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Abstract We establish a formalism for working with incidence algebras of posets with symmetries, and we develop equivariant Kazhdan–Lusztig–Stanley theory within this formalism. This gives a new way of thinking about the equivariant Kazhdan–Lusztig polynomial and equivariant Z-polynomial of a matroid.

1. Introduction

The incidence algebra of a locally finite poset was first introduced by Rota, and has proved to be a natural formalism for studying such notions as Möbius inversion [11], generating functions [4], and Kazhdan–Lusztig–Stanley polynomials [12, Section 6].

A special class of Kazhdan–Lusztig–Stanley polynomials that have received a lot of attention recently is that of Kazhdan–Lusztig polynomials of matroids, where the relevant poset is the lattice of flats [5, 9]. If a finite group W acts on a matroid M (and therefore on the lattice of flats), one can define the W-equivariant Kazhdan–Lusztig polynomial of M [7]. This is a polynomial whose coefficients are virtual representations of W, and has the property that taking dimensions recovers the ordinary Kazhdan–Lusztig polynomial of M. In the case of the uniform matroid of rank d on n elements, it is actually much easier to describe the S_n-equivariant Kazhdan–Lusztig polynomial, which admits a nice description in terms of partitions of n, than it is to describe the non-equivariant Kazhdan–Lusztig polynomial [7, Theorem 3.1].

While the definition of Kazhdan–Lusztig–Stanley polynomials is greatly clarified by the language of incidence algebras, the definition of the equivariant Kazhdan–Lusztig polynomial of a matroid is completely *ad hoc* and not nearly as elegant. The purpose of this note is to define the equivariant incidence algebra of a poset with a finite group of symmetries, and to show that the basic constructions of Kazhdan–Lusztig–Stanley theory make sense in this more general setting. In the case of a matroid, we show that this approach recovers the same equivariant Kazhdan–Lusztig polynomials that were defined in [7].

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2. The equivariant incidence algebra

Fix once and for all a field k. Let P be a locally finite poset equipped with the action of a finite group W. We consider the category $\mathcal{C}^W(P)$ whose objects consist of

- a k-vector space V
- a direct product decomposition $V = \prod_{x \leqslant y \in P} V_{xy}$, each V_{xy} finite dimensional
- an action of W on V compatible with the decomposition.

More concretely, for any $\sigma \in W$ and any $x \leqslant y \in P$, we have a linear map

$$\varphi^\sigma_{xy} : V_{xy} \to V_{\sigma(x)\sigma(y)},$$

and we require that $\varphi^e_{xy} = \mathrm{id}_{V_{xy}}$ and that $\varphi^{\sigma'}_{\sigma(x)\sigma(y)} \circ \varphi^\sigma_{xy} = \varphi^{\sigma'\sigma}_{xy}$. Morphisms in $\mathcal{C}^W(P)$ are defined to be linear maps that are compatible with both the decomposition and the action. This category admits a monoidal structure, with tensor product given by

$$(U \otimes V)_{xz} \coloneqq \bigoplus_{x \leqslant y \leqslant z} U_{xy} \otimes V_{yz}.$$

Let $I^W(P)$ be the Grothendieck ring of $\mathcal{C}^W(P)$; we call $I^W(P)$ the *equivariant incidence algebra* of P with respect to the action of W.

Example 2.1. If W is the trivial group, then $I^W(P)$ is isomorphic to the usual incidence algebra of P with coefficients in $\mathbb{Z}$. That is, it is isomorphic as an abelian group to a direct product of copies of $\mathbb{Z}$, one for each interval in P, and multiplication is given by convolution.

Remark 2.2. If W acts on P and $\psi : W' \to W$ is a group homomorphism, then ψ induces a functor $F_\psi : \mathcal{C}^W(P) \to \mathcal{C}^{W'}(P)$ and a homomorphism $R_\psi : I^W(P) \to I^{W'}(P)$.

