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A construction of pairs of non-commutative rank 8 association schemes from non-symmetric rank 3 association schemes

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Abstract We construct a pair of non-commutative rank 8 association schemes from a rank 3 non-symmetric association scheme. For the pair, two association schemes have the same character table but different Frobenius–Schur indicators. This situation is similar to that for the dihedral and quaternion groups of order 8. We also determine the structures of adjacency algebras of the rank 8 schemes over the rational number field.

1. Introduction

From a rank 3 non-symmetric association scheme of order $n-1$, we construct a pair of association schemes $(\mathcal{D}, \mathcal{Q})$ with the following properties.

- $\mathcal{D}$ and $\mathcal{Q}$ are non-commutative, of order $4n$, and of rank 8.
- $\mathcal{D}$ and $\mathcal{Q}$ have the same character tables but their Frobenius–Schur indicators are different.

These properties are similar to the pair of the dihedral group D_8 and the quaternion group Q_8 of order 8.

In the theory of association schemes, the adjacency algebras are mainly considered over the complex number field. In this paper, we determine the structures of adjacency algebras of $\mathcal{D}$ and $\mathcal{Q}$ over the rational number field $\mathbb{Q}$. We prove

$$\mathbb{Q}\mathcal{D} \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathrm{M}_2(\mathbb{Q}),$$
$$\mathbb{Q}\mathcal{Q} \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}(-1, -n+1),$$

where $\mathrm{M}_2(\mathbb{Q})$ is the full matrix algebra of degree 2 and $\mathbb{Q}(-1, -n+1)$ is the quaternion division algebra.

It is known that a rank 3 non-symmetric association scheme of order $n-1$ exists if and only if there exists a skew-Hadamard matrix of order n with $n \equiv 0 \pmod 4$. There is a conjecture that such a matrix of order n exists for an arbitrary $n \equiv 0 \pmod 4$.

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Keywords. Character table, quaternion algebras, association schemes.

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2. Preliminaries

For a field K, we denote by $\mathrm{M}_n(K)$ the full matrix algebra of degree n over K. For a matrix M, the transposed matrix of M will be denoted by ${}^tM$. By I_n, we denote the identity matrix of degree n. By J_n, we denote the $n \times n$ matrix with all entries 1.

We define an association scheme in matrix form. Let $\mathcal{S} = \{A_0, \dots, A_d\}$ be a set of non-zero $n \times n$ matrices with entries in $\{0, 1\}$. Then the set $\mathcal{S}$ is called an *association scheme* of *order* n and *rank* $d+1$ if

(1) $A_0 = I_n$,
(2) $\sum_{i=0}^{d} A_i = J_n$, and
(3) for any $0 \leqslant i, j \leqslant d$, ${}^tA_i$ and A_iA_j are linear combinations of $A_0, \dots, A_d$.

By definition, all rows of A_i contain the same number of 1. We call this number the *valency* of A_i and denote it by n_i. The association scheme $\mathcal{S}$ is said to be *symmetric* if ${}^tA_i = A_i$ for all $0 \leqslant i \leqslant d$ and *non-symmetric* otherwise. The association scheme $\mathcal{S}$ is said to be *commutative* if $A_iA_j = A_jA_i$ for all $0 \leqslant i, j \leqslant d$ and *non-commutative* otherwise. For a field K, $K\mathcal{S} := \bigoplus_{i=0}^{d} KA_i$ is a $(d+1)$-dimensional K-algebra by the condition (3). We call $K\mathcal{S}$ the *adjacency algebra* of $\mathcal{S}$ over K. It is known that $K\mathcal{S}$ is semisimple if the characteristic of K is 0 [5, Theorem 4.1.3 (ii)]. A *representation* of $\mathcal{S}$ over K means a K-algebra homomorphism from $K\mathcal{S}$ to a full matrix algebra $\mathrm{M}_t(K)$ for some positive integer t.

