ALGEBRAIC COMBINATORICS

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ABSTRACT We define a new category analogous to **FI** for the 0-Hecke algebra $H_n(0)$ called the 0-Hecke category, $\mathcal{H}$, indexing sequences of representations of $H_n(0)$ as n varies under suitable compatibility conditions. We establish a new type of representation stability in this setting and prove it is implied by being a finitely generated $\mathcal{H}$-module. We then provide examples of $\mathcal{H}$-modules and discuss further desirable properties these modules possess.

1. INTRODUCTION

The category **FI** of finite sets and injections, first defined in [3], and its variants have been of great interest recently. Being a finitely generated **FI**-module implies many desirable properties that are often very difficult to prove on their own about sequences of symmetric group representations, such as representation stability and polynomial growth. The study of this combinatorial category and its modules has been fruitful, providing tools to prove a variety of stability results about spaces such as $H^i(\mathrm{Conf}_n(M); \mathbf{Q})$ the cohomology of configuration space of n distinct ordered points on a connected, oriented manifold M and many others [3]. In this vein, a variety of other combinatorial categories have been defined. The most relevant categories to us that have seen a surge of interest in recent years are **FI** and **OI**, for a general survey of results we refer the reader to [3, 35, 37, 38]. Recently, in [11] the authors establish a structure theory for **OI**-modules similar to the structure theory for **FI**-modules in [37] by studying $\mathcal{J}$ the monoid of increasing functions.

One can loosely think of **FI**-modules as sequences of compatible vector spaces with a complete symmetry present, and similarly **OI**-modules can be thought of as sequences of compatible vector spaces with no requirement of symmetry. A similar story plays out in the analogous field of symmetric function theory as seen in [12], where the ring of symmetric functions **Sym** plays the role of **FI** and the ring of nonsymmetric functions plays the role of **OI**. Interpolating between these two rings is the ring of quasisymmetric functions **QSym**. One of the goals of this paper is to define the analogous category that interpolates between **FI** and **OI** and explore its surprising structure.

This category turns out to be the categorical analogue of **FI** for the 0-Hecke algebra. We denote it by $\mathcal{H}$ and call it the 0-*Hecke category*. We begin by defining $\mathcal{H}$ and

discussing how it can be viewed as a quotient of the braid category as seen in [32, §1.2]. We then prove a variety of structural results about $\mathcal{H}$-modules. Most notably, despite the 0-Hecke algebra not being semi-simple, one can say a surprising amount about the underlying $H_n(0)$ representation theory of a finitely generated $\mathcal{H}$-module. In particular, we are able to construct an explicit basis for the Grothendieck group of $\mathcal{H}$-modules $\mathcal{G}(\mathrm{Mod}_{\mathcal{H}})$ consisting of *padded induced modules* $M(\alpha, k)$ where α is a composition and $-1 \leqslant k \leqslant |\alpha|$ is an integer. These modules are quotients of the standard induced modules. This ultimately allows us to prove that finitely generated $\mathcal{H}$-modules satisfy a new form of representation stability,

THEOREM 1.1. *For any finitely generated $\mathcal{H}$-module V, we have a unique finite decomposition in the Grothendieck group $\mathcal{G}(\mathrm{Mod}_{\mathcal{H}})$,*

$$[V_n] = \sum_{i,j} c_{\alpha_i, k_j} [M(\alpha_i, k_j)_n]$$

where the coefficients c_{α_i, k_j} do not depend on n.

As with the standard form of representation stability for **FI**, this theorem implies that a finite list of data, namely $\{(\alpha_i, k_i)\}$, completely describe all the irreducible $H_n(0)$ representations that will occur in V_n for $n \gg 0$. We also describe the exact process to go from the finite list of data to the irreducible representations one desires.

We then study an important statistic often associated with these combinatorial categories Gabriel–Krull dimension due to [9]. One can think of this as the analogue of Krull dimension in commutative algebra, it gives a rough measure of the complexity of the category. We prove,

THEOREM 1.2. *The Gabriel–Krull dimension of $\mathcal{H}$ is infinite.*

In this respect, $\mathcal{H}$ bears more similarity to **OI** which [11] shows has infinite Gabriel–Krull dimension. We dedicate the remainder of the paper to exploring examples of $\mathcal{H}$-modules. We expect there are many more examples,

1) In Section 9 we explore one of the more basic but tractable finitely $\mathcal{H}$-modules $[n] \mapsto k[x_1, \ldots, x_n]_d$ the degree d polynomials in n variables. It is not hard to show this forms a $\mathcal{H}$-module where $H_n(0)$ acts by Demazure operators. This example is instructive because for small values of d we can explicitly see the representation stability predicted by Theorem 1.1.

2) In Section 10 we present one of the motivating examples for this paper. From work of [33] and [13] it was known that the group cohomology $H^i(B(n,q), \mathbf{F}_q)$, where $B(n,q)$ is the Borel subgroup of $\mathbf{GL}(n,q)$, carried an action of $H_n(0)$ for any fixed $i \geqslant 0$. It does not, however, have any natural complete symmetry, i.e. an action of $\mathcal{S}_n$. In this setting it is natural to ask if a $\mathcal{H}$-module structure is present, we are able to prove

THEOREM 1.3. *The assignment $[n] \mapsto H^i(B(n,q), \mathbf{F}_q)$ with compatibility maps described in Section* 10 *forms a $\mathcal{H}$-module.*

We believe this defines a finitely generated $\mathcal{H}$-module, but this problem is still open.

3) In Section 11 we study one of the other motivating examples for this paper, $H_i(B(n,q), \mathbf{F}_q)$. By a spectral sequence argument $H_i(B(n,q), \mathbf{F}_q)$ is equal to $H_i(U(n,q), \mathbf{F}_q)$, where $U(n,q)$ is the unipotent subgroup. This motivated Putman, Sam and Snowden to prove that the assignment $[n] \mapsto H_i(U(n,q), \mathbf{F}_q)$ with inclusion maps induced by inclusion of unipotent subgroups is a finitely generated **OI**-module [29]. We define a new $H_n(0)$-action on $H_i(U(n,q), \mathbf{F}_q)$ and prove

THEOREM 1.4. *The assignment* $[n] \mapsto H_i(U(n, q), \mathbf{F}_q)$ *with compatibility maps*

$$\Phi_n \colon H_i(U(n, q), \mathbf{F}_q) \to H_i(U(n+1, q), \mathbf{F}_q)$$

induced by the natural inclusion $U(n, q) \hookrightarrow U(n+1, q)$ *is a finitely generated* $\mathcal{H}$-*module.*

It turns out that the restriction to **OI** of the $\mathcal{H}$-action gives an action equivalent to the one defined in [29]. This result is a strengthening of their result because $\mathcal{H}$ has more structure. In particular, Theorem 1.4 implies

THEOREM 1.5. *Fix* $i \geqslant 0$, *the sequence of* $H_n(0)$-*modules* $\{H_i(U(n, q), \mathbf{F}_q)\}_{n \geqslant 0}$ *is representation stable.*

4) In Section 12 we explain how we can turn the collection of Stanley Reisner rings of the Boolean algebra into a $\mathcal{H}$-module using work of [14]. We then prove this $\mathcal{H}$-module is finitely generated and so is representation stable in our new sense.

5) Finally, in Section 13 we make the connection to **QSym** more explicit and explore how a new class of modules defined by [41] which map to quasisymmetric Schur functions as defined in [12] under the quasisymmetric characteristic map have a finitely generated $\mathcal{H}$-module structure. These modules provide another example of a $\mathcal{H}$-module that cannot be a **FI**-module.

1.1. OUTLINE OF ARGUMENT. The proof of Theorem 1.1 breaks into the following steps:

1) For each composition α, we first define an important class of $\mathcal{H}$-modules called induced modules $M(\alpha)$ in Section 3 and use this to define padded induced modules $M(\alpha, k)$ in Section 4.

2) Every finitely generated $\mathcal{H}$-module has eventually polynomial growth, the degree of the polynomial that eventually describes the growth is called the *polynomial degree* of the $\mathcal{H}$-module. We prove that padded induced modules have the property that any proper quotient of them has strictly smaller polynomial degree in Theorem 4.6.

3) We then argue in Theorem 5.1 that $M(\alpha, k)$ is the smallest polynomial degree k quotient of $M(\alpha)$, i.e. any other polynomial degree k quotient of $M(\alpha)$ must contain $M(\alpha, k)$.

4) The above allows us to deduce that $M(\alpha, k)$ is simple in the Serre quotient $\mathcal{H}_k$ of polynomial degree $\leqslant k$ objects by polynomial degree $\leqslant k-1$ objects and that every simple is of this form. This can be seen in Lemma 5.2.

5) Lemma 5.2 means that the $M(n)$ are all finite length, and every finitely generated $\mathcal{H}$-module is a quotient of a direct sum of the $M(n)$, so we can argue that every object in the Serre quotient has finite length (Lemma 5.3). This allows us to find a finite length filtration of any finitely generated polynomial degree k $\mathcal{H}$-module V in $\mathcal{H}_k$ with successive quotients isomorphic to $M(\alpha_i, k)$ (Theorem 5.4).

6) We then use this filtration to show that the isomorphism classes $[M(\alpha, k)]$ as α ranges over all compositions and $k \in \mathbf{Z}$ ranges $-1 \leqslant k \leqslant |\alpha|$ span the Grothendieck group of finitely generated $\mathcal{H}$-modules, $\mathcal{G}(\mathrm{Mod}_{\mathcal{H}})$ in Theorem 5.5.

7) Finally, we construct functions on the Grothendieck group to argue that the $[M(\alpha, k)]$ are in fact linearly independent and hence form a basis for $\mathcal{G}(\mathrm{Mod}_{\mathcal{H}})$ (Theorem 5.6). This ultimately implies our new form of representation stability.

1.2. RELATION TO PREVIOUS WORK. We show in Theorem 2.13 that $\mathcal{H}$ is another example of an ever growing class of noetherian categories. In [38] the authors describe a Gröbner method for proving noetherianity of combinatorial categories which we apply here. This underlying idea fits into a broader area of interest called *noetherianity up to symmetry*. For a nice introduction we recommend [5]. Ultimately, one works with

a space or object on which a group or algebra acts and proves finite generation up to the action of this group or algebra. Noetherianity up to symmetry is important in [26, 35, 36, 37], where the authors explore various manifestations of this idea to prove finite generation results for various representations of categories and twisted commutative algebras. These ideas are also present in [3, 6, 20, 34, 40, 42] and many other recent papers.

The 0-Hecke algebras $H_n(0)$ are also an example of a growing class of diagram algebras that satisfy some form of representation stability. Diagram algebras and their representations have been of great interest to researchers for many years. Brauer originally introduced the Brauer algebra in [2] to study Schur–Weyl duality for the orthogonal group. Since then, many other diagram algebras appeared in the literature, we refer the reader to [16, 17, 21, 24, 43]. Recently, Patzt in [28] showed that three of the most important and well studied diagram algebras, the Partition, Brauer and Temperly–Lieb diagram algebras, also satisfy a form of representation stability. The Brauer algebra, and its representation theory, has also received considerable attention in [39].

As we have mentioned one can view $\mathcal{H}$ as interpolating between **FI** and **OI**. This paper opens the door to many questions explored in [3] for **FI**-modules and [38, 11] for **OI**-modules.

$\mathcal{H}$ has concrete connections to **QSym**, the ring of quasisymmetric functions, as seen in Section 13. This ring has seen a resurgence of interest as of late, for a general survey we refer the reader to [25]. The connection between **FI** and the ring of symmetric functions is very explicit, in particular the authors in [37] prove that the Grothendieck group of **FI**-modules is isomorphic to two copies of the ring of symmetric functions. It would be interesting to see if a similar result were true about $\mathcal{H}$ and what else the category could tell us about **QSym**.

This paper also provides a more systematic approach to studying natural sequences of representations of the 0-Hecke algebra. Recent such examples addressed in this paper include the Stanley–Reisner ring of the Boolean algebra studied in [14], quasisymmetric Schur modules studied in [41], and $H^i(B(n,q), \mathbf{F}_q)$ as studied in [13, 33]. There are many other examples we are working on in a future paper including ordered set partitions studied in [15].

Quillen computed the cohomology of the general linear group in non-defining characteristic, $H^i(\mathbf{GL}(n,q), \mathbf{F}_\ell)$, using Sylow subgroups as a key ingredient [30]. In defining characteristic, $B(n,q)$ is a Sylow-q subgroup of $\mathbf{GL}(n,q)$, and so researchers have naturally sought to understand $H^i(B(n,q); \mathbf{F}_q)$. Quillen's methods depend on working over non-defining characteristic and the pre-existing **FI** literature does not naturally apply because there is no natural symmetric group action. The category $\mathcal{H}$ provides the missing categorical framework to study the group homology of the Borel group in defining characteristic.

1.3. Conventions. For the majority of the paper **k** denotes a field of arbitrary characteristic because the representation theory of $H_n(0)$ does not depend on characteristic. The majority of the notation for this paper is outlined in Section 2.

2. Background

2.1. Compositions. A *composition* α of n is a list of positive natural numbers $(\alpha_1, \ldots, \alpha_k)$ such that $\alpha_1 + \cdots + \alpha_k = n$. There are 2^{n-1} compositions of n. We will denote that α is a composition of n by writing $\alpha \vDash n$. Compositions of n isomorphically correspond to *descent sets* on $[n-1]$, subsets of increasing integers in $[n-1]$, in the following way. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ we write

$D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$. It is also easy to reverse this process. Given a descent set D, we denote by $C(D)$ the corresponding composition.

We can view compositions combinatorially as ribbon diagrams. A *ribbon diagram* is an edgewise connected skew diagram with no 2 by 2 boxes. The composition $\alpha \vDash n$ given by $(\alpha_1, \ldots, \alpha_k)$ corresponds to the ribbon diagram with α_i boxes in row i. For example, $(2, 1, 3)$ corresponds to the ribbon diagram
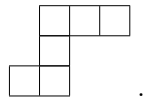
Given a composition α, let $\ell(\alpha)$, the *length* of α, denote the number of entries in α. Combinatorially, this is the number of rows in the ribbon diagram of shape α. We denote by $w(\alpha)$ the *width* of α. This gives the number of columns in the ribbon diagram of α.

PROPOSITION 2.1. *For any composition* $\alpha \vDash n$, $w(\alpha) = n - \ell(\alpha) + 1$.

Proof. To see why this is the case, notice that the number of columns in the ribbon diagram is exactly,

$$(\alpha_1) + (\alpha_2 - 1) + \cdots + (\alpha_k - 1) = (\alpha_1 + \cdots + \alpha_k) - k + 1 = n - \ell(\alpha) + 1. \qquad \square$$

Given two compositions α, β we define their *sum* to be $\alpha + \beta = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell)$ and their *join* as $\alpha \lhd \beta = (\alpha_1, \ldots, \alpha_k + \beta_1, \beta_2, \ldots, \beta_\ell)$.

In a ribbon diagram, we enumerate boxes from left to right and bottom to top. So by the i-th box, we mean that i-th box counted in this manner. We will often discuss adding boxes to positions of a ribbon diagram, we make this concrete here. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha \vDash n$, adding a box to the *i-th position* of α means that we add a box above the $(i-1)$-st box in the ribbon diagram corresponding to α and shift the corresponding boxes up to ensure we still have a ribbon diagram. Explicitly, if the $(i-1)$-st box is in α_j, i.e. $\alpha_1 + \cdots + \alpha_{j-1} < i - 1 \leqslant \alpha_1 + \cdots + \alpha_j$, then the ribbon diagram where we add a box to position $i - 1$ is exactly

$$(\alpha_1, \ldots, \alpha_{j-1}, \alpha_j - \beta_i, 1) \lhd (\beta_i, \alpha_{j+1}, \ldots, \alpha_k),$$

where β_i is a correction factor so that $\alpha_1 + \cdots + \alpha_j - \beta_i = i - 1$.

From this we can see that in a composition of n, it makes sense to add a box to any position i with $1 \leqslant i \leqslant n + 1$. We say that $\alpha \vDash n$ has $n + 1$ positions, so position$(\alpha) = |\alpha| + 1$. Combinatorially this is because we are allowed to add a box to the left of the first box as well.

2.2. 0-HECKE ALGEBRA. The 0-*Hecke algebra* $H_n(0)$ is the associative algebra generated by $\pi_1, \ldots, \pi_{n-1}$ over an arbitrary field $\mathbf{F}$ where the generators satisfy the relations

$$\begin{cases} \pi_i^2 = \pi_i & 1 \leqslant i \leqslant n - 1 \\ \pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} & 1 \leqslant i \leqslant n - 2 \\ \pi_i \pi_j = \pi_j \pi_i & |i - j| > 1. \end{cases}$$

We will most often use this generating set. The last two relations are commonly called the *braid relations* and the first relation is sometimes called the *skein relation*. $H_n(0)$ is a deformation of the symmetric group $\mathfrak{S}_n$ and one of the most well studied degenerate deformations. Recall that the symmetric group has Coxeter generators $\{s_1, \ldots, s_{n-1}\}$ where the $s_i = (i, i+1)$ is the adjacent transposition. These generators satisfy the braid relations, but not the skein relation, instead $s_i^2 = 1$. If $w \in \mathfrak{S}_n$ is a permutation and $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression for w in the Coxeter

generators we define the $H_n(0)$ algebra element $\pi_w := \pi_{i_1} \cdots \pi_{i_k} \in H_n(0)$. One can show that the set $\{\pi_w \mid w \in \mathfrak{S}_n\}$ forms a basis for $H_n(0)$ as an **F**-vector space.

It is well known that $H_n(0)$ has another algebra generating set $\{\overline{\pi}_1, \ldots, \overline{\pi}_{n-1}\}$ subject to the relations

$$
\begin{cases}
\overline{\pi}_i^2 = -\overline{\pi}_i & 1 \leqslant i \leqslant n-1 \\
\overline{\pi}_i \overline{\pi}_{i+1} \overline{\pi}_i = \overline{\pi}_{i+1} \overline{\pi}_i \overline{\pi}_{i+1} & 1 \leqslant i \leqslant n-2 \\
\overline{\pi}_i \overline{\pi}_j = \overline{\pi}_j \overline{\pi}_i & |i-j| > 1.
\end{cases}
$$

Here $\overline{\pi}_i = \pi_i - 1$ for all i.

In [27] the author characterizes all simple and projective $H_n(0)$ modules. In particular, they are indexed by compositions $\alpha \vDash n$. The complete list of irreducible modules is given by $\{\mathbf{C}_\alpha\}_\alpha$ where $\mathbf{C}_\alpha$ is a one dimensional $H_n(0)$-module spanned by v_α where for $i = 1, \ldots, n-1$,

$$
\pi_i v_\alpha = \begin{cases}
0 & i \in D(\alpha) \\
v_\alpha & i \notin D(\alpha).
\end{cases}
$$

We will denote the complete list of projective modules by $\{\mathbf{P}_\alpha\}$. For more details we refer the reader to [27]. As a quick example we note that $\mathbf{C}_n = \mathbf{P}_n$ is the trivial representation where all generators π_i act by 1 because $D(n) = \{\}$.

2.3. SYMMETRIC FUNCTION THEORY. Let $X = (x_1, x_2, \ldots)$ be a totally ordered infinite set of variables. Then we denote the **Z**-algebra of *symmetric functions* in X with coefficients in **Z** by Sym. There is a clear **N**-grading on this algebra, its degree n component has basis given by the *Schur functions* $\{s_\lambda \mid \lambda \vdash n\}$, i.e. λ is a partition of n. The Schur function s_λ can be expressed as

$$
(1) \qquad s_\lambda = \sum_T \mathbf{x}^T
$$

where the sum is over all semi-standard tableau T of shape λ and $\mathbf{x}^T$ is the monomial

$$
\mathbf{x}^T := x_1^{c_1(T)} x_2^{c_2(T)} \cdots
$$

where $c_i(T)$ is the number of times i appears in T. Given partitions $\mu \subset \lambda$, we also let $s_{\lambda/\mu} \in$ Sym denote the *skew Schur function*. This function is defined via Equation (1), where we sum over all skew-tableau of shape λ/μ. The most important example of a skew Schur function for us is a *ribbon Schur function*, where λ/μ is a ribbon tableau. We will index these by s_α where α is a composition corresponding to the ribbon λ/μ. For further reading on symmetric functions we refer the reader to [23].

There is a coproduct structure on Sym given by replacing the variables $x_1, x_2, \ldots$ with $x_1, x_2, \ldots, y_1, y_2, \ldots$ so that Sym becomes a graded Hopf algebra which is self dual under the basis $\{s_\lambda\}$ [10, §2].