We now give a second, more down to earth description of $I^W(P)$. Let $\mathrm{VRep}(W)$ denote the ring of finite dimensional virtual representations of W over the field k. A group homomorphism $\psi : W' \to W$ induces a ring homomorphism

$$\Lambda_\psi : \mathrm{VRep}(W) \to \mathrm{VRep}(W').$$

For any $x \in P$, let $W_x \subset W$ be the stabilizer of x. We also define $W_{xy} \coloneqq W_x \cap W_y$ and $W_{xyz} \coloneqq W_x \cap W_y \cap W_z$. Note that, for any $x, y \in P$ and $\sigma \in W$, conjugation by σ gives a group isomorphism

$$\psi^\sigma_{xy} : W_{xy} \to W_{\sigma(x)\sigma(y)},$$

which induces a ring isomorphism

$$\Lambda_{\psi^\sigma_{xy}} : \mathrm{VRep}(W_{\sigma(x)\sigma(y)}) \to \mathrm{VRep}(W_{xy}).$$

An element $f \in I^W(P)$ is uniquely determined by a collection

$$\{f_{xy} \mid x \leqslant y \in P\},$$

where $f_{xy} \in \mathrm{VRep}(W_{xy})$ and for any $\sigma \in W$ and $x \leqslant y \in P$, $f_{xy} = \Lambda_{\psi^\sigma_{xy}}\left(f_{\sigma(x)\sigma(y)}\right)$. The unit $\delta \in I^W(P)$ is characterized by the property that δ_{xx} is the 1-dimensional trivial representation of W_x for all $x \in P$ and $\delta_{xy} = 0$ for all $x < y \in P$. The following proposition describes the product structure on $I^W(P)$ in this representation.

Proposition 2.3. *For any $f, g \in I^W(P)$,*

$$(fg)_{xz} \coloneqq \sum_{x \leqslant y \leqslant z} \frac{|W_{xyz}|}{|W_{xz}|} \mathrm{Ind}^{W_{xz}}_{W_{xyz}} \left(\left(\mathrm{Res}^{W_{xy}}_{W_{xyz}} f_{xy}\right) \otimes \left(\mathrm{Res}^{W_{yz}}_{W_{xyz}} g_{yz}\right)\right).$$

Remark 2.4. It may be surprising to see the fraction $\frac{|W_{xyz}|}{|W_{xz}|}$ in the statement of Proposition 2.3, since $\mathrm{VRep}(W_{xy})$ is not a vector space over the rational numbers. We could in fact replace the sum over $[x, z]$ with a sum over one representative of each W_{xz}-orbit in $[x, z]$ and then eliminate the factor of $\frac{|W_{xyz}|}{|W_{xz}|}$. Including the fraction in the equation allows us to avoid choosing such representatives.

Remark 2.5. Proposition 2.3 could be taken as the definition of $I^W(P)$. It is not so easy to prove associativity directly from this definition, though it can be done with the help of Mackey's restriction formula (see for example [3, Corollary 32.2]).

Remark 2.6. Suppose that $\psi : W' \to W$ is a group homomorphism, and for any $x, y \in P$, consider the induced group homomorphism $\psi_{xy} : W'_{xy} \to W_{xy}$. For any $f \in I^W(P)$, we have, $R_\psi(f)_{xy} = \Lambda_{\psi_{xy}}(f_{xy})$. In particular, if W' is the trivial group, then $R_\psi(f)_{xy}$ is equal to the dimension of the virtual representation $f_{xy} \in \mathrm{VRep}(W_{xy})$.

Before proving Proposition 2.3, we state the following standard lemma in representation theory.

Lemma 2.7. *Suppose that $E = \bigoplus_{s\in S} E_s$ is a vector space that decomposes as a direct sum of pieces indexed by a finite set S. Suppose that G acts linearly on E and acts by permutations on S such that, for all $s \in S$ and $\gamma \in G$, $\gamma \cdot E_s = E_{\gamma\cdot s}$. For each $x \in S$, let $G_x \subset G$ denote the stabilizer of s. Then there exists an isomorphism*

$$E \cong \bigoplus_{s\in S} \frac{|G_s|}{|G|} \mathrm{Ind}_{G_s}^{G}\left(E_s\right)$$

of representations of G.[(1)]

Proof of Proposition 2.3. By linearity, it is sufficient to prove the proposition in the case where we have objects U and V of $\mathcal{C}^W(P)$ with $f = [U]$ and $g = [V]$. This means that, for all $x \leqslant y \leqslant z \in P$, $f_{xy} = [U_{xy}] \in \mathrm{VRep}(W_{xy})$, $g_{yz} = [V_{yz}] \in \mathrm{VRep}(W_{yz})$, and

$$(fg)_{xz} = \left[(U \otimes V)_{xz}\right] = \left[\bigoplus_{x\leqslant y\leqslant z} U_{xy} \otimes V_{yz}\right] \in \mathrm{VRep}(W_{xz}).$$

The proposition then follows from Lemma 2.7 by taking $E = (U \otimes V)_{xz}$, $S = [x, z]$, and $G = W_{xz}$. ☐

Let R be a commutative ring. Given an element $f \in I^W(P) \otimes R$ and a pair of elements $x \leqslant y \in P$, we will write f_{xy} to denote the corresponding element of $\mathrm{VRep}(W_{xy}) \otimes R$.

