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Abstract Friezes with coefficients are maps assigning numbers to the edges and diagonals of a regular polygon such that all Ptolemy relations for crossing diagonals are satisfied. Among these, the classic Conway–Coxeter friezes are the ones where all values are positive integers and all edges have value 1. Every subpolygon of a Conway–Coxeter frieze yields a frieze with coefficients over the positive integers. In this paper we give a complete arithmetic criterion for which friezes with coefficients appear as subpolygons of Conway–Coxeter friezes. This generalizes a result of our earlier paper with Peter Jørgensen from triangles to subpolygons of arbitrary size.

1. Introduction

Frieze patterns are infinite configurations of numbers introduced by Coxeter [2] in the 1970s, the shape of which is reminiscent of friezes which appeared in architecture and decorative art for centuries. The entries in a frieze pattern have to satisfy a specific rule for each neighbouring 2×2-determinant. This frieze pattern rule is for example implicitly contained in the structure of smooth toric varieties [8, 1.6] and has been essential in the study of continued fractions more than a century earlier [9, § 51]. It also reappeared some 30 years after Coxeter's definition as the exchange condition in Fomin and Zelevinsky's cluster algebras, mathematical structures which became highly influential in many areas of modern mathematics. This connection to cluster algebras initiated an intensive renewed interest in frieze patterns in recent years, see [7]. Whereas classic frieze patterns are bounded by rows of 1's, to capture cluster algebras with coefficients more general boundary rows and a modified rule for 2×2-determinants are needed. The resulting frieze patterns with coefficients have been suggested by Propp [10] and recently their fundamental properties have been studied in [5]. Among other things, it is proven in [5] that a frieze pattern with coefficients can be viewed as a map on edges and diagonals of a regular polygon (with values in a suitable number system) satisfying the Ptolemy relations for any pair of crossing diagonals; we then speak of a frieze with coefficients to distinguish these viewpoints.

For classic frieze patterns this viewpoint was well-known, not least for classic frieze patterns over positive integers, where a beautiful result of Conway and Coxeter [1] shows that such frieze patterns are in bijection with triangulations of regular polygons. Any subpolygon of a Conway–Coxeter frieze yields a frieze with coefficients over the positive integers. The natural question arises which friezes with coefficients actually

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appear as subpolygons of Conway–Coxeter friezes. A solution would give new insight into the subtle arithmetic relations of entries in Conway–Coxeter friezes, and hence triangulations of polygons.

It is a special property of a frieze with coefficients to appear in a Conway–Coxeter frieze. For instance, we observed in [5] that for every triangle in a Conway–Coxeter frieze the greatest common divisors of any two of the three numbers must be equal. This already rules out many friezes with coefficients.

Still, there are many friezes with coefficients where the condition on the greatest common divisors holds for all triangles and then it is a priori difficult to determine whether such a frieze with coefficients appears in a Conway–Coxeter frieze, or not. As one main result of [5] we have shown that for triangles this happens if and only if the three numbers are all odd or do not have the same 2-valuation.

Recall that for a prime number p and a positive integer m, the p-valuation $\nu_p(m)$ is the maximal non-negative integer ℓ such that p^ℓ divides m but $p^{\ell+1}$ does not divide m.

The aim of this paper is to give a complete solution to this problem for polygons of arbitrary size, that is, we present a characterization of those friezes with coefficients which appear as subpolygons in Conway–Coxeter friezes.

Theorem. *Let $\mathcal{C}$ be a frieze with coefficients on an n-gon over the positive integers. Then $\mathcal{C}$ appears as a subpolygon of some Conway–Coxeter frieze if and only if the following conditions are satisfied:*

(1) *For any triangle (a,b,c) in $\mathcal{C}$ we have* $\gcd(a,b)=\gcd(b,c)=\gcd(a,c)$.
(2) *Let $p<n$ be a prime number. Then for each $(p+1)$-subpolygon $\mathcal{D}$ of $\mathcal{C}$ the labels of edges and diagonals in $\mathcal{D}$ are either all not divisible by p or they do not all have the same p-valuation.*

Combining this result with Proposition 4.2, we obtain the following consequence (where $k\cdot\mathcal{E}$ denotes the frieze with coefficients obtained by multiplying the label of each edge and diagonal of $\mathcal{E}$ by k).

Corollary. *Let $\mathcal{C}$ be a frieze with coefficients on an n-gon over the positive integers. Assume that we have* $\gcd(a,b)=\gcd(b,c)=\gcd(a,c)$ *for any triangle (a,b,c) in $\mathcal{C}$. Then there exists a Conway–Coxeter frieze $\mathcal{E}$ such that $\mathcal{C}$ is a subpolygon of $k\cdot\mathcal{E}$ for some positive integer k.*

The two directions of the if and only if statement of the theorem are proven separately in Section 4. The proof of sufficiency is constructive, that is, we give an explicit algorithm to compute a Conway–Coxeter frieze containing a given frieze with coefficients satisfying Conditions (1) and (2) as a subpolygon. Section 5 contains a detailed example. In general, these Conway–Coxeter friezes are not unique. However, our algorithm yields all possible Conway–Coxeter friezes that contain a given frieze with coefficients, because each step in the induction allows choices to be made and this can lead to several different extensions.

2. Frieze patterns with coefficients

In this section we collect the necessary definitions and fundamental properties of frieze patterns with coefficients. This concept appeared in a preprint by Propp in 2005 which has recently been published [10], but it can even be found earlier in a book by Coxeter [3]. A general theory of frieze patterns with coefficients has recently been developed in [5].