There is a larger algebra which contains Sym given by loosening the requirement that the functions be symmetric. Once again if we let X be a totally ordered set of variables, we can define the **Z**-algebra of *quasisymmetric functions* **QSym** as the power series of bounded degree in X which are upward shift invariant in the sense that the coefficient of the monomial $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ is equal to the coefficient of $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ for any increasing sequence of integers $i_1 < i_2 < \cdots < i_k$. Notice that every symmetric function satisfies this property, but there are quasisymmetric functions that are not symmetric, for example

$$
x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2
$$

is a quasisymmetric function in 3 variables but is not symmetric. This means that we naturally have **Sym** $\subset$ **QSym**. The algebra **QSym** has a basis consisting of *monomial*

quasisymmetric functions M_α. For a composition $\alpha \vDash n$ with $\alpha = \{\alpha_1, \ldots, \alpha_k\}$

$$M_\alpha := \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

We can use this basis to define one of the most important bases for **QSym**, the *Gessel fundamental quasisymmetric functions*, denoted $\{F_\alpha\}$. To begin we let $F_0 = 1$ and for any composition α,

$$F_\alpha = \sum_{\beta \preceq \alpha} M_\beta$$

where $\beta \preceq \alpha$ if we can obtain α by adding together adjacent elements of β. For example $(1, 1, 2, 3, 2) \preceq (2, 2, 5)$. This is sometimes called the *refinement order* on compositions.

The final algebra we consider is **NSym** the graded algebra of *noncommutative symmetric functions*. This is the free unital associative noncommutative algebra of noncommutative functions in X invariant under the natural symmetric group action. Alternatively, we can define it as the free unital associative noncommutative algebra $\mathbf{Z}\langle h_1, h_2, \ldots \rangle$ generated over $\mathbf{Z}$ by the symbols $h_1, h_2, \ldots$ where h_d has degree d. The degree n component of **NSym** has $\mathbf{Z}$-basis given by $\{\mathbf{h}_\alpha \mid \alpha \vDash n\}$ where

$$\mathbf{h}_\alpha = h_{\alpha_1} \cdots h_{\alpha_k}.$$

Remember this is a noncommutative product, so considering compositions instead of partitions is important. One of the most important bases for the degree n component of **NSym** consists of the noncommutative ribbon Schur functions, $\{s_\alpha \mid \alpha \vDash n\}$. We note that these are not the same as the ribbon Schur functions mentioned above, because they are not commutative, their commutative image will be one of the ribbon Schur functions. In terms of the h_α we define these as

$$s_\alpha := \sum_{\beta \preceq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} \mathbf{h}_\beta.$$

As in the symmetric function case there are coproducts on **QSym** and **NSym** but they are not self dual, instead they are dual to each other as Hopf algebras [10, §5].

2.4. QUASICHARACTERISTIC MAPS. Let A be a finite dimensional algebra over a field $\mathbf{F}$. We will often work with the *Grothendieck group* $\mathcal{G}(A)$ of finitely generated A-modules. This group is the quotient of the free abelian group generated by isomorphism classes of finitely generated A-modules $[M]$ by the relation $[M] - [L] - [N]$ if there is a short exact sequence of finitely generated A-modules

$$0 \to L \to M \to N \to 0.$$

If we let $A = H_n(0)$, $\mathcal{G}(H_n(0))$ has free basis given by the collection of isomorphism classes of irreducible $H_n(0)$-modules, i.e. the $\{[C_\alpha]\}$ for all $\alpha \vDash n$. We will not use it much for now, but it is worth mentioning that there is another Grothendieck group $\mathcal{K}(A)$ consisting of all finitely-generated projective A-modules which has free basis given by the collection of isomorphism classes of indecomposable projective $H_n(0)$-modules. These were also characterized in [27] and correspond to compositions of n as well.

In the following we assume some representation theory background, a good resource for these results is [7]. The symmetric group algebra $\mathbf{Q}[\mathfrak{S}_n]$ is semisimple and has irreducible representations V_λ indexed by partitions $\lambda \vdash n$. In this setting, the Grothendieck group $\mathcal{G}(\mathbf{Q}[\mathfrak{S}_\bullet])$ of the tower

$$\mathbf{Q}[\mathfrak{S}_n]: \mathbf{Q}[\mathfrak{S}_0] \hookrightarrow \mathbf{Q}[\mathfrak{S}_1] \hookrightarrow \cdots$$

of symmetric group algebras is the direct sum of the $\mathcal{G}(\mathbf{Q}[\mathfrak{S}_n])$ for all $n \geqslant 0$. It can be given the structure of a graded Hopf algebra where the product and coproduct are respectively induction and restriction of representations along the natural embedding $\mathfrak{S}_n \otimes \mathfrak{S}_m \hookrightarrow \mathfrak{S}_{n+m}$. The *Frobenius characteristic map* Fch of a finite dimensional $\mathbf{Q}[\mathfrak{S}_n]$-modules V is defined first on the basis for $\mathcal{G}(\mathbf{Q}[\mathfrak{S}_n])$ consisting of isomorphism classes of Specht modules $[\mathbf{V}_\lambda]$

$$\mathrm{Fch}([\mathbf{V}_\lambda]) = s_\lambda$$

where s_λ is the Schur function corresponding to the partition λ, we then extend linearly. Incredibly, the map Fch gives a graded Hopf algebra isomorphism between $\mathcal{G}(\mathbf{Q}[\mathfrak{S}_\bullet]) \cong \mathrm{Sym}$ [10, §4.4]. This connection has been incredibly useful as it allows us to study the representation theory of the symmetric group by working with symmetric functions and vice versa. One important result coming from this connection is the decomposition of tensor products $V_\lambda \otimes V_n$ of Specht modules considered now as a representation of $\mathfrak{S}_{|\lambda|+n}$ into its irreducible components via the Pieri rule. For further details we refer the reader to [23, I 5].

It turns out that there are two analogous characteristic maps Ch and **ch** defined by Krob and Thibon [18], which make it possible to study representations of $H_n(0)$ through the rings **QSym** and **NSym** defined above. We will recall their construction because it will be very important in the coming sections.

As discussed, the two Grothendieck groups $\mathcal{G}(H_n(0))$ and $\mathcal{K}(H_n(0))$ have bases given by $\{[\mathbf{C}_\alpha] \mid \alpha \vDash n\}$ and $\{[\mathbf{P}_\alpha] \mid \alpha \vDash n\}$ respectively. Similar to the symmetric group algebra case, we have Grothendieck groups $\mathcal{G}(H_\bullet(0)) := \bigoplus_{n \geqslant 0} \mathcal{G}(H_n(0))$ and $\mathcal{K}(H_\bullet(0)) := \bigoplus_{n \geqslant 0} \mathcal{K}(H_n(0))$ associated to the tower of algebras

$$H_\bullet(0) \colon H_0(0) \hookrightarrow H_1(0) \hookrightarrow H_2(0) \hookrightarrow \cdots .$$

These groups are both graded Hopf algebras with product and coproduct given by induction and restriction of representations along the natural embedding $H_n(0) \otimes H_m(0) \hookrightarrow H_{n+m}(0)$. Furthermore, they are dual to each other via the pairing $\langle [\mathbf{P}_\alpha], [\mathbf{C}_\beta] \rangle = \delta_{\alpha,\beta}$ where this is the Kronecker-Delta function.

Krob and Thibon [18] define two linear characteristic maps

$$\mathrm{Ch} \colon \mathcal{G}(H_\bullet(0)) \to \mathbf{QSym} \qquad \text{and} \qquad \mathbf{ch} \colon \mathcal{K}(H_\bullet(0)) \to \mathbf{NSym},$$

by $\mathrm{Ch}([\mathbf{C}_\alpha]) := F_\alpha$ and $\mathbf{ch}(\mathbf{P}_\alpha) := s_\alpha$ where F_α is the fundamental quasisymmetric function and s_α is the noncommutative ribbon Schur function. They then show these maps are graded isomorphisms of Hopf algebras. For more information we refer the reader to [18].

2.5. THE 0-HECKE CATEGORY $\mathcal{H}$. In [1] the author describes a natural way to view $H_n(0)$ as a quotient of the braid group B_n. Recall that a *geometric braid* is a disjoint union of n edges called *strands*, in $D \times I$ where $I = [0,1]$ and D is a closed disk. The set of endpoints of the strands is required to be $\{p_1, \ldots, p_n\} \times \{0,1\}$, and each strand is required to intersect each disk cross-section exactly once. Two geometric braids are said to be equivalent if it is possible to continuously deform one to get the other. The elements of B_n are equivalence classes of braids under this continuous deformation equivalence relation. For more we refer the reader to [1]. From now on, we will refer to geometric braids as braids.

DEFINITION 2.2. *A crossing in a braid is called* positive *if the strand on top in the crossing goes from top left to bottom right, otherwise it is called* negative.

We can encode each π_i as a braid with strands connecting vertices i to $i + 1$ in the top row to vertices $i + 1$ to i in the bottom row with a positive crossing. Then every

element of $H_n(0)$ can be represented as a composition of positive crossing diagrams and hence can be viewed as a braid diagram with strictly positive crossings, meaning the strand originating at the first vertex crosses above any vertex it crosses. The strand originating from the second vertex can only cross below the strand originating from vertex 1 and above all other strands etc.

DEFINITION 2.3. *We call a braid diagram from* [n] *vertices to* [m] *vertices* order preserving *if there are no crossings.*

We will define $\mathcal{H}$ as the quotient of the Braid Category in [32, §1.2] by the skein relation, i.e. that $\pi_i^2 = \pi_i$. Graphically, this is the relation
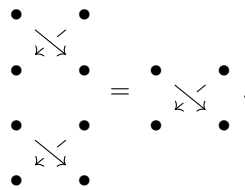

More explicitly, the objects of $\mathcal{H}$ consist of sets $[n]$ for $n \in \mathbf{Z}_{\geqslant 0}$ and $\hom_{\mathcal{H}}([n],[m])$ for $n \leqslant m$ is given by all positive crossing braid diagrams from n vertices to m vertices.

To compose two maps $f \in \hom_{\mathcal{H}}([n],[m])$ and $g \in \hom_{\mathcal{H}}([m],[\ell])$ we view both maps as elements of $H_\ell(0)$ by padding the top rows of the corresponding diagrams with *dummy vertices*, called "free" ends in [32], so that every row has ℓ vertices. We then connect the dummy vertices to the vertices in the bottom row that have degree 0 so that we do not introduce any crossings between the edges we introduce and so that every crossing we must create is positive. The edges adjacent to dummy vertices are called *dummy edges*. Now compose as we would in $H_\ell(0)$, i.e. by following paths and applying the skein relations to ensure we have positive crossings. We then remove all dummy vertices and the dummy edges from the resulting composition.

Notice that the number of dummy vertices we add to f does not matter after we add $m - n$ because after that each dummy vertex is mapped identically to itself.

The following are an important list of relations that result from the above composition rules:
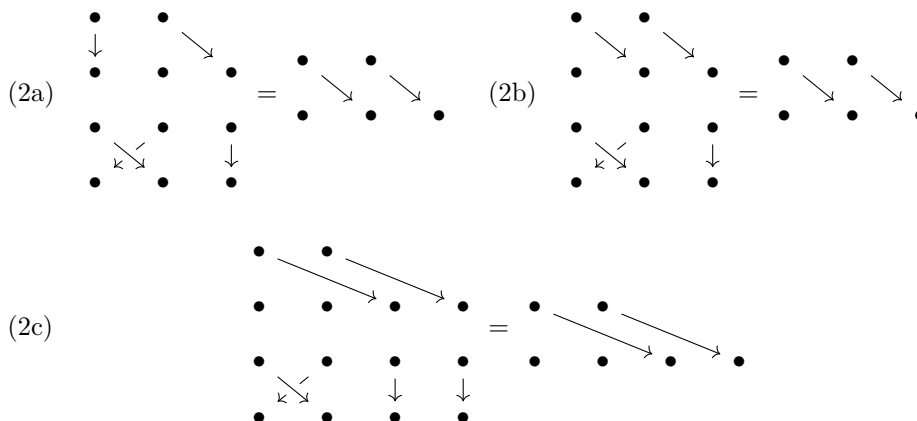
REMARK 2.4. It is easy to check that all of these relations arise from our padded composition in $\mathcal{H}$. We will illustrate how to check Relation (2b) and leave the rest for the reader. Once again, dummy vertices are denoted by $\odot$ and dummy edges are

dashed, Relation (2b) can be seen as,
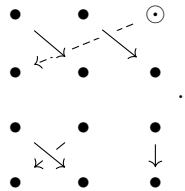
In $H_3(0)$ this is $\pi_1\pi_1\pi_2 = \pi_1\pi_2$ by the skein relation. Recall that composition is read from right to left and with our diagrams we read from top to bottom. As a result when we remove the dummy vertex and its connected edge we recover the original diagram.

PROPOSITION 2.5. **OI** *is a subcategory of* $\mathcal{H}$.

Proof. We have a natural embedding of objects. For maps, every order preserving injection corresponds to an order preserving braid diagram. It remains to check that given two order preserving maps, they compose as they would in **OI** using our composition rules.

Let f and g be two order preserving maps, $f \in \hom_{\mathcal{H}}([n],[m])$, $g \in \hom_{\mathcal{H}}([m],[\ell])$. To compose we will pad f with $\ell - n$ dummy vertices and g with $\ell - m$ dummy vertices. When we add the dummy edges by virtue of the fact that the original maps were order preserving there will never be crossings between non-dummy edges in the resulting composition in $H_\ell(0)$. So when we remove the dummy vertices and edges, we will recover the exact composition we would expect in **OI**. $\qquad\square$

PROPOSITION 2.6. *Composition in* $\mathcal{H}$ *is well defined and associative.*

Proof. One way to see this is that composition is well defined in the Braid Category and hence is well defined in the quotient. This also follows immediately from the fact that composition is well defined and associative in $H_n(0)$ for any n. Since when we compose two maps $f \in \hom_{\mathcal{H}}([n],[m])$ and $g \in \hom_{\mathcal{H}}([m],[\ell])$ we first pad them and proceed to compose the corresponding padded diagrams in $H_\ell(0)$. Associativity in $H_\ell(0)$ implies that no matter how we compose we get the same diagram, so when we remove dummy vertices and edges we get the same well defined diagram in $\hom_{\mathcal{H}}([n],[\ell])$. $\qquad\square$

DEFINITION 2.7. *We call the natural injection of* $H_n(0)$ *into* $H_m(0)$ *sending* $\pi_i \mapsto \pi_i$ *for* $i = 1, \ldots, n-1$ *the* principal injection *and denote it by* $\iota_{n,m}$. *This is represented by the order preserving braid diagram that sends vertex i to vertex i.*

Notice that $H_m(0)$ acts on $\hom_{\mathcal{H}}([n],[m])$ via post-composition. We can see that the element $\iota_{n,m}$ generates all other elements under this action, but the action is not transitive as in the **FI** case, which we would expect because there are no inverses in $H_m(0)$. We can also see that the annihilator of $\iota_{n,m}$ is isomorphic to $H_{m-n}(0)$ because it is generated by all the maps $\overline{\pi}_{n+1}, \ldots, \overline{\pi}_m$. From this we can conclude,

PROPOSITION 2.8. *As a* $H_m(0)$-*module,*

$$\hom_{\mathcal{H}}([n],[m]) \cong H_m(0)/H_{m-n}(0).$$

Proof. Define a $H_m(0)$-equivariant map

$$\phi\colon H_m(0) \to \hom_{\mathcal{H}}([n],[m])$$

via the action of the element on $\iota_{n,m}$, so $\phi(\pi_i) = \pi_i \iota_{n,m}$ and we extend so ϕ is an algebra homomorphism. From the above observations, this map is surjective. By the first isomorphism theorem we have

$$H_m(0)/\ker(\phi) \cong \hom_{\mathcal{H}}([n],[m]).$$

But by definition $\ker(\phi)$ is exactly the annihilator of $\iota_{n,m}$, and this is isomorphic to $H_{m-n}(0)$. □

Using this, we can derive a $\mathcal{H}$-module criterion.

THEOREM 2.9 ($\mathcal{H}$-module criterion). *Suppose that $\{W_n\}$ is a sequence of $H_n(0)$ representations with $H_n(0)$-equivariant maps $\phi_n \colon W_n \to W_{n+1}$, where the action on W_{n+1} is given by embedding $H_n(0)$ into $H_{n+1}(0)$ via the principal injection $\iota_{n,n+1}$. Let $K \cong H_{m-n}(0)$ be the stabilizer of $\iota_{n,m}$ under the action of $H_m(0)$ by post composition. Then $\{W_n\}$ can be promoted to a $\mathcal{H}$-module with $(\iota_{n,n+1})_* = \phi_n$ if and only if for all $n < m$*

$$\pi \cdot v = v \qquad \forall \pi \in K \qquad and \qquad v \in \text{image}((\iota_{n,m})_*).$$

Proof. First if we assume that $\{W_n\}$ is a $\mathcal{H}$-module, we clearly have $\pi \cdot v = v$ for every $\pi \in K$ and $v \in \text{image}((\iota_{n,m})_*)$ by the rules for composition, since we know the stabilizer will be generated by $\pi_{n+1}, \ldots, \pi_m$ and all of these element act by the identity after $\iota_{n,m}$ so by functoriality we must have $\pi_*(\iota_{n,m})_* = (\pi \cdot \iota_{n,m})_* = (\iota_{n,m})_*$.

For the other direction, we can factor any map $f \in \hom_{\mathcal{H}}([n],[m])$ as $f = T_w \iota_{n,m}$ where T_w is a word in the generators of $H_m(0)$. We will define the $\mathcal{H}$ action by letting $(\iota_{n,n+1})_* = \phi_n$ and then $f_* = T_w \phi_{m-1} \cdots \phi_n$. It remains to check that functoriality is satisfied.

Suppose we have $f \in \hom_{\mathcal{H}}([n],[m])$ and $g \in \hom_{\mathcal{H}}([m],[\ell])$, then we claim

$$(g \circ f)_* = T_p \phi_{\ell-1} \cdots \phi_m T_w \phi_{m-1} \cdots \phi_n = g_* f_*.$$

In order for the above to be well defined and true we need any permutation of strictly dummy vertices in the composition $g \circ f$ to act by the identity on any vector in the image of $(\iota_{n,\ell})_*$. This is guaranteed by our assumption because such permutations are exactly the stabilizer of $\iota_{n,\ell}$. □

REMARK 2.10. We quickly notice that checking the above is equivalent to checking that every element that annihilates $\iota_{n,m}$ acts by 0. This is because if π stabilizes, then $\pi - 1$ annihilates and vice versa.

Theorem 2.9 classifies the sequences of 0-Hecke representations that can be "promoted" to a $\mathcal{H}$-module, a similar result for **FI**-modules appears in [3, Remark 3.3.1]. We will now prove a few more properties of this category. First, we will show it is noetherian.

DEFINITION 2.11. *We say that a category of modules is* locally noetherian *if any submodule of a finitely generated module is finitely generated.*

DEFINITION 2.12. *Given two categories $\mathcal{C}$ and $\mathcal{C}'$, we say that a functor $\Phi \colon \mathcal{C} \to \mathcal{C}'$ satisfies* property (F) *if the following condition holds: given any object x of $\mathcal{C}'$ there exist finitely many objects $y_1, \ldots, y_n$ of $\mathcal{C}$ and morphisms $f_i \colon x \to \Phi(y_i)$ in $\mathcal{C}'$ such that for any object y of $\mathcal{C}$ and any morphism $f \colon x \to \Phi(y)$ in $\mathcal{C}'$, there exists a morphism $g \colon y_i \to y$ in $\mathcal{C}$ such that $f = \Phi(g) \circ f_i$.*

This property is important because it is equivalent to the pullback functor $\Phi^* \colon \text{Rep}_{\mathbf{k}}(\mathcal{C}') \to \text{Rep}_{\mathbf{k}}(\mathcal{C})$ preserving finite generation. For more on this we direct the reader to [38, §3].

Theorem 2.13. *The category of $\mathcal{H}$-modules, $\mathrm{Mod}_{\mathcal{H}}$, is locally noetherian.*

Proof. We use Gröbner methods outlined in [38]. In particular, we will show that $\mathcal{H}$ is quasi-Gröbner which proves that it is locally noetherian. In [38] the authors show that **OI** is a Gröbner category, i.e. that it is locally noetherian. So it suffices to produce a functor $\Phi\colon \mathbf{OI} \to \mathcal{H}$ that satisfies property (F). The functor Φ will be the natural inclusion outlined in Proposition 2.5.