Proposition 2.8. *An element $f \in I^W(P) \otimes R$ is (left or right) invertible if and only if $f_{xx} \in \mathrm{VRep}(W_x) \otimes R$ is invertible for all $x \in P$. In this case, the left and right inverses are unique and they coincide.*

Proof. By Proposition 2.3, an element g is a right inverse to f if and only if $g_{xx} = f_{xx}^{-1}$ for all $x \in P$ and

$$\sum_{x\leqslant y\leqslant z} \frac{|W_{xyz}|}{|W_{xz}|} \mathrm{Ind}_{W_{xyz}}^{W_{xz}}\left(\left(\mathrm{Res}_{W_{xyz}}^{W_{xy}} f_{xy}\right) \otimes \left(\mathrm{Res}_{W_{xyz}}^{W_{yz}} g_{yz}\right)\right) = 0$$

(1) As in Remark 2.4, we may eliminate the fraction at the cost of choosing one representative of each W-orbit in S.

for all $x < z \in P$.[(2)] The second condition can be rewritten as

$$\left(\operatorname{Res}^{W_x}_{W_{xz}} f_{xx}\right) \otimes g_{xz} = - \sum_{x<y\leqslant z} \frac{|W_{xyz}|}{|W_{xz}|} \operatorname{Ind}^{W_{xz}}_{W_{xyz}} \left(\left(\operatorname{Res}^{W_{xy}}_{W_{xyz}} f_{xy}\right) \otimes \left(\operatorname{Res}^{W_{yz}}_{W_{xyz}} g_{yz}\right)\right).$$

If f_{xx} is invertible in $\operatorname{VRep}(W_x) \otimes R$, then $\operatorname{Res}^{W_x}_{W_{xz}} f_{xx}$ is invertible in $\operatorname{VRep}(W_{xz}) \otimes R$, and this equation has a unique solution for g. Thus f has a right inverse if and only if $f_{xx} \in \operatorname{VRep}(W_x) \otimes R$ is invertible for all $x \in P$. The argument for left inverses is identical, so it remains only to show that left and right inverses coincide.

Let g be right inverse to f. Then g is also left inverse to some function, which we will denote h. We then have

$$f = f\delta = f(gh) = (fg)h = \delta h = h,$$

so g is left inverse to f, as well. □

3. Equivariant Kazhdan–Lusztig–Stanley theory

In this section we take R to be the ring $\mathbb{Z}[t]$ and for each $f \in I^W(P) \otimes \mathbb{Z}[t]$ and $x \leqslant y \in P$, we write $f_{xy}(t)$ for the corresponding component of f. One can regard $f_{xy}(t)$ as a polynomial whose coefficients are virtual representations of W_{xy}, or equivalently as a graded virtual representation of W_{xy}. We assume that P is equipped with a W-invariant *weak rank function* in the sense of [2, Section 2]. This is a collection of natural numbers $\{r_{xy} \in \mathbb{N} \mid x \leqslant y \in P\}$ with the following properties:

- $r_{xy} > 0$ if $x < y$
- $r_{xy} + r_{yz} = r_{xz}$ if $x \leqslant y \leqslant z$
- $r_{xy} = r_{\sigma(x)\sigma(y)}$ if $x \leqslant y$ and $\sigma \in W$.

Following the notation of [9, Section 2.1], we define

$$\mathscr{I}^W(P) \coloneqq \left\{ f \in I^W(P) \otimes \mathbb{Z}[t] \;\middle|\; \deg f_{xy}(t) \leqslant r_{xy} \text{ for all } x \leqslant y \right\}$$

along with

$$\mathscr{I}^W_{1/2}(P) \coloneqq \left\{ f \in I^W(P) \otimes \mathbb{Z}[t] \;\middle|\; \deg f_{xy}(t) < r_{xy}/2 \text{ and } f_{xx}(t) = \delta_{xx}(t) \right\}.$$

Note that $\mathscr{I}^W(P)$ is a subalgebra of $I^W(P)$, and we define an involution $f \mapsto \overline{f}$ of $\mathscr{I}^W(P)$ by putting $\overline{f}_{xy}(t) \coloneqq t^{r_{xy}} f_{xy}(t^{-1})$. An element $\kappa \in \mathscr{I}^W(P)$ is called a *P-kernel* if $\kappa_{xx}(t) = \delta_{xx}(t)$ for all $x \in P$ and $\overline{\kappa} = \kappa^{-1}$.

