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Abstract Based upon a previous work of Manjunath and Sturmfels for a finite, complete, undirected graph, and a refined algorithm by Eröcal, Motsak, Schreyer and Steenpaß for computing syzygies, we display a free resolution of the lattice ideal associated to a finite, strongly connected, weighted, directed graph. Moreover, the resolution is minimal precisely when the digraph is strongly complete.

1. Introduction

The aim of this paper is to present a free resolution of the lattice ideal $I(\mathcal{L})$ associated to the lattice $\mathcal{L}$ spanned by the columns of the Laplacian matrix L of a finite, strongly connected, weighted, directed graph $\mathbb{G}$, and to show that this resolution is minimal if and only if the graph $\mathbb{G}$ is strongly complete. Our techniques are a novel mixture of a mild generalization of previous work by Manjunath and Sturmfels for finite, complete, weighted, undirected graphs and a recent refined algorithm for computing syzygies due to Eröcal, Motsak, Schreyer and Steenpaß. (The meaning of the terminology we use is presented in Section 2.)

First we present some background.

The Abelian Sandpile Model (ASM) is a game played on a finite, weighted, connected, undirected graph $\mathbb{G}$ with n vertices, that realizes the dynamics implicit in the discrete Laplacian matrix L of the graph, this matrix being an integer matrix that is symmetric. Each configuration is a mapping from the vertices of the graph into the set of non-negative integers. The value of the mapping at a vertex may be considered as the number of grains of sand on a sandpile placed at the vertex. The game's evolution is given by a 'toppling' rule: each vertex containing at least as many grains as it has neighbours distributes one grain to each of them. (This process has also been called 'sand-firing' or 'chip-firing'.)

The ASM was introduced by Bak, Tang and Wiesenfeld [3] in the context of self-organized critical phenomena in statistical physics and has been studied extensively since. In the seminal paper [7], Cori, Rossin and Salvy, enumerated the vertices of the undirected graph $\mathbb{G}$ using a natural metric, considered the lattice $\mathcal{L}$ spanned over $\mathbb{Z}$ by the rows of the symmetric Laplacian matrix L and introduced a so-called

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toppling (or lattice) ideal $I(\mathcal{L})$ in the polynomial ring $\mathbb{Q}[x] := \mathbb{Q}[x_1, \dots, x_n]$. This ideal encodes configurations with monomials and topplings with binomials. In particular, they showed that the set of toppling binomials constituted a Gröbner basis for $I(\mathcal{L})$ with respect to a natural monomial ordering in $\mathbb{Q}[x]$, which may be taken to be the graded reverse lexicographic ordering.

In their recent survey paper [22], Perkinson, Perlman and Wilmes associated to the Laplacian matrix L of a directed graph $\mathbb{G}$ a collection of binomial equations, and noted that these binomials span the lattice ideal $I(\mathcal{L})$ in $\mathbb{C}[x_1, \dots, x_n]$, where $\mathcal{L}$ again is the lattice spanned over $\mathbb{Z}$ by the rows of the (now non-necessarily symmetric) Laplacian matrix L, using the theory of lattice ideals extensively in their discussion.

More recently, and for complete undirected graphs, i.e. any two distinct vertices are connected by an edge, Manjunath and Sturmfels [18] gave a minimal free resolution of $\mathbb{K}[x]/I(\mathcal{L})$, $\mathbb{K}$ an arbitrary field, showing also that the binomials in question formed a minimal generating set and a minimal Gröbner basis of $I(\mathcal{L})$ in the graded reverse lexicographic ordering. Subsequently, Manjunath, Schreyer and Wilmes[17], and Mohammadi and Shokrieh [19], gave a minimal free resolution of $\mathbb{K}[x]/I(\mathcal{L})$ in the case where the underlying undirected graph was, more generally, connected. We remark here that the techniques used in these papers are a mix of results from combinatorial graph theory and the theory of Gröbner bases (e.g. the Schreyer algorithm as a tool to proving the exactness of a complex).

The transition from undirected to directed graphs introduces extra technicalities, and means that we have to rely heavily on techniques from the theory of Gröbner bases and homogeneous Commutative Algebra. Our general setting is the following: $\mathbb{G}$ is a finite, strongly connected, weighted, directed graph, L stands for its Laplacian matrix, $\mathcal{L}$ is the lattice spanned by the columns of L and $I(\mathcal{L})$ is the lattice ideal associated to $\mathcal{L}$.

First of all, we see that, having again enumerated the vertices of the graph $\mathbb{G}$ using a generalized natural metric, $I(\mathcal{L})$ is homogeneous in a weighted reverse lexicographic order on $\mathbb{K}[x]$, $\mathbb{K}$ an arbitrary field, the weighting coming from the structure of $\mathrm{adj}(L)$ (see Remark 2.8). We then use basic aspects of homogeneous Commutative Algebra and previous work of ours to show that $I(\mathcal{L})$ is a Cohen-Macaulay ideal of dimension 1, so that $\mathrm{projdim}_{\mathbb{K}[x]}(\mathbb{K}[x]/I(\mathcal{L})) = n - 1$: see Corollary 2.10.

This earlier work shows up as follows. We find (see Sections 2.2, 2.3) that the matrices L we deal with are what we called Critical Binomial Matrices (CB matrices, in brief) in previous work, and that $\mathbb{G}$ is, further, strongly complete precisely when the matrices L are what we previously called Positive Critical Binomial Matrices (PCB matrices, in brief): see [20], [21]. In fact, as we shall see in Section 2, since we treat the case where $\mathbb{G}$ is strongly connected, the CB matrix L has a further property which is precisely Irreducibility, hence the matrices L are Irreducible Critical Binomial Matrices (ICB matrices, in brief), whose properties are developed in the paper: see, for example, Sections 2.6, 2.7, 4.1. Wherever possible, we give new proofs and develop new approaches to this previous work: see, for example, Propositions 2.3, 2.9.

There are two main ingredients in the proof of our main theorem (Theorem 6.1). The first is the fact that the resolving complex Cyc_n of [18] (treated with a little care in our more general context, as the Laplacian matrix L need no longer be symmetric) continues to be a complex and, moreover, is exact in degree zero: see Sections 3.5, 4.5. The second ingredient is a refined choice of Schreyer syzygies: see Section 5.2. We find, via an inductive proof that keeps track of the leading terms of the Manjunath-Sturmfels boundaries, that in a very satisfying yet surprising manner, these boundaries pair off with appropriate Schreyer syzygies, enabling one to deduce in Theorem 6.1

that Cyc_n is in fact acyclic. It then follows easily that Cyc_n is minimal if and only if $\mathbb{G}$ is, further, strongly complete (see Corollary 6.16).

We finish this introduction by emphasizing that, usually, and alternatively to the works of, e.g. Perkinson, Perlman and Wilmes in [22], or Backman and Manjunath in [2], we have taken $\mathcal{L}$ to be the lattice spanned by the columns of the Laplacian matrix L and not the rows (see Sections 2.1 and 2.5). Note that, for the undirected graph case this is an irrelevant matter, because L being a symmetric matrix, then the lattice spanned by the columns or the lattice spanned by the rows do coincide. It is worth to mention here that studying the case of the lattice spanned by the columns can be related to a column chip-firing game. This is done by Asadi and Backman in [1, Introduction and Section 3.2]. An interesting future research would be to pass from columns to rows. As a possible approach one could consider the lattice $\mathcal{L}^\top$, spanned by the columns of the transpose of the Laplacian matrix $L^\top$, and then proceed in a similar vein as in [21]. See also the connection with Eulerian digraphs (the indegree and outdegree coincide for every vertex) in the aforementioned work of Asadi and Backman ([1, Theorem 3.18]).

Throughout, we use the explicit case $n = 4$ as a running illustrative example.

Liam O'Carroll died on October 2017, while this manuscript was under revision. Liam and I started to work together several years ago. To me, all these years of collaboration with him have been, above all, years of enjoying our friendship. I will greatly miss him.

2. Notations and preliminaries

2.1. General setting. The following notations will remain in force throughout the paper. We denote the set of non-negative integers by $\mathbb{N}$ and the set of positive integers by $\mathbb{N}_+$. Let n be a positive integer with $n \geqslant 3$, to avoid trivialities, and let $[n] = \{1, \dots, n\}$. Any vector $\alpha \in \mathbb{Z}^n$ can be written uniquely as $\alpha = \alpha^+ - \alpha^-$, where $\alpha^+ = \max(\alpha, 0) \in \mathbb{N}^n$ and $\alpha^- = -\min(\alpha, 0) \in \mathbb{N}^n$ have disjoint support. Set $\mathbb{1} = (1, \dots, 1)$.

Let $\mathbb{K}$ be a field and let $\mathbb{A} = \mathbb{K}[x_1, \dots, x_n] = \mathbb{K}[x]$ be the polynomial ring over $\mathbb{K}$ in indeterminates $x_1, \dots, x_n$. The maximal ideal generated by $x_1, \dots, x_n$ will be denoted $\mathfrak{m} = (x_1, \dots, x_n)$. For any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write x^α as shorthand for $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. By a binomial in $\mathbb{A}$, we understand a polynomial of $\mathbb{A}$ with at most two terms, say $\lambda x^\alpha - \mu x^\beta$, where $\lambda, \mu \in \mathbb{K}$ and $\alpha, \beta \in \mathbb{N}^n$. A binomial ideal of $\mathbb{A}$ is an ideal generated by binomials.

Let M be an $n \times s$ integer matrix, $M = (m_{i,j})$; that is, $m_{i,j} \in \mathbb{Z}$, for all $i = 1, \dots, n$, $j = 1, \dots, s$. We will say that M is non-negative, and write $M \geqslant 0$, if all the entries in M are non-negative, i.e. $m_{i,j} \geqslant 0$, for all i, j. Similarly, M is said to be positive, $M > 0$, if $m_{i,j} > 0$, for all i, j.

We will denote by $m_{i,*}$ and $m_{*,j}$ the i-th row and j-th column, repectively, of M. We will denote by $\mathcal{M} \subset \mathbb{Z}^n$ the additive subgroup of $\mathbb{Z}^n$ spanned by the columns of M, commonly called the lattice of $\mathbb{Z}^n$ defined by the columns of M.

Furthermore, $f_{m_{*,j}} = x^{(m_{*,j})^+} - x^{(m_{*,j})^-}$ will denote the binomial of $\mathbb{A}$ defined by the j-th column of M. The ideal $I(M) = (f_{m_{*,j}} \mid j = 1, \dots, s)$, generated by the binomials $f_{m_{*,j}}$, is called the binomial ideal associated to the matrix M. The binomial ideal $I(\mathcal{M}) = (x^{m^+} - x^{m^-} \mid m \in \mathcal{M})$ is called the lattice ideal associated to $\mathcal{M}$. Clearly $I(M) \subseteq I(\mathcal{M})$.

By a permutation of a square $n \times n$ matrix M, we understand a permutation of the rows of M together the same permutation of the columns. Equivalently, a permutation of M is a square matrix $P^\top MP$, where P is a permutation matrix resulting from a permutation of the columns of the $n\times n$ identity matrix I.

2.2. (Positive) critical binomial matrices and ideals. Recall that a critical binomial matrix, CB matrix for short, is an $n\times n$ integer matrix defined as follows:

$$L=\begin{pmatrix} a_{1,1} & -a_{1,2} & \dots & -a_{1,n} \\ -a_{2,1} & a_{2,2} & \dots & -a_{2,n} \\ \vdots & \vdots & & \vdots \\ -a_{n,1} & -a_{n,2} & \dots & a_{n,n} \end{pmatrix}, \tag{1}$$

where $a_{i,i}>0$, $a_{i,i}=\sum_{j\neq i} a_{i,j}$, $a_{i,j}\geqslant 0$, for all $i,j=1,\dots,n$, and, for each column of L, at least one off-diagonal entry is nonzero (see [21]). Alternatively, an $n\times n$ integer matrix L is a CB matrix whenever $L=D-B$, where D is a positive, diagonal $n\times n$ integer matrix and B is a non-negative $n\times n$ integer matrix with zero entries in the diagonal, nonzero columns $b_{*,1},\dots,b_{*,n}$, and such that $(D^{-1}B)\mathbb{1}^\top=\mathbb{1}^\top$. In particular, $D^{-1}B$ is an stochastic matrix. The expression $L=D-B$, which is clearly unique, will be called the "(D,B) form of L".

Observe that a permutation $P^\top LP$ of a CB matrix L is again a CB matrix. Indeed, writing $L=D-B$ in its (D,B) form, then $P^\top LP=P^\top DP-P^\top BP$, and $P^\top DP>0$ is diagonal and $P^\top BP\geqslant 0$, with zero entries in the diagonal, nonzero columns and $(P^\top DP)^{-1}(P^\top BP)\mathbb{1}^\top=\mathbb{1}^\top$.

Since $a_{i,i}>0$ and $b_{*,1},\dots,b_{*,n}$ are nonzero, for each column $l_{*,1},\dots,l_{*,n}$ of an $n\times n$ CB matrix L, the binomial

$$f_{l_{*,j}}=x^{(l_{*,j})^+}-x^{(l_{*,j})^-}=x_j^{a_{j,j}}-x_1^{a_{1,j}}\cdots x_{j-1}^{a_{j-1,j}}x_{j+1}^{a_{j+1,j}}\cdots x_n^{a_{n,j}},$$

has two non-constant terms. The binomial ideal $I(L)$, generated by the $f_{l_{*,j}}$, is called the critical binomial ideal, CB ideal for short, associated to L (see [21]).

A special class of CB matrices is the following. When all the coefficients $a_{i,j}$ are positive, the matrix L is said to be a positive critical binomial matrix (PCB matrix, for short). The ideal $I(L)$ is called the positive critical binomial ideal (PCB ideal, for short) associated to L. These ideals were extensively studied in [20] (see also [21]).

2.3. Irreducible binomial matrices. A square $n\times n$ matrix $M=(m_{i,j})$ is called reducible if the set of indices $[n]=\{1,\dots,n\}$ can be split into two non-empty disjoint sets I,J such that $m_{i,j}=0$, for all $i\in I$ and $j\in J$. The matrix M is called irreducible if it is not reducible. Equivalently, M is reducible if and only if there exists a permutation matrix P such that

$$P^\top MP=\begin{pmatrix} M_{I,I} & 0 \\ M_{J,I} & M_{J,J}\end{pmatrix}, \tag{2}$$

where $M_{I,I}$ and $M_{J,J}$ are square matrices (see, e.g. [13, Chapter XIII, p.50]; see also [5, § 3.2]). Here we understand $M_{I,J}$ as the submatrix of the matrix M defined by the set of rows I and the set of columns J. Of course, this is equivalent to saying that there exists a permutation matrix Q that permutes columns and then corresponding rows, such that

$$Q^\top MQ=\begin{pmatrix} M_{J,J} & M_{J,I} \\ 0 & M_{I,I}\end{pmatrix}.$$

Clearly, adding a diagonal matrix, multiplying by a diagonal matrix, or transposing, are three operations that preserve the condition of irreducibility on a matrix.

We will say that L is an irreducible critical binomial matrix (ICB matrix, for short), if L is a CB matrix which is irreducible. Write $L = D - B$ in its (D, B) form. If L is irreducible, then $(D^{-1}B)^\top$ is also irreducible, and conversely.

Clearly, we have the chain of implications linking these three properties: PCB $\Rightarrow$ ICB $\Rightarrow$ CB. (When $n = 2$, the three classes coincide.) If $n = 3$, then the condition ICB is equivalent to the condition CB, but there are ICB matrices which are not PCB. If $n = 4$, it is easy to see that the three classes are distinct.

Remark 2.1. Let L be a reducible CB matrix, $n \geqslant 4$. After a permutation, we can suppose that L can be written as

$$L = \begin{pmatrix} L_{I,I} & 0 \\ L_{J,I} & L_{J,J} \end{pmatrix}, \tag{3}$$

where $L_{I,I}$ is a square $r \times r$ integer matrix, $L_{J,J}$ is a square $s \times s$ integer matrix, with $r + s = n$ and $1 \leqslant r, s \leqslant n$. Note that $L_{I,I} \mathbb{1}^\top = 0$. In particular, since the sum of the entries of the first row is zero and $a_{1,1} > 0$, then $r \geqslant 2$. Similarly, $s \geqslant 2$, otherwise the last column of L would not have an off-diagonal nonzero entry.

2.4. The digraphs we will deal with. All graphs $\mathbb{G} = (\mathbb{V}, \mathbb{E}, w)$ we will deal with are assumed to be finite, weighted, directed graphs. That is, $\mathbb{V} = \{v_1, \dots, v_n\}$ is the finite set of vertices; $\mathbb{E}$ is the set of directed arcs of $\mathbb{G}$, which are ordered pairs (v_i, v_j), with $i \neq j$; and w is a weight function that associates to a every arc e in $\mathbb{E}$ a weight $w_e \in \mathbb{N}_+$. Moreover, we will always assume that $\mathbb{G}$ has no loops, sources or sinks. That is, there are no arcs from v_i to v_i, for any i; and for every vertex v_i, there exists at least a $j \in \{1, \dots, n\} \setminus \{i\}$ and an arc $(v_j, v_i) \in \mathbb{E}$, and there exists at least a $k \in \{1, \dots, n\} \setminus \{i\}$ and an arc $(v_i, v_k) \in \mathbb{E}$. (See, e.g. [5] or [14], for all unexplained notations on graph theory.)

Assumption 2.2. *Thus, from now on, a digraph $\mathbb{G}$ will mean a finite, weighted, directed graph, without loops, sources or sinks.*

A digraph $\mathbb{G}$ is said to be strongly connected if any two (distinct) vertices can be joined by a (directed) path. A directed path from vertex v_i to a distinct vertex v_j will be denoted by $v_i \to \cdots \to v_j$. The *unweighted length* of the path $v_i \to \cdots \to v_j$ is the number of arcs which constitute it, without reference to their weights. The *unweighted distance* $\mathrm{d}^\vee(v_i, v_j)$ between distinct vertices v_i and v_j is defined to be the minimum unweighted length of a directed path connecting v_i with v_j. Thus, $\mathrm{d}^\vee(v_i, v_j) = 1$ if and only if $(v_i, v_j) \in \mathbb{E}$. It may happen that $\mathrm{d}^\vee(v_i, v_j) \neq \mathrm{d}^\vee(v_j, v_i)$.

A directed graph $\mathbb{G}$ is said to be strongly complete if any two (distinct) vertices can be joined by an arc, that is, $(v_i, v_j) \in \mathbb{E}$, for all $i, j = 1, \dots, n$, $i \neq j$.

The *adjacency matrix* $A(\mathbb{G})$ of $\mathbb{G}$ is the $n \times n$ integer matrix given by $A(\mathbb{G})_{i,j} = w_e$ if $e = (v_i, v_j) \in \mathbb{E}$, and $A(\mathbb{G})_{i,j} = 0$ otherwise. Since $\mathbb{G}$ has no loops, $A(\mathbb{G})_{i,i} = 0$; since $\mathbb{G}$ has no sources, for each column of $A(\mathbb{G})$, at least one off-diagonal entry is nonzero; since $\mathbb{G}$ has no sinks, for each row of $A(\mathbb{G})$, at least one off-diagonal entry is nonzero. The *(out)degree matrix* $D(\mathbb{G})$ of $\mathbb{G}$ is the $n \times n$ integer diagonal matrix defined by $D(\mathbb{G})_{i,i} = \sum_{e \in O(v_i)} w_e$, the weighted (out)degree of the vertex v_i, where $O(v_i)$ is the set of vertices v_j, $j \neq i$, such that $(v_i, v_j) \in \mathbb{E}$. Since $\mathbb{G}$ has no sinks, $D(\mathbb{G})_{i,i} \neq 0$ for every $i = 1, \dots, n$. The *Laplacian matrix* $L(\mathbb{G})$ of $\mathbb{G}$ is defined as $L(\mathbb{G}) = D(\mathbb{G}) - A(\mathbb{G})$.

2.5. Our dictionary: CB matrices - digraphs. Clearly, the Laplacian matrix $L(\mathbb{G})$ of a digraph $\mathbb{G}$ is a CB matrix, sometimes also denoted $L_{\mathbb{G}}$ (recall that we are using Assumption 2.2). Conversely, given an $n \times n$ CB matrix $L = D - B$, one can define a digraph $\mathbb{G}_L$ of n vertices $\mathbb{V} = \{v_1, \dots, v_n\}$ with adjacency matrix B in an obvious way, namely, there exists a directed edge $v_i \to v_j$ between distinct vertices v_i and v_j if and only if $a_{i,j} > 0$, in which case the edge $v_i \to v_j$ is assigned weight $a_{i,j}$.

Moreover, L is an ICB matrix if and only if $\mathbb{G}_L$ is strongly connected (see [8], for a recent related work). Indeed, first remark that a permutation of L is just a re-enumeration of the vertex set of $\mathbb{G}$ and a corresponding relabelling of the directed arcs. Now suppose that L is reducible and let I, J be the partition on $[n]$ such that $a_{i,j} = 0$, for all $i \in I$ and all $j \in J$. Then it follows that there is no directed path from a vertex v_i to a vertex v_j, with $i \in I$ and $j \in J$. Conversely, suppose that $\mathbb{G}$ is not strongly connected and, after a re-enumeration of the vertices and corresponding relabelling of the directed arcs, suppose there is no directed path from v_1 to v_n. Let $I = \{1\} \cup \{i \in [n] \mid \exists\, v_1 \to \cdots \to v_i\}$ and let $J = [n] \setminus I$. After a re-enumeration of the vertices and corresponding relabelling of the directed arcs, we can suppose that $I = \{1, 2, \dots, r\}$ and $J = \{r+1, \dots, n\}$. Clearly, there is not an arc from v_i, $i \in I$, to v_j, $j \in J$ (otherwise there would be a directed path $v_1 \to \cdots \to v_i \to v_j$ and j would belong to I, a contradiction). Therefore $L_{I,J} = 0$ and L is reducible (see, e.g. [5, Theorem 3.2.1], for more details).

Finally, L is a PCB matrix if and only if $\mathbb{G}_L$ is strongly complete.

Observe that if $\mathbb{G}$ is undirected, then $L_{\mathbb{G}}$ is symmetric, and the discussion above can be specialized to this case in an obvious manner (see, e.g. [18]).

2.6. Irreducible critical binomial matrices and their adjugate matrices. Given an $n \times n$ integer matrix L, let $L_{i,j}$ denote the $(n-1) \times (n-1)$ matrix obtained from L by eliminating the i-th row and the j-th column of L. Set $\operatorname{adj}(L) = ((-1)^{i+j}|L_{j,i}|)$ to be the adjugate matrix of L, sometimes also called the adjoint matrix, where $|L_{j,i}|$ stands for the determinant $\det(L_{j,i})$. For convenience, we write $\epsilon_{i,j}$ to denote $(-1)^{i+j}$.

In the next set of results, invertibility, ranks or linear subspaces are thought of over $\mathbb{Q}$. Observe that if $\operatorname{rank}(L) = n$, then $\operatorname{adj}(L)$ is invertible (since $L \cdot \operatorname{adj}(L) = |L| \cdot \mathrm{I}$). If $\operatorname{rank}(L) = n-1$, then, by definition, at least one $(n-1) \times (n-1)$ minor of L is nonzero, thus $\operatorname{adj}(L) \neq 0$; moreover, $\operatorname{rank}(\operatorname{adj}(L)) = 1$ because $L \cdot \operatorname{adj}(L) = |L| \cdot \mathrm{I} = 0$ and so all the columns of $\operatorname{adj}(L)$ belong to the nullspace of L, which is of dimension 1. If $\operatorname{rank}(L) \leqslant n-2$, then all the $(n-1) \times (n-1)$ minors of L are zero, so $\operatorname{adj}(L) = 0$. In particular, L is invertible if and only if $\operatorname{adj}(L)$ is invertible.

The next result can be deduced from [21, Theorem 7.6]. We present here a more direct proof, that does not involve a wider class of matrices.

Proposition 2.3. *Let L be an $n \times n$ integer matrix and let $\operatorname{adj}(L)$ be its adjugate matrix.*

(a) *If $L\mathbb{1}^\top = 0$, then all the rows of $\operatorname{adj}(L)$ are equal.*
(b) *If L is a CB matrix, then, further, $\operatorname{adj}(L) \geqslant 0$.*
(c) *If L is an ICB matrix, then, further, $\operatorname{rank}(L) = n-1$ and $\operatorname{adj}(L) > 0$.*

Proof. If $\operatorname{rank}(L) \leqslant n-2$, then $\operatorname{adj}(L) = 0$ and all the rows are equal. Suppose that $\operatorname{rank}(L) \geqslant n-1$. Since $L\mathbb{1}^\top = 0$, then $\operatorname{rank}(L) = n-1$ and so $\operatorname{rank}(\operatorname{adj}(L)) = 1$. Moreover, $L \cdot \operatorname{adj}(L) = 0$, so every column of $\operatorname{adj}(L)$ is in the $\mathbb{Q}$-linear subspace $\operatorname{Nullspace}(L)$, which is generated by $\mathbb{1}^\top$. Hence there exist $\mu_1, \dots, \mu_n \in \mathbb{Q}$ such that the columns of $\operatorname{adj}(L)$ are equal to $\mu_1\mathbb{1}^\top, \dots, \mu_n\mathbb{1}^\top$:

$$\operatorname{adj}(L) = \begin{pmatrix} \epsilon_{1,1}|L_{1,1}| & \epsilon_{1,2}|L_{2,1}| & \dots & \epsilon_{1,n}|L_{n,1}| \\ \epsilon_{2,1}|L_{1,2}| & \epsilon_{2,2}|L_{2,2}| & \dots & \epsilon_{2,n}|L_{n,2}| \\ \vdots & \vdots & & \vdots \\ \epsilon_{n,1}|L_{1,n}| & \epsilon_{n,2}|L_{2,n}| & \dots & \epsilon_{n,n}|L_{n,n}| \end{pmatrix} = \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & \vdots & & \vdots \\ \mu_1 & \mu_2 & \dots & \mu_n \end{pmatrix}. \tag{4}$$

Thus each row of $\operatorname{adj}(L)$ is equal to $\mu := (\mu_1, \dots, \mu_n)$, which proves (a). Note that, since $\operatorname{adj}(L)$ is an integer matrix, then necessarily $\mu_1, \dots, \mu_n \in \mathbb{Z}$.

The proof of (b) is an adaptation of [20, Lemma 2.1] (see also [21, Theorem 6.3, (c)]). Fix $i \in \{1, \dots, n\}$. Let us prove that $\mu_i = |L_{i,i}|$ is non-negative. By the Gershgorin Circle Theorem, every eigenvalue λ of $L_{i,i}$ lies within at least one of the discs $\{z \in \mathbb{C} \mid |z - a_{j,j}| \leqslant R_j\}$, $j \neq i$, where $R_j = \sum_{k \neq i,j} |-a_{j,k}|$. Observe that $R_j \leqslant a_{j,j}$ due to the fact that L is a CB matrix. If $\lambda \in \mathbb{R}$, then $\lambda \geqslant 0$. If $\lambda \notin \mathbb{R}$, then since $L_{i,i}$ is a real matrix, the conjugate $\overline{\lambda}$ must also be an eigenvalue of $L_{i,i}$. Since $|L_{i,i}|$ is the product of the $n-1$ (possibly repeated) eigenvalues of $L_{i,i}$, it follows that $\mu_i \geqslant 0$ and so $\operatorname{adj}(L) \geqslant 0$.

We now turn to the proof of (c). Let us first see that $\operatorname{adj}(L) \neq 0$. Write $L = D - B$, with $D > 0$ diagonal and $D^{-1}B$ stochastic. In particular, the spectral radius $\rho(D^{-1}B)$ of $D^{-1}B$ is 1 ([13, Chapter XIII, p. 83]). Since L is irreducible, $(D^{-1}B)^\top$ is an irreducible, non-negative, $n \times n$ integer matrix with $\rho((D^{-1}B)^\top) = 1$. By the Perron-Frobenius Theorem (see, e.g. [13, Chapter XIII, p. 53], [14, Theorem 8.8.1, p. 178]), $\rho((D^{-1}B)^\top)$ is realized by a simple eigenvalue; in particular, $\operatorname{Nullspace}((D^{-1}B)^\top - \mathrm{I})$ has dimension 1. Since $\operatorname{Nullspace}((D^{-1}B)^\top - \mathrm{I})$ coincides with $\operatorname{Nullspace}(L^\top)$, it follows that $L^\top$ and L have rank $n-1$. In particular, $\operatorname{adj}(L)$ has rank 1 and $\operatorname{adj}(L) \neq 0$.