A subset $\mathcal{T}$ of an association scheme $\mathcal{S}$ is called a *closed subset* if $e_{\mathcal{T}} := n_{\mathcal{T}}^{-1} \sum_{A_i \in \mathcal{T}} A_i$ is an idempotent, where $n_{\mathcal{T}} := \sum_{A_i \in \mathcal{T}} n_i$. A closed subset $\mathcal{T}$ of $\mathcal{S}$ is said to be *normal* if $e_{\mathcal{T}}$ is a central element of the complex adjacency algebra $\mathbb{C}\mathcal{S}$. A closed subset $\mathcal{T}$ of $\mathcal{S}$ is said to be *strongly normal* if ${}^tA_jA_iA_j \in \bigoplus_{A_\ell \in \mathcal{T}} \mathbb{C}A_\ell$ for $A_i \in \mathcal{T}$ and $A_j \in \mathcal{S}$. It is known that a strongly normal closed subset is normal. The *thin radical* of $\mathcal{S}$ is $O_\theta(\mathcal{S}) = \{A_i \mid n_i = 1\}$ and is known to be a closed subset of $\mathcal{S}$. The *thin residue* $O^\theta(\mathcal{S})$ of $\mathcal{S}$ is the intersection of all strongly normal closed subsets of $\mathcal{S}$ and is known to be a strongly normal closed subset of $\mathcal{S}$. For details, the reader is referred to [5] or [2].

Now we consider the adjacency algebra over the complex number field $\mathbb{C}$. By Wedderburn's theorem [4, 3.5 Theorem], $\mathbb{C}\mathcal{S} \cong \bigoplus_{i=1}^{\ell} \mathrm{M}_{t_i}(\mathbb{C})$ for some $t_1, \dots, t_\ell$. The set of projections $\mathbb{C}\mathcal{S} \to \mathrm{M}_{t_i}(\mathbb{C})$ $(i = 1, \dots, \ell)$ is a complete set of representatives of equivalence classes of irreducible representations of $\mathbb{C}\mathcal{S}$. The matrix trace of a representation is called the *character* of the representation. By $\mathrm{Irr}(\mathcal{S}) = \{\chi_1, \dots, \chi_\ell\}$, we denote the set of irreducible characters of $\mathbb{C}\mathcal{S}$. The $\ell \times (d+1)$ matrix $(\chi_i(A_j))$ is called the *character table* of $\mathcal{S}$. Since $\mathbb{C}\mathcal{S}$ is defined as a matrix algebra, the map $A_i \mapsto A_i$ is a representation. We call it the *standard representation* and its character the *standard character* of $\mathcal{S}$. The decomposition of the standard character into the irreducibles is written as $\sum_{\chi \in \mathrm{Irr}(\mathcal{S})} m_\chi \chi$. We call the coefficient m_χ the *multiplicity* of χ.

The *Frobenius–Schur indicator* $\nu(\chi)$ of $\chi \in \mathrm{Irr}(\mathcal{S})$ is defined by

$$\nu(\chi) := \frac{m_\chi}{n\ \chi(A_0)} \sum_{i=0}^{d} \frac{1}{n_i} \chi({A_i}^2).$$

An irreducible character is said to be of the *first kind* if it is afforded by a real representation, of the *second kind* if it is afforded by a representation which is equivalent to its complex conjugate but is not of the first kind, and of the *third kind* if it is not of the first or second kind. The following theorem is known.

THEOREM 2.1 ([3, (7,5), (7.6)]).

(1) *For* $\chi \in \mathrm{Irr}(\mathcal{S})$,

$$\nu(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is of the first kind,} \\ -1 & \text{if } \chi \text{ is of the second kind,} \\ 0 & \text{if } \chi \text{ is of the third kind.} \end{cases}$$

(2) $\sum_{\chi \in \mathrm{Irr}(\mathcal{S})} \nu(\chi)\chi(A_0) = \sharp\{A_i \in \mathcal{S} \mid {}^tA_i = A_i\}$.

3. CONSTRUCTION

Let $\{A_0 = I_{n-1}, A_1, A_2 = {}^tA_1\}$ be a non-symmetric rank 3 association scheme of order $n-1$. Remark that $n \equiv 0 \pmod 4$ for such an association scheme to exist. We will construct a pair $(\mathcal{D}, \mathcal{Q})$ of association schemes with the properties described in the Introduction.

Set $a \coloneqq n-1$ and $b \coloneqq (n-2)/2$.

The following lemma is well-known.

LEMMA 3.1.

(1) $A_1^2 = \frac{b-1}{2}A_1 + \frac{b+1}{2}A_2$.
(2) $A_2^2 = \frac{b+1}{2}A_1 + \frac{b-1}{2}A_2$.
(3) $A_1A_2 = A_2A_1 = bA_0 + \frac{b-1}{2}A_1 + \frac{b-1}{2}A_2$.