Theorem 3.1. *If $\kappa \in \mathscr{I}^W(P)$ is a P-kernel, there exists a unique pair of functions $f, g \in \mathscr{I}^W_{1/2}(P)$ such that $\overline{f} = \kappa f$ and $\overline{g} = g\kappa$.*

Proof. We follow the proof in [9, Theorem 2.2]. We will prove existence and uniqueness of f; the proof for g is identical. Fix elements $x < w \in P$. Suppose that $f_{yw}(t)$ has been defined for all $x < y \leqslant w$ and that the equation $\overline{f} = \kappa f$ holds where defined. Let

$$Q_{xw}(t) \coloneqq \sum_{x<y\leqslant w} \frac{|W_{xyw}|}{|W_{xw}|} \operatorname{Ind}^{W_{xw}}_{W_{xyw}} \left(\left(\operatorname{Res}^{W_{xy}}_{W_{xyw}} \kappa_{xy}(t)\right) \otimes \left(\operatorname{Res}^{W_{yw}}_{W_{xyw}} f_{yw}(t)\right)\right),$$

which is an element of $\operatorname{VRep}(W_{xw}) \otimes \mathbb{Z}[t]$. The equation $\overline{f} = \kappa f$ for the interval $[x, w]$ translates to

$$\overline{f}_{xw}(t) - f_{xw}(t) = Q_{xw}(t).$$

(2) If the ring R has integer torsion, then we rewrite this equation without the fractions as described in Remark 2.4.

It is clear that there is at most one polynomial $f_{xw}(t)$ of degree strictly less than $r_{xw}/2$ satisfying this equation. The existence of such a polynomial is equivalent to the statement

$$t^{r_{xw}} Q_{xw}(t^{-1}) = -Q_{xw}(t).$$

To prove this, we observe that

$$\begin{aligned}
&t^{r_{xw}} Q_{xw}(t^{-1}) \\
&= t^{r_{xw}} \sum_{x<y\leqslant w} \frac{|W_{xyw}|}{|W_{xw}|} \operatorname{Ind}_{W_{xyw}}^{W_{xw}} \left(\left(\operatorname{Res}_{W_{xyw}}^{W_{xy}} \kappa_{xy}(t^{-1})\right) \otimes \left(\operatorname{Res}_{W_{xyw}}^{W_{yw}} f_{yw}(t^{-1})\right)\right) \\
&= \sum_{x<y\leqslant w} \frac{|W_{xyw}|}{|W_{xw}|} \operatorname{Ind}_{W_{xyw}}^{W_{xw}} \left(\left(\operatorname{Res}_{W_{xyw}}^{W_{xy}} t^{r_{xy}} \kappa_{xy}(t^{-1})\right) \otimes \left(\operatorname{Res}_{W_{xyw}}^{W_{yw}} t^{r_{yw}} f_{yw}(t^{-1})\right)\right) \\
&= \sum_{x<y\leqslant w} \frac{|W_{xyw}|}{|W_{xw}|} \operatorname{Ind}_{W_{xyw}}^{W_{xw}} \left(\left(\operatorname{Res}_{W_{xyw}}^{W_{xy}} \overline{\kappa}_{xy}(t)\right) \otimes \left(\operatorname{Res}_{W_{xyw}}^{W_{yw}} \overline{f}_{yw}(t)\right)\right) \\
&= \sum_{x<y\leqslant w} \frac{|W_{xyw}|}{|W_{xw}|} \operatorname{Ind}_{W_{xyw}}^{W_{xw}} \left(\left(\operatorname{Res}_{W_{xyw}}^{W_{xy}} \overline{\kappa}_{xy}(t)\right) \otimes \left(\operatorname{Res}_{W_{xyw}}^{W_{yw}} (\kappa f)_{yw}(t)\right)\right).
\end{aligned}$$

