

It follows from Equation (4) that the eigenvalues of $\mathcal{H}(n, 2)$ satisfy

$$(6) \quad p_r(n - s) = (-1)^r p_r(s) \quad \text{and} \quad p_{n-r}(s) = (-1)^s p_r(s),$$

for $r, s = 0, \dots, n$.

The proof of Lemma 3.3 uses the following equations, which are Propositions 2.1(3) and 2.3 of [6].

PROPOSITION 3.1. *The eigenvalues of $\mathcal{H}(n, 2)$ satisfy*

- (a) $p_r(s + 1) - p_r(s) = -p_{r-1}(s + 1) - p_{r-1}(s)$, for $s = 0, \dots, n - 1$, $r = 1, \dots, n$ and
- (b) $p_{r-1}(s) - p_{r-1}(s + 2) = 4 \sum_h (-2)^h \binom{n-2-h}{r-2-h} \binom{s}{h}$, for $s = 0, \dots, n - 2$ and $r = 1, \dots, n$.

Note that the Kronecker product of two complex Hadamard matrices is a complex Hadamard matrix. Hence for $\epsilon \in \{-1, 1\}$,

$$[I_2 + \epsilon i(J_2 - I_2)]^{\otimes n} = \sum_{j=0}^n (\epsilon i)^j A_j$$

is a complex Hadamard matrix in the Bose–Mesner algebra of $\mathcal{H}(n, 2)$.

Suppose $A(X)$ belongs to the Bose–Mesner algebra of $\mathcal{H}(n, 2)$ and

$$A(X)E_s = \theta_s E_s, \quad \text{for } s = 0, \dots, n.$$

It follows from Equations (3) and (4) that

$$\sqrt{2^n} e^{-i\tau A(X)} = e^{i\beta} [I_2 + \epsilon i(J_2 - I_2)]^{\otimes n}$$

if and only if

$$\begin{aligned} \sqrt{2^n} e^{-i\tau\theta_s} &= e^{i\beta} (1 + \epsilon i)^{n-s} (1 - \epsilon i)^s \\ &= \sqrt{2^n} e^{i\beta} e^{\epsilon i\pi(n-2s)/4}, \quad \text{for } s = 0, \dots, n. \end{aligned}$$

This system of equations holds exactly when

$$e^{i\beta} = e^{-i\tau\theta_0 - \epsilon i\pi n/4}$$

and

$$e^{-i\tau(\theta_s - \theta_0)} = e^{-\epsilon i\pi s/2}, \quad \text{for } s = 0, \dots, n.$$

LEMMA 3.2. *Suppose $A(X)$ belongs to the Bose–Mesner algebra of $\mathcal{H}(n, 2)$ and $A(X)E_s = \theta_s E_s$, for $s = 0, \dots, n$. If there exist k and $\epsilon \in \{-1, 1\}$ satisfying*

$$\theta_s - \theta_0 \equiv \epsilon s 2^{k-1} \pmod{2^{k+1}}, \quad \text{for } s = 0, \dots, n,$$

then there exists $\beta \in \mathbb{R}$ such that

$$\sqrt{2^n} e^{-i\frac{\pi}{2^k} A(X)} = e^{i\beta} [I_2 + \epsilon i(J_2 - I_2)]^{\otimes n}.$$

That is, X admits instantaneous uniform mixing at time $\pi/2^k$.

LEMMA 3.3. Let $r \geq 1$. Let α be the largest integer such that $\binom{n-1}{r-1}$ is divisible by 2^α . Suppose

$$\binom{n-2-h}{r-2-h} \equiv 0 \pmod{2^{\alpha+1-h}}, \quad \text{for } h = 0, \dots, \alpha.$$

Then there exists $\beta \in \mathbb{R}$ such that

$$\sqrt{2^n} e^{-i \frac{\pi}{2^{\alpha+2}} A_r} = e^{i\beta} [I_2 + \epsilon i(J_2 - I_2)]^{\otimes n},$$

where $\epsilon \in \{-1, 1\}$ satisfies

$$\binom{n-1}{r-1} \equiv -\epsilon 2^\alpha \pmod{2^{\alpha+2}}.$$

In particular, X_r admits instantaneous uniform mixing at time $\pi/2^{\alpha+2}$.

Further, if n is even and r is odd, then there exists $\beta' \in \mathbb{R}$ such that

$$\sqrt{2^n} e^{-i \frac{\pi}{2^{\alpha+2}} A_{n-r}} = e^{i\beta'} [I_2 + (-1)^{\frac{n+2}{2}} \epsilon i(J_2 - I_2)]^{\otimes n}.$$

In particular, X_{n-r} admits instantaneous uniform mixing at time $\pi/2^{\alpha+2}$.

Proof. Since $2^{\alpha+3}$ divides the right-hand side of Proposition 3.1 (b), we have

$$p_{r-1}(s+2) \equiv p_{r-1}(s) \pmod{2^{\alpha+3}}, \quad \text{for } s = 0, \dots, n-2.$$

Applying this congruence repeatedly gives, for $s = 0, \dots, n-1$,

$$-p_{r-1}(s+1) - p_{r-1}(s) \equiv -p_{r-1}(1) - p_{r-1}(0) \pmod{2^{\alpha+3}}.$$

It follows from Equation (5) that $-p_{r-1}(1) - p_{r-1}(0) = -2\binom{n-1}{r-1}$, which is divisible by $2^{\alpha+1}$ but not by $2^{\alpha+2}$. Let $\epsilon \in \{-1, 1\}$ satisfy

$$\binom{n-1}{r-1} \equiv -\epsilon 2^\alpha \pmod{2^{\alpha+2}}.$$

Then

$$-p_{r-1}(1) - p_{r-1}(0) \equiv \epsilon 2^{\alpha+1} \pmod{2^{\alpha+3}}$$

and

$$-p_{r-1}(s+1) - p_{r-1}(s) \equiv \epsilon 2^{\alpha+1} \pmod{2^{\alpha+3}}, \quad \text{for } s = 0, \dots, n-1.$$

By Proposition 3.1 (a), we have

$$p_r(s+1) - p_r(s) \equiv \epsilon 2^{\alpha+1} \pmod{2^{\alpha+3}}$$

and therefore

$$(7) \quad p_r(s) - p_r(0) \equiv \epsilon s 2^{\alpha+1} \pmod{2^{\alpha+3}}, \quad \text{for } s = 0, \dots, n.$$

By Lemma 3.2,

$$\sqrt{2^n} e^{-i \frac{\pi}{2^{\alpha+2}} A_r} = e^{i\beta} [I_2 + \epsilon i(J_2 - I_2)]^{\otimes n},$$

for some $\beta \in \mathbb{R}$, and X_r admits instantaneous uniform mixing at time $\pi/2^{\alpha+2}$.