Although in this paper we are only dealing with frieze patterns over the positive integers, we reproduce the basic definition from [5] in a more general form allowing arbitrary complex numbers as entries.

Definition 2.1. *Let $R \subseteq \mathbb{C}$ be a subset of the complex numbers. Let $n \in \mathbb{Z}_{\geqslant 0}$.*

A frieze pattern with coefficients of height n *over R is an infinite array of the form*

$$\begin{array}{ccccccccccccc}
 & & \ddots & & & & & & \ddots & & & & \\
0 & c_{i-1,i} & c_{i-1,i+1} & c_{i-1,i+2} & \cdots & & \cdots & c_{i-1,n+i} & c_{i-1,n+i+1} & 0 & & & \\
 & 0 & c_{i,i+1} & c_{i,i+2} & c_{i,i+3} & \cdots & & \cdots & c_{i,n+i+1} & c_{i,n+i+2} & 0 & & \\
 & & 0 & c_{i+1,i+2} & c_{i+1,i+3} & c_{i+1,i+4} & \cdots & & \cdots & c_{i+1,n+i+2} & c_{i+1,n+i+3} & 0 & \\
 & & & & \ddots & & & & & \ddots & & &
\end{array}$$

where we also set $c_{i,i} = 0 = c_{i,n+i+3}$ for all $i \in \mathbb{Z}$, such that the following holds:

(1) *$c_{i,j} \in R$ for all $i \in \mathbb{Z}$ and $i < j < n+i+3$.*
(2) *$c_{i,i+1} \neq 0$ for all $i \in \mathbb{Z}$.*
(3) *For every (complete) adjacent 2×2-submatrix $\begin{pmatrix} c_{i,j} & c_{i,j+1} \\ c_{i+1,j} & c_{i+1,j+1} \end{pmatrix}$ we have*

$$c_{i,j}c_{i+1,j+1} - c_{i,j+1}c_{i+1,j} = c_{i+1,n+i+3}c_{j,j+1}. \tag{$E_{i,j}$}$$

Remark 2.2.

(1) Classic frieze patterns, as introduced by Coxeter [2], are those frieze patterns with coefficients with $c_{i,i+1} = 1$ for all $i \in \mathbb{Z}$. A Conway–Coxeter frieze pattern is a classic frieze pattern over $\mathbb{Z}_{>0}$. A fundamental result of Conway and Coxeter states that these frieze patterns are in bijection with triangulations of regular polygons, see [1].
(2) There is a close connection between frieze patterns and Fomin and Zelevinsky's cluster algebras. Namely, starting with a set of indeterminates on a row in the frieze pattern, the frieze conditions $(E_{i,j})$ produce the cluster variables of the cluster algebra of Dynkin type A. Whereas the classic Conway–Coxeter frieze patterns correspond to cluster algebras without coefficients, the more general frieze patterns with coefficients correspond to the cluster algebras of Dynkin type A with boundary edges acting as frozen variables. From the cluster algebras perspective this is the main motivation to study frieze patterns with coefficients.

In general, there are too many frieze patterns with coefficients to expect a satisfactory theory, even in the case of classic frieze patterns, see [4, Section 3] for an illustration of the case of wild SL_3-frieze patterns. Therefore, it is very common in the literature to restrict to tame frieze patterns. Many interesting frieze patterns are tame, e.g. all frieze patterns without zero entries, see [5, Proposition 2.4] for a proof.

Definition 2.3. *Let $\mathcal{C}$ be a frieze pattern with coefficients as in Definition* 2.1*. Then $\mathcal{C}$ is called* tame *if every complete adjacent 3×3-submatrix of $\mathcal{C}$ has determinant 0.*

The entries of a tame frieze pattern with coefficients are closely linked by many remarkable equations (in addition to the defining equations $(E_{i,j})$ in Definition 2.1). We restate some results from [5] which are relevant for the present paper.

First, the entries in a tame frieze pattern are invariant under a glide symmetry.

Proposition 2.4 ([5, Theorem 2.4]). *Let $R \subseteq \mathbb{C}$ be a subset. Let $\mathcal{C} = (c_{i,j})$ be a tame frieze pattern with coefficients over R of height n. Then for all entries of $\mathcal{C}$ we have $c_{i,j} = c_{j,n+i+3}$.*

This implies that the triangular region shown in Figure 1 yields a fundamental domain for the action of the glide symmetry. Note that the indices of the entries are in bijection with the edges and diagonals of a regular $(n+3)$-gon (viewed as pairs of vertices). This means that we can view every tame frieze pattern with coefficients of height n over R as a map on the edges and diagonals of a regular $(n+3)$-gon with values in R.

In the case of Conway–Coxeter frieze patterns the diagonals which are mapped to 1 give a triangulation of the $(n+3)$-gon.

$$\begin{array}{cccccccc}
 & \ddots & & & & & \ddots & \\
0 & c_{1,2} & c_{1,3} & \cdots & \cdots & \cdots & c_{1,n+3} & 0 \\
 & 0 & c_{2,3} & c_{2,4} & \cdots & \cdots & c_{2,n+3} & \ddots \\
 & & \ddots & \ddots & \ddots & & \vdots & \\
 & & & \ddots & \ddots & \ddots & \vdots & \\
 & & & & 0 & c_{n+1,n+2} & c_{n+1,n+3} & \\
 & & & & & 0 & c_{n+2,n+3} & \ddots \\
 & & & & & & 0 & \ddots \\
 & & & & & & & 0
\end{array}$$

Figure 1. Fundamental domain for the glide symmetry of a frieze pattern with coefficients.