This is formally equal to the expression for $(\overline{\kappa}(\kappa f))_{xw} - (\kappa f)_{xw}$, which by associativity is equal to the expression for

$$((\overline{\kappa}\kappa) f)_{xw} - (\kappa f)_{xw} = f_{xw} - (\kappa f)_{xw}.$$

Thus we have

$$\begin{aligned}
t^{r_{xw}} Q_{xw}(t^{-1}) &= - \sum_{x<y\leqslant w} \frac{|W_{xyw}|}{|W_{xw}|} \operatorname{Ind}_{W_{xyw}}^{W_{xw}} \left(\left(\operatorname{Res}_{W_{xyw}}^{W_{xy}} \kappa_{xy}(t)\right) \otimes \left(\operatorname{Res}_{W_{xyw}}^{W_{yw}} f_{yw}(t)\right)\right) \\
&= -Q_{xw}(t).
\end{aligned}$$

Thus there is a unique choice of polynomial $f_{xw}(t)$ consistent with the equation $\overline{f} = \kappa f$ on the interval $[x, w]$. ☐

We will refer to the element $f \in \mathscr{I}^W_{1/2}(P)$ from Theorem 3.1 is the *right equivariant KLS-function* associated with κ, and to g as the *left equivariant KLS-function* associated with κ. For any $x \leqslant y$, we will refer to the graded virtual representations $f_{xy}(t)$ and $g_{xy}(t)$ as (right or left) *equivariant KLS-polynomials*. When W is the trivial group, these definitions specialize to the ones in [9, Section 2].

Example 3.2. Let $\zeta \in \mathscr{I}^W(P)$ be the element defined by letting $\zeta_{xy}(t)$ be the trivial representation of W_{xy} in degree zero for all $x \leqslant y$, and let $\chi := \zeta^{-1}\overline{\zeta}$. The function χ is called the *equivariant characteristic function* of P with respect to the action of W. We have $\chi^{-1} = \overline{\zeta}^{-1}\zeta = \overline{\chi}$, so χ is a P-kernel. Since $\overline{\zeta} = \zeta\chi$, ζ is equal to the left KLS-function associated with χ. However, the right KLS-function f associated with χ is much more interesting! See Propositions 4.1 and 4.3 for a special case of this construction.

We next introduce the equivariant analogue of the material in [9, Section 2.3]. If κ is a P-kernel with right and left KLS-functions f and g, we define $Z := g\kappa f \in \mathscr{I}^W(P)$, which we call the *equivariant Z-function* associated with κ. For any $x \leqslant y$, we will refer to the graded virtual representation $Z_{xy}(t)$ as an *equivariant Z-polynomial*.

Proposition 3.3. *We have $\overline{Z} = Z$.*

Proof. Since $\overline{g} = g\kappa$, we have $Z = g\kappa f = \overline{g} f$. Since $\overline{f} = \kappa f$, we have $Z = g\kappa f = g\overline{f}$. Thus $\overline{Z} = \overline{\overline{g} f} = \overline{\overline{g}}\,\overline{f} = g\overline{f} = Z$. ☐

Remark 3.4. Suppose that $\kappa \in I^W(P)$ is a P-kernel and $f, g, Z \in I^W(P)$ are the associated equivariant KLS-functions and equivariant Z-function. It is immediate from the definitions that, if $\psi : W' \to W$ is a group homomorphism, then $R_\psi(f), R_\psi(g), R_\psi(Z) \in I^{W'}(P)$ are the equivariant KLS-functions and equivariant Z-function associated with the P-kernel $R_\psi(\kappa) \in I^{W'}(P)$. In particular, if we take W' to be the trivial group, then Remark 2.6 tells us that the ordinary KLS-polynomials and Z-polynomials are recovered from the equivariant KLS-polynomials and Z-polynomials by sending virtual representations to their dimensions.

4. Matroids

Let M be a matroid, let L be the lattice of flats of M equipped with the usual weak rank function, and let W be a finite group acting on L. Let $OS_M^W(t)$ be the Orlik–Solomon algebra of M [8], regarded as a graded representation of W. Following [7, Section 2], we define

$$H_M^W(t) := t^{\operatorname{rk} M} OS_M^W(-t^{-1}) \in \operatorname{VRep}(W) \otimes \mathbb{Z}[t].$$

If W is trivial, then $H_M^W(t) \in \mathbb{Z}[t]$ is equal to the characteristic polynomial of M. For any $F \leqslant G \in L$, let M_{FG} be the minor of M with lattice of flats $[F, G]$ obtained by deleting the complement of G and contracting F; this matroid inherits an action of the stabilizer group $W_{FG} \subset W$. Define $H \in \mathscr{I}^W(L)$ by putting $H_{FG}(t) = H_{M_{FG}}^{W_{FG}}(t)$ for all $F \leqslant G$.