Suppose n is even and r is odd. By Lemma 3.2, it suffices to show

$$p_{n-r}(s) - p_{n-r}(0) \equiv (-1)^{\frac{n+2}{2}} \epsilon s 2^{\alpha+1} \pmod{2^{\alpha+3}}, \quad \text{for } s = 0, \dots, n.$$

When s is even, $2^{\alpha+2}$ divides $s 2^{\alpha+1}$ and $(-1)^{(n+2)/2} \epsilon s 2^{\alpha+1} \equiv \epsilon s 2^{\alpha+1} \pmod{2^{\alpha+3}}$. Applying Equations (6) and (7), we have

$$\begin{aligned} p_{n-r}(s) - p_{n-r}(0) &= p_r(s) - p_r(0) \\ &\equiv (-1)^{\frac{n+2}{2}} \epsilon s 2^{\alpha+1} \pmod{2^{\alpha+3}}. \end{aligned}$$

When s is odd, Equation (6) gives $p_{n-r}(s) - p_{n-r}(0) = -p_r(s) - p_r(0)$. Applying Equations (5) and (7), we get

$$(8) \quad p_r(1) - p_r(0) = \frac{-2r}{n} \binom{n}{r} \equiv \epsilon 2^{\alpha+1} \pmod{2^{\alpha+3}},$$

so $2^{\alpha+1}$ is the largest power of 2 that divides $\frac{2r}{n} \binom{n}{r}$.

If $n \equiv 0 \pmod{4}$, then $2^{\alpha+3}$ divides $2 \binom{n}{r} = 2p_r(0)$ and

$$\begin{aligned} p_{n-r}(s) - p_{n-r}(0) &= -[p_r(s) - p_r(0)] - 2p_r(0) \\ &\equiv -[p_r(s) - p_r(0)] \pmod{2^{\alpha+3}} \\ &\equiv (-1)^{\frac{n+2}{2}} \epsilon s 2^{\alpha+1} \pmod{2^{\alpha+3}}. \end{aligned}$$

Suppose $n \equiv 2 \pmod{4}$. By Equation (5),

$$2p_r(s) = \sum_j (-1)^j 2^{j+1} \binom{n-j}{r-j} \binom{s}{j}.$$

The hypothesis of this lemma ensures that $2^{\alpha+3}$ divides $2^{j+1} \binom{n-j}{r-j} \binom{s}{j}$ for $j \geq 2$. Thus

$$2p_r(s) \equiv 2 \binom{n}{r} - 2^2 \binom{n-1}{r-1} s \pmod{2^{\alpha+3}}.$$

We see from Equation (8) that $2^{\alpha+1}$ is the highest power of 2 that divides $\frac{2r}{n} \binom{n}{r}$. Since r is odd and $n \equiv 2 \pmod{4}$, $2^{\alpha+1}$ is the largest power of 2 that divides $\binom{n}{r}$. Using our assumption on $\binom{n-1}{r-1}$,

$$2p_r(s) \equiv 2^{\alpha+2} (\gamma_1 - \gamma_2) \pmod{2^{\alpha+3}},$$

for some odd integers γ_1 and γ_2 . Therefore, $2p_r(s)$ is divisible by $2^{\alpha+3}$ and

$$\begin{aligned} p_{n-r}(s) - p_{n-r}(0) &= [p_r(s) - p_r(0)] - 2p_r(s) \\ &\equiv (-1)^{\frac{n+2}{2}} \epsilon s 2^{\alpha+1} \pmod{2^{\alpha+3}}. \end{aligned}$$

By Lemma 3.2, there exists $\beta' \in \mathbb{R}$ such that

$$\sqrt{2^n} e^{-i \frac{\pi}{2^{\alpha+2}} A_{n-r}} = e^{i \beta'} [I_2 + (-1)^{\frac{n+2}{2}} \epsilon i (J_2 - I_2)]^{\otimes n},$$

and instantaneous uniform mixing occurs in X_{n-r} at time $2^{\alpha+2}$. $\square$

To find the n 's and r 's that satisfy the condition in Lemma 3.3, we need the following results from number theory, due to Lucas and Kummer, respectively (see Chapter IX of [9]).

THEOREM 3.4. *Let p be a prime. Suppose the representation of N and M in base p are $n_k \dots n_1 n_0$ and $m_k \dots m_1 m_0$, respectively.*

Then

$$\binom{N}{M} \equiv \binom{n_k}{m_k} \dots \binom{n_0}{m_0} \pmod{p}.$$

THEOREM 3.5. *Let p be a prime. The largest integer k such that p^k divides $\binom{N}{M}$ is the number of carries in the addition of $N - M$ and M in base p representation.*

Let 2^α be the highest power of 2 that divides $\binom{n-1}{r-1}$. That is, there are exactly α carries in the addition of $n - r$ and $r - 1$ in base 2 representation. If both n and r are even, then no carry takes place in the right-most digit. Therefore, there are exactly α carries in the addition of $n - r$ and $r - 2$ in base 2 representation. Similarly, when n is odd and r is even, there are exactly $\alpha - 1$ carries in the addition of $n - r$ and $r - 2$ in base 2 representation. In both cases, $2^{\alpha+1}$ does not divide $\binom{n-2}{r-2}$, so the hypothesis of Lemma 3.3 does not hold when r is even.

COROLLARY 3.6. Suppose n is even. If r is an odd positive integer with $1 \leq r \leq n$, and

$$\binom{n-1}{r-1} \equiv 1 \pmod{2},$$

then there exist $\beta, \beta' \in \mathbb{R}$ such that

$$\sqrt{2^n} e^{-i \frac{\pi}{4} A_r} = e^{i\beta} [I_2 + \epsilon i(J_2 - I_2)]^{\otimes n}$$

and

$$\sqrt{2^n} e^{-i \frac{\pi}{4} A_{n-r}} = e^{i\beta'} [I_2 + (-1)^{\frac{n+2}{2}} \epsilon i(J_2 - I_2)]^{\otimes n},$$

where $\epsilon \in \{-1, 1\}$ satisfies $\binom{n-1}{r-1} \equiv -\epsilon \pmod{4}$.