Let us now prove that $\operatorname{adj}(L) > 0$. Suppose not, i.e. there exists some μ_j which is zero. Since $\operatorname{adj}(L) \neq 0$, by permuting L, we can suppose that $\mu_1, \dots, \mu_r$ are positive and $\mu_{r+1}, \dots, \mu_n$ are zero, where $1 \leqslant r < n$. Setting $I = \{1, \dots, r\}$ and $J = \{r+1, \dots, n\}$, L can be written as

$$L = \begin{pmatrix} L_{I,I} & L_{I,J} \\ L_{J,I} & L_{J,J} \end{pmatrix}, \text{ where } L_{I,J} = -\begin{pmatrix} a_{1,r+1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{r,r+1} & \dots & a_{r,n} \end{pmatrix},$$

and $L_{I,I}$ and $L_{J,J}$ are two square matrices. Write $\mu = (\mu_I, \mu_J)$, with $\mu_I = (\mu_1, \dots, \mu_r)$ and $\mu_J = (\mu_{r+1}, \dots, \mu_n) = 0$. Since $\operatorname{adj}(L) \cdot L = 0$, then $\mu L = 0$. It follows that $\mu_I L_{I,J} = 0$. But $\mu_1, \dots, \mu_r > 0$ and $a_{i,j} \geqslant 0$, for all $i = 1, \dots, r$ and all $j = r+1, \dots, n$, so $L_{I,J} = 0$. Thus L is reducible, a contradiction. $\square$

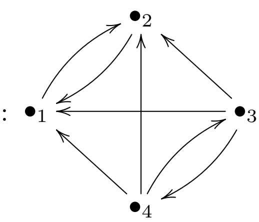
Notation 2.4. Given L an ICB matrix, we will denote the last (and so any) row of $\operatorname{adj}(L)$ by $\mu = \mu(L) = (\mu_1, \dots, \mu_n) \in \mathbb{N}_+^n$. If $d = \gcd(\mu)$, set $\nu(L) := \mu(L)/d$.

Corollary 2.5. *Let L be an $n \times n$ integer matrix. Then L is an* ICB *matrix if and only if L is a* CB *matrix and any set of $n-1$ rows of L is linearly independent.*

Proof. Let $F = \langle l_{1,*}, \dots, l_{n,*} \rangle$ be the $\mathbb{Q}$-linear subspace spanned by the rows of a CB matrix L. Writing $\operatorname{adj}(L)$ as in (4), and since $\operatorname{adj}(L) \cdot L = 0$, it follows that $\mu_1 l_{1,*} + \dots + \mu_n l_{n,*} = 0$, with each $\mu_i \geqslant 0$. Suppose first of all that L is irreducible. Fix $i \in \{1, \dots, n\}$. By Proposition 2.3, $\operatorname{rank}(L) = n-1$ and $\mu_i > 0$, for all $i = 1, \dots, n$. Therefore, F has dimension $n-1$ and $\langle l_{1,*}, \dots \widehat{l_{i,*}}, \dots, l_{n,*} \rangle$ equals F. Hence the set of rows $\{l_{1,*}, \dots \widehat{l_{i,*}}, \dots, l_{n,*}\}$ is linearly independent. Conversely, if $n = 3$, the condition CB implies the condition ICB. Suppose $n \geqslant 4$ and that L is reducible. After a permutation, one can suppose that L can be written as in (3), where $L_{I,I}$ is an $r \times r$ matrix, $2 \leqslant r \leqslant n-2$, such that $L_{I,I} \mathbb{1}^\top = 0$. In particular, $\operatorname{rank}(L_{I,I}^\top) = \operatorname{rank}(L_{I,I}) \leqslant r-1$. Therefore the first r rows of L are linearly dependent. $\square$

Remark 2.6. Clearly, an easy adaptation of the same argument shows that if L is an ICB matrix, then L is a CB matrix and any set of $n-1$ columns of L is linearly independent. However, the converse is not true, as the example below shows:

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}, \text{ corresponding to } \mathbb{G}:$$



2.7. Homogeneity of lattice ideals of strongly connected digraphs.

Assumption 2.7. *Let L be an* ICB *matrix (or, equivalently, let $\mathbb{G}_L$ be a strongly connected digraph). Let $\nu = \nu(L)$ be defined as is Notation* 2.4. *From now on, we will always consider the $\mathbb{N}$-grading on $\mathbb{A} = \mathbb{K}[x_1, \dots, x_n]$ where each x_i is given weight ν_i. Thus for a monomial $x^\alpha \in \mathbb{A}$, we define the degree of x^α as follows:* $\deg(x^\alpha) = \nu_1\alpha_1 + \cdots + \nu_n\alpha_n$.

Remark 2.8. With this grading, the matrix ideal $I(L) = (x^{l^+_{*,1}} - x^{l^-_{*,1}}, \dots, x^{l^+_{*,n}} - x^{l^-_{*,n}})$ and the lattice ideal $I(\mathcal{L}) = (x^{l^+} - x^{l^-} \mid l \in \mathcal{L})$ are homogeneous ideals of $\mathbb{A}$ (cf. Definitions in Section 2.1). Indeed, let $l \in \mathcal{L}$. So there exists $b \in \mathbb{Z}^n$ with $l^\top = Lb^\top$, a linear combination of the columns of L. Then $\nu l^\top = \nu L b^\top = 0$, so $\nu(l^+)^\top = \nu(l^-)^\top$ and $\deg(x^{l^+}) = \nu(l^+)^\top = \nu(l^-)^\top = \deg(x^{l^-})$. Thus $f_l = x^{l^+} - x^{l^-}$ is homogeneous. (See [20, Definition 2.2] or [21, Remark 5.6].)

Note that the hypothesis that $\mathbb{G}$ be strongly connected is essential to define a $\mathbb{N}$-grading on $\mathbb{A} = \mathbb{K}[x_1, \dots, x_n]$ and to ensure that $I(L)$ is homogeneous. See, for instance, the example in Remark 2.6.

The following result can be deduced from [21, Proposition 7.6]. Here we give a shorter direct proof.

Proposition 2.9. *Let L be an* ICB *matrix. Then* $\operatorname{rad}(I(L), x_i) = \mathfrak{m}$, *for all* $i = 1, \dots, n$.

Proof. Fix $i \in \{1, \dots, n\}$. Let $\mathbb{G}_L$ be the corresponding strongly connected digraph. Let $v_{i_1} \to \cdots \to v_{i_N}$ be a directed path passing through all the vertices and begining in $v_{i_1} = v_i$ (there might be possibly repeated vertices, though adjacent vertices will not be repeated). To simplify notations, write f_i to denote the binomial defined by the i-th column of L. Thus $I(L) = (f_{i_N}, \dots, f_{i_1})$. Write $f_{i_1} = x_{i_1}^{a_{i_1,i_1}} - g_1$, where $g_1 \in \mathbb{A}$ is a monomial divisible by at least one variable x_j different from x_{i_1}. Since there is a directed arc from the vertex v_{i_1} to the vertex v_{i_2}, then $a_{i_1,i_2} \neq 0$ and $f_{i_2} = x_{i_2}^{a_{i_2,i_2}} - x_{i_1}^{a_{i_1,i_2}} g_2$, where $g_2 \in \mathbb{A}$ denotes a monomial, possibly constant. Then $\operatorname{rad}(f_{i_2}, x_{i_1}) = (x_{i_2}, x_{i_1})$. Similarly, $\operatorname{rad}(f_{i_3}, x_{i_2}) = (x_{i_3}, x_{i_2})$ and so on. Therefore

$$\begin{aligned} &\operatorname{rad}(I(L), x_i) = \operatorname{rad}(f_{i_N}, \dots, f_{i_2}, f_{i_1}, x_{i_1}) = \operatorname{rad}(f_{i_N}, \dots, f_{i_2}, x_{i_1}, g_1) = \\ &\operatorname{rad}(f_{i_N}, \dots, f_{i_3}, x_{i_2}, x_{i_1}, g_1) = \ldots = \operatorname{rad}(x_{i_N}, \dots, x_{i_1}, g_1) = \operatorname{rad}(\mathfrak{m}, g_1) = \mathfrak{m}. \end{aligned}$$

□

We write $\operatorname{Hull}(I)$ for the intersection of the isolated primary components of I (see, e.g. [20] or [21]).

Corollary 2.10. *Let L be an* ICB *matrix. Then $I(L)$ and $I(\mathcal{L})$ have height $n-1$. Moreover, $I(\mathcal{L}) = \operatorname{Hull}(I(L))$ is the intersection of the isolated primary components of $I(L)$. In particular, $I(\mathcal{L})$ is a Cohen-Macaulay ideal of dimension* 1 *and projective dimension* $n - 1$.

Proof. That $I(L)$ has height $n-1$ follows from Remark 2.8, Proposition 2.9 and [21, Lemma 5.1]. That $I(\mathcal{L})$ has height $n-1$ and is the Hull of $I(L)$ follows from [21, Proposition 5.7]. In particular, $I(\mathcal{L})$ is (homogeneous) unmixed and $\mathbb{A}/I(\mathcal{L})$ is a (graded) Cohen-Macaulay ring of dimension 1. By [6, Corollary 2.2.15], $I(\mathcal{L})$ is perfect and so the projective dimension of $I(\mathcal{L})$ is $n-1$. □

3. The Cyc complex associated to a directed graph

3.1. The set of cyclically ordered partitions and an enumeration. Let us now recall some material from [18, Section 2], treating it however in a more general context that the one considered there. Set $[n]=\{1,\ldots,n\}$, take $1\leqslant k\leqslant n$, and let $\mathrm{Cyc}_{n,k}$ denote the set of cyclically ordered partitions of the set $[n]$ into k blocks, which we shall always take to be non-empty. Each element of $\mathrm{Cyc}_{n,k}$ has the form $(I_1,\ldots,I_k)$, where $I_1\cup\ldots\cup I_k=[n]$ is a partition, and we regard the $(I_1,\ldots,I_k)$ as formal symbols subject to the identifications

$$(I_1,I_2,\ldots,I_k)=(I_2,I_3,\ldots,I_k,I_1)=\ldots=(I_k,I_1,\ldots,I_{k-2},I_{k-1}).$$

Each such symbol clearly specifies an equivalence class, and we pick a unique representative of the corresponding equivalence class by assuming, after relabelling of suffices, that $n\in I_k$.

The cardinality of the set $\mathrm{Cyc}_{n,k}$ is $|\mathrm{Cyc}_{n,k}|=(k-1)!\cdot S_{n,k}$, where $S_{n,k}$ is the Stirling number of the second kind. Note that $|\mathrm{Cyc}_{n,1}|=1$ and $|\mathrm{Cyc}_{n,n}|=(n-1)!$. When the value of n is understood from the context, it will be convenient to denote $|\mathrm{Cyc}_{n,k}|$ by r_{k-1} (see Definition 3.3).

We take the opportunity to introduce a convenient enumeration of $\mathrm{Cyc}_{n,k}$, which will be used subsequently in the following sections.

Remark 3.1. Given a subset I of $[n]$ of cardinality $|I|$, let χ_I denote the incidence vector of I, so that $\chi_I^\top=(\chi_I(1),\ldots,\chi_I(n))\in\{0,1\}^n\subset\mathbb{Z}^n$, with $\chi_I(i)=1$, if $i\in I$, and $\chi_I(i)=0$, otherwise. If J and $I\subseteq[n]$ are two different subsets of $[n]$, we will use the notation $J\prec I$ and say that "J precedes I" (or alternatively, "I succeeds J"), whenever $|J|>|I|$, or whenever $|J|=|I|$ and the difference $\chi_J^\top-\chi_I^\top$ is $(*,\ldots,*,1,0\ldots,0)$; that is, the rightmost non-zero coefficient of $\chi_J^\top-\chi_I^\top$ is 1. Given $(J_1,\ldots,J_k)$ and $(I_1,\ldots,I_k)$, two different cyclically ordered partitions of $[n]$ into k blocks, with $n\in J_k,I_k$, we will use the notation $(J_1,\ldots,J_k)\prec(I_1,\ldots,I_k)$ and say that "$(J_1,\ldots,J_k)$ precedes $(I_1,\ldots,I_k)$" (or alternatively, "$(I_1,\ldots,I_k)$ succeeds $(J_1,\ldots,J_k)$"), whenever, for some $s\in\{1,\ldots,k-1\}$, $J_1=I_1,\ldots,J_{s-1}=I_{s-1}$ and J_s precedes I_s. This is a "size-reverse lexicographic enumeration", so we will refer to it as an srle (for a discussion on this subject see, e.g. [16]). Any reference made below without qualification to an enumeration will refer to the enumeration employing the srle. (In this regard, note the word of caution given in Remark 6.18.)

Example 3.2. Set $n=4$. For the sake of brevity, write $(123,4)$ for $(\{1,2,3\},\{4\})$. Understanding that, in the display, elements on the left precede elements on the right and employing the srle, $\mathrm{Cyc}_{4,k}$ is enumerated as follows:

$$\begin{aligned}
\mathrm{Cyc}_{4,2}&=\{(123,4),(23,14),(13,24),(12,34),(3,124),(2,134),(1,234)\};\\
\mathrm{Cyc}_{4,3}&=\{(23,1,4),(13,2,4),(12,3,4),(3,12,4),(3,2,14),(3,1,24),\\
&\qquad(2,13,4),(2,3,14),(2,1,34),(1,23,4),(1,3,24),(1,2,34)\};\\
\mathrm{Cyc}_{4,4}&=\{(3,2,1,4),(3,1,2,4),(2,3,1,4),(2,1,3,4),(1,3,2,4),(1,2,3,4)\}.
\end{aligned}$$

3.2. The Cyc sequence associated to a digraph or to a CB matrix. Let $\mathbb{G}$ be a digraph of n vertices or, equivalently, consider an $n\times n$ CB matrix L (recall Assumption 2.2). For disjoint non-empty subsets I and J of $[n]$, we define

$$x^{I\to J}=\prod_{i\in I}x_i^{\sum_{j\in J}a_{i,j}}.$$

If I or J is the empty set, we put $x^{I\to J}$ equal to 1.

Definition 3.3. *Let $\mathcal{C}_{\mathbb{G}}$ (also denoted by $\mathcal{C}_L$,* Cyc *or just $\mathcal{C}$) be the sequence of homomorphisms of $\mathbb{A}$-modules defined as follows:*

$$\mathcal{C}:0\leftarrow\mathcal{C}_0\xleftarrow{\partial_1}\mathcal{C}_1\xleftarrow{\partial_2}\cdots\xleftarrow{\partial_{n-2}}\mathcal{C}_{n-2}\xleftarrow{\partial_{n-1}}\mathcal{C}_{n-1}\leftarrow 0,$$

where $\mathcal{C}_k=\mathbb{K}[x]^{\mathrm{Cyc}_{n,k+1}}$ is the free $\mathbb{K}[x]$-module of rank $r_k:=|\mathrm{Cyc}_{n,k+1}|$ and with basis $\mathcal{B}_k$, the set of elements of $\mathrm{Cyc}_{n,k+1}$. The elements of $\mathcal{B}_k$ are enumerated employing the srle*. With this enumeration, they will be denoted by $\mathcal{B}_k=\{e_{k,1},\dots,e_{k,r_k}\}$. Their images under ∂_k will be denoted by $\partial_k(\mathcal{B}_k)=\{f_{k-1,1},\dots,f_{k-1,r_k}\}$, with $f_{k-1,j}:=\partial_k(e_{k,j})$. The module $\mathcal{C}_0$ is $\mathbb{K}[x]^{\mathrm{Cyc}_{n,1}}=\mathbb{K}[x]$. Here the trivial 1-block $([n])$ will be identified with the unit element of $\mathbb{K}[x](=\mathbb{A})$. The boundary map $\partial_k:\mathcal{C}_k\to\mathcal{C}_{k-1}$, in slightly simplified notation that we use throughout, is given by the formula*

$$\begin{aligned}\partial_k(I_1,\dots,I_{k+1})&=\sum_{s=1}^{k}(-1)^{s-1}x^{I_s\to I_{s+1}}(I_1,\dots,I_s\cup I_{s+1},\dots,I_{k+1})\\&\quad -x^{I_{k+1}\to I_1}(I_2,\dots,I_k,I_1\cup I_{k+1}).\end{aligned}\tag{5}$$

Remark 3.4. Observe that the k terms of the first addition in (5) are enumerated according to the srle; this is not the case for the last term, which depends on the relationship between I_1 and I_2. However, provided that, for all $i=1,\dots,n-1$, $a_{n,i}>0$, the last term is distinguished from the others because it is the only one whose coefficient contains the variable x_n. This follows from the fact that $n\in I_{k+1}$.

3.3. The degree zero component of Cyc. Let $\mathbb{G}$ be a digraph of n vertices or, equivalently, consider an $n\times n$ CB matrix L. Let $I(L)$ be the CB ideal associated to L and let $I(\mathcal{L})$ be the lattice ideal associated to the lattice $\mathcal{L}$ spanned by the columns of L (see Section 2.1).

Remark 3.5. Let $(I,\overline{I})\in\mathcal{C}_1$, with $I\subseteq[n-1]$, $I\neq\varnothing$ and $\overline{I}=[n]\setminus I$. Let χ_I be the incidence vector of I. Then

$$x^{I\to\overline{I}}=x^{(L\chi_I)^+},\quad x^{\overline{I}\to I}=x^{(L\chi_I)^-}\ \text{ and }\ \partial_1(I,\overline{I})=x^{(L\chi_I)^+}-x^{(L\chi_I)^-}.\tag{6}$$

If follows that $I(L)\subseteq\partial_1(\mathcal{C}_1)\subseteq I(\mathcal{L})$ is a chain of inclusions of three ideals of $\mathbb{K}[x]=\mathbb{A}$. Furthermore, if L is an ICB matrix, then $I(L)$, $\partial_1(\mathcal{C}_1)$ and $I(\mathcal{L})$ are homogeneous ideals.

Proof. If $i\in I$, the i-th coordinate of $L\chi_I=\sum_{j\in I}l_{*,j}$ is $(L\chi_I)_i=a_{i,i}-\sum_{j\in I\setminus\{i\}}a_{i,j}=\sum_{j\in\overline{I}}a_{i,j}$, which is positive. Therefore, in this case, $(L\chi_I)_i=((L\chi_I)^+)_i$. If $i\notin I$, the i-th coordinate of $L\chi_I$ is $(L\chi_I)_i=-\sum_{j\in I}a_{i,j}$, which is non-positive. Therefore, in this case, $((L\chi_I)^+)_i=0$. Hence

$$x^{(L\chi_I)^+}=\prod_{i\in I}x_i^{\sum_{j\in\overline{I}}a_{i,j}},$$

which coincides with the definition of $x^{I\to\overline{I}}$. Clearly $\chi_I+\chi_{\overline{I}}=\mathbb{1}^\top$. Therefore, $L\chi_{\overline{I}}=-L\chi_I$ and $(L\chi_{\overline{I}})^+=(-L\chi_I)^+=(L\chi_I)^-$. Thus $x^{\overline{I}\to I}=x^{(L\chi_{\overline{I}})^+}=x^{(L\chi_I)^-}$.

We know that $\partial_1(I,\overline{I}) = x^{I\to\overline{I}}(I\cup\overline{I}) - x^{\overline{I}\to I}(I\cup\overline{I})$. As we mentioned before, we make the identification $(I\cup\overline{I}) \equiv ([n])$ with the unit element in $\mathbb{K}[x] = \mathcal{C}_0$. So

$$\partial_1(I,\overline{I}) = x^{I\to\overline{I}} - x^{\overline{I}\to I} = x^{(L\chi_I)^+} - x^{(L\chi_I)^-},$$

which proves (6). Since $L\chi_I \in \mathcal{L}$, then $x^{(L\chi_I)^+} - x^{(L\chi_I)^-}$ is in $I(\mathcal{L})$. This proves $\partial_1(\mathcal{C}_1) \subseteq I(\mathcal{L})$.

On the other hand, for $i = 1,\dots,n-1$, the i-th column $l_{*,i}$ of L can be expressed as $l_{*,i} = L\chi_{\{i\}}$. So, by (6), $x^{l^+_{*,i}} - x^{l^-_{*,i}} = \partial_1(\{i\},\overline{\{i\}}) \in \partial_1(\mathcal{C}_1)$. Recalling that our partitions $(I,\overline{I})$ of $[n]$ have the property $n \in \overline{I}$, then $l_{*,n}$ can be expressed as $l_{*,n} = -(l_{*,1}+\cdots+l_{*,n-1}) = -L\chi_{\{1,\dots,n-1\}}$. Taking into account that, when $\beta = -\alpha \in \mathbb{Z}^n$, then $x^{\beta^+} - x^{\beta^-} = -(x^{\alpha^+} - x^{\alpha^-})$, and using (6) again,

$$\begin{aligned} x^{l^+_{*,n}} - x^{l^-_{*,n}} &= -(x^{(L\chi_{\{1,\dots,n-1\}})^+} - x^{(L\chi_{\{1,\dots,n-1\}})^-}) \\ &= -\partial_1(\{1,\dots,n-1\},\{n\}) \in \partial_1(\mathcal{C}_1). \end{aligned}$$

Therefore, $I(L) = (x^{l^+_{*,1}} - x^{l^+_{*,1}},\dots,x^{l^+_{*,n}} - x^{l^+_{*,n}}) \subseteq \partial_1(\mathcal{C}_1)$.

Now, suppose that L is an ICB matrix. Since $L\chi_I \in \mathcal{L}$, it follows, as in the proof of Remark 2.8, that $x^{(L\chi_I)^+} - x^{(L\chi_I)^-}$ is homogeneous (in the grading considered in Assumption 2.7). This proves that $\partial_1(\mathcal{C}_1)$ is homogeneous and $\partial_1(\mathcal{C}_1) \subseteq I(\mathcal{L})$. By Remark 2.8, we already know that $I(L)$ and $I(\mathcal{L})$ are homogeneous. $\square$

Notation 3.6. For ease of notation, if $(C,\overline{C}) \in \mathcal{C}_1$ (with $C \subseteq [n-1]$, $C \neq \varnothing$ and $\overline{C} = [n] \smallsetminus C$) is a partition of $[n]$, we set $m_C := x^{C\to\overline{C}}$, $m_{\overline{C}} := x^{\overline{C}\to C}$, and $f_C := m_C - m_{\overline{C}} = \partial_1(C,\overline{C})$.

Definition 3.7. *We define $\partial_0 : \mathcal{C}_0 = \mathbb{K}[x] \to \mathbb{K}[x]/I(\mathcal{L})$ to be the natural projection onto the quotient ring, so that $\partial_0 \circ \partial_1 = 0$.*

3.4. The Cyc sequence in four variables.

Let us display the sequence of homomorphims $\mathcal{C}$ of Definition 3.3 for a digraph $\mathbb{G}$ of four vertices or, equivalently, for a 4×4 CB matrix L.

Example 3.8. Let $n = 4$. To simplify notations, write x,y,z,t instead of x_1,x_2,x_3,x_4, respectively. As in Example 3.2, write

$$(i_1i_2\dots i_r, j_1j_2\dots j_s) \text{ instead of } (\{i_1,i_2,\dots,i_r\},\{j_1,j_2,\dots,j_s\}).$$

The resulting complex $\mathcal{C}$ has the following form.

$$\mathcal{C} : 0 \leftarrow \mathcal{C}_0 = \mathbb{K}[x] \xleftarrow{\partial_1} \mathcal{C}_1 = \mathbb{K}[x]^7 \xleftarrow{\partial_2} \mathcal{C}_2 = \mathbb{K}[x]^{12} \xleftarrow{\partial_3} \mathcal{C}_3 = \mathbb{K}[x]^6 \leftarrow 0.$$

The differentials go as follows. Recall that the elements of the basis $\mathcal{B}_k$ are enumerated employing the srle (see Definition 3.3 and Example 3.2; the underlined terms will be the leading ones in an ordering specified subsequently in Assumption 4.9). For ∂_1:

$$\begin{aligned}
f_{0,1} &= \partial_1(e_{1,1}) = \partial_1(123,4) = \underline{x^{a_{1,4}}y^{a_{2,4}}z^{a_{3,4}}} - t^{a_{4,4}},\\
f_{0,2} &= \partial_1(e_{1,2}) = \partial_1(23,14) = \underline{y^{a_{2,1}+a_{2,4}}z^{a_{3,1}+a_{3,4}}} - x^{a_{1,2}+a_{1,3}}t^{a_{4,2}+a_{4,3}},\\
f_{0,3} &= \partial_1(e_{1,3}) = \partial_1(13,24) = \underline{x^{a_{1,2}+a_{1,4}}z^{a_{3,2}+a_{3,4}}} - y^{a_{2,1}+a_{2,3}}t^{a_{4,1}+a_{4,3}},\\
f_{0,4} &= \partial_1(e_{1,4}) = \partial_1(12,34) = \underline{x^{a_{1,3}+a_{1,4}}y^{a_{2,3}+a_{2,4}}} - z^{a_{3,1}+a_{3,2}}t^{a_{4,1}+a_{4,2}},\\
f_{0,5} &= \partial_1(e_{1,5}) = \partial_1(3,124) = \underline{z^{a_{3,3}}} - x^{a_{1,3}}y^{a_{2,3}}t^{a_{4,3}},\\
f_{0,6} &= \partial_1(e_{1,6}) = \partial_1(2,134) = \underline{y^{a_{2,2}}} - x^{a_{1,2}}z^{a_{3,2}}t^{a_{4,2}},\\
f_{0,7} &= \partial_1(e_{1,7}) = \partial_1(1,234) = \underline{x^{a_{1,1}}} - y^{a_{2,1}}z^{a_{3,1}}t^{a_{4,1}}.
\end{aligned}$$

For ∂_2 we have:

$$\begin{aligned}
f_{1,1} &= \partial_2(e_{2,1}) = \partial_2(23,1,4)\\
&= y^{a_{2,1}}z^{a_{3,1}}(123,4) - \underline{x^{a_{1,4}}(23,14)} - t^{a_{4,2}+a_{4,3}}(1,234),\\
f_{1,2} &= \partial_2(e_{2,2}) = \partial_2(13,2,4)\\
&= x^{a_{1,2}}z^{a_{3,2}}(123,4) - \underline{y^{a_{2,4}}(13,24)} - t^{a_{4,1}+a_{4,3}}(2,134),\\
f_{1,3} &= \partial_2(e_{2,3}) = \partial_2(12,3,4)\\
&= x^{a_{1,3}}y^{a_{2,3}}(123,4) - \underline{z^{a_{3,4}}(12,34)} - t^{a_{4,1}+a_{4,2}}(3,124),\\
f_{1,4} &= \partial_2(e_{2,4}) = \partial_2(3,12,4)\\
&= z^{a_{3,1}+a_{3,2}}(123,4) - \underline{x^{a_{1,4}}y^{a_{2,4}}(3,124)} - t^{a_{4,3}}(12,34),\\
f_{1,5} &= \partial_2(e_{2,5}) = \partial_2(3,2,14)\\
&= z^{a_{3,2}}(23,14) - \underline{y^{a_{2,1}+a_{2,4}}(3,124)} - x^{a_{1,3}}t^{a_{4,3}}(2,134),\\
f_{1,6} &= \partial_2(e_{2,6}) = \partial_2(3,1,24)\\
&= z^{a_{3,1}}(13,24) - \underline{x^{a_{1,2}+a_{1,4}}(3,124)} - y^{a_{2,3}}t^{a_{4,3}}(1,234),\\
f_{1,7} &= \partial_2(e_{2,7}) = \partial_2(2,13,4)\\
&= y^{a_{2,1}+a_{2,3}}(123,4) - \underline{x^{a_{1,4}}z^{a_{3,4}}(2,134)} - t^{a_{4,2}}(13,24),\\
f_{1,8} &= \partial_2(e_{2,8}) = \partial_2(2,3,14)\\
&= y^{a_{2,3}}(23,14) - \underline{z^{a_{3,1}+a_{3,4}}(2,134)} - x^{a_{1,2}}t^{a_{4,2}}(3,124),\\
f_{1,9} &= \partial_2(e_{2,9}) = \partial_2(2,1,34)\\
&= y^{a_{2,1}}(12,34) - \underline{x^{a_{1,3}+a_{1,4}}(2,134)} - z^{a_{3,2}}t^{a_{4,2}}(1,234),\\
f_{1,10} &= \partial_2(e_{2,10}) = \partial_2(1,23,4)\\
&= x^{a_{1,2}+a_{1,3}}(123,4) - \underline{y^{a_{2,4}}z^{a_{3,4}}(1,234)} - t^{a_{4,1}}(23,14),\\
f_{1,11} &= \partial_2(e_{2,11}) = \partial_2(1,3,24)\\
&= x^{a_{1,3}}(13,24) - \underline{z^{a_{3,2}+a_{3,4}}(1,234)} - y^{a_{2,1}}t^{a_{4,1}}(3,124),\\
f_{1,12} &= \partial_2(e_{2,12}) = \partial_2(1,2,34)\\
&= x^{a_{1,2}}(12,34) - \underline{y^{a_{2,3}+a_{2,4}}(1,234)} - z^{a_{3,1}}t^{a_{4,1}}(2,134).
\end{aligned}$$

Finally, for ∂_3:

$$\begin{aligned}
f_{2,1} &= \partial_3(e_{3,1}) = \partial_3(3,2,1,4)\\
&= z^{a_{3,2}}(23,1,4) - y^{a_{2,1}}(3,12,4) + \underline{x^{a_{1,4}}(3,2,14)} - t^{a_{4,3}}(2,1,34),\\
f_{2,2} &= \partial_3(e_{3,2}) = \partial_3(3,1,2,4)\\
&= z^{a_{3,1}}(13,2,4) - x^{a_{1,2}}(3,12,4) + \underline{y^{a_{2,4}}(3,1,24)} - t^{a_{4,3}}(1,2,34),\\
f_{2,3} &= \partial_3(e_{3,3}) = \partial_3(2,3,1,4)\\
&= y^{a_{2,3}}(23,1,4) - z^{a_{3,1}}(2,13,4) + \underline{x^{a_{1,4}}(2,3,14)} - t^{a_{4,2}}(3,1,24),\\
f_{2,4} &= \partial_3(e_{3,4}) = \partial_3(2,1,3,4)\\
&= y^{a_{2,1}}(12,3,4) - x^{a_{1,3}}(2,13,4) + \underline{z^{a_{3,4}}(2,1,34)} - t^{a_{4,2}}(1,3,24),\\
f_{2,5} &= \partial_3(e_{3,5}) = \partial_3(1,3,2,4)\\
&= x^{a_{1,3}}(13,2,4) - z^{a_{3,2}}(1,23,4) + \underline{y^{a_{2,4}}(1,3,24)} - t^{a_{4,1}}(3,2,14),\\
f_{2,6} &= \partial_3(e_{3,6}) = \partial_3(1,2,3,4)\\
&= x^{a_{1,2}}(12,3,4) - y^{a_{2,3}}(1,23,4) + \underline{z^{a_{3,4}}(1,2,34)} - t^{a_{4,1}}(2,3,14).
\end{aligned}$$

3.5. Cyc is a complex for any CB matrix. Let L be a CB matrix or, equivalently, let $\mathbb{G}$ be a digraph. Let $\mathcal{C}_{\mathbb{G}}$ be the sequence of homomorphisms as in Definition 3.3.

