ALGEBRAIC COMBINATORICS

Dennis Jahn, Robert Löwe & Christian Stump

Minkowski decompositions for generalized associahedra of acyclic type

Volume 4, issue 5 (2021), p. 757-775.

<http://alco.centre-mersenne.org/item/ALCO_2021__4_5_757_0>

Algebraic Combinatorics
Volume 4, issue 5 (2021), p. 757–775
https://doi.org/10.5802/alco.177

Minkowski decompositions for generalized associahedra of acyclic type

Dennis Jahn, Robert Löwe & Christian Stump

Abstract We give an explicit subword complex description of the generators of the type cone of the g-vector fan of a finite type cluster algebra with acyclic initial seed. This yields in particular a description of the Newton polytopes of the F-polynomials in terms of subword complexes as conjectured by S. Brodsky and the third author. We then show that the cluster complex is combinatorially isomorphic to the totally positive part of the tropicalization of the cluster variety as conjectured by D. Speyer and L. Williams.

1. Introduction and main results

A generalized associahedron for a cluster algebra of finite type is a simple polytope whose face lattice is dual to that of the cluster complex. Constructing such generalized associahedra has been a fruitful area of mathematical research since the introduction of cluster algebras by S. Fomin and A. Zelevinsky in the early 2000s. We refer to [9, 5, 11, 17, 12] in this chronological order for some of the milestones and history. This paper is a continuation of [3] and builds on recent results from [2, 1] and from [16].

The paper has three major results, two of which resolve conjectures by S. Brodsky and the third author and, respectively, by D. Speyer and L. Williams. Theorem 1.1 gives a self-contained combinatorial construction of the rays of the type cone of the g-vector fan of a finite type cluster algebra with acyclic initial seed via subword complexes and brick polytopes. Using this construction together with recent results from [2, 1] and [16], Theorem 1.3 yields that this construction also describes the Newton polytopes of the F-polynomials of the cluster algebra. This description was conjectured in [3, Conjecture 2.12]. The appearance of the F-polynomials is then as well used to derive Theorem 1.4 showing that the totally positive part of the tropical cluster variety is, modulo its lineality space, linearly isomorphic to the g-vector fan. As the g-vector fan is combinatorially isomorphic to the cluster complex, this affirmatively answers [19, Conjecture 8.1] for finite type cluster algebras with principal coefficients and acyclic initial seed.

In order to precisely state the results, let $\Delta \subseteq \Phi^+ \subseteq \Phi_{\geqslant -1} \subseteq \Phi$ denote a finite crystallographic root system with fundamental weights ∇ and let M denote an initial mutation matrix with principal coefficients for a cluster algebra $\mathcal{A}(\mathsf{M})$ of type Φ

Manuscript received 20th June 2020, revised 25th February 2021, accepted 15th April 2021.

Keywords. Cluster algebras, F-polynomials, subword complex, type cone.

Acknowledgements. Robert Löwe is supported by the DFG grant SFB-TRR 109 "Discretization in Geometry and Dynamics". Dennis Jahn and Christian Stump are supported by the DFG Heisenberg grant STU 563/4-1 "Noncrossing phenomena in Algebra and Geometry".

ISSN: 2589-5486

with cluster variables $\{u_\beta(\mathbf{x},\mathbf{y}) \mid \beta \in \Phi_{\geqslant -1}\}$ and cluster complex $S(\mathsf{M})$ given by the set of compatible cluster variables. The cluster variables have the form $u_\beta(\mathbf{x},\mathbf{y}) = p(\mathbf{x},\mathbf{y})/\mathbf{x}^\beta$ with $\mathbf{x} = (x_1,\dots,x_n)$ and $\mathbf{y} = (y_1,\dots,y_n)$ for $p(\mathbf{x},\mathbf{y}) \in \mathbb{N}[\mathbf{x},\mathbf{y}]$ and $\mathbf{x}^\beta = x_1^{\beta_1}\cdots x_n^{\beta_n}$ with $\beta = \beta_1\alpha_1+\cdots+\beta_n\alpha_n$ expanded in the root basis $\Delta = \{\alpha_1,\dots,\alpha_n\}$. Its F-polynomials are denoted by $\left\{F_\beta = u_\beta(\mathbf{1},\mathbf{y}) \;\middle|\; \beta \in \Phi^+\right\}$ and its g-vector fan $\mathcal{F}_g(\mathsf{M})$ is given by the cones over compatible sets of g-vectors $g_\beta = g_1\omega_1+\cdots+g_n\omega_n$ such that $u_\beta(\mathbf{x},\mathbf{0}) = x_1^{g_1}\cdots x_n^{g_n}$ expanded in the weight basis $\nabla = \{\omega_1,\dots,\omega_n\}$. It is well-known that these cones indeed define a complete simplicial fan which is, by definition, isomorphic to the cluster complex $S(\mathsf{M})$. Let $(W,\mathcal{S})$ denote the Coxeter system generated by $\mathcal{S} = \{s_\alpha \mid \alpha \in \Delta\}$ and let $c \in W$ be a standard Coxeter element given by the product of the reflections in $\mathcal{S}$ in some order. One may associate to this data an acyclic initial mutation matrix M_c with principal coefficients, and as well a brick polytope $\mathsf{Asso}\,(\mathsf{M}_\mathrm{c})$ with normal fan given by the g-vector fan $\mathcal{F}_g(\mathsf{M}_\mathrm{c})$ of $\mathcal{A}(\mathsf{M}_\mathrm{c})$. In particular, $\mathsf{Asso}\,(\mathsf{M}_\mathrm{c})$ is a generalized associahedron for $\mathcal{A}(\mathsf{M}_\mathrm{c})$. Brick polytopes for subword complexes come with natural Minkowski decompositions which in the present context may be written in the form

$$\mathsf{Asso}\,(\mathsf{M}_\mathrm{c}) = \sum_{\beta\in\Phi^+} \mathsf{Asso}_\beta\,(\mathsf{M}_\mathrm{c})\,. \tag{1}$$

The type cone $\mathbb{TC}(\mathcal{F}_g(\mathsf{M}_\mathrm{c}))$ of the g-vector fan is the space of all its polytopal realizations. We thus have

$$\mathsf{Asso}\,(\mathsf{M}_\mathrm{c}) \in \mathbb{TC}\big(\mathcal{F}_g(\mathsf{M}_\mathrm{c})\big). \tag{2}$$

While motivated by beautiful constructions in [2] and [16], the following result is entirely self-contained and only uses properties of brick polytopes developed in [17] and [3].

Theorem 1.1. *For an acyclic initial mutation matrix M_c with principal coefficients, the type cone of the g-vector fan $\mathcal{F}_g(\mathsf{M}_\mathrm{c})$ is the open simplicial cone generated by the natural Minkowski summands of the brick polytope $\mathsf{Asso}\,(\mathsf{M}_\mathrm{c})$,*

$$\mathbb{TC}\big(\mathcal{F}_g(\mathsf{M}_\mathrm{c})\big) = \operatorname{cone}\left\{\mathsf{Asso}_\beta\,(\mathsf{M}_\mathrm{c}) \;\middle|\; \beta \in \Phi^+\right\}.$$

Remark 1.2. This theorem and its proof are combinatorial and do not use any representation theory. The definition of a generalized associahedron $\mathsf{Asso}\,(\mathsf{M}_\mathrm{c})$ in [11, 17] extends verbatim to the noncrystallographic finite types $I_2(m)$ for $m \notin \{3,4,6\}$ and H_3, H_4. The theorem also holds for noncrystallographic types when replacing the left-hand side by the type cone of weak Minkowski summands of $\mathsf{Asso}\,(\mathsf{M}_\mathrm{c})$ even though mutations of cluster variables, g-vectors and F-polynomials in these types do not behave combinatorially nicely [14].

Combining [2, Theorem 3] (simply-laced types) and [1, Theorem 6.1] (multiply-laced types) with [16, Theorem 2.26], one obtains that the rays of the type cone of the g-vector fan are also equal to the Newton polytopes of the F-polynomials,

$$\mathbb{TC}\big(\mathcal{F}_g(\mathsf{M}_\mathrm{c})\big) = \operatorname{cone}\left\{\operatorname{Newton}(F_\beta) \;\middle|\; \beta \in \Phi^+\right\}, \tag{$\star$}$$

where the exponent vectors are written in the root basis Δ, and in particular that

$$\sum_{\beta\in\Phi^+} \operatorname{Newton}(F_\beta) \;\in\; \mathbb{TC}\big(\mathcal{F}_g(\mathsf{M}_\mathrm{c})\big) \tag{$\star\star$}$$

is a generalized associahedron for $\mathcal{A}(\mathsf{M}_\mathrm{c})$. According to [16], H. Thomas announced that a future version of [2] will generalize ($\star$) also to cyclic finite types. In this case, ($\star\star$) was conjectured by S. Brodsky and the third author in [3, Conjecture 2.22]. Combining this with Theorem 1.1 and known properties of F-polynomials, we obtain the second main result describing Newton polytopes of F-polynomials for acyclic initial seeds in terms of subword complexes.

Theorem 1.3 ([3, Conjecture 2.12]). *Let* M_c *be an acyclic initial mutation matrix with principal coefficients. For any positive root* $\beta \in \Phi^+$*, we have*

$$\operatorname{Newton}(F_\beta) = \mathsf{Asso}_\beta(\mathsf{M}_c).$$

In [19] the authors associate to the cluster algebra $\mathcal{A}(\mathsf{M})$ a polyhedral fan $\operatorname{Trop}^+ \operatorname{Spec} \mathcal{A}(\mathsf{M})$ by tropicalizing the positive part of the affine variety $\operatorname{Spec} \mathcal{A}(\mathsf{M})$. Using $(\star\star)$, we finally derive the following theorem[(1)].

Theorem 1.4. *For acyclic initial mutation matrix* M_c *with principal coefficients, the totally positive part of the tropical variety associated to the cluster algebra* $\mathcal{A}(\mathsf{M}_c)$ *is, modulo its lineality space* $\mathcal{L}$*, linearly isomorphic to the* g*-vector fan,*

$$\operatorname{Trop}^+ \operatorname{Spec} \mathcal{A}(\mathsf{M}_c)/\mathcal{L} \;\cong\; \mathcal{F}_g(\mathsf{M}_c).$$

As the g-vector fan is combinatorially isomorphic to the cluster complex, this affirmatively answers a conjecture by D. Speyer and L. Williams in this situation.

Corollary 1.5 ([19, Conjecture 8.1]). *In the situation of Theorem 1.4, the cluster complex* $S(\mathsf{M})$ *is combinatorially isomorphic to the polyhedral fan* $\operatorname{Trop}^+ \operatorname{Spec} \mathcal{A}(\mathsf{M}_c)$.

1.1. Acknowledgements. The third author would like to thank Thomas Lam, Arnau Padrol, Markus Reineke, Raman Sanyal and Hugh Thomas for valuable discussions concerning various parts of this paper.

2. A natural Minkowski decomposition of generalized associahedra

We follow the notions from [3] and refer to Section 2 therein for details.