In particular, X_r and X_{n-r} admit instantaneous uniform mixing at time $\pi/4$.

Proof. When $r = 1$, we have $\binom{n-2}{r-2} = 0$. For $r \geq 3$, both $n-r$ and $r-2$ are odd, there is at least one carry (in the rightmost digit) in the addition of $n-r$ and $r-2$ in base 2 representation. By Theorem 3.5, 2 divides $\binom{n-2}{r-2}$. The result follows from applying Lemma 3.3 with $\alpha = 0$. $\square$

COROLLARY 3.7. Let $n = 2^m(2l+1)$, for integers $l \geq 0$ and $m \geq 1$. For each odd r satisfying $1 \leq r < 2^m$, there exist $\beta, \beta' \in \mathbb{R}$ such that

$$\sqrt{2^n} e^{-i \frac{\pi}{4} A_r} = e^{i\beta} [I_2 + \epsilon i(J_2 - I_2)]^{\otimes n}$$

and

$$\sqrt{2^n} e^{-i \frac{\pi}{4} A_{n-r}} = e^{i\beta'} [I_2 + (-1)^{\frac{n+2}{2}} \epsilon i(J_2 - I_2)]^{\otimes n},$$

where $\epsilon \in \{-1, 1\}$ satisfies $\binom{n-1}{r-1} \equiv -\epsilon \pmod{4}$.

In particular, X_r and X_{n-r} admit instantaneous uniform mixing at time $\pi/4$.

Proof. Let r be an odd integer between 1 and 2^m . In base 2 representation, let $(n-1)$ and $(r-1)$ be $v_k \dots v_0$ and $u_k \dots u_0$, respectively. Then $v_j = 1$ for $j \leq m-1$ and $u_h = 0$ for $h \geq m$, so $\binom{v_j}{u_j} = 1$ for all j . By Lucas' Theorem, we have

$$\binom{n-1}{r-1} \equiv 1 \pmod{2}.$$

The result follows from Corollary 3.6. $\square$

We are now ready to show the existence of graphs that admit instantaneous uniform mixing earlier than time $\pi/4$.

THEOREM 3.8. Let $n = 2^{k+2} - 8$, for some $k \geq 2$. For $j = 1, 3, 5, 7$, there exists $\beta_j \in \mathbb{R}$ such that

$$(9) \quad \sqrt{2^n} e^{-i \frac{\pi}{2^k} A_{(2^{k+1}-j)}} = e^{i\beta_j} [I_2 + \epsilon_j i(J_2 - I_2)]^{\otimes n},$$

where $\epsilon_j \in \{-1, 1\}$ satisfies

$$\binom{n-1}{(2^{k+1}-j)-1} \equiv -\epsilon_j 2^{k-2} \pmod{2^k}.$$

That is, $X_{2^{k+1}-1}$, $X_{2^{k+1}-3}$, $X_{2^{k+1}-5}$ and $X_{2^{k+1}-7}$ in $\mathcal{H}(2^{k+2} - 8, 2)$ admit instantaneous uniform mixing at time $\pi/2^k$.

Proof. Let $n = 2^{k+2} - 8$ and $r = \frac{n}{2} - 1$. Then

$$n - r = 2^{k+1} - 3 = 2^k + 2^{k-1} + \dots + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$$

and

$$r - 1 = 2^{k+1} - 6 = 2^k + 2^{k-1} + \dots + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0.$$

There are $(k-2)$ carries in the addition of $n-r$ and $r-1$ in base 2 representation. By Kummer's Theorem, the highest power of 2 that divides $\binom{n-1}{r-1}$ is 2^{k-2} .

We want to show that 2^{k-1-h} divides $\binom{n-2-h}{r-2-h}$, for $0 \leq h \leq k-2$. When $h = 0$,

$$r-2 = 2^{k+1} - 7 = 2^k + 2^{k-1} + \cdots + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0,$$

so there are $(k-1)$ carries in the addition of $n-r$ and $r-2$ in base 2 representation. By Kummer's Theorem, 2^{k-1} divides $\binom{n-2}{r-2}$.

Similarly, there are $(k-2)$ carries in the addition of $n-r$ and $r-3$ in base 2 representation, so 2^{k-2} divides $\binom{n-3}{r-3}$.

As h increments by 1, the number of 1's in the leftmost $(k-2)$ digits in the base 2 representation of $r-2-h$ decreases by at most one. Hence there are at least $k-1-h$ carries in the addition of $n-r$ and $r-2-h$ in base 2 representation, and 2^{k-1-h} divides $\binom{n-2-h}{r-2-h}$, for $h = 0, \dots, k-2$.

Applying Lemma 3.3 with $r = 2^{k+1} - 5$ and $\alpha = k-2$, Equation (9) holds for $j = 5$ and $j = 3$, and $X_{2^{k+1}-5}$ and $X_{2^{k+1}-3}$ admit instantaneous uniform mixing at time $\pi/2^k$.

A similar analysis shows that Equation (9) holds for $j = 1$ and $j = 7$, and instantaneous uniform mixing occurs in $X_{2^{k+1}-1}$ and $X_{2^{k+1}-7}$ at the same time. $\square$

4. PERFECT STATE TRANSFER

Let u and w be distinct vertices in X . We say that *perfect state transfer* occurs from u to w in the continuous-time quantum walk on X at time τ if

$$|(e^{-i\tau A(X)})_{u,w}| = 1.$$

We say that X is *periodic* at u with period τ if

$$|(e^{-i\tau A(X)})_{u,u}| = 1.$$

If $A(X)$ belongs to the Bose-Mesner algebra of an association scheme $\mathcal{A}$ and X is periodic at some vertex u , then X is periodic at every vertex because $I \in \mathcal{A}$. In this case, we simply say that X is *periodic*.

Consider X_r in the Hamming scheme $\mathcal{H}(2^m, 2)$ when r is odd. We see from the proof of Corollary 3.7 that $\binom{2^m-1}{r-1}$ is odd. It follows from Theorem 2.3 of [5] that perfect state transfer occurs in X_r at time $\pi/2$. Moreover, let $1 \leq r' \leq 2^m$ be an odd integer distinct from r , then the graph $X_r \cup X_{r'}$ is periodic with period $\pi/2$.