Convention. We use the notion (tame) *frieze pattern with coefficients* for an infinite array as in Definition 2.1 and the notion (tame) *frieze with coefficients* for a corresponding map from edges and diagonals of a regular polygon.

Secondly, the entries in a frieze (pattern) with coefficients satisfy Ptolemy relations, as visualized in Figure 2.

Definition 2.5. *Let $\mathcal{C} = (c_{i,j})$ be a tame frieze with coefficients over $R \subseteq \mathbb{C}$ on a regular m-gon. We say that $\mathcal{C}$ satisfies the Ptolemy relation for the indices $1 \leqslant i \leqslant j \leqslant k \leqslant \ell \leqslant m$ if the following equation holds:*

$$c_{i,k}c_{j,\ell} = c_{i,\ell}c_{j,k} + c_{i,j}c_{k,\ell}. \qquad (E_{i,j,k,\ell})$$

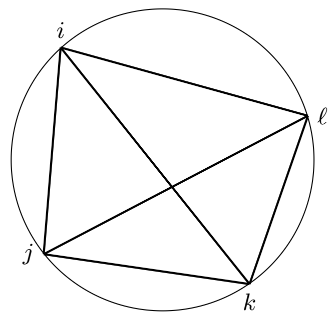

Figure 2. The Ptolemy relation $(E_{i,j,k,\ell})$.

An old result by Coxeter (see [2, Equation (5.7)]) states that classic friezes satisfy all Ptolemy relations and this can be extended to friezes with coefficients.

Proposition 2.6 ([5, Theorem 2.6]). *Every tame frieze with coefficients over some subset $R \subseteq \mathbb{C}$ satisfies all Ptolemy relations.*

3. n-gons in Conway–Coxeter friezes

From now on we consider frieze patterns with coefficients over positive integers.

Let us take any classic Conway–Coxeter frieze $\mathcal{C}$ on an n-gon, that is, a map from edges and diagonals of a regular polygon to the positive integers such that all edges of the n-gon are mapped to 1. Restricting this map to any subpolygon of the n-gon yields a frieze with coefficients. In fact, the restricted map still satisfies all Ptolemy relations of the subpolygon. See Figure 3 for an example.

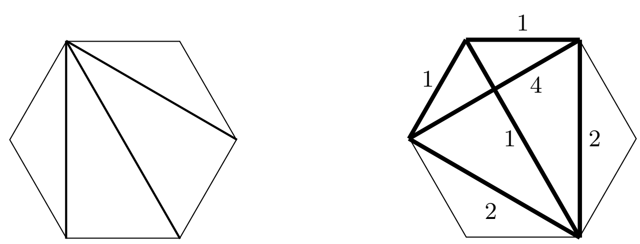

Figure 3. A frieze with coefficients cut out of a Conway–Coxeter frieze.

In [5] we addressed the fundamental question which friezes with coefficients actually appear as subpolygons of Conway–Coxeter friezes and obtained the following complete answer for the special case of triangles.

Theorem 3.1 ([5, Theorem 5.12]). *Let $a, b, c \in \mathbb{N}$. The triple (a, b, c) appears as labels of a triangle in some Conway–Coxeter frieze if and only if the following two conditions are satisfied:*

(1) $\gcd(a, b) = \gcd(b, c) = \gcd(a, c)$.
(2) $\nu_2(a) = \nu_2(b) = \nu_2(c) = 0$ *or* $|\{\nu_2(a), \nu_2(b), \nu_2(c)\}| > 1$ *where* $\nu_2(\cdot)$ *denotes the 2-valuation, that is, the numbers a, b, c are either all odd or do not all have the same 2-valuation.*

The main aim of this paper is a generalization of the previous theorem to arbitrary subpolygons in Conway–Coxeter friezes. That is, we give arithmetic conditions on the entries of a frieze with coefficients which characterize whether or not the frieze with coefficients appears as a subpolygon in some Conway–Coxeter frieze. The following theorem is the main result of this paper.

Theorem 3.2. *Let $\mathcal{C}$ be a frieze with coefficients on an n-gon over the positive integers. Then $\mathcal{C}$ appears as a subpolygon of some Conway–Coxeter frieze if and only if the following conditions are satisfied:*

(1) *For any triangle (a, b, c) in $\mathcal{C}$ we have* $\gcd(a, b) = \gcd(b, c) = \gcd(a, c)$.
(2) *Let $p < n$ be a prime number. Then for each $(p + 1)$-subpolygon $\mathcal{D}$ of $\mathcal{C}$ the labels of edges and diagonals in $\mathcal{D}$ are either all not divisible by p or they do not all have the same p-valuation.*

Note that for the special case $n = 3$ this gives precisely the criterion of Theorem 3.1. Actually, our proof of the main result Theorem 3.2 does not need the previous result on triangles from [5], so we get Theorem 3.1 as a proper corollary of the new result.

Example 3.3. There are friezes with coefficients where each triangle appears as a subpolygon of a Conway–Coxeter frieze, but the entire frieze does not. For instance, consider the square with labels as in Figure 4. This gives a frieze with coefficients since the Ptolemy relation is satisfied. All triangles satisfy the conditions from Theorem 3.1.

However, for the square itself Condition (2) of Theorem 3.2 fails for $p = 3$, so this square can not appear as a subpolygon of a Conway–Coxeter frieze.

This example was first discovered by Grobe in his Master's thesis [6], by a different argument not using Theorem 3.2.