Proposition 3.9. *The sequence $\mathcal{C}_{\mathbb{G}}$ is a chain complex of free $\mathbb{K}[x]$-modules.*

Proof. Fix a basis element $(I_1, I_2, ..., I_{k+1}) \in \mathcal{B}_k$ of $\mathcal{C}_k$, where $2 \leqslant k \leqslant n-1$. We need to show that $\partial_{k-1}(\partial_k(I_1, I_2, ..., I_{k+1})) = 0$. Clearly, $\partial_{k-1}(\partial_k(I_1, \ldots, I_{k+1}))$ is expressed as a linear combination of $k(k+1)$ elements of $\mathcal{C}_{k-2}$. Indeed:

$$\partial_{k-1}(\partial_k(I_1, \ldots, I_{k+1})) = \sum_{s=1}^{k} (-1)^{s-1} x^{I_s \to I_{s+1}} \partial_{k-1}(I_1, \ldots, I_s \cup I_{s+1}, \ldots, I_{k+1}) - x^{I_{k+1} \to I_1} \partial_{k-1}(I_2, \ldots, I_k, I_1 \cup I_{k+1}). \tag{7}$$

For $s = 1$, the expression for $\partial_{k-1}(I_1 \cup I_2, \ldots, I_{k+1})$ is equal to:

$$x^{I_1 \cup I_2 \to I_3}(I_1 \cup I_2 \cup I_3, \ldots, I_{k+1}) + \sum_{t=3}^{k} (-1)^{t-2} x^{I_t \to I_{t+1}} (I_1 \cup I_2, \ldots, I_t \cup I_{t+1}, \ldots, I_{k+1}) - x^{I_{k+1} \to I_1 \cup I_2}(I_3, \ldots, I_k, I_1 \cup I_2 \cup I_{k+1}). \tag{8}$$

For $s = 2$, the expression for $\partial_{k-1}(I_1, I_2 \cup I_3, \ldots, I_{k+1})$ is:

$$x^{I_1 \to I_2 \cup I_3}(I_1 \cup I_2 \cup I_3, \ldots, I_{k+1}) - x^{I_2 \cup I_3 \to I_4}(I_1, I_2 \cup I_3 \cup I_4, \ldots, I_{k+1}) + \sum_{t=4}^{k} (-1)^{t-2} x^{I_t \to I_{t+1}} (I_1, I_2 \cup I_3, \ldots, I_t \cup I_{t+1}, \ldots, I_{k+1}) - x^{I_{k+1} \to I_1}(I_2 \cup I_3, \ldots, I_k, I_1 \cup I_{k+1}). \tag{9}$$

For $s \in \{3, \ldots, k-2\}$, the expression for $\partial_{k-1}(I_1, \ldots, I_s \cup I_{s+1}, \ldots, I_{k+1})$ is:

$$\begin{aligned} &\sum_{t=1}^{s-2} (-1)^{t-1} x^{I_t \to I_{t+1}} (I_1, \ldots, I_t \cup I_{t+1}, \ldots, I_s \cup I_{s+1}, \ldots, I_{k+1}) \\ &+ \sum_{t=s+2}^{k} (-1)^{t-2} x^{I_t \to I_{t+1}} (I_1, \ldots, I_s \cup I_{s+1}, \ldots, I_t \cup I_{t+1}, \ldots, I_{k+1}) \\ &+ (-1)^{s-2} x^{I_{s-1} \to I_s \cup I_{s+1}} (I_1, \ldots, I_{s-1} \cup I_s \cup I_{s+1}, \ldots, I_{k+1}) \\ &+ (-1)^{s-1} x^{I_s \cup I_{s+1} \to I_{s+2}} (I_1, \ldots, I_s \cup I_{s+1} \cup I_{s+2}, \ldots, I_{k+1}) \\ &- x^{I_{k+1} \to I_1} (I_2, \ldots, I_s \cup I_{s+1}, \ldots, I_1 \cup I_{k+1}). \end{aligned} \tag{10}$$

For $s = k-1$, the expression for $\partial_{k-1}(I_1, \ldots, I_{k-1} \cup I_k, I_{k+1})$ is equal to:

$$\begin{aligned} &\sum_{t=1}^{k-3} (-1)^{t-1} x^{I_t \to I_{t+1}} (I_1, \ldots, I_t \cup I_{t+1}, \ldots, I_{k-1} \cup I_k, I_{k+1}) \\ &+ (-1)^{k-3} x^{I_{k-2} \to I_{k-1} \cup I_k} (I_1, \ldots, I_{k-2} \cup I_{k-1} \cup I_k, I_{k+1}) \\ &+ (-1)^{k-2} x^{I_{k-1} \cup I_k \to I_{k+1}} (I_1, \ldots, I_{k-1} \cup I_k \cup I_{k+1}) \\ &- x^{I_{k+1} \to I_1} (I_2, \ldots, I_{k-1} \cup I_k, I_1 \cup I_{k+1}). \end{aligned} \tag{11}$$

For $s=k$, we have $\partial_{k-1}(I_1,\dots,I_k\cup I_{k+1})$ equal to

$$\begin{aligned}(12)\quad &\sum_{t=1}^{k-2}(-1)^{t-1}x^{I_t\to I_{t+1}}(I_1,\dots,I_t\cup I_{t+1},\dots,I_k\cup I_{k+1})\\ &\qquad +(-1)^{k-2}x^{I_{k-1}\to I_k\cup I_{k+1}}(I_1,\dots,I_{k-1}\cup I_k\cup I_{k+1})\\ &\qquad\qquad -x^{I_k\cup I_{k+1}\to I_1}(I_2,\dots,I_1\cup I_k\cup I_{k+1}).\end{aligned}$$

As for the final term, $\partial_{k-1}(I_2,\dots,I_k,I_1\cup I_{k+1})$ is equal to:

$$\begin{aligned}(13)\quad &\sum_{t=2}^{k-1}(-1)^{t-2}x^{I_t\to I_{t+1}}(I_2,\dots,I_t\cup I_{t+1},\dots,I_1\cup I_{k+1})\\ &\qquad +(-1)^{k-2}x^{I_k\to I_1\cup I_{k+1}}(I_2,\dots,I_1\cup I_k\cup I_{k+1})\\ &\qquad\qquad -x^{I_1\cup I_{k+1}\to I_2}(I_3,\dots,I_1\cup I_2\cup I_{k+1}).\end{aligned}$$

The basis elements of $\mathcal{B}_{k-2}$ that appear in these linear combinations, from (8) to (13), solely involve partitions obtained from $(I_1,\dots,I_{k+1})$ by coalescing twice two adjacent blocks. Any such basis element appears twice, obtained by doing two different mergings done in two different orders. For instance, the element $(I_1\cup I_2\cup I_3,\dots,I_{k+1})$ appears in the first summand in (8) and in (9). In (8), one has first merged I_1 with I_2, obtaining $x^{I_1\to I_2}(I_1\cup I_2,\dots,I_{k+1})$ and then, subsequently, one has merged $I_1\cup I_2$ with I_3, giving $x^{I_1\to I_2}x^{I_1\cup I_2\to I_3}(I_1\cup I_2\cup I_3,\dots,I_{k+1})$. On the other hand, the first summand in (9) is obtained by first merging I_2 with I_3, getting $(-1)x^{I_2\to I_3}(I_1,I_2\cup I_3,\dots,I_{k+1})$, and afterwards, on merging I_1 with $I_2\cup I_3$, one obtains $(-1)x^{I_2\to I_3}x^{I_1\to I_2\cup I_3}(I_1\cup I_2\cup I_3,\dots,I_{k+1})$. Observe that both monomial coefficients are equal, but opposite in sign, so they cancel each with the other.

We list now the complete set of partitions appearing as basis elements in the expression for $\partial_{k-1}(\partial_k(I_1,\dots,I_{k+1}))$. Concretely, and enumerated accordingly to the srle, the first $k-1$ are:

$$(I_1\cup I_2\cup I_3,\dots,I_{k+1}),(I_1\cup I_2,I_3\cup I_4,\dots,I_{k+1}),\dots,(I_1\cup I_2,\dots,I_k\cup I_{k+1}).$$

These elements appear in (8) and in (9).

Without mentioning where they come from, we have the following, still enumerated with the srle.

Next we have a set of $k-2$, all of them beginning with $(I_1,I_2\cup I_3\square)$, thus succeeding the $k-1$ above:

$$\begin{aligned}&(I_1,I_2\cup I_3\cup I_4,\dots,I_{k+1}),(I_1,I_2\cup I_3,I_4\cup I_5,\dots,I_{k+1}),\dots,\\ &\qquad\qquad (I_1,I_2\cup I_3,\dots,I_k\cup I_{k+1}).\end{aligned}$$

Then we have the set of $k-3$, beginning with $(I_1,I_2,I_3\cup I_4\square)$, and so on. In the second last place, we have the set of 2 beginning with $(I_1,\dots,I_{k-2}\cup I_{k-1}\square)$, that is, $(I_1,\dots,I_{k-2}\cup I_{k-1}\cup I_k,I_{k+1})$ and $(I_1,\dots,I_{k-2},I_{k-1}\cup I_k\cup I_{k+1})$, and finally we have $(I_1,\dots I_{k-2},I_{k-1}\cup I_k\cup I_{k+1})$. Altogether then, these amount in total to $(k-2)+(k-3)+\cdots+2+1$ partitions.

In the expression for $\partial_{k-1}(\partial_k(I_1,\dots,I_{k+1}))$, there are still k further basis elements of $\mathcal{B}_{k-2}$ involved. Concretely, these are the final terms in (8), (9), (10), (11) and (12), and all the terms in (13). It is not possible to enumerate them together with the former ones (employing the srle) unless more information is given. Each involves the leftmost block I_1 being merged with the rightmost block I_{k+1}. These basis elements are also characterized by the fact that the corresponding coefficient term contains the variable x_n (possibly with exponent zero). Enumerated among themselves (in the srle), the first $k-1$ are as follows:

$$(I_2 \cup I_3, \ldots, I_1 \cup I_{k+1}), (I_2, I_3 \cup I_4, \ldots, I_1 \cup I_{k+1}), \ldots,\\ (I_2, \ldots, I_{k-1} \cup I_k, I_1 \cup I_{k+1}), (I_2, \ldots, I_1 \cup I_k \cup I_{k+1}).$$

Finally, we have the last term $(I_3, \ldots, I_1 \cup I_2 \cup I_{k+1})$.

Thus a total amount of distinct $(k-1) + \sum_{i=1}^{k-2} i + k = k(k+1)/2$ basis elements of $\mathcal{B}_{k-2}$ appear in $\partial_{k-1}(\partial_k(I_1, \ldots, I_{k+1}))$. As mentioned before, each of them appears twice. This is consistent with the total number of $k(k+1)$ elements in the linear expression for $\partial_{k-1}(\partial_k(I_1, \ldots, I_{k+1}))$.

To finish the proof, one can check that the two terms involving each basis element have the same monomial coefficient, but with opposite sign (alternatively, one can use (15) below). This will prove that $\partial_{k-1}(\partial_k(I_1, \ldots, I_{k+1})) = 0$.

- Terms $(I_1, \ldots, I_s \cup I_{s+1} \cup I_{s+2}, \ldots, I_{k+1})$, with $1 \leqslant s \leqslant k-1$. The two monomial coefficients involving this basis element are:
$$(-1)^{s-1} x^{I_s \to I_{s+1}} (-1)^{s-1} x^{I_s \cup I_{s+1} \to I_{s+2}} \text{ and}\\ (-1)^s x^{I_{s+1} \to I_{s+2}} (-1)^{s-1} x^{I_s \to I_{s+1} \cup I_{s+2}}.$$
- Terms $(I_1, \ldots, I_s \cup I_{s+1}, \ldots, I_t \cup I_{t+1}, \ldots, I_{k+1})$, with $1 \leqslant s$, $s+1 < t$ and $t+1 \leqslant k+1$. The two monomial coefficients involving this basis element are:
$$(-1)^{s-1} x^{I_s \to I_{s+1}} (-1)^{t-2} x^{I_t \to I_{t+1}} \text{ and}\\ (-1)^{t-1} x^{I_t \to I_{t+1}} (-1)^{s-1} x^{I_s \to I_{s+1}}.$$
- Terms $(I_2, \ldots, I_s \cup I_{s+1}, \ldots, I_1 \cup I_{k+1})$, with $2 \leqslant s \leqslant k-1$. Here, the two monomial coefficients are:
$$(-1) x^{I_{k+1} \to I_1} (-1)^{s-2} x^{I_s \to I_{s+1}} \text{ and}\\ (-1)^{s-1} x^{I_s \to I_{s+1}} (-1) x^{I_{k+1} \to I_1}.$$
- Terms $(I_2, \ldots, I_1 \cup I_k \cup I_{k+1})$. The two monomial coefficients are:
$$(-1)^{k-1} x^{I_k \to I_{k+1}} (-1) x^{I_k \cup I_{k+1} \to I_1} \text{ and}\\ (-1) x^{I_{k+1} \to I_1} (-1)^{k-2} x^{I_k \to I_1 \cup I_{k+1}}.$$
- Finally, the two monomial coefficients of $(I_3, \ldots, I_1 \cup I_2 \cup I_{k+1})$ are
$$(-1) x^{I_{k+1} \to I_1} (-1) x^{I_{k+1} \cup I_1 \to I_2} \text{ and}\\ x^{I_1 \to I_2} (-1) x^{I_{k+1} \to I_1 \cup I_2}.$$

It is easy to check that they are equal in pairs, but opposite in sign. This finishes the proof. ☐

Remark 3.10. If $\mathbb{G}$ is strongly complete, or equivalently L is a PCB matrix, then $\partial_k(\mathcal{C}_k) \subseteq \mathfrak{m}\mathcal{C}_{k-1}$, since $a_{i,j} > 0$, for all $i, j = 1, \ldots, n$, ensures the containment $x^{I_t \to I_{t+1}} \in (x_1, \ldots, x_n) = \mathfrak{m}$.

4. Gröbner bases for lattice ideals associated to strongly connected digraphs

4.1. Block echelon form of an ICB matrix. We start by defining what we will consider to be a block echelon form of an ICB matrix. In order to simplify the exposition, the expression "I-rows" will stand for the set of rows whose row subindex is in I, and analogously as regards the expression "J-columns". As in (2), given $I, J \subseteq [n]$, let $M_{I,J}$ be the submatrix of M defined by the I-rows and the J-columns.

Definition 4.1. *Let L be an $n\times n$ ICB matrix and let δ be a positive integer, $1\leqslant \delta\leqslant n-1$. Given $q_1,\dots,q_\delta\in\mathbb{N}_+$, with $q_1+\dots+q_\delta=n-1$, let $(I_1,\dots,I_\delta)$ be the partition of $[n-1]$ into δ blocks of cardinalities $q_1,\dots,q_\delta$, defined by $I_1=[q_1]$, $I_1\cup I_2=[q_1+q_2],\dots,I_1\cup\ldots\cup I_\delta=[n-1]$. Set $I_{\delta+1}=\{n\}$. Then L is said to be in δ-block echelon form if $L_{I_i,I_j}=0$, for every i and j such that $1\leqslant j\leqslant \delta-1$ and $j+2\leqslant i\leqslant \delta+1$, and each column of L_{I_{j+1},I_j} is nonzero, for every $j=1,\dots,\delta$.*

Example 4.2. Let L be the ICB matrix below which is in δ-block echelon form, where $\delta=3$, and $q_1=1$, $q_2=2$ and $q_3=2$. Then $(I_1,I_2,I_3)=(\{1\},\{2,3\},\{4,5\})$. Note that the 4 blocks on the diagonal have diagonal positive entries. The 3 blocks below the diagonal blocks have nonzero columns. All the remaining blocks below the aforementioned are zero.

$$L=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1\\ 0 & 1 & -1 & 0 & 0 & 0\\ -1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0 & 0\\ 0 & -1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

Remark 4.3. Any $n\times n$ ICB matrix L can be reduced by permutations to a δ-block echelon form.

Proof. (We remark that it may help to consider the shape of the matrix in the previous example.)

Indeed, set $I_{\delta+1}=\{n\}$. The notation J_δ-columns, with $J_\delta\subseteq[n-1]$, will denote the set of columns with column subindex in $[n-1]$, having a nonzero entry in the last row. Since L is irreducible, then, necessarily, in the last row of L there are other nonzero entries apart from $a_{n,n}$ (this follows too from the fact that since L is, in particular, a CB matrix, $0<a_{n,n}=\sum_{j\neq n}a_{n,j}$). Therefore $|J_\delta|:=q_\delta\geqslant 1$. Move these q_δ columns to the outer right of the first $n-1$ columns; we use the notation I_δ-columns to denote the corresponding set of columns. Then perform the corresponding permutation of rows, so preserving the structure of a CB matrix, while in the last row, which remains untouched by the row permutations just carried out, we maintain the property that all its $q_\delta+1$ nonzero entries are in the set of $I_\delta\cup I_{\delta+1}$-columns.

Since L is irreducible, then there must be at least one column to the left of the set of $I_\delta\cup I_{\delta+1}$-columns having a nonzero entry in the set of $I_\delta\cup I_{\delta+1}$-rows, these being strictly below the diagonal. For, if not, setting $I=[n]\setminus I_\delta\cup I_{\delta+1}$ and $J=I_\delta\cup I_{\delta+1}$, one would have a partition (I,J) of $[n]$ into two non-empty disjoint sets and such that $L_{J,I}=0$. Thus, L would be reducible, a contradiction (recall the definition of irreducible in Section 2.3). So, denote by $J_{\delta-1}\subseteq[n]\setminus I_\delta\cup I_{\delta+1}$ the set of column indices to the left of the set of $I_\delta\cup I_{\delta+1}$-columns which have a nonzero entry in the set of $I_\delta\cup I_{\delta+1}$-rows, these being strictly below the diagonal, where $|J_{\delta-1}|:=q_{\delta-1}\geqslant 1$. Now proceed recursively. □

Remark 4.4. Let L be an ICB matrix in δ-block echelon form. Let $C\subseteq[n-1]$, $C\neq\varnothing$. Then the rightmost nonzero coefficient of $(L\chi_C)^\top$ is negative.

Proof. Keeping the notations in Definition 4.1, set $C_i=C\cap I_i$ and let $m\geqslant 1$ be the maximum integer with $C_m\neq\varnothing$. Clearly $\chi_C=\sum_{i=1}^{\delta}\chi_{C_i}=\sum_{i=1}^{m}\chi_{C_i}$ and $L\chi_C=\sum_{i=1}^{m}L\chi_{C_i}$. Since all the columns to the left of I_m have zero I_{m+1}-rows, it is enough to see that $L\chi_{C_m}\leqslant 0$ and different from zero. This follows from the fact that $L_{I_{m+1},I_m}\leqslant 0$ and that each column of L_{I_{m+1},I_m} is nonzero. □

To finish this section, let us prove that the δ-block-echelon form reflects the natural partition of the set of vertices into subsets of elements whose distance from the last vertex is constant. This brings out another aspect of the relationship between ICB matrices L and δ-block echelon form, this time arising from the natural metric $\mathrm{d}^\vee$ on $\mathbb{G}_L$.

Lemma 4.5. *Let L be an $n\times n$ ICB matrix and let $\mathbb{G}_L=(\mathbb{V},\mathbb{E})$ its associated digraph. Suppose that L is in δ-block echelon form and let $(I_1,\dots,I_\delta)$ be the associated partition of $[n-1]$ into δ blocks, with $I_{\delta+1}=\{n\}$. For $i=1,\dots,\delta+1$, set $W_i:=\{v_k\in\mathbb{V}\mid k\in I_i\}$, so that $W_{\delta+1}=\{v_n\}$. Then $(W_1,\dots,W_\delta)$ is a partition of $\mathbb{V}\smallsetminus\{v_n\}$ into δ non-empty subsets of $\mathbb{V}$. Moreover, for each $i=1,\dots,\delta$, the set W_i coincides with the set $V_i:=\{v\in\mathbb{V}\mid \mathrm{d}^\vee(v_n,v)=\delta+1-i\}$.*

Proof. Clearly, since $(I_1,\dots,I_\delta)$ is a partition of $[n-1]$ into δ non-empty blocks, then $(W_1,\dots,W_\delta)$ is a partition of $\mathbb{V}\smallsetminus\{v_n\}$ into δ non-empty subsets of $\mathbb{V}$. Let us prove that $W_i=V_i$ by decreasing induction on $i=1,\dots,\delta+1$. Clearly $W_{\delta+1}=\{v_n\}=V_{\delta+1}$.

Fix i, with $1\leqslant i\leqslant\delta$. Suppose that $W_{i+1}=V_{i+1},\dots,W_{\delta+1}=V_{\delta+1}$ and let us prove $W_i=V_i$.

Take $v_{k_i}\in W_i$, so that $k_i\in I_i$. Since L_{I_{i+1},I_i} has non-zero columns, there exists $k_{i+1}\in I_{i+1}$, with $a_{k_{i+1},k_i}>0$, so there is an arc $v_{k_{i+1}}\to v_{k_i}$ in $\mathbb{G}$. Similarly, since $L_{I_{i+2},I_{i+1}}$ has non-zero columns, there exists $k_{i+2}\in I_{i+2}$, with $a_{k_{i+2},k_{i+1}}>0$, and an arc $v_{k_{i+2}}\to v_{k_{i+1}}$ in $\mathbb{G}$. Recursively, there are vertices $v_{k_j}\in W_j$ and a directed path $v_n\to v_{k_\delta}\to\cdots\to v_{k_{i+1}}\to v_{k_i}$. In particular, $\mathrm{d}^\vee(v_n,v_{k_i})\leqslant\delta+1-i$ and $v_{k_i}\in V_i\cup\ldots\cup V_{\delta+1}$. On the other hand, since $(W_1,\dots,W_{\delta+1})$ is a partition of $\mathbb{V}$ and $v_{k_i}\in W_i$, then $v_{k_i}\notin W_{i+1}\cup\ldots\cup W_{\delta+1}=V_{i+1}\cup\ldots\cup V_{\delta+1}$. Thus $v_{k_i}\in V_i$ and it follows that $W_i\subseteq V_i$.

Now, take $v_k\in V_i$. Then there exist $v_n\to v_{k_\delta}\to\cdots\to v_{k_{i+1}}\to v_k$, a directed path of minimum unweighted length $\delta+1-i$. In particular, $\mathrm{d}^\vee(v_n,v_{k_{i+1}})\leqslant\delta-i$ and, in fact, this is an equality, otherwise $\mathrm{d}^\vee(v_n,v_k)<\delta+1-i$. Hence $v_{k_{i+1}}\in V_{i+1}=W_{i+1}$ and $k_{i+1}\in I_{i+1}$. The existence of the arc $v_{k_{i+1}}\to v_k$ ensures $a_{k_{i+1},k}>0$. Since L is in δ-block echelon form, the index k must sit inside $I_i\cup\ldots\cup I_{\delta+1}$. If $k\in I_{i+1}\cup\ldots\cup I_{\delta+1}$, then $v_k\in W_{i+1}\cup\ldots\cup W_{\delta+1}=V_{i+1}\cup\ldots\cup V_{\delta+1}$, a contradiction with the fact that V_i is clearly disjoint with $V_{i+1}\cup\ldots\cup V_{\delta+1}$. Thus $k\in I_i$ and $v_k\in V_i$ and we derive the opposite inclusion $V_i\subseteq W_i$. $\square$

4.2. A convenient enumeration of the vertices. Taking into acccount the former lemma, let us define a special enumeration on the set of vertices and arcs of a strongly connected digraph, and hence on the set of indeterminates $x_1,\dots,x_n$ (see [7, p. 7], [19, Section 2.2.3], for the case where $\mathbb{G}$ is undirected, or, equivalently, $L_{\mathbb{G}}$ is symmetric).

Definition 4.6. *Let $\mathbb{G}=(\mathbb{V},\mathbb{E})$ be a strongly connected digraph. Fix a distinguished vertex ω in $\mathbb{V}$. Since $\mathbb{G}$ is finite, $\delta:=\max\{\mathrm{d}^\vee(\omega,v)\mid v\in\mathbb{V}\}$ is finite and $\delta\geqslant1$. Consider the partition $(V_1,\dots,V_\delta)$ of $\mathbb{V}\smallsetminus\{\omega\}$ into δ non-empty subsets defined by $V_i=\{v\in\mathbb{V}\mid\mathrm{d}^\vee(\omega,v)=\delta+1-i\}$, for $i=1,\dots,\delta$. Set $V_{\delta+1}=\{\omega\}$. Let $q_i=|V_i|$ be the cardinality of each V_i. Take the partition $(I_1,\dots,I_\delta)$ of $[n-1]$ into δ blocks of cardinalities $q_1,\dots,q_\delta$, defined by $I_1=[q_1]$, $I_1\cup I_2=[q_1+q_2],\dots,I_1\cup\ldots\cup I_\delta=[n-1]$. Set $I_{\delta+1}=\{n\}$. The (ω,δ)-enumeration of the vertices and arcs of $\mathbb{G}$ is determined by assigning the set of indices I_i to the set of vertices V_i. That is, $V_i=\{v_k\in\mathbb{V}\mid k\in I_i\}$, for $i=1,\dots,\delta$. The index n is assigned to the vertex ω, so that ω becomes v_n. In particular, if $i<j$, then $\mathrm{d}^\vee(v_n,v_i)\geqslant\mathrm{d}^\vee(v_n,v_j)$. On the other hand, if $\mathrm{d}^\vee(v_n,v_i)>\mathrm{d}^\vee(v_n,v_j)$, then $i<j$. Clearly, whenever $\mathrm{d}^\vee(v_n,v_i)=\mathrm{d}^\vee(v_n,v_j)$, the three possibilities $i<j$, $i=j$ or $i>j$ may happen.*

In Lemma 4.5 we have seen that if L is an $n\times n$ ICB matrix in δ-block echelon form, then the corresponding digraph $\mathbb{G}_L$ has an (ω,δ)-enumeration (where ω is taken to be the last vertex). Now let us prove that a digraph $\mathbb{G}$ with an (ω,δ)-enumeration has the corresponding ICB matrix $L_{\mathbb{G}}$ in δ-block echelon form.

Lemma 4.7. *Let $\mathbb{G}=(\mathbb{V},\mathbb{E})$ be a strongly connected digraph with an (ω,δ)-enumeration. Let $L=L_{\mathbb{G}}$ be the* ICB *matrix associated to $\mathbb{G}$. Then L is in δ-block echelon form.*

Proof. With the notation in Definition 4.6,

$$\{v_k\in\mathbb{V}\mid k\in I_i\}=\{v_k\in\mathbb{V}\mid \mathrm{d}^{\vee}(\omega,v_k)=\delta+1-i\}=:V_i,$$

for $i=1,\dots,\delta$, and $I_{\delta+1}=\{n\}$ and $\omega=v_n$. We want to prove that $L_{I_i,I_j}=0$, for every i and j such that $1\leqslant j\leqslant\delta-1$ and $j+2\leqslant i\leqslant\delta+1$, and each column of L_{I_{j+1},I_j} is nonzero, for every $j=1,\dots,\delta$.

So, fix j with $1\leqslant j\leqslant\delta-1$ and $v_k\in V_j$. Then, we need to prove that

$$a_{i,k}=0,\text{ for all } i\in I_{j+2}\cup\ldots\cup I_{\delta+1}\ ;\ a_{i,k}\neq 0,\text{ for some } i\in I_{j+1}. \tag{14}$$

Note that, for $j=\delta$, then $I_{j+1}=I_{\delta+1}=\{n\}$ and $a_{i,k}=a_{n,k}\neq 0$, for all $k\in I_\delta$, since the vertices $v_k\in V_\delta$ are such that $\mathrm{d}^{\vee}(\omega,v_k)=1$.