Let X be one of the graphs considered in Corollary 3.7 or Theorem 3.8. At the time τ of instantaneous uniform mixing in X , we have

$$e^{-i\tau A(X)} = \frac{e^{i\beta}}{\sqrt{2^n}} \begin{pmatrix} 1 & \epsilon i \\ \epsilon i & 1 \end{pmatrix}^{\otimes n}, \quad \text{for some } \beta \in \mathbb{R} \text{ and } \epsilon \in \{-1, 1\}.$$

Observe that, for $\epsilon, \epsilon' \in \{-1, 1\}$,

$$(10) \quad \begin{pmatrix} 1 & \epsilon i \\ \epsilon i & 1 \end{pmatrix} \begin{pmatrix} 1 & \epsilon' i \\ \epsilon' i & 1 \end{pmatrix} = \begin{cases} 2 \begin{pmatrix} 0 & \epsilon i \\ \epsilon i & 0 \end{pmatrix} & \text{if } \epsilon = \epsilon', \\ 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \epsilon \neq \epsilon'. \end{cases}$$

We see that

$$e^{-i2\tau A(X)} = e^{2\beta i} \begin{pmatrix} 0 & \epsilon i \\ \epsilon i & 0 \end{pmatrix}^{\otimes n},$$

and X has perfect state transfer at time 2τ .

We generalize the above observation by applying Equation (10) to the union of two graphs in $\mathcal{H}(n, 2)$.

LEMMA 4.1. Let X and X' be graphs in $\mathcal{H}(n, 2)$ such that $E(X) \cap E(X') = \emptyset$, and there exist $\beta, \beta' \in \mathbb{R}$ and $\epsilon, \epsilon' \in \{-1, 1\}$ such that

$$e^{-i\tau A(X)} = \frac{e^{i\beta}}{\sqrt{2^n}} \begin{pmatrix} 1 & \epsilon i \\ \epsilon i & 1 \end{pmatrix}^{\otimes n} \quad \text{and} \quad e^{-i\tau A(X')} = \frac{e^{i\beta'}}{\sqrt{2^n}} \begin{pmatrix} 1 & \epsilon' i \\ \epsilon' i & 1 \end{pmatrix}^{\otimes n}.$$

If $\epsilon = \epsilon'$ then $X \cup X'$ has perfect state transfer at time τ . Otherwise, $X \cup X'$ is periodic at time τ .

Proof. As $A(X)$ and $A(X')$ commute, it follows from Equation (10) that

$$e^{-i\tau A(X \cup X')} = e^{-i\tau A(X)} e^{-i\tau A(X')} = \begin{cases} e^{(\beta+\beta')i} \begin{pmatrix} 0 & \epsilon i \\ \epsilon i & 0 \end{pmatrix}^{\otimes n} & \text{if } \epsilon = \epsilon', \\ e^{(\beta+\beta')i} I_{2^n} & \text{otherwise.} \end{cases}$$

□

With the help of the following result in number theory, Theorem 1 of [4], we find graphs in $\mathcal{H}(2^m, 2)$ and $\mathcal{H}(2^{k+2} - 8, 2)$ that have perfect state transfer earlier than $\pi/2$.

THEOREM 4.2. Let p be prime, n and k be positive integers. If p^k divides n then

$$\binom{n-1}{s} \equiv (-1)^{s-\lfloor s/p \rfloor} \binom{n/p-1}{\lfloor s/p \rfloor} \pmod{p^k},$$

for $s = 0, \dots, n-1$.

PROPOSITION 4.3. For $m \geq 3$, and for odd integers r and r' satisfying

$$(11) \quad 1 \leq r < r' < 2^{m-1} \quad \text{or} \quad 2^{m-1} < r < r' < 2^m,$$

perfect state transfer occurs in the graph $X_r \cup X_{r'}$ of $\mathcal{H}(2^m, 2)$ at time $\pi/4$.

Proof. Let r be an odd integer between 2^b and 2^{b+1} for some $b \leq m-1$. Let $s_0 = r-1$ and $s_i = \lfloor s_{i-1}/2 \rfloor$, for $i = 1, \dots, b$. Let $n = 2^m$. Applying Theorem 4.2 repeatedly gives

$$\binom{n-1}{r-1} \equiv (-1)^{s_0-s_i} \binom{2^{m-i}-1}{s_i} \pmod{2^{m-i+1}}, \quad \text{for } 1 \leq i \leq b.$$

Since $s_b = 1$ and $m-b+1 \geq 2$, applying the above equation with $i = b$ yields

$$\binom{n-1}{r-1} \equiv (-1)^{r-2} (2^{m-b} - 1) \pmod{4}.$$

If $r < 2^{m-1}$, we have $b \leq m-2$ and

$$\binom{n-1}{r-1} \equiv 1 \pmod{4}.$$

If $2^{m-1} < r$, we have $b = m-1$ and

$$\binom{n-1}{r-1} \equiv -1 \pmod{4}.$$

It follows from Corollary 3.7 that there exist $\beta, \beta' \in \mathbb{R}$ such that

$$e^{-i\frac{\pi}{4}A_r} = \frac{e^{i\beta}}{\sqrt{2^n}} \begin{pmatrix} 1 & \epsilon i \\ \epsilon i & 1 \end{pmatrix}^{\otimes n} \quad \text{and} \quad e^{-i\frac{\pi}{4}A_{r'}} = \frac{e^{i\beta'}}{\sqrt{2^n}} \begin{pmatrix} 1 & \epsilon' i \\ \epsilon' i & 1 \end{pmatrix}^{\otimes n},$$

where

$$\epsilon = \begin{cases} -1 & \text{if } r \text{ and } r' \text{ are odd integers between } 1 \text{ and } 2^{m-1}, \\ 1 & \text{if } r \text{ and } r' \text{ are odd integers between } 2^{m-1} \text{ and } 2^m. \end{cases}$$

By Lemma 4.1, perfect state transfer occurs in $X_r \cup X_{r'}$ at time $\frac{\pi}{4}$. □

PROPOSITION 4.4. *For integer $k \geq 2$, perfect state transfer occurs in graphs*

$$X_{2^{k+1}-5} \cup X_{2^{k+1}-7} \quad \text{and} \quad X_{2^{k+1}-1} \cup X_{2^{k+1}-3}$$

of $\mathcal{H}(2^{k+2}-8, 2)$ at time $\pi/2^k$.