It is clear that Φ is essentially surjective, so it remains to show that it satisfies property (F). Let $x = [n]$ be a given object in $\mathcal{H}$. Every map $f \in \hom_{\mathcal{H}}([n], [m])$ can be factored as $i(n,m)T_w$ where $T_w \in H_n(0)$ and $i(n,m)$ is an order preserving injection from $[n]$ to $[m]$. If we take $y_1, \ldots, y_{n!} = [n]$ and $f_i\colon [n] \to \Phi(y_i)$ to be the ith element of $H_n(0)$ under any enumeration it is clear that Φ satisfies property (F). $\qquad\square$

3. Induced Modules

We will now consider the *induced modules* $M(\mathbf{C}_\alpha)$ where $\mathbf{C}_\alpha$ is the simple one dimensional $H_n(0)$-module defined in the Section 2. $M(\mathbf{C}_\alpha)$ is defined as

$$M(\mathbf{C}_\alpha)_m := \mathbf{k}[\hom_{\mathcal{H}}([n],[m])] \otimes_{\mathbf{k}[H_n(0)]} \mathbf{C}_\alpha,$$

where the action of $\mathcal{H}$ is composition on the left tensor factor. It is not hard to see that

$$M(\mathbf{C}_\alpha)_m \cong \mathrm{Ind}_{H_n(0) \times H_{m-n}(0)}^{H_m(0)} \mathbf{C}_\alpha \boxtimes \mathbf{C}_{m-n}.$$

We also define $M(n)_m := \mathbf{k}[\hom_{\mathcal{H}}([n],[m])]$. There is a convenient way to index basis elements of $M(\alpha)$ in each degree.

Lemma 3.1. *For a fixed $\alpha \vDash n$ and degree d a basis for $M(\alpha)_d$ is indexed by all order preserving injections of n into d. Explicitly this basis is given by $\{g \otimes v_\alpha\}$ where g ranges over all order preserving injections of n into d and v_α spans $\mathbf{C}_\alpha$.*

Proof. For fixed d and n, we may assume $d \geqslant n$, the space $M(\alpha)_d$ is exactly

$$M(n)_d \otimes_{\mathbf{k}[H_n(0)]} \mathbf{C}_\alpha.$$

Pick any element $f \otimes v_\alpha$ where v_α spans $\mathbf{C}_\alpha$. We may assume by linearity that f is a single map and not a sum of maps. If f has any crossings, we can factor f as an element of $H_n(0)$ followed by an order preserving injection g, i.e. $f = gT_w$. We can then pass the element of $H_n(0)$ through the tensor and have it act on v_α. This implies that this vector $f \otimes v_\alpha = g \otimes (T_w v_\alpha)$. But we know T_w acts by either 0 or 1 on v_α so either the original vector is zero or it equals $g \otimes v_\alpha$ for some order preserving injection g. $\qquad\square$

Corollary 3.2. *For a fixed $\alpha \vDash n$, $\dim(M(\alpha)_d) = \binom{d}{n}$.*

Proof. This follows immediately from Lemma 3.1 because there are $\binom{d}{n}$ order preserving injections of $[n]$ into $[d]$. $\qquad\square$

Under the quasisymmetric characteristic map $\mathrm{Ch}\colon \mathcal{G}(\mathcal{H}) \to \mathrm{QSym}$ defined in Section 2, the isomorphism class of the simple element $[\mathcal{C}_\alpha]$ corresponds to the fundamental quasisymmetric function F_α. We can use this characteristic map to decompose the isomorphism class $[M(\mathbf{C}_\alpha)]_n$, in particular it corresponds to the analogous variant of the Pieri rule for fundamental quasisymmetric functions which we now recall.

Given two words w and v of length n and m respectively, we define their *shuffle set*, denoted by $w \odot v$ as the set of all words, $u_1 \cdots u_{m+n}$, where for some subset of disjoint indices $\{i_1, \ldots, i_n\}$ and $\{j_1, \ldots, j_m\}$ with $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_m$,

we have $u_{i_k} = w_k$ and $u_{j_k} = v_k$. Given a word w on an ordered set such as $\mathbf{Z}_{\geqslant 0}$ we define the *descent set* of w as

$$D(w) = \{i \mid w_i > w_{i+1}\}.$$

As a quick example consider the word $w = 14253$, then $D(w) = \{2, 4\}$.

For fundamental quasisymmetric functions F_α and F_β we have the following multiplication rule, for more details on this rule we refer the reader to [19],

$$F_\alpha F_\beta = \sum_{u \in w(\alpha) \odot w(\beta)} F_{C(D(u))}$$

where $w(\alpha)$ is some word on $\mathbf{Z}_{\geqslant 0}$ with descent set $D(w(\alpha)) = D(\alpha)$ and the same for $w(\beta)$ with the condition that $w(\alpha)$ and $w(\beta)$ must consist of disjoint natural numbers. We recall that $C(D(u))$ is the composition corresponding to the descent set of the word u.

We are specifically concerned with the case where $\beta = m$. In this case, combinatorially if we interpret $\alpha \vDash n$ as a ribbon diagram we sum over all F_γ with $\gamma \vDash m + n$ a ribbon diagram constructed from α by adding m boxes in any way to any of the positions of α. This multiplication rule allows us to discover which simple modules are in each degree of our induced modules. Consider the following example,

EXAMPLE 3.3. Consider the following induced $\mathcal{H}$-module, $M\left(\begin{smallmatrix} & \boxed{2} \\ \boxed{1} & \boxed{3} \end{smallmatrix}\right)$. We will explicitly compute the simples in degree 4,

$$M\left(\begin{smallmatrix} & \boxed{2} \\ \boxed{1}\boxed{3} \end{smallmatrix}\right)_4 = \begin{smallmatrix} \boxed{1} \\ \boxed{3} \\ \boxed{2}\boxed{4} \end{smallmatrix} + \begin{smallmatrix} \boxed{1}\boxed{3} \\ \boxed{2}\boxed{4} \end{smallmatrix} + \begin{smallmatrix} & \boxed{3} \\ \boxed{1}\boxed{4} \\ \boxed{2} \end{smallmatrix} + \begin{smallmatrix} & & \boxed{3} \\ \boxed{1}\boxed{2}\boxed{4} \end{smallmatrix} .$$

We number the boxes to illustrate the connection to descent words and the multiplication rule above. Each diagram corresponds to the descent word where we read from bottom to top, left to right. Here $w(\beta) = 1$ and $w(\alpha) = 243$. We then sum over all shuffles, so we get the descent words $\{2431, 2413, 2143, 1243\}$ which correspond to inserting 1 into all possible places.

From this point forward, we will no longer number the boxes because the descent set is recoverable from the diagrams themselves. We placed the numbers in the above example purely to illustrate the connection between the multiplication rule for fundamental quasisymmetric functions and adding boxes to certain positions of α.
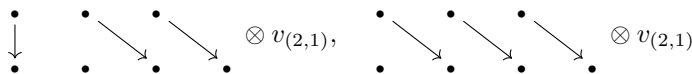
We also note that the action of the 0-Hecke algebra $H_n(0)$ on the simples in the $\mathcal{H}$-degree n component of a $\mathcal{H}$-module $M(\alpha)$ corresponds to shifting added boxes down to a lower position in the diagram. This perspective will be very useful in the coming sections. We will often switch perspectives depending on which allows for the easiest proof. To see what we mean, consider the following example.

EXAMPLE 3.4. Once again if we consider the induced module $M((2,1))$, we can see that each of the diagrams in the above example correspond to a basis element, which are indexed by order preserving injections. In degree 4 we have

$$M\left(\square\square\right)_4 = \square\square + \square\square + \square\square + \square\square ,$$

where the marked boxes are the box we add to the ribbon tableau corresponding to $(2,1)$. These diagrams correspond respectively to the basis elements

$$\downarrow \quad \downarrow \quad \downarrow \qquad \otimes v_{(2,1)}, \qquad \downarrow \quad \downarrow \quad \searrow \qquad \otimes v_{(2,1)},$$

where we read from left to right then top to bottom and $v_{(2,1)}$ is the basis element that spans $\mathbf{C}_{(2,1)}$. To see this correspondence, notice that the final diagram is preserved by the $H_4(0)$ action where π_1 acts by 1, π_2 acts by 1, and π_3 acts by 0. The corresponding descent set is precisely $\{3\}$ and so the corresponding composition of 4 is $(3,1)$. Notice this is the diagram where the box is placed in the lowest possible position.

When we quotient by this simple, one can check that the next simple that we can inject into this module is precisely the third diagram where we add a box to the second position. Under the $H_4(0)$ action, we can map the third diagram to the final diagram where we add a box to the 1st position by applying π_1. In this same way, it is possible to move any box we have added in a higher row to a lower row under this $H_4(0)$ action since adding boxes to the i-th row corresponds to having that many degree 0 vertices before the i-th strand. This implies that if a submodule contains a simple in $M(\alpha)$ that corresponds to adding a box to a higher position of α, the submodule must also contain all the simple modules corresponding to adding the same boxes to lower positions. This will be extremely useful in the coming sections.

PROPOSITION 3.5. *The simple objects in* $\mathrm{Mod}_{\mathcal{H}}$ *are precisely the modules equal to* C_α *in degree* $|\alpha|$ *and zero elsewhere.*

Proof. Suppose V is a simple $\mathcal{H}$-module. Let d be the first degree where $V_d \neq 0$. Then by simplicity we must have $V_{\geqslant d+1} = 0$, otherwise $V_{\geqslant d+1}$ would be a nonzero submodule. Furthermore, V_d must be a simple $H_d(0)$ module otherwise it would have a submodule. This means $V_d = \mathbf{C}_\alpha$ for some composition $\alpha \vDash d$. □

4. Padded Induced Modules

We say that a finitely generated $\mathcal{H}$-module V has *polynomial degree* d if $\dim(V_n)$ is given by a degree d polynomial for sufficiently large n.

Given a composition $\alpha \vDash n$, for $0 \leqslant i \leqslant n$, let $\eta_i(\alpha)$ consist of the first i blocks of α and $\tau_j(\alpha)$ consist of the last j blocks of α. We refer to these $\eta_i(\alpha)$ as *heads* of α and $\tau_j(\alpha)$ as *tails* of α. We explain what a head looks like explicitly and the tail is similar. If $\alpha = (\alpha_1, \ldots, \alpha_k)$ then $\eta_i(\alpha) = (\alpha_1, \ldots, \alpha_\ell - \beta)$ where β is a potential correction term to make sure that $\eta_i(\alpha)$ is a composition of i. We say that ρ and γ are *summands* of α if we can write $\alpha = \rho + \gamma$ where ρ and γ are both compositions. We say that $\eta_i(\alpha)$ or $\tau_j(\alpha)$ is *split* if we need a correction term β, combinatorially this is if the i-th block reading from left to right is not at the end of a row. If $\eta_i(\alpha)$ is split then $\alpha = \eta_i(\alpha) + \tau_{n-i}(\alpha)$, if not then $\alpha = \eta_i(\alpha) \lhd \tau_{n-i}(\alpha)$.

We now define a new module $M(\alpha, k)$ where α is a composition of n and k is an integer $-1 \leqslant k \leqslant n$. First we let $M(\alpha, n) := M(\alpha)$ and $M(\alpha, -1) := \mathbf{C}_\alpha$ these are both quotients of $M(\alpha)$. Then for $0 \leqslant k < n$, we let $M(\alpha, k)$ be the quotient of $M(\alpha)$ of degree k defined as follows. Let $A_{\alpha, k}$ be the submodule of $M(\alpha)$ generated by the basis element in degree $|\alpha| + 1$ indexed by the order preserving injection $g_k : [n] \to [n+1]$,

$$g_k(i) = \begin{cases} i+1 & i \geqslant |\alpha| - k, \\ i & \text{else.} \end{cases}$$

When we quotient by this module, the remaining simples in degree m of $M(\alpha, k)$ are indexed by $M(\tau_k(\alpha))_{m-n+k}$. To find the simples of $M(\alpha, k)$ corresponding to the compositions $\{\beta_{i,m-n+k}\}_i$ of $M(\tau_k(\alpha))_{m-n+k}$ we either get $\eta_{n-k}(\alpha) + \beta_{i,m-n+k}$ or $\eta_{n-k}(\alpha) \lhd \beta_{i,m-n+k}$. The first case occurs if $\eta_{n-k}(\alpha)$ is split or if we add a box to the first row of $\tau_k(\alpha)$ in $\beta_{i,m-n+k}$. The second case occurs if $\eta_{n-k}(\alpha)$ is not split

and we do not add a box to the first row of $\tau_k(\alpha)$. We explain this further below in Lemma 4.4. We call these *degree k induced α modules*.

PROPOSITION 4.1. *The module $M(\alpha, k)$ where $|\alpha| = n$, has Hilbert function $\binom{d-n+k}{k}$ and so has polynomial degree k.*

Proof. The simples that occur in $M(\alpha, k)_d$ are precisely those obtained from α by adding $d - n$ boxes to the top $k + 1$ positions of α. There are $\binom{d-n+k}{k}$ ways to do this and each simple is one dimensional. $\qquad\square$

REMARK 4.2. Notice, this result recovers the previous statement that $\dim(M(\alpha)_d) = \binom{d}{n}$, as $M(\alpha) = M(\alpha, n)$ if $|\alpha| = n$. Comparing the proof of this result with Corollary 3.2 further illustrates the connection between the basis of order preserving injections and the combinatorial interpretation of computing simples that occur using ribbon diagrams.
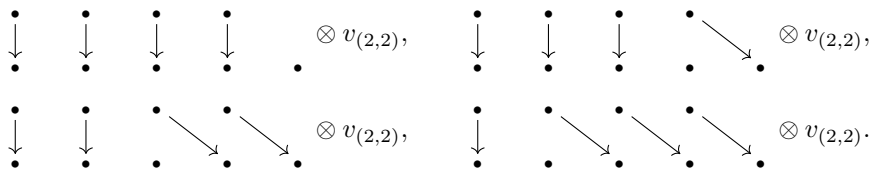
EXAMPLE 4.3. Consider the example of $M((2,2), 3)$. We define this as a quotient of $M((2,2))$ by the submodule generated by

$$\searsearrow\searrow\searrow\searrow \otimes v_{(2,2)},$$
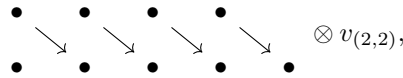
where recall $v_{(2,2)}$ spans $\mathbf{C}_{(2,2)}$. The above map is precisely the map indexed by the order preserving injection g described above. In this case $|\alpha| - 3 = 1$. Notice that α_1 is not split, i.e. $\alpha_1 = (1)$ and $\alpha^3 = (1, 2)$. From the above all the remaining simples should be generated by adjoining the simples in $M((1,2))$ to α_1. In degree 4 this quotient is spanned by

$$\downarrow\downarrow\downarrow\downarrow \otimes v_{(2,2)}.$$
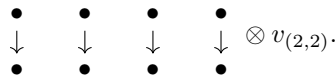
This corresponds to the simple $\mathbf{C}_{2,2}$, which we can realize as $\alpha_1 \lhd (1, 2)$ where $(1, 2)$ is the only simple in $M((1,2))_3$. In degree 5 the quotient is spanned by

$$\downarrow\downarrow\downarrow\downarrow \otimes v_{(2,2)}, \qquad \downarrow\downarrow\downarrow\searrow \otimes v_{(2,2)},$$

$$\downarrow\downarrow\searrow\searrow \otimes v_{(2,2)}, \qquad \downarrow\searrow\searrow\searrow \otimes v_{(2,2)}.$$

Notice the restriction the quotient places on the remaining basis elements is that the first arrow must go directly down. If it does, the basis element will not be in the quotient, if it does not the basis element is forced into the quotient. The quotient can be viewed as all simples where we place at least one block into the first position of the diagram for α, so all the remaining diagrams are exactly the ones where we never have a block in that position. These correspond to the following compositions $\alpha_1 \lhd (1, 2, 1)$, $\alpha_1 \lhd (1, 1, 2)$, $\alpha_1 \lhd (1, 3)$ and $\alpha_1 + (2, 2)$. The compositions on the right are exactly those present in $M((1,2))_4$. Notice α_1 is split, but in the final diagram we add a box to the first position of $(1, 2)$ so we add instead of join.

The above illustrates that the representation theory of $M(\alpha, k)$ is very similar to that of $M(\tau_k(\alpha))$. To make this more explicit,

LEMMA 4.4. *Submodules of $M(\alpha, k)$ are in bijective correspondence with submodules of $M(\tau_k(\alpha))$. This bijection respects the $\mathcal{H}$-module structure and preserves polynomial degree.*

Proof. Simples are indexed by their sole basis element, as we have seen above (Lemma 3.1). We will give a bijective correspondence between these basis elements that respects the $\mathcal{H}$-module structure. This bijection then extends to a bijective correspondence of submodules, as every submodule can be described by the basis elements that generate it as an $\mathcal{H}$-module. A bijective correspondence of basis elements also clearly respects the polynomial degree.

Given a basis element of $M(\tau_k(\alpha))_d$, to get the corresponding basis element of $M(\alpha, k)_{d+|\alpha|-k}$ we prepend $|\alpha| - k$ vertices to the top and bottom row of the string diagram corresponding to the basis element and connect them directly downwards. By construction this will be nonzero in the quotient so our map has a nonzero image in $M(\alpha, k)_{d+|\alpha|-k}$. Furthermore, each basis element has a unique image and this map respects the action of $H_d(0)$ where we embed $H_d(0) \hookrightarrow H_{d+|\alpha|-k}(0)$ by $\pi_i \mapsto \pi_{|\alpha|-k+i}$. In particular, we see that if the basis vectors corresponding to simples $\{\mathbf{C}_{\beta^i}\}$ generate a submodule of $M(\tau_k(\alpha))$, the basis vectors corresponding to the simples $\{\mathbf{C}_{\gamma^i}\}$ where we prepend the missing part of α, i.e. $\eta_{|\alpha|-k}(\alpha)$, to β^i will generate a corresponding submodule of $M(\alpha, k)$.

The inverse map is the restriction map that ignores the first $|\alpha| - k$ fixed vertical edges that must occur in any basis element of the quotient module $M(\alpha, k)$. This map is also an injection because by construction every basis element that occurs in any degree of $M(\alpha, k)$ has $|\alpha| - k$ fixed vertical edges, so the only new information is what occurs within the final d pairs of vertices. Notice, this also respects the $H_d(0)$ action.

Finally, by construction the only way to generate new basis elements in $M(\alpha, k)$ from a given element is via the $H_d(0)$ action on the final d vertices. As a result, generating sets for submodules are completely determined up to this action. Since the above bijection respects this $H_d(0)$ action, it gives us a bijective correspondence of submodules. $\square$

REMARK 4.5. The above shows that when dealing with proofs about polynomial degree of submodules it suffices to work with the $M(\alpha)$. It is important to note that this does not show that $M(\tau_k(\alpha))$ and $M(\alpha, k)$ are isomorphic as $\mathcal{H}$-modules, this is very much not the case. In particular, we are not trying to say that there is an isomorphism of submodules, just a bijective correspondence that respects the $\mathcal{H}$-module structure. We show that the simples occurring in submodules of $M(\alpha, k)$ are controlled by $M(\tau_k(\alpha))$ and hence so is the dimension because all simples are one dimensional. To understand this lemma further, look back at Example 4.3.

We now prove an important theorem about quotients of $M(\alpha, k)$.

THEOREM 4.6. *Let α be a composition with $|\alpha| = n$. Then any proper quotient of $M(\alpha, k)$ has polynomial degree $< k$. Equivalently, any proper submodule of $M(\alpha, k)$ has Hilbert polynomial of the same degree with the same leading term as $M(\alpha, k)$.*

Lemma 3.1 allows us to place a total order on the basis elements of $M(\alpha)$ for any α. In particular, we say that $g_1 \otimes v_\alpha \geqslant g_2 \otimes v_\alpha$ if g_1 has image in $[m]$ while g_2 has image in $[n]$ with $m > n$ or if g_1 and g_2 both have image in $[n]$ and $(g_1(1), g_1(2), \ldots, g_1(n)) \geqslant (g_2(1), g_2(2), \ldots, g_2(n))$ in the lexicographic order. It is not hard to see this is indeed a total order.

LEMMA 4.7. *Order preserving maps preserve the lexicographic order on basis elements of the same degree.*

Proof. Given two basis elements $g_1 \otimes v_\alpha \geqslant g_2 \otimes v_\alpha$ with $g_1, g_2 \in M(n)_d$ order preserving and another order preserving map $f \in \hom_{\mathcal{H}}([d], [m])$ we know that f acts on these basis elements by composition in the first component and we need to show $f(g_1 \otimes v_\alpha) \geqslant$

$f(g_2 \otimes v_\alpha)$. Equivalently we need to show,

$$(fg_1) \otimes v_\alpha \geqslant (fg_2) \otimes v_\alpha.$$

fg_1 and fg_2 both inject into $[m]$ so we have to show

$$(fg_1(1), fg_1(2), \ldots, fg_1(n)) \geqslant (fg_2(1), fg_2(2), \ldots, fg_2(n)).$$

This is clear because f is order preserving. $\qquad\square$

Every element we are concerned with will be a sum of basis elements in the same degree, because we consider degrees separately. The above result implies that order preserving injections between degrees respect the total order on each degree. With these results, we are ready to prove our theorem,

Proof of Theorem 4.6. It suffices to prove this result for $M(\alpha) = M(\alpha, |\alpha|)$ as α varies by Lemma 4.4. Let V be a proper submodule of $M(\alpha)$ in $\mathrm{Mod}_{\mathcal{H}}$. Suppose that $d > n$ is the first degree where $V_d \neq 0$. Then because V is a submodule of $M(\alpha)$ we know V_d will contain a vector v with the following vector as a nonzero summand,
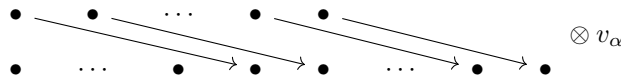
where v_α is a vector spanning $\mathbf{C}_\alpha$ and there are n vertices on top, d on the bottom and the first $d - n$ vertices are not mapped to. We will call the above vector u. This vector will appear because we can apply the moves in (2a) to obtain such a v having u as a nonzero summand. In degree d, $u = f \otimes v_\alpha$ is the largest element with respect to the lex order on basis elements.