Proposition 4.1. *The function H is the equivariant characteristic function of L.*

Proof. It is proved in [7, Lemma 2.5] that $\zeta H = \overline{\zeta}$. Multiplying on the left by ζ^{-1}, we have $H = \zeta^{-1}\overline{\zeta}$, which is the definition of the equivariant characteristic function. □

Remark 4.2. The proof of [7, Lemma 2.5] is surprisingly difficult.(3) Consequently, Proposition 4.1 is a deep fact about Orlik–Solomon algebras, not just a formal consequence of the definitions.

The *equivariant Kazhdan–Lusztig polynomial* $P_M^W(t) \in \operatorname{VRep}(W) \otimes \mathbb{Z}[t]$ was introduced in [7, Section 2.2]. Define $P \in \mathscr{I}_{1/2}^W(L)$ by putting $P_{FG}(t) = P_{M_{FG}}^{W_{FG}}(t)$ for all $F \leqslant G$. The defining recursion for $P_M^W(t)$ in [7, Theorem 2.8] translates to the formula $\overline{P} = HP$, which immediately implies the following proposition.

Proposition 4.3. *The function P is the right equivariant KLS-function associated with H.*

The *equivariant Z-polynomial* $Z_M^W(t) \in \operatorname{VRep}(W) \otimes \mathbb{Z}[t]$ was introduced in [10, Section 6]. Define $Z \in \mathscr{I}^W(L)$ by putting $Z_{FG}(t) = Z_{M_{FG}}^{W_{FG}}(t)$ for all $F \leqslant G$. The defining recursion for $Z_M^W(t)$ in [10, Section 6] translates to the formula $Z = \overline{\zeta}P$.

Proposition 4.4. *The function Z is the Z-function associated with H.*

Proof. Example 3.2 tells us that the right KLS-function associated with H is ζ and Proposition 4.3 tells us that the left KLS-function associated with H is P, thus the Z-function is equal $\zeta H P = \overline{\zeta}P = Z$. □

The following corollary was asserted without proof in [10, Section 6], and follows immediately from Propositions 3.3 and 4.4.

Corollary 4.5. *The polynomial $Z_M^W(t)$ is palindromic. That is,*

$$t^{\operatorname{rk} M} Z_M^W(t^{-1}) = Z_M^W(t).$$

(3)The difficult part appears in the proof of Lemma 2.4, which is then used to prove Lemma 2.5.

When W is the trivial group, Gao and Xie define polynomials $Q_M(t)$ and $\widehat{Q}_M(t) = (-1)^{\operatorname{rk} M} Q_M(t)$ with the property that $\left(P^{-1}\right)_{FG}(t) = \widehat{Q}_{M_{FG}}(t)$ [6]. If $\widehat{0}$ and $\widehat{1}$ are the minimal and maximal flats of M, this is equivalent to the statement that $Q_M(t) = (-1)^{\operatorname{rk} M}\left(P^{-1}\right)_{\hat{0}\hat{1}}(t)$. The polynomial $Q_M(t)$ is called the *inverse Kazhdan–Lusztig polynomial of* M.[4] Using the machinery of this paper, we may extend their definition to the equivariant setting by defining the *equivariant inverse Kazhdan–Lusztig polynomial*

$$Q_M^W(t) \coloneqq (-1)^{\operatorname{rk} M}\left(P^{-1}\right)_{\hat{0}\hat{1}}(t).$$

If we then define $\widehat{Q} \in \mathscr{I}^W_{1/2}(L)$ by putting $\widehat{Q}_{FG}(t) = (-1)^{r_{FG}} Q^{W_{FG}}_{M_{FG}}(t)$ for all $F \leqslant G$, we immediately obtain the following proposition.

PROPOSITION 4.6. *The functions P and $\widehat{Q}$ are mutual inverses in $I^W(L)$.*

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[4]The reason for bestowing this name on $Q_M(t)$ rather than $\widehat{Q}_M(t)$ is that $Q_M(t)$ has non-negative coefficients; this was conjectured in [6, Conjecture 4.1] and proved in [1, Theorem 1.4].