Proof. Let $n = 2^{k+2} - 8$ and $m = \frac{n}{8}$. Let $\epsilon_1, \epsilon_3, \epsilon_5, \epsilon_7$ be the integers defined in Theorem 3.8.

Consider $4m - 1 = 2^{k+1} - 5$ and $4m - 3 = 2^{k+1} - 7$. From

$$\binom{8m-1}{4m-4} = \left[1 - 4 \frac{5m}{(4m+3)(2m+1)} \right] \binom{8m-1}{4m-2},$$

we get

$$\begin{aligned} \binom{n-1}{(2^{k+1}-7)-1} &= \left[1 - 4 \frac{5m}{(4m+3)(2m+1)} \right] \binom{n-1}{(2^{k+1}-5)-1} \\ &\equiv \left[1 - 4 \frac{5m}{(4m+3)(2m+1)} \right] (-\epsilon_5 2^{k-2}) \pmod{2^k}. \end{aligned}$$

Since $4m+3$ and $2m+1$ are coprime with 2^k , we have

$$\binom{n-1}{(2^{k+1}-7)-1} \equiv -\epsilon_5 2^{k-2} \pmod{2^k},$$

and $\epsilon_7 = \epsilon_5$. It follows from Theorem 3.8 and Lemma 4.1 that perfect state transfer occurs in $X_{2^{k+1}-5} \cup X_{2^{k+1}-7}$ at time $\pi/2^k$.

For $X_{2^{k+1}-3}$ and $X_{2^{k+1}-1}$, we have $4m+1 = 2^{k+1}-3$ and $4m+3 = 2^{k+1}-1$. From

$$\binom{8m-1}{4m+2} = \left[1 - 4 \frac{3m}{(4m+1)(2m+1)} \right] \binom{8m-1}{4m},$$

we have

$$\begin{aligned} \binom{n-1}{(2^{k+1}-1)-1} &= \left[1 - 4 \frac{3m}{(4m+1)(2m+1)} \right] \binom{n-1}{(2^{k+1}-3)-1} \\ &\equiv \left[1 - 4 \frac{3m}{(4m+1)(2m+1)} \right] (-\epsilon_3 2^{k-2}) \pmod{2^k}. \end{aligned}$$

Since $4m+1$ and $2m+1$ are coprime with 2^k , we have

$$\binom{n-1}{(2^{k+1}-1)-1} \equiv -\epsilon_3 2^{k-2} \pmod{2^k},$$

and $\epsilon_1 = \epsilon_3$. It follows from Theorem 3.8 and Lemma 4.1 that perfect state transfer occurs in $X_{2^{k+1}-1} \cup X_{2^{k+1}-3}$ at time $\pi/2^k$. $\square$

5. HALVED n -CUBE

The n -cube X is a connected bipartite graph of diameter n . When $n \geq 2$, X_2 has two components, one of which has the set $\mathcal{E}$ of binary words of even weights as its vertex set. The *halved n -cube*, denoted by $\widehat{X}$, is the subgraph of X_2 induced by $\mathcal{E}$. It is a distance regular graph on 2^{n-1} vertices with diameter $\lfloor \frac{n}{2} \rfloor$. The intersection numbers of $\widehat{X}$ are

$$\widehat{a}_j = 2j(n-2j), \quad \widehat{b}_j = \frac{(n-2j)(n-2j-1)}{2} \quad \text{and} \quad \widehat{c}_j = j(2j-1),$$

for $j = 0, \dots, \lfloor \frac{n}{2} \rfloor$, and the eigenvalues of $\widehat{X}$ are $p_2(0), p_2(1), \dots, p_2(\lfloor n/2 \rfloor)$.

Let $\widehat{\mathcal{A}} = \{I, \widehat{A}_1, \dots, \widehat{A}_{\lfloor n/2 \rfloor}\}$ where $\widehat{A}_r = A(\widehat{X}_r)$. We use $\widehat{p}_r(s)$ to denote the eigenvalues of $\widehat{\mathcal{A}}$ and let $\widehat{p}_{-1}(s) = 0$. Equation (11) on page 128 of [3] states that, for $r, s = 0, \dots, \lfloor n/2 \rfloor$,

$$\widehat{p}_1(s)\widehat{p}_r(s) = \widehat{c}_{r+1}\widehat{p}_{r+1}(s) + \widehat{a}_r\widehat{p}_r(s) + \widehat{b}_{r-1}\widehat{p}_{r-1}(s).$$

It is straightforward to verify that $\widehat{p}_r(s) = p_{2r}(s)$ satisfies these recursions, so the eigenvalues of $\widehat{\mathcal{A}}$ are

$$(12) \quad \widehat{p}_r(s) = p_{2r}(s), \quad \text{for } r, s = 0, \dots, \lfloor \frac{n}{2} \rfloor.$$

For more information on the halved n -cube, please see Sections 4.2 and 9.2D of [3].

When $n = 2m + 1$, Equation (4) yields

$$\sum_{h=0}^n p_h(s)i^h = (1+i)^{2m+1-s}(1-i)^s = 2^m i^{m-s}(1+i), \quad \text{for } s = 0, \dots, n.$$

The real part of this sum is

$$(13) \quad \sum_{r=0}^m p_{2r}(s)(-1)^r = \sum_{r=0}^m \widehat{p}_r(s)(-1)^r = \begin{cases} 2^m & \text{if } m-s \equiv 0 \pmod{4} \text{ or } m-s \equiv 3 \pmod{4}, \\ -2^m & \text{otherwise.} \end{cases}$$

By Proposition 2.1, $\sum_{r=0}^m (-1)^r \widehat{A}_r$ is a (complex) Hadamard matrix.