If we consider all the order preserving maps that factor through f in later degrees they will be linearly independent because they are basis elements as described in Lemma 3.1. Applying order preserving injections preserves the lex order as seen in Lemma 4.7, so in any order preserving injection of v the image of u will be the largest vector that appears as a summand.

Consider a collection of order preserving injections of v that inject u to unique vectors (e.g. we want the final n vertices of the d total vertices to have unique images under each map and we want the first $d - n$ vertices to map directly downward so there are no repetitions in the mapping of u). We claim the images of v under these order preserving injections are linearly independent. If there was some nontrivial linear combination of these order preserving injections of v, consider the images of u. The coefficient of the largest image of v, i.e. the one that contains the largest image of u with respect to the lexicographic order, must be zero because the image of u will be the largest possible vector in that degree with respect to the lexicographic order and no other selected order preserving injection of v contains that vector as a summand by construction. We then continue in that fashion to find that all the coefficients must be zero.

This means that every order preserving injection we have selected of the vector v will be linearly independent. If we map to degree $m \geqslant d \geqslant n$ there are $\binom{m-d+n}{n}$ such injections. Each will be given by selecting n vertices from the final $m - d + n$ vertices in degree m. So if we consider the submodule generated by the vector v, it gives a lower bound on the dimension of the original submodule V in each degree. We know $\dim(M(\alpha)_m)$ gives an upper bound on the dimension of V_m. All of these submodules have eventually polynomial growth because they are finitely generated by Noetherianity (Theorem 2.13). In total we have shown that

$$\binom{m}{n} \geqslant \dim(V_m) \geqslant \binom{m - d + n}{n},$$

and $\dim(V_m)$ is eventually polynomial. All of this implies that $\dim(V_m)$ must eventually equal a polynomial of degree n with leading coefficient $\frac{1}{n!}$. As a result, the quotient module $M(\alpha)/V$ has degree $< n$ because the leading terms will cancel. $\qquad\square$

5. Grothendieck Group

Let $\mathcal{H}_{\leqslant d} \coloneqq (\mathrm{Mod}_{\mathcal{H}})_{\leqslant d}$ the $\mathcal{H}$-modules of polynomial degree $\leqslant d$. Consider the successive Serre quotient category where we consider every polynomial degree d module modulo polynomial degree $< d$ objects, we will denote this category of polynomial degree d objects modulo degree $\leqslant d-1$ objects by $\mathcal{H}_d \coloneqq \mathcal{H}_{\leqslant d}/\mathcal{H}_{\leqslant d-1}$. It turns out that the $M(\alpha, k)$ are simple in the quotient by lower polynomial degree submodules.

THEOREM 5.1. *Any nonzero polynomial degree k quotient of $M(\alpha)$ must contain $M(\alpha, k)$, i.e. $M(\alpha, k)$ is the smallest nonzero polynomial degree k quotient of $M(\alpha)$.*

Proof. For any α and k, this is equivalent to saying that if $M(\alpha, k) = M(\alpha)/A_{\alpha,k}$ that any other quotient $M(\alpha)/B$ of polynomial degree k has $B \subseteq A_{\alpha,k}$.

If $k = -1$, $M(\alpha, -1) = \mathbf{C}_\alpha$ which is clearly the smallest polynomial degree -1 nonzero quotient of $M(\alpha)$ as any further quotient would have to be identically zero.

Assume $k \geqslant 0$. Suppose there were a submodule B with $A_{\alpha,k} \subset B$ such that $M(\alpha)/B$ had polynomial degree k. This implies the existence of a nonzero submodule of $M(\alpha, k)$ such that the quotient still has polynomial degree k which is impossible by Theorem 4.6, as the polynomial degree must drop to $k-1$.

The other option is that $A_{\alpha,k}$ and B are not comparable in the containment order. Suppose this is the case with $k \geqslant 0$, and that $M(\alpha)/B$ has polynomial degree k.

Let b span a simple in B that is not contained in $A_{\alpha,k}$. For b to not be in $A_{\alpha,k}$, the corresponding simple module must not have a box added to the first $|\alpha|-k$ positions of α. This is because by construction $A_{\alpha,k}$ contains every such simple module. Suppose b is in $\mathcal{H}$-degree ℓ. Without loss we can assume we add all ℓ boxes to the $(|\alpha|-k+1)$-st position since the submodule generated by any other choice of simple contains this one because we can always shift boxes down.

In the quotient of $M(\alpha)$ by the submodule generated by b, the remaining simples correspond to diagrams where we add no more than $\ell-1$ boxes to the first $|\alpha|-k+1$ positions of α. The number of ways to place $\ell - 1$ boxes in the first $|\alpha| - k + 1$ positions of α is $\binom{|\alpha|-k+1+\ell-2}{\ell-2}$ which is a constant. There are a total of $|\alpha| + 1$ positions we could add boxes to, and once we fix a way to place at most $\ell-1$ boxes in the first $|\alpha| - k + 1$ positions, we can add the remaining boxes in any way to the remaining k positions. In degree d, we could choose to add at most $d - |\alpha|$ boxes to the top k positions, there are at most $\binom{d-|\alpha|+k-1}{k-1}$ ways to do this. The quotient then has dimension $\leqslant \binom{|\alpha|-k+1+\ell-2}{\ell-2}\binom{d-|\alpha|+k-1}{k-1}$, which is a degree $k-1$ polynomial in d. B clearly contains the submodule generated by b and so this implies the quotient $M(\alpha)/B$ has smaller polynomial degree and hence polynomial degree $\leqslant k-1$ which is a contradiction. $\qquad\square$

LEMMA 5.2. *The module $M(\alpha, k)$ is finitely generated and simple in $\mathcal{H}_k$. Furthermore, every simple module is of this form.*

Proof. Finite generation is inherited in the quotient category. To see that $M(\alpha, k)$ is simple, Theorem 4.6 implies that if V is any nonzero submodule it must have the same polynomial degree and leading coefficient as $M(\alpha, k)$ and so the quotient has smaller degree. This implies that $M(\alpha, k) = V$ in the quotient so $M(\alpha, k)$ is simple.

For the final statement, let $L \in \mathcal{H}_k$ be simple. Denote by $\widetilde{L}$ a lift of L to an object in $\mathrm{Mod}_{\mathcal{H}}$. We may assume that $\widetilde{L}$ is a quotient of $M(\beta)$ for some choice of β. Indeed,

we know that $\widetilde{L}$ must be generated by a single element v in degree d. By simplicity, v must span L_d. If this were not the case, there would be some $\pi_j(v) = w$ with w not a scalar multiple of v. Consider the submodule M of L spanned by w. There are two options, one is that M has polynomial degree k, in which case L is not simple because M does not contain v, a contradiction. Otherwise, M has polynomial degree $< k$, so that $L = L/M$ in $\mathcal{H}_k$. So we may in general assume that $\widetilde{L}$ is generated by a single element in lowest degree and that in this degree that element spans. This means that $\widetilde{L}_d$ is an irreducible representation of $\mathcal{H}$, say $\mathbf{C}_\beta$. We then get a natural surjection $M(\beta) \twoheadrightarrow L$. Any simple quotient of $M(\beta)$ must be isomorphic to $M(\beta, k)$ for some k in $\mathcal{H}_k$. Indeed, Theorem 5.1 shows $M(\beta, k)$ is the smallest degree k quotient of $M(\beta)$, so $M(\beta, k)$ is a submodule of the quotient, but because the quotient is simple they must be equal. $\square$

We define $\mathrm{Mod}_K = \bigoplus_{k \geqslant 0} \mathcal{H}_k$. That is, we consider all polynomial degree d objects modulo lower degree objects.

LEMMA 5.3. *Every object of* Mod_K *has finite length.*

Proof. Every module V is a quotient of a finite direct sum of $M(n)$ which have finite length by Lemma 5.2, so V is finite length. $\square$

THEOREM 5.4. *Every finitely generated* $\mathcal{H}$-*module* V *with polynomial degree* d *has a finite filtration in* $\mathcal{H}_d$
$$0 = F_1 \subset F_2 \subset \cdots \subset F_n = V,$$
where F_i/F_{i-1} *is isomorphic to* $M(\alpha_i, d)$.

Proof. V has a finite composition series from Lemma 5.3, because V has polynomial degree d we know the simples must as well otherwise we would be able to shorten the composition series. This implies that F_i/F_{i-1} is isomorphic to a simple module of degree d, so a simple module in $\mathcal{H}_d$. Lemma 5.2 implies each of the simples are isomorphic to $M(\alpha_i, d)$. $\square$

COROLLARY 5.5. *The isomorphism classes* $[M(\alpha, k)]$ *span* $\mathcal{G}(\mathrm{Mod}_{\mathcal{H}})$.

Proof. Given any finitely generated $\mathcal{H}$-module V of polynomial growth d, we claim that its isomorphism class can be expressed as a finite sum
$$[V] = \sum_j [M(\alpha_j, k_j)]$$
where $k_j \leqslant d$. We will proceed by induction on the polynomial growth of V. If $d = -1$, V is torsion and even without using Theorem 5.4 we can express V as a sum
$$[V] = \sum_i [M(\alpha_i, -1)]$$
by starting in the highest $\mathcal{H}$-degree and injecting the corresponding torsion modules. Now suppose the result holds for degrees $\leqslant d - 1$ and consider some $d \geqslant 0$. From Theorem 5.4 we can write
$$[V] = \sum_i [M(\alpha_i, d)] + [W]$$
where $[W]$ is a module with smaller polynomial degree $\leqslant d - 1$ and the first sum is finite. By induction we have
$$[W] = \sum_j [M(\alpha_j, k_j)]$$
where $k_j \leqslant d - 1$. This completes the proof. $\square$

We now wish to strengthen this result to show that the $[M(\alpha, k)]$ actually form a basis.

THEOREM 5.6. *The $[M(\alpha, k)]$ form a basis for $\mathcal{G}(\mathrm{Mod}_{\mathcal{H}})$.*

Proof. Corollary 5.5 implies the $[M(\alpha, k)]$ span $\mathcal{G}(\mathrm{Mod}_{\mathcal{H}})$, it remains to prove they are linearly independent. To do this, we will construct functions.

To prove linear independence, it suffices to prove that $M(\alpha, k)$ are linearly independent for a fixed $k \geqslant -1$. This follows from considering the corresponding Hilbert polynomials. If there were a finite linear combination,

$$\sum_{i,j} c_{i,j} [M(\alpha_i, j)] = 0,$$

if we consider the corresponding sum of Hilbert polynomials for sufficiently large $\mathcal{H}$-degree, that sum must be zero as well. This implies that the coefficients of each polynomial degree must be zero. By considering the largest degree, say ℓ, such coefficients can only appear from the $[M(\alpha_i, \ell)]$. If we can show linear independence here, it implies the above sum can be zero if and only if the coefficients $c_{i,\ell} = 0$. We can continue in this fashion to show that every coefficient must be zero. So to show linear independence in general, it suffices to prove it for each fixed $k \geqslant -1$.

Fix some $k \geqslant -1$. For a finitely generated $\mathcal{H}$-module V, let $\Gamma_\alpha([V])$ be the number of times $\mathbf{C}_\alpha$ appears in the composition series of the $V_{|\alpha|}$. This clearly respects short exact sequences and so is well defined on the Grothendieck group. Notice that for a fixed $k \geqslant -1$, $\Gamma_\alpha([M(\beta, k)])$ satisfies

$$\Gamma_\alpha([M(\beta, k)]) = \begin{cases} 0 & |\beta| > |\alpha|, \text{ or } |\alpha| = |\beta| \text{ and } \alpha \neq \beta, \\ 1 & \beta = \alpha, \\ * & \text{else}, \end{cases}$$

where $*$ could be any non-negative integer. First, if $|\beta| > |\alpha|$ the lowest nonzero $\mathcal{H}$-degree of $M(\beta, k)$ is $|\beta|$ so $\mathbf{C}_\alpha$ could not possibly occur because it only appears in $\mathcal{H}$-degree $|\alpha|$. If $|\alpha| = |\beta|$, there are two cases to consider. If $\alpha = \beta$, then $\mathbf{C}_\alpha$ appears in the lowest degree of $M(\alpha, k)$ precisely once, so $\Gamma_\alpha(M(\alpha, k)) = 1$. If $\alpha \neq \beta$, then $M(\beta, k)$ has only $\mathbf{C}_\beta$ in degree $|\beta| = |\alpha|$ and so $\mathbf{C}_\alpha$ also cannot occur. It is possible that $\mathbf{C}_\alpha$ could appear multiple times in some $M(\beta, k)$ with $|\beta| < |\alpha|$, this is the remaining case, but this will not matter.

Order the compositions of i arbitrarily, then place all compositions in increasing order according to i. Call this list of all compositions of any non-negative integer $\{\alpha_i\}$. The order described is equivalent to saying that if $i \leqslant j$ we have $|\alpha_i| \leqslant |\alpha_j|$ so the size of the compositions is weakly increasing in our list. Consider the matrix whose (i, j) entry is $\Gamma_{\alpha_i}(M(\beta_j, k))$. The above implies that this matrix will be lower uni-triangular because $\Gamma_{\alpha_i}(M(\alpha_j, k))$ is potentially nonzero precisely when $j \leqslant i$ as here we could have $|\alpha_j| < |\alpha_i|$, and when $j = i$ $\Gamma_{\alpha_i}(M(\alpha_i, k)) = 1$. When $j > i$, we either have $|\alpha_i| = |\alpha_j|$ but $\alpha_i \neq \alpha_j$ in which case the (i, j)-entry is zero, or $|\alpha_j| > |\alpha_i|$ in which case again the (i, j)-entry is zero.

This implies that the matrix is invertible and so there is some change of basis in which we can express the Kronecker-Delta functions $\delta_{\alpha, k}$ as a linear combination of the Γ_α, where recall that

$$\delta_{\alpha, k}(M(\beta, k)) := \begin{cases} 1 & (\alpha, k) = (\beta, k), \\ 0 & \text{else}. \end{cases}$$

The existence of such functions implies that the $[M(\alpha, k)]$ are linearly independent for a fixed $k \geqslant -1$. Indeed, suppose there were some finite linear combination

$$\sum_i c_{\alpha_i}[M(\alpha_i, k)] = 0,$$

then applying $\delta_{\alpha,k}$ implies $c_{\alpha,k} = 0$. As this holds for every choice of α, this implies every coefficient is zero and so they must be linearly independent. This argument holds for any fixed $k \geqslant -1$ and so implies that the $[M(\alpha, k)]$ are linearly independent for any such fixed k. By the above, we have that the $[M(\alpha, k)]$ are linearly independent as α ranges over all compositions and k an integer with $k \geqslant -1$. $\qquad\square$

Lemma 5.2 describes the simple modules in $\mathcal{H}_d = \mathcal{H}_{\leqslant d}/\mathcal{H}_{\leqslant d-1}$, for the coming results we wish to strengthen this slightly.

LEMMA 5.7. *Any simple object in $\mathcal{H}/\mathcal{H}_{\leqslant d-1}$ is isomorphic to $M(\alpha, d)$ for some choice of α.*

Proof. Any simple object in $\mathcal{H}_{\leqslant d}/\mathcal{H}_{\leqslant d-1}$ is isomorphic to $M(\alpha, d)$ by Lemma 5.2. So the $M(\alpha, d)$ are simple, it remains to consider higher polynomial degree modules. Given any other simple object L that has polynomial degree $> d$, we can lift L to $\widetilde{L}$ in $\mathrm{Mod}_{\mathcal{H}}$. As in Lemma 5.2, we may assume that $\widetilde{L}$ is a quotient of $M(\beta)$ for some choice of β. Any quotient of $M(\beta)$ must contain $M(\beta, k)$ for an appropriate choice of $k > d$ by Theorem 4.6. Hence it cannot possibly be simple unless it equals $M(\beta, k)$ in the localization. But $M(\beta, k)$ is only simple in the quotient by $\mathcal{H}_{\leqslant k-1}$, otherwise it has a nontrivial quotient corresponding to $M(\beta, k-1)$ which implies the existence of a nontrivial submodule of polynomial degree $> d - 1$. This means that the only simples in $\mathcal{H}/\mathcal{H}_{\leqslant d-1}$ are exactly $M(\beta, d)$ as β varies over all compositions. $\qquad\square$

THEOREM 5.8. *A finitely generated $\mathcal{H}$-module has polynomial degree $\leqslant d$ if and only if it has Gabriel–Krull dimension $\leqslant d$.*

Proof. Let $\mathcal{H}^{\leqslant d}$ be the objects that are finite length in the quotient $\mathcal{H}/\mathcal{H}^{\leqslant d-1}$. We will prove by induction on d that $\mathcal{H}^{\leqslant d} = \mathcal{H}_{\leqslant d}$.

For $d = -1$ the statement is true because in each case we get torsion modules. These are the only polynomial degree -1 modules by definition and they are also the only finite length modules in $\mathcal{H}$ by Lemma 3.5.

Now if we assume the statement for $\leqslant d-1$, we will prove it for d where $d \geqslant 0$. We aim to show that a finitely generated $\mathcal{H}$-module V has polynomial degree $\leqslant d$ if and only if the image of V in $\mathcal{H}/\mathcal{H}^{\leqslant d-1}$ is finite length. By induction this is equivalent to showing that V has polynomial degree $\leqslant d$ if and only if the image of V in $\mathcal{H}/\mathcal{H}_{\leqslant d-1}$ is finite length because we have $\mathcal{H}_{\leqslant d-1} = \mathcal{H}^{\leqslant d-1}$.

If V has polynomial degree $\leqslant d$ then it is is finite length in $\mathcal{H}/\mathcal{H}_{\leqslant d-1}$ by Lemma 5.3.

Suppose instead that V is finite length in $\mathcal{H}/\mathcal{H}_{\leqslant d-1}$. Consider the finite composition series for V in $\mathcal{H}/\mathcal{H}_{\leqslant d}$. Lemma 5.7 implies $[V]$ is a sum of $[M(\alpha_i, d)]$ modulo lower polynomial degree terms as these are precisely the simple modules in $\mathcal{H}/\mathcal{H}_{\leqslant d}$. This means V has polynomial degree $\leqslant d$ as each of the terms in the expression of $[V]$ in the Grothendieck group does. The result then follows by induction. $\qquad\square$

COROLLARY 5.9. *The Gabriel–Krull dimension of $\mathcal{H}$ is infinite.*

Proof. This follows immediately from Theorem 5.8 and the fact that we have $\mathcal{H}$-modules with arbitrarily large polynomial degree, e.g. $M((d))$ is a degree d $\mathcal{H}$-module. $\qquad\square$

Heuristically, $\mathcal{H}$ can be thought of as somewhere between **OI** and **FI** but closer to **OI**. In particular, we know that **OI** is a subcategory of $\mathcal{H}$ under the map $\Phi\colon \mathbf{OI} \to \mathcal{H}$ described in Proposition 2.5. This gives rise to a pullback functor $\Phi^*\colon \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathbf{OI}}$ which then induces a pullback functor on Grothendieck groups which by abuse of notation we will also call Φ^*. It is natural to ask where the $[M(\alpha, k)]$ land.

We will use the notation outlined in [11]. In their paper, Güntürkün and Snowden study the representation theory of the increasing monoid, denoted $\mathcal{J}$, which they show also describes the representation theory of **OI**-modules. We recall the essential definitions from their paper, for further details consult [11].

One of the most important classes of $\mathcal{J}$-modules are called *standard modules*, denoted by E^λ where λ is a finite word on the alphabet $\{a, b\}$. Given such a word $\lambda = \lambda_1 \ldots \lambda_r$ the module E^λ associated to λ is defined as follows: E^λ has a basis consisting of elements of the form $e_{i_1, \ldots, i_r}$ where $1 \leqslant i_1 < \cdots < i_r$ subject to the constraint that if $\lambda_k = b$ then $i_k - i_{k-1} = 1$ (with the convention that $i_0 = 0$). The increasing monoid acts by $\sigma e_{i_1, \ldots, i_r} = e_{\sigma(i_1), \ldots, \sigma(i_r)}$ if this is a basis element of E^λ and 0 otherwise.