Set $x \coloneqq (1,2,3,4)$, $y \coloneqq (1,2)(3,4)$, permutations of degree 4, and $G \coloneqq \langle x, y\rangle \cong D_8$. We identify the elements of G with the corresponding permutation matrices, namely

$$x = \begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \end{pmatrix}, \quad y = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix},$$

and so on. The next lemma is clear by definition.

LEMMA 3.2. *As matrices,* $1 + x^2 = xy + x^3y$.

We keep the above notations in the rest of this article.

3.1. THE ASSOCIATION SCHEME $\mathcal{D}$.

We define $n \times n$ matrices by

$$E \coloneqq \left(\begin{array}{c|c} 0 & 1 \ \ldots \ 1 \\ \hline 1 & \\ \vdots & O \\ 1 & \end{array}\right), \quad \tilde{A}_i \coloneqq \left(\begin{array}{c|c} 0 & 0 \ \ldots \ 0 \\ \hline 0 & \\ \vdots & A_i \\ 0 & \end{array}\right) \quad (i = 1, 2).$$

By Lemma 3.1 and direct calculations, we have the following lemma.

LEMMA 3.3.

(1) $E^2 + \tilde{A}_1\tilde{A}_2 + \tilde{A}_2\tilde{A}_1 = aI_n + b\tilde{A}_1 + b\tilde{A}_2$.
(2) $\tilde{A}_1^2 + \tilde{A}_2^2 = b\tilde{A}_1 + b\tilde{A}_2$.
(3) $E\tilde{A}_1 + \tilde{A}_2E = E\tilde{A}_2 + \tilde{A}_1E = bE$.

We consider a subgroup $H \coloneqq C_G(y) = \{1, x^2, y, x^2y\} \cong C_2 \times C_2$ of G. It is easy to see that $\sum_{h\in H} h = \sum_{g \in G \smallsetminus H} g = J_4$. We define $4n \times 4n$ matrices by

$$\sigma_h \coloneqq I_n \otimes h \quad \text{for } h \in H$$

and

$$\begin{aligned}\mu_g &:= E\otimes g + \tilde{A}_1\otimes gy + \tilde{A}_2\otimes gx^2y\\ &= (I_n\otimes g)(E\otimes 1 + \tilde{A}_1\otimes y + \tilde{A}_2\otimes x^2y)\\ &= (E\otimes 1 + \tilde{A}_1\otimes x^2y + \tilde{A}_2\otimes y)(I_n\otimes g) \quad \text{for } g\in G\setminus H.\end{aligned}$$

We will show that $\mathcal{D} := \{\sigma_h \mid h\in H\}\cup\{\mu_g \mid g\in G\setminus H\}$ forms an association scheme.

Lemma 3.4. *The set $\mathcal{D}$ is closed under the transposition and*

$$\sum_{h\in H}\sigma_h + \sum_{g\in G\setminus H}\mu_g = J_{4n}.$$

Proof. We have ${}^t\sigma_h = \sigma_h$ for $h\in H$, ${}^t\mu_{xy} = \mu_{xy}$, ${}^t\mu_{x^3y} = \mu_{x^3y}$ and ${}^t\mu_x = \mu_{x^3}$. Thus $\mathcal{D}$ is closed under the transposition.

Since $\sum_{h\in H} h = \sum_{g\in G\setminus H} g = J_4$ and $E + \tilde{A}_1 + \tilde{A}_2 = J_n - I_n$, we have $\sum_{h\in H}\sigma_h + \sum_{g\in G\setminus H}\mu_g = J_{4n}$. □

Lemma 3.5.

(1) $\sigma_h\sigma_{h'} = \sigma_{hh'}$ *for* $h, h'\in H$.
(2) $\sigma_h\mu_g = \mu_{hg}$ *and* $\mu_g\sigma_h = \mu_{gh}$ *for* $h\in H$ *and* $g\in G\setminus H$.
(3) $\mu_g\mu_{g'} = a\sigma_{gg'} + b\mu_{gg'x} + b\mu_{gg'x^3}$ *for* $g, g'\in G\setminus H$.

Proof. It is easy to show that (1) and (2) hold.