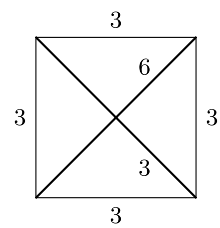

Figure 4. A frieze with coefficients which is not a subpolygon of a Conway–Coxeter frieze, but all of whose triangles do appear in Conway–Coxeter friezes.

Example 3.4. Condition (2) requires to check all prime numbers $p < n$ and the corresponding $(p+1)$-subpolygons. This is indeed necessary, as the following examples show. Note that for $p = 3$ this is Example 3.3 above.

Let p be any odd prime number. We consider the Conway–Coxeter frieze on a $(p + 1)$-gon given by a fan triangulation, that is, all diagonals start at the same vertex; see Figure 5 for the case $p = 11$.

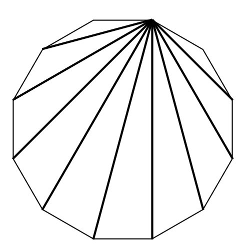

Figure 5. A fan on a dodecagon.

Using Ptolemy relations one checks that the maximal label of a diagonal in this frieze is $p - 1$ (actually, for each diagonal its label is one more than the number of diagonals of the fan triangulation it crosses). Let $\mathcal{C}$ be the frieze with coefficients on a $(p + 1)$-gon obtained by multiplying the above Conway–Coxeter frieze by p. Then the labels of all edges and diagonals have p-valuation 1. By Theorem 3.2 we see that $\mathcal{C}$ is not a subpolygon of a Conway–Coxeter frieze. However, for any prime number $q < p$, all $(q + 1)$-subpolygons of $\mathcal{C}$ do appear as subpolygons of Conway–Coxeter friezes, again by Theorem 3.2; in fact, the corresponding $(q + 1)$-subpolygons in the Conway–Coxeter frieze clearly satisfy the conditions of Theorem 3.2 and the validity of these conditions is not affected by multiplication with p, since $q < p$ are prime numbers.

4. Proof of the main result

The aim of this section is to prove Theorem 3.2. For clarity, the two directions of the if and only if statement are shown separately.

4.1. Necessity. We recall from [5] a basic property of Conway–Coxeter friezes, namely that every triangle in a Conway–Coxeter frieze satisfies Condition (1) of Theorem 3.2.

Lemma 4.1 ([5, Lemma 4.3]). *Let $\mathcal{C}$ be a Conway–Coxeter frieze and $i \leqslant j \leqslant k$. Then the greatest common divisor of any two of the numbers $c_{i,j}$, $c_{j,k}$ and $c_{i,k}$ divides the third number. In particular,* $\gcd(c_{i,j}, c_{j,k}) = \gcd(c_{i,k}, c_{j,k}) = \gcd(c_{i,j}, c_{i,k})$.

The next step in the proof is to notice that the condition on the gcd's from Lemma 4.1 has implications for the situation where $(p+1)$-subpolygons with the same p-valuations exist.

Proposition 4.2. *Let $\mathcal{C}$ be a frieze with coefficients on an n-gon over the positive integers. Assume that we have*

$$\gcd(a,b) = \gcd(b,c) = \gcd(a,c) \tag{4.1}$$

for any triangle (a,b,c) in $\mathcal{C}$ and that $\mathcal{C}$ contains a $(p+1)$-subpolygon $\mathcal{D}$ for a prime number p such that the labels of all edges and diagonals of $\mathcal{D}$ have the same p-valuation m. Then the label of every edge and diagonal of $\mathcal{C}$ is divisible by p^m.

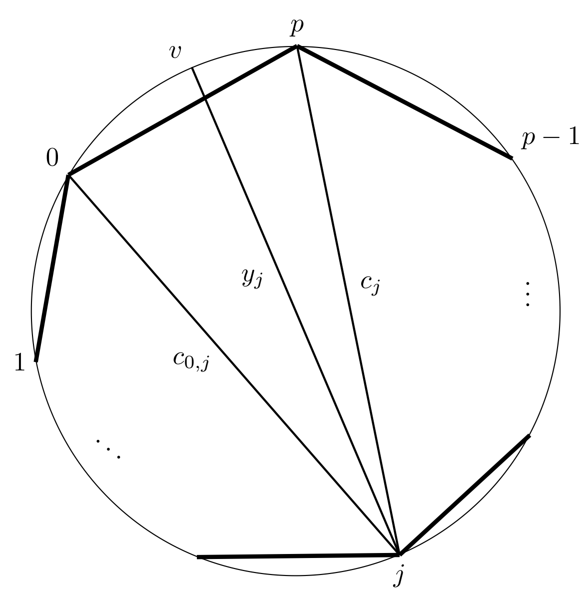

Figure 6. A $(p+1)$-subpolygon in a larger frieze.

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be as above; we denote the vertices of $\mathcal{D}$ by $0, \dots, p$. We proceed by induction on m. If $m = 0$, then the claim is trivial, so consider $m > 0$.

Assume first that every diagonal (i, v) for $i = 0, \dots, p$ and v not a vertex of $\mathcal{D}$ is divisible by p. Then if v, w are vertices of $\mathcal{C}$ not in $\mathcal{D}$, then the label of the diagonal (v, w) is divisible by p as well by assumption (4.1) since $(c_{0,v}, c_{v,w}, c_{0,w})$ is a triangle. Dividing the labels of all edges and diagonals of $\mathcal{C}$ by p we obtain a frieze with coefficients $\mathcal{C}'$ satisfying the assumption of the proposition with $m-1$ instead of m, thus we are finished by induction.

We may thus now assume without loss of generality that there exists a vertex v such that the label of the diagonal (v, p) is not divisible by p, see Figure 6. For $j = 0, 1, \dots, p$ we set $c_j := c_{p,j}$ and $y_j := c_{v,j}$ for abbreviation.