In [11, Theorem 12.1] the authors show that the isomorphism classes of standard modules form an integral basis for the Grothendieck group of finitely generated $\mathcal{J}$-modules. To understand $\Phi^*([M(\alpha, k)])$ it thus suffices to express the image in terms of the isomorphism classes of these standard modules.

PROPOSITION 5.10. *Under the pullback functor we have* $\Phi^*([M(\alpha, k)]) = [\underline{E}^{b^{|\alpha|-k}a^{k+1}}]$.

Proof. The pullback functor forgets the action of $H_n(0)$ in each $\mathcal{H}$-degree and only allows order preserving injections between degrees. This means that when we consider basis elements for $M(\alpha, k)$ indexed by order preserving injections, we can forget the vector v_α that spans $\mathbf{C}_\alpha$ as this only keeps track of the 0-Hecke action in each degree. We then see that in each degree d, a basis for $M(\alpha, k)$ is given exactly by order preserving injections $g\colon [|\alpha|] \to [d]$ where the first $|\alpha| - k$ numbers must map to themselves. Such order preserving injections are in bijective correspondence with basis elements of $\underline{E}^{b^{|\alpha|-k}a^{k+1}}$ where we send the injection $g\colon [|\alpha|] \to [d]$ to the basis element $e_{g(1), \ldots, g(|\alpha|), d+1}$. Notice by construction $g(i) = i$ for $i = 1, \ldots, |\alpha| - k$ which is exactly mandated by the constraint word $b^{|\alpha|-k}a^{k+1}$.

This map also respects the transition maps between degrees because if such a map does not fix the first $|\alpha| - k$ entries, it will be zero in both $M(\alpha, k)$ and $\underline{E}^{b^{|\alpha|-k}a^{k+1}}$. Otherwise, the remaining entries are free to be sent anywhere in both modules. $\square$

We show in Theorem 5.6 that the $[M(\alpha, k)]$ form a basis for the Grothendieck group of $\mathcal{H}$. Proposition 5.10 shows that $\Phi^*([M(\alpha, k)]) = \Phi^*([M(\beta, k)])$ so long as $|\alpha| = |\beta|$. Heuristically, this is because the only difference between these two modules is the 0-Hecke action in each degree. When we forget this action, they become the same **OI**-module.

The fact that Φ^* is not surjective can be explained intuitively by $\mathcal{H}$-modules allowing us to move basis vector upwards within degrees. The basis elements we miss are the $[E^\lambda]$ where λ has b elements interspersed within a elements. This would correspond to an order preserving injection where we force $g(k) = g(k-1) + 1$. This condition cannot be preserved by a $\mathcal{H}$-action unless we only fix the beginning elements. This is illustrative of the statement that every $\mathcal{H}$-module is an **OI**-module but not every **OI**-module is a $\mathcal{H}$-module. Also, it illustrates that there is a difference between the categories **OI** and $\mathcal{H}$.

6. Representation Stability

With the previous section established, we are ready to state some immediate representation stability results for sequences of representations of $H_n(0)$-modules.

Theorem 6.1. *For any finitely generated $\mathcal{H}$-module V, we have a unique finite decomposition in the Grothendieck group $\mathcal{G}(\mathrm{Mod}_{\mathcal{H}})$,*

$$[V] = \sum_{i,j} c_{\alpha^i, k_j} [M(\alpha^i, k_j)]$$

where the coefficients c_{α^i, k_j} are integers.

Proof. This follows immediately from Theorem 5.6. □

We will say that a sequence of $H_n(0)$ representations is *representation stable* if there is such a decomposition in the Grothendieck group. The above shows that being finitely generated as a $\mathcal{H}$-module implies representation stability. We can make this more explicit in terms of the simple modules that can occur in finitely generated $\mathcal{H}$-modules for sufficiently large n.

Theorem 6.2. *For any finitely generated $\mathcal{H}$-module V, for sufficiently large n the simples that appear in V_n are exactly those that appear in the finite sum,*

$$\bigoplus_{i,j} c_{\alpha^i, k_j} M(\alpha^i, k_j)_n$$

where the coefficient c_{α^i, k_j} is independent of n.

Proof. This is a direct consequence of 6.1. □

Remark 6.3. For any finitely generated $\mathcal{H}$-module V, we have for sufficiently large n a unique finite list of pairs $\{(\alpha^i, k_i)\}$ of compositions with integers $k \geqslant -1$ that dictate exactly which simples occur in V_n.

To see what we mean in this theorem, notice that in $M(\alpha, k)$ only $\mathbf{C}_\alpha$ occurs in degree $|\alpha|$. After this, to find the simples that occur in degree $d \geqslant |\alpha|$ you add $d - |\alpha|$ boxes to the top $k + 1$ positions of α in all ways possible.

We often refer to stability occurring for sufficiently large n. As of now, we do not have explicit bounds on when this stability begins to occur. This is another potential direction for further research, as such bounds have been investigated in the **FI**-setting. In [3], they refer to the n for which this stability occurs as the *stable range*. They prove that for a sequence of S_n-representations $\{V_n, \phi_n\}$ coming from a **FI**-module V, stabilization occurs for $n \geqslant \mathrm{weight}(V) + \mathrm{stab\text{-}deg}(V)$. For explicit definitions of weight and stability degree, we refer the reader to [3, Definition 3.2.1] and [3, Definition 3.1.3] respectively. One can define analogous definitions of weight and stability degree for $\mathcal{H}$-modules, but our method of proving representation stability is largely different from the **FI** case, so it is not immediately obvious how one could find a similar lower bound on stabilization. We do, however, believe one exists and are actively working to find it.

7. The Shift Functor

As in the case of **FI** it makes sense to define a shift functor Σ for $\mathcal{H}$. Given a $\mathcal{H}$-module V, we define its *first shift* ΣV to be the $\mathcal{H}$-module with $(\Sigma V)_n = V_{n+1}$ on sets. For any $f \in \hom_{\mathcal{H}}([n], [m])$, the map $\Sigma V(f) \colon V_{n+1} \to V_{m+1}$ is the map $V(F)$ where F agrees with f on $[n]$ and maps $n + 1$ to $m + 1$. We denote by Σ_a the ath iterate of Σ. From this definition it is not hard to see that

$$(\Sigma V)_n \cong \mathrm{Res}^{H_{n+1}(0)}_{H_n(0)} V_{n+1}.$$

Also, notice there is an inclusion map $V \to \Sigma V$ induced by the natural inclusion $\iota_n \colon [n] \to [n+1]$ with $\iota_n(i) = i$ for all i. We define the *derivative* of V, denoted by ΔV as

$$\Delta V := \operatorname{coker}(V \to \Sigma V).$$

In the case of **FI** and many other categories of interest these functors play an important role and have many nice properties. For further reading on this subject we refer the reader to [31, 3, 4, 8]. We will see that many of these desirable properties are also present for $\mathcal{H}$.

Lemma 7.1. *Given any padded induced module $M(\alpha, k)$ with $k \geqslant 1$ we have*

$$\Sigma M(\alpha, k) \cong M(\alpha, k) \oplus M(\alpha_{|\alpha|-1}, k-1).$$

If $k = 0$, $\Sigma M(\alpha, 0) = M(\alpha, 0)$. If $k = -1$, $\Sigma M(\alpha, -1) = 0$.

Proof. To prove this, fix an $\mathcal{H}$-degree d. In this degree we know $(\Sigma M(\alpha, k))_d = M(\alpha, k)_{d+1}$ as a set, where the action comes from restriction. Take the basis for $M(\alpha, k)$ as in Lemma 3.1. In $M(\alpha, k)_{d+1}$ we can separate the basis elements into two groups, the first consisting of all elements indexed by an order preserving injection with $g(|\alpha|) = d + 1$ and the second consisting of the remaining elements where $g(|\alpha|) \leqslant d$. In $\operatorname{Res}_{H_d(0)}^{H_{d+1}(0)} M(\alpha, k)_{d+1}$ the first group becomes a sub-representation isomorphic to $M(\alpha_{|\alpha|-1}, k-1)_d$ and the second becomes a sub-representation isomorphic to $M(\alpha, k)_d$. This is clear because when we restrict we ignore π_d. Under the action of $\pi_1, \ldots, \pi_{d-1}$ the first collection of elements consists of all order preserving injections from $[|\alpha| - 1]$ to $[d]$ and the action of $\pi_1, \ldots, \pi_{d-1}$ is dictated by $\mathbf{C}_{\alpha_{|\alpha|-1}}$. The second collection of basis elements consists of all order preserving injections from $[|\alpha|]$ to $[d]$ and the action of $\pi_1, \ldots, \pi_{d-1}$ is dictated by $\mathbf{C}_\alpha$.

Furthermore, because in $\Sigma M(\alpha, k)$ we must map the final vertex to the final vertex these groupings of basis elements are preserved under all transition maps, so they do form $\mathcal{H}$-submodules. The above shows that these submodules are isomorphic to $M(\alpha, k)$ and $M(\alpha_{|\alpha|-1}, k-1)$. It is also clear from the above that we have the short exact sequence

$$0 \to M(\alpha_{|\alpha|-1}, k-1) \to \Sigma M(\alpha, k) \to M(\alpha, k) \to 0$$

and that it splits, where the splitting is exactly the inclusion of $M(\alpha, k)$ into $\Sigma M(\alpha, k)$.
$\square$

Corollary 7.2. *If V is a $\mathcal{H}$-module finitely generated in degree d, ΔV is finitely generated in degrees $\leqslant d - 1$.*

Proof. Every finitely generated $\mathcal{H}$-module is a quotient of a direct sum of $M(d)$, so it suffices to check the result holds there. Lemma 7.1 and its proof imply that $\Delta(M(d)) \cong M(d-1)$ which is generated in degree $d - 1$. The result follows. $\square$

8. Regularity

One can also define a notion of regularity for $\mathcal{H}$-modules. Let V be a $\mathcal{H}$-module. We define $\operatorname{Tor}_0(M)$ to be the $\mathcal{H}$-module that assigns to the set S the quotient of $V(S)$ by the sum of the images of the $V(T)$ as T varies over all proper subsets of S. This is analogous with $H_0(V)$ the 0-th homology. It is easy to see that Tor_0 is a right-exact functor, so we consider its left derived functors $\operatorname{Tor}_\bullet$. We will also sometimes refer to these as $H_\bullet$ the $\mathcal{H}$-homology. We let $t_i(V)$ be the maximum $\mathcal{H}$-degree occurring in $\operatorname{Tor}_i(V)$. We use the convention $t_i(V) = -1$ if $\operatorname{Tor}_i(V) = 0$. We define the *Castelnuovo–Mumford regularity* of V, denoted $\operatorname{reg}(M)$, to be the minimum

integer r such that $t_i(V) \leqslant r + i$ for all i. We will show that every finitely generated $\mathcal{H}$-module has finite regularity.

To do this we will use the theory of shift functors established in [8]. Corollary 7.2 shows that the shift functor we define in Section 7 is a generic shift functor in the language of [8]. We will make use of the following result

COROLLARY 8.1 ([8, Corollary 1.11]). *Suppose that k is a commutative Noetherian ring and $\underline{C}$ has a generic shift functor Σ. If $\mathrm{reg}(\Sigma M(s)) \leqslant s$ for all $s \geqslant 0$, then $\mathrm{reg}(V) < \infty$ for every finitely generated $\underline{C}$-module V.*

$\mathcal{H}$ satisfies all the necessary properties outlined by Gan and Li in [8, §1.2], so we may apply their results in our setting. We can use Corollary 8.1 in combination with Lemma 7.1 to conclude,

THEOREM 8.2. *Every finitely generated $\mathcal{H}$-module V has finite regularity.*

Proof. We established that $\mathcal{H}$ has a generic shift functor Σ as defined in Section 7. Lemma 7.1 implies that $\mathrm{reg}(\Sigma M(s)) = \mathrm{reg}(M(s) \oplus M(s-1)) \leqslant s$. Corollary 8.1 then immediately implies every finitely generated $\mathcal{H}$-module V has finite regularity. $\qquad\square$

REMARK 8.3. We want to thank an anonymous referee for pointing out that we can use the shift functor to directly prove finite regularity. One can also immediately deduce local noetherianity of $\mathcal{H}$ from the existence of this shift functor in combination with Lemma 7.1 as seen in [8]. We use the methods developed by Sam and Snowden in [38] to prove Theorem 2.13 because it requires less setup at the start and because we make use of the explicit connection between **OI** and $\mathcal{H}$.

9. DEGREE d POLYNOMIALS IN n VARIABLES

The first example we consider is probably the simplest. Consider the $\mathcal{H}$-module V with $V_n = \mathbf{k}[x_1, \ldots, x_n]_d$, that is degree d polynomials in n variables. There is a $H_n(0)$ action on V_n via *Demazure operations*, in particular the element τ_i of $H_n(0)$ acts by the operator π_i where

$$\pi_i(f) := \partial_i(x_i f) = \frac{x_i f - s_i(x_i f)}{x_i - x_{i+1}},$$

where ∂_i is the *divided difference operator* and s_i is the transposition $(i, i+1)$ acting on indices.

Given a map $f \in \mathrm{hom}_{\mathcal{H}}([n], [m])$, we can factor f as a series of π_i in $H_n(0)$ followed by the injection of variables $\iota_{n,m}$ followed by a series of π_i in $H_m(0)$. f acts as the Demazure operators corresponding to the π_i followed by the injection of variables $\iota_{n,m}$ then the other Demazure operators corresponding to the remaining π_i in $H_m(0)$.

For example, consider the map in $\mathrm{hom}_{\mathcal{H}}([3], [5])$ corresponding to the diagram
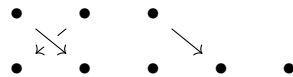
we can factor this as the map π_1 followed by $\iota_{3,5}$ followed by π_3. When we apply this to the element x_1 in the case where $d = 1$ we get $x_1 + x_2$ because $\pi_1(x_1) = x_1 + x_2$ and $\pi_3(x_1 + x_2) = x_1 + x_2$.

We can use our Criterion 2.9 to verify this is a $\mathcal{H}$-module. In this case the natural inclusion map will be an inclusion of variables. We just have to verify that for the

natural inclusion of variables $\mathbf{k}[x_1, \ldots, x_n] \to \mathbf{k}[x_1, \ldots, x_m]$ the Demazure operators $\pi_{n+1}, \ldots, \pi_{m-1}$ act by 1. This is clearly the case since

$$\frac{x_{i+1} f(x_1, \ldots, x_n, 0, \ldots, 0) - x_{i+2} f(x_1, \ldots, x_n, 0, \ldots, 0)}{x_{i+1} - x_{i+2}} = f$$

for all $i = n, \ldots, m-1$. Notice that V is finitely generated as a $\mathcal{H}$-module in degrees $\leqslant d$. This is because once we have d variables, we can get every composition of d, so we have all unique exponent vectors of monomials. We then have to inject to higher degrees to get all other monomials. Theorem 6.1 implies we should be able to decompose V in the Grothendieck group $\mathcal{G}(\mathrm{Mod}_K)$. We will show how to do this explicitly in the case that $d = 1$ and $d = 2$.

When $d = 1$ we have $V_1 = \mathrm{Span}_{\mathbf{F}}(x_1)$, $V_2 = \mathrm{Span}_{\mathbf{F}}(x_1, x_2)$ etc. We will now explicitly decompose these V_i in the Grothendieck group. We will do this for $i = 2$ because it illustrates the process in general. V_2 has a one dimensional subspace spanned by x_2 where $\pi_1(x_2) = 0$. This corresponds to $\mathbf{C}_{(1,1)}$. When we quotient by this space $\pi_1(x_1) = x_1$ so in the quotient x_1 spans a subspace isomorphic to $\mathbf{C}_{(2)}$. Hence we have

$$[V]_2 = \square\!\square_{\text{(column)}} + \square\square.$$

If we continue to decompose in a similar fashion we find

$$[V]_1 = \square, \qquad\qquad [V]_2 = \square\square + \square\!\square_{\text{(column)}},$$

$$[V]_3 = (\text{diagrams}), \quad [V]_4 = (\text{diagrams}).$$

It is actually not hard to see that $V = M(\square)$. According to Theorem 6.1, in $\mathcal{G}(\mathrm{Mod}_{\mathcal{H}})$ we have uniquely that,

$$[V_n] = [M(\square)_n].$$

When $d = 2$, we have $V_1 = \mathrm{Span}_{\mathbf{F}}(x_1^2)$, $V_2 = \mathrm{Span}_{\mathbf{F}}(x_1^2, x_2^2, x_1 x_2)$, etc. The $\mathcal{H}$ action is described by injecting variables $x_i \mapsto x_i$ and applying the corresponding Demazure operators. For example take $f \in \hom_{\mathcal{H}}([2], [3])$,
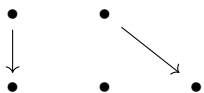
The corresponding map $f_*(x_1^2) = x_1^2$ because it will correspond to $\pi_2(x_1^2)$ because the map f factors as $\pi_2 \iota_{2,3}$. $f_*(x_2^2) = \pi_2(x_2^2) = x_2^2 + x_2 x_3 + x_3^2$. Finally $f_*(x_1 x_2) = x_1 x_2 + x_1 x_3$. From this example it is not hard to see this $\mathcal{H}$ module is finitely generated in degree 2. It is not generated in degree 1 because the images of x_1^2 are only x_1^2 and $x_1^2 + x_1 x_2 + x_2^2$. This is also expected from an easy dimension count.

If we carry out the decomposition in the Grothendieck group, we find that

$$[V]_1 = \square, \qquad [V]_2 = 2\,\square\square + \square\!\square_{\text{(col)}}, \qquad [V]_3 = 2\,(\text{diag}) + 2\,(\text{diag}) + 2\,(\text{diag}),$$

$$[V]_4 = 2\,(\text{diag}) + 3\,(\text{diag}) + 2\,(\text{diag}) + 2\,(\text{diag}) + (\text{diag}).$$

One can then check that in the Grothendieck group $\mathcal{G}(\mathrm{Mod}_{\mathcal{H}})$ we get the decomposition,

$$[V] = [M(\square\square)] + [M(\square)]$$

so when $n \geqslant 3$ all of these modules are nonzero in that degree and we have

$$[V_n] = [M(\square\square)_n] + [M(\square)_n].$$

It is easy to see that this captures all of the diagrams that can occur. We will now describe this decomposition in some detail because this is one of the few cases where we can completely work it out. In general it is very hard to determine.

The first submodule F_1 will be exactly the submodule spanned by the elements $x_i x_j$. It is clear that injections between degrees preserve these elements that the Demazure operators also respect these elements. One can check that the sub-$\mathcal{H}$-module spanned by these elements is isomorphic to $M(\square\square)$.

The quotient V/F_1 is spanned by x_i^2 in each degree. It is not hard to check explicitly that this quotient module is isomorphic to $M(\square)$.

This is a particularly nice example because V is actually isomorphic to a direct sum of $M(\alpha)$, but in general this will not be the case.

10. Cohomology of Borel Groups

In the following example, we assume some knowledge of group cohomology. We refer the reader to [33, 22] for a rigorous treatment of the background in relation to this example. Let G be a group and let M be a left G-module. Thus M is a representation of G. Then we can describe the cohomology of G with coefficients in M using either homogeneous or nonhomogeneous cochains. In this section, we will only consider homogeneous cochains. We denote the i-th cohomology group of G with coefficients in M is given by $H^i(G, M)$.

We will now discuss how the 0-Hecke algebra $H_n(0)$ acts on $H^i(B(n, q), \mathbf{F}_q)$ where $B(n, q) \subset \mathbf{GL}_n(\mathbf{F}_q)$ is the group of upper-triangular matrices over $\mathbf{F}_q$. We will then show that one can upgrade the sequence of 0-Hecke algebra representations $[n] \mapsto H^i(B(n, q), \mathbf{F}_q)$ to a $\mathcal{H}$-module.

There is a much broader theory for the following, but we will restrict to the case that matters to us. For a more rigorous treatment, including proofs that everything is well defined we again refer the reader to [33, 22].

DEFINITION 10.1. *Let G be a group. Two subgroups Γ and Γ' are said to be* commensurable *if*

$$[\Gamma : \Gamma \cap \Gamma'] < \infty, \qquad [\Gamma' : \Gamma \cap \Gamma'] < \infty,$$

that is, if $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ'.

It is not hard to show that this defines an equivalence relation on subgroups of G. With this in mind we can define the *commensurator* of a subgroup Γ in G as follows

$$\widetilde{\Gamma} := \{\alpha \in G \mid \alpha^{-1}\Gamma\alpha \sim \Gamma\}.$$

Let Δ be a sub-semigroup of the group G and let $C(\Delta)$ denote the set of mutually commensurable subgroups Γ of G such that

$$\Gamma \subseteq \Delta \subseteq \widetilde{\Gamma}.$$

Given some $\Gamma \in C(\Delta)$ and R a commutative ring with identity, we denote by $\mathcal{H}_R(\Gamma; \Delta)$ the free R module generated by the double cosets $\Gamma\alpha\Gamma$ with $\alpha \in \Delta$ and call this the *Hecke algebra* associated to Γ and Δ over R. So every element can be written as

$$\sum_{\alpha \in \Delta} c_\alpha \Gamma\alpha\Gamma.$$

where $c_\alpha \in R$ are zero except for a finite number of α.