Suppose $g, g'\in G\setminus H$. Remark that $gg'\in H$. By $g'x^2 = x^2g'$, $yg' = g'x^2y$, Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned}\mu_g\mu_{g'} &= (I_n\otimes g)(E\otimes 1 + \tilde{A}_1\otimes y + \tilde{A}_2\otimes x^2y)(E\otimes 1 + \tilde{A}_1\otimes x^2y + \tilde{A}_2\otimes y)(I_n\otimes g')\\ &= (I_n\otimes g)(aI_n\otimes 1 + b(\tilde{A}_1+\tilde{A}_2)\otimes(1+x^2) + bE\otimes(y+x^2y))(I_n\otimes g')\\ &= a\sigma_{gg'} + b(\tilde{A}_1+\tilde{A}_2)\otimes g(1+x^2)g' + bE\otimes g(y+x^2y)g'\\ &= a\sigma_{gg'} + b(\tilde{A}_1+\tilde{A}_2)\otimes gg'(1+x^2) + bE\otimes gg'x^2(y+x^2y)\\ &= a\sigma_{gg'} + b(\tilde{A}_1+\tilde{A}_2)\otimes gg'(xy+x^3y) + bE\otimes gg'x^2y(xy+x^3y)\\ &= a\sigma_{gg'} + b(\tilde{A}_1+\tilde{A}_2)\otimes gg'(xy+x^3y) + bE\otimes gg'(x+x^3)\\ &= a\sigma_{gg'} + b\mu_{gg'x} + b\mu_{gg'x^3}.\end{aligned}$$

Now (3) holds. □

Theorem 3.6. *The set $\mathcal{D} = \{\sigma_h \mid h\in H\}\cup\{\mu_g \mid g\in G\setminus H\}$ forms an association scheme.*

Proof. This is clear by Lemma 3.4 and Lemma 3.5. □

We summarize basic properties of the association scheme $\mathcal{D}$ by Lemma 3.4 and Lemma 3.5.

Proposition 3.7. *For the association scheme $\mathcal{D} = \{\sigma_1, \sigma_{x^2}, \sigma_y, \sigma_{x^2y}, \mu_x, \mu_{x^3}, \mu_{xy}, \mu_{x^3y}\}$, the following properties hold.*

(1) *The valencies of σ_1, σ_{x^2}, σ_y, σ_{x^2y} are 1 and the valencies of μ_x, μ_{x^3}, μ_{xy}, μ_{x^3y} are $n-1$.*
(2) *The thin radical is $O_\theta(\mathcal{D}) = \{\sigma_1, \sigma_{x^2}, \sigma_y, \sigma_{x^2y}\}$, and is normal in $\mathcal{D}$.*
(3) *The thin residue is $O^\theta(\mathcal{D}) = \{\sigma_1, \sigma_{x^2}, \mu_x, \mu_{x^3}\}$.*
(4) *The matrices σ_1, σ_{x^2}, σ_y, σ_{x^2y}, μ_{xy}, μ_{x^3y} are symmetric and μ_x, μ_{x^3} are non-symmetric.*

3.2. The association scheme $\mathcal{Q}$. We define $n \times n$ matrices by

$$\hat{A}_1 \coloneqq \left(\begin{array}{c|c} 0 & 1 \ \dots \ 1 \\ \hline 0 & \\ \vdots & A_1 \\ 0 & \end{array}\right), \quad \hat{A}_2 \coloneqq \left(\begin{array}{c|c} 0 & 0 \ \dots \ 0 \\ \hline 1 & \\ \vdots & A_2 \\ 1 & \end{array}\right).$$

By Lemma 3.1 and direct calculations, we have the following lemma.

Lemma 3.8.

(1) $\hat{A}_1^2 + \hat{A}_2^2 = b(J_n - I_n)$.
(2) $\hat{A}_1\hat{A}_2 + \hat{A}_2\hat{A}_1 = bJ_n + (a-b)I_n$.

We consider a subgroup $K \coloneqq C_G(x) = \{1, x, x^2, x^3\} \cong C_4$ of G. It is easy to see that $\sum_{k\in K} k = \sum_{g\in G\smallsetminus K} g = J_4$. We define $4n \times 4n$ matrices by

$$\sigma_k \coloneqq I_n \otimes k \quad \text{for } k \in K$$

and

$$\begin{aligned}\tau_g &\coloneqq \hat{A}_1 \otimes g + \hat{A}_2 \otimes gx^2 \\ &= (I_n \otimes g)(\hat{A}_1 \otimes 1 + \hat{A}_2 \otimes x^2) \\ &= (\hat{A}_1 \otimes 1 + \hat{A}_2 \otimes x^2)(I_n \otimes g) \quad \text{for } g \in G \smallsetminus K.\end{aligned}$$

We will show that $\mathcal{Q} \coloneqq \{\sigma_k \mid k \in K\} \cup \{\tau_g \mid g \in G \smallsetminus K\}$ forms an association scheme.