For $j = 1, \dots, p-1$ the Ptolemy relation for the crossing diagonals $(0, p)$ and (v, j) of $\mathcal{C}$ gives

$$c_0 y_j = y_0 c_j + c_{0,j} y_p.$$

Dividing this equation by p^m leads to

$$c'_0 y_j = y_0 c'_j + c'_{0,j} y_p \tag{4.2}$$

where $c'_0 = \frac{c_0}{p^m}$, $c'_j = \frac{c_j}{p^m}$ and $c'_{0,j} = \frac{c_{0,j}}{p^m}$. By assumption on $\mathcal{D}$, none of these three positive integers is divisible by p. In addition, note that y_j is not divisible by p by assumption (4.1), since (y_p, y_j, c_j) are the labels of a triangle in $\mathcal{C}$ and $y_p = c_{v,p}$ is not divisible by p. Then Equation (4.2) implies

$$y_0 \not\equiv -(c'_j)^{-1} c'_{0,j} y_p \pmod{p} \qquad \text{for all } j = 1, \dots, p-1. \tag{4.3}$$

On the other hand, for any $1 \leqslant i < j \leqslant p-1$, dividing the Ptolemy relation for the crossing diagonals $(0, j)$ and (p, i) by p^{2m} yields

$$c'_{0,j} c'_i - c'_{0,i} c'_j = c'_0 c'_{i,j} \not\equiv 0 \pmod{p}.$$

That is, the residue classes modulo p appearing on the right of (4.3) are pairwise different for $j = 1, \dots, p-1$. Hence the conditions in (4.3) rule out all nonzero residue classes modulo p for y_0, but since y_0 is not divisible by p, this leaves no choice for y_0. This is a contradiction and thus this case cannot occur. □

We now show that Conditions (1) and (2) are necessary for a frieze with coefficients to appear as a subpolygon of a Conway–Coxeter frieze. So assume that $\mathcal{C}$ is a frieze with coefficients that appears as a subpolygon of a Conway–Coxeter frieze $\mathcal{E}$.

By Lemma 4.1, Condition (1) is satisfied in $\mathcal{E}$, thus satisfied in $\mathcal{C}$ as well.

Now assume that $\mathcal{C}$ contains a $(p+1)$-subpolygon $\mathcal{D}$ for a prime number p such that the labels of all edges and diagonals of $\mathcal{D}$ have the same p-valuation m. Proposition 4.2 tells us that then the labels of all edges and diagonals of $\mathcal{E}$ are divisible by p^m. Since the edges of the Conway–Coxeter frieze $\mathcal{E}$ are labelled by 1, we obtain $m = 0$, that is, the labels of all edges and diagonals of $\mathcal{D}$ are not divisible by p, and Condition (2) holds.

4.2. Sufficiency. It remains to prove the sufficiency statement of Theorem 3.2. Let $\mathcal{C}$ be a frieze with coefficients over $\mathbb{Z}_{>0}$ on an n-gon satisfying Conditions (1) and (2). We have to show that $\mathcal{C}$ can be extended to a Conway–Coxeter frieze.

If all boundary edges have label 1 then $\mathcal{C}$ is itself a Conway–Coxeter frieze and we are done. So assume that $\mathcal{C}$ has a boundary edge with label $c_0 > 1$. The idea of the proof is to proceed inductively. That is, we aim to construct a frieze with coefficients $\widetilde{\mathcal{C}}$ over $\mathbb{Z}_{>0}$ on an $(n+1)$-gon with the following properties:

(i) $\widetilde{\mathcal{C}}$ contains $\mathcal{C}$ as a subpolygon.
(ii) The edges attached to the new vertex have labels 1 and y_0 where $0 < y_0 < c_0$.
(iii) $\widetilde{\mathcal{C}}$ still satisfies Conditions (1) and (2).

Carrying out this procedure inductively for each boundary edge of $\mathcal{C}$ eventually produces a frieze with coefficients with all boundary edges having label 1, that is, a Conway–Coxeter frieze containing $\mathcal{C}$ as a subpolygon. We will give an explicit algorithm to determine such a frieze with coefficients $\widetilde{\mathcal{C}}$, that is, the proof of this direction is constructive.

We label the vertices of the n-gon by $0, 1, \dots, n-1$ in counterclockwise order, such that the edge with label c_0 has vertices 0 and $n-1$, see Figure 7.

We set $c_j := c_{j,n-1}$ for $0 \leqslant j \leqslant n-2$, see the ultra thick lines in Figure 7. We aim to find suitable labels $y_j := c_{j,n}$ for the new edges and diagonals in the larger frieze with coefficients $\widetilde{\mathcal{C}}$ (the dashed lines in Figure 7) such that all Ptolemy relations in $\widetilde{\mathcal{C}}$ are satisfied.

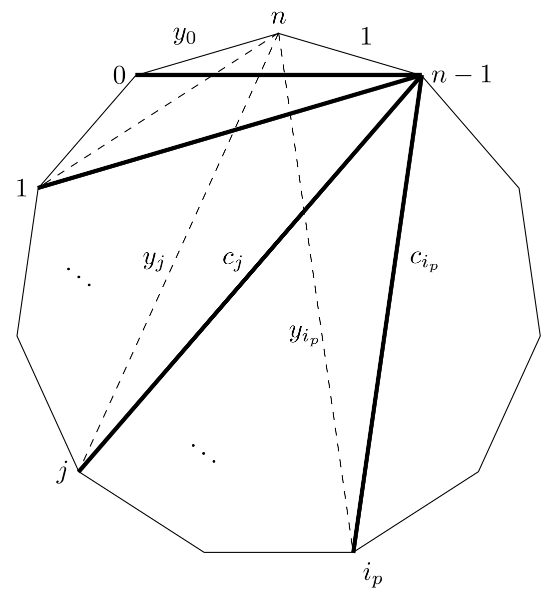

Figure 7. Extending a frieze with coefficients.