The Hecke algebra $\mathcal{H}(\Gamma; \widetilde{\Gamma})$ acts on the cohomology $H^i(\Gamma, A)$ where Γ is a subgroup of a group G and A is a unitary left $\mathbf{Z}[G]$-module. If we take $G = \mathbf{GL}(n, q)$, $\Gamma = B_n$ the Borel subgroup and $A = \mathbf{F}_q$, then $\widetilde{\Gamma} = \mathbf{GL}(n, q)$ and $\mathcal{H}(B_n; \mathbf{GL}(n, q)) = \mathcal{H}_n(q)$. If we restrict scalars to a field of characteristic q then we recover the 0-Hecke algebra. To see this explicitly we refer the reader to [13, Pages 6-7]

The action is defined in the following way. Given a homogeneous cochain $\psi \colon \Gamma^{i+1} \to M$ and a double coset $\Gamma \alpha \Gamma$ with $\alpha \in \widetilde{\Gamma}$, we define a **k**-linear map

$$\mathfrak{T}(\alpha) \colon \mathfrak{C}^i(\Gamma, M) \to \mathfrak{C}^i(\Gamma, M)$$

by

$$(\mathfrak{T}(\alpha)\psi)(\gamma_0, \dots, \gamma_i) \coloneqq \sum_{j=1}^d \alpha_j^{-1}\psi(\xi_j(\gamma_0), \dots, \xi_j(\gamma_i))$$

where $\Gamma \alpha \Gamma = \bigcup_{1 \leqslant j \leqslant d} \Gamma \alpha_j$ and $\alpha_j \gamma_k = \xi_j(\gamma_k)\alpha_{\iota(\gamma_k)}$. This gives a well defined action of $\mathcal{H}(\Gamma; \widetilde{\Gamma})$ on the cohomology $H^i(\Gamma, A)$ for any i as seen in [33, 22].

As mentioned above $\mathcal{H}(B(n, q), \mathbf{GL}(n, q)) \cong H_n(0)$, where $\mathfrak{T}(E_{i,i+1})$ is identified with the generator $\overline{\pi}_i$, $E_{i,i+1}$ being the permutation matrix swapping i and $i+1$ [13]. So the above gives a well defined action of $H_n(0)$ on $H^i(B(n, q), \mathbf{F}_q)$ for any choice of $i \geqslant 0$. The main theorem we will prove in this section is:

THEOREM 10.2. *For any fixed $i \geqslant 0$, the assignment $[n] \mapsto H^i(B(n, q), \mathbf{F}_q)$ is a $\mathcal{H}$-module.*

Before we are ready to prove this theorem, we need a few lemmas. First we will explicitly define the embeddings from $H^i(B(n, q)) \to H^i(B(n+1, q))$. Notice there is a group homomorphism $r_{n+1,n} \colon B(n+1, q) \to B(n, q)$ given by restricting to the first n rows and n columns of any matrix in $B(n+1, q)$. It is not hard to check that this is indeed a group homomorphism. It is well known that such a group homomorphism induces a map on cohomology

$$\Phi_n \colon H^i(B(n, q)) \to H^i(B(n+1, q))$$

where the indexing is swapped because cohomology is contravariant. Explicitly this map on the level of cochains is defined by

$$\Phi_n(\phi)(\gamma_0, \dots, \gamma_i) = \phi(r_{n+1,n}(\gamma_0), \dots, r_{n+1,n}(\gamma_i)),$$

where $\phi \in \mathfrak{C}^i(\Gamma_n, \mathbf{F}_q)$ and $\gamma_j \in B(n+1, q)$. We ultimately wish to show that these maps in combination with the above $H_n(0)$-action combine to define a $\mathcal{H}$-module structure. We will prove this in stages, ultimately working towards invoking Theorem 2.9. First we must prove some structure theorems about the cosets that appear in our decomposition.

LEMMA 10.3. *For a fixed permutation matrix $E_{i,i+1}$, if we consider the decomposition $\Gamma_n E_{i,i+1}\Gamma_n = \sqcup_i \Gamma_n \alpha_i$ there are exactly q cosets independent of n and we can take as coset representatives matrices M with $M_{i+1,i+1} = x$, $M_{i,i+1} = M_{i+1,i} = 1$, $M_{j,j} = 1$ for $j \neq i, i+1$ and $M_{j,k} = 0$ else including $j = k = i$, where x ranges over all elements of $\mathbf{F}_q$.*

Proof. If we were to take left cosets instead of right this coset decomposition would correspond to finding all distinct complete flags that are equivalent to the complete flag corresponding to $E_{i,i+1}\Gamma_n$ up to a left action of Γ_n. This is precisely because Γ_n is the stabilizer of the right $\mathbf{GL}(n, q)$-action on complete flags of an n-dimensional vector space, so left cosets correspond to distinct complete flags. If we act on the left instead of the right $B(n, q)$ acts by upward row operations instead of column operations. So

we can once again view cosets as complete flags if we identify the bottom row with the first spanning vector, the $(n-1)$st row with the second spanning vector etc. Then the left $B(n,q)$ action stabilizes these complete flags.

So we must now find all complete flags equivalent to the complete flag corresponding to $\Gamma_n E_{i,i+1}$ up to a right action of Γ_n. The complete flag corresponding to $\Gamma_n E_{i,i+1}$ is exactly

$$V_0 \subset V_1 \subset \cdots \subset V_{n-(i+1)} \subset V_{n-i} \subset \cdots \subset V_{n-1}$$

with

$$\begin{cases} V_j = \mathrm{Span}_{\mathbf{F}_q}(e_n,\ldots,e_{n-j}) & j = 0,\ldots,n-(i+2) \\ V_j = \mathrm{Span}_{\mathbf{F}_q}(e_n,\ldots,e_{i+2},e_i) & j = n-(i+1) \\ V_j = \mathrm{Span}\,\mathbf{F}_q(e_n,\ldots,e_{i+2},e_i,e_{i+1}) & j = n-i \\ V_j = \mathrm{Span}_{\mathbf{F}_q}(e_n,\ldots,e_{n-j},e_i,e_{i+1},e_{i-1},\ldots,e_{n-j}) & j = n-i+1,\ldots,n-1. \end{cases}$$

All of these equivalent distinct flags are exactly in correspondence with the M_x described above. That is $M_{i+1,i+1} = x$, $M_{i,i+1} = M_{i+1,i} = 1$, $M_{i,i} = 0$ and all other entries are zero, where x ranges over all elements of $\mathbf{F}_q$. This is clear because the only difference in flags is what multiple of the vector e_{i+1} we add to e_i to get $V_{n-(i+1)} = \mathrm{Span}_{\mathbf{F}_q}(e_n,\ldots,e_{i+2},e_i+xe_{i+1})$ and $V_{n-i} = \mathrm{Span}_{\mathbf{F}_q}(e_n,\ldots,e_{i+2},e_i+xe_{i+1},e_{i+1})$ in the complete flag

$$V_0 \subset \cdots V_{n-(i+1)} \subset V_{n-i} \subset \cdots \subset V_{n-1}.$$

There are precisely q such distinct choices. $\qquad\qquad\square$

LEMMA 10.4. *The map* $\Phi_n \colon H^i(B(n,q)) \to H^i(B(n+1,q))$ *is* $H_n(0)$-*equivariant where* $H_n(0)$ *acts via the natural embedding into* $H_{n+1}(0)$ *on* $H^i(B(n+1,q))$.

Proof. The action is defined on the level of cochains, so we can prove equivariance there and it will imply equivariance on cohomology. Given a cochain map $\phi \in \mathfrak{C}^i(\Gamma_n, \mathbf{F}_q)$ and any $\pi_i \in H_n(0)$ acting by $\mathfrak{T}(E_{i,i+1})$ we wish to show $\Phi_n\mathfrak{T}(E_{i,i+1})\phi = \mathfrak{T}(E_{i,i+1})\Phi_n\phi$. If we compute the left hand side we get,

$$\mathfrak{T}(E_{i,i+1})\Phi_n\phi(\gamma_0,\ldots,\gamma_i) = \sum_{j=1}^{q} M_{x_j}^{-1}\phi(r_{n+1,n}(\xi_j(\gamma_0)),\ldots,r_{n,n+1}(\xi_j(\gamma_i))),$$

where here we are using the explicit definition of the map $\mathfrak{T}(E_{i,i+1})$ along with Lemma 10.3, so each M_{x_j} is the $(n+1) \times (n+1)$ matrix described there. If we compute the right hand side,

$$\Phi_n\mathfrak{T}(E_{i,i+1})\phi = \sum_{j=1}^{q} (M_{x_j})^{-1}_{n+1,n+1}\phi(\xi_j(r_{n+1,n}(\gamma_0)),\ldots,\xi_j(r_{n+1,n}(\gamma_i))),$$

where in this case the coset representatives we get are exactly the $n \times n$ minors of M_{x_j} where we delete the last row and column, this is $(M_{x_j})_{n+1,n+1}$. This is only true because we are considering $\mathfrak{T}(E_{i,i+1})$ for $i = 1,\ldots,n-1$.

First notice that each of these matrices acts trivially because we take coefficients in the ground field. To prove these two are equal, it suffices to show $\xi_j(r_{n+1,n}(\gamma_k)) = r_{n+1,n}(\xi_j(\gamma_k))$.

By definition $\xi_j(r_{n+1,n}(\gamma_k))$ is the matrix determined by

$$(3) \qquad\qquad (M_{x_j})_{n+1,n+1}r_{n+1,n}(\gamma_k) = \xi_j(r_{n+1,n}(\gamma_k))(M_{x_m})_{n+1,n+1}$$

and $\xi_j(\gamma_k)$ is determined by

$$(4) \qquad\qquad\qquad M_{x_j}\gamma_k = \xi_j(\gamma_k)M_{x_\ell}.$$

In Equation (3) on the left hand side, we are multiplying minors, so one can see that this is precisely the multiplication that occurs in Equation (4) on the left hand side

if you remove the last row and column of each matrix since restriction is a group homomorphism. This implies that the right hand sides must be equal as well if we restrict the right hand side of Equation (4) to only consider the $n \times n$ minors where we delete the last row and column, i.e. $\xi_j(r_{n+1,n}(\gamma_k)) = r_{n+1,n}(\xi_j(\gamma_k))$. From the above this implies equivariance. $\qquad \square$

LEMMA 10.5. *If we take any element in the image of* $\Phi_{n,m} = \Phi_m \Phi_{m-1} \cdots \Phi_n$, $\mathfrak{T}(E_{i,i+1})$ *will act by 0 on this element for* $i = n+1, \ldots, m$.

Proof. Once again we can work at the level of cochains. Given a cochain map $\phi \in \mathfrak{C}(\Gamma_n, \mathbf{F}_q)$ we will show that $\mathfrak{T}(E_{i,i+1})\Phi_{n,m}\phi = 0$. To do this, we can explicitly compute this map,

$$\mathfrak{T}(E_{i,i+1})\Phi_{n,m}\phi(\gamma_0, \ldots, \gamma_i) = \sum_{j=1}^{q} M_{x_j}^{-1} \phi(r_{m,n}(\xi_j(\gamma_0)), \ldots, r_{m,n}(\xi_j(\gamma_i))),$$

where once again M_{x_j} is the $m \times m$ matrix described in Lemma 10.3. This detail will actually not matter for this proof, the important fact here is that there are exactly q cosets. Since each $M_{x_j}^{-1}$ acts trivially as we take coefficients in the ground field, we can simplify this equation,

$$\mathfrak{T}(E_{i,i+1})\Phi_{n,m}\phi(\gamma_0, \ldots, \gamma_i) = \sum_{j=1}^{q} \phi(r_{m,n}(\xi_j(\gamma_0)), \ldots, r_{m,n}(\xi_j(\gamma_i))).$$

Again recall that $\xi_j(\gamma_k)$ is defined by

$$M_{x_j}\gamma_k = \xi_j(\gamma_k)M_{x_m}$$

where each of the M_{x_ℓ} have $(M_{x_\ell})_{i+1,i+1} = x_\ell$, $(M_{x_\ell})_{i,i+1} = (M_{x_\ell})_{i+1,i} = 1$ and $M_{a,b} = 0$ else. In particular if we consider the $n \times n$ minor where we delete the last $m - n$ rows and columns we get the identity matrix since $i \geqslant n+1$. As a result, if we apply this restriction map $r_{m,n}$ to both sides of the above equation because it is a group homomorphism we find

$$r_{m,n}(\gamma_k) = r_{m,n}(\xi_j(\gamma_k)),$$

where notice this does not depend on j. As a result,

$$\mathfrak{T}(E_{i,i+1})\Phi_{n,m}\phi(\gamma_0, \ldots, \gamma_i) = \sum_{j=1}^{q} \phi(r_{m,n}(\gamma_0), \ldots, r_{m,n}(\gamma_i))$$
$$= q\phi(r_{m,n}(\gamma_0), \ldots, r_{m,n}(\gamma_i)) = 0. \qquad \square$$

We are now ready to prove our main theorem.

THEOREM 10.6. *For any fixed* $i \geqslant 0$, *if we let* $\pi_j \in H_n(0)$ *act by* $\mathfrak{T}(E_{j,j+1}) + \mathrm{id}$ *and we take as our transition maps the* $\Phi_n \colon H^i(B(n,q), \mathbf{F}_q) \to H^i(B(n+1,q), \mathbf{F}_q)$ *then the assignment* $[n] \mapsto H^i(B(n,q), \mathbf{F}_q)$ *is a* $\mathcal{H}$-*module*.

Proof. According to Theorem 2.9 it suffices to show that the transition maps Φ_n are $H_n(0)$-equivariant and that the $\mathfrak{T}(E_{j,j+1}) + \mathrm{id}$ act via the identity on the image of $\Phi_{n,m}$ for $j = n+1, \ldots, m$. The first statement is exactly what we show in Lemma 10.4 and the second statement follows immediately from Lemma 10.5. $\qquad \square$

REMARK 10.7. We believe this is a finitely generated $\mathcal{H}$ module, but have not been able to prove it yet.

11. Homology of Borel Groups

In [29] the authors study the group homology of the unipotent group and prove in particular that the assignment $[n] \mapsto H_i(U_n, \mathbf{F}_q)$ is a finitely generated **OI**-module for any fixed $i \geqslant 0$. Although being an **OI**-module does provide some insight into the behavior of these homology groups, it is natural to ask if there is a more rigid categorical structure present. In this section we will discuss how this **OI**-module structure can be extended to a $\mathcal{H}$-module structure. We then use [29] to show that the $\mathcal{H}$-module is also finitely generated. This then implies we have representation stability.

Our ultimate goal is to gain a deeper understanding of the homology $H_i(B(n, q), \mathbf{F}_q)$, but it is equivalent to the homology of $H_i(U(n, q), \mathbf{F}_q)$ the unipotent subgroup, so we will study this. To see why this is the case notice we have a short exact sequence

$$0 \to \mathrm{Diag}(n, q) \to B(n, q) \to U(n, q) \to 0$$

where $\mathrm{Diag}(n, q)$ are the diagonal matrices. From the Hochschild–Serre spectral sequence we have

$$H_i(U(n, q), H_j(\mathrm{Diag}(n, q), \mathbf{F}_q)) \implies H_{i+j}(B(n, q), \mathbf{F}_q).$$

Since the order of $\mathrm{Diag}(n, q)$ is prime to q, the input vanishes when $j > 0$, so the spectral sequence immediately degenerates and gives

$$H_i(U(n, q), H_0(\mathrm{Diag}(n, q), \mathbf{F}_q)) = H_i(B(n, q), \mathbf{F}_q).$$

$\mathrm{Diag}(n, q)$ acts trivially on $\mathbf{F}_q$, so the left side is $H_i(U(n, q), \mathbf{F}_q)$.

We will define a 0-Hecke action on $U(n, q)$, this action will then induce an action on homology. We will then show this action is finitely generated via the arguments in [29].

Given an element $M \in U(n, q)$ we will define the action of $\pi_i \in H_n(0)$ for $i = 1, \ldots, n-1$. π_i will act on M by replaces the entry in columns $i + 1$ with the entry in column i for rows $1, \ldots, i-1$, it will set $M_{i,i+1} = 0$ and it will replace the entry in row $i + 1$ with the entry in row i for columns $i + 2, \ldots, n$.

PROPOSITION 11.1. *The action defined above is a well defined $H_n(0)$ action on $U(n, q)$.*

Proof. We first check that the π_i satisfy the necessary axioms. First we can see $\pi_i \pi_j = \pi_j \pi_i$ for $|i - j| > 1$. We may assume $i < j$. In this case the only interaction between π_i and π_j is in the four entries $(i, j), (i, j+1), (i+1, j), (i+1, j+1)$ where we would be adding rows i and $i + 1$ as well as the columns j and $j + 1$. However, if we labeled the entries as $(i, j) = a$, $(i, j+1) = b$, $(i+1, j) = c$ and $(i+1, j+1) = d$ then regardless of the order of composition, $(i+1, j+1) = a$ and all the other entries becomes zero.

Next we must check that $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ for $i = 1, \ldots, n-2$. If we fix some row j with $1 \leqslant j \leqslant i-1$ the three entries affected in that row are columns $i, i+1$ and $i+2$. Let entry $(j, i) = a$, $(j, i+1) = b$ and $(j, i+2) = c$. Regardless of the order of composition we have entry $(j, i) = (j, i+1) = (j, i+2) = a$. The only other part of the matrix we need to check is entries (k, ℓ) with $k \leqslant \ell$ and $k = i, i+1, i+2$, $\ell = i, i+1, i+2$. However in this case we always have entries $(i, i+1) = (i, i+2) = (i+1, i+2) = 0$ and $(i, i) = (i+1, i+1) = 1$ regardless of the order of composition. Checking the columns is similar.

Finally we verify that $\pi_i^2 = \pi_i$ for $i = 1, \ldots, n-1$. This is not hard to see from definition.

Next we need to verify that $\pi_i(AB) = \pi_i(A)\pi_i(B)$. Fix a row $1 \leqslant j \leqslant i-1$ and consider the entries in columns i and $i+1$. $(AB)_{j,i} = \sum_{k=1}^{n} A_{j,k} B_{k,i}$ and $(AB)_{j,i+1} =$

$\sum_{k=1}^{n} A_{j,k} B_{k,i+1}$. So $\pi_i(AB)_{j,i} = \sum_{k=1}^{n} A_{j,k} B_{k,i} = (\pi_i(A)\pi_i(B))_{j,i}$ and

$$\pi_i(AB)_{j,i+1} = \sum_{k=1}^{n} A_{j,k} B_{k,i} = (\pi_i(A)\pi_i(B))_{j,i+1}$$

Checking the appropriate columns is similar. $\qquad \square$

The above action then induces an action on homology by considering the action on homogeneous chains $\pi_i[g_1, \ldots, g_k] = [\pi_i g_1, \ldots, \pi_i g_k]$. This action is well defined by Proposition 11.1. Furthermore, this action commutes with the differential, once again by Proposition 11.1, so it induces an action on homology.

Furthermore, there is a natural embedding of groups $\phi_{n,n+1} \colon U(n,q) \hookrightarrow U(n+1,q)$ where we add a new row and column to the bottom of any matrix in $U(n,q)$ with a 1 on the diagonal. This induces a map on homology $\Phi_n \colon H^i(U(n,q), \mathbf{F}_q) \to H^i(U(n+1,q), \mathbf{F}_q)$. Explicitly on the level of chains, we embed the element $[g_1, \ldots, g_k] \to [\phi_{n,n+1}(g_1)| \cdots |\phi_{n,n+1}(g_k)]$.

We will now argue that the above action and Φ_n endow the homology of the unipotent group with a finitely generated $\mathcal{H}$-module structure.

LEMMA 11.2. *The map* $\Phi_n \colon H_i(B(n,q), \mathbf{F}_q) \to H_i(B(n+1,q), \mathbf{F}_q)$ *is* $H_n(0)$-*equivariant where* $H_n(0)$ *acts via the natural embedding into* $H_{n+1}(0)$ *on* $H_i(B(n+1,q), \mathbf{F}_q)$.