Lemma 3.9. *The set $\mathcal{Q}$ is closed under the transposition and $\sum_{k\in K}\sigma_k + \sum_{g\in G\smallsetminus K}\tau_g = J_{4n}$.*

Proof. We have ${}^t\sigma_1 = \sigma_1$, ${}^t\sigma_{x^2} = \sigma_{x^2}$, ${}^t\sigma_x = \sigma_{x^3}$, ${}^t\tau_y = \tau_{x^2y}$, and ${}^t\tau_{xy} = \tau_{x^3y}$. Thus $\mathcal{D}$ is closed under the transposition.

Since $\sum_{k\in K} k = \sum_{g\in G\smallsetminus K} g = J_4$ and $I_n + \hat{A}_1 + \hat{A}_2 = J_n$, we have $\sum_{k\in K}\sigma_k + \sum_{g\in G\smallsetminus K}\tau_g = J_{4n}$. ☐

Lemma 3.10.

(1) *$\sigma_k\sigma_{k'} = \sigma_{kk'}$ for $k, k' \in K$.*
(2) *$\sigma_k\tau_g = \tau_{kg}$ and $\tau_g\sigma_k = \tau_{gk}$ for $k \in K$ and $g \in G \smallsetminus K$.*
(3) *$\tau_g\tau_{g'} = a\sigma_{gg'x^2} + b\tau_{gg'xy} + b\tau_{gg'x^3y}$ for $g, g' \in G \smallsetminus K$.*

Proof. It is easy to show that (1) and (2) hold.

Suppose $g, g' \in G \smallsetminus K$. Remark that $gg' \in K$. By Lemma 3.2 and Lemma 3.8, we have

$$\begin{aligned}\tau_g\tau_{g'} &= (I_n \otimes g)(\hat{A}_1 \otimes 1 + \hat{A}_2 \otimes x^2)^2(I_n \otimes g') \\ &= (I_n \otimes gg')(\hat{A}_1{}^2 \otimes 1 + (\hat{A}_1\hat{A}_2 + \hat{A}_2\hat{A}_1) \otimes x^2 + \hat{A}_1{}^2 \otimes 1) \\ &= (I_n \otimes gg')(b(J_n - I_n) \otimes 1 + (bJ_n + (a-b)I_n) \otimes x^2) \\ &= (I_n \otimes gg')(aI_n \otimes x^2 + b(J_n - I_n) \otimes (1 + x^2)) \\ &= (I_n \otimes gg')(aI_n \otimes x^2 + b(\hat{A}_1 + \hat{A}_2) \otimes (1 + x^2)) \\ &= (I_n \otimes gg')(aI_n \otimes x^2 + b(\hat{A}_1 + \hat{A}_2) \otimes (xy + x^3y)) \\ &= a\sigma_{gg'x^2} + b\tau_{gg'xy} + b\tau_{gg'x^3y}.\end{aligned}$$

Now (3) holds. ☐

Theorem 3.11. *The set $\mathcal{Q} = \{\sigma_k \mid k \in K\} \cup \{\tau_g \mid g \in G \smallsetminus K\}$ forms an association scheme.*

Proof. This is clear by Lemma 3.9 and Lemma 3.10. □

We summarize basic properties of the association scheme $\mathcal{Q}$ by Lemma 3.9 and Lemma 3.10.