2.1. Generalized associahedra for acyclic type. Let $(W, \mathcal{S})$ be a finite type Coxeter system of rank n and let $\Delta \subseteq \Phi^+ \subseteq \Phi_{\geqslant -1} \subseteq \Phi \subseteq V$ be a finite root system for $(W, \mathcal{S})$ inside an Euclidean vector space V, with simple roots $\Delta = \{\alpha_s \mid s \in \mathcal{S}\}$, positive roots Φ^+ and almost positive roots $\Phi_{\geqslant -1} = \Phi^+ \sqcup -\Delta$. Denote by $N = |\Phi^+|$ the number of positive roots and $n + N = |\Phi_{\geqslant -1}|$. Let $C = (a_{st})_{s,t \in \mathcal{S}}$ denote the corresponding *Cartan matrix* given by $s(\alpha_t) = \alpha_t - a_{st}\alpha_s$ and set $\nabla = \{\omega_s \mid s \in \mathcal{S}\} \subseteq V$ to be the fundamental weights given by

$$\alpha_s = \sum_{t \in \mathcal{S}} a_{ts}\omega_t.$$

One then has $s(\omega_t) = \omega_t - \delta_{s=t}\alpha_s$ for $s, t \in \mathcal{S}$. Throughout this paper, we consider $V \cong \mathbb{R}^\Delta$ to have fixed basis Δ, though in the examples we simultaneously consider the vector space with standard basis and standard inner product.

We consider a fixed Coxeter element $c \in W$ and a reduced word $\mathrm{c} = \mathrm{s}_1 \cdots \mathrm{s}_n$ for c. To avoid double indices we write α_i for α_{s_i} and $\omega_i = \omega_{s_i}$. The initial mutation matrix $\mathsf{M}_c = (m_{ij})$ is then obtained from the Cartan matrix by

$$m_{ij} = \begin{cases} 0 & \text{if } i = j, \\ -a_{st} & \text{if } s = s_i \text{ appears before } t = s_j \text{ in the reduced word c}, \\ a_{st} & \text{if } s = s_i \text{ appears after } t = s_j \text{ in the reduced word c}, \end{cases}$$

for $1 \leqslant i, j \leqslant n$, together with an identity matrix below.

(1) In response to a first preprint, Thomas Lam informed us that a more general version of this theorem also follows from [1, Theorems 4.1 & 4.2] which implies the first part of [19, Conjecture 8.1].

cluster variable	d-vector	g-vector	F-polynomial
Example A_3			
x_1	$\overline{1}00_\Delta$	$100_\nabla = \frac{1}{4}321_\Delta$	
x_2	$0\overline{1}0_\Delta$	$010_\nabla = \frac{1}{4}121_\Delta$	
x_3	$00\overline{1}_\Delta$	$001_\nabla = \frac{1}{4}123_\Delta$	
$x_4 = \frac{x_2+y_1}{x_1}$	100_Δ	$\overline{1}10_\nabla = \frac{1}{4}\overline{2}00_\Delta$	$F_{100_\Delta} = y_1 + 1$
$x_5 = \frac{x_1y_1y_2+x_3y_1+x_2x_3}{x_1x_2}$	110_Δ	$\overline{1}01_\nabla = \frac{1}{4}\overline{2}02_\Delta$	$F_{110_\Delta} = y_1y_2 + y_1 + 1$
$x_6 = \frac{x_1x_2y_1y_2y_3+x_1y_1y_2+x_3y_1+x_2x_3}{x_1x_2x_3}$	111_Δ	$\overline{1}00_\nabla = \frac{1}{4}\overline{321}_\Delta$	$F_{111_\Delta} = y_1y_2y_3 + y_1y_2 + y_1 + 1$
$x_7 = \frac{x_1y_2+x_3}{x_2}$	010_Δ	$0\overline{1}1_\nabla = \frac{1}{4}002_\Delta$	$F_{010_\Delta} = y_2 + 1$
$x_8 = \frac{x_1x_2y_2y_3+x_1y_2+x_3}{x_2x_3}$	011_Δ	$0\overline{1}0_\nabla = \frac{1}{4}\overline{121}_\Delta$	$F_{011_\Delta} = y_2y_3 + y_2 + 1$
$x_9 = \frac{x_2y_3+1}{x_3}$	001_Δ	$00\overline{1}_\nabla = \frac{1}{4}\overline{123}_\Delta$	$F_{001_\Delta} = y_3 + 1$
Example B_2			
x_1	$\overline{1}0_\Delta$	$10_\nabla = 11_\Delta$	
x_2	$0\overline{1}_\Delta$	$01_\nabla = \frac{1}{2}12_\Delta$	
$x_3 = \frac{x_2^2+y_1}{x_1}$	10_Δ	$\overline{1}2_\nabla = 01_\Delta$	$F_{10_\Delta} = y_1 + 1$
$x_4 = \frac{x_1y_1y_2+y_1+x_2^2}{x_1x_2}$	11_Δ	$\overline{1}1_\nabla = \frac{1}{2}\overline{1}0_\Delta$	$F_{11_\Delta} = y_1y_2 + y_1 + 1$
$x_5 = \frac{x_1^2y_1y_2^2+2x_1y_1y_2+y_1+x_2^2}{x_1x_2^2}$	12_Δ	$\overline{1}0_\nabla = \overline{11}_\Delta$	$F_{12_\Delta} = y_1y_2^2 + 2y_1y_2 + y_1 + 1$
$x_6 = \frac{x_1y_2+1}{x_2}$	01_Δ	$0\overline{1}_\nabla = \frac{1}{2}\overline{12}_\Delta$	$F_{01_\Delta} = y_2 + 1$

Figure 1. The cluster variables with its d-vectors, g-vectors and F-polynomials for the initial mutation matrices in Examples 2.1 and 2.2.

Example 2.1 (A_3-example). Take $W = \mathfrak{S}_4$ the symmetric group with adjacent transpositions as simple generators

$$\mathcal{S} = \big\{s_1 = (1,2),\ s_2 = (2,3),\ s_3 = (3,4)\big\}$$

acting on $V = \big\{(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \in \mathbb{R}^4 \mid \lambda_1+\lambda_2+\lambda_3+\lambda_4 = 0\big\} \cong \mathbb{R}^4/\mathbb{R}(1,1,1,1)$, equipped with the standard inner product, by permuting the standard basis. Here and below we write shorthand $\overline{\lambda} := -\lambda$ for scalars λ. We choose

$$\Delta = \big\{\alpha_1 = 1\overline{1}00,\ \alpha_2 = 01\overline{1}0,\ \alpha_3 = 001\overline{1}\big\}$$

as a basis of V. We may express an element in V as $10\overline{1}0 = 110_\Delta = \alpha_1 + \alpha_2$ where the first expression $10\overline{1}0 = (1,0,\overline{1},0)$ is the genuine element in $V \subset \mathbb{R}^4$ and the second expression $110_\Delta = (1,1,0)_\Delta$ is in the chosen basis Δ in the given order. We obtain

$$\begin{aligned}\Phi^+ &= \big\{1\bar{1}00\,,\ 10\bar{1}0\,,\ 100\bar{1}\,,\ 01\bar{1}0\,,\ 010\bar{1}\,,\ 001\bar{1}\,\big\}\\ &= \big\{100_\Delta,\ 110_\Delta,\ 111_\Delta,\ 010_\Delta,\ 011_\Delta,\ 001_\Delta\big\},\end{aligned}$$

and finally $\Phi_{\geqslant -1} = \Phi^+ \sqcup -\Delta$ and $\Phi = \Phi^+ \sqcup -\Phi^+$. In this case, $n = |\mathcal{S}| = 3$ and $N = |\Phi^+| = 6$. The corresponding Cartan matrix is

$$C = \begin{pmatrix} 2 & \bar{1} & 0 \\ \bar{1} & 2 & \bar{1} \\ 0 & \bar{1} & 2 \end{pmatrix}$$

and the fundamental weights are

$$\nabla = \big\{\omega_1 = 1000 = \tfrac{1}{4}(321_\Delta), \quad \omega_2 = 1100 = \tfrac{1}{4}(121_\Delta), \quad \omega_3 = 1110 = \tfrac{1}{4}(123_\Delta)\big\}.$$

Fix the Coxeter element $c = (1,2,3,4) \in \mathfrak{S}_4$ to be the long cycle with reduced word $c = \mathrm{s}_1\mathrm{s}_2\mathrm{s}_3$. Figure 1 shows cluster variables, d- and g-vectors and F-polynomials for the initial mutation matrix

$$\mathsf{M}_\mathrm{c} = \begin{pmatrix} 0 & 1 & 0 \\ \bar{1} & 0 & 1 \\ 0 & \bar{1} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 2.2 (B_2-example). Take $W = \mathfrak{S}_2^B$ the group of signed permutations with simple generators

$$\mathcal{S} = \big\{s_1 = (1,2),\ s_2 = (2,\bar{2})\big\}$$

where s_1 is the usual adjacent transposition interchanging the standard basis elements e_1 and e_2, and where s_2 interchanges e_2 and $-e_2$. W acts on $V = \mathbb{R}^2$, equipped with the standard inner product. We choose

$$\Delta = \big\{\alpha_1 = 2\bar{2},\ \alpha_2 = 02\big\}$$

as a basis of V. With notation as above, we obtain

$$\begin{aligned}\Phi^+ &= \big\{2\bar{2}\ \ ,\ 20\ \ ,\ 22\ \ ,\ 02\ \ \big\}\\ &= \big\{10_\Delta,\ 11_\Delta,\ 12_\Delta,\ 01_\Delta\big\},\end{aligned}$$

and finally $\Phi_{\geqslant -1} = \Phi^+ \sqcup -\Delta$ and $\Phi = \Phi^+ \sqcup -\Phi^+$. In this case, $n = |\mathcal{S}| = 2$ and $N = |\Phi^+| = 4$. The corresponding Cartan matrix is

$$C = \begin{pmatrix} 2 & \bar{1} \\ \bar{2} & 2 \end{pmatrix}$$

and the fundamental weights are

$$\nabla = \big\{\omega_1 = 20 = 11_\Delta, \quad \omega_2 = 11 = \tfrac{1}{2}(12_\Delta)\big\}.$$

Fix the Coxeter element $c = (1,2,\bar{1},\bar{2}) \in \mathfrak{S}_2^B$ to be the long cycle with reduced word $\mathrm{c} = \mathrm{s}_1\mathrm{s}_2$. Figure 1 shows cluster variables, d- and g-vectors and F-polynomials for the initial mutation matrix

$$\mathsf{M}_\mathrm{c} = \begin{pmatrix} 0 & 1 \\ \bar{2} & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

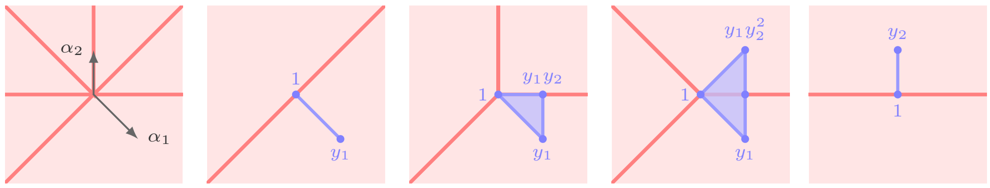

Figure 2. The g-vector fan of type B_2 from Example 2.2 (left) is the common refinement of the normal fans of the Newton polytopes (blue) of the F-polynomials.

For later reference, Figure 2 shows the g-vector fan in the weight basis and the Newton polytopes of the F-polynomials in the root basis in this case.