THEOREM 5.1. *For $n \geq 3$, the adjacency algebra of the halved n -cube contains a complex Hadamard matrix if and only if n is odd.*

Proof. Suppose $n = 2m$. Using Proposition 2.2, it is sufficient to show that

$$\left[\sum_{r=0}^m |\widehat{p}_r(m-1)| \right]^2 < 2^{2m-1}, \quad \text{for } m \geq 2.$$

It follows from Equations (4) and (12) that for $r \geq 0$,

$$\begin{aligned} \widehat{p}_r(m-1) &= [x^{2r}](1+x)^{m+1}(1-x)^{m-1} \\ &= [x^{2r}](1+2x+x^2)(1-x^2)^{m-1} \\ &= (-1)^r \left[\binom{m-1}{r} - \binom{m-1}{r-1} \right]. \end{aligned}$$

Hence

$$|\widehat{p}_r(m-1)| = \begin{cases} \binom{m-1}{r} - \binom{m-1}{r-1} & \text{if } 0 \leq r \leq \frac{m}{2} \\ \binom{m-1}{r-1} - \binom{m-1}{r} & \text{if } \frac{m}{2} < r \leq m \end{cases}$$

and

$$\begin{aligned} \sum_{r=0}^m |\widehat{p}_r(m-1)| &= \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \left[\binom{m-1}{r} - \binom{m-1}{r-1} \right] + \sum_{r=\lfloor \frac{m}{2} \rfloor + 1}^m \left[\binom{m-1}{r-1} - \binom{m-1}{r} \right] \\ &= 2 \binom{m-1}{\lfloor \frac{m}{2} \rfloor}. \end{aligned}$$

A simple mathematical induction on m shows that $4 \binom{m-1}{\lfloor \frac{m}{2} \rfloor}^2 < 2^{2m-1}$, for $m \geq 2$.

When n is odd, $\sum_{r=0}^m (-1)^r \widehat{A}_r$ is a complex Hadamard matrix. $\square$

THEOREM 5.2. *For $n \geq 3$, the halved n -cube admits instantaneous uniform mixing if and only if n is odd.*

Proof. From the above theorem, the halved n -cube does not admit instantaneous uniform mixing when $n \geq 4$ is even.

Suppose $n = 2m + 1$ and $e^{-2i\tau} \in \{-i, i\}$. For $s = 0, \dots, m$, we have

$$\widehat{p}_1(s) = 2(m-s)(m-s+1) - m$$

and

$$\begin{aligned} e^{-i\tau\widehat{p}_1(s)} &= (e^{-2i\tau})^{(m-s)(m-s+1)} e^{i\tau m} \\ &= \begin{cases} e^{i\tau m} & \text{if } m-s \equiv 0 \pmod{4} \text{ or } m-s \equiv 3 \pmod{4}, \\ -e^{i\tau m} & \text{otherwise.} \end{cases} \end{aligned}$$

We see from Equation (13) that

$$2^m e^{-i\tau\widehat{p}_1(s)} = e^{i\tau m} \sum_{r=0}^m (-1)^r \widehat{p}_r(s), \quad \text{for } s = 0, \dots, m.$$

Since $|e^{i\tau m}(-1)^r| = 1$, it follows from Proposition 2.3 that $\widehat{X}_1$ admits instantaneous uniform mixing at time $\frac{\pi}{4}$. $\square$

The halved 2-cube is the complete graph on two vertices and it admits instantaneous uniform mixing (see [1]).

When $n \geq 3$, the halved n -cube is isomorphic to the cubelike graph of $\mathbb{Z}_2^{n-1}$ with connection set

$$C = \{\mathbf{a} : \text{weight of } \mathbf{a} \text{ is } 1 \text{ or } 2\}.$$

Applying Theorem 2.3 of [5] to the halved n -cube with even n , we see that perfect state transfer occurs from $\mathbf{a}$ to $\mathbf{a} \oplus \mathbf{1}$ at time $\pi/2$. But this graph does not have instantaneous uniform mixing.

6. FOLDED n -CUBE

Let Γ be a distance regular graph on v vertices with diameter d and intersection array $\{b_0, b_1, \dots, b_{d-1}; c_1, \dots, c_d\}$. We say Γ is *antipodal* if Γ_d is a union of complete graph K_R 's, for some fixed R . The vertex sets of the K_R 's in Γ_d form an equitable partition $\mathcal{P}$ of Γ and the quotient graph of Γ with respect to $\mathcal{P}$ is called the *folded graph* $\widetilde{\Gamma}$ of Γ . When $d > 2$, $\widetilde{\Gamma}$ is a distance regular graph on $\frac{v}{R}$ vertices with diameter $\lfloor \frac{d}{2} \rfloor$, see Proposition 4.2.2 (ii) of [3]. Moreover $\widetilde{\Gamma}$ has intersection numbers $\widetilde{a}_j = a_j$, $\widetilde{b}_j = b_j$ and $\widetilde{c}_j = c_j$ for $j = 0, \dots, \lfloor \frac{d}{2} \rfloor - 1$ and

$$\widetilde{c}_{\lfloor \frac{d}{2} \rfloor} = \begin{cases} c_{\lfloor \frac{d}{2} \rfloor} & \text{if } d \text{ is odd,} \\ Rc_{\frac{d}{2}} & \text{if } d \text{ is even.} \end{cases}$$

From Proposition 4.2.3 (ii) of [3], we see that if the eigenvalues of Γ are $p_1(0) \geq p_1(1) \geq \dots \geq p_1(d)$, then $\widetilde{\Gamma}$ has eigenvalues $\widetilde{p}_1(j) = p_1(2j)$ for $j = 0, \dots, \lfloor \frac{d}{2} \rfloor$. The eigenvalues for $\widetilde{A}_j$'s and A_j 's satisfy the same recursive relation (Equation (11) on Page 128 of [3]) for $j = 0, \dots, \lfloor \frac{d}{2} \rfloor$ when d is odd and for $j = 0, \dots, \frac{d}{2} - 1$ when d is even. When d is even, $\widetilde{p}_{\frac{d}{2}}(s) = \frac{1}{R}p_{\frac{d}{2}}(2s)$. Therefore

$$(14) \quad \widetilde{p}_r(s) = \begin{cases} p_r(2s) & \text{if } 0 \leq r < \lfloor \frac{d}{2} \rfloor, \\ p_{\lfloor \frac{d}{2} \rfloor}(2s) & \text{if } d \text{ is odd and } r = \lfloor \frac{d}{2} \rfloor, \\ \frac{1}{R}p_{\frac{d}{2}}(2s) & \text{if } d \text{ is even and } r = \frac{d}{2}. \end{cases}$$