Proof. We verify this on the level of chains, this will then imply the result for homology. Fix some $\pi_j \in H_n(0)$, so $j = 1, \ldots, n-1$ and some chain $[g_1, \ldots, g_i]$ with $g_\ell \in U(n,q)$. Consider

$$\Phi_n(\pi_j([g_1, \ldots, g_i])) = \Phi_n([\pi_j(g_1), \ldots, \pi_j(g_i)]) = [\phi_{n,n+1}\pi_j(g_1), \ldots, \phi_{n,n+1}\pi_j(g_i)].$$

Notice that $\phi_{n,n+1}\pi_j(g_\ell) = \pi_j\phi_{n,n+1}(g_\ell)$ because in $\phi_{n,n+1}(g_\ell)$ we only add a column of zeroes in rows $1, \ldots, n$, so this will not affect the action of π_j since $1 \leqslant j \leqslant n-1$. $\qquad \square$

To show that the assignment $[n] \mapsto H_i(B(n,q), \mathbf{F}_q)$ with transition maps given by Φ_n is a $\mathcal{H}$-module it remains to show the following

LEMMA 11.3. *If we take any element in the image of* $\Phi_{n,m} = \Phi_m\Phi_{m-1}\cdots\Phi_n$, π_j *will act by the identity on this element for* $j = n+1, \ldots, m$.

Proof. Once again we verify this on the level of chains. Fix some $\pi_j \in H_m(0)$ for $n+1 \leqslant j \leqslant m$ and some chain $[g_1, \ldots, g_i]$ for $g_\ell \in U(n,q)$. Notice that the embedded chain element $\Phi_{n,m}([g_1, \ldots, g_i])$ has each g_ℓ with only nonzero entries on the diagonal below row n and they are all equal to 1. Hence when we apply π_j it can only act by the identity because $n+1 \leqslant j \leqslant m$. $\qquad \square$

THEOREM 11.4. *For any fixed* $i \geqslant 0$, *if we let* $\pi_j \in H_n(0)$ *act as described above and we take as our transition maps the* $\Phi_n \colon H_i(B(n,q), \mathbf{F}_q) \to H_i(B(n+1,q), \mathbf{F}_q)$ *then the assignment* $[n] \mapsto H_i(B(n,q), \mathbf{F}_q)$ *is a* $\mathcal{H}$-*module.*

Proof. According to Theorem 2.9 it suffices to show that the transition maps Φ_n are $H_n(0)$-equivariant and that the π_j act via the identity on the image of $\Phi_{n,m}$ for $j = n+1, \ldots, m$. The first statement is exactly what we show in Lemma 11.2 and the second statement follows immediately from Lemma 11.3. $\qquad \square$

In [29] the authors study $[n] \mapsto H_i(U(n,q), \mathbf{F}_q)$ as an **OI**-module. They ultimately show that it is a finitely generated **OI**-module. We will now show that when we restrict our $\mathcal{H}$-action it agrees with their **OI**-module action. Their finite generation result then implies that we have finite generation as a $\mathcal{H}$-module and so the homology of the Unipotent subgroup in defining characteristic is representation stable.

We will first describe their action explicitly. For their purposes, they did not need an explicit description, so it is not in their paper, but they do describe one implicitly. Given an order preserving injection $\iota \colon [n] \hookrightarrow [m]$ we describe the **OI**-action on the level of chains. Suppose $\{\alpha_1, \ldots, \alpha_{m-n}\} = [m] \smallsetminus \iota([n])$, then for $g \in U(n, q)$, $\iota_* g$ is the matrix where $g_{i,j} = (\iota_* g)_{\iota(i), \iota(j)}$, $(\iota_* g)_{\alpha_i, \alpha_i} = 1$ and all other entries are zero.

We can see this is the action the authors define in [29, §4.4]. We will now argue that it agrees with our $\mathcal{H}$-module action. The key is to realize that we still have an induced **OI**-group structure given by $\mathbf{U}_n = U(n, q)$ and maps given by the $\mathcal{H}$-module structure described above. Let $\mathbf{U}' = \Sigma(\mathbf{U})$. It is still the case that $U(n + 1, q)$ is the semi-direct product $U_n \ltimes \mathbf{F}_q^n$ and most importantly that this description is still functorial.

To verify this, let $\mathbf{E}$ be the **OI**-group given by $\mathbf{E}_n = \mathbf{F}_q^n$ as described in [29]. We then have homomorphisms of **OI**-groups $i \colon \mathbf{U} \to \mathbf{U}'$ and $p \colon \mathbf{U}' \to \mathbf{U}$ with $pi = \mathrm{id}$ and $\ker(p) = \mathbf{E}$. Where i is given by the natural embedding and p forgets the final column. This functorial decomposition is the only place where the specific **OI**-module structure is used in the proof of finite generation in [29].

LEMMA 11.5. *The restriction of the $\mathcal{H}$-module structure on $[n] \mapsto H_k(U(n, q), \mathbf{F}_q)$ to* **OI** *is a finitely generated* **OI**-*module.*

Proof. This follows formally from the proof of Theorem 1.4 in [29, §6] by the remarks in the previous two paragraphs. The key step being that we still have the same functorial decomposition as described above, so Proposition 6.3 from [29] holds. $\qquad\square$

See the following example to illuminate this proof.

EXAMPLE 11.6. Consider the order preserving injection $\iota(1) = 1$, $\iota(i) = i + 1$ for $i = 2, 3$. This corresponds to the map $\pi_2 \pi_3 \Phi_{3,4}$. The action described in [29] sends the matrix

$$M = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So the action on the final column is the embedding of the entries from the third column into positions 1 and 3. Comparing this to our action,

$$\pi_3 \Phi_{3,4}(M) = \begin{pmatrix} 1 & a & b & b \\ 0 & 1 & c & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \pi_2 \pi_3 \Phi_{3,4}(M) = \begin{pmatrix} 1 & a & a & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Chains are determined up to a left action of $U_n(q)$. So each matrix in our chain will be of this form, and we can apply the element

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{to get} \qquad \begin{pmatrix} 1 & a & a & b \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Checking for other maps is similar. So it is not hard to see that the action on chains is equivalent in the final column, which is precisely the kernel of the map p, called $\mathbf{E}$ in [29].

THEOREM 11.7. *The $\mathcal{H}$-module $[n] \mapsto H_i(U(n, q), \mathbf{F}_q)$ with transition maps described in Theorem* 11.4 *is finitely generated as a $\mathcal{H}$-module.*

Proof. Lemma 11.5 in combination with [29, Theorem 1.4] imply this $\mathcal{H}$-module is finitely generated with respect to the **OI**-substructure. We already showed there is functor $\Phi\colon \mathbf{OI} \to \mathcal{H}$ satisfying property (F) in Theorem 2.13, so being finitely generated with respect to the restriction to **OI** is equivalent to being finitely generated with respect to the full $\mathcal{H}$-action. $\square$

COROLLARY 11.8. *For any fixed $i \geqslant 0$ sequence of $H_n(0)$-modules $\{H_i(U(n,q), \mathbf{F}_q)\}_{n\geqslant 0}$ is representation stable.*

Proof. This follows immediately from Theorems 11.7 and 6.1. $\square$

COROLLARY 11.9. *For any fixed $i \geqslant 0$ sequence of $H_n(0)$-modules $\{H_i(B(n,q), \mathbf{F}_q)\}_{n\geqslant 0}$ is representation stable.*

Proof. This follows formally from the introduction to this section. Since

$$H_i(U(n,q), \mathbf{F}_q) = H_i(B(n,q), \mathbf{F}_q),$$

Corollary 11.8 implies this result. $\square$

12. Graded Components of Stanley Reisner Rings

We first recall some necessary definitions. An *abstract simplicial complex* Δ on a vertex set V is a collection of finite subsets of V, called *faces*, such that any subset of a face is also a face. The *dimension* of a face F is $|F|-1$, so that one vertex faces have dimension zero. The *dimension of a simplicial complex* is the maximum dimension of its faces. We say that a $(d-1)$-dimensional simplicial complex is *balanced* if there exists a coloring map $r\colon V \to [d]$ such that every face consists of vertices of distinct colors. The reason we call this coloring map r is because we call the *rank set* of a face F, denoted by $r(F)$, all the colors of all of its vertices.

The *Stanley–Reisner ring* $\mathbf{F}[\Delta]$ of a simplicial complex Δ over a field $\mathbf{F}$ is

$$\mathbf{F}[\Delta] := \mathbf{F}[y_v \mid v \in V]/I_\Delta,$$

where $I_\Delta := \langle y_u y_v \mid u, v \in V, \{u,v\} \notin \Delta \rangle$. So a monomial $y_{v_1} \cdots y_{v_d}$ is nonzero if and only if $v_1, \ldots, v_d$ all belong to the same face of Δ. This ideal does not equate monomials, it just makes some of them zero, so we can see that all nonzero monomials form a $\mathbf{F}$-basis for $\mathbf{F}[\Delta]$.

If Δ is balanced then its Stanley–Reisner ring $\mathbf{F}[\Delta]$ is multigraded, that is any nonzero monomial $m = y_{v_1} \cdots y_{v_k}$ has a *rank multiset* $r(m)$

12.1. Stanley Reisner Ring of the Boolean Algebra. The Boolean algebra $\mathcal{B}_n$ is the ranked poset of all subsets of $[n]$ ordered by inclusion with minimum element $\varnothing$ and maximum element $[n]$. The rank of an element is defined as the cardinality of the corresponding set, where clearly $|\varnothing| = 0$. Following the definition of the Stanley–Reisner ring above, if we take $\mathcal{B}_n$ as our simplicial complex where the vertices are subsets of $[n]$ and the faces are chains of subsets, we see that $\mathbf{F}[\mathcal{B}_n] = \mathbf{F}[y_A \mid A \subset [n]]/I_\Delta$. In this case

$$I_\Delta = \langle y_A y_B \mid A, B \text{ incomparable} \rangle,$$

in other words A is not a subset of B and B is not a subset of A. For example if we take $A = \{1,3,4\}$ and $B = \{2,4,5\}$. So the nonzero monomials are exactly given by weakly increasing chains of subsets, sometimes called *multichains*. This means that we have an $\mathbf{F}$-basis $\{y_M\}$ indexed by multichains M in $\mathcal{B}_n$. This basis is multigraded by the rank multisets $r(M)$ of the multichains M. For example the multichain $\{2\} \subseteq \{2\} \subset \{1,2,4\} \subset [5]$ has $r(M) = \{1,1,3,5\}$.

There is a natural way of encoding multichains, which [14] uses to construct a $H_n(0)$ action on $\mathbf{F}[\mathcal{B}_n]$ which we recall now. For more details we refer the reader

to [14]. Let $M = (A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k)$ be an arbitrary multichain of length k in $\mathcal{B}_n$, set $A_0 = \varnothing$ and $A_{k+1} = [n]$ by convention. For any such multichain M of length k in $\mathcal{B}_n$, let $p_i(M) = \min\{j \in [k+1] \mid i \in A_j\}$, this records the first position where i occurs in M. By definition it must appear in every position after this, so the collection $p(M) = (p_1(M), \ldots, p_n(M))$ encodes the multichain M. The map $M \mapsto p(M)$ gives a bijection between the set of multichains of length k in $\mathcal{B}_n$ and the set $[k+1]^n$ of all words of length n on the alphabet $[k+1]$ for any fixed integer $k \geqslant 0$.

In Section 9 we saw how $H_n(0)$ can act on the polynomial ring via Demazure operators using the generating set π_i. In this example, it is easier to use the generating set $\overline{\pi_i}$, once again we recall that $\pi_i = \overline{\pi_i} + 1$. Let $M = (A_1 \subseteq \cdots \subseteq A_k)$ be a multichain in $\mathcal{B}_n$, then Huang defined

$$\overline{\pi_i}(y_M) := \begin{cases} -y_M, & p_i(M) > p_{i+1}(M), \\ 0, & p_i(M) = p_{i+1}(M), \\ s_i(y_M), & p_i(M) < p_{i+1}(M), \end{cases}$$

for $i = 1, \ldots, n-1$. Huang shows this is a well defined action that respects the multigrading. We can use this to see how the π_i should act,

$$\pi_i(y_M) := \begin{cases} 0, & p_i(M) > p_{i+1}(M), \\ y_M, & p_i(M) = p_{i+1}(M), \\ s_i(y_M) + y_M, & p_i(M) < p_{i+1}(M). \end{cases}$$

We will now define a $\mathcal{H}$ module where $[n] \mapsto \mathbf{F}[\mathcal{B}_n]$ and the inclusion map $\iota_{n,n+1}$ acts by sending a multichain $M = (A_1 \subseteq \cdots \subseteq A_k)$ to the multichain with $A_{k+1} = [n+1]$ instead of $[n]$. The crossings act by the π_i we just defined on multichains, not the $\overline{\pi_i}$. It is clear that the inclusion maps are $H_n(0)$-equivariant. From Theorem 2.9 it remains to check that if we apply $(\iota_{n,m})_*$ that $\pi_{n+1}, \ldots, \pi_m$ act by 1, or equivalently that $\overline{\pi}_{n+1}, \ldots, \overline{\pi}_m$ act by 0.

By definition of our embedding, elements in the image of $(\iota_{n,m})_*$ will be a polynomial in y_A where every set A is a subset of $[n]$, i.e. it will consist of monomials y_M in $\mathcal{B}_m$ where the multichain M has $A_i \subset [n]$ and $A_{k+1} = [m]$. This implies that $p_i(M) = k+1 = p_{i+1}(M)$ for every $i = n+1, \ldots, m-1$. As a result, $\pi_i(y_M) = y_M$ for every $i = n+1, \ldots, m-1$ so Theorem 2.9 implies the following:

THEOREM 12.1. *The assignment* $[n] \mapsto \mathbf{F}[\mathcal{B}_n]$, *the Stanley–Reisner ring of the Boolean algebra, with transition maps as defined above is a $\mathcal{H}$-module.*

This module cannot possibly be finitely generated because it does not grow like a polynomial. However, there is a multigrading present, where we say that a multichain M has multigrading $g = (r_1, \ldots, r_k)$ if the multichain is of the form $A_1 \subseteq \cdots \subseteq A_k$ where $|A_i| = r_i$. The $H_n(0)$ action respects the multigrading [14], so we can consider the $\mathcal{H}$-submodule given by restricting to homogeneous polynomials whose monomials correspond to multichains of a fixed length k that correspond to a fixed composition $(\alpha_1, \ldots, \alpha_k)$, that is the set sizes remain fixed as well. When we apply the transfer map $\tau \colon \mathbf{F}[\mathcal{B}_n] \to \mathbf{F}[X]$ defined by

$$\tau(y_M) := \prod_{1 \leqslant i \leqslant k} \prod_{j \in A_j} x_j,$$

for all multichains $M = (A_1 \subseteq \cdots \subseteq A_k)$ in $\mathcal{B}_n$, these correspond exactly to degree $r_1(M) + \cdots + r_k(M)$ monomials. This map is not a ring homomorphism, but it restricts to an isomorphism $\tau \colon \mathbf{F}[\mathcal{B}_n^*] \cong \mathbf{F}[X]$ of $H_n(0)$-modules. For more details we refer the reader to [14, §3.4].

Theorem 12.2. *If we fix any multigrading g, the assignment $[n] \mapsto \mathbf{F}[\mathcal{B}_n]_g$ defines a homogeneous multigraded $\mathcal{H}$-module that is finitely generated.*

Proof. First, because the action as defined above respects the multigrading and so do our embedding maps $(\iota_{n,m})_*$ Theorem 12.1 implies that this is a $\mathcal{H}$-module.

To see that it is finitely generated, notice from the discussion above that if we have fixed multigrading $r(M) = (a_1, \ldots, a_k)$ the monomial corresponding to the multichain

$$[a_1] \subseteq [a_2] \subseteq \cdots \subseteq [a_k],$$

in lowest degree a_k will generate all other monomials because after we embed to a higher degree we can apply permutations to get any multichain. In terms of monomials, this multichain corresponds to

$$x_1^{r_1(M)} x_2^{r_2(M)} \cdots x_{a_k}^{r_{a_k}(M)}$$

where $r_i(M)$ is the number of times i occurs in the multichain M. We can also encode this monomial using its exponent vector as

$$(r_1(M), r_2(M), \ldots, r_{a_k}(M)) = (\gamma_1, \ldots, \gamma_{a_k}).$$

In a higher degree, in order for us to stay in the same multigrading, the corresponding monomial must have exponent vector corresponding to the above vector where we can insert 0s and permute the γ_i since this is equivalent to the multiset sizes remaining the same. The action of the 0-Hecke algebra elements $\overline{\pi_i}$ on these exponent vectors is via sorting. If we have an exponent vector $(a_1, \ldots, a_n)$, $\overline{\pi_i}$ will swap a_i and a_j if $a_i > a_j$, it will be zero if $a_i = a_j$ and it will act by -1 is $a_i < a_j$. By construction our original multichain corresponds to the exponent vector with $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_k$.

Given any other exponent vector $\beta = (\beta_1, \ldots, \beta_d)$ in a higher degree d of the same multigrading, we can first inject $(\gamma_1, \ldots, \gamma_{a_k})$ into that degree to get $(\gamma_1, \ldots, \gamma_{a_k}, 0, \ldots, 0)$ where there are $d - a_k$ zeroes. Let $(\beta_{i_1}, \ldots, \beta_{i_{a_k}})$ be the nonzero entries of the exponent vector β. Use the $\overline{\pi_i}$ to sort the γ_i into the order in which the nonzero β_{i_j} appear. Algorithmically do this by looking at the first entry, if $\beta_{i_1} = \gamma_1$ move on to β_{i_2}, if not this means $\beta_{i_1} = \gamma_j < \gamma_1$. Since this is the case, we can move γ_j to the first position as it will be smaller than everything to its left. We then consider β_{i_2}, if $\beta_{i_2} = \gamma_1$ we move on to β_{i_3}, otherwise we perform the same procedure to place the correct γ_j into the second spot. We continue in this way until the γ_i are in the correct order.

We can then use the $\overline{\pi_i}$ to sort the γ_i into the nonzero entries of the exponent vector $(\beta_1, \ldots, \beta_d)$, we can do this because all the other entries will be zero so we can shift any nonzero entries to the right as much as we want. This shows that we can generate any monomial with the fixed multigrading g, which completes the proof. $\square$

Corollary 12.3. *For a fixed multigrading g, and n a sufficiently large positive integer, $F[\mathcal{B}_n]_g$ is representation stable. That is, there is a finite list of compositions α_i paired with finitely many integers $k_j \in \mathbf{Z}_{\geqslant -1}$ such that*

$$[F[\mathcal{B}_n]_g] = \bigoplus_{i,j} c_{\alpha_i, k_j} [M(\alpha_i, k_j)]$$

where the non-negative integer integers c_{α_i, k_j} are independent of n.

Proof. This follows directly from Theorem 6.1 in combination with Theorem 12.2. $\square$

Another way to state this theorem is that for any fixed multigrading g and for n sufficiently large, there is a finite list $\{(\alpha_i, k_i)\}$ of compositions paired with integers $k_i \in \mathbf{Z}_{\geqslant -1}$ that completely control the simple $H_n(0)$-modules that can occur in $F[\mathcal{B}_n]_g$.

13. Quasisymmetric Schur Modules

In this section, we pursue an example of a $\mathcal{H}$-module that is not a **FI**-module. These modules arise in a natural way. Through the Frobenius characteristic map, we get an isomorphism between the irreducible representations of the symmetric group up to isomorphism and the ring of symmetric functions. In this setting, the Specht modules V_λ map to s_λ the Schur polynomial.

There is a similar picture for representations of the 0-Hecke algebra. As we have discussed, there is a noncommutative and quasisymmetric characteristic map. The quasisymmetric characteristic map provides an isomorphism between the Grothendieck group of finitely generated $H_n(0)$-modules for all n and the ring of quasisymmetric functions. This map sends the irreducible module $\mathbf{C}_\alpha$ to the fundamental quasisymmetric function F_α. For many years, these were thought of as the analogue of Schur functions in **QSym**. Although they do have a multiplication rule as we have seen and studied above, they do not naturally lift many of the well known properties of Schur functions to the ring of quasisymmetric functions (expression in terms of monomial symmetric functions, Pieri rule, etc.).

In [12], the authors discovered and defined this appropriate analogue which they aptly named quasisymmetric Schur functions. For more details on why these functions are a natural refinement of Schur functions in the quasisymmetric setting we refer the reader to [12]. It then became a natural question to ask if there were representations of $H_n(0)$ that realize these quasisymmetric Schur functions under the quasisymmetric characteristic map. Recently, in [41], the authors define a collection of $H_n(0)$ modules for varying n and prove their image is exactly the quasisymmetric Schur functions. We will now define these modules and show how it is possible to put a $\mathcal{H}$-module structure on suitable collections of them. The construction of these modules also illustrates the type of symmetry that $\mathcal{H}$ preserves, namely an upward symmetry, as opposed to **FI** which can only act when the corresponding objects have complete symmetry.