Let $w_\circ \in W$ be the unique longest element in weak order. For a given word $\mathrm{Q} = \mathrm{q}_1 \cdots \mathrm{q}_m$ in the simple system $\mathcal{S}$ define the *(spherical) subword complex* $\mathcal{SC}(\mathrm{Q})$ as the simplicial complex of sets of (positions of) letters in Q whose complement contains a reduced word of $w_\circ$. A more general version of these complexes were introduced by A. Knutson and E. Miller in [13]. By definition, the facets of $\mathcal{SC}(\mathrm{Q})$ are subwords of Q whose complements are reduced words for $w_\circ$. We consider facets as sorted lists of indices, written in set notation. Moreover define I_{g} and I_{ag} to be the lexicographically first and last facets, respectively, and call them *greedy facet* and *antigreedy facet*. The following notions were introduced and studied for general subword complexes in [4, 17]. For $\mathrm{Q} = \mathrm{q}_1 \cdots \mathrm{q}_m$ and any facet $I \in \mathcal{SC}(\mathrm{Q})$ associate a *root function* $\mathsf{r}(I, \cdot) : [m] \to \Phi = W(\Delta) \subseteq V$ and a *weight function* $\mathsf{w}(I, \cdot) : [m] \to W(\nabla) \subseteq V$ defined by

$$\mathsf{r}(I,k) = \Pi \mathrm{Q}_{[k-1]\smallsetminus I}(\alpha_{q_k}) \quad \text{and} \quad \mathsf{w}(I,k) = \Pi \mathrm{Q}_{[k-1]\smallsetminus I}(\omega_{q_k}),$$

where $\Pi \mathrm{Q}_X$ denotes the product of the simple reflections $q_x \in \mathrm{Q}$, for $x \in X \subseteq [m]$, in the order given by Q. It is well known, see [13, Theorem 3.7], that $\mathcal{SC}(\mathrm{Q})$ is a simplicial sphere, thus for a given facet I and index $i \in I$ there exists a unique adjacent facet J and index $j \in J$ with $I \smallsetminus i = J \smallsetminus j$. We call the transition from I to J the *flip* of i in I and if $i < j$ such a flip is called *increasing*, in which case we write $I \prec J$. This yields a poset structure on the set of facets of $\mathcal{SC}(\mathrm{Q})$ with I_{g} as unique minimal element and I_{ag} as unique maximal element.

Following [4], the (abstract) cluster complex $S(\mathsf{M}_{\mathrm{c}})$ can be seen as a subword complex as follows. Denote by $\mathrm{w}_\circ(\mathrm{c})$ the *Coxeter-sorting word* (or *c-sorting word*) of $w_\circ$, i.e. the lexicographically first subword of c^N that is a reduced word for $w_\circ$. The notion of Coxeter-sorting words was introduced by N. Reading in [18] and is an essential ingredient in the combinatorial descriptions of finite type cluster algebras and, in particular, in the description of cluster complexes in terms of subword complexes. In this setting we get the cluster complex as

$$S(\mathsf{M}_{\mathrm{c}}) \cong \mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big). \tag{3}$$

Example 2.3 (A_3-example). For the Coxeter element $c = s_1s_2s_3$ with fixed reduced word $\mathrm{c} = \mathrm{s}_1\mathrm{s}_2\mathrm{s}_3$ we identify the letter s_i with its index i. The c-sorting word of $w_\circ$ then is $\mathrm{w}_\circ(\mathrm{c}) = 123121$ and we obtain $\mathrm{cw}_\circ(\mathrm{c}) = 123123121$ for the subword complex

$\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$. The values of the root function are given by

I	$\mathsf{r}(I,\cdot)$								
	1	2	3	4	5	6	7	8	9
$123 = I_{\mathrm{g}}$	$1\bar{1}00$	$01\bar{1}0$	$001\bar{1}$	$1\bar{1}00$	$10\bar{1}0$	$100\bar{1}$	$01\bar{1}0$	$010\bar{1}$	$001\bar{1}$
129	$1\bar{1}00$	$01\bar{1}0$	$001\bar{1}$	$1\bar{1}00$	$100\bar{1}$	$10\bar{1}0$	$010\bar{1}$	$01\bar{1}0$	$00\bar{1}1$
137	$1\bar{1}00$	$01\bar{1}0$	$010\bar{1}$	$10\bar{1}0$	$1\bar{1}00$	$100\bar{1}$	$0\bar{1}10$	$010\bar{1}$	$001\bar{1}$
178	$1\bar{1}00$	$01\bar{1}0$	$010\bar{1}$	$10\bar{1}0$	$100\bar{1}$	$1\bar{1}00$	$001\bar{1}$	$0\bar{1}01$	$001\bar{1}$
189	$1\bar{1}00$	$01\bar{1}0$	$010\bar{1}$	$10\bar{1}0$	$100\bar{1}$	$1\bar{1}00$	$001\bar{1}$	$01\bar{1}0$	$001\bar{1}$
234	$1\bar{1}00$	$10\bar{1}0$	$001\bar{1}$	$\bar{1}100$	$10\bar{1}0$	$100\bar{1}$	$01\bar{1}0$	$010\bar{1}$	$001\bar{1}$
249	$1\bar{1}00$	$10\bar{1}0$	$001\bar{1}$	$\bar{1}100$	$100\bar{1}$	$10\bar{1}0$	$010\bar{1}$	$01\bar{1}0$	$001\bar{1}$
345	$1\bar{1}00$	$10\bar{1}0$	$100\bar{1}$	$01\bar{1}0$	$\bar{1}010$	$100\bar{1}$	$01\bar{1}0$	$010\bar{1}$	$001\bar{1}$
357	$1\bar{1}00$	$10\bar{1}0$	$100\bar{1}$	$01\bar{1}0$	$\bar{1}100$	$100\bar{1}$	$0\bar{1}10$	$010\bar{1}$	$001\bar{1}$
456	$1\bar{1}00$	$10\bar{1}0$	$100\bar{1}$	$01\bar{1}0$	$001\bar{1}$	$\bar{1}001$	$01\bar{1}0$	$010\bar{1}$	$001\bar{1}$
469	$1\bar{1}00$	$10\bar{1}0$	$100\bar{1}$	$01\bar{1}0$	$001\bar{1}$	$\bar{1}010$	$010\bar{1}$	$01\bar{1}0$	$00\bar{1}1$
567	$1\bar{1}00$	$10\bar{1}0$	$100\bar{1}$	$01\bar{1}0$	$010\bar{1}$	$\bar{1}001$	$0\bar{1}10$	$010\bar{1}$	$001\bar{1}$
678	$1\bar{1}00$	$10\bar{1}0$	$100\bar{1}$	$01\bar{1}0$	$010\bar{1}$	$\bar{1}100$	$001\bar{1}$	$0\bar{1}01$	$001\bar{1}$
$689 = I_{\mathrm{ag}}$	$1\bar{1}00$	$10\bar{1}0$	$100\bar{1}$	$01\bar{1}0$	$010\bar{1}$	$\bar{1}100$	$001\bar{1}$	$0\bar{1}10$	$00\bar{1}1$

and the values of the weight function are given by

I	$\mathsf{w}(I,\cdot)$								
	1	2	3	4	5	6	7	8	9
$123 = I_{\mathrm{g}}$	1000	1100	1110	1000	1100	1110	0100	0110	0010
129	1000	1100	1110	1000	1100	1101	0100	0101	0001
137	1000	1100	1110	1000	1010	1110	0010	0110	0010
178	1000	1100	1110	1000	1010	1011	0010	0011	0010
189	1000	1100	1110	1000	1010	1011	0010	0011	0001
234	1000	1100	1110	0100	1100	1110	0100	0110	0010
249	1000	1100	1110	0100	1100	1101	0100	0101	0001
345	1000	1100	1110	0100	0110	1110	0100	0110	0010
357	1000	1100	1110	0100	0110	1110	0010	0110	0010
456	1000	1100	1110	0100	0110	0111	0100	0110	0010
469	1000	1100	1110	0100	0110	0111	0100	0101	0001
567	1000	1100	1110	0100	0110	0111	0010	0110	0010
678	1000	1100	1110	0100	0110	0111	0010	0011	0010
$689 = I_{\mathrm{ag}}$	1000	1100	1110	0100	0110	0111	0010	0011	0001

Example 2.4 (B_2-example). For the Coxeter element $c = s_1 s_2$ with reduced word $\mathrm{c} = \mathrm{s}_1\mathrm{s}_2$ we identify the letters s_i with its index i. The c-sorting word of $w_\circ$ then is $\mathrm{w}_\circ(\mathrm{c}) = 1212$ and we obtain $\mathrm{cw}_\circ(\mathrm{c}) = 121212$ for the subword complex $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$.

The values of the root and weight function are given by

I	$\mathsf{r}(I,\cdot)$						$\mathsf{w}(I,\cdot)$					
	1	2	3	4	5	6	1	2	3	4	5	6
$12 = I_{\mathrm{g}}$	$2\bar{2}$	02	$2\bar{2}$	20	22	02	20	11	20	11	02	$\bar{1}1$
16	$2\bar{2}$	02	22	20	$2\bar{2}$	$0\bar{2}$	20	11	20	$1\bar{1}$	$0\bar{2}$	$\bar{1}\bar{1}$
23	$2\bar{2}$	20	$\bar{2}2$	20	22	02	20	11	02	11	02	$\bar{1}1$
34	$2\bar{2}$	20	22	$\bar{2}0$	22	02	20	11	02	$\bar{1}1$	02	$\bar{1}1$
45	$2\bar{2}$	20	22	02	$\bar{2}\bar{2}$	02	20	11	02	$\bar{1}1$	$\bar{2}0$	$\bar{1}1$
$56 = I_{\mathrm{ag}}$	$2\bar{2}$	20	22	02	$\bar{2}2$	$0\bar{2}$	20	11	02	$\bar{1}1$	$\bar{2}0$	$\bar{1}\bar{1}$

As described in [17], one may construct a generalized associahedron as follows by means of subword complexes and brick polytopes. Define the *brick vector* of the facet I of $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ as

$$\mathsf{b}(I) = \sum_{k=1}^{N} \big(\mathsf{w}(I, n+k) - \mathsf{w}(I_{\mathrm{ag}}, n+k)\big) \in V, \tag{4}$$

and the *brick polytope* $\mathsf{Asso}\,(\mathsf{M}_{\mathrm{c}})$ in V as the convex hull of all brick vectors of $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$, that is,

$$\mathsf{Asso}\,(\mathsf{M}_{\mathrm{c}}) = \mathrm{conv}\left\{\mathsf{b}(I) \;\middle|\; I \text{ facet of } \mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)\right\}.$$

It was shown in [17, Corollary 6.10] that this polytope is indeed a generalized associahedron.

As explained in [3], we consider g-vectors to be expressed in the weight basis. That is, we embed a g-vector $(g_1, \ldots, g_n)$ into the vector space V as $g_1\omega_1 + \cdots + g_n\omega_n \in V$. With this convention, we have the following previously known proposition, see [17, Corollary 6.36]. We briefly provide an alternative proof that does not rely on properties of *Cambrian fans* but on the direct relation between g-vectors of clusters and the weight function of the corresponding facet of the subword complex.