For each vertex $\mathbf{a}$ in the n -cube X , $\mathbf{1} \oplus \mathbf{a}$ is the unique vertex at distance n from $\mathbf{a}$. Therefore X_n is a union of K_2 's. The folded n -cube $\tilde{X}$ has 2^{n-1} vertices, diameter $\lfloor \frac{n}{2} \rfloor$, and eigenvalues

$$(15) \quad \tilde{p}_r(s) = \begin{cases} [x^r](1+x)^{n-2s}(1-x)^{2s} & \text{if } 0 \leq r < \lfloor \frac{n}{2} \rfloor, \\ [x^{\lfloor \frac{n}{2} \rfloor}](1+x)^{n-2s}(1-x)^{2s} & \text{if } n \text{ is odd and } r = \lfloor \frac{n}{2} \rfloor, \\ [x^{\frac{n}{2}}]_{\frac{1}{2}}(1+x)^{n-2s}(1-x)^{2s} & \text{if } n \text{ is even and } r = \frac{n}{2}. \end{cases}$$

The folded n -cube is isomorphic to the graph obtained from an $(n-1)$ -cube by adding the perfect matching in which a vertex $\mathbf{a}$ is adjacent to $\mathbf{1} \oplus \mathbf{a}$. Best et al. proved the following result, see Theorem 1 of [2].

THEOREM 6.1. *For $n \geq 3$, the folded n -cube admits instantaneous uniform mixing if and only if n is odd.*

In particular, the adjacency algebra of the folded n -cube contains a complex Hadamard matrix when n is odd.

THEOREM 6.2. *For $n \geq 3$, the adjacency algebra of the folded n -cube contains a complex Hadamard matrix if and only if n is odd.*

Proof. Suppose $n = 4m$, for some $m \geq 1$. We have, for $r = 0, \dots, 2m-1$,

$$\begin{aligned} \tilde{p}_r(m) &= [x^r](1+x)^{2m}(1-x)^{2m} \\ &= \begin{cases} (-1)^{\frac{r}{2}} \binom{2m}{\frac{r}{2}} & \text{if } r \text{ is even,} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\tilde{p}_{2m}(m) = (-1)^m \frac{1}{2} \binom{2m}{m}.$$

Now

$$\begin{aligned} \sum_{r=0}^{2m} |\tilde{p}_r(m)| &= \sum_{r=0}^{m-1} \binom{2m}{r} + \frac{1}{2} \binom{2m}{m} \\ &= \frac{1}{2} \left[\sum_{r=0}^{2m} \binom{2m}{r} \right] \\ &= 2^{2m-1}. \end{aligned}$$

We have $\left[\sum_{s=0}^{2m} |\tilde{p}_s(m)| \right]^2 < 2^{4m-1}$. By Proposition 2.2, the adjacency algebra of the folded $4m$ -cube does not contain a complex Hadamard matrix.

Suppose $n = 4m + 2$. By Equation (15),

$$\tilde{p}_r(m) = \begin{cases} 1 & \text{if } r = 0, \\ (-1)^{\lfloor \frac{r}{2} \rfloor} 2 \binom{2m}{\lfloor \frac{r}{2} \rfloor} & \text{if } 1 \leq r < 2m \text{ is odd,} \\ (-1)^{\frac{r}{2}} \left[\binom{2m}{\frac{r}{2}} - \binom{2m}{\frac{r}{2}-1} \right] & \text{if } 2 \leq r \leq 2m \text{ is even,} \\ (-1)^m \binom{2m}{m} & \text{if } r = 2m + 1. \end{cases}$$

Now

$$\begin{aligned} \sum_{s=0}^{2m+1} |\tilde{p}_s(m)| &= 1 + \sum_{r=0}^{m-1} 2 \binom{2m}{r} + \sum_{r=1}^m \left[\binom{2m}{r} - \binom{2m}{r-1} \right] + \binom{2m}{m} \\ &= 2^{2m} + \binom{2m}{m}. \end{aligned}$$

A simple mathematical induction on m shows that $[2^{2m} + \binom{2m}{m}]^2 < 2^{4m+1}$, for all integer $m \geq 2$. We conclude that the adjacency algebra of the folded $(4m+2)$ -cube does not contain a complex Hadamard matrix, for $m \geq 2$.

The folded 6-cube has eigenvalues

$$\begin{aligned} p_0(1) = p_0(2) = 1, & & p_1(1) = -p_1(2) = 2, \\ p_2(1) = p_2(2) = -1 & \text{and} & p_3(1) = -p_3(2) = -2. \end{aligned}$$

Let $W = \sum_{j=0}^3 t_j \tilde{A}_j$ be a type II matrix. Adding the equations in Proposition 2.1 for $s = 1$ and $s = 2$ gives

$$-\left(\frac{t_0}{t_2} + \frac{t_2}{t_0}\right) - 4\left(\frac{t_1}{t_3} + \frac{t_3}{t_1}\right) = 22.$$

The left-hand side is at most ten if $|t_0| = |t_1| = |t_2| = |t_3| = 1$. Therefore, the adjacency algebra of the folded 6-cube does not contain a complex Hadamard matrix. $\square$

The folded 2-cube is the complete graph on two vertices and it admits instantaneous uniform mixing (see [1]).

7. FOLDED HALVED $2m$ -CUBE

According to Page 141 of [3], the halved $2m$ -cube $\hat{X}$ is antipodal with antipodal classes of size two and the folded $2m$ -cube $\tilde{X}$ is bipartite for $m \geq 2$. In addition, the folded graph of $\hat{X}$ is isomorphic to the halved graph of $\tilde{X}$. We use $\mathcal{X}$ to denote the folded graph of $\hat{X}$ which is a distance regular graph on 2^{2m-2} vertices with diameter $\lfloor \frac{m}{2} \rfloor$. Let $\mathcal{A}_r = A(\mathcal{X}_r)$, for $r = 0, \dots, \lfloor \frac{m}{2} \rfloor$.