We begin by making the necessary definitions. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n, we define its *reverse composition diagram* which we will denote by α as an array of left-justified boxes with α_i boxes in row i from the top. Notice, this is very different from the ribbon tableau representation of a composition α. The reverse composition diagram is more akin to Young diagrams. We say that a box is in position (i,j) if it is i rows down from the top and j columns in from left to right. We are now ready to make a key definition,

DEFINITION 13.1. *Given a composition $\alpha \vDash n$, we can define a* standard reverse composition tableau*, abbreviated SRCT τ of shape α and size n to be a bijective filling $\tau \colon \alpha \to \{1, \ldots, n\}$ of the cells (i,j) of the reverse composition diagram α subject to the conditions*

(1) *The entries in each row must be decreasing when read from left to right,*
(2) *The entries in the first column must be increasing when read from top to bottom,*
(3) *The filling must satisfy the* triple rule*, namely, if $i < j$ and $\tau(i, k) > \tau(j, k+1)$, then $\tau(i, k+1)$ exists and $\tau(i, k+1) > \tau(j, k+1)$.*

We denote the set of all SRCTs by $\mathrm{SRCT}(\alpha)$ maintaining the notation in [41]. For more information on the triple rule, and the definitions and constructions we refer the reader to [41].

EXAMPLE 13.2. Let $\alpha = (2, 1, 4)$ be a composition of 7, then an example of an element of SRCT(α) is

$$
\begin{array}{|c|c|c|c|}
\hline 2 & 1 \\
\cline{1-2} 3 \\
\cline{1-1} 7 & 6 & 5 & 4 \\
\hline
\end{array}.
$$

A nonexample is

$$
\begin{array}{|c|c|c|c|}
\hline 6 & 1 \\
\cline{1-2} 3 \\
\cline{1-1} 7 & 5 & 4 & 2 \\
\hline
\end{array}
$$

because this does not satisfy the triple rule and the first column is not increasing. In particular, we see that $\tau(1, 1) > \tau(3, 2)$, but $\tau(1, 2) = 1 < \tau(3, 2) = 5$.

Given a SRCT τ there is a notion of a corresponding descent set

$$\mathrm{Des}(\tau) = \{i \mid i + 1 \text{ appears weakly right of } i\} \subseteq [n - 1].$$

From this we can construct a *descent composition* of τ, $\mathrm{comp}(\tau) = \mathrm{comp}(\mathrm{Des}(\tau))$. The collection of standard reverse composition tableaux is important because it is used to define the quasisymmetric Schur function $\mathcal{S}_\alpha$. Namely,

DEFINITION 13.3. *Let $\alpha \vDash n$ be a composition. Then the* quasisymmetric Schur function $\mathcal{S}_\alpha$ *is defined by $S_\varnothing = 1$ and*

$$S_\alpha = \sum_{\tau \in \mathrm{SRCT}(\alpha)} F_{\mathrm{comp}(\tau)}$$

where $\mathrm{comp}(\tau)$ was defined above.

Now we are almost ready to define a $H_n(0)$ action on SRCTs, we just need the notion of attacking blocks.

DEFINITION 13.4. *Given $\tau \in \mathrm{SRCT}(\alpha)$ for a composition $\alpha \vDash n$, and any positive integer i with $1 \leqslant i \leqslant n - 1$ we say that i and $i + 1$ are* attacking *if one of the following is true*

(1) *i and $i + 1$ are in the same column of τ, or*
(2) *i and $i + 1$ are in adjacent columns of τ, with $i + 1$ positioned strictly down and to the right of i.*

Given $\tau \in \mathrm{SRCT}(\alpha)$ for some $\alpha \vDash n$, and a positive integer $1 \leqslant i \leqslant n - 1$, let $s_i(\tau)$ denote the filling obtained by interchanging the positions of entries i and $i + 1$ in τ. Now we are ready to define a 0-Hecke action on SRCT(α) as follows, for $1 \leqslant i \leqslant n - 1$ let

$$
\overline{\pi}_i(\tau) = \begin{cases} -\tau & i \notin \mathrm{Des}(\tau) \\ 0 & i \in \mathrm{Des}(\tau), \ i \text{ and } i + 1 \text{ attacking} \\ s_i(\tau) & i \in \mathrm{Des}(\tau), \ i \text{ and } i + 1 \text{ non-attacking.} \end{cases}
$$

The action that the authors in [41] describe differs from ours by a sign, we will denote their generators by $\tilde{\pi}_i$. We choose to introduce the sign because this action is more amenable to an $\mathcal{H}$-module structure and is more in line with usual 0-Hecke actions. It is also not hard to check that this is still a well defined action and that the partial order described below is the same as the one in [41]. We will, however, do this explicitly. We omit some details that overlap with the proofs in [41]. For all of the following, τ will denote some SRCT of size n.

LEMMA 13.5. *For $1 \leqslant i \leqslant n - 1$ we have $\overline{\pi}_i^2 = -\overline{\pi}_i$.*

Proof. If $i \notin \mathrm{Des}(\tau)$, then $\overline{\pi}_i(\tau) = -\tau$. So we see that $\overline{\pi}_i^2(\tau) = \tau = -\overline{\pi}_i(\tau)$.

If $i \in \mathrm{Des}(\tau)$ then this is roughly the same proof as in [41]. $\qquad\square$

LEMMA 13.6. *For $1 \leqslant i, j \leqslant n - 1$ with $|i - j| \geqslant 2$, we have $\overline{\pi}_i \overline{\pi}_j = \overline{\pi}_j \overline{\pi}_i$.*

Proof. If neither i nor j belong to $\mathrm{Des}(\tau)$ then $\overline{\pi}_i(\tau) = \overline{\pi}_j(\tau) = -\tau$ and so $\overline{\pi}_i \overline{\pi}_j(\tau) = \tau = \overline{\pi}_j \overline{\pi}_i(\tau)$.

Suppose that $i \in \mathrm{Des}(\tau)$. If i and $i+1$ are attacking, then this is the same proof as in [41]. Otherwise we can assume i and $i + 1$ are non-attacking. This means $\overline{\pi}_i(\tau) = s_i(\tau)$. If $j \notin \mathrm{Des}(\tau)$ then because $|i - j| \geqslant 2$ we also have $j \notin \mathrm{Des}(s_i(\tau))$. As a result $\overline{\pi}_j \overline{\pi}_i(\tau) = \overline{\pi}_j(s_i(\tau)) = -s_i(\tau) = \overline{\pi}_i \overline{\pi}_j(\tau)$. If $j \in \mathrm{Des}(\tau)$ our generators agree with [41]. $\qquad\square$

LEMMA 13.7. *For $1 \leqslant i \leqslant n - 2$, we have $\overline{\pi}_i \overline{\pi}_{i+1} \overline{\pi}_i = \overline{\pi}_{i+1} \overline{\pi}_i \overline{\pi}_{i+1}$.*

Proof. We will proceed in cases. The first case is if $i, i + 1 \notin \mathrm{Des}(\tau)$. This means $\overline{\pi}_i(\tau) = \overline{\pi}_{i+1}(\tau) = -\tau$ so the desired identity clearly holds.

If $i \notin \mathrm{Des}(\tau)$ but $i + 1 \in \mathrm{Des}(\tau)$ we have $\overline{\pi}_i(\tau) = -\tau$. If $i + 1$ and $i + 2$ are attacking, then $\overline{\pi}_{i+1}(\tau) = 0$. This implies that $\overline{\pi}_i \overline{\pi}_{i+1} \overline{\pi}_i(\tau) = \overline{\pi}_{i+1} \overline{\pi}_i \overline{\pi}_{i+1}(\tau) = 0$. As a result we may assume that $i + 1$ and $i + 2$ are non-attacking, or equivalently that $\overline{\pi}_{i+1}(\tau) = s_{i+1}(\tau)$. We then have three possibilities

(1) If $i \notin \mathrm{Des}(s_{i+1}(\tau))$ then $\overline{\pi}_i \overline{\pi}_{i+1}(\tau) = -s_{i+1}(\tau)$, so $\overline{\pi}_{i+1} \overline{\pi}_i \overline{\pi}_{i+1} = -\overline{\pi}_{i+1} \overline{\pi}_{i+1}(\tau) = s_{i+1}(\tau)$ by Lemma 13.5. By assumption $\overline{\pi}_i(\tau) = -\tau$ so $\overline{\pi}_i \overline{\pi}_{i+1} \overline{\pi}_i(\tau) = s_{i+1}(\tau)$ as well.

(2) If $i \in \mathrm{Des}(s_{i+1}(\tau))$ with i and $i+1$ attacking in $s_{i+1}(\tau)$ then $\overline{\pi}_{i+1} \overline{\pi}_i \overline{\pi}_{i+1}(\tau) = 0 = \overline{\pi}_i \overline{\pi}_{i+1} \overline{\pi}_i(\tau)$.

(3) Finally if $i \in \mathrm{Des}(s_{i+1}(\tau))$ with i and $i + 1$ non-attacking in $s_{i+1}(\tau)$ then $\overline{\pi}_i \overline{\pi}_{i+1}(\tau) = s_i s_{i+1} \tau$. As in [41] $i + 1$ is not a descent in $s_i s_{i+1}$ so $\overline{\pi}_{i+1} \overline{\pi}_i \overline{\pi}_{i+1}(\tau) = -s_i s_{i+1}(\tau)$. This is precisely $\overline{\pi}_i \overline{\pi}_{i+1} \overline{\pi}_i(\tau)$ since $\overline{\pi}_i(\tau) = -\tau$.

The next case to consider is if $i \in \mathrm{Des}(\tau)$ and $i + 1 \notin \mathrm{Des}(\tau)$. In this case we have $\overline{\pi}_{i+1}(\tau) = -\tau$. For the exact same reason as above we may assume that i and $i+1$ are non-attacking otherwise both identities vanish. This means $\overline{\pi}_i \overline{\pi}_{i+1} = -s_i(\tau)$. Again we have three possibilities

(1) If $i + 1 \notin \mathrm{Des}(s_i(\tau))$ then $\overline{\pi}_{i+1}(s_i(\tau)) = -s_i(\tau)$. This implies that $\overline{\pi}_{i+1} \overline{\pi}_i \overline{\pi}_{i+1} = s_i(\tau) = \overline{\pi}_i \overline{\pi}_{i+1} \overline{\pi}_i(\tau)$ since $\overline{\pi}_i \overline{\pi}_{i+1} \overline{\pi}_i(\tau) = -\overline{\pi}_i(s_i(\tau)) = -\overline{\pi}_i^2(\tau) = s_i(\tau)$.

(2) If $i + 1 \in \mathrm{Des}(s_i(\tau))$ with $i + 1$, $i + 2$ attacking in $s_i(\tau)$ we have $\overline{\pi}_{i+1} \overline{\pi}_i(\tau) = \overline{\pi}_{i+1}(s_i(\tau)) = 0$. This implies that both identities are zero.

(3) Finally, if $i + 1 \in \mathrm{Des}(s_i(\tau))$ with $i + 1$ and $i + 2$ non-attacking in $s_i(\tau)$ this means $\overline{\pi}_{i+1} \overline{\pi}_i(\tau) = s_{i+1} s_i(\tau)$. Again notice $i \notin \mathrm{Des}(s_{i+1} s_i(\tau))$ so $\overline{\pi}_i \overline{\pi}_{i+1} \overline{\pi}_i(\tau) = -s_{i+1} s_i(\tau)$. Now $\overline{\pi}_{i+1} \overline{\pi}_i \overline{\pi}_{i+1}(\tau) = -s_{i+1} s_i(\tau)$ as well because $\overline{\pi}_{i+1}(\tau) = -\tau$.

For the final case, suppose $i, i + 1 \in \mathrm{Des}(\tau)$. In this case, the proof in [41] applies to our case because there is never a point where an element is not in a descent set. $\qquad\square$

The above proves that our definition of the $\overline{\pi}_i$ defines a $H_n(0)$ action on $\mathrm{SRCT}(\alpha)$ with $\alpha \vDash n$. When defining our $\mathcal{H}$-module, we prefer to use the generators π_i, which we describe explicitly in this case,

$$\pi_i(\tau) = \begin{cases} 0 & i \notin \mathrm{Des}(\tau) \\ \tau & i \in \mathrm{Des}(\tau), \ i \text{ and } i + 1 \text{ attacking} \\ s_i(\tau) + \tau & i \in \mathrm{Des}(\tau), \ i \text{ and } i + 1 \text{ non-attacking.} \end{cases}$$

In [41] the authors then place a partial order on elements of $\mathrm{SRCT}(\alpha)$ called $\preceq_\alpha$, where $\tau_1 \preceq_\alpha \tau_2$ if and only if there exists an element $\pi_\sigma \in H_n(0)$ such that $\pi_\sigma(\tau_1) = \tau_2$. Notice, this would not be possible for **FI**. An important part of establishing that partial order is well defined, i.e. the anti-symmetry, follows from the fact that $H_n(0)$ preserves upward symmetry.

They then arbitrarily extend this partial order to a total order $\preceq_\alpha^t$ and define new modules $\mathcal{V}_{\tau_i}$ for some SRCT $\tau_i \in \mathrm{SRCT}(\alpha)$ with

$$\mathcal{V}_{\tau_i} = \mathrm{Span}\{\tau_j \mid \tau_i \preceq_\alpha^t \tau_j\} \qquad \text{for } 1 \leqslant i \leqslant m.$$

One can think of this as the poset ideal generated by τ_i. They prove that for any choice of τ_i, $\mathcal{V}_{\tau_i}$ is a $H_n(0)$-module. Once again, we note that there is no natural Σ_n action on these modules, or on SRCTs in general. The reason for studying these $\mathcal{V}_{\tau_i}$ is that if we take τ as the minimal element of $\mathrm{SRCT}(\alpha)$ with respect to $\preceq_\alpha^t$ then $\mathrm{Ch}(\mathcal{V}_\tau) = \mathcal{S}_\alpha$ the quasisymmetric Schur function.

We want to note that changing π_i does not affect $\mathcal{V}_{\tau_i}$ as defined in [41]. This follows immediately from the observation that we only get a new nonzero SRCT if we fall into the third case where $i \in \mathrm{Des}(\tau)$ with i and $i+1$ non-attacking. So suppose we have

$$\tilde{\pi}_\sigma(\tau_1) = \tau_2$$

then $\pi_\sigma(\tau_1) = \tau_2 + \sum_i \pi_{\sigma_i}(\tau_1)$ where σ_i is the first i generators in σ. By definition this tail will be in $\mathcal{V}_{\tau_1}$ and so $\tau_2 \in \mathcal{V}_{\tau_1}$. Conversely suppose we have $\pi_\sigma(\tau_1) = \tau_2$, then $\tilde{\pi}_\sigma(\tau_1) = \tau_2 - \sum_i \pi_{\sigma_i}(\tau_1)$ so the definition of $\mathcal{V}_\tau$ does not change if we change the $H_n(0)$ action as above.

Now with our definition of V_τ we claim there is a natural $H_n(0)$ equivariant embedding $\Phi_n \colon \mathrm{SRCT}(\alpha) \to \mathrm{SRCT}(\alpha')$ for $\alpha \vDash n$ and $\alpha' \vDash n+1$ where $\alpha' = \alpha + (1)$ given by filling in α' with the elements of α and placing $n+1$ in the new square. First, notice this is $H_n(0)$ equivariant because the descent set does not change and we do not shift the elements of α so $\pi_1, \ldots, \pi_{n-1}$ act in the same way.

Now define a $\mathcal{H}$-module $\mathcal{H}(\mathcal{V}_\tau)$ with $\tau \vDash n$ via the assignment $[i] \mapsto \mathcal{V}_{\tau+(1^{i-n})}$ for $i \geqslant n$ and $[i] \mapsto 0$ for $i < n$. And define the map from degree i to degree $i+1$ via the map in the previous paragraph.

THEOREM 13.8. *For a composition $\tau \vDash n$, the assignment $[i] \mapsto \mathcal{V}_{\tau+(1^{i-n})}$ for $i \geqslant n$ and $[i] \mapsto 0$ for $i < n$, with the order preserving injection $\iota_{i,i+1}$ from degree i to degree $i+1$ defined via Φ_i is a $\mathcal{H}$-module.*

Proof. By Theorem 2.9 it remains to check in degree $m > n$ that $\pi_{n+1}, \ldots, \pi_{m-1}$ act by the identity on the image of the above map. Notice that any SRCT in the image has $n+1, \ldots, m-1$ descending in the first column. This implies that they are all in the descent set and attacking, so by definition π_i acts by the identity for $i = n+1, \ldots, m-1$. $\square$

EXAMPLE 13.9. In practice, consider the example

If we wish to embed to a higher degree, say 8 we get

$$
\begin{array}{|c|c|}
\hline
2 & 1 \\
\hline
3 \\
\cline{1-1}
5 & 4 \\
\hline
6 \\
\cline{1-1}
7 \\
\cline{1-1}
8 \\
\cline{1-1}
\end{array}
$$
.

We can view these modules $\mathcal{H}(\mathcal{V}_\tau)$ as analogues of $L_\lambda^{\geqslant D}$ in [37], but in this case rather than growth in the first row we have growth in the first column.

PROPOSITION 13.10. *The $\mathcal{H}$-module $\mathcal{H}(\mathcal{V}_\tau)$ is finitely generated for any choice of composition τ.*

Proof. It is not hard to see that the images of the embeddings from the lowest degree generate the entire $\mathcal{H}$-module. This is because in any SRCT of shape $(\tau, 1^d)$ we must fill the 1^d section with the d largest natural numbers in $[|\tau| + d]$, so the number of SRCTs of shape $(\tau, 1^d)$ is actually the same as the number of SRCTs of shape τ. $\square$

COROLLARY 13.11. *For any $n \geqslant 0$ and fixed composition $\tau \vDash n$, the sequence of $H_j(0)$-modules $\{\mathcal{V}_{\tau+(1^{i-n})}\}_{i \geqslant n}$ satisfies representation stability.*

Proof. This follows immediately from Proposition 13.10 and Theorem 6.1. $\square$

REMARK 13.12. It is important to remember here that we follow the notation in [41, 12] where they encode compositions as standard reverse composition tableau. This should not be confused with the above encoding of compositions as ribbon tableau. The above states that for any $i \gg 0$, there is a finite fixed list of compositions $\{(\alpha^j, k_j)\}$ that completely encodes all the irreducible representations that will appear in $\mathcal{V}_{\tau+(1^{i-n})}$.

14. FURTHER QUESTIONS

This work was largely motivated by Sections 10, 11 and the work in [11, 29]. In particular, the sequence of modules $H^i(B(n, q), \mathbf{F}_q)$ are a natural object to study, but they do not have any natural **FI**-module structure. This is largely due to the fact that there is no apparent complete symmetry, only an upward symmetry. There are other sequences of modules with this same property such as the quasisymmetric Schur modules seen in Section 13 that have received a large amount of attention recently. Heuristically one can think of such modules as the quasisymmetric functions that are not symmetric. We believe there are many more natural examples of such modules and are already investigating a few. Some immediate further questions are summarized as follows,

1) Is the $\mathcal{H}$-module $[n] \mapsto H^i(B(n, q), \mathbf{F}_q)$ studied in Section 10 finite generated as a $\mathcal{H}$-module? We believe it is, but have not been able to prove it.
2) What other properties of **FI**-module are also satisfied by $\mathcal{H}$-modules? We have a notion of representation stability and finite regularity, is there a notion of depth or weight?
3) Is there any hope of bounding the regularity? We suspect this is impossible because it seems like the ability to do this depends on the Gabriel–Krull dimension being finite, which it is not for $\mathcal{H}$-modules.
4) Is there a refined version of representation stability if we only consider sequences of projective $H_n(0)$-modules?

5) Is there a more concrete connection between $\mathcal{H}$ and the ring of quasisymmetric functions? The Grothendieck group of **FI**-modules is isomorphic to two copies of the ring of symmetric functions [36]; is there some analogue for $\mathcal{G}(\mathrm{Mod}_{\mathcal{H}})$? We suspect there might be an infinite analogue.

6) Symmetric function theory suggests a deeper connection between **FI** and $\mathcal{H}$. In particular, every symmetric function is quasisymmetric, and every space with complete symmetry also has partial symmetry. It then becomes natural to ask, does every **FI**-module has a natural $\mathcal{H}$-module structure?

   To make this more concrete, there is another category $\mathbf{FI}_q$, the q-deformation of **FI**, over the ring $\mathbf{C}[q]$, where the $q = 1$ fiber gives **FI** and the $q = 0$ fiber recovers $\mathcal{H}$. In this way, we get a correspondence between Grothendieck groups $\mathbf{K}(\mathrm{Mod}_{\mathbf{FI}}) \leftarrow \mathbf{K}(\mathrm{Mod}_{\mathbf{FI}_q}) \rightarrow \mathbf{K}(\mathrm{Mod}_{\mathcal{H}})$. Evidence suggests the first functor, taking the fiber at $q = 1$, is close to an equivalence if one considers flat $\mathbf{FI}_q$-modules. This would then give a map in the desired direction.

7) What can one say about the extensions of $\mathcal{H}$-modules?

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