PROPOSITION 2.5. *The normal fan of* $\mathsf{Asso}\,(\mathsf{M}_{\mathrm{c}})$ *is the g-vector fan. That is,*

$$\mathsf{Asso}\,(\mathsf{M}_{\mathrm{c}}) \in \mathbb{TC}\big(\mathcal{F}_g(\mathsf{M}_{\mathrm{c}})\big).$$

Proof. It is shown in [17, Proposition 6.6] that the facet normals of all facets of $\mathsf{Asso}\,(\mathsf{M}_{\mathrm{c}})$ containing a given brick vector $\mathsf{b}(I)$ for some facet I of $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ are $\{\mathsf{w}(I,i) \mid i \in I\}$. With the above embedding of the g-vectors into V, it was then shown in [3, Corollary 2.10] that this set coincides with the set of g-vectors inside the cluster of $\mathcal{A}(\mathsf{M}_{\mathrm{c}})$ corresponding to I inside $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ under the isomorphism in (3) which is also explained in more detail in Remark 2.6. □

The given definition of the brick polytope differs from the definition given in [17] by a translation and is chosen so that the brick vector $\mathsf{b}(I_{\mathrm{ag}})$ of the antigreedy facet is the origin. This translation corresponds to the shifted weight function as used in [3, Conjecture 2.12]. Furthermore, we have for any facet I of $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ that $\mathsf{w}(I,k) = \mathsf{w}(I_{\mathrm{ag}},k)$ for all $1 \leqslant k \leqslant n$. This clarifies why we do not consider the first n weight vectors in the summation in (4).

The root function of the greedy facet provides a bijection between the set of positive roots and the positions $n+1, \ldots, n+N$. That is, $\{\mathsf{r}(I_{\mathrm{g}}, n+k) \mid 1 \leqslant k \leqslant N\} = \Phi^+$. As observed in [3, Lemma 3.7], we moreover have $\mathsf{r}(I_{\mathrm{g}}, n+k) = \mathsf{w}(I_{\mathrm{g}}, n+k) - \mathsf{w}(I_{\mathrm{ag}}, n+k)$

for all $1 \leqslant k \leqslant N$. For $\beta = \mathsf{r}(I_{\mathrm{g}}, n+k) \in \Phi^+$ and a facet I, we sometimes write $\mathsf{w}(I, \beta) := \mathsf{w}(I, n+k)$ for simplicity, and define

$$\mathsf{Asso}_\beta\left(\mathsf{M}_{\mathrm{c}}\right) = \operatorname{conv}\left\{\mathsf{w}(I,\beta) - \mathsf{w}(I_{\mathrm{ag}},\beta) \;\middle|\; I \text{ facet of } \mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)\right\}.$$

Remark 2.6. This identification of the positions $n+1, \dots, n+N$ and Φ^+ is the same as the isomorphism in (3) in the following sense. As known since [8], sending a cluster variable $u_\beta(\mathbf{x}, \mathbf{y})$ to its d-vector β is a bijection between cluster variables and almost positive roots $\Phi_{\geqslant -1}$. Identifying the positions $1, \dots, n$ with the simple negative roots $-\alpha_1, \dots, -\alpha_n$ in this order and the above identification between positions $n+1, \dots, n+N$ and Φ^+ is a bijection between cluster variables and positions $1, \dots, n, n+1, \dots, n+N$ and this bijection induces the bijection used in (3). In particular, the polytope $\mathsf{Asso}_\beta\left(\mathsf{M}_{\mathrm{c}}\right)$ naturally correspond to the cluster variable u_β. This correspondence turns out to be a structural correspondence as discussed in Section 3 where we show that $\mathsf{Asso}_\beta\left(\mathsf{M}_{\mathrm{c}}\right) = \mathrm{Newton}\left(F_\beta\right) = \mathrm{Newton}\left(u_\beta(\mathbf{1}, \mathbf{y})\right)$ is the Newton polytope of the F-polynomial associated to this cluster variable.

Example 2.7 (A_3-example). We display the shifted weight function for positions $n+1, \dots, n+N$ and the brick vector in the following shifted weight table.

I	$\mathsf{w}(I,\cdot) - \mathsf{w}(I_{\mathrm{ag}},\cdot)$						$\mathsf{b}(I)$
	4	5	6	7	8	9	
123	$1\bar{1}00 = 100_\triangle$	$10\bar{1}0 = 110_\triangle$	$100\bar{1} = 111_\triangle$	$01\bar{1}0 = 010_\triangle$	$010\bar{1} = 011_\triangle$	$001\bar{1} = 001_\triangle$	$31\bar{1}\bar{3} = 343_\triangle$
129	$1\bar{1}00 = 100_\triangle$	$10\bar{1}0 = 110_\triangle$	$10\bar{1}0 = 110_\triangle$	$01\bar{1}0 = 010_\triangle$	$01\bar{1}0 = 010_\triangle$	$0000 = 000_\triangle$	$31\bar{4}0 = 340_\triangle$
137	$1\bar{1}00 = 100_\triangle$	$1\bar{1}00 = 100_\triangle$	$100\bar{1} = 111_\triangle$	$0000 = 000_\triangle$	$010\bar{1} = 011_\triangle$	$001\bar{1} = 001_\triangle$	$3\bar{1}1\bar{3} = 323_\triangle$
178	$1\bar{1}00 = 100_\triangle$	$1\bar{1}00 = 100_\triangle$	$1\bar{1}00 = 100_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$001\bar{1} = 001_\triangle$	$3\bar{3}1\bar{1} = 301_\triangle$
189	$1\bar{1}00 = 100_\triangle$	$1\bar{1}00 = 100_\triangle$	$1\bar{1}00 = 100_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$3\bar{3}00 = 300_\triangle$
234	$0000 = 000_\triangle$	$10\bar{1}0 = 110_\triangle$	$100\bar{1} = 111_\triangle$	$01\bar{1}0 = 010_\triangle$	$010\bar{1} = 011_\triangle$	$001\bar{1} = 001_\triangle$	$22\bar{1}\bar{3} = 243_\triangle$
249	$0000 = 000_\triangle$	$10\bar{1}0 = 110_\triangle$	$10\bar{1}0 = 110_\triangle$	$01\bar{1}0 = 010_\triangle$	$01\bar{1}0 = 010_\triangle$	$0000 = 000_\triangle$	$22\bar{4}0 = 240_\triangle$
345	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$100\bar{1} = 111_\triangle$	$01\bar{1}0 = 010_\triangle$	$010\bar{1} = 011_\triangle$	$001\bar{1} = 001_\triangle$	$120\bar{3} = 133_\triangle$
357	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$100\bar{1} = 111_\triangle$	$0000 = 000_\triangle$	$010\bar{1} = 011_\triangle$	$001\bar{1} = 001_\triangle$	$111\bar{3} = 123_\triangle$
456	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$01\bar{1}0 = 010_\triangle$	$010\bar{1} = 011_\triangle$	$001\bar{1} = 001_\triangle$	$020\bar{2} = 022_\triangle$
469	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$01\bar{1}0 = 010_\triangle$	$01\bar{1}0 = 010_\triangle$	$0000 = 000_\triangle$	$02\bar{2}0 = 020_\triangle$
567	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$010\bar{1} = 011_\triangle$	$001\bar{1} = 001_\triangle$	$011\bar{2} = 012_\triangle$
678	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$001\bar{1} = 001_\triangle$	$001\bar{1} = 001_\triangle$
689	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$	$0000 = 000_\triangle$

Especially, the gray marked column corresponds to the polytope

$$\mathsf{Asso}_{111_\triangle}(\mathsf{M}_{\mathrm{c}}) = \operatorname{conv}\left\{111_\triangle,\ 110_\triangle,\ 100_\triangle,\ 000_\triangle\right\}.$$

Example 2.8 (B_2-example). We display the shifted weight function for positions $n+1, \dots, n+N$ and the brick vector in the following shifted weight table.

I	$\mathsf{w}(I,\cdot) - \mathsf{w}(I_{\mathrm{ag}},\cdot)$				$\mathsf{b}(I)$
	3	4	5	6	
12	$2\bar{2} = 10_\triangle$	$20 = 11_\triangle$	$22 = 12_\triangle$	$02 = 01_\triangle$	$62 = 34_\triangle$
16	$2\bar{2} = 10_\triangle$	$2\bar{2} = 10_\triangle$	$2\bar{2} = 10_\triangle$	$00 = 00_\triangle$	$6\bar{6} = 30_\triangle$
23	$00 = 00_\triangle$	$20 = 11_\triangle$	$22 = 12_\triangle$	$02 = 01_\triangle$	$44 = 24_\triangle$
34	$00 = 00_\triangle$	$00 = 00_\triangle$	$22 = 12_\triangle$	$02 = 01_\triangle$	$24 = 13_\triangle$
45	$00 = 00_\triangle$	$00 = 00_\triangle$	$00 = 00_\triangle$	$02 = 01_\triangle$	$02 = 01_\triangle$
56	$00 = 00_\triangle$	$00 = 00_\triangle$	$00 = 00_\triangle$	$00 = 00_\triangle$	$00 = 00_\triangle$

The gray marked column corresponds to the polytope

$$\mathsf{Asso}_{12_\Delta}(\mathsf{M}_\mathrm{c}) = \mathrm{conv}\left\{12_\Delta,\ 10_\Delta,\ 00_\Delta\right\},$$

and the brick polytope $\mathsf{Asso}(\mathsf{M}_\mathrm{c})$ can be seen in Figure 3.

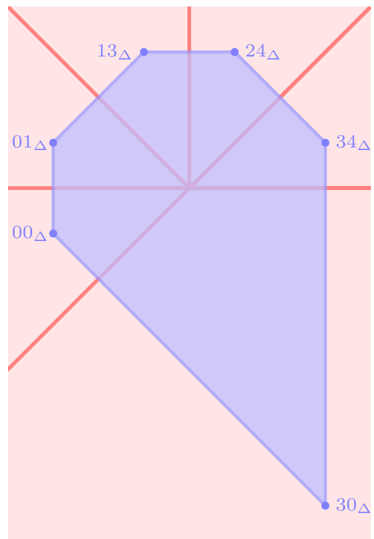

Figure 3. The brick polytope $\mathsf{Asso}(\mathsf{M}_\mathrm{c})$ of type B_2 from Example 2.8 and its outer normal fan, centered at $\frac{1}{2}(34_\Delta)$.

We next collect several properties of the polytopes $\mathsf{Asso}_\beta(\mathsf{M}_\mathrm{c})$. We first recall the following crucial lemma.

Lemma 2.9 ([17, Lemmas 4.4 & 4.5]). *Let I, J be two adjacent facets of $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ with $I \smallsetminus i = J \smallsetminus j$ for $i < j$. For any $k \in \{1, \ldots, n+N\}$ we have*

$$\mathsf{w}(J,k) = \mathsf{w}(I,k) - \lambda \mathsf{r}(I,i) \text{ for some } \lambda \in \mathbb{Z}_{\geqslant 0}.$$

Moreover for the brick vectors we obtain

$$\mathsf{b}(J) = \mathsf{b}(I) - \lambda \mathsf{r}(I,i) \text{ for some } \lambda \in \mathbb{Z}_{>0}.$$

For a set $X \subseteq \Phi^+$ of positive roots, we set $\mathsf{b}_X(I) = \sum_{\beta \in X} \big(\mathsf{w}(I,\beta) - \mathsf{w}(I_{\mathrm{ag}},\beta)\big)$ and define the polytope $\mathsf{Asso}_X(\mathsf{M}_\mathrm{c}) \subset V$ as

$$\mathsf{Asso}_X(\mathsf{M}_\mathrm{c}) := \mathrm{conv}\left\{\mathsf{b}_X(I) \;\middle|\; I \text{ facet of } \mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)\right\}.$$

We state the following mild generalization of [17, Proposition 5.17] for the present context. The proof given there also applies in the present generality and indeed for all *root-independent* subword complexes as briefly defined in Section 2.2.1 below.