For computing suitable positive integers y_j, we consider each prime power divisor of c_0 separately and eventually use the Chinese Remainder Theorem.

Let p be a prime divisor of c_0 and $\ell := \nu_p(c_0)$ be the p-valuation (that is, p^ℓ divides c_0 but $p^{\ell+1}$ does not divide c_0). We set

$$m := \min\{\nu_p(c_i) \mid 0 \leqslant i \leqslant n-2\},$$

and we choose a vertex i_p with $\nu_p(c_{i_p}) = m$. Note that for every vertex j in $\mathcal{C}$ we have $p^m \mid c_j$ (by minimality of m) and also $p^m \mid c_{i,j}$ for $i \neq j$ (by Condition (1) for $\mathcal{C}$).

For any positive integer u we define u' by $u = p^{\nu_p(u)}u'$.

We first want to determine a suitable label y_{i_p}.

Lemma 4.3. *With the above notation there are positive integers y_{i_p} such that the following conditions are satisfied.*

(i) $y_{i_p} \not\equiv 0 \pmod{p}$.

(ii) *If $m > 0$, then for every vertex j such that $p \nmid \frac{c_{i_p,j}}{p^m}$ and $p \nmid \frac{c_j}{p^m}$ we have*

$$\begin{cases} c'_j y_{i_p} - c'_{i_p,j} \not\equiv 0 \pmod{p} & \text{if } j < i_p, \\ c'_j y_{i_p} + c'_{i_p,j} \not\equiv 0 \pmod{p} & \text{if } i_p < j. \end{cases} \tag{4.4}$$

Proof. If $m = 0$ then only part (i) applies and we are done.

So from now on we assume that $m > 0$. We consider the nonzero residue classes modulo p and show that for the elements in at least one residue class the conditions of the lemma hold. Let j be a vertex such that $p \nmid \frac{c_{i_p,j}}{p^m}$ and $p \nmid \frac{c_j}{p^m}$. Then the second condition in the lemma rules out the residue class $\pm(c'_j)^{-1}c'_{i_p,j} \pmod{p}$ to be chosen for y_{i_p}.

Claim. *Let vertices i and j both satisfy the assumptions in the second condition of the lemma. Then (4.4) rules out the same residue class modulo p if $p \mid \frac{c_{i,j}}{p^m}$ and different residue classes modulo p otherwise.*

Proof of the claim. We can assume $i < j$. There are different cases according to the location of the vertex i_p. We give the details for the case $i < j < i_p$, the other cases $i < i_p < j$ and $i_p < i < j$ are completely analogous.

Dividing the Ptolemy relation for the crossing diagonals (i, i_p) and $(j, n-1)$ by p^{2m} yields

$$c'_j c'_{i,i_p} = c'_i c'_{j,i_p} + c'_{i_p} \frac{c_{i,j}}{p^m}.$$

This implies

$$(c'_i)^{-1} c'_{i,i_p} - (c'_j)^{-1} c'_{j,i_p} \equiv (c'_i)^{-1} (c'_j)^{-1} c'_{i_p} \frac{c_{i,j}}{p^m} \pmod p$$

which is congruent to 0 if and only if p divides $\frac{c_{i,j}}{p^m}$. Since the summands on the left hand side are the values ruled out for y_{i_p} by (4.4), the claim follows.

By assumption, the frieze with coefficients $\mathcal{C}$ satisfies Condition (2) of Theorem 3.2. This means that there cannot be $p-1$ different vertices $j_1, \dots, j_{p-1}$ satisfying the assumptions in Condition (ii) and such that $p \nmid \frac{c_{j_r,j_s}}{p^m}$ for all $r \neq s$ (in fact, otherwise the subpolygon of $\mathcal{C}$ with vertices $j_1, \dots, j_{p-1}, i_p, n-1$ would be a $(p+1)$-subpolygon where alle edges and diagonals have the same p-valuation $m > 0$, contradicting Condition (2)). Using the above claim this implies that not all residue classes modulo p are ruled out by Condition (4.4) and hence we can choose positive integers y_{i_p} as claimed. $\square$

Using a suitable value for y_{i_p} as in Lemma 4.3 we now want to look for suitable values for the other new diagonals y_j, such that the Ptolemy relations in the larger polygon $\widetilde{\mathcal{C}}$ can be satisfied. Recall that above we have defined $\ell = \nu_p(c_0)$.

Lemma 4.4. *We keep the above notation and fix a positive integer y_{i_p} satisfying the conditions in Lemma 4.3. Then for every integer y_j in the residue class*

$$\begin{cases} (c'_{i_p})^{-1} \left(\frac{c_j}{p^m} y_{i_p} - \frac{c_{i_p,j}}{p^m} \right) \pmod{p^\ell} & \text{if } j < i_p, \\ (c'_{i_p})^{-1} \left(\frac{c_j}{p^m} y_{i_p} + \frac{c_{i_p,j}}{p^m} \right) \pmod{p^\ell} & \text{if } i_p < j, \end{cases}$$

the following holds.