Suppose that $a_{i,k}\neq 0$ for some $i\in I_{j+2}\cup\ldots\cup I_{\delta+1}$. Let $v_n\to v_{i_1}\to\cdots\to v_{i_l}\to v_i$ be a directed path of minimum unweighted length $l+1$, where $l+1\leqslant\delta-j-1$ (because $i\in I_{j+2}\cup\ldots\cup I_{\delta+1}$). Since $a_{i,k}\neq 0$, there is an arc $v_i\to v_k$ in $\mathbb{G}$, and so a directed path from v_n to v_k of unweighted length at most $\delta-j$, a contradiction, since $\mathrm{d}^{\vee}(v_n,v_k)=\delta+1-j$ (recall that $v_k\in V_j$). This proves the first part of (14).

Now, take a directed path $v_n\to v_{i_1}\to\cdots\to v_{i_{\delta-j}}\to v_k$ of minimum unweighted length $\delta+1-j$. In particular, $\mathrm{d}^{\vee}(\omega,v_{i_{\delta-j}})=\delta-j$. Thus $i_{\delta-j}=:i\in I_{j+1}$ and $a_{i,k}\neq 0$, since $v_i\to v_k$ is an arc of $\mathbb{G}$. This proves the second part of (14). ☐

Assumption 4.8. *Let L be an* ICB *matrix or, equivalently, let $\mathbb{G}_L$ be a strongly connected digraph. From now on, we will assume that L is always in δ-block echelon form or, equivalently, that $\mathbb{G}$ has a (ω,δ)-enumeration (see Lemmas* 4.5 *and* 4.7*).*

Note that, if L is a PCB *matrix or, equivalently, $\mathbb{G}$ is strongly complete, this assumption is vacuous.*

4.3. The weighted reverse lexicographic order on the polynomial ring. Let L be an ICB matrix or, equivalently, let $\mathbb{G}_L$ be a strongly connected digraph. Recall that, by Assumption 2.7, $\deg(x_i)=\nu_i$ and $\deg(x^\alpha)=\nu_1\alpha_1+\cdots+\nu_n\alpha_n$.

Assumption 4.9. *From now on, we will always suppose that $\mathbb{A}=\mathbb{K}[x]=\mathbb{K}[x_1,\dots,x_n]$ is endowed with the weighted reverse lexicographic order (*wrlo*, for short). Concretely, given $\alpha,\beta\in\mathbb{N}^n$, we set $x^\alpha>x^\beta$ precisely when either $\deg(x^\alpha)>\deg(x^\beta)$, or $\deg(x^\alpha)=\deg(x^\beta)$ and there exists $i\in\{1,\dots,n\}$ such that $\alpha_n=\beta_n,\dots,\alpha_{i+1}=\beta_{i+1}$ and $\alpha_i<\beta_i$. This last condition will be written $\alpha-\beta=(*,\dots,*,-,0,\dots,0)$. See, e.g,* [15, p. 13]*, where this ordering is denoted by $\mathbf{wp}(\nu_1,\dots,\nu_n)$.*

We insert a note of caution: it need no longer be the case that, under this wrlo, $x_1>\cdots>x_n$, since, for example, we have $\deg(x_2)>\deg(x_1)$ whenever $\nu_2>\nu_1$.

Given a non-zero polynomial $f\in\mathbb{A}$, we let $\mathbf{L}\mathrm{c}(f)$, $\mathbf{L}\mathrm{m}(f)$ and $\mathbf{L}\mathrm{t}(f)$ denote the leading coefficient, the leading monomial and the leading term, respectively, of f with respect to the wrlo (see, e.g. [9, § 2.2, Definition 7]).

Recall that $f_C:=m_C-m_{\overline{C}}=\partial_1(C,\overline{C})$, where $C\subseteq[n-1]$, $C\neq\varnothing$ (see Notation 3.6).

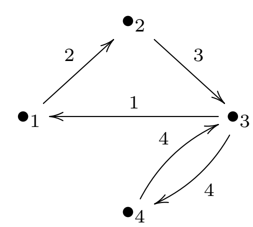
Lemma 4.10. *Let $(C,\overline{C})\in\mathcal{C}_1$ and set $f_C:=m_C-m_{\overline{C}}=\partial_1(C,\overline{C})$. Then f_C is homogeneous and $\mathbf{L}\mathrm{m}(f_C)=m_C$. In particular, $\mathbf{L}\mathrm{t}(f_C)=\mathbf{L}\mathrm{m}(f_C)$.*

Proof. That $f_C = m_C - m_{\overline{C}}$ is homogeneous is shown in Remark 3.5. So $\operatorname{grade}(m_C) = \operatorname{grade}(x^{C\to\overline{C}})$ is equal to $\operatorname{grade}(m_{\overline{C}}) = \operatorname{grade}(x^{\overline{C}\to C})$. Also by Remark 3.5, we know that, $m_C = x^{C\to\overline{C}} = x^{(L\chi_C)^+}$ and $m_{\overline{C}} = x^{\overline{C}\to C} = x^{(L\chi_C)^-}$, where χ_C is the incidence vector of C (see Remark 3.1). Then $(L\chi_C)^+ - (L\chi_C)^- = L\chi_C$. So we must see that the rightmost nonzero coefficient of $(L\chi_C)^\top$ is negative. But this follows from Remark 4.4. $\square$

If the variables $x_1, \ldots, x_n$ are not enumerated according to Assumption 4.8, then Lemma 4.10 may fail.

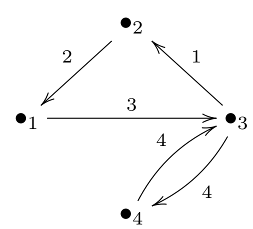
Example 4.11. Consider the following CB matrices L and L':

$$L = \begin{pmatrix} 2 & -2 & 0 & 0 \\ 0 & 3 & -3 & 0 \\ -1 & 0 & 5 & -4 \\ 0 & 0 & -4 & 4 \end{pmatrix}, \text{ corresponding to } \mathbb{G}_L :$$



;

$$L' = \begin{pmatrix} 3 & 0 & -3 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & -1 & 5 & -4 \\ 0 & 0 & -4 & 4 \end{pmatrix}, \text{ corresponding to } \mathbb{G}_{L'} :$$



.

Observe that L' arises from L by the transposition of rows and columns 1 and 2 of L. The corresponding digraphs $\mathbb{G}_L$ and $\mathbb{G}_{L'}$ are the same, but with a re-enumeration of vertices and arcs. Since $\mathbb{G}_L$ and $\mathbb{G}_{L'}$ are strongly connected, L and L' are ICB matrices. Note that L' is in 3-block echelon form, whereas L is not.

A simple check shows that

$$\begin{aligned} &\mu(L) = (12, 8, 24, 24), \text{ so } \nu(L) = (3, 2, 6, 6), \text{ and} \\ &\mu(L') = (8, 12, 24, 24), \text{ so } \nu(L') = (2, 3, 6, 6). \end{aligned}$$

Imposing the usual wrlo on our polynomial ring $\mathbb{K}[x, y, z, t]$, with x_1, x_2, x_3, x_4 being replaced by x, y, z, t, respectively, then $\deg(x) = 3$, $\deg(y) = 2$, $\deg(z) = 6$ and $\deg(t) = 6$, when working with L. Considering L and $C := \{2\}$, we have: $f_C = m_C - m_{\overline{C}}$, $m_C = y^3$, $m_{\overline{C}} = x^2$. Note that $y^3 < x^2$ because $(0, 3, 0, 0) - (2, 0, 0, 0) = (-2, 3, 0, 0)$, which is consistent with $(L\chi_C)^\top = (-2, 3, 0, 0)$. Thus $\mathbf{L}\mathrm{m}(f_C) = x^2 = m_{\overline{C}}$.

4.4. A Gröbner basis for the image of the first differential.

Let L be an ICB matrix or, equivalently, let $\mathbb{G}_L$ be a strongly connected digraph. Recall that $\mathcal{B}_k$ is the set of elements of $\mathrm{Cyc}_{n,k+1}$, enumerated employing the srle (see Definition 3.3), forming a basis of the free $\mathbb{K}[x]$-module $\mathcal{C}_k = \mathbb{K}[x]^{\mathrm{Cyc}_{n,k+1}}$. Concretely, in degree zero:

$$\begin{aligned} \partial_1(\mathcal{B}_1) &= \{\partial_1(e_{1,1}), \ldots, \partial_1(e_{1,r_1})\} \\ &= \{f_{0,1}, \ldots, f_{0,r_1}\} = \{f_C \mid C \subsetneq [n-1], C \neq \varnothing\} \subset \mathcal{C}_0. \end{aligned}$$

The purpose of this subsection is to prove that $\partial_1(\mathcal{B}_1)$ is a Gröbner basis of $\partial_1(\mathcal{C}_1)$ (see also Notation 3.6). We begin with some useful notations and remarks.

Notation 4.12. Given three disjoint subsets $A, B, C \subset [n]$, note that:

$$x^{A\cup B\to C} = x^{A\to C}x^{B\to C} \text{ and } x^{A\to B\cup C} = x^{A\to B}x^{A\to C}. \tag{15}$$

We set:

$$x^{(A\to B,C)^+} := \frac{\mathrm{LCM}(x^{A\to B}, x^{A\to C})}{x^{A\to C}} = \frac{\prod_{i\in A} x_i^{\max(\sum_{j\in C} a_{i,j}, \sum_{j\in B} a_{i,j})}}{\prod_{i\in A} x_i^{\sum_{j\in C} a_{i,j}}} = \prod_{i\in A} x_i^{(\sum_{j\in B} a_{i,j} - \sum_{j\in C} a_{i,j})^+}. \tag{16}$$

In particular,

$$\mathrm{LCM}(x^{A\to B}, x^{A\to C}) = x^{(A\to B,C)^+}x^{A\to C} = x^{(A\to C,B)^+}x^{A\to B}. \tag{17}$$

Notation 4.13. Let C and D be two non-empty subsets of $[n-1]$. Write $E = C\cap D$, $F = C\smallsetminus E$, $G = D\smallsetminus E$, $U = C\cup D$ and $V = \overline{U} = [n]\smallsetminus U$. In particular, $\overline{C} = [n]\smallsetminus C = G\cup V$ and $\overline{D} = [n]\smallsetminus D = F\cup V$. The following picture may help in reading everything that will follow.

$\cap$	D	$\overline{D}$
C	E	F
$\overline{C}$	G	$V.$

Lemma 4.14. *The restriction of the map $\partial_1 : \mathcal{C}_1 \to \mathcal{C}_0$ to the set $\mathcal{B}_1$ is injective. Concretely, given C, D, two non-empty subsets of $[n-1]$, then $f_C = f_D$ if and only if $C = D$.*

Proof. Suppose that $f_C = f_D$. By Lemma 4.10, $m_C = \mathbf{L}\mathrm{t}(f_C) = \mathbf{L}\mathrm{t}(f_D) = m_D$. Thus $m_C = m_D$ and hence $m_{\overline{C}} = m_{\overline{D}}$. Suppose that $C \neq D$. Since the initial hypotheses are symmetrical in C and D, we may assume without loss of generality that $C \not\subseteq D$, that is, that $F \neq \varnothing$, with Notation 4.13 in force. Using (15), one has:

$$\begin{aligned} f_C &= m_C - m_{\overline{C}} = x^{C\to\overline{C}} - x^{\overline{C}\to C} \\ &= x^{E\to G}x^{E\to V}x^{F\to G}x^{F\to V} - x^{G\to E}x^{G\to F}x^{V\to E}x^{V\to F} \end{aligned}$$

and

$$\begin{aligned} f_D &= m_D - m_{\overline{D}} = x^{D\to\overline{D}} - x^{\overline{D}\to D} \\ &= x^{E\to F}x^{E\to V}x^{G\to F}x^{G\to V} - x^{F\to E}x^{F\to G}x^{V\to E}x^{V\to G}. \end{aligned}$$

It follows from the equality $m_C = m_D$ that $x^{F\to G\cup V} = 1$, i.e. that $a_{i,j} = 0$, for all $i \in F$ and $j \in \overline{C} = G\cup V$. We deduce that $x^{F\to E\cup G} = 1$ from the equality $m_{\overline{C}} = m_{\overline{D}}$, i.e. that $a_{i,j} = 0$, for all $i \in F$ and $j \in D = E \cup G$. Thus $a_{i,j} = 0$, for all $i \in F$ and $j \in E \cup G \cup V = \overline{F}$. Therefore there is no directed path joining any vertex in $F \neq \varnothing$ to the vertex $v_n \in \overline{F}$, which is a contradiction to the hypothesis that $\mathbb{G}_L$ is a strongly connected digraph. $\square$

Let us calculate the S-polynomial $S(f_C, f_D)$ of $f_C = \partial_1(C, \overline{C})$ and $f_D = \partial_1(D, \overline{D})$. Observe that if $C = D$, then $f_C = f_D$ and $S(f_C, f_D) = 0$.

Lemma 4.15. *Let C and D be two non-empty distinct subsets of $[n-1]$. Consider the monomials $l_{C,D}$ and $l_{D,C}$, defined as follows:*

$$l_{C,D} = x^{(E\to G,F)^+}x^{F\to G}x^{V\to D} \quad \textit{and} \quad l_{D,C} = x^{(E\to F,G)^+}x^{G\to F}x^{V\to C}.$$

(*a*) *Suppose that $C \not\subset D$ and that $D \not\subset C$. Then F, G are non-empty subsets of $[n-1]$, so $(F, \overline{F})$ and $(G, \overline{G})$ are in $\mathcal{C}_1$. Then the S-polynomial of f_C and f_D is*

$$S(f_C, f_D) = l_{C,D} f_F - l_{D,C} f_G, \tag{18}$$

with $f_F = \partial_1(F, \overline{F})$ and $f_G = \partial_1(G, \overline{G})$. Moreover,

$$\mathbf{L}\mathrm{m}(S(f_C, f_D)) \geqslant \mathbf{L}\mathrm{m}(l_{C,D} f_F), \mathbf{L}\mathrm{m}(l_{D,C} f_G).$$

(*b*) *Suppose, without loss of generality, that $C \subsetneq D$. Then $F = \varnothing$ and G is a non-empty subset of $[n-1]$, so $(G, \overline{G}) \in \mathcal{C}_1$. Then the S-polynomial of f_C and f_D is*

$$S(f_C, f_D) = -l_{D,C} f_G, \tag{19}$$

with $f_G = \partial_1(G, \overline{G})$. Moreover, $\mathbf{L}\mathrm{m}(S(f_C, f_D)) \geqslant \mathbf{L}\mathrm{m}(l_{D,C} f_G)$.

Proof. By definition (see, e.g. [9, § 2.6, Definition 4]), $S(f_C, f_D) = m_{D,C} f_C - m_{C,D} f_D$, where

$$m_{D,C} = \frac{\mathrm{LCM}(\mathbf{L}\mathrm{m}(f_D), \mathbf{L}\mathrm{m}(f_C))}{\mathbf{L}\mathrm{t}(f_C)} \quad \text{and} \quad m_{C,D} = \frac{\mathrm{LCM}(\mathbf{L}\mathrm{m}(f_C), \mathbf{L}\mathrm{m}(f_D))}{\mathbf{L}\mathrm{t}(f_D)}. \tag{20}$$

Suppose that $C \not\subset D$ and $D \not\subset C$, so that $F, G \neq \varnothing$. By Lemma 4.10, $\mathbf{L}\mathrm{t}(f_C) = \mathbf{L}\mathrm{m}(f_C) = m_C = x^{C \to \overline{C}}$ and $\mathbf{L}\mathrm{t}(f_D) = \mathbf{L}\mathrm{m}(f_D) = m_D = x^{D \to \overline{D}}$. Using (15) and (17), and the fact that $\mathrm{LCM}(fg, fh) = f \cdot \mathrm{LCM}(g, h)$ for elements $f, g, h \in \mathbb{A} = \mathbb{K}[x]$, one has:

$$\begin{aligned} m_{D,C} &= \frac{\mathrm{LCM}(\mathbf{L}\mathrm{m}(f_D), \mathbf{L}\mathrm{m}(f_C))}{\mathbf{L}\mathrm{t}(f_C)} \\ &= \frac{\mathrm{LCM}(x^{E \to F}, x^{E \to G})}{x^{E \to G}} x^{G \to F} x^{G \to V} = x^{(E \to F, G)^+} x^{G \to F} x^{G \to V}. \end{aligned} \tag{21}$$

In a symmetric manner,

$$m_{C,D} = \frac{\mathrm{LCM}(\mathbf{L}\mathrm{m}(f_C), \mathbf{L}\mathrm{m}(f_D))}{\mathbf{L}\mathrm{t}(f_D)} = x^{(E \to G, F)^+} x^{F \to G} x^{F \to V}. \tag{22}$$

Set $g_{G,F} := x^{(E \to F, G)^+} x^{G \to F}$ and $g_{F,G} := x^{(E \to G, F)^+} x^{F \to G}$. Therefore $m_{D,C} = g_{G,F} x^{G \to V}$ and $l_{D,C} = g_{G,F} x^{V \to C}$. Analogously, $m_{C,D} = g_{F,G} x^{F \to V}$ and $l_{C,D} = g_{F,G} x^{V \to D}$. Then

$$\begin{aligned} S(f_C, f_D) &= g_{G,F} x^{G \to V} f_C - g_{F,G} x^{F \to V} f_D \\ &= g_{G,F} x^{G \to V} m_C - g_{G,F} x^{G \to V} m_{\overline{C}} - g_{F,G} x^{F \to V} m_D + g_{F,G} x^{F \to V} m_{\overline{D}}. \end{aligned}$$

Set

$$\begin{aligned} m_1 &:= g_{G,F} x^{G \to V} m_C \quad , \quad m_2 := g_{G,F} x^{G \to V} m_{\overline{C}} \, , \\ m_3 &:= g_{F,G} x^{F \to V} m_D \quad , \quad m_4 := g_{F,G} x^{F \to V} m_{\overline{D}} \, , \end{aligned}$$

so that $S(f_C, f_D) = m_1 - m_2 - m_3 + m_4$. An easy computation using (15) shows that

$$m_1 - m_3 = (g_{G,F} x^{C \to G} - g_{F,G} x^{D \to F}) x^{U \to V}.$$

However, using (15) and then (17), we get

$$\begin{aligned} & g_{G,F} x^{C \to G} - g_{F,G} x^{D \to F} \\ &= x^{(E \to F, G)^+} x^{G \to F} x^{E \to G} x^{F \to G} - x^{(E \to G, F)^+} x^{F \to G} x^{E \to F} x^{G \to F} \\ &= (x^{(E \to F, G)^+} x^{E \to G} - x^{(E \to G, F)^+} x^{E \to F}) x^{F \to G} x^{G \to F} = 0. \end{aligned} \tag{23}$$

Therefore, $S(f_C, f_D) = -m_2 + m_4$. On the other hand,

$$l_{C,D} f_F - l_{D,C} f_G = l_{C,D} m_F - l_{C,D} m_{\overline{F}} - l_{D,C} m_G + l_{D,C} f_{\overline{G}}.$$

Set

$$\begin{aligned} m_5 &:= l_{C,D} m_F = g_{F,G} x^{V \to D} m_F \quad , \quad m_6 := l_{C,D} m_{\overline{F}} = g_{F,G} x^{V \to D} m_{\overline{F}} \ , \\ m_7 &:= l_{D,C} m_G = g_{G,F} x^{V \to C} m_G \quad , \quad m_8 := l_{D,C} m_{\overline{G}} = g_{G,F} x^{V \to C} m_{\overline{G}} \ , \end{aligned}$$

so that $l_{C,D} f_F - l_{D,C} f_G = m_5 - m_6 - m_7 + m_8$. An easy computation using (15) shows that

$$-m_6 + m_8 = (-g_{F,G} x^{D \to F} + g_{G,F} x^{C \to G}) x^{V \to U}.$$

The situation is symmetrical in C and D. Swapping C and D in an expression forces a swap in F and G, with E, U and V invariant. Thus $-g_{F,G} x^{D \to F} + g_{G,F} x^{C \to G} = 0$ follows by symmetry from the equality (23). Therefore, $-m_6 + m_8 = 0$ and $l_{C,D} f_F - l_{D,C} f_G = m_5 - m_7$. An easy computation using (15) shows that $m_2 = m_7$ and, by symmetry, $m_4 = m_5$. Therefore, $S(f_C, f_D) = -m_2 + m_4 = m_5 - m_7 = l_{C,D} f_F - l_{D,C} f_G$.

Observe that one can consider $(F, \overline{F}), (G, \overline{G}) \in \mathcal{C}_1$ due to the hypothesis $F, G \neq \varnothing$. Using Lemma 4.10, it follows that $\mathbf{Lm}(f_F) = m_F$ and $\mathbf{Lm}(f_G) = m_G$. Since $l_{C,D}$ and $l_{D,C}$ are monomials, then $\mathbf{Lm}(l_{C,D} f_F) = l_{C,D} m_F = m_5$ and $\mathbf{Lm}(l_{D,C} f_G) = l_{D,C} m_G = m_7$ (see, e.g. [9, §2.2, Exercise 11]). On the other hand, $S(f_C, f_D) = -m_2 + m_4$. Thus, one of the two monomials, either m_2, or else m_4, is the leading monomial of $S(f_C, f_D)$. If $m_2 < m_4$, then $\mathbf{Lm}(l_{C,D} f_F) = m_5 = m_4 = \mathbf{Lm}(S(f_C, f_D))$ and $\mathbf{Lm}(l_{D,C} f_G) = m_7 = m_2 < m_4 = \mathbf{Lm}(S(f_C, f_D))$. On the contrary, if $m_4 < m_2$, then $\mathbf{Lm}(l_{C,D} f_F) = m_5 = m_4 < m_2 = \mathbf{Lm}(S(f_C, f_D))$ and $\mathbf{Lm}(l_{D,C} f_G) = m_7 = m_2 = \mathbf{Lm}(S(f_C, f_D))$.

Finally, assume, without loss of generality, that $C \subsetneq D$. Then $F = \varnothing$ and $f_F = 0$. Hence $g_{G,F} = 1$ and $g_{F,G} = x^{C \to G}$; $m_{D,C} = x^{G \to V}$ and $l_{D,C} = x^{V \to C}$; $m_{C,D} = x^{C \to G}$ and $l_{C,D} = x^{C \to G} x^{V \to D}$. As before, $m_1 - m_3 = 0$ and $S(f_C, f_D) = -m_2 + m_4$. On the other hand, since $F = \varnothing$, then $f_F = 0$ and $l_{C,D} f_F - l_{D,C} f_G = -l_{D,C} f_G$, which is readily seen to be equal to $-m_2 + m_4$. Moreover, trivially then, one has $\mathbf{Lm}(S(f_C, f_D)) \geqslant \mathbf{Lm}(l_{D,C} f_G)$. □

Proposition 4.16. *The set $\partial_1(\mathcal{B}_1) = \{f_C \mid C \subseteq [n-1], C \neq \varnothing\}$ is a Gröbner basis for the ideal $\partial_1(\mathcal{C}_1)$.*

Proof. This follows from Lemma 4.15 and the criterion of Buchberger (see, e.g. [9, § 2.6, Theorem 6 and § 2.9, Theorem 3]). □

4.5. The Cyc complex is exact in degree zero. Let L be an ICB matrix or, equivalently, let $\mathbb{G}$ be a strongly connected digraph. Recall that $\partial_0 : \mathcal{C}_0 = \mathbb{K}[x] \to \mathbb{K}[x]/I(\mathcal{L})$ is defined to be the natural projection onto the quotient ring, so that $\partial_1(\mathcal{C}_1) \subseteq I(\mathcal{L})$ (Definition 3.7).

Proposition 4.17. *We have $\partial_1(\mathcal{C}_1) : x_n = \partial_1(\mathcal{C}_1)$ and $\partial_1(\mathcal{C}_1) = I(\mathcal{L}) = \ker(\partial_0)$.*

Proof. According to Definition 3.3, let

$$\mathcal{B}_1 = \{e_{1,1}, \dots, e_{1,r_1}\} \text{ and } \partial_1(\mathcal{B}_1) = \{f_{0,1}, \dots, f_{0,r_1}\} \subset \mathbb{K}[x],$$

enumerated employing the srle. By Proposition 4.16, $\partial_1(\mathcal{B}_1)$ is a Gröbner basis for the ideal $\partial_1(\mathcal{C}_1)$. Now, take $f \in \partial_1(\mathcal{C}_1) : x_n$. Using the Division Algorithm, f can be written as

$$f = g_1 f_{0,1} + \dots + g_{r_1} f_{0,r_1} + h, \tag{24}$$

where either $h = 0$, or else $h \neq 0$ and no term in h is divisible by any one of $\mathbf{Lt}(f_{0,i})$, for $i = 1, \dots, r_1$ (see, e.g. [9, § 2.3, Theorem 3]). Suppose that $h \neq 0$. On multiplying by x_n in (24), one deduces that $x_n h \in \partial_1(\mathcal{C}_1)$ and so $\mathbf{Lt}(x_n h) \in \mathbf{Lt}(\partial_1(\mathcal{C}_1))$. Since $f_{0,1}, \dots, f_{0,r_1}$ is a Gröbner basis of $\partial_1(\mathcal{C}_1)$, $\mathbf{Lt}(\partial_1(\mathcal{C}_1)) = \langle \mathbf{Lt}(f_{0,1}), \dots, \mathbf{Lt}(f_{0,r_1}) \rangle$. By, e.g. [9, § 2.4, Lemma 2], $\mathbf{Lt}(x_n h) = x_n \mathbf{Lt}(h)$ is a multiple of some $\mathbf{Lt}(f_{0,i})$. Since x_n does not occur in any $\mathbf{Lt}(f_{0,i})$, it follows that $\mathbf{Lt}(h)$ is a multiple of some $\mathbf{Lt}(f_{0,i})$, a contradiction. Therefore $\partial_1(\mathcal{C}_1) : x_n = \partial_1(\mathcal{C}_1)$. In particular, $\partial_1(\mathcal{C}_1) : x_n^m = \partial_1(\mathcal{C}_1)$, for all $m \geqslant 1$.

By Remark 3.5, $I(L) \subseteq \partial_1(\mathcal{C}_1) \subseteq I(\mathcal{L})$. By [21, Proposition 5.7], $I(\mathcal{L}) = I(L) : x_n^m$, for some $m \gg 0$. Therefore

$$I(\mathcal{L}) = I(L) : x_n^m \subseteq \partial_1(\mathcal{C}_1) : x_n^m = \partial_1(\mathcal{C}_1) \subseteq I(\mathcal{L}),$$

so $I(\mathcal{L}) = \partial_1(\mathcal{C}_1)$ as desired. $\square$

Putting together Propositions 4.16 and 4.17, we get the following result.

Corollary 4.18. *The set $\partial_1(\mathcal{B}_1) \subset \mathcal{C}_0$ is a Gröbner basis of $\partial_1(\mathcal{C}_1) = I(\mathcal{L}) = \ker(\partial_0)$.*

4.6. Minimality in the strongly complete case. Let L be an ICB matrix or, equivalently, let $\mathbb{G}$ be a strongly connected digraph.

Proposition 4.19. *Suppose that $\mathbb{G}$ is strongly complete. Then $\partial_1(\mathcal{B}_1)$ is a minimal Gröbner basis and a minimal homogeneous system of generators of $I(\mathcal{L})$.*

Proof. Recall that $\partial_1(\mathcal{B}_1) = \{f_{0,1}, \dots, f_{0,r_1}\} = \{f_C \mid C \subseteq [n-1], C \neq \varnothing\}$. By Proposition 4.18, $\partial_1(\mathcal{B}_1)$ is a Gröbner basis of $I(\mathcal{L})$. Let us see that it is a minimal such basis (see, e.g. [9, § 2.7. Definition 4]). Fix f_C, $C \subseteq [n-1]$, $C \neq \varnothing$. Clearly, the leading coefficient of f_C is 1. Moreover, the leading term of f_C is $\mathbf{Lt}(f_C) = m_C = x^{C \to \overline{C}}$ (see Remark 3.6 and Lemma 4.10). Suppose that $\mathbf{Lt}(f_C)$ lies in the ideal $\langle \mathbf{Lt}(f_D) \mid D \subseteq [n-1], D \neq C, D \neq \varnothing \rangle$. Since the latter is a monomial ideal, it follows that $\mathbf{Lt}(f_D)$ divides $\mathbf{Lt}(f_C)$, for some $D \subseteq [n-1]$, $D \neq C$, $D \neq \varnothing$. Now, use that G is strongly complete, or equivalently, L is a PCB matrix. Therefore, $a_{i,j} > 0$, for all $i, j = 1, \dots, n$. Since

$$x^{D \to \overline{D}} = \prod_{i \in D} x_i^{\sum_{j \in \overline{D}} a_{i,j}} \qquad \text{divides} \qquad x^{C \to \overline{C}} = \prod_{i \in C} x_i^{\sum_{j \in \overline{C}} a_{i,j}}$$

and $\sum_{j \in \overline{D}} a_{i,j} > 0$, it follows that $D \subsetneq C$, and so $\overline{C}$ is a proper subset of $\overline{D}$. However, if $i \in D$, then the exponent of x_i in m_D is $\sum_{j \in \overline{D}} a_{i,j}$, whereas the exponent of x_i in m_C is $\sum_{j \in \overline{C}} a_{i,j}$, which is strictly smaller, because $\overline{C} \subsetneq \overline{D}$, a contradiction.