Proposition 2.10. *We have the Minkowski decomposition*

$$\mathsf{Asso}_X(\mathsf{M}_\mathrm{c}) = \sum_{\beta \in X} \mathsf{Asso}_\beta(\mathsf{M}_\mathrm{c}).$$

Proof. We may neglect the contributions of the shifts by $\mathsf{w}(I_{\mathrm{ag}},\cdot)$, as these cancel in all considerations. By definition we have

$$\mathsf{Asso}_X(\mathsf{M}_\mathrm{c}) \subseteq \sum_{\beta \in X} \mathsf{Asso}_\beta(\mathsf{M}_\mathrm{c}).$$

To obtain equality we show that every vertex of $\sum_{\beta \in X} \mathsf{Asso}_\beta\,(\mathsf{M}_\mathrm{c})$ is also a vertex of $\mathsf{Asso}_X(\mathsf{M}_\mathrm{c})$. Consider a linear functional $f : V \to \mathbb{R}$. For two adjacent facets $I \smallsetminus i = J \smallsetminus j$ of $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ and a positive root $\beta \in X$ we have by Lemma 2.9 that either $f(\mathsf{w}(I,\beta)) = f(\mathsf{w}(J,\beta))$ or $f(\mathsf{w}(I,\beta)) - f(\mathsf{w}(J,\beta))$ has the same sign as $f(\mathsf{b}_X(I)) - f(\mathsf{b}_X(J))$. Therefore a facet I_f maximizes $f(\mathsf{b}_X(\cdot))$ among all facets if and only if it maximizes $f(\mathsf{w}(\cdot,\beta))$ for every $\beta \in X$.

Let now v be a vertex of the Minkowski sum $\sum_{\beta \in X} \mathsf{Asso}_\beta\,(\mathsf{M}_\mathrm{c})$ and let $f : V \to \mathbb{R}$ be a linear functional maximized at v. Thus, $v = \sum_{\beta \in X} v_\beta$ such that v_β maximizes f for $\mathsf{Asso}_\beta\,(\mathsf{M}_\mathrm{c})$.

On the other hand, f is also maximized by some vertex $\mathsf{b}_X(I_f)$ of $\mathsf{Asso}_X(\mathsf{M}_\mathrm{c})$. By the previous consideration, f thus maximizes $\mathsf{w}(I_f,\beta)$ for every $\beta \in X$ and we obtain $v_\beta = \mathsf{w}(I_f,\beta)$. Hence $v = \sum_{\beta \in X} \mathsf{w}(I_f,\beta) = \mathsf{b}_X(I_f)$. ☐

The description of the Minkowski decomposition of the brick polytope in the previous proposition also yields the following corollary.

Corollary 2.11. *The set of vertices of* $\mathsf{Asso}_X(\mathsf{M}_\mathrm{c})$ *is* $\big\{\mathsf{b}_X(I) \,\big|\, I \text{ facet of } \mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)\big\}$.

Example 2.12 (A_3-example). For $X = \Phi^+ \smallsetminus \{111_\Delta\}$ the polytope $\mathsf{Asso}_X(\mathsf{M}_\mathrm{c})$ is given by

$$\begin{aligned}\mathsf{Asso}_X(\mathsf{M}_\mathrm{c}) = {}& \operatorname{conv}\big\{21\bar{1}\bar{2},\ 21\bar{3}0,\ 2\bar{1}1\bar{2},\ 2\bar{2}1\bar{1},\ 2\bar{2}00,\ 12\bar{1}\bar{2},\ 12\bar{3}0,\\ & \qquad 020\bar{2},\ 011\bar{2},\ 020\bar{2},\ 02\bar{2}0,\ 011\bar{2},\ 001\bar{1},\ 0000\big\}\\ = {}& \operatorname{conv}\big\{232_\Delta, 230_\Delta, 212_\Delta, 201_\Delta, 200_\Delta, 132_\Delta, 130_\Delta,\\ & \qquad 022_\Delta, 012_\Delta, 022_\Delta, 020_\Delta, 012_\Delta, 001_\Delta, 000_\Delta\big\}.\end{aligned}$$

For later reference we note that $\mathsf{r}(I_\mathrm{g},6) = 111_\Delta$ and

$$\mathsf{b}_X(\{345\}) = \mathsf{b}_X(\{456\}) = 022_\Delta, \quad \mathsf{b}_X(\{357\}) = \mathsf{b}_X(\{567\}) = 012_\Delta.$$

Example 2.13 (B_2-example). For $X = \Phi^+ \smallsetminus \{12_\Delta\}$ the polytope $\mathsf{Asso}_X(\mathsf{M}_\mathrm{c})$ is given by

$$\begin{aligned}\mathsf{Asso}_X(\mathsf{M}_\mathrm{c}) = {}& \operatorname{conv}\big\{40,\quad 4\bar{4},\quad 22,\quad 02,\quad 02,\quad 00\quad\big\}\\ = {}& \operatorname{conv}\big\{22_\Delta,\ 20_\Delta,\ 12_\Delta,\ 01_\Delta,\ 01_\Delta,\ 00_\Delta\big\}.\end{aligned}$$

For later reference we note that $\mathsf{b}_X(\{34\}) = \mathsf{b}_X(\{45\}) = 01_\Delta$ and $12_\Delta = \mathsf{r}(I_\mathrm{g},5)$.

We next introduce the following canonical long flip sequence in the subword complex $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ from the greedy to the antigreedy facet,

$$I_\mathrm{g} = I_0 \prec I_1 \prec \cdots \prec I_N = I_\mathrm{ag}$$

where $I_{\ell+1}$ is obtained from I_ℓ by flipping the unique index i in I_ℓ such that $I_{\ell+1} \smallsetminus \{\ell+1+n\} = I_\ell \smallsetminus \{i\}$. Indeed, up to commutation of consecutive commuting letters, the index i is the smallest index that yields an increasing flip. Indeed, there is some flexibility in defining this sequence—any sequence of flips corresponding to *source mutations* in the associated cluster algebra would work.

Example 2.14 (A_3-example). For $\mathrm{cw}_\circ(\mathrm{c}) = 123123121$ the canonical long flip sequence is given by

$$I_\mathrm{g} = \{1,2,3\} \prec \{2,3,4\} \prec \{3,4,5\} \prec \{4,5,6\} \prec \{5,6,7\} \prec \{6,7,8\} \prec \{6,8,9\} = I_\mathrm{ag}.$$

Example 2.15 (B_2-example). For $\mathrm{cw}_\circ(\mathrm{c}) = 121212$ the canonical long flip sequence is given by

$$I_\mathrm{g} = \{1,2\} \prec \{2,3\} \prec \{3,4\} \prec \{4,5\} \prec \{5,6\} = I_\mathrm{ag}.$$

This flip sequence already appeared in [17, Proposition 6.7] and in the proof of [3, Lemma 3.7], where in particular the following property was used.

Lemma 2.16. *For every index $j \in \{n+1, \dots, n+N\}$ there exists a unique pair $I_\ell \prec I_{\ell+1}$ in the canonical long flip sequence and an index i such that $I_\ell \smallsetminus i = I_{\ell+1} \smallsetminus j$. Moreover, in this case the weight function $\mathsf{w}(I_{\ell+1}, \cdot)$ is obtained from $\mathsf{w}(I_\ell, \cdot)$ by*

$$\mathsf{w}(I_{\ell+1}, k) = \begin{cases} \mathsf{w}(I_\ell, k) - \mathsf{r}(I_\ell, i) & \text{if } k = j, \\ \mathsf{w}(I_\ell, k) & \text{otherwise.} \end{cases}$$

In particular, $\mathsf{w}(I_\ell, \cdot)$ and $\mathsf{w}(I_{\ell+1}, \cdot)$ only differ for the index j.

Proof. Up to commutations of consecutive commuting letters in the word $\mathrm{cw}_\circ(\mathrm{c}) = \mathrm{q}_1\mathrm{q}_2 \dots \mathrm{q}_{n+N}$, the facet I_ℓ consists of the letters $\mathrm{q}_{\ell+1} \dots \mathrm{q}_{\ell+n}$. Indeed, we may assume without loss of generality that for each $0 \leqslant \ell < N$ we have $I_{\ell+1} \smallsetminus I_\ell = \{\ell+n+1\}$. Moreover, $\{q_{\ell+1}, \dots, q_{\ell+n}\} = \mathcal{S}$ (this follows, for example, from [4, Theorem 2.7]) and the facets $I_\ell \prec I_{\ell+1}$ may be visualized inside the word $\mathrm{cw}_\circ(\mathrm{c})$ as

$$I_\ell = \mathrm{q}_1 \dots \mathrm{q}_\ell \ \widehat{\mathrm{q}}_{\ell+1}\widehat{\mathrm{q}}_{\ell+2} \dots \widehat{\mathrm{q}}_{\ell+n} \ \mathrm{q}_{\ell+n+1} \ \mathrm{q}_{\ell+n+2} \ \dots \mathrm{q}_{n+N}$$

$$I_{\ell+1} = \mathrm{q}_1 \dots \mathrm{q}_\ell \ \mathrm{q}_{\ell+1}\widehat{\mathrm{q}}_{\ell+2} \dots \widehat{\mathrm{q}}_{\ell+n} \ \widehat{\mathrm{q}}_{\ell+n+1} \ \mathrm{q}_{\ell+n+2} \ \dots \mathrm{q}_{n+N}$$

where the letters with hats are omitted and where we assumed, again without loss of generality, that $q_{\ell+1} = q_{\ell+n+1}$. The statement of the lemma now follows with

$$\mathsf{r}(I_\ell, \ell+1) = \mathsf{r}(I_\ell, \ell+n+1) = \mathsf{r}(I_{\ell+1}, \ell+1) = -\mathsf{r}(I_{\ell+1}, \ell+n+1) = q_1 \dots q_\ell(s_{q_{\ell+1}}). \qquad \square$$

This lemma yields an interesting combinatorial property of the polytopes $\mathsf{Asso}_\beta(\mathsf{M}_\mathrm{c})$ that we do not use further below.