(a) $p \nmid \gcd(y_j, c_j)$.
(b) *For every $0 \leqslant i < j$ we have $c_i y_j \equiv c_j y_i + c_{i,j} \pmod{p^\ell}$.*

Proof. We have a congruence

$$c'_{i_p} y_j \equiv \frac{c_j}{p^m} y_{i_p} \mp \frac{c_{i_p,j}}{p^m} \pmod{p^\ell}. \tag{4.5}$$

(a) We consider various cases.

Case 1. Suppose $p \mid \frac{c_{i_p,j}}{p^m}$. Then Condition (1) for $\mathcal{C}$ implies that $p \nmid \frac{c_j}{p^m}$. Moreover, $p \nmid y_{i_p}$ by Lemma 4.3. Then (4.5) gives $p \nmid y_j$. In particular, $p \nmid \gcd(y_j, c_j)$.

Case 2. Suppose $p \mid \frac{c_j}{p^m}$. Then Condition (1) for $\mathcal{C}$ implies that $p \nmid \frac{c_{i_p,j}}{p^m}$. Then (4.5) yields that $p \nmid y_j$. In particular, $p \nmid \gcd(y_j, c_j)$.

Case 3. Suppose $p \nmid \frac{c_{i_p,j}}{p^m}$ and $p \nmid \frac{c_j}{p^m}$. If $m = 0$ then $p \nmid c_j$ by assumption and hence $p \nmid \gcd(y_j, c_j)$. If $m > 0$ then according to the choice of y_{i_p} in Lemma 4.3, the right hand side of (4.5) is invertible modulo p^ℓ. Hence the left hand side is invertible as well. Thus, $p \nmid y_j$ and in particular $p \nmid \gcd(y_j, c_j)$.

(b) Due to the signs in the definition of y_j, there are separate cases. We present the argument for the case $i < i_p < j$, the other cases $i < j < i_p$ and $i_p < i < j$ are very similar.

By definition of y_j we have

$$\begin{aligned} c_i y_j - c_j y_i &\equiv c_i (c'_{i_p})^{-1} \left(\frac{c_j}{p^m} y_{i_p} + \frac{c_{i_p,j}}{p^m} \right) - c_j (c'_{i_p})^{-1} \left(\frac{c_i}{p^m} y_{i_p} - \frac{c_{i_p,i}}{p^m} \right) \pmod{p^\ell} \\ &\equiv (c'_{i_p})^{-1} \left(c_i \frac{c_{i_p,j}}{p^m} + c_j \frac{c_{i_p,i}}{p^m} \right) \pmod{p^\ell}. \end{aligned}$$

The Ptolemy relation in $\mathcal{C}$ for the crossing diagonals (i,j) and $(i_p, n-1)$ reads

$$c_{i,j} c_{i_p} = c_i c_{i_p,j} + c_{i_p,i} c_j.$$

Dividing this equation by p^m and plugging it into the above congruence gives

$$c_i y_j - c_j y_i \equiv (c'_{i_p})^{-1} c_{i,j} \frac{c_{i_p}}{p^m} \equiv (c'_{i_p})^{-1} c_{i,j} c'_{i_p} \equiv c_{i,j} \pmod{p^\ell},$$

as claimed. □

We have now constructed residue classes for $y_0, y_1, \ldots, y_{n-2}$ modulo $p^{\nu_p(c_0)}$ for each prime divisor p of c_0, satisfying the conditions in Lemma 4.3 and Lemma 4.4. Then the Chinese Remainder Theorem yields residue classes for $y_0, y_1, \ldots, y_{n-2}$ modulo c_0, which according to Lemma 4.4 (b) in particular satisfy

$$c_0 y_j \equiv c_j y_0 + c_{0,j} \pmod{c_0}$$

for all $j = 1, \ldots, n-2$. For y_0 we choose the smallest positive representative in this residue class, that is, we have $0 < y_0 < c_0$. (In fact, by Lemma 4.4 (a) we have that y_0 and c_0 are coprime, in particular, y_0 is nonzero.) Recall that this is needed to make the inductive strategy work. In particular, for this choice we have that for each vertex $j = 1, \ldots, n-2$ the number

$$y_j = \frac{c_j y_0 + c_{0,j}}{c_0} \tag{4.6}$$

is a positive integer.

Finally, to make the inductive strategy work, we have to show that $\widetilde{\mathcal{C}}$ is indeed a frieze with coefficients and that $\widetilde{\mathcal{C}}$ satisfies Conditions (1) and (2) of Theorem 3.2.

Proposition 4.5. *With the above notations and definitions, the following hold.*

(a) *All Ptolemy relations in $\widetilde{\mathcal{C}}$ are satisfied, that is, $\widetilde{\mathcal{C}}$ is a frieze with coefficients over $\mathbb{Z}_{>0}$.*

(b) *$\widetilde{\mathcal{C}}$ satisfies Conditions* (1) *and* (2) *of Theorem* 3.2.

Proof. (a) The Ptolemy relations not involving any of the new diagonals with label y_j are Ptolemy relations of $\mathcal{C}$ and hold by assumption since $\mathcal{C}$ is a frieze with coefficients.

For crossings of diagonals labelled y_j with the diagonal with label c_0 the Ptolemy relation holds by definition of y_j in (4.6).