By Corollary 4.18, $\partial_1(\mathcal{B}_1)$ is a homogeneous system of generators for $\partial_1(\mathcal{C}_1) = I(\mathcal{L})$. Let us see that it is a minimal one. Suppose that

$$u_1 f_{0,1} + \cdots + u_{r_1} f_{0,r_1} = 0, \tag{25}$$

for some polynomials $u_i \in \mathbb{A} = \mathbb{K}[x]$. In other words, $(u_1, \dots, u_{r_1}) \in \mathbb{A}^{r_1}$ is in the kernel of $\varphi : \mathbb{A}^{r_1} \to \partial_1(\mathcal{C}_1)$, the free $\mathbb{A}$-linear presentation of $\partial_1(\mathcal{C}_1)$, sending each element of the canonical basis e_i to $f_{0,i}$. Write each u_i in the form $u_i = v_i + x_n w_i$, $v_i, w_i \in \mathbb{K}[x]$, where v_i contains no term involving x_n. Set $x_n = 0$ in the equality (25) above. Then we get

$$v_1 \mathbf{Lt}(f_{0,1}) + \cdots + v_{r_1} \mathbf{Lt}(f_{0,r_1}) = 0,$$

so $(v_1, \dots, v_{r_1})$ is a syzygy on the leading terms $\mathbf{Lt}(f_{0,1}), \dots, \mathbf{Lt}(f_{0,r_1})$. By [9, § 2.9. Proposition 8, see also Definition 5], $(v_1, \dots, v_{r_1})$ is of the form $\sum_{i<j} v_{i,j} S_{i,j}$, where

$$S_{i,j} = \frac{\mathrm{LCM}(\mathbf{Lm}(f_{0,j}), \mathbf{Lm}(f_{0,i}))}{\mathbf{Lt}(f_{0,i})} e_i - \frac{\mathrm{LCM}(\mathbf{Lm}(f_{0,i}), \mathbf{Lm}(f_{0,j}))}{\mathbf{Lt}(f_{0,j})} e_j.$$

We have $f_{0,i} = f_C$ and $f_{0,j} = f_D$, for some $C, D \subseteq [n-1]$, $C, D \neq \varnothing$ and $C \neq D$. Then

$$\frac{\mathrm{LCM}(\mathbf{L}\mathrm{m}(f_{0,j}), \mathbf{L}\mathrm{m}(f_{0,i}))}{\mathbf{L}\mathrm{t}(f_{0,i})} = \frac{\mathrm{LCM}(\mathbf{L}\mathrm{m}(f_D), \mathbf{L}\mathrm{m}(f_C))}{\mathbf{L}\mathrm{t}(f_C)} = m_{D,C} \text{ and}$$
$$\frac{\mathrm{LCM}(\mathbf{L}\mathrm{m}(f_{0,i}), \mathbf{L}\mathrm{m}(f_{0,j}))}{\mathbf{L}\mathrm{t}(f_{0,j})} = \frac{\mathrm{LCM}(\mathbf{L}\mathrm{m}(f_C), \mathbf{L}\mathrm{m}(f_D))}{\mathbf{L}\mathrm{t}(f_D)} = m_{C,D},$$

with the notations as in the equality (20). Writing $E = C \cap D$, $F = C \setminus E$, $G = D \setminus E$, $U = C \cup D$ and $V = \overline{U} = [n] \setminus U$ and using the equalities (21) and (22) in the proof of Lemma 4.15, then

$$m_{D,C} = x^{(E \to F, G)^+} x^{G \to F} x^{G \to V} \text{ and } m_{C,D} = x^{(E \to G, F)^+} x^{F \to G} x^{F \to V}.$$

Suppose that $C \not\subset D$ and $D \not\subset C$. Then F and G are non-empty. Since $a_{i,j} > 0$, for all $i, j = 1, \ldots, n$, we deduce that $x^{G \to F}$ and $x^{F \to G}$, and so $m_{D,C}$ and $m_{C,D}$, are in the irrelevant maximal ideal $\mathfrak{m}$. Suppose, without loss of generality, that $C \subsetneq D$. Then $m_{D,C} = x^{G \to V}$ and $m_{C,D} = x^{C \to G}$, where G, V and C are non-empty (see the last paragraph in the proof of Lemma 4.15). Again, we deduce that $m_{D,C}$ and $m_{C,D}$ are in $\mathfrak{m}$. Hence each u_i also lies in $\mathfrak{m}$ and $(u_1, \ldots, u_{r_1}) \in \mathfrak{m}\mathbb{A}^{r_1}$. On tensoring the short exact sequence

$$0 \to \ker(\varphi) \to \mathbb{A}^{r_1} \xrightarrow{\varphi} \partial_1(\mathcal{C}_1) \to 0$$

by $\mathbb{A}/\mathfrak{m} = k$, we deduce that the induced map $\overline{\varphi} : \mathbb{A}^{r_1}/\mathfrak{m}\mathbb{A}^{r_1} \to \partial_1(\mathcal{C}_1)/\mathfrak{m}\partial_1(\mathcal{C}_1)$ is an isomorphism of k-vector spaces. Applying the graded Nakayama Lemma, we deduce that $\partial_1(\mathcal{B}_1)$ is a minimal homogeneous system of generators for $\partial_1(\mathcal{C}_1)$ (see, e.g. [6, Exercise 1.5.24]). $\square$

Remark 4.20. The minimality for the strongly complete case will also follow from Corollary 6.16, since $\ker(\partial_1) = \partial_2(\mathcal{C}_2) \subset \mathfrak{m}\mathcal{C}_1$ and so $\mathcal{C}_1/\mathfrak{m}\mathcal{C}_1 \cong \partial_1(\mathcal{C}_1)/\mathfrak{m}\partial_1(\mathcal{C}_1)$.

4.7. Non-minimality in the non-strongly complete case. Let L be an ICB matrix or, equivalently, let $\mathbb{G}$ be a strongly connected digraph. It is easy to give an example where, if $\mathbb{G}$ is not strongly complete, then $\partial_1(\mathcal{B}_1)$ is not a minimal system of generators of $I(\mathcal{L})$.

Example 4.21. Consider the following ICB matrix L, which is not a PCB matrix:

$$L = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \text{ corresponding to } \mathbb{G}_L : \begin{tikzcd} & \bullet_2 \arrow[dl, "1"'] & \\ \bullet_1 \arrow[dr, "1"'] & & \bullet_3 \arrow[ul, "1"'] \\ & \bullet_4 \arrow[ur, "1"'] & \end{tikzcd}.$$

Then $\partial_1(\mathcal{B}_1) = \{x-t, y-t, xz-yt, x-z, z-t, y-z, x-y\}$, whereas $I(L) = (x-y, y-z, z-t, -x+t)$, which is clearly minimally generated by $\{x-y, y-z, z-t\}$. Thus $I(L)$ is a complete intersection ideal. In particular, it is unmixed and $I(L) = I(\mathcal{L})$. Therefore $\partial_1(\mathcal{B}_1)$ cannot be a minimal homogeneous system of generators of $I(\mathcal{L})$.

5. Computing syzygies of a Gröbner basis

5.1. Preliminary notations. Let us recall some preliminary notations. We follow [12, Section 2], [4] and [11, § 15].

Definitions 5.1. *Let $F = \mathbb{A}^r$ be the free $\mathbb{A} = \mathbb{K}[x]$-module of rank r, and let $e_1, \ldots, e_r$ be the canonical basis of F.*

(1) <u>Monomials</u>. *A monomial me_i in F is the product of a monomial m in $\mathbb{A} = \mathbb{K}[x]$ with a basis element e_i. A term λme_i in F is the product of a monomial me_i in F with a scalar λ in $\mathbb{K}$.*

(2) <u>Divisibility</u>. *A monomial m_1e_i divides a monomial m_2e_j if $i = j$ and m_1 divides m_2; in this case, m_2e_j/m_1e_i is defined as m_2/m_1. Similarly, a monomial m_1 divides m_2e_j if m_1 divides m_2 and in this case m_2e_j/m_1 is defined as $(m_2/m_1)e_j$. The least common multiple of two monomials m_1e_i and m_2e_j of F is $\mathrm{LCM}(m_1e_i, m_2e_j) = \mathrm{LCM}(m_1, m_2)e_i$, if $i = j$, or 0 otherwise.*

(3) <u>Orderings</u>. *A (global) monomial ordering on F is a total ordering $>$ on the set of monomials of F such that if m_1e_i and m_2e_j are monomials in F and m is a monomial in $\mathbb{A}$, then $m_1e_i > m_2e_j \Rightarrow (m\cdot m_1)e_i > (m\cdot m_2)e_j$. Moreover, $me_i > e_i$ for all i and all monomials $m \neq 1$, and we require in addition that $m_1e_i > m_2e_i \Leftrightarrow m_1e_j > m_2e_j$, for all i, j.*

(4) <u>Leading terms</u>. *Let $>$ be a monomial ordering on F and $f = \lambda me_i +$ lower order terms, $f \in F \smallsetminus \{0\}$, where $\lambda \neq 0$. Then the leading coefficient, the leading monomial and the leading term of f are $\mathbf{Lc}(f) = \lambda$, $\mathbf{Lm}(f) = me_i$ and $\mathbf{Lt}(f) = \lambda me_i$, respectively. For any subset $S \subset F$, the leading module of S is defined as $\mathbf{L}(S) := \langle \mathbf{Lm}(f) \mid f \in S \smallsetminus \{0\}\rangle$.*

Definitions 5.2. *Let $F_0 = \mathbb{A}^s$ be the free $\mathbb{A}$-module of rank s and let $G_0 = \{f_1, \ldots, f_r\}$ be a finite subset of $F_0\smallsetminus\{0\}$. Take $F_1 = \mathbb{A}^r$, the free $\mathbb{A}$-module of rank r, the cardinality of G_0, and let $\varphi_1 : F_1 \to F_0$ be defined by $\varphi_1(e_i) = f_i$.*

(5) <u>Syzygies</u>. *The (first) syzygy module of G_0 is $\mathrm{Syz}(G_0) := \ker(\varphi_1)$. An element of $\ker(\varphi_1)$ is called a syzygy of G_0.*

(6) <u>Induced orderings</u>. *Given a monomial ordering $>$ on F_0, the induced ordering on F_1 (w.r.t. $>$ and G_0) is the monomial ordering $>$ defined by $m_1e_i > m_2e_j \Leftrightarrow \mathbf{Lm}(m_1f_i) > \mathbf{Lm}(m_2f_j)$, or $\mathbf{Lm}(m_1f_i) = \mathbf{Lm}(m_2f_j)$ and $i > j$.*

(7) <u>The S-vectors</u>. *For $i, j \in \{1, \ldots, r\}$, the S-vector of f_i and f_j is defined as*

$$S(f_i, f_j) = m^1_{j,i}f_i - m^1_{i,j}f_j \in \langle G_0\rangle \subset F_0, \text{ where } m^1_{j,i} := \frac{\mathrm{LCM}(\mathbf{Lm}(f_j), \mathbf{Lm}(f_i))}{\mathbf{Lt}(f_i)}.$$

In particular, $\mathbf{Lt}(m^1_{j,i}f_i) = m^1_{j,i}\mathbf{Lt}(f_i) = \mathrm{LCM}(\mathbf{Lm}(f_j), \mathbf{Lm}(f_i))$, which, by symmetry, will be equal to $\mathbf{Lt}(m^1_{i,j}f_j)$. Therefore $\mathbf{Lm}(m^1_{j,i}f_i) = \mathbf{Lm}(m^1_{i,j}f_j) > \mathbf{Lm}(S(f_i, f_j))$.

(8) <u>Standard expression</u>. *Let $g \in F_0$. An equality $g = g_1f_1 + \cdots + g_rf_r + h$, with $g_i \in \mathbb{A}$ and $h \in F_0$, is a standard expression for g with remainder h (and w.r.t. $>$ and G_0) if the following two conditions are satisfied:*
   (a) *$\mathbf{Lm}(g) \geqslant \mathbf{Lm}(g_if_i)$, for all $i = 1, \ldots, r$, whenever both g and g_if_i are nonzero.*
   (b) *If h is nonzero, then $\mathbf{Lt}(h)$ is not divisible by any $\mathbf{Lt}(f_i)$.*

Discussion 5.3. Let $F_0 = \mathbb{A}^s$ be the free $\mathbb{A}$-module of rank s and let $G_0 = \{f_1, \ldots, f_r\}$ be a finite subset of $F_0 \smallsetminus \{0\}$. Take $F_1 = \mathbb{A}^r$, the free $\mathbb{A}$-module of rank r, the cardinality of G_0, and let $\varphi_1 : F_1 \to F_0$ be defined by $\varphi_1(e_i) = f_i$. Suppose that G_0 is a Gröbner basis w.r.t. some ordering $>$. By the Buchberger Criterion (see, e.g. [11, Theorem 15.8]), there are standard expressions with remainder zero:

$$S(f_i, f_j) = m^1_{j,i}f_i - m^1_{i,j}f_j = g^{(i,j)}_1 f_1 + \cdots + g^{(i,j)}_r f_r. \tag{26}$$

In particular, $\mathbf{Lm}(m^1_{j,i}f_i) = \mathbf{Lm}(m^1_{i,j}f_j) > \mathbf{Lm}(S(f_i, f_j)) \geqslant \mathbf{Lm}(g^{(i,j)}_s f_s)$, for all $s = 1, \ldots, r$, whenever both $S(f_i, f_j)$ and the particular $g^{(i,j)}_s f_s$ are nonzero.

Each such expression (26) defines a syzygy of G_0 in F_1, associated to the S-vector of f_i and f_j:

$$m^1_{j,i}e_i - m^1_{i,j}e_j - (g^{(i,j)}_1 e_1 + \cdots + g^{(i,j)}_r e_r). \tag{27}$$

This element will be called a τ-syzygy associated to the S-vector of f_i and f_j. Note that, unless the standard expression (26) is a "determinate division with reminder", the elements $g^{(i,j)}_s$ are not necessarily uniquely determined (see [4, Theorem 1.3], [10, Theorem 2.2.12]).

However, if $i > j$, since $\mathbf{L}\mathrm{m}(m^1_{j,i}f_i) = \mathbf{L}\mathrm{m}(m^1_{i,j}f_j)$, then $m^1_{j,i}e_i > m^1_{i,j}e_j$. Moreover, since $\mathbf{L}\mathrm{m}(m^1_{i,j}f_j) > \mathbf{L}\mathrm{m}(S(f_i,f_j)) \geqslant \mathbf{L}\mathrm{m}(g^{(i,j)}_s f_s)$, then $m^1_{i,j}e_j > \mathbf{L}\mathrm{m}(g^{(i,j)}_s)e_s$, for all $s = 1,\dots,r$, whenever the particular $g^{(i,j)}_s f_s$ is nonzero.

Thus, if $\sigma \in \ker(\varphi_1)$ is a τ-syzygy associated to the S-vector of two elements of G_0, obtained as described above, setting e_i (respectively, e_j) to be the unique basis element involved in $\mathbf{L}\mathrm{t}(\sigma)/\mathbf{L}\mathrm{c}(\sigma)$ (respectively, $\mathbf{L}\mathrm{t}(\sigma-\mathbf{L}\mathrm{t}(\sigma))/\mathbf{L}\mathrm{c}(\sigma-\mathbf{L}\mathrm{t}(\sigma))$), one deduces that σ is a τ-syzygy associated to the S-vector of $f_i = \varphi_1(e_i)$ and $f_j = \varphi_1(e_j)$, with $i > j$. In particular, $\mathbf{L}\mathrm{t}(\sigma) = m^1_{j,i}e_i$ (see [4, p.6], [10, p.68] or [12, Remark 3.6]).

A τ-syzygy associated to the S-vector of f_i and f_j will be denoted by $\tau_1(e_i,e_j)$. However, this notation can be somewhat misleading, in the following sense. Suppose that f_i, f_j, f_p, f_q are four elements of G_0, with $i,j,p,q \in \{1,\dots,r\}$. The equality $(i,j) = (p,q)$ does not ensure $\tau_1(e_i,e_j)$ is equal to $\tau_1(e_p,e_q)$, because of the non-uniqueness in the choice of the elements $g^{(i,j)}_s$ (unless "determinate division with remainder" is used, as mentioned above). Nevertheless, the equality $\tau_1(e_i,e_j) = \tau_1(e_p,e_q)$ does ensure that $e_i = e_p$ and that $e_j = e_q$, as seen above.

For ease of reference, we highlight the property:

$$\mathbf{L}\mathrm{t}(\tau_1(e_i,e_j)) = m^1_{j,i}e_i. \tag{28}$$

5.2. The Schreyer algorithm to compute the syzygies of a Gröbner basis. We recall the Schreyer algorithm to compute the syzygies of a Gröbner basis (see [12, Theorem 3.2 and Proposition 3.5], [10, Corollary 2.3.19], [4, Corollary 1.11], [11, Theorem 15.10, Corollary 15.11]). We will refer to [12] particularly for easy quoting. First we introduce some more notations.

Notation 5.4. Let $F_0 = \mathbb{A}^s$ be the free $\mathbb{A}$-module of rank s and let $G_0 = \{f_1,\dots,f_r\}$ be a finite subset of $F_0 \setminus \{0\}$. Recall that $m^1_{j,i} = \mathrm{LCM}(\mathbf{L}\mathrm{m}(f_j),\mathbf{L}\mathrm{m}(f_i))/\mathbf{L}\mathrm{t}(f_i)$. Let $\{M_i(G_0)\}_{1\leqslant i\leqslant r}$ be the set of ideals of $\mathbb{A} = \mathbb{K}[x]$ defined as follows: $M_1(G_0) = 0$, and for each $i = 2,\dots,r$,

$$\begin{aligned} M_i(G_0) &= (\mathbf{L}\mathrm{t}(f_1),\dots,\mathbf{L}\mathrm{t}(f_{i-1})) : \mathbf{L}\mathrm{t}(f_i) \\ &= \langle \mathbf{L}\mathrm{t}(f_1)\rangle : \mathbf{L}\mathrm{t}(f_i) + \cdots + \langle \mathbf{L}\mathrm{t}(f_{i-1})\rangle : \mathbf{L}\mathrm{t}(f_i). \end{aligned}$$

Note that, for each $j = 1,\dots,i-1$,

$$\begin{aligned} \langle \mathbf{L}\mathrm{t}(f_j)\rangle : \mathbf{L}\mathrm{t}(f_i) &= \langle \mathbf{L}\mathrm{m}(f_j)\rangle : \mathbf{L}\mathrm{m}(f_i) \\ &= \langle \frac{\mathrm{LCM}(\mathbf{L}\mathrm{m}(f_j),\mathbf{L}\mathrm{m}(f_i))}{\mathbf{L}\mathrm{m}(f_i)} \rangle \\ &= \langle \frac{\mathrm{LCM}(\mathbf{L}\mathrm{m}(f_j),\mathbf{L}\mathrm{m}(f_i))}{\mathbf{L}\mathrm{t}(f_i)} \rangle = (m^1_{j,i}). \end{aligned}$$

Therefore, $M_i(G_0) = (m^1_{1,i},\dots,m^1_{i-1,i})$. Observe that, in fact, the $M_i(G_0)$ are monomial ideals.

Theorem 5.5. ([12, Proposition 3.5]) *Let $F_0 = \mathbb{A}^s$ be the free $\mathbb{A}$-module of rank s and let $G_0 = \{f_1,\dots,f_r\}$ be a finite subset of $F_0 \setminus \{0\}$. Suppose that G_0 is a Gröbner basis* *of* $\langle G_0 \rangle$ *w.r.t. some monomial ordering* $>$ *on* F_0*. Let* $F_1 = \mathbb{A}^r$ *be the free* $\mathbb{A}$*-module of rank* r*, the cardinality of* G_0*, and with canonical basis* $e_1, \dots, e_r$*. Let* $\varphi_1 : F_1 \to F_0$ *be defined by* $\varphi_1(e_i) = f_i$*. For each* $i = 2, \dots, r$*, and for each minimal generator* x^α *of* $M_i(G_0)$*, choose exactly one index* $j = j(i, \alpha)$*, with* $1 \leqslant j < i$*, such that* $m^1_{j,i}$ *divides* x^α*, and compute a standard expression for* $S(f_i, f_j)$ *with remainder zero:*

$$S(f_i, f_j) = m^1_{j,i} f_i - m^1_{i,j} f_j = g_1^{(i,j)} f_1 + \cdots + g_r^{(i,j)} f_r.$$

Let $\tau_1(e_i, e_j) = m^1_{j,i} e_i - m^1_{i,j} e_j - (g_1^{(i,j)} e_1 + \cdots + g_r^{(i,j)} e_r) \in \ker(\varphi_1)$ *be the corresponding* τ*-syzygy associated to the* S*-vector of* f_i *and* f_j*. Then*

$$G_1 = \bigcup_{i=2}^{r} \{\tau_1(e_i, e_j) \,|\, x^\alpha \text{ a minimal generator of } M_i(G_0) \text{ and } j = j(i, \alpha) \text{ with } m^1_{j,i} | x^\alpha\}$$

is a Gröbner basis of $\ker(\varphi_1) \subset F_1$ *w.r.t. the monomial ordering on* F_1 *induced by* $>$ *and* G_0*. In particular,* $\ker(\varphi_1) = \langle G_1 \rangle$*.*

Remark 5.6. Fix $i \in \{2, \dots r\}$. As remarked above, $M_i(G_0) = (m^1_{j,i}, | \ 1 \leqslant j < i)$. By definition, x^α is a minimal generator of $M_i(G_0)$. Hence if we choose an index $j = j(i, \alpha)$ with $1 \leqslant j < i$ such that the generator $m^1_{j,i}$ of $M_i(G_0)$ divides x^α, then necessarily $m^1_{j,i}$ equals x^α up to a nonzero scalar. In the situation where the $m^1_{j,i}$ are monomials up to $\pm$, which will be the case in our applications of Theorem 5.5, then $m^1_{j,i} = \pm x^\alpha$.

Remark 5.7. Suppose that we have a τ-syzygy $\tau_1(e_i, e_j)$ such that $m^1_{j,i}$ is a non-minimal generator of $M_i(G_0)$ and let G_1 be as in Theorem 5.5. Then

$$\tilde{G}_1 = G_1 \cup \{\tau_1(e_i, e_j)\}$$

is still a Gröbner basis of $\ker(\varphi_1)$. This follows from the definition of Gröbner basis. Indeed, since G_1 is a Gröbner basis, then $\mathbf{L}(G_1) = \mathbf{L}(\ker(\varphi_1))$. Since $\tau_1(e_i, e_j) \in \ker(\varphi_1)$, then $\mathbf{L}\mathrm{t}(\tau_1(e_i, e_j)) \in \mathbf{L}(\ker(\varphi_1))$ and $\mathbf{L}(\tilde{G}_1)$ is still equal to $\mathbf{L}(\ker(\varphi_1))$.

Remark 5.8. We will consider in G_1 the following enumeration: let $\tau_1(e_p, e_q)$ and $\tau_1(e_i, e_j)$ be two elements of G_1, with $i, j, p, q \in \{1, \dots, r\}$, such that $(p, q) \neq (i, j)$. That is, $\tau_1(e_p, e_q) \neq \tau_1(e_i, e_j)$ (cf. Discussion 5.3). We will use the notation $\tau_1(e_p, e_q) \prec \tau_1(e_i, e_j)$ and say that "$\tau_1(e_p, e_q)$ precedes $\tau_1(e_i, e_j)$", or alternatively "$\tau_1(e_i, e_j)$ succeeds $\tau_1(e_p, e_q)$", whenever $p < i$, or whenever $p = i$ and $q < j$.

6. Exactness of the Cyc complex

6.1. Statement of the main result.

The main result of the paper is the following.

Theorem 6.1. *Let* $\mathbb{G}$ *be a strongly connected digraph of* n *vertices or, equivalently, let* L *be an* $n \times n$ ICB *matrix. Let* $\mathcal{C}_\mathbb{G}$ *be the associated* Cyc *complex. Then*

$$\mathcal{C}_\mathbb{G} : \ 0 \leftarrow \mathbb{K}[x]/I(\mathcal{L}) \leftarrow \mathcal{C}_0 = \mathbb{K}[x] \xleftarrow{\partial_1} \mathcal{C}_1 = \mathbb{K}[x]^{r_1} \xleftarrow{\partial_2} \cdots$$
$$\cdots \xleftarrow{\partial_{n-1}} \mathcal{C}_{n-1} = \mathbb{K}[x]^{r_{n-1}} \leftarrow 0$$

is a free resolution of $\mathbb{K}[x]/I(\mathcal{L})$*.*

Remark 6.2. Recall that we assume that $\mathbb{G}$ is a finite, weighted, directed graph, without loops, sources or sinks (cf. Assumption 2.2). In particular, its Laplacian matrix L is a CB matrix and, since $\mathbb{G}$ is strongly connected, then L is an ICB matrix (see Subsection 2.5). By Proposition 3.9, $\mathcal{C}_\mathbb{G}$ is a chain complex of free $\mathbb{A}$-modules. Moreover, we suppose in $\mathbb{A} = \mathbb{K}[x]$ the $\mathbb{N}$-grading given by $\nu(L)$ (cf. Notation 2.4 and

Assumption 2.7). Furthermore, we assume that $\mathbb{G}$ has a (ω,δ)-enumeration or, equivalently, that L is in δ-block echelon form (see 4.8). Finally, we suppose that $\mathcal{C}_0=\mathbb{K}[x]$ is endowed with the wrlo (Assumption 4.9).

Recall also that $\partial_0:\mathcal{C}_0=\mathbb{K}[x]\to\mathbb{K}[x]/I(\mathcal{L})$ is defined to be the natural projection onto the quotient ring (see Definition 3.7). Observe that we may interpret $\mathcal{C}_n=0$ and $\partial_n=0$.

Let us define a monomial ordering on each module $\mathcal{C}_k$ of the Cyc complex.

Remark 6.3. Recall that the elements of the enumerated free basis $\mathcal{B}_k$ of the free module $\mathcal{C}_k$ are denoted by $e_{k,1},\dots,e_{k,r_k}$, where $r_k=\operatorname{rank}(\mathcal{C}_k)=|\mathrm{Cyc}_{n,k+1}|$. Their images under ∂_k are denoted by $f_{k-1,1},\dots,f_{k-1,r_k}$, with $f_{k-1,j}:=\partial_k(e_{k,j})$ (Definition 3.3). Let

$$G_{k-1}:=\partial_k(\mathcal{B}_k)=\{\partial_k(e_{k,1}),\dots,\partial_k(e_{k,r_k})\}=\{f_{k-1,1},\dots,f_{k-1,r_k}\}\subset\mathcal{C}_{k-1}.$$

By Assumption 4.9, we endow $\mathcal{C}_0=\mathbb{K}[x]$ with the wrlo. Using Definitions 5.2, where $G_0=\partial_1(\mathcal{B}_1)$, with $\mathcal{B}_1$ a basis of $\mathcal{C}_1$ and $\partial_1:\mathcal{C}_1\to\mathcal{C}_0$, we endow $\mathcal{C}_1$ with the monomial ordering induced by the one in $\mathcal{C}_0$ and the subset G_0. Taking into account that $G_{k-1}=\partial_k(\mathcal{B}_k)$, with $\mathcal{B}_k$ a basis of $\mathcal{C}_k$ and $\partial_k:\mathcal{C}_k\to\mathcal{C}_{k-1}$, we can proceed recursively, and endow each $\mathcal{C}_k$ with the monomial ordering induced by the one in $\mathcal{C}_{k-1}$ and the subset G_{k-1}.

Notation 6.4. With the notations as above in Remark 6.3, set

$$m^k_{j,i}:=\frac{\mathrm{LCM}(\mathbf{L}\mathrm{m}(\partial_k(e_{k,j})),\mathbf{L}\mathrm{m}(\partial_k(e_{k,i})))}{\mathbf{L}\mathrm{t}(\partial_k(e_{k,i}))}=\frac{\mathrm{LCM}(\mathbf{L}\mathrm{m}(f_{k-1,j}),\mathbf{L}\mathrm{m}(f_{k-1,i}))}{\mathbf{L}\mathrm{t}(f_{k-1,i})}.$$

Similarly to Notation 5.4, $M_i(G_{k-1})$ are the monomial ideals of $\mathbb{A}=\mathbb{K}[x]$ defined as follows: $M_1(G_{k-1})=0$, and for each $i=2,\dots,r_{k-1}$,

$$\begin{aligned}&M_i(G_{k-1})=(\mathbf{L}\mathrm{t}(f_{k-1,1}),\dots,\mathbf{L}\mathrm{t}(f_{k-1,i-1})):\mathbf{L}\mathrm{t}(f_{k-1,i})=\\&\langle\mathbf{L}\mathrm{t}(f_{k-1,1})\rangle:\mathbf{L}\mathrm{t}(f_{k-1,i})+\dots+\langle\mathbf{L}\mathrm{t}(f_{k-1,i-1})\rangle:\mathbf{L}\mathrm{t}(f_{k-1,i})=(m^k_{1,i},\dots,m^k_{i-1,i}).\end{aligned}$$

Note that, if $\mathbf{L}\mathrm{m}(f_{k-1,j})$ and $\mathbf{L}\mathrm{m}(f_{k-1,i})$ are two monomials of $\mathcal{C}_{k-1}$ which are multiple of different basis elements, then $\langle\mathbf{L}\mathrm{t}(f_{k-1,j})\rangle:\mathbf{L}\mathrm{t}(f_{k-1,i})=0$. In such a case, we understand $m^k_{j,i}=0$ (see Definition 5.1)

Notation 6.5. We will prove Theorem 6.1 by induction on the degree of the component of the complex. We specify now the statements to be shown.