Corollary 2.17. *For every $\beta \in \Phi^+$ the segment connecting $\mathbf{0}$ and β is an edge of $\mathsf{Asso}_\beta(\mathsf{M}_\mathrm{c})$.*

Proof. As the brick polytope $\mathsf{Asso}(\mathsf{M}_\mathrm{c})$ realizes $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ its edges are in one-to-one correspondence to flips in $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$. Combining Lemma 2.9 and Proposition 2.10 we obtain a similar result for $\mathsf{Asso}_\beta(\mathsf{M}_\mathrm{c})$ saying its edges are in one-to-one correspondence with flips that change the weight function $\mathsf{w}(\cdot, \beta)$. Applying Lemma 2.16 to the canonical long flip sequence

$$I_\mathrm{g} = I_0 \prec I_1 \prec \dots \prec I_{N-1} \prec I_N = I_\mathrm{ag}$$

we obtain for $\beta = \mathsf{r}(I_\mathrm{g}, n+i)$ that

$$\mathsf{w}(I_\mathrm{g}, \beta) = \mathsf{w}(I_1, \beta) = \dots = \mathsf{w}(I_{i-1}, \beta),$$

and

$$\mathsf{w}(I_i, \beta) = \dots = \mathsf{w}(I_{N-1}, \beta) = \mathsf{w}(I_\mathrm{ag}, \beta).$$

As $\mathsf{w}(I_\mathrm{g}, \beta) - \mathsf{w}(I_\mathrm{ag}, \beta) = \beta$ we conclude the statement. $\square$

2.2. Generators of the type cone. The following definitions mostly follow [16]. Let $\mathcal{F}$ be an essential complete simplicial fan in $\mathbb{R}^d$. A *polytopal realization* of $\mathcal{F}$ is a convex polytope in $\mathbb{R}^d$ whose outer normal fan is $\mathcal{F}$. The space of all polytopal realizations of $\mathcal{F}$ is called the *type cone* of $\mathcal{F}$, denoted by $\mathbb{TC}(\mathcal{F})$, see also [15]. A parametrization of $\mathbb{TC}(\mathcal{F})$ is commonly described as follows. Denote by $G \in \mathbb{R}^{m \times d}$ the matrix whose rows generate the rays of $\mathcal{F}$. Each height vector $h \in \mathbb{R}^m$ defines a polytope

$$P_h \ = \ \left\{x \in \mathbb{R}^d \mid Gx \leqslant h\right\}.$$

Now the type cone of $\mathcal{F}$ can be parametrized as the open polyhedral cone

$$\mathbb{TC}(\mathcal{F}) = \{h \in \mathbb{R}^m \mid P_h \text{ has normal fan } \mathcal{F}\}.$$

We write $P_h \in \mathbb{TC}(\mathcal{F})$ by identifying a polytope P_h with its height vector $h \in \mathbb{R}^m$. With this definition, $\mathbb{TC}(\mathcal{F})$ has d-dimensional lineality space corresponding to translations in $\mathbb{R}^d$. More specifically, for $P_h \in \mathbb{TC}(\mathcal{F})$ and a translation vector $b \in \mathbb{R}^d$ we have

$$P_h + b = P_{h+Gb} \in \mathbb{TC}(\mathcal{F}) \ ,$$

Thus the lineality space of $\mathbb{TC}(\mathcal{F})$ is given by the image of the matrix G. We identify $\mathbb{TC}(\mathcal{F})$ with its pointed quotient $\mathbb{TC}(\mathcal{F})/G\mathbb{R}^d$. The closure $\overline{\mathbb{TC}}(\mathcal{F})$ is called the *closed type cone*. The faces of $\overline{\mathbb{TC}}(\mathcal{F})$ correspond to (weak) Minkowski summands of P with the same normal fan (which are coarsenings of $\mathcal{F}$). In particular, the (extremal) generators of $\overline{\mathbb{TC}}(P)$ correspond to the indecomposable Minkowski summands of P.

We aim at the description of the type cone $\mathbb{TC}(\mathcal{F}_g(\mathsf{M}_\mathrm{c}))$ of the g-vector fan $\mathcal{F}_g(\mathsf{M}_\mathrm{c})$ given in Theorem 1.1. We first state the following lemma which we then use to understand the rays of the type cone.

Lemma 2.18. *Let $C \subset \mathbb{R}^m$ be a full-dimensional closed polyhedral cone and let $x = x_1 + \cdots + x_m$ for $x_1, \ldots, x_m \in C$ with*

(i) *x is an interior point of C and*
(ii) *$x - x_i$ is contained in the boundary of C for every $i \in \{1, \ldots, m\}$.*

Then $C = \operatorname{cone}\{x_1, \ldots, x_m\}$. In particular, the cone C is simplicial.

One crucial ingredient in the proof is the following observation for polyhedral cones. Consider a face F of C, i.e. $F = \{x \in C \mid f(x) = 0\}$ for some linear functional $f : \mathbb{R}^m \to \mathbb{R}$ with $f(y) \geqslant 0$ for all $y \in C$. For any $y_1, \ldots, y_k \in C$, one then has

$$(\dagger) \qquad y_1, \ldots, y_k \in F \Longleftrightarrow y_1 + \cdots + y_k \in F.$$

Proof of Lemma 2.18. Write $X = \{x_1, \ldots, x_m\}$. We first show that X is linearly independent. Assuming the contrary, one may express some x_i in terms of $X \smallsetminus \{x_i\}$. By condition (ii), $x - x_i$ is in some proper boundary face F of C. By (†), we obtain $X \smallsetminus \{x_i\} \subset F$ and thus $x_i \in F$. This would mean that $x = (x - x_i) + x_i \in F$, which contradicts condition (i) saying that this point is in the interior of C. It follows that $\operatorname{cone}(X)$ is a simplicial full-dimensional cone inside C. As condition (ii) implies that its boundary is also contained in the boundary of C, we conclude the statement. □

Proof of Theorem 1.1. The g-vector fan $\mathcal{F}_g(\mathsf{M}_\mathrm{c})$ is an essential complete simplicial fan in $\mathbb{R}^n$ with $n + N$ rays. Therefore, after passing to the quotient by its n-dimensional lineality space, the closed type cone $\overline{\mathbb{TC}}(\mathcal{F}_g(\mathsf{M}_\mathrm{c}))$ is an N-dimensional pointed polyhedral cone. We aim at applying Lemma 2.18 using the points $\{\mathsf{Asso}_\beta(\mathsf{M}_\mathrm{c}) \mid \beta \in \Phi^+\}$. We have seen in (2) that

$$\mathsf{Asso}(\mathsf{M}_\mathrm{c}) = \sum_{\beta \in \Phi^+} \mathsf{Asso}_\beta(\mathsf{M}_\mathrm{c})$$

is an interior point of $\overline{\mathbb{TC}}(\mathcal{F}_g(\mathsf{M}_\mathrm{c}))$. Therefore, it suffices to show that for each $\gamma \in \Phi^+$ the polytope $\mathsf{Asso}_{\Phi^+ \smallsetminus \{\gamma\}}(\mathsf{M}_\mathrm{c})$ is contained in the boundary of $\overline{\mathbb{TC}}(\mathcal{F}_g(\mathsf{M}_\mathrm{c}))$.

Let $\gamma \in \Phi^+$ and let $j \in \{n{+}1, \ldots, n{+}N\}$ be the unique index such that $r(I_g, j) = \gamma$. Lemma 2.16 ensures the existence of a unique index ℓ such that j is contained in $I_{\ell+1}$ but not in I_ℓ in the canonical long flip sequence $I_0 \prec \cdots \prec I_N$. Since $\mathsf{w}(I_\ell, \cdot)$ and $\mathsf{w}(I_{\ell+1}, \cdot)$ only differ for the index j, it follows that

$$\mathsf{b}_{\Phi^+ \smallsetminus \{\gamma\}}(I_\ell) = \mathsf{b}_{\Phi^+ \smallsetminus \{\gamma\}}(I_{\ell+1}).$$

Proposition 2.10 and the second part of Lemma 2.9 now show that the number of vertices of $\mathsf{Asso}_{\Phi^+ \smallsetminus \{\gamma\}}(\mathsf{M}_\mathrm{c})$ is strictly less than the number of vertices of $\mathsf{Asso}(\mathsf{M}_\mathrm{c})$.

This means that it is a proper weak Minkowski summand and it is thus not contained in the interior of $\overline{\mathbb{TC}}(\mathcal{F}_g(\mathsf{M}_c))$. Invoking Lemma 2.18 yields the proposed statement

$$\overline{\mathbb{TC}}(\mathcal{F}_g(\mathsf{M}_c)) = \text{cone}\left\{\mathsf{Asso}_\beta\left(\mathsf{M}_c\right) \mid \beta \in \Phi^+\right\}$$

and that the type cone is in particular simplicial. □

2.2.1. *Generators of the type cone for general spherical subword complexes.* We close this section with a brief discussion of properties of type cones for examples of general subword complexes. It turns out that the situation for cluster complexes is particularly special. Most importantly, the conclusion of Lemma 2.16 does not hold in general for spherical subword complexes.

The complex $\mathcal{SC}\big(\mathrm{cw}_\circ(\mathrm{c})\big)$ is known to have the following properties. For a word Q, we call a spherical subword complex $\mathcal{SC}(\mathrm{Q})$ *root-independent* if the multiset

$$\mathsf{R}(I) = \left\{\!\!\left\{\mathsf{r}(I,i) \;\middle|\; i \in I\right\}\!\!\right\}$$

is linearly independent for any (and thus every) facet I and it is *of full support* if every position in Q is contained in some facet (meaning that all elements of the ground set are indeed vertices). Observe that spherical subword complexes of full support are also full-dimensional, meaning that $\mathsf{R}(I)$ generates V for any facet I. This is an immediate consequence of [17, Proposition 3.8].

We conjecture that these properties identify cluster complexes among spherical subword complexes.

Conjecture 2.19. *Let* Q *be a word in* $\mathcal{S}$. *The following statements are equivalent:*

(i) *Up to commutations of consecutive commuting letters* $\mathrm{Q} = \mathrm{cw}_\circ(\mathrm{c})$ *for some Coxeter element* c.

(ii) $\mathcal{SC}(\mathrm{Q})$ *is root-independent and of full support.*

Remark that the first property was shown, for words of length $n+N$, to be equivalent to the so-called SIN-property in [4, Theorem 2.7]. Furthermore they conjecture these subword complexes to maximize the number of facets among subword complexes with words of this length [4, Conjecture 9.8].

We next show that relaxing one of the two conditions yields examples for which the conclusion of Theorem 1.1 does not hold. We denote by P_i the polytope corresponding to the i-th column in the shifted weight table,

$$P_i = \text{conv}\left\{\mathsf{w}(I,i) - \mathsf{w}(I_{\mathrm{ag}},i) \;\middle|\; I \text{ facet of } \mathcal{SC}(\mathrm{Q})\right\}$$

for the given subword complex $\mathcal{SC}(\mathrm{Q})$.

Example 2.20 (B_2-example). For $\mathrm{Q} = 1212121$, the subword complex $\mathcal{SC}(\mathrm{Q})$ is of full support but not root-independent as $\mathsf{R}(I_{\mathrm{g}}) = \mathsf{R}(\{1,2,3\}) = \{\!\!\{10_\Delta,\ 01_\Delta,\ 10_\Delta\}\!\!\}$. The list of facets is

$$\big\{\{1,2,3\},\ \{1,2,6\},\ \{1,3,4\},\ \{1,4,5\},\ \{1,5,6\},\ \{1,6,7\},\\ \{2,3,7\},\ \{3,4,7\},\ \{4,5,7\},\ \{5,6,7\}\big\}.$$

One may easily check that its brick polytope is the permutahedron of type B_2 whose normal fan is the Coxeter fan. The complete list of polytopes is

$$\begin{aligned}
P_1 = P_2 &= \text{conv}\{00_\Delta\}\\
P_3 = P_7 &= \text{conv}\{00_\Delta,\ 10_\Delta\}\\
P_4 &= \text{conv}\{00_\Delta,\ 10_\Delta,\ 11_\Delta\}\\
P_6 &= \text{conv}\{00_\Delta,\ 01_\Delta,\ 11_\Delta\}\\
P_5 = P_3 + P_? &= \text{conv}\{00_\Delta,\ 10_\Delta,\ 12_\Delta,\ 22_\Delta\},
\end{aligned}$$

where $P_? = \operatorname{conv}\{00_\Delta,\ 12_\Delta\}$ is a missing generator of the type cone. Furthermore the sum of P_4 and P_6 can be decomposed into

$$P_4 + P_6 = \operatorname{conv}\{00_\Delta,\ 10_\Delta\} + \operatorname{conv}\{00_\Delta,\ 01_\Delta\} + \operatorname{conv}\{00_\Delta,\ 11_\Delta\}.$$

In particular, the type cone of the brick polytope is not simplicial.