Proposition 3.12. *For the association scheme* $\mathcal{Q}=\{\sigma_1,\sigma_{x^2},\sigma_x,\sigma_{x^3},\tau_{xy},\tau_{x^3y},\tau_y,\tau_{x^2y}\}$, *the following properties hold.*

(1) *The valencies of* σ_1, σ_{x^2}, σ_x, σ_{x^3} *are* 1 *and the valencies of* τ_{xy}, τ_{x^3y}, τ_y, τ_{x^2y} *are* $n-1$.
(2) *The thin radical is* $O_\theta(\mathcal{Q})=\{\sigma_1,\sigma_{x^2},\sigma_x,\sigma_{x^3}\}$, *and is normal in* $\mathcal{Q}$.
(3) *The thin residue is* $O^\theta(\mathcal{Q})=\{\sigma_1,\sigma_{x^2},\tau_{xy},\tau_{x^3y}\}$.
(4) *The matrices* σ_1, σ_{x^2} *are symmetric and* σ_x, σ_{x^3}, τ_{xy}, τ_{x^3y}, τ_y, τ_{x^2y} *are non-symmetric.*

4. The character tables of $\mathcal{D}$ and $\mathcal{Q}$

In this section, we will determine the character tables of $\mathcal{D}$ and $\mathcal{Q}$. Consequently, we can see that the tables are the same. Moreover, we will show that their Frobenius–Schur indicators are different.

Proposition 4.1. *The character table of* $\mathcal{D}$ *is*

	σ_1	σ_{x^2}	σ_y	σ_{x^2y}	μ_x	μ_{x^3}	μ_{xy}	μ_{x^3y}	m_{χ_i}
χ_1	1	1	1	1	$n-1$	$n-1$	$n-1$	$n-1$	1
χ_2	1	1	-1	-1	$n-1$	$n-1$	$-n+1$	$-n+1$	1
χ_3	1	1	1	1	-1	-1	-1	-1	$n-1$
χ_4	1	1	-1	-1	-1	-1	1	1	$n-1$
χ_5	2	-2	0	0	0	0	0	0	n

The Frobenius–Schur indicators are $\nu(\chi_i)=1$ $(i=1,2,3,4,5)$.

Proof. By Proposition 3.7 (2), the thin radical $O_\theta(\mathcal{D})=\{\sigma_1,\sigma_{x^2},\sigma_y,\sigma_{x^2y}\}$ is a normal closed subset of $\mathcal{D}$. By [1, Theorem 3.5], we can determine χ_1 and χ_3. By Proposition 3.7 (3), the thin residue $O^\theta(\mathcal{D})=\{\sigma_1,\sigma_{x^2},\mu_x,\mu_{x^3}\}$ is a strongly normal closed subset of $\mathcal{D}$ and χ_2 is determined. By [2, Theorem 3.5], $\chi_4\coloneqq\chi_2\chi_3$ is an irreducible character. Now, by $\sum_{i=1}^5 m_{\chi_i}\chi_i(\rho)=0$ for $\sigma_1\neq\rho\in\mathcal{D}$ [5, Chap. 4], we can determine χ_5.

Since χ_i $(i=1,2,3,4)$ are rational characters of degree 1, they are of the first kind. There are 6 symmetric matrices by Proposition 3.7 (4). By Theorem 2.1 (2),

$$6=\sum_{i=1}^{5}\nu(\chi_i)\chi_i(1)=1+1+1+1+2\nu(\chi_5)$$

and we have $\nu(\chi_5)=1$. □

Proposition 4.2. *The character table of* $\mathcal{Q}$ *is*

	σ_1	σ_{x^2}	σ_x	σ_{x^3}	τ_{xy}	τ_{x^3y}	τ_y	τ_{x^2y}	m_{φ_i}
φ_1	1	1	1	1	$n-1$	$n-1$	$n-1$	$n-1$	1
φ_2	1	1	-1	-1	$n-1$	$n-1$	$-n+1$	$-n+1$	1
φ_3	1	1	1	1	-1	-1	-1	-1	$n-1$
φ_4	1	1	-1	-1	-1	-1	1	1	$n-1$
φ_5	2	-2	0	0	0	0	0	0	n

The Frobenius–Schur indicators are $\nu(\varphi_i)=1$ $(i=1,2,3,4)$ *and* $\nu(\varphi_5)=-1$.

Proof. By Proposition 3.12 (2), the thin radical $O_\theta(\mathcal{Q}) = \{\sigma_1, \sigma_{x^2}, \sigma_x, \sigma_{x^3}\}$ is a normal closed subset of $\mathcal{Q}$. We can determine φ_1 and φ_3. By Proposition 3.12 (3), the thin residue $O^\theta(\mathcal{Q}) = \{\sigma_1, \sigma_{x^2}, \tau_{xy}, \tau_{x^3y}\}$ is a strongly normal closed subset of $\mathcal{Q}$ and φ_2 is determined. Now the character table can be calculated in the same way as in Proposition 4.1.