<u>Statement $\mathbf{H}_0$</u>:

(a) $G_0=\partial_1(\mathcal{B}_1)$ is a Gröbner basis of $\ker(\partial_0)\subset\mathcal{C}_0$.

(b) $\mathbf{L}\mathrm{t}(\partial_1(C,\overline{C}))=(-1)^0x^{C\to\overline{C}}(C\cup\overline{C})$, for each $(C,\overline{C})\in\mathcal{B}_1$.

<u>Statement $\mathbf{H}_k$</u>, for $1\leqslant k\leqslant n-2$ (recall that $n\geqslant 3$):

(a) $G_k=\partial_{k+1}(\mathcal{B}_{k+1})$ is a Gröbner basis of $\ker(\partial_k)\subset\mathcal{C}_k$ w.r.t. the monomial ordering defined as in Remark 6.3.

(b) $\mathbf{L}\mathrm{t}(\partial_{k+1}(I_1,\dots,I_{k+2}))=(-1)^kx^{I_{k+1}\to I_{k+2}}(I_1,\dots,I_{k+1}\cup I_{k+2})$, for each basis element $(I_1,\dots,I_{k+2})\in\mathcal{B}_{k+1}$.

<u>Statement $\mathbf{H}_{n-1}$</u>: $\partial_{n-1}:\mathcal{C}_{n-1}\to\partial_{n-2}$ is injective.

Remark 6.6. We already know that $\mathbf{H}_0$ holds. Indeed, by Corollary 4.18, $G_0=\partial_1(\mathcal{B}_1)$ is a Gröbner basis of $\operatorname{Im}(\partial_1)=I(\mathcal{L})=\ker(\partial_0)$. Moreover, by Lemma 4.10,

$$\mathbf{L}\mathrm{t}(\partial_1(C,\overline{C}))=\mathbf{L}\mathrm{t}(f_C)=m_C=x^{C\to\overline{C}}=(-1)^0x^{C\to\overline{C}}(C\cup\overline{C}),$$

where, according to Definition 3.3, $(C\cup\overline{C})\equiv([n])$ is identified with the unit element of $\mathcal{C}_0=\mathbb{K}[x]$.

Remark 6.7. If $\mathbf{H}_{n-2}$ holds, then $\mathbf{H}_{n-1}$ holds. Indeed, for each $i = 2, \ldots, r_{n-1}$, the basis element $e_{n-1,i} = (I_1, \ldots, I_n) \in \mathcal{B}_{n-1}$ is a cyclically ordered partition of $[n]$ into n blocks, with $n \in I_n$. Thus, $(I_1, \ldots, I_n) = (\{i_1\}, \ldots, \{i_{n-1}\}, \{n\})$, where $\{i_1, \ldots, i_{n-1}, n\} = [n]$. Any different basis element $e_{n-1,j} \in \mathcal{B}_{n-1}$, with $1 \leqslant j < i$, will be of the same form: $e_{n-1,j} = (\{j_1\}, \ldots, \{j_{n-1}\}, \{n\}) \in \mathcal{B}_{n-1}$. By part (b) of $\mathbf{H}_{n-2}$,

$$\begin{aligned}&\mathbf{L}\mathrm{t}(f_{n-1,i}) = \mathbf{L}\mathrm{t}(\partial_{n-1}(e_{n-1,i})) =\\ &(-1)^{n-2} x^{I_{n-1} \to \{n\}} (\{i_1\}, \ldots, \{i_{n-2}\}, \{i_{n-1}, n\})\end{aligned}$$

and, similarly,

$$\begin{aligned}&\mathbf{L}\mathrm{t}(f_{n-1,j}) = \mathbf{L}\mathrm{t}(\partial_{n-1}(e_{n-1,j})) =\\ &(-1)^{n-2} x^{J_{n-1} \to \{n\}} (\{j_1\}, \ldots, \{j_{n-2}\}, \{j_{n-1}, n\}).\end{aligned}$$

One sees that, since $e_{n-1,i}$ and $e_{n-1,j}$ are different, then $\mathbf{L}\mathrm{t}(f_{n-1,i})$ and $\mathbf{L}\mathrm{t}(f_{n-1}, j)$ are multiple of different basis elements. Thus $\langle \mathbf{L}\mathrm{t}(f_{n-1,j}) \rangle : \mathbf{L}\mathrm{t}(f_{n-1,i}) = 0$ and $M_i(G_{n-1}) = 0$, for all $i = 1, \ldots, r_{n-1}$ (see Notation 6.4). By Theorem 5.5, we deduce that $\ker(\partial_{n-1}) = 0$.

Remark 6.8. If $\mathbf{H}_k$ holds, for every $k = 0, \ldots, n-1$, then $\mathcal{C}_{\mathbb{G}}$ is an exact complex. Indeed, by Remark 6.6 above, $\mathrm{Im}(\partial_1) = \ker(\partial_0)$. For $1 \leqslant k \leqslant n-2$, by part (a) of $\mathbf{H}_k$,

$$\mathrm{Im}(\partial_{k+1}) = \langle \partial_{k+1}(\mathcal{B}_{k+1}) \rangle = \langle \text{Gröbner basis of } \ker(\partial_k) \rangle = \ker(\partial_k).$$

Finally, by definition, $\partial_n = 0$, so $\mathrm{Im}(\partial_n) = 0$. On the other hand, ∂_{n-1} is injective by $\mathbf{H}_{n-1}$.

Purpose 6.9. In the light of Remarks 6.6, 6.7 and 6.8, the proof of the main theorem is reduced to show "$\mathbf{H}_{k-1} \Rightarrow \mathbf{H}_k$, for $k = 1, \ldots, n-2$".

Before that, we need two important results. These are presented in the next two sections.

6.2. Identifying superfluous monomial generators of the module quotients. Given a basis element $e_{k,i} = (I_1, \ldots, I_k, I_{k+1}) \in \mathcal{B}_k$, we want to identify superfluous elements $m^k_{j,i}$ in $M_i(G_{k-1})$, where $1 \leqslant j < i \leqslant r_k$ (recall Notation 6.4 and Remark 5.6).

Lemma 6.10. *Fix an integer k, with $1 \leqslant k \leqslant n-2$. Suppose that $\mathbf{H}_{k-1}$ holds. Fix a basis element $e_{k,i} = (I_1, \ldots, I_k, I_{k+1})$ in $\mathcal{B}_k$. Let $e_{k,j} = (J_1, \ldots, J_k, J_{k+1}) \in \mathcal{B}_k$, for some $1 \leqslant j < i \leqslant r_k$.*

(a) *If for some $1 \leqslant s \leqslant k-1$, $J_s \neq I_s$, then $m^k_{j,i} = 0$.*

Suppose, on the contrary, that $e_{k,j} = (I_1, \ldots, I_{k-1}, J_k, J_{k+1})$, so that $J_k \cup J_{k+1} = I_k \cup I_{k+1}$.

(b) *Then,*

$$m^k_{j,i} = (-1)^{k-1} x^{(J_k \cap I_k \to J_{k+1}, I_{k+1})^+} x^{J_k \cap I_{k+1} \to J_{k+1}}.$$

In particular, if $J_k \supsetneq I_k$ (and so $J_{k+1} \subsetneq I_{k+1}$), then

$$m^k_{j,i} = (-1)^{k-1} x^{J_k \cap I_{k+1} \to J_{k+1}}. \tag{29}$$

(c) *If $J_k \not\supset I_k$, let $1 \leqslant l < j$ be such that $e_{k,l} = (I_1, \ldots, I_{k-1}, J_k \cup I_k, J_{k+1} \cap I_{k+1})$. Then $m^k_{l,i} \mid m^k_{j,i}$, so $m^k_{j,i}$ is superfluous.*

Proof. Note that (a) is vacuous if $k=1$. So, suppose momentarily, that $k \geqslant 2$ and that $J_s \neq I_s$, for some $1 \leqslant s \leqslant k-1$. By the hypothesis that $\mathbf{H}_{k-1}$ holds, we have that

$$\mathbf{L}\mathrm{t}(\partial_k(e_{k,j})) = (-1)^{k-1} x^{J_k \to J_{k+1}} (J_1, \dots, J_{k-1}, J_k \cup J_{k+1}) \tag{30}$$

and

$$\mathbf{L}\mathrm{t}(\partial_k(e_{k,i})) = (-1)^{k-1} x^{I_k \to I_{k+1}} (I_1, \dots, I_{k-1}, , I_k \cup I_{k+1}),$$

where $(J_1, \dots, J_k \cup J_{k+1}) \neq (I_1, \dots, I_k \cup I_{k+1})$. Then (a) follows from the fact that the least common multiple of two monomial terms is zero provided the two basis elements involved are different (see Definition 5.1, (2)).

Suppose that $e_{k,j} = (I_1, \dots, I_{k-1}, J_k, J_{k+1})$, with $J_k \cup J_{k+1} = I_k \cup I_{k+1}$. Note that this means that (J_k, J_{k+1}) and (I_k, I_{k+1}) are each partitions of the same set $J_k \cup J_{k+1} = I_k \cup I_{k+1}$. Now the two basis elements involved in $\mathbf{L}\mathrm{m}(\partial_k(e_{k,j}))$ and $\mathbf{L}\mathrm{m}(\partial_k(e_{k,i}))$ are the same (see (30) above). Therefore, using Definition 5.1, (2), again:

$$\begin{aligned} m^k_{j,i} &= \frac{\mathrm{LCM}(x^{J_k \to J_{k+1}}, x^{I_k \to I_{k+1}})(I_1, \dots, I_{k-1}, I_k \cup I_{k+1})}{(-1)^{k-1} x^{I_k \to I_{k+1}} (I_1, \dots, I_{k-1}, I_k \cup I_{k+1})} \\ &= \frac{\mathrm{LCM}(x^{J_k \to J_{k+1}}, x^{I_k \to I_{k+1}})}{(-1)^{k-1} x^{I_k \to I_{k+1}}}. \end{aligned}$$

Now, using (15), we see that

$$\begin{aligned} x^{J_k \to J_{k+1}} &= x^{J_k \cap I_k \to J_{k+1}} x^{J_k \cap I_{k+1} \to J_{k+1}} \text{ and} \\ x^{I_k \to I_{k+1}} &= x^{J_k \cap I_k \to I_{k+1}} x^{J_{k+1} \cap I_k \to I_{k+1}}. \end{aligned}$$

Thus

$$\begin{aligned} &\mathrm{LCM}(x^{J_k \to J_{k+1}}, x^{I_k \to I_{k+1}}) \\ &\quad = \mathrm{LCM}(x^{J_k \cap I_k \to J_{k+1}} x^{J_k \cap I_{k+1} \to J_{k+1}}, x^{J_k \cap I_k \to I_{k+1}} x^{J_{k+1} \cap I_k \to I_{k+1}}) \\ &\quad\quad = \mathrm{LCM}(x^{J_k \cap I_k \to J_{k+1}}, x^{J_k \cap I_k \to I_{k+1}}) x^{J_k \cap I_{k+1} \to J_{k+1}} x^{J_{k+1} \cap I_k \to I_{k+1}}. \end{aligned}$$

Recalling (17) in Notation 4.12, we deduce:

$$\frac{\mathrm{LCM}(x^{J_k \to J_{k+1}}, x^{I_k \to I_{k+1}})}{(-1)^{k-1} x^{I_k \to I_{k+1}}} = (-1)^{k-1} x^{(J_k \cap I_k \to J_{k+1}, I_{k+1})^+} x^{J_k \cap I_{k+1} \to J_{k+1}}.$$

In particular, if $J_k \supsetneq I_k$, then $J_{k+1} \subsetneq I_{k+1}$ and $x^{(J_k \cap I_k \to J_{k+1}, I_{k+1})^+} = 1$. Therefore

$$m^k_{j,i} = (-1)^{k-1} x^{J_k \cap I_{k+1} \to J_{k+1}}.$$

This proves (b). Suppose now that $J_k \not\supset I_k$, so $J_{k+1} \not\subset I_{k+1}$ and $J_{k+1} \cap I_{k+1} \subsetneq J_{k+1}$. Let $1 \leqslant l < j < i$ be such that $e_{k,l} = (I_1, \dots, I_{k-1}, J_k \cup I_k, J_{k+1} \cap I_{k+1})$. Using (b), we have that:

$$\begin{aligned} m^k_{l,i} &= (-1)^{k-1} x^{J_k \cap I_{k+1} \to J_{k+1} \cap I_{k+1}} \text{ and} \\ m^k_{j,i} &= (-1)^{k-1} x^{(J_k \cap I_k \to J_{k+1}, I_{k+1})^+} x^{J_k \cap I_{k+1} \to J_{k+1}}. \end{aligned}$$

In particular, $m^k_{l,i}$ divides $m^k_{j,i}$, which proves (c). □

The following is an easy remark but will prove to be very useful in organizing the basis elements of $\mathcal{B}_k$.

Remark 6.11. Fix an integer k, with $1 \leqslant k \leqslant n-2$. Let $e_{k,i} = (I_1, \dots, I_k, I_{k+1})$ be a basis element of $\mathcal{B}_k$, where $1 \leqslant i \leqslant r_k$. Set

$$\mathcal{B}_{k,i} := \{(I_1, \dots, I_{k-1}, J_k, J_{k+1}) \in \mathcal{B}_k \mid J_k \supsetneq I_k\}.$$

The following conditions hold.

(a) The set $\mathcal{B}_{k,i}$ is empty if and only if $I_{k+1} = \{n\}$. (For instance, for the first basis element

$$e_{k,1} = (\{k, \ldots, n-1\}, \{k-1\}, \ldots, \{2\}, \{1\}, \{n\})$$

of $\mathcal{B}_k$, considering the srle, we have $\mathcal{B}_{k,1} = \varnothing$.)

(b) The cardinality of $\mathcal{B}_{k,i}$ is $|\mathcal{B}_{k,i}| = 2^{|I_{k+1}|-1} - 1$. (This formula holds trivially if $I_{k+1} = \{n\}$.)

(c) If $\mathcal{B}_{k,i} \neq \varnothing$ and $e_{k,j} = (I_1, \ldots, I_{k-1}, J_k, J_{k+1})$ in $\mathcal{B}_{k,i}$, then $1 \leqslant j < i$. Moreover $\mathcal{B}_{k,j} \subsetneq \mathcal{B}_{k,i}$.

(d) If $\mathcal{B}_{k,i} \neq \varnothing$, let $\varrho_{k,i} : \mathcal{B}_{k,i} \to \mathcal{B}_{k+1}$ be defined by

$$\varrho_{k,i}(e_{k,j}) := (I_1, \ldots, I_k, J_k \smallsetminus I_k, J_{k+1}),$$

while if $\mathcal{B}_{k,i} = \varnothing$, we set $\varrho_{k,i}(\mathcal{B}_{k,i}) := \varnothing$. Then $\varrho_{k,i}$ is well-defined and injective.

(e) If $i \neq j$, then $\varrho_{k,i}(\mathcal{B}_{k,i})$ and $\varrho_{k,j}(\mathcal{B}_{k,j})$ are two disjoint sets.

(f) Then $\mathcal{B}_{k+1} = \bigcup_{i=2}^{r_k} \varrho_{k,i}(\mathcal{B}_{k,i})$. In particular, $r_{k+1} = \sum_{i=2}^{r_k} |\mathcal{B}_{k,i}|$.

Proof. Suppose that $I_{k+1} = \{n\}$ and let $(I_1, \ldots, I_{k-1}, J_k, J_{k+1}) \in \mathcal{B}_k$, with $J_k \supseteq I_k$. Then

$$J_k \cup J_{k+1} = [n] \smallsetminus \cup_{s=1}^{k-1} I_s = I_k \cup I_{k+1} = I_k \cup \{n\}.$$

Since $n \in J_{k+1}$, it follows that $J_k = I_k$ and so there does not exist any element in $\mathcal{B}_{k,i}$. On the other hand, if $I_{k+1} \supsetneq \{n\}$ and $m \in I_{k+1}$ with $m \neq n$, just take $J_k = I_k \cup \{m\}$ and $J_{k+1} = I_{k+1} \smallsetminus \{m\}$. Then $(I_1, \ldots, I_{k-1}, J_k, J_{k+1})$ is in $\mathcal{B}_{k,i}$. This proves (a).

Suppose that $\mathcal{B}_{k,i} \neq \varnothing$, i.e. $I_{k+1} \supsetneq \{n\}$. So $I_k \subsetneq [n-1] \smallsetminus \cup_{s=1}^{k-1} I_s$. Then the elements $(I_1, \ldots, I_{k-1}, J_k, J_{k+1})$ of $\mathcal{B}_{k,i}$ are in one-to-one correspondence with the non-empty subsets $J_k \smallsetminus I_k$ of the set $[n-1] \smallsetminus \cup_{s=1}^{k} I_s = I_{k+1} \smallsetminus \{n\}$. This proves ($b$).

Let $e_{k,j} = (I_1, \ldots, I_{k-1}, J_k, J_{k+1}) \in \mathcal{B}_{k,i}$, that is, $J_k \supsetneq I_k$. Then, according to the srle, $e_{k,j}$ precedes $e_{k,i}$ and so $j < i$. Moreover, if $e_{k,h} = (I_1, \ldots, I_{k-1}, H_k, H_{k+1})$ is in $\mathcal{B}_{k,j}$, then $H_k \supsetneq J_k \supsetneq I_k$, and so $e_{k,h} \in \mathcal{B}_{k,i}$. Furthermore, $e_{k,j} \in \mathcal{B}_{k,i}$, but $e_{k,j} \notin \mathcal{B}_{k,j}$. This proves ($c$).

Clearly, if $e_{k,j} = (I_1, \ldots, I_{k-1}, J_k, J_{k+1}) \in \mathcal{B}_{k,i}$, then $J_k \supsetneq I_k$ and $J_k \smallsetminus I_k \neq \varnothing$. Thus $\varrho_{k,i}(e_{k,j})$ is indeed in $\mathcal{B}_{k+1}$. Take $e_{k,j} = (I_1, \ldots, I_{k-1}, J_k, J_{k+1})$ and $e_{k,h} = (I_1, \ldots, I_{k-1}, H_k, H_{k+1})$ in $\mathcal{B}_{k,i}$ such that $\varrho_{k,i}(e_{k,j}) = \varrho_{k,i}(e_{k,h})$. Then $J_{k+1} = H_{k+1}$ and, so $J_k = (J_k \cup J_{k+1}) \smallsetminus J_{k+1} = (H_k \cup H_{k+1}) \smallsetminus H_{k+1} = H_k$, and $e_{k,j} = e_{k,h}$, which proves (d).

Next we prove (e). Let $e_{k,i} = (I_1, \ldots, I_k, I_{k+1})$ and $e_{k,j} = (J_1, \ldots, J_k, J_{k+1})$, with $\mathcal{B}_{k,i} \neq \varnothing$ and $\mathcal{B}_{k,j} \neq \varnothing$, respectively. Suppose that $e_{k,p} = (I_1, \ldots, I_{k-1}, P_k, P_{k+1})$ is in $\mathcal{B}_{k,i}$ and $e_{k,q} = (J_1, \ldots, J_{k-1}, Q_k, Q_{k+1})$ is in $\mathcal{B}_{k,j}$. Suppose also that $\varrho_{k,i}(e_{k,p}) = \varrho_{k,j}(e_{k,q})$. Then $I_s = J_s$, for $s = 1, \ldots, k$. In particular, $I_{k+1} = [n] \smallsetminus \cup_{s=1}^{k} I_s = [n] \smallsetminus \cup_{s=1}^{k} J_s = J_{k+1}$. Thus $e_{k,i} = e_{k,j}$. The rest follows by (d).

Finally, we prove that (f) holds. Consider $(H_1, \ldots, H_{k+2}) \in \mathcal{B}_{k+1}$. Let us find two basis elements $e_{k,i} = (I_1, \ldots, I_{k+1}) \in \mathcal{B}_k$ and $e_{k,j}$ in $\mathcal{B}_{k,i}$, so that $e_{k,j} = (I_1, \ldots, I_{k-1}, J_k, J_{k+1})$, with $J_k \supsetneq I_k$, and such that

$$(H_1, \ldots, H_{k+2}) = \varrho_{k,i}(e_{k,j}) = (I_1, \ldots, I_k, J_k \smallsetminus I_k, J_{k+1}).$$

This forces $I_s = H_s$, for $s = 1, \ldots, k$, $J_k \smallsetminus I_k = H_{k+1}$ and $J_{k+1} = H_{k+2}$. Therefore I_{k+1} must be equal to $[n] \smallsetminus \cup_{s=1}^{k} I_s = [n] \smallsetminus \cup_{s=1}^{k} H_s = H_{k+1} \cup H_{k+2}$ and $J_k = I_k \cup H_{k+1} = H_k \cup H_{k+1}$. Thus, both

$$e_{k,i} = (I_1, \ldots, I_k, I_{k+1}) = (H_1, \ldots, H_k, H_{k+1} \cup H_{k+2}) \in \mathcal{B}_k \text{ and}$$
$$e_{k,j} = (I_1, \ldots, I_{k-1}, J_k, J_{k+1}) = (H_1, \ldots, H_{k-1}, H_k \cup H_{k+1}, H_{k+2}) \in \mathcal{B}_{k,i},$$

are uniquely determined by $(H_1, \ldots, H_{k+2}) \in \mathcal{B}_{k+1}$. $\square$

Having in mind Notation 6.4, Theorem 5.5 and Remark 5.7, then the preceding Lemma 6.10 and Remark 6.11 can be summarised as follows. Note that Lemma 6.10 discards elements $m^k_{j,i}$ which are known to be superfluous for certain. However, it does not ensure that we have discarded all of them. In fact, we will see that this is precisely characterised by the digraph $\mathbb{G}$ being strongly complete (see Corollary 6.16).

Proposition 6.12. *Fix an integer k, with $1 \leqslant k \leqslant n-2$. Suppose that $\mathbf{H}_{k-1}$ holds. For each $i = 2, \dots, r_k$, then*

$$\{m^k_{j,i} \mid e_{k,j} \in \mathcal{B}_{k,i}\}$$

is a generating set for the module quotient $M_i(G_{k-1})$. Let $\tau_k(e_{k,i}, e_{k,j})$ stand for a τ-syzygy associated to the S-vector of $f_{k-1,i} = \partial_k(e_{k,i})$ and $f_{k-1,j} = \partial_k(e_{k,j})$, where $e_{k,j} \in \mathcal{B}_{k,i}$. Then there is a Gröbner basis of $\ker(\partial_k) \subset \mathcal{C}_k$ (w.r.t. the monomial ordering on $\mathcal{C}_k$ induced by the monomial ordering on $\mathcal{C}_{k-1}$ and the Gröbner basis G_{k-1}) of the form

$$\bigcup_{i=2}^{r_k} \{\tau_k(e_{k,i}, e_{k,j}) \mid e_{k,j} \in \mathcal{B}_{k,i}\}. \tag{31}$$

The cardinality of this Gröbner basis is $\sum_{i=2}^{r_k} |\mathcal{B}_{k,i}|$, which coincides with r_{k+1}, the rank of $\mathcal{C}_{k+1}$.

Proof. Indeed, Lemma 6.10 ensures that in order to find a generating set of $M_i(G_{k-1})$ we only need to consider the $m^k_{j,i}$ with $e_{k,j} \in \mathcal{B}_{k,i}$. By Theorem 5.5 and Remark 5.7, we deduce that indeed there is a Gröbner basis of $\ker(\partial_k)$ of the form (31).

By Discussion 5.3, we know that if $\tau_k(e_{k,i}, e_{k,j}) = \tau_k(e_{k,p}, e_{k,q})$, then $e_{k,i} = e_{k,p}$ and $e_{k,j} = e_{k,q}$. In particular, the union in (31) is of disjoint sets and

$$|\{\tau_k(e_{k,i}, e_{k,j}) \mid e_{k,j} \in \mathcal{B}_{k,i}\}| = |\mathcal{B}_{k,i}|.$$

Thus, the cardinality of this Gröbner basis is $\sum_{i=2}^{r_k} |\mathcal{B}_{k,i}|$. By Remark 6.11, (f), $\sum_{i=2}^{r_k} |\mathcal{B}_{k,i}| = r_{k+1}$. □

Remark 6.13. With the notations above, if $\mathbb{G}$ is strongly complete or, equivalently, if L is a PCB matrix, then

$$|\mathcal{B}_{k,i}| = |\{m^k_{j,i} \mid e_{k,j} \in \mathcal{B}_{k,i}\}|.$$

For in the strongly complete case, the map $\mathcal{B}_{k,i} \to \mathcal{C}_{k-1}$ defined by $e_{k,j} \mapsto m^k_{j,i}$ is injective. Indeed, if $e_{k,j} \in \mathcal{B}_{k,i}$, then $m^k_{j,i} = (-1)^{k-1} x^{J_k \cap I_{k+1} \to J_{k+1}}$ (see (29) in Lemma 6.10), where $e_{k,i} = (I_1, \dots, I_{k+1})$ and $e_{k,j} = (I_1, \dots, I_{k-1}, J_k, J_{k+1})$, with $J_k \supsetneq I_k$. Since $a_{i,j} > 0$, for all i, j, it follows that $m^k_{j,i}$ determines $J_k \cap I_{k+1}$. Hence $J_k = (J_k \cap I_k) \cup (J_k \cap I_{k+1}) = I_k \cup (J_k \cap I_{k+1})$, and so J_{k+1} and hence $e_{k,j}$ are uniquely determined by $m^k_{j,i}$.

6.3. The images of basis elements are syzygies associated to S-vectors.

Let us show that the differential ∂_{k+1} of the Cyc complex maps each basis element of $\mathcal{B}_{k+1}$ to a τ-syzygy associated to an S-vector, up to a $\pm$ sign (see Discussion 5.3).

Proposition 6.14. *Fix an integer k, with $1 \leqslant k \leqslant n-2$. Suppose that $\mathbf{H}_{k-1}$ holds. Fix a basis element $(I_1, \dots, I_{k+2})$ in $\mathcal{B}_{k+1}$. Then*

$$\partial_{k+1}(I_1, \dots, I_{k+2}) = -\tau_k(e_{k,i}, e_{k,j}),$$

where $e_{k,i} = (I_1, \dots, I_{k+1} \cup I_{k+2})$ and $e_{k,j} = (I_1, \dots, I_k \cup I_{k+1}, I_{k+2})$. That is, $-\partial_{k+1}(I_1, \dots, I_{k+2})$ is a τ-syzygy associated to the S-vector of $f_{k-1,i} = \partial_k(e_{k,i})$ and $f_{k-1,j} = \partial_k(e_{k,j})$.

Proof. Note that, according to the srle, $e_{k,j}$ does indeed precede $e_{k,i}$ (see Remark 3.1) and it follows from the definition that $e_{k,j} \in \mathcal{B}_{k,i}$ (see Remark 6.11). By Definition 3.3, we have:

$$\begin{aligned}\partial_{k+1}(I_1,\dots,I_{k+2}) =&(-1)^k x^{I_{k+1}\to I_{k+2}}(I_1,\dots,I_{k+1}\cup I_{k+2})\\ &+(-1)^{k-1}x^{I_k\to I_{k+1}}(I_1,\dots,I_k\cup I_{k+1},I_{k+2})\\ &+\sum_{s=1}^{k-1}(-1)^{s-1}x^{I_s\to I_{s+1}}(I_1,\dots,I_s\cup I_{s+1},\dots,I_{k+2})\\ &-x^{I_{k+2}\to I_1}(I_2,\dots,I_{k+1},I_1\cup I_{k+2}).\end{aligned}$$

Set $e_{k,j_s} := (I_1,\dots,I_s\cup I_{s+1},\dots,I_{k+2})$, for some j_s, with $1\leqslant j_1<\cdots<j_s<\cdots<j_{k-1}<j<i$. Moreover, set $e_{k,l} := (I_2,\dots,I_{k+1},I_1\cup I_{k+2})$, for some $l\in\{1,\dots,r_k\}$. Then

$$\begin{aligned}\partial_{k+1}(I_1,\dots,I_{k+2}) &= (-1)^k x^{I_{k+1}\to I_{k+2}}e_{k,i}\\ &+(-1)^{k-1}x^{I_k\to I_{k+1}}e_{k,j}+\sum_{s=1}^{k-1}(-1)^{s-1}x^{I_s\to I_{s+1}}e_{k,j_s}-x^{I_{k+2}\to I_1}e_{k,l}.\end{aligned}\tag{32}$$

Since $\partial_k\circ\partial_{k+1}=0$, and $\partial_k(e_{k,u})=f_{k-1,u}$, for each u, we obtain:

$$\begin{aligned}(-1)^{k-1}x^{I_{k+1}\to I_{k+2}}f_{k-1,i}&-(-1)^{k-1}x^{I_k\to I_{k+1}}f_{k-1,j}\\ &=\sum_{s=1}^{k-1}(-1)^{s-1}x^{I_s\to I_{s+1}}f_{k-1,j_s}-x^{I_{k+2}\to I_1}f_{k-1,l}.\end{aligned}\tag{33}$$

Let us see that the left hand side of (33) coincides with the S-vector of $f_{k-1,i}$ and $f_{k-1,j}$.