Example 2.21 (B_2-example). For $\mathrm{Q} = 212212$ the subword complex $\mathcal{SC}(\mathrm{Q})$ is root-independent and full-dimensional but not of full support as $\mathsf{R}(I_{\mathrm{g}}) = \mathsf{R}(\{1,3\}) = \{\!\{10_\Delta,\ 12_\Delta\}\!\}$ and the list of facets is

$$\{\{1,3\},\ \{1,4\},\ \{3,6\},\ \{4,6\}\}.$$

The positions 2 and 5 are not contained in any facet. The complete list of polytopes is

$$\begin{aligned}
P_1 = P_2 \quad &= \operatorname{conv}\{00_\Delta\} \\
P_3 = P_6 \quad &= \operatorname{conv}\{00_\Delta,\ 01_\Delta\} \\
P_5 = P_3 + P_6 &= \operatorname{conv}\{00_\Delta,\ 02_\Delta\} \\
P_4 = P_3 + P_? &= \operatorname{conv}\{00_\Delta,\ 01_\Delta,\ 11_\Delta,\ 12_\Delta\},
\end{aligned}$$

where $P_? = \operatorname{conv}\{00_\Delta,\ 11_\Delta\}$ is the missing generator of the type cone.

3. Newton polytopes of F-polynomials from subword complexes

Let $\mathcal{A}(\mathsf{M}_{\mathrm{c}})$ be the finite type cluster algebra with acyclic initial mutation matrix M_{c} with principal coefficients and denote by $\mathcal{F}_g(\mathsf{M}_{\mathrm{c}})$ its g-vector fan. We have seen in Theorem 1.1 that the type cone $\mathbb{TC}(\mathcal{F}_g(\mathsf{M}_{\mathrm{c}}))$ is generated by the natural Minkowski summands of the brick polytope $\mathsf{Asso}\,(\mathsf{M}_{\mathrm{c}})$,

$$\mathbb{TC}\big(\mathcal{F}_g(\mathsf{M}_{\mathrm{c}})\big) = \operatorname{cone}\left\{\mathsf{Asso}_\beta\,(\mathsf{M}_{\mathrm{c}}) \;\middle|\; \beta \in \Phi^+\right\}.$$

A description of the generators of $\mathbb{TC}\big(\mathcal{F}_g(\mathsf{M}_{\mathrm{c}})\big)$ was also obtained by combining results from [2, 1] and [16] as follows. In [2, Theorem 1] the authors provide polytopal realizations of $\mathcal{F}_g(\mathsf{M}_{\mathrm{c}})$. This construction produces a generalized associahedron $\mathcal{X}_p$ for each $p \in \mathbb{R}_{>0}^{\Phi^+}$. It was then shown in [2, Theorem 3] (simply-laced types) and in [1, Theorem 6.1] (multiply-laced types) that $\mathcal{X}_p$ for $p = e_\beta$ and $\beta \in \Phi^+$ equals the Newton polytope of the F-polynomial F_β. In [16, Theorem 2.26], the authors explain that within the latter constuction $\mathbb{R}_{>0}^{\Phi^+}$ can be understood as (a linear transformation of) the type cone $\mathbb{TC}\big(\mathcal{F}_g(\mathsf{M}_{\mathrm{c}})\big)$. In particular, this establishes the fact that the Newton polytopes of the F-polynomials generate the type cone,

$$\mathbb{TC}(\mathcal{F}_g(\mathsf{M}_{\mathrm{c}})) = \operatorname{cone}\left\{\operatorname{Newton}(F_\beta) \;\middle|\; \beta \in \Phi^+\right\}.$$

In order to prove Theorem 1.3, it remains to properly identify which Newton polytope of an F-polynomial corresponds to which Minkowski summand of the brick polytope. This is done using the following property of F-polynomials.

Proposition 3.1 ([7] (simply-laced types), [6] (multiply-laced types)). *For every $\beta \in \Phi^+$, the F-polynomial $F_\beta = F_\beta(\mathbf{y})$ has constant term* 1 *and a unique componentwise highest exponent vector given by β. In particular,* $\mathbf{0}$ *and β are both vertices of* $\operatorname{Newton}(F_\beta)$.

Proof. For simply-laced types, this is [7, Proposition 3.1 & Theorem 5.1] and for multiply-laced types, this is [6, Proposition 9.4]. $\square$

This proposition can be rechecked in types A_3 and B_2 in Figure 1. Now we are ready to prove our second main result.

Proof of Theorem 1.3. Since $\mathbb{TC}\big(\mathcal{F}_g(\mathsf{M}_c)\big)$ is a simplicial cone of dimension $N = |\Phi^+|$ we already know that the two sets of generators,

$$\big\{\mathrm{Newton}\,(F_\beta) \;\big|\; \beta \in \Phi^+\big\} \;\text{ and }\; \big\{\mathsf{Asso}_\beta\,(\mathsf{M}_c) \;\big|\; \beta \in \Phi^+\big\}\,,$$

are non-redundant and coincide up to scalar factors. Let $\beta \in \Phi^+$. By Proposition 3.1 the unique maximal and minimal vertices of Newton (F_β) are β and $\mathbf{0}$, respectively. Since $\beta = \mathsf{b}_{\{\beta\}}(I_g)$ and $\mathbf{0} = \mathsf{b}_{\{\beta\}}(I_{ag})$, these vectors are by Proposition 2.10 vertices of $\mathsf{Asso}_\beta\,(\mathsf{M}_c)$ as well. Applying Lemma 2.9 we see that they are the maximal and minimal vertices of $\mathsf{Asso}_\beta\,(\mathsf{M}_c)$, respectively. Thus the polytopes Newton (F_β) and $\mathsf{Asso}_\beta\,(\mathsf{M}_c)$ coincide. □

4. The tropical positive cluster variety

In this section, we prove Theorem 1.4 starting from the type cone description ($\star\star$) on page 758 in terms of Newton polytopes of F-polynomials. It is independent of the subword complex description and does not make use of it. We again emphasize that a more general version of Theorem 1.4 follows from [1, Theorems 4.1 & 4.2].

Following [19], we start with the needed notions from tropical geometry. Let $E \subset \mathbb{Z}^d_{\geqslant 0}$ be non-empty and finite and let $f = \sum_{e \in E} f_e \mathbf{u}^e \in \mathbb{Q}[\mathbf{u}]$ with $f_e \neq 0$ for all $e \in E$ be a rational polynomial supported on E. For each weight $w \in \mathbb{R}^d$ we define

$$E(w) = \arg\max_{e \in E}(w \cdot e).$$

That is, $E(w)$ is the intersection of E with the face of Newton $(f) = \mathrm{conv}(E)$ that is maximized in direction w. The *tropical hypersurface* $\mathrm{Trop}(f) \subset \mathbb{R}^d$ is the collection of those weights $w \in \mathbb{R}^d$ for which $E(w)$ contains at least two elements. $\mathrm{Trop}(f)$ naturally carries the structure of a polyhedral fan, whose cones are formed by those weights $w \in \mathrm{Trop}(f)$ that yield the same $E(w)$. This fan thus agrees with the codimension-one skeleton of the normal fan of Newton (f).

The *positive part* $\mathrm{Trop}^+(f)$ of the tropical hypersurface was introduced in [19] and is defined as follows. Split $E = E_f^+ \sqcup E_f^-$ according to the signs of the coefficients of f. That is,

$$E_f^+ = \{e \in E \mid f_e > 0\}, \quad E_f^- = \{e \in E \mid f_e < 0\}.$$

Now $\mathrm{Trop}^+(f)$ is defined as the subfan of $\mathrm{Trop}(f)$ consisting of those weights for which neither $E(w) \cap E_f^+$ nor $E(w) \cap E_f^-$ is empty,

$$\mathrm{Trop}^+(f) = \Big\{w \in \mathbb{R}^d \;\Big|\; E(w) \cap E_f^+ \neq \varnothing \;\text{ and }\; E(w) \cap E_f^- \neq \varnothing\Big\}.$$

For any ideal $\mathcal{I} \subset \mathbb{Q}[\mathbf{u}]$ the *positive tropical variety* $\mathrm{Trop}^+(\mathcal{I})$ is defined as the intersection of all positive tropical hypersurfaces $\mathrm{Trop}^+(f)$ for $f \in \mathcal{I}$.

We next move to the positive tropical variety considered here. Let $\mathcal{A}(\mathsf{M})$ be a finite type cluster algebra of rank n with (not necessarily acyclic) initial mutation matrix M with principal coefficients. We denote by $X_\Delta = \{x_1, \ldots, x_n\}$ the set of initial cluster variables and by $X_{\Phi^+} = \{x_\beta \mid \beta \in \Phi^+\}$ the set of non-initial cluster variables. Thus the set of all cluster variables is the disjoint union $X = X_\Delta \sqcup X_{\Phi^+}$. Furthermore, let $Y = \{y_1, \ldots, y_n\}$ be the set of principle coefficient variables. Recall that each non-initial cluster variable $x_\beta \in X_{\Phi^+}$ is expressed in terms of the initial seed by

$$x_\beta = \frac{p_\beta(\mathbf{x}, \mathbf{y})}{\mathbf{x}^\beta}$$

where $p_\beta(\mathbf{x}, \mathbf{y})$ is a subtraction-free polynomial in the initial cluster and coefficient variables and $\mathbf{x}^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n}$ for $\beta = (\beta_1, \cdots, \beta_n)_\Delta \in \mathbb{R}^\Delta$.

Following [19] we embed $\operatorname{Spec}\mathcal{A}(\mathsf{M})$ as the affine variety $V(\mathcal{I}_\mathsf{M}) \subset \mathbb{Q}^{X\sqcup Y}$, where $\mathcal{I}_\mathsf{M}$ is the ideal generated by the non-initial cluster variables, i.e.

$$\mathcal{I}_\mathsf{M} = \left\langle x_\beta \cdot \mathbf{x}^\beta - p_\beta(\mathbf{x},\mathbf{y}) \;\middle|\; \beta \in \Phi^+ \right\rangle.$$

Note that in this case the special form of the generators immediately yields a subtraction-free parametrization $\Psi : (\mathbb{Q}^*)^{X_\Delta \sqcup Y} \to V(\mathcal{I}_\mathsf{M}) \cap (\mathbb{Q}^*)^{X\sqcup Y}$ given as the graph of the map

$$\begin{aligned} (\mathbb{Q}^*)^{X_\Delta \sqcup Y} &\longrightarrow (\mathbb{Q}^*)^{X_{\Phi^+}}, \\ (\mathbf{x},\mathbf{y}) &\longmapsto \left(\frac{p_\beta(\mathbf{x},\mathbf{y})}{\mathbf{x}^\beta}\right)_{\beta\in\Phi^+} \end{aligned}$$

We denote by $\operatorname{Trop}\Psi : \mathbb{R}^{X_\Delta\sqcup Y} \to \mathbb{R}^{X\sqcup Y}$ the *tropicalization* of the map Ψ. This is the piecewise linear map obtained by replacing every $\times$ in Ψ with a $+$, every $/$ with a $-$ and every $+$ with a max. The following result is an immediate consequence of [19, Proposition 2.6].