Since φ_i $(i = 1, 2, 3, 4)$ are rational characters of degree 1, they are of the first kind. There are 2 symmetric matrices by Proposition 3.12 (4). By Theorem 2.1 (2),

$$2 = \sum_{i=1}^{5} \nu(\varphi_i)\varphi_i(1) = 1 + 1 + 1 + 1 + 2\nu(\varphi_5)$$

and we have $\nu(\varphi_5) = -1$. □

5. Irreducible representations and rational adjacency algebras

We will determine irreducible representations and the structures of rational adjacency algebras of $\mathcal{D}$ and $\mathcal{Q}$, respectively.

Proposition 5.1. *The map $T : \mathcal{D} \to \mathrm{M}_2(\mathbb{C})$ defined by*

$$\sigma_{x^2} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mu_x \mapsto \begin{pmatrix} 0 & -1 \\ n-1 & 0 \end{pmatrix}$$

is an irreducible representation of $\mathcal{D}$ affording χ_5.

Proof. We can check all products in Lemma 3.5. □

Similarly, we have the following proposition.

Proposition 5.2. *The map $T' : \mathcal{Q} \to \mathrm{M}_2(\mathbb{C})$ defined by*

$$\sigma_x \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \tau_y \mapsto \begin{pmatrix} 0 & -1 \\ n-1 & 0 \end{pmatrix}$$

is an irreducible representation of $\mathcal{Q}$ affording φ_5.

Now we can determine the structures of rational adjacency algebras. To describe the result, we define (generalized) quaternion algebras [4, 1.6]. For a field F and $r, s \in F \smallsetminus \{0\}$, a *quaternion algebra* $F(r, s)$ is a four dimensional F-algebra with basis 1, $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ with the products

$$\boldsymbol{i}^2 = r, \quad \boldsymbol{j}^2 = s, \quad \boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}.$$

If r and s are negative rational numbers, then $\mathbb{Q}(r, s)$ is a division algebra.

Proposition 5.3. *As $\mathbb{Q}$-algebras, we have the following isomorphisms.*

1. $\mathbb{Q}\mathcal{D} \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathrm{M}_2(\mathbb{Q})$.
2. $\mathbb{Q}\mathcal{Q} \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}(-1, -n+1)$. *($\mathbb{Q}(-1, -n+1)$ is a division algebra.)*

Proof. Since χ_i $(i = 1, 2, 3, 4)$ have rational values on $\mathcal{D}$, we can determine $\mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Q}$. For the representation T in Proposition 5.1, it is easy to see that $T(\mathbb{Q}\mathcal{D}) = \mathrm{M}_2(\mathbb{Q})$. Thus (1) holds.

Since φ_i $(i = 1, 2, 3, 4)$ have rational values on $\mathcal{Q}$, we can determine $\mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Q}$. For the representation T' in Proposition 5.2, the set consisting of

$$T'(\sigma_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad T'(\sigma_x) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix},$$
$$T'(\tau_y) = \begin{pmatrix} 0 & -1 \\ n-1 & 0 \end{pmatrix}, \qquad T'(\tau_{xy}) = \begin{pmatrix} 0 & -\sqrt{-1} \\ -(n-1)\sqrt{-1} & 0 \end{pmatrix}$$

is a $\mathbb{Q}$-basis of $T'(\mathbb{Q}\mathcal{Q})$. We can see that

$$T'(\sigma_x)^2 = -T'(\sigma_1), \qquad T'(\tau_y)^2 = -(n-1)T'(\sigma_1),$$
$$T'(\sigma_x)T'(\tau_y) = -T'(\tau_y)T'(\sigma_x) = T'(\tau_{xy}).$$

This shows that $T'(\mathbb{Q}\mathcal{Q}) \cong \mathbb{Q}(-1, -n+1)$ and (2) holds. □

6. Remark

Our pair $(\mathcal{D}, \mathcal{Q})$ has similar properties to the pair (D_8, Q_8), the dihedral and quaternion groups of order 8. Our association schemes have order $4n$ and $n \equiv 0 \pmod 4$, and thus (D_8, Q_8) is not obtained by our construction. If we set $n = 2$ and $A_1 = A_2 = O$ and apply our construction, then we can construct the pair (D_8, Q_8) and all arguments are valid for it.

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