By Equations (12) and (14), the eigenvalues of the folded halved $2m$ -cube are

$$(16) \quad \mathcal{P}_r(s) = \begin{cases} p_{2r}(2s) & \text{if } 0 \leq r < \lfloor \frac{m}{2} \rfloor, \\ p_{2\lfloor \frac{m}{2} \rfloor}(2s) & \text{if } m \text{ is odd and } r = \lfloor \frac{m}{2} \rfloor, \\ \frac{1}{2}p_m(2s) & \text{if } m \text{ is even and } r = \frac{m}{2}. \end{cases}$$

THEOREM 7.1. *The adjacency algebra of the folded halved $2m$ -cube contains a complex Hadamard matrix if and only if m is even.*

Proof. Suppose $m = 2u + 1$. Then

$$\begin{aligned} \mathcal{P}_r(u) &= [x^{2r}](1 + 2x + x^2)(1 - x^2)^{2u} \\ &= \begin{cases} 1 & \text{if } r = 0 \\ (-1)^r \binom{2u}{r} + (-1)^{r-1} \binom{2u}{r-1} & \text{if } 1 \leq r \leq u. \end{cases} \end{aligned}$$

Then

$$\sum_{r=0}^u |\mathcal{P}_r(u)| = 1 + \sum_{r=1}^u \left[\binom{2u}{r} - \binom{2u}{r-1} \right] = \binom{2u}{u}.$$

Hence

$$\left[\sum_{r=0}^u |\mathcal{P}_r(u)| \right]^2 < \left[\sum_{r=0}^{2u} \binom{2u}{r} \right]^2 = 2^{4u}.$$

By Proposition 2.2, the adjacency algebra of the folded halved $(4u+2)$ -cube does not contain a complex Hadamard matrix.

Suppose $m = 2u$. By Equations (16) and (6),

$$\begin{aligned} \sum_{r=0}^u (-1)^r \mathcal{P}_r(s) &= \sum_{r=0}^{u-1} (-1)^r p_{2r}(2s) + \frac{1}{2} (-1)^u p_{2u}(2s) \\ &= \frac{1}{2} \sum_{r=0}^{u-1} (-1)^r p_{2r}(2s) + \frac{1}{2} (-1)^u p_{2u}(2s) + \frac{1}{2} \sum_{r=0}^{u-1} (-1)^r (-1)^{2s} p_{4u-2r}(2s) \\ &= \frac{1}{2} \sum_{r=0}^{2u} (-1)^r p_{2r}(2s), \end{aligned}$$

which is equal to the real part of $\frac{1}{2} \sum_{j=0}^{4u} i^j p_j(2s)$. By Equation (4),

$$(17) \quad \frac{1}{2} \sum_{j=0}^{4u} i^j p_j(2s) = \frac{1}{2} (1+i)^{4u-2s} (1-i)^{2s} = (-1)^{u-s} 2^{2u-1}.$$

By Proposition 2.1, $\sum_{s=0}^u (-1)^s \mathcal{A}_s$ is a complex Hadamard matrix. $\square$

THEOREM 7.2. *The folded halved $2m$ -cube admits instantaneous uniform mixing if and only if m is even.*

Proof. Suppose $m = 2u$ and $e^{-8i\tau} = -1$. For $s = 0, \dots, u$,

$$\mathcal{P}_1(s) = 8(u-s)^2 - 2u$$

and

$$2^{2u-1} e^{-i\tau \mathcal{P}_1(s)} = 2^{2u-1} (-1)^{(u-s)^2} e^{2iu\tau},$$

which is equal to $e^{2iu\tau} \sum_{r=0}^u (-1)^r \mathcal{P}_r(s)$ from Equation (17). By Proposition 2.3, the folded halved $4u$ -cube admits instantaneous uniform mixing at time $\pi/8$. $\square$

Acknowledgements. The author would like to thank Chris Godsil, Natalie Mullin and Aidan Roy for many interesting discussions. The author is grateful for Akihiro Munemasa's advice on the exposition.

REFERENCES

- [1] Amir Ahmadi, Ryan Belk, Christino Tamon, and Carolyn Wendler, *On mixing in continuous-time quantum walks on some circulant graphs*, Quantum Inf. Comput. **3** (2003), no. 6, 611–618.
- [2] Ana Best, Markus Kliegl, Shawn Mead-Gluchacki, and Christino Tamon, *Mixing of quantum walks on generalized hypercubes*, International Journal of Quantum Information **6** (2008), no. 6, 1135–1148.
- [3] Andries E. Brouwer, Arjeh M. Cohen, and Arnold Neumaier, *Distance-regular graphs*, Springer-Verlag, Berlin, 1989.
- [4] Tian Xin Cai and Andrew Granville, *On the residues of binomial coefficients and their products modulo prime powers*, Acta Math. Sin. (Engl. Ser.) **18** (2002), no. 2, 277–288.
- [5] Wang-Chi Cheung and Chris Godsil, *Perfect state transfer in cubelike graphs*, Linear Algebra Appl. **435** (2011), no. 10, 2468–2474.
- [6] Laura Chihara and Dennis Stanton, *Zeros of generalized Krawtchouk polynomials*, J. Approx. Theory **60** (1990), no. 1, 43–57.
- [7] Andrew M. Childs, *Universal computation by quantum walk*, Phys. Rev. Lett. **102** (2009), no. 18, 180501 (4 pages).
- [8] Matthias Christandl, Nilanjana Datta, Tony Dorlas, Artur Ekert, Alastair Kay, and Andrew J. Landahl, *Perfect transfer of arbitrary states in quantum spin networks*, Phys. Rev. A **71** (2005), no. 3, 032312 (11 pages).
- [9] Leonard Eugene Dickson, *History of the theory of numbers. Vol. I: Divisibility and primality*, Chelsea Publishing Co., New York, 1966.
- [10] Edward Farhi and Sam Gutmann, *Quantum computation and decision trees*, Phys. Rev. A (3) **58** (1998), no. 2, 915–928.
- [11] Chris Godsil, *Generalized Hamming schemes*, <https://arxiv.org/abs/1011.1044>, 2010.
- [12] ———, *State transfer on graphs*, Discrete Math. **312** (2012), no. 1, 129–147.

- [13] Julia Kempe, *Quantum random walks: an introductory overview*, Contemporary Physics **44** (2003), no. 4, 307–327.
- [14] Cristopher Moore and Alexander Russell, *Quantum walks on the hypercube*, in Randomization and approximation techniques in computer science, Lecture Notes in Comput. Sci., vol. 2483, Springer, Berlin, 2002, pp. 164–178.

ADA CHAN, York University, Dept. of Mathematics and Statistics, 4700 Keele Street, Toronto,
Ontario, M3J 1P3, Canada
E-mail : `ssachan@yorku.ca`