Since $\mathbf{H}_{k-1}$ holds, we have

$$\begin{aligned}\mathbf{L}\mathrm{t}(f_{k-1,j})&=\mathbf{L}\mathrm{t}(\partial_k(e_{k,j}))=\mathbf{L}\mathrm{t}(\partial_k(I_1,\dots,I_k\cup I_{k+1},I_{k+2}))\\ &=(-1)^{k-1}x^{I_k\cup I_{k+1}\to I_{k+2}}(I_1,\dots,I_k\cup I_{k+1}\cup I_{k+2}).\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{L}\mathrm{t}(f_{k-1,i})&=\mathbf{L}\mathrm{t}(\partial_k(e_{k,i}))=\mathbf{L}\mathrm{t}(\partial_k(I_1,\dots,I_{k+1}\cup I_{k+2}))\\ &=(-1)^{k-1}x^{I_k\to I_{k+1}\cup I_{k+2}}(I_1,\dots,I_k\cup I_{k+1}\cup I_{k+2}).\end{aligned}$$

Then

$$\begin{aligned}&\mathrm{LCM}(\mathbf{L}\mathrm{m}(f_{k-1,j}),\mathbf{L}\mathrm{m}(f_{k-1,i}))\\ &\qquad=\mathrm{LCM}(x^{I_k\cup I_{k+1}\to I_{k+2}},x^{I_k\to I_{k+1}\cup I_{k+2}})(I_1,\dots,I_k\cup I_{k+1}\cup I_{k+2}).\end{aligned}$$

Using (15) in Notation 4.12,

$$x^{I_k\cup I_{k+1}\to I_{k+2}}=x^{I_k\to I_{k+2}}x^{I_{k+1}\to I_{k+2}}\text{ and }x^{I_k\to I_{k+1}\cup I_{k+2}}=x^{I_k\to I_{k+1}}x^{I_k\to I_{k+2}}.$$

Therefore,

$$\mathrm{LCM}(x^{I_k\cup I_{k+1}\to I_{k+2}},x^{I_k\to I_{k+1}\cup I_{k+2}})=x^{I_k\to I_{k+2}}x^{I_{k+1}\to I_{k+2}}x^{I_k\to I_{k+1}}.$$

Hence

$$m^k_{j,i}=\frac{\mathrm{LCM}(x^{I_k\cup I_{k+1}\to I_{k+2}},x^{I_k\to I_{k+1}\cup I_{k+2}})}{(-1)^{k-1}x^{I_k\to I_{k+1}\cup I_{k+2}}}=(-1)^{k-1}x^{I_{k+1}\to I_{k+2}},$$

and, similarly, $m_{i,j}^k = (-1)^{k-1}x^{I_k \to I_{k+1}}$. Therefore, the S-vector of $f_{k-1,i}$ and $f_{k-1,j}$ is equal to

$$\begin{aligned}S(f_{k-1,i}, f_{k-1,j}) &= m_{j,i}^k f_{k-1,i} - m_{i,j}^k f_{k-1,j} \\ &= (-1)^{k-1}x^{I_{k+1}\to I_{k+2}} f_{k-1,i} - (-1)^{k-1}x^{I_k \to I_{k+1}} f_{k-1,j},\end{aligned}$$

which is the left hand side of (33).

Now, let us prove that the right hand side of (33) is a standard expression for the S-vector $S(f_{k-1,i}, f_{k-1,j})$ w.r.t. the Gröbner basis $G_{k-1} = \{f_{k-1,1}, \dots, f_{k-1,r_k}\}$, with remainder zero. Note that G_{k-1} is indeed a Gröbner basis of $\ker(\partial_{k-1})$ due to hypothesis $\mathbf{H}_{k-1}$. First, observe that the only coincident terms among the summands of

$$m_{j,i}^k f_{k-1,i} = (-1)^{k-1}x^{I_{k+1}\to I_{k+2}} f_{k-1,i} \quad \text{and} \quad m_{i,j}^k f_{k-1,j} = (-1)^{k-1}x^{I_k \to I_{k+1}} f_{k-1,j}$$

are precisely the leading terms. Indeed:

$$\begin{aligned}m_{j,i}^k f_{k-1,i} &= (-1)^{k-1}x^{I_{k+1}\to I_{k+2}} f_{k-1,i} = (-1)^{k-1}x^{I_{k+1}\to I_{k+2}}\partial_k(e_{k,i}) \\ &= (-1)^{k-1}x^{I_{k+1}\to I_{k+2}}\partial_k(I_1, \dots, I_{k+1}\cup I_{k+2}) \\ &= (-1)^{k-1}x^{I_{k+1}\to I_{k+2}}\sum_{s=1}^{k-1}(-1)^{s-1}x^{I_s\to I_{s+1}}(I_1, \dots \\ &\qquad\qquad \dots, I_s\cup I_{s+1}, \dots, I_{k+1}\cup I_{k+2}) \\ &\quad + \underline{(-1)^{k-1}x^{I_{k+1}\to I_{k+2}}(-1)^{k-1}x^{I_k\to I_{k+1}\cup I_{k+2}}(I_1, \dots, I_k\cup I_{k+1}\cup I_{k+2})} \\ &\quad - (-1)^{k-1}x^{I_{k+1}\to I_{k+2}}x^{I_{k+1}\cup I_{k+2}\to I_1}(I_2, \dots, I_1\cup I_{k+1}\cup I_{k+2})\end{aligned} \tag{34}$$

whereas

$$\begin{aligned}m_{i,j}^k f_{k-1,j} &= (-1)^{k-1}x^{I_k\to I_{k+1}} f_{k-1,j} = (-1)^{k-1}x^{I_k\to I_{k+1}}\partial_k(e_{k,j}) \\ &= (-1)^{k-1}x^{I_k\to I_{k+1}}\partial_k(I_1, \dots, I_k\cup I_{k+1}, I_{k+2}) \\ &= (-1)^{k-1}x^{I_k\to I_{k+1}}\sum_{s=1}^{k-2}(-1)^{s-1}x^{I_s\to I_{s+1}}(I_1, \dots \\ &\qquad\qquad \dots, I_s\cup I_{s+1}, \dots I_k\cup I_{k+1}, I_{k+2}) \\ &\quad + (-1)^{k-1}x^{I_k\to I_{k+1}}(-1)^{k-2}x^{I_{k-1}\to I_k\cup I_{k+1}}(I_1, \dots, I_{k-1}\cup I_k\cup I_{k+1}, I_{k+2}) \\ &\quad + \underline{(-1)^{k-1}x^{I_k\to I_{k+1}}(-1)^{k-1}x^{I_k\cup I_{k+1}\to I_{k+2}}(I_1, \dots, I_k\cup I_{k+1}\cup I_{k+2})} \\ &\quad - (-1)^{k-1}x^{I_k\to I_{k+1}}x^{I_{k+2}\to I_1}(I_2, \dots, I_k\cup I_{k+1}, I_1\cup I_{k+2})\end{aligned} \tag{35}$$

Excluding the (underlined) coincident leading terms (note that hypothesis $\mathbf{H}_{k-1}$ holds), the basis elements involved in (34) end either in $I_{k+1}\cup I_{k+2}$ or else in $I_1\cup I_{k+1}\cup I_{k+2}$, while the basis elements involved in (35) end either in I_{k+2} or else in $I_1\cup I_{k+2}$. Therefore, the S-vector of $f_{k-1,i}$ and $f_{k-1,j}$ cancels the leading terms of these expressions, but no other summand of $m_{j,i}^k f_{k-1,i}$ is cancelled by any other summand of $m_{i,j}^k f_{k-1,j}$. In particular,

$$\mathbf{L}\mathrm{m}(S(f_{k-1,i}, f_{k-1,j})) \geqslant \mathbf{L}\mathrm{m}(m_{j,i}^k f_{k-1,i} - \mathbf{L}\mathrm{t}(m_{j,i}^k f_{k-1,i})). \tag{36}$$

Now, we need to identify the leading monomials of the summands on the right hand side of (33). For $1 \leqslant s \leqslant k-1$, since the hypothesis $\mathbf{H}_{k-1}$ holds,

$$\begin{aligned}(37)\quad \mathbf{Lm}(x^{I_s \to I_{s+1}} f_{k-1,j_s}) &= x^{I_s \to I_{s+1}} \mathbf{Lm}(\partial_k(e_{k,j_s}))\\ &= x^{I_s \to I_{s+1}} \mathbf{Lm}(\partial_k(I_1, \dots, I_s \cup I_{s+1}, \dots, I_{k+2}))\\ &= x^{I_s \to I_{s+1}} x^{I_{k+1} \to I_{k+2}} (I_1, \dots, I_s \cup I_{s+1}, \dots, I_{k+1} \cup I_{k+2}).\end{aligned}$$

Similarly, the leading monomial of the last summand on the right hand side of (33) is:

$$\begin{aligned}(38)\qquad \mathbf{Lm}(x^{I_{k+2} \to I_1} f_{k-1,l}) &= x^{I_{k+2} \to I_1} \mathbf{Lm}(\partial_k(e_{k,l}))\\ &= x^{I_{k+2} \to I_l} \mathbf{Lm}(\partial_k(I_2, \dots, I_{k+1}, I_1 \cup I_{k+2}))\\ &= x^{I_{k+2} \to I_1} x^{I_{k+1} \to I_1 \cup I_{k+2}} (I_2, \dots, I_1 \cup I_{k+1} \cup I_{k+2}).\end{aligned}$$

All the monomials terms in (37) and in (38) appear in (34) and are different from the underlined leading term $\mathbf{Lt}(m^k_{j,i} f_{k-1,i})$. (By (15), $x^{I_{k+2} \to I_1} x^{I_{k+1} \to I_1 \cup I_{k+2}} = x^{I_{k+1} \to I_{k+2}} x^{I_{k+1} \cup I_{k+2} \to I_1}$.) Thus

$$(39)\quad \mathbf{Lm}(m^k_{j,i} f_{k-1,i} - \mathbf{Lt}(m^k_{j,i} f_{k-1,i})) \geqslant \mathbf{Lm}(x^{I_s \to I_{s+1}} f_{k-1,j_s}), \mathbf{Lm}(x^{I_{k+2} \to I_1} f_{k-1,l}).$$

Concatenating (36) and (39), we deduce that the right hand side of (33) is indeed a standard expression for $S(f_{k-1,i}, f_{k-1,j})$ with remainder zero and w.r.t. the Gröbner basis G_{k-1}.

Hence, the corresponding τ-syzygy $\tau_k(e_{k,i}, e_{k,j})$ associated to the S-vector of the elements $f_{k-1,i} = \partial_k(e_{k,i})$ and $f_{k-1,j} = \partial_k(e_{k,j})$ is

$$\begin{aligned}(40)\qquad \tau_k(e_{k,i}, e_{k,j}) &= (-1)^{k-1} x^{I_{k+1} \to I_{k+2}} e_{k,i}\\ &-(-1)^{k-1} x^{I_k \to I_{k+1}} e_{k,j} - \sum_{s=1}^{k-1} (-1)^{s-1} x^{I_s \to I_{s+1}} e_{k,j_s} + x^{I_{k+2} \to I_1} e_{k,l}.\end{aligned}$$

Comparing (40) with (32), we get $\partial_{k+1}(I_1, \dots, I_{k+2}) = -\tau_k(e_{k,i}, e_{k,j})$, the desired equality. □

As an immediate consequence of Proposition 6.14, we get the following result. The second part can be seen as a natural generalisation of Lemma 4.14.

Corollary 6.15. *Fix an integer k, with $1 \leqslant k \leqslant n-2$. Suppose that $\mathbf{H}_{k-1}$ holds. Then*

(*a*) $\mathbf{Lt}(\partial_{k+1}(I_1, \dots, I_{k+1}, I_{k+2})) = (-1)^k x^{I_{k+1} \to I_{k+2}} (I_1, \dots, I_k, I_{k+1} \cup I_{k+2})$.

(*b*) *Moreover, the restriction of the map $\partial_{k+1} : \mathcal{C}_{k+1} \to \mathcal{C}_k$ over the basis set $\mathcal{B}_{k+1}$ is injective. In particular, since $G_k := \partial_{k+1}(\mathcal{B}_{k+1})$, then*

$$|G_k| = |\partial_{k+1}(\mathcal{B}_{k+1})| = |\mathcal{B}_{k+1}| = r_{k+1}.$$

Proof. By Proposition 6.14, and with the notations used there, and by (28) in Discussion 5.3,

$$\begin{aligned}\mathbf{Lt}(\partial_{k+1}(I_1, \dots, I_{k+2})) &= -\mathbf{Lt}(\tau_k(e_{k,i}, e_{k,j})) = -m^k_{j,i} e_{k,i}\\ &= (-1)(-1)^{k-1} x^{I_{k+1} \to I_{k+2}} (I_1, \dots, I_{k+1} \cup I_{k+2})\\ &= (-1)^k x^{I_{k+1} \to I_{k+2}} (I_1, \dots, I_{k+1} \cup I_{k+2}),\end{aligned}$$

which proves (*a*).

Suppose that $\partial_{k+1}(I_1, \dots, I_{k+2}) = \partial_{k+1}(J_1, \dots, J_{k+2})$. Hence their leading terms coincide. Using the first part (*a*), it follows that

$$(-1)^k x^{I_{k+1} \to I_{k+2}} (I_1, \dots, I_k, I_{k+1} \cup I_{k+2}) = (-1)^k x^{J_{k+1} \to J_{k+2}} (J_1, \dots, J_k, J_{k+1} \cup J_{k+2}).$$

Therefore, $I_s = J_s$, for every $1 \leqslant s \leqslant k$, and moreover, $I_{k+1} \cup I_{k+2} = J_{k+1} \cup J_{k+2}$. Consider the summand

$$(-1)^{k-1} x^{I_k \to I_{k+1}} (I_1, \dots, I_{k-1}, I_k \cup I_{k+1}, I_{k+2})$$

of $\partial_{k+1}(I_1, \dots, I_{k+2})$, preceding the leading term and different from

$$-x^{I_{k+2} \to I_1} (I_2, \dots, I_{k+1}, I_1 \cup I_{k+2})$$

(see (5) in Definition 3.3 and Remark 3.4). On the one hand, this element coincides with

$$(-1)^{k-1} x^{I_k \to I_{k+1}} (J_1, \dots, J_{k-1}, I_k \cup I_{k+1}, I_{k+2}).$$

On the other hand, this element must coincide with a summand of $\partial_{k+1}(J_1, \dots, J_{k+2})$. It follows that either $I_k \cup I_{k+1} = J_k \cup J_{k+1}$ and $I_{k+2} = J_{k+2}$, or else, $I_k \cup I_{k+1} = J_k$ and $I_{k+2} = J_{k+1} \cup J_{k+2}$. However, the second case cannot occur, because, since $I_k = J_k$, it would follow that $I_{k+1} = \varnothing$, a contradiction. Thus the first case holds, which clearly implies $I_{k+1} = J_{k+1}$ and $I_{k+2} = J_{k+2}$. □

6.4. Proof of the main theorem. Now, we have all the ingredients to prove Theorem 6.1.

Proof. Acording to Purpose 6.9, we have to show "$\mathbf{H}_{k-1} \Rightarrow \mathbf{H}_k$, for $k = 1, \dots, n-2$". So fix an integer k, with $1 \leqslant k \leqslant n-2$ and suppose that $\mathbf{H}_{k-1}$ holds. Let us prove that $\mathbf{H}_k$ holds. By Proposition 6.12, we know that there is a Gröbner basis $\tilde{G}_k$, say, of $\ker(\partial_k)$ of the form:

$$\tilde{G}_k := \bigcup_{i=2}^{r_k} \{\tau_k(e_{k,i}, e_{k,j}) \mid e_{k,j} \in \mathcal{B}_{k,i}\}. \tag{41}$$

This is a Gröbner basis of $\ker(\partial_k)$ w.r.t. the monomial ordering on $\mathcal{C}_k$ induced by the monomial ordering on $\mathcal{C}_{k-1}$ and the Gröbner basis G_{k-1} (see also Remark 6.3). Note that Proposition 6.12 ensures that the cardinality of $\tilde{G}_k$ is r_{k+1}.

In Proposition 6.14, we have shown that, for any $(I_1, \dots, I_{k+2}) \in \mathcal{B}_{k+1}$,

$$\partial_{k+1}(I_1, \dots, I_{k+2}) = -\tau_k(e_{k,i}, e_{k,j}),$$

where $e_{k,i} = (I_1, \dots, I_{k+1} \cup I_{k+2})$ and $e_{k,j} = (I_1, \dots, I_k \cup I_{k+1}, I_{k+2})$. That is, $-\partial_{k+1}(I_1, \dots, I_{k+2})$ is a τ-syzygy associated to the S-vector of $f_{k-1,i} = \partial_k(e_{k,i})$ and $f_{k-1,j} = \partial_k(e_{k,j})$. In other words, any element of $-G_k = -\partial_{k+1}(\mathcal{B}_{k+1})$ is an element of $\tilde{G}_k$. Observe that, by Corollary 6.15, (b), the cardinality of G_k is also r_{k+1}. Since the set $-G_k$ is included in the Gröbner basis $\tilde{G}_k$ of $\ker(\partial_k)$, and both G_k and $\tilde{G}_k$ have the same cardinality, it follows that $G_k = \partial_{k+1}(\mathcal{B}_{k+1})$ is a Gröbner basis of $\ker(\partial_k)$.

(To avoid the preceding discussion using cardinalities, we can argue as follows. Take $\tau_k(e_{k,i}, e_{k,j})$, any element in $\tilde{G}_k$: that is, a τ-syzygy associated to the S-vector of $f_{k-1,i} = \partial_k(e_{k,i})$ and $f_{k-1,j} = \partial_k(e_{k,j})$, with $e_{k,j} \in \mathcal{B}_{k,i}$. Remark that, by Discussion 5.3, both basis elements $e_{k,i}$ and $e_{k,j}$ are uniquely determined by the syzygy $\tau_k(e_{k,i}, e_{k,j})$. Write $e_{k,i} = (I_1, \dots, I_k, I_{k+1})$ and $e_{k,j} = (I_1, \dots, I_{k-1}, J_k, J_{k+1})$, with $J_k \supsetneq I_k$. Consider the map $\varrho_{k,i} : \mathcal{B}_{k,i} \to \mathcal{B}_{k+1}$ introduced in Remark 6.11. Take the element

$$\varrho_{k,i}(e_{k,j}) = (I_1, \dots, I_k, J_k \smallsetminus I_k, J_{k+1}) =: (H_1, \dots, H_k, H_{k+1}, H_{k+2}) \in \mathcal{B}_{k+1}.$$

Observe that $H_s = I_s$, for all $s = 1, \dots, k$, that $H_{k+1} = J_k \smallsetminus I_k$ and that $H_{k+2} = J_{k+1}$. Thus $H_{k+1} \cup H_{k+2} = (J_k \smallsetminus I_k) \cup J_{k+1} = I_{k+1}$ and $H_k \cup H_{k+1} = I_k \cup (J_k \smallsetminus I_k) = J_k$.

Hence

$$(H_1, \ldots, H_k, H_{k+1} \cup H_{k+2}) = (I_1, \ldots, I_k, I_{k+1}) = e_{k,i} \text{ and}$$
$$(H_1, \ldots, H_{k-1}, H_k \cup H_{k+1}, H_{k+2}) = (I_1, \ldots, I_{k-1}, J_k, J_{k+1}) = e_{k,j}.$$

Then, by Proposition 6.14 again, $\partial_{k+1}(\varrho_{k,i}(e_{k,j})) = -\tau_k(e_{k,i}, e_{k,j})$. That is, any element of $\tilde{G}_k$ is an element of $-G_k = -\partial_{k+1}(\mathcal{B}_{k+1})$. In conclusion, $G_k = \partial_{k+1}(\mathcal{B}_{k+1})$ is a Gröbner basis of $\ker(\partial_k)$.)

Since $\mathbf{H}_{k-1}$ holds, it follows that part (a) of $\mathbf{H}_k$ also holds (recall Notation 6.5). Part (b) of $\mathbf{H}_k$ follows from Corollary 6.15, (a), and the hypothesis that $\mathbf{H}_{k-1}$ holds. This completes the proof. □

We characterize now when $\mathcal{C}_{\mathbb{G}}$ is a minimal resolution of $\mathbb{K}[x]/I(\mathcal{L})$. This result generalises the first main result in [18], namely Theorem 2 there. In a follow-up paper, we shall investigate applications of our results, beginning with generalisations of the remaining results of [18], before branching out more widely.

Corollary 6.16. *Let $\mathbb{G}$ be a strongly connected digraph or, equivalently, let L be an* ICB *matrix. Let $\mathcal{C}_{\mathbb{G}}$ be its associated* Cyc *complex. Then*

$$\mathcal{C}_{\mathbb{G}}: \quad 0 \leftarrow \mathbb{K}[x]/I(\mathcal{L}) \leftarrow \mathcal{C}_0 = \mathbb{K}[x] \xleftarrow{\partial_1} \mathcal{C}_1 = \mathbb{K}[x]^{r_1} \xleftarrow{\partial_2} \cdots$$
$$\cdots \xleftarrow{\partial_{n-1}} \mathcal{C}_{n-1} = \mathbb{K}[x]^{r_{n-1}} \leftarrow 0,$$

is a minimal free resolution of $\mathbb{K}[x]/I(\mathcal{L})$ if and only if $\mathbb{G}$ is strongly complete.

Proof. Suppose that $\mathbb{G}$ is strongly complete. That $\mathcal{C}_{\mathbb{G}}$ is a minimal free resolution of $\mathbb{K}[x]/I(\mathcal{L})$ follows from Theorem 6.1 and Remark 3.10.

Conversely, suppose that $\mathcal{C}_{\mathbb{G}}$ is a minimal free resolution of $\mathbb{K}[x]/I(\mathcal{L})$. Consider $i, j \in [n-1]$ with $i \neq j$, together with a basis element of the form

$$(\{i\}, \{j\}, I_3, \ldots, I_{k+1}) \in \mathcal{B}_k.$$

Then $\pm x_i^{a_{i,j}}$ appears as a coefficient in the expression for $\partial_k(\{i\}, \{j\}, I_3, \ldots, I_{k+1})$. By hypothesis, $x_i^{a_{i,j}} \in \mathfrak{m}$, and it follows that $a_{i,j} > 0$. Similarly, $a_{j,i} > 0$.

Next consider a basis element of the form $(I_1, \ldots, I_{k-1}, \{i\}, \{n\}) \in \mathcal{B}_k$. A similar argument shows that $a_{i,n} > 0$.

Finally, consider a basis element of the form $(\{i\}, I_2, \ldots, I_k, \{n\}) \in \mathcal{B}_k$. Again, a similar argument shows that $a_{n,i} > 0$, and the result follows. □

6.5. More on enumerations. Since $\partial_{k+1} : \mathcal{B}_{k+1} \to \mathcal{C}_k$ is injective, it is natural to consider in $\partial_{k+1}(\mathcal{B}_{k+1})$ the enumeration inherited by the srle enumeration of $\mathcal{B}_{k+1}$ (see Corollary 6.15, (b)). This is not a minor matter as regards this paper. Indeed, to choose a "correct" enumeration in $\mathcal{B}_k$, and hence in $G_{k-1} = \partial_k(\mathcal{B}_k)$, has been seen to be crucial in the proof of Lemma 6.10 and hence in the proof of the main theorem, Theorem 6.1. In the light of this comment, the next remark (and Remark 6.18 below) may help to clarify ideas.

Remark 6.17. Let us consider in $\partial_{k+1}(\mathcal{B}_{k+1})$ the enumeration inherited by $\mathcal{B}_{k+1}$. On the other hand, let us also consider in $\Upsilon := \bigcup_{i=2}^{r_k} \{\tau_k(e_{k,i}, e_{k,j}) \mid e_{k,j} \in \mathcal{B}_{k,i}\}$ the enumeration defined in Remark 5.8 (where Υ is possibly larger than the set of syzygies considered in Remark 5.8). Then both enumerations agree, considering that $-\partial_{k+1}(\mathcal{B}_{k+1}) = \Upsilon$ (see the proof of Theorem 6.1).

Proof. Let $e_{k+1,l} = (I_1, \ldots, I_{k+2}) \in \mathcal{B}_{k+1}$ and $\partial_{k+1}(e_{k+1,l}) = -\tau_k(e_{k,i}, e_{k,j})$, with

$$e_{k,i} = (I_1, \ldots, I_k, I_{k+1} \cup I_{k+2}) \text{ and } e_{k,j} = (I_1, \ldots, I_{k-1}, I_k \cup I_{k+1}, I_{k+2}).$$

Analogously, let $e_{k+1,h} = (H_1, \dots, H_{k+2}) \in \mathcal{B}_{k+1}$ and $\partial_{k+1}(e_{k+1,h}) = -\tau_k(e_{k,p}, e_{k,q})$, with

$$e_{k,p} = (H_1, \dots, H_k, H_{k+1} \cup H_{k+2}) \text{ and } e_{k,q} = (H_1, \dots H_{k-1}, H_k \cup H_{k+1}, H_{k+2}).$$

(See Proposition 6.14.) Suppose that $\partial_{k+1}(e_{k+1,h})$ precedes $\partial_{k+1}(e_{k+1,l})$ in the inherited enumeration by $\mathcal{B}_{k+1}$, that is, $e_{k+1,h}$ precedes $e_{k+1,l}$. By Remark 3.1, there exists an integer $s \in \{1, \dots, k+1\}$, such that $H_1 = I_1, \dots, H_{s-1} = I_{s-1}$ and H_s precedes I_s. If $s \leqslant k$, then $e_{k,p}$ precedes $e_{k,i}$. If $s = k+1$, then $e_{k,p} = e_{k,i}$ and $e_{k,q}$ precedes $e_{k,j}$. Therefore, $\tau_k(e_{k,p}, e_{k,q})$ precedes $\tau_k(e_{k,i}, e_{k,j})$, according to the enumeration considered in Remark 5.8.

Conversely, suppose that $\tau_k(e_{k,p}, e_{k,q})$ precedes $\tau_k(e_{k,i}, e_{k,j})$. If $e_{k,p}$ precedes $e_{k,i}$, then there exists an integer $s \in \{1, \dots, k\}$, such that $H_1 = I_1, \dots, H_{s-1} = I_{s-1}$ and H_s precedes I_s. This clearly implies that $e_{k+1,h}$ precedes $e_{k+1,l}$. On the other hand, if $e_{k,p} = e_{k,i}$ and $e_{k,q}$ precedes $e_{k,p}$, then, since $H_1 = I_1, \dots, H_k = I_k$, necessarily $H_k \cup H_{k+1}$ precedes $I_k \cup I_{k+1}$. But, since $H_k = I_k$, it follows that H_{k+1} precedes I_{k+1} and so $e_{k+1,h}$ precedes $e_{k+1,l}$, and $\partial_{k+1}(e_{k+1,h})$ precedes $\partial_{k+1}(e_{k+1,l})$. □

Remark 6.18. If we attempt to prove Theorem 6.1 by means of Theorem 5.5, only now using an alternative but seemingly natural enumeration of the basis elements of the successive free modules in the resolution, we can find that the module quotients $M_i(G_{k-1})$ have an 'excessive' number of generators, and as a result, the free resolution we obtain is no longer the Cyc complex.

For example, consider the $n \times n$ PCB matrix L considered in [18, Example 1, 3]:

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

Here $\nu(L) = (1,1,1,1)$. Thus each variable x, y, z, t is given degree 1 and we endow $A = \mathbb{K}[x,y,z,t]$ with the (weighted) reverse lexicographic ordering (see Assumptions 2.7 and 4.9). Now, instead of enumerating according to the srle, we enumerate $\mathrm{Cyc}_{4,2}$ as follows (see Example 3.2):

$$\mathrm{Cyc}_{4,2} = \{(1,234), (2,134), (3,124), (123,4), (12,34), (13,24), (23,14)\}.$$

Note that in Corollary 4.18, the enumeration considered in $\mathcal{B}_1$ is not essential for its proof. Thus it follows that $G_0 = \{f_{0,1}, f_{0,2}, f_{0,3}, f_{0,4}, f_{0,5}, f_{0,5}, f_{0,6}, f_{0,7}\}$ is a Gröbner basis of $I(\mathcal{L})$, where

$$\begin{aligned}
f_{0,1} &= \partial_1(e_{1,1}) = \partial_1(1,234) = \underline{x^3} - yzt, \\
f_{0,2} &= \partial_1(e_{1,2}) = \partial_1(2,134) = \underline{y^3} - xzt, \\
f_{0,3} &= \partial_1(e_{1,3}) = \partial_1(3,124) = \underline{z^3} - xyt, \\
f_{0,4} &= \partial_1(e_{1,4}) = \partial_1(123,4) = \underline{xyz} - t^3, \\
f_{0,5} &= \partial_1(e_{1,5}) = \partial_1(12,34) = \underline{x^2y^2} - z^2t^2, \\
f_{0,6} &= \partial_1(e_{1,6}) = \partial_1(13,24) = \underline{x^2z^2} - y^2t^2 \text{ and} \\
f_{0,7} &= \partial_1(e_{1,7}) = \partial_1(23,14) = \underline{y^2z^2} - x^2t^2.
\end{aligned}$$

It is a simple check to see that $(M_1(G_0) = 0)$:

$$\begin{aligned}
&M_2(G_0) = (x^3)\ ,\ M_3(G_0) = (x^3, y^3)\ ,\ M_4(G_0) = (x^2, y^2, z^2)\ , \\
&M_5(G_0) = M_6(G_0) = M_7(G_0) = (x, y, z) = \mathfrak{m}.
\end{aligned}$$

That gives 15 minimal generators x^α of the ideal quotients $M_i(G_0)$, for $i = 2, \dots, 7$, in Theorem 5.5. Thus the Gröbner basis G_1 of the syzygies of G_0 would contain 15 elements and the corresponding resolution would begin with

$$0 \leftarrow \mathbb{K}[x]/I(\mathcal{L}) \leftarrow \mathbb{K}[x] \leftarrow \mathbb{K}[x]^7 \leftarrow \mathbb{K}[x]^{15} \leftarrow \cdots,$$

and so it would not coincide with the complex $\mathcal{C}_L$.