Proposition 4.1. *The map* $\operatorname{Trop}\Psi : \mathbb{R}^{X_\Delta\sqcup Y} \to \mathbb{R}^{X\sqcup Y}$ *is a piecewise linear parametrization of the positive tropical variety* $\operatorname{Trop}^+(\mathcal{I}_\mathsf{M})$.

It should be mentioned here that in [19] the authors are working over the field of complex Puiseux series, one of the prototypical examples of a field with non-trivial valuation. This is standard in tropical geometry since it reveals strong connections between classical algebraic geometry and tropical geometry. The ideal $\mathcal{I}_\mathsf{M}$ in Proposition 4.1 is understood over the complex Puiseux series and for the definition of a positive tropical hypersurface for a complex Puiseux polynomial we refer to [19]. However, the map $\operatorname{Trop}\Psi$ stays unchanged when working over $\mathbb{Q}$.

The domains of linearity of $\operatorname{Trop}\Psi$ form a polyhedral fan in $\mathbb{R}^{X_\Delta\sqcup Y}$, which we denote by $\mathcal{F}_\Psi$. Following [19, Definition 4.2] we equip $\operatorname{Trop}^+(\mathcal{I}_\mathsf{M})$ with the fan structure obtained by applying $\operatorname{Trop}\Psi$ to $\mathcal{F}_\Psi$. Whenever we refer to $\operatorname{Trop}^+(\mathcal{I}_\mathsf{M})$ as a polyhedral fan we consider this fan structure.

Proof of Theorem 1.4. By Proposition 4.1 the map $\operatorname{Trop}\Psi : \mathbb{R}^{X_\Delta\sqcup Y} \to \operatorname{Trop}^+(\mathcal{I}_\mathsf{M})$ is a piecewise linear parametrization of $\operatorname{Trop}^+(\mathcal{I}_\mathsf{M})$. Moreover, $\operatorname{Trop}^+(\mathcal{I}_\mathsf{M})$ is piecewise linearly isomorphic to $\mathcal{F}_\Psi$ by construction . For each $\beta \in \Phi^+$ denote by $\mathcal{F}_\beta$ the normal fan of $\operatorname{Newton}(p_\beta)$. The domains of linearity of the coordinate function $\operatorname{Trop}\Psi_\beta(\mathbf{x},\mathbf{y}) = \operatorname{Trop}(p_\beta(\mathbf{x},\mathbf{y})/\mathbf{x}^\beta)$ are the maximal cones of $\mathcal{F}_\beta$. Thus the domains of linearity of $\operatorname{Trop}\Psi$, and hence the fan structure $\mathcal{F}_\Psi$ of the positive tropical variety $\operatorname{Trop}^+(\mathcal{I}_\mathsf{M})$, are given by the common refinement of these fans $\mathcal{F}_\beta$ for $\beta \in \Phi^+$.

It follows from [10, Corollary 6.3] that there exists an affine transformation $T : \mathbb{R}^Y \to \mathbb{R}^{X_\Delta\sqcup Y}$ such that for all $\beta \in \Phi^+$ we have

$$\operatorname{Newton}(p_\beta) = T\left(\operatorname{Newton}(F_\beta)\right).$$

Conversely, $\operatorname{Newton}(F_\beta)$ is obtained from $\operatorname{Newton}(p_\beta)$ by the coordinate projection

$$\mathbb{R}^{X_\Delta\sqcup Y} \to \mathbb{R}^Y.$$

Therefore each fan $\mathcal{F}_\beta$ is linearly isomorphic to the normal fan of $\operatorname{Newton}(F_\beta)$. By Theorem 1.3 the common refinement of these normal fans is the g-vector fan $\mathcal{F}_g$. This shows that $\mathcal{F}_\Psi$ is piecewise linearly isomorphic to $\mathcal{F}_g$. $\square$

Example 4.2 (B_2-example). We continue Example 2.2. We denote by $X_\Delta = \{x_1, x_2\}$ the initial cluster variables and by $Y = \{y_1, y_2\}$ the principle coefficient variables.

This yields the non-initial cluster variables

$$x_3 = x_{10_\triangle} = (x_2^2 + y_1)/x_1$$
$$x_4 = x_{11_\triangle} = (x_1y_1y_2 + x_2^2 + y_1)/x_1x_2$$
$$x_5 = x_{12_\triangle} = (x_1^2y_1y_2^2 + 2x_1y_1y_2 + x_2^2 + y_1)/x_1x_2^2$$
$$x_6 = x_{01_\triangle} = (x_1y_2 + 1)/x_2$$

as given in the above example. The piecewise linear map $\operatorname{Trop}\Psi : \mathbb{R}^{X_\triangle \sqcup Y} \to \mathbb{R}^{X \sqcup Y}$ has non-trivial coordinate functions

$$\operatorname{Trop}\Psi_{10_\triangle} = \max(2x_2\,,\, y_1) - x_1$$
$$\operatorname{Trop}\Psi_{11_\triangle} = \max(x_1 + y_1 + y_2\,,\, 2x_2\,,\, y_1) - x_1 - x_2$$
$$\operatorname{Trop}\Psi_{12_\triangle} = \max(2x_1 + y_1 + 2y_2\,,\, x_1 + y_1 + y_2\,,\, 2x_2\,,\, y_1) - x_1 - 2x_2$$
$$\operatorname{Trop}\Psi_{01_\triangle} = \max(x_1 + y_2\,,\, 0) - x_2.$$

The domains of linearity of $\operatorname{Trop}\Psi$ define a complete four-dimensional polyhedral fan $\mathcal{F}_\Psi$ in $\mathbb{R}^{X_\triangle \sqcup Y} = \mathbb{R}^4$ with two-dimensional lineality space. By intersecting $\mathcal{F}_\Psi$ with the coordinate plane $\mathbb{R}^Y$ by setting $x_1 = x_2 = 0$ we obtain an essential 2-dimensional fan. This fan is the the common coarsening of the normal fans of the F-polynomials $F_{10_\triangle}, F_{11_\triangle}, F_{12_\triangle}, F_{01_\triangle}$, see Figures 2 and 4. The normal fans are depicted in the dual basis of the root basis, known as the coweight basis, which in this case is given as

$$\nabla^\vee = \{\omega_1^\vee,\ \omega_2^\vee\} = \left\{\tfrac{1}{2}(10),\ \tfrac{1}{2}(11)\right\}.$$

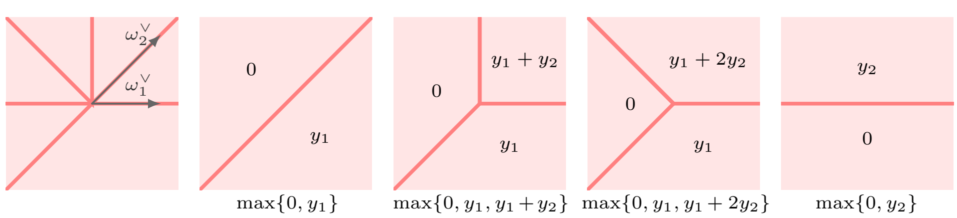

Figure 4. The g-vector fan of type B_2 (left) is the common refinement of the domains of linearity of the coordinate functions of $\operatorname{Trop}\Psi$ from Example 4.2 after intersecting with the (y_1, y_2)-plane.

References

[1] Nima Arkani-Hamed, Song He, and Thomas Lam, *Cluster configuration spaces of finite type*, Preprint, available at `arxiv.org/abs/2005.11419`, 2020.

[2] Véronique Bazier-Matte, Guillaume Douville, Kaveh Mousavand, Hugh Thomas, and Emine Yıldırım, *ABHY associahedra and Newton polytopes of F-polynomials for finite type cluster algebras*, Preprint, available at `arxiv.org/abs/1808.09986`, 2018.

[3] Sarah B. Brodsky and Christian Stump, *Towards a uniform subword complex description of acyclic finite type cluster algebras*, Algebr. Comb. **1** (2018), no. 4, 545–572.

[4] Cesar Ceballos, Jean-Philippe Labbé, and Christian Stump, *Subword complexes, cluster complexes, and generalized multi-associahedra*, J. Algebraic Combin. **39** (2014), no. 1, 17–51.

[5] Frédéric Chapoton, Sergey Fomin, and Andrei Zelevinsky, *Polytopal realizations of generalized associahedra*, Canad. Math. Bull. **45** (2002), no. 4, 537–566.

[6] Laurent Demonet, *Mutations of group species with potentials and their representations. Applications to cluster algebras*, Preprint, available at `arxiv.org/abs/1003.5078`, 2010.

[7] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky, *Quivers with potentials and their representations II: Applications to cluster algebras*, J. Amer. Math. Soc. **23** (2010), no. 3, 749–790.

[8] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. II. Finite type classification*, Invent. Math. **154** (2003), no. 1, 63–121.

[9] ———, *Y-systems and generalized associahedra*, Ann. of Math. (2) **158** (2003), no. 3, 977–1018.
[10] ———, *Cluster algebras. IV. Coefficients*, Compos. Math. **143** (2007), no. 1, 112–164.
[11] Christophe Hohlweg, Carsten E. M. C. Lange, and Hugh Thomas, *Permutahedra and generalized associahedra*, Adv. Math. **226** (2011), no. 1, 608–640.
[12] Christophe Hohlweg, Vincent Pilaud, and Salvatore Stella, *Polytopal realizations of finite type* $\mathbf{g}$*-vector fans*, Adv. Math. **328** (2018), 713–749.
[13] Allen Knutson and Ezra Miller, *Subword complexes in Coxeter groups*, Adv. Math. **184** (2004), no. 1, 161–176.
[14] Philipp Lampe, *On the approximate periodicity of sequences attached to non-crystallographic root systems*, Exp. Math. **27** (2018), no. 3, 265–271.
[15] Peter McMullen, *Representations of polytopes and polyhedral sets*, Geometriae Dedicata **2** (1973), 83–99.
[16] Arnau Padrol, Yann Palu, Vincent Pilaud, and Pierre-Guy Plamondon, *Associahedra for finite type cluster algebras and minimal relations between* $\mathbf{g}$*-vectors*, Preprint, available at `arxiv.org/abs/1906.06861`, 2019.
[17] Vincent Pilaud and Christian Stump, *Brick polytopes of spherical subword complexes and generalized associahedra*, Adv. Math. **276** (2015), 1–61.
[18] Nathan Reading, *Sortable elements and Cambrian lattices*, Algebra Universalis **56** (2007), no. 3-4, 411–437.
[19] David Speyer and Lauren Williams, *The tropical totally positive Grassmannian*, J. Algebraic Combin. **22** (2005), no. 2, 189–210.

Dennis Jahn, Fakultät für Mathematik, Ruhr-Universität Bochum, Germany
E-mail : `dennis.jahn@rub.de`

Robert Löwe, Institut für Mathematik, TU Berlin, Germany
E-mail : `loewe@math.tu-berlin.de`

Christian Stump, Fakultät für Mathematik, Ruhr-Universität Bochum, Germany
E-mail : `christian.stump@rub.de`