7. An example: the four-dimensional case

In this section we write down the case $n = 4$ just as an illustrative example. This will enable the reader to visualise the details of the proofs of our main results for this particular case.

Recall that, when $n = 4$, the complex $\mathcal{C}$ has the following form (cf. Example 3.8 for more details).

$$\mathcal{C} : 0 \leftarrow \mathcal{C}_0 = \mathbb{K}[x] \xleftarrow{\partial_1} \mathcal{C}_1 = \mathbb{K}[x]^7 \xleftarrow{\partial_2} \mathcal{C}_2 = \mathbb{K}[x]^{12} \xleftarrow{\partial_3} \mathcal{C}_3 = \mathbb{K}[x]^6 \leftarrow 0,$$

where we write $\mathbb{K}[x] \equiv \mathbb{K}[x, y, z, t]$. Moreover, Theorem 6.1 says that this is a free resolution of $\mathbb{K}[x]/I(\mathcal{L})$. By Corollary 4.18, $G_0 := \partial_1(\mathcal{B}_1) \subset \mathcal{C}_0$ is a Gröbner basis of $\partial_1(\mathcal{C}_1) = I(\mathcal{L}) = \ker(\partial_0)$. Using Definition 3.3 (see also Example 3.8), on enumerating according to the srle and underlining leading terms, the elements of G_0 are:

$$\begin{aligned}
f_{0,1} &= \partial_1(e_{1,1}) = \partial_1(123, 4) = \underline{x^{a_{1,4}} y^{a_{2,4}} z^{a_{3,4}}} - t^{a_{4,4}},\\
f_{0,2} &= \partial_1(e_{1,2}) = \partial_1(23, 14) = \underline{y^{a_{2,1}+a_{2,4}} z^{a_{3,1}+a_{3,4}}} - x^{a_{1,2}+a_{1,3}} t^{a_{4,2}+a_{4,3}},\\
f_{0,3} &= \partial_1(e_{1,3}) = \partial_1(13, 24) = \underline{x^{a_{1,2}+a_{1,4}} z^{a_{3,2}+a_{3,4}}} - y^{a_{2,1}+a_{2,3}} t^{a_{4,1}+a_{4,3}},\\
f_{0,4} &= \partial_1(e_{1,4}) = \partial_1(12, 34) = \underline{x^{a_{1,3}+a_{1,4}} y^{a_{2,3}+a_{2,4}}} - z^{a_{3,1}+a_{3,2}} t^{a_{4,1}+a_{4,2}},\\
f_{0,5} &= \partial_1(e_{1,5}) = \partial_1(3, 124) = \underline{z^{a_{3,3}}} - x^{a_{1,3}} y^{a_{2,3}} t^{a_{4,3}},\\
f_{0,6} &= \partial_1(e_{1,6}) = \partial_1(2, 134) = \underline{y^{a_{2,2}}} - x^{a_{1,2}} z^{a_{3,2}} t^{a_{4,2}},\\
f_{0,7} &= \partial_1(e_{1,7}) = \partial_1(1, 234) = \underline{x^{a_{1,1}}} - y^{a_{2,1}} z^{a_{3,1}} t^{a_{4,1}}.
\end{aligned}$$

Let us calculate the ideal quotients $M_i(G_0) = (m^1_{1,i}, \dots, m^1_{i-1,i})$ of Notation 6.4. To do that, we need to calculate $m^1_{j,i} = \mathrm{LCM}(\mathbf{L}\mathrm{m}(f_{0,j}), \mathbf{L}\mathrm{m}(f_{0,i}))/\mathbf{L}\mathrm{t}(f_{0,i})$, for $1 \leqslant j < i \leqslant 7$. By definition $M_1(G_0) = 0$. The subsequent $M_2(G_0)$ is equal to $(m^1_{1,2}) = (x^{a_{1,4}})$. To calculate $M_3(G_0) = (m^1_{1,3}, m^1_{2,3})$, observe that $m^1_{2,3} = y^{a_{2,1}+a_{2,4}} z^{(a_{3,1}-a_{3,2})^+}$ is a multiple of $m^1_{1,3} = y^{a_{2,4}}$, so it follows that $M_3(G_0) = (y^{a_{2,4}})$. Note that this is consistent with Lemma 6.10 and Proposition 6.12. Indeed, $e_{1,3} = (13, 14)$ and

$$\mathcal{B}_{1,3} = \{(J_1, J_2) \in \mathcal{B}_1 \mid J_1 \supsetneq \{1, 3\}\}.$$

Since $\{1, 2, 3\} \supsetneq \{1, 3\}$, then $e_{1,1} = (123, 4) \in \mathcal{B}_{1,3}$ and, since $\{2, 3\} \not\supset \{1, 3\}$, then $e_{1,2} \notin \mathcal{B}_{1,3}$. In other words, $m^1_{2,3}$ is superfluous. The next $m^1_{j,i}$ are presented in the following tables:

$m^1_{j,i}$	$i = 4$	$i = 5$
$j = 1$	$z^{a_{3,4}}$	$x^{a_{1,4}} y^{a_{2,4}}$
$j = 2$	$y^{(a_{2,1}-a_{2,3})^+} z^{a_{3,1}+a_{3,4}}$	$y^{a_{2,1}+a_{2,4}}$
$j = 3$	$x^{(a_{1,2}-a_{1,3})^+} z^{a_{3,2}+a_{3,4}}$	$x^{a_{1,2}+a_{1,4}}$
$j = 4$		$x^{a_{1,3}+a_{1,4}} y^{a_{2,3}+a_{2,4}}$
$j = 5$		
$j = 6$		

$m^1_{j,i}$	$i=6$	$i=7$
$j=1$	$x^{a_{1,4}}z^{a_{3,4}}$	$y^{a_{2,4}}z^{a_{3,4}}$
$j=2$	$z^{a_{3,1}+a_{3,4}}$	$y^{a_{2,1}+a_{2,4}}z^{a_{3,1}+a_{3,4}}$
$j=3$	$x^{a_{1,2}+a_{1,4}}z^{a_{3,2}+a_{3,4}}$	$z^{a_{3,2}+a_{3,4}}$
$j=4$	$x^{a_{1,3}+a_{1,4}}$	$y^{a_{2,3}+a_{2,4}}$
$j=5$	$z^{a_{3,3}}$	$z^{a_{3,3}}$
$j=6$		$y^{a_{2,2}}$

Similarly, $M_4(G_0)=(m^1_{1,4},m^1_{2,4},m^1_{3,4})$ is generated by the monomials in the column labelled "$i=4$". But $m^1_{2,4}$ and $m^1_{3,4}$ are multiples of $m^1_{1,4}$. Therefore, $M_4(G_0)=(z^{a_{3,4}})$. Again, this is consistent with Proposition 6.12, since $e_{1,2}$ and $e_{1,3}$ are not in $\mathcal{B}_{1,4}$, so they are superfluous. As for $M_5(G_0)$, note that $e_{1,1},e_{1,2},e_{1,3}$ are in $\mathcal{B}_{1,5}$, but $e_{1,4}$ is not. Thus $M_5(G_0)=(x^{a_{1,4}}y^{a_{2,4}},y^{a_{2,1}+a_{2,4}},x^{a_{1,2}+a_{1,4}})$. Similarly, $M_6(G_0)=(x^{a_{1,4}}z^{a_{3,4}},z^{a_{3,1}+a_{3,4}},x^{a_{1,3}+a_{1,4}})$ and $M_7(G_0)=(y^{a_{2,4}}z^{a_{3,4}},z^{a_{3,2}+a_{3,4}},y^{a_{2,3}+a_{2,4}})$. Observe that there are exactly twelve monomial generators, while $\mathcal{C}_2$ is precisely the free module of rank 12, which is consistent with the equality $r_2=\sum_{i=2}^{r_1}|\mathcal{B}_{1,i}|$ in Proposition 6.12. Concretely, the set of 12 monomial generators of the ideal quotients $M_i(G_0)$ are:

$$\bigcup_{i=2}^{7}\{m^1_{j,i}\mid e_{1,j}\in\mathcal{B}_{1,i}\}$$
$$=\{m^1_{1,2},m^1_{1,3},m^1_{1,4},m^1_{1,5},m^1_{2,5},m^1_{3,5},m^1_{1,6},m^1_{2,6},m^1_{4,6},m^1_{1,7},m^1_{3,7},m^1_{4,7}\}.$$

Observe that all these $m^1_{j,i}$ are different provided that L is a PCB matrix, which is coherent with Remark 6.13. On the other hand, if L is an ICB matrix, which is not a PCB matrix, then there may be coincidences (e.g. if $a_{1,4}=0$ and $a_{2,4}=0$, then $m^1_{1,2}$ and $m^1_{1,3}$ are equal to 1).

Each monomial generator $m^1_{j,i}$ of $M_i(G_0)$ induces a corresponding τ-syzygy element $\tau_1(e_{1,i},e_{1,j})$ associated to the S-vector of $f_{0,i}$ and $f_{0,j}$. Then Theorem 5.5 states that

$$\{\tau_1(e_{1,2},e_{1,1}),\tau_1(e_{1,3},e_{1,1}),\tau_1(e_{1,4},e_{1,1}),\tau_1(e_{1,5},e_{1,1}),\tau_1(e_{1,5},e_{1,2}),\tau_1(e_{1,5},e_{1,3}),$$
$$\tau_1(e_{1,6},e_{1,1}),\tau_1(e_{1,6},e_{1,2}),\tau_1(e_{1,6},e_{1,4}),\tau_1(e_{1,7},e_{1,1}),\tau_1(e_{1,7},e_{1,3}),\tau_1(e_{1,7},e_{1,4})\}$$

is a Gröbner basis of $\ker(\partial_1)\subset\mathcal{C}_1$ w.r.t. the monomial ordering on $\mathcal{C}_1$ induced by the wrlo on $\mathcal{C}_0=\mathbb{K}[x]$ and the Gröbner basis G_0. Note that we have enumerated this Gröbner basis according to Remark 5.8. The proof of Theorem 6.1 and Remark 6.17 say that the following is an equality of enumerated sets:

$$\begin{aligned}G_1:=\partial_2(\mathcal{B}_2)=-\{\,&\tau_1(e_{1,2},e_{1,1}),\tau_1(e_{1,3},e_{1,1}),\tau_1(e_{1,4},e_{1,1}),\tau_1(e_{1,5},e_{1,1}),\\&\tau_1(e_{1,5},e_{1,2}),\tau_1(e_{1,5},e_{1,3}),\tau_1(e_{1,6},e_{1,1}),\tau_1(e_{1,6},e_{1,2}),\\&\tau_1(e_{1,6},e_{1,4}),\tau_1(e_{1,7},e_{1,1}),\tau_1(e_{1,7},e_{1,3}),\tau_1(e_{1,7},e_{1,4})\}.\end{aligned}$$

More concretely (recall Example 3.8), and underlining the leading terms (using either (28) or Theorem 6.1):

$$\begin{aligned}f_{1,1}&=\partial_2(e_{2,1})=\partial_2(23,1,4)\\&=y^{a_{2,1}}z^{a_{3,1}}e_{1,1}-\underline{x^{a_{1,4}}e_{1,2}}-t^{a_{4,2}+a_{4,3}}e_{1,7}=-\tau_1(e_{1,2},e_{1,1}),\\f_{1,2}&=\partial_2(e_{2,2})=\partial_2(13,2,4)\\&=x^{a_{1,2}}z^{a_{3,2}}e_{1,1}-\underline{y^{a_{2,4}}e_{1,3}}-t^{a_{4,1}+a_{4,3}}e_{1,6}=-\tau_1(e_{1,3},e_{1,1}),\\f_{1,3}&=\partial_2(e_{2,3})=\partial_2(12,3,4)\\&=x^{a_{1,3}}y^{a_{2,3}}e_{1,1}-\underline{z^{a_{3,4}}e_{1,4}}-t^{a_{4,1}+a_{4,2}}e_{1,5}=-\tau_1(e_{1,4},e_{1,1}),\end{aligned}$$

$$\begin{aligned}
f_{1,4} &= \partial_2(e_{2,4}) = \partial_2(3,12,4)\\
&= z^{a_{3,1}+a_{3,2}}e_{1,1} - \underline{x^{a_{1,4}}y^{a_{2,4}}e_{1,5}} - t^{a_{4,3}}e_{1,4} = -\tau_1(e_{1,5},e_{1,1}),\\
f_{1,5} &= \partial_2(e_{2,5}) = \partial_2(3,2,14)\\
&= z^{a_{3,2}}e_{1,2} - \underline{y^{a_{2,1}+a_{2,4}}e_{1,5}} - x^{a_{1,3}}t^{a_{4,3}}e_{1,6} = -\tau_1(e_{1,5},e_{1,2}),\\
f_{1,6} &= \partial_2(e_{2,6}) = \partial_2(3,1,24)\\
&= z^{a_{3,1}}e_{1,3} - \underline{x^{a_{1,2}+a_{1,4}}e_{1,5}} - y^{a_{2,3}}t^{a_{4,3}}e_{1,7} = -\tau_1(e_{1,5},e_{1,3}),\\
f_{1,7} &= \partial_2(e_{2,7}) = \partial_2(2,13,4)\\
&= y^{a_{2,1}+a_{2,3}}e_{1,1} - \underline{x^{a_{1,4}}z^{a_{3,4}}e_{1,6}} - t^{a_{4,2}}e_{1,3} = -\tau_1(e_{1,6},e_{1,1}),\\
f_{1,8} &= \partial_2(e_{2,8}) = \partial_2(2,3,14)\\
&= y^{a_{2,3}}e_{1,2} - \underline{z^{a_{3,1}+a_{3,4}}e_{1,6}} - x^{a_{1,2}}t^{a_{4,2}}e_{1,5} = -\tau_1(e_{1,6},e_{1,2}),\\
f_{1,9} &= \partial_2(e_{2,9}) = \partial_2(2,1,34)\\
&= y^{a_{2,1}}e_{1,4} - \underline{x^{a_{1,3}+a_{1,4}}e_{1,6}} - z^{a_{3,2}}t^{a_{4,2}}e_{1,7} = -\tau_1(e_{1,6},e_{1,4}),\\
f_{1,10} &= \partial_2(e_{2,10}) = \partial_2(1,23,4)\\
&= x^{a_{1,2}+a_{1,3}}e_{1,1} - \underline{y^{a_{2,4}}z^{a_{3,4}}e_{1,7}} - t^{a_{4,1}}e_{1,2} = -\tau_1(e_{1,7},e_{1,1}),\\
f_{1,11} &= \partial_2(e_{2,11}) = \partial_2(1,3,24)\\
&= x^{a_{1,3}}e_{1,3} - \underline{z^{a_{3,2}+a_{3,4}}e_{1,7}} - y^{a_{2,1}}t^{a_{4,1}}e_{1,5} = -\tau_2(e_{1,7},e_{1,3}),\\
f_{1,12} &= \partial_2(e_{2,12}) = \partial_2(1,2,34)\\
&= x^{a_{1,2}}e_{1,4} - \underline{y^{a_{2,3}+a_{2,4}}e_{1,7}} - z^{a_{3,1}}t^{a_{4,1}}e_{1,6} = -\tau_1(e_{1,7},e_{1,4}).
\end{aligned}$$

Let us focus on the first equality. Since Cyc is a complex, $\partial_1(f_{1,1}) = \partial_1(\partial_2(e_{2,1})) = 0$. Hence

$$x^{a_{1,4}}\partial_1(e_{1,2}) - y^{a_{2,1}}z^{a_{3,1}}\partial_1(e_{1,1}) = -t^{a_{4,2}+a_{4,3}}\partial_1(e_{1,7}).$$

In other words,

$$x^{a_{1,4}}f_{0,2} - y^{a_{2,1}}z^{a_{3,1}}f_{0,1} = -t^{a_{4,2}+a_{4,3}}f_{0,7}. \tag{42}$$

On one hand, $x^{a_{1,4}}f_{0,2} - y^{a_{2,1}}z^{a_{3,1}}f_{0,1} = S(f_{0,2},f_{0,1})$ because $m^1_{1,2} = x^{a_{1,4}}$ and $m^1_{2,1} = y^{a_{2,1}}z^{a_{3,1}}$. On the other hand, (42) is a standard expression of $S(f_{0,2},f_{0,1})$ w.r.t. the wrlo on $\mathcal{C}_0$ and with respect to G_0, and with remainder zero. Indeed, the S-vector of $f_{0,2}$ and $f_{0,1}$ is defined so as to cancel the leading terms of $f_{0,2}$ and $f_{0,1}$. Thus on the left hand side we have just a binomial, and hence similarly on the right hand side. Therefore, the leading term of the right hand side must necessarily be equal to the leading term of the left hand side. This proves that (42) is a standard expression. One deduces that $\tau_1(e_{1,2},e_{1,1}) = x^{a_{1,4}}e_{1,2} - y^{a_{2,1}}z^{a_{3,1}}e_{1,1} + t^{a_{4,2}+a_{4,3}}e_{1,7}$ is a τ-syzygy associated to the S-vector of $f_{0,2}$ and $f_{0,1}$. So $f_{1,1} = \partial_2(e_{2,1}) = -\tau_1(e_{1,2},e_{1,1})$, which is consistent with Proposition 6.14 and Theorem 6.1.

The remainder of the equalities can be treated in the same manner, so we deduce that $G_1 = \partial_2(\mathcal{B}_2)$ is a Gröbner basis of $\ker(\partial_1)$ and

$$\operatorname{Im}(\partial_2) = \langle \partial_2(\mathcal{B}_2)\rangle = \langle \tau_1(e_{1,i},e_{1,j}) \mid i = 2,\dots,7 \text{ and } e_{1,j} \in \mathcal{B}_{1,i}\rangle = \ker(\partial_1),$$

and the complex $\mathcal{C}$ is exact in degree 1, as shown in Theorem 6.1.

We next find the syzygies of $G_1 = \partial_2(\mathcal{B}_2) = \{f_{1,1},\dots,f_{1,12}\}$. To do that, we compute the corresponding module quotients of Notation 6.4. By definition, $M_1(G_1) = 0$. Since the least common multiple of two monomial terms with different basis elements is zero, it follows that $M_2(G_1)$, $M_3(G_1)$ and $M_4(G_1)$ are zero. For the same reason, $M_7(G_1) = 0$ and $M_{10}(G_1) = 0$. Note that $M_5(G_1)$ is generated by $m^2_{4,5} = -x^{a_{1,4}}$.

(Remark this negative sign, which will be the reason for the negative sign in the subsequent equality $\partial_3(e_{3,1}) = -\tau_2(e_{2,5}, e_{2,4})$.) Next, $M_6(G_1) = (m^2_{4,6}, m^2_{5,6})$. Since $e_{2,5} \notin \mathcal{B}_{2,6}$, then $m^2_{5,6}$ is superfluous. Thus $M_6(G_1) = (m^2_{4,6}) = (y^{a_{2,4}})$. It is easy to check that the nonzero module quotients $M_i(G_1)$ are:

$$\begin{aligned}
&M_5(G_1) = (m^2_{4,5}) = (x^{a_{1,4}})\ ,\ M_6(G_1) = (m^2_{4,6}) = (y^{a_{2,4}})\ ,\\
&M_8(G_1) = (m^2_{7,8}) = (x^{a_{1,4}})\ ,\ M_9(G_1) = (m^2_{7,9}) = (z^{a_{3,4}})\ ,\\
&M_{11}(G_1) = (m^2_{10,11}) = (y^{a_{2,4}}) \text{ and } M_{12}(G_1) = (m^2_{10,12}) = (z^{a_{3,4}}).
\end{aligned}$$

Observe that there are exactly six monomial generators, while $\mathcal{C}_3$ is precisely the free module of rank 6, which is consistent with the equality $r_3 = \sum_{i=2}^{r_2} |\mathcal{B}_{2,i}|$ in Proposition 6.12. Concretely, the set of 6 monomial generators of the ideal quotients $M_i(G_1)$ are:

$$\bigcup_{i=2}^{12} \{m^2_{j,i} \mid e_{2,j} \in \mathcal{B}_{2,i}\} = \{m^2_{4,5}, m^2_{4,6}, m^2_{7,8}, m^2_{7,9}, m^2_{10,11}, m^2_{10,12}\}.$$

Each monomial generator $m^2_{j,i}$ of $M_i(G_1)$ produces a corresponding τ-syzygy element $\tau_2(e_{2,i}, e_{2,j})$ associated to the S-vector of $f_{1,i}$ and $f_{1,j}$. Then Theorem 5.5 affirms that

$$\begin{aligned}
&\{\tau_2(e_{2,5}, e_{2,4}), \tau_2(e_{2,6}, e_{2,4}), \tau_2(e_{2,8}, e_{2,7}),\\
&\tau_2(e_{2,9}, e_{2,7}), \tau_2(e_{2,11}, e_{2,10}), \tau_2(e_{2,12}, e_{2,10})\}
\end{aligned}$$

is a Gröbner basis of $\ker(\partial_2) \subset \mathcal{C}_2$. Again, we have enumerated this Gröbner basis according to Remark 5.8. Moreover, Theorem 6.1 and Remark 6.17 say that the following is an equality of enumerated sets:

$$\begin{aligned}
G_2 := \partial_3(\mathcal{B}_3) = -\{\ &\tau_2(e_{2,5}, e_{2,4}), \tau_2(e_{2,6}, e_{2,4}), \tau_2(e_{2,8}, e_{2,7}),\\
&\tau_2(e_{2,9}, e_{2,7}), \tau_2(e_{2,11}, e_{2,10}), \tau_2(e_{2,12}, e_{2,10})\}.
\end{aligned}$$

More concretely (recall Example 3.8), and underlining the leading terms (using either (28) or Theorem 6.1):

$$\begin{aligned}
f_{2,1} &= \partial_3(e_{3,1}) = \partial_3(3,2,1,4)\\
&= z^{a_{3,2}} e_{2,1} - y^{a_{2,1}} e_{2,4} + \underline{x^{a_{1,4}} e_{2,5}} - t^{a_{4,3}} e_{2,9} = -\tau_2(e_{2,5}, e_{2,4}),\\
f_{2,2} &= \partial_3(e_{3,2}) = \partial_3(3,1,2,4)\\
&= z^{a_{3,1}} e_{2,2} - x^{a_{1,2}} e_{2,4} + \underline{y^{a_{2,4}} e_{2,6}} - t^{a_{4,3}} e_{2,12} = -\tau_2(e_{2,6}, e_{2,4}),\\
f_{2,3} &= \partial_3(e_{3,3}) = \partial_3(2,3,1,4)\\
&= y^{a_{2,3}} e_{2,1} - z^{a_{3,1}} e_{2,7} + \underline{x^{a_{1,4}} e_{2,8}} - t^{a_{4,2}} e_{2,6} = -\tau_2(e_{2,8}, e_{2,7}),\\
f_{2,4} &= \partial_3(e_{3,4}) = \partial_3(2,1,3,4)\\
&= y^{a_{2,1}} e_{2,3} - x^{a_{1,3}} e_{2,7} + \underline{z^{a_{3,4}} e_{2,9}} - t^{a_{4,2}} e_{2,11} = -\tau_2(e_{2,9}, e_{2,7}),\\
f_{2,5} &= \partial_3(e_{3,5}) = \partial_3(1,3,2,4)\\
&= x^{a_{1,3}} e_{2,2} - z^{a_{3,2}} e_{2,10} + \underline{y^{a_{2,4}} e_{2,11}} - t^{a_{4,1}} e_{2,5} = -\tau_2(e_{2,11}, e_{2,10}),\\
f_{2,6} &= \partial_3(e_{3,6}) = \partial_3(1,2,3,4)\\
&= x^{a_{1,2}} e_{2,3} - y^{a_{2,3}} e_{2,10} + \underline{z^{a_{3,4}} e_{2,12}} - t^{a_{4,1}} e_{2,8} = -\tau_2(e_{2,12}, e_{2,10}).
\end{aligned}$$

Again, let us focus on the first equality. Since Cyc is a complex, $\partial_2(f_{2,1}) = \partial_2(\partial_3(e_{3,1})) = 0$. Hence

$$-x^{a_{1,4}} \partial_2(e_{2,5}) + y^{a_{2,1}} \partial_2(e_{2,4}) = z^{a_{3,2}} \partial_2(e_{2,1}) - t^{a_{4,3}} \partial_2(e_{2,9}).$$

In other words,

$$-x^{a_{1,4}} f_{1,5} + y^{a_{2,1}} f_{1,4} = z^{a_{3,2}} f_{1,1} - t^{a_{4,3}} f_{1,9}. \tag{43}$$

Note that $-x^{a_{1,4}}f_{1,5}+y^{a_{2,1}}f_{1,4}=S(f_{1,5},f_{1,4})$, because $m^2_{4,5}=-x^{a_{1,4}}$ and $m^2_{5,4}=-y^{a_{2,1}}$. On the other hand, (43) is a standard expression of $S(f_{1,5},f_{1,4})$ with respect to G_1, with remainder zero. Indeed, the S-vector of $f_{1,5}$ and $f_{1,4}$ is defined so as to cancel the leading terms of $f_{1,5}$ and $f_{1,4}$. Recall that

$$\begin{aligned} f_{1,5} &= z^{a_{3,2}}e_{1,2} - \underline{y^{a_{2,1}+a_{2,4}}e_{1,5}} - x^{a_{1,3}}t^{a_{4,3}}e_{1,6},\\ f_{1,4} &= z^{a_{3,1}+a_{3,2}}e_{1,1} - \underline{x^{a_{1,4}}y^{a_{2,4}}e_{1,5}} - t^{a_{4,3}}e_{1,4},\\ f_{1,1} &= y^{a_{2,1}}z^{a_{3,1}}e_{1,1} - \underline{x^{a_{1,4}}e_{1,2}} - t^{a_{4,2}+a_{4,3}}e_{1,7},\\ f_{1,9} &= y^{a_{2,1}}e_{1,4} - \underline{x^{a_{1,3}+a_{1,4}}e_{1,6}} - z^{a_{3,2}}t^{a_{4,2}}e_{1,7}. \end{aligned}$$

Note that the leading term of $z^{a_{3,2}}f_{1,1}$ and the leading term of $t^{a_{4,3}}f_{1,9}$ coincide with the two summands of $-x^{a_{1,4}}f_{1,5}-\mathbf{L}\mathrm{t}(-x^{a_{1,4}}f_{1,5})$, as stated in the proof Proposition 6.14. It follows that (43) is a standard expression, so one deduces that

$$\tau_2(e_{2,5},e_{2,4}) = -x^{a_{1,4}}e_{2,5}+y^{a_{2,1}}e_{2,4}-z^{a_{3,2}}e_{2,1}+t^{a_{4,3}}e_{2,9}$$

is a τ-syzygy associated to the S-vector of $f_{1,5}$ and $f_{1,4}$. Thus, $f_{2,1}=-\tau_2(e_{2,5},e_{2,4})$, which is consistent with Proposition 6.14 and Theorem 6.1.

The remainder of the equalities can be treated in the same manner, so we deduce that $G_2=\partial_3(\mathcal{B}_3)$ is a Gröbner basis of $\ker(\partial_2)$ and

$$\mathrm{Im}(\partial_3)=\langle\partial_3(\mathcal{B}_3)\rangle=\langle\tau_2(e_{2,i},e_{2,j}) \mid i=2,\dots,12 \text{ and } e_{2,j}\in\mathcal{B}_{2,i}\rangle=\ker(\partial_2),$$

and the complex $\mathcal{C}$ is exact in degree 2, as shown in Theorem 6.1.

Finally, observe that, since all the leading terms of $f_{2,i}$ have distinct basis elements, then the module quotients $M_i(G_2)$ are all zero (see Notation 6.4). Therefore, there are no syzygies among the elements of G_2 and $\partial_3:\mathcal{C}_3\to\mathcal{C}_2$ is injective, from which follows the exactness of the $\mathcal{C}$ complex in degree 3, which is consistent with Remark 6.7.

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