Let (i,k) be a diagonal in $\mathcal{C}$ crossing the new diagonal with label y_j. Using the formula in (4.6) and Ptolemy relations in $\mathcal{C}$ we get

$$\begin{aligned} y_k c_{i,j} + y_i c_{j,k} &= \frac{c_k y_0 + c_{0,k}}{c_0} c_{i,j} + \frac{c_i y_0 + c_{0,i}}{c_0} c_{j,k} \\ &= \frac{1}{c_0} (y_0 (c_k c_{i,j} + c_i c_{j,k}) + c_{0,k} c_{i,j} + c_{0,i} c_{j,k}) \\ &= \frac{1}{c_0} (y_0 c_j c_{i,k} + c_{0,j} c_{i,k}) = \frac{c_j y_0 + c_{0,j}}{c_0} c_{i,k} = y_j c_{i,k}. \end{aligned}$$

Note that in particular we also obtain $c_i y_j = c_j y_i + c_{i,j}$ for all i, j.

(b) For Condition (1) we have to consider the triangles in $\widetilde{\mathcal{C}}$ which are not already in $\mathcal{C}$. There are different types of triangles to consider.

The triangle $(1, y_0, c_0)$ satisfies Condition (1) by Lemma 4.4 (a). For a triangle $(1, y_j, c_j)$ with $j \neq 0$ we know again by Lemma 4.4 (a) that $p \nmid \gcd(y_j, c_j)$ for all prime divisors p of c_0. Suppose q is a prime number dividing y_j and c_j but $q \nmid c_0$. Then $q \mid c_{0,j}$ by (4.6). Thus q is a common divisor of $c_{0,j}$ and c_j. But the triangle $(c_0, c_{0,j}, c_j)$ in $\mathcal{C}$ satisfies Condition (1), so $q \mid c_0$, a contradiction. Thus we have shown that $\gcd(y_j, c_j) = 1$ and the triangle $(1, y_j, c_j)$ satisfies Condition (1).

The other new triangles in $\widetilde{\mathcal{C}}$ are of the form $(y_i, c_{i,j}, y_j)$. We use the Ptolemy relation $c_i y_j = c_j y_i + c_{i,j}$. Let d be a common divisor of y_i and $c_{i,j}$. Then d divides $c_i y_j$. But y_i and c_i are coprime as shown in the previous paragraph, so d divides y_j, as desired. Similarly, if d is a common divisor of $c_{i,j}$ and y_j, then d divides y_i. Finally, if d is a common divisor of y_i and y_j then d divides $c_{i,j}$.

So Condition (1) holds for all triangles in $\widetilde{\mathcal{C}}$.

For Condition (2) we have to consider all possible $(q+1)$-gons in $\widetilde{\mathcal{C}}$ for all prime numbers $q < n+1$. The subpolygons in $\mathcal{C}$ satisfy Condition (2) by assumption. So it suffices to consider $(q+1)$-gons $\mathcal{D}$ involving the new vertex n and q vertices of $\mathcal{C}$. Suppose that all edges and diagonals in $\mathcal{D}$ have the same positive q-valuation. Note that in $\widetilde{\mathcal{C}}$ there is a boundary edge with label 1 attached to $\mathcal{D}$. But we have shown in the proof of necessity in Subsection 4.1 that such a configuration leads to a contradiction. Therefore, $\widetilde{\mathcal{C}}$ satisfies Condition (2), as needed for the inductive procedure to work.

This completes the proof of the sufficiency direction in Theorem 3.2. ☐

5. A worked example

The proof of our main Theorem 3.2 is constructive. In this section we go through an explicit example to illustrate how the methods in the proof of the previous section yield an algorithm to determine a Conway–Coxeter frieze having a given frieze with coefficients as a subpolygon.

Let $\mathcal{C}$ be the frieze with coefficients given in Figure 8.

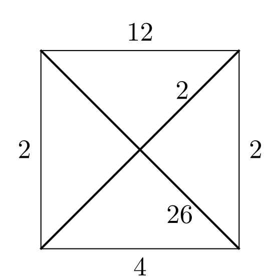

Figure 8. A frieze with coefficients on a square.

One checks that $\mathcal{C}$ satisfies Conditions (1) and (2) of Theorem 3.2, therefore $\mathcal{C}$ can be realized as a subpolygon of some Conway–Coxeter frieze. We illustrate here how to determine such a Conway–Coxeter frieze using the methods from the previous section.

Each boundary edge of $\mathcal{C}$ has to be extended. We start with the boundary edge with label 12. With the notation as in the previous section we set $c_0 = 12$, and hence $c_1 = 2$ and $c_2 = 2$. We consider each prime divisor of c_0 separately.

For $p = 2$ we have $m = \min\{\nu_2(c_i) \mid 0 \leqslant i \leqslant 2\} = 1$, and we choose the vertex $i_2 = 2$. We want to determine a suitable value for y_{i_2}, using Lemma 4.3. One checks that no restriction occurs here, so we can choose $y_{i_2} = 1$.

For $p=3$ we have $m=\min\{\nu_3(c_i)\mid 0\leqslant i\leqslant 2\}=0$, and we choose $i_3=2$. Since $m=0$ here, by Lemma 4.3 we can choose for y_{i_3} any positive integer not divisible by $p=3$. So we choose $y_{i_3}=1$.

The next step now is to compute a suitable value for $y_0 \pmod{c_0}$ by using Lemma 4.4.

For $p=2$, we have

$$y_0 \equiv (c'_{i_2})^{-1}\left(\frac{c_0}{2}y_{i_2}-\frac{c_{2,0}}{2}\right)\equiv 1\cdot(6\cdot 1-13)\equiv 1 \pmod 4.$$

Similarly, for $p=3$ one gets

$$y_0 \equiv (c'_{i_3})^{-1}\left(\frac{c_0}{1}y_{i_3}-\frac{c_{2,0}}{1}\right)\equiv 2\cdot(12\cdot 1-26)\equiv 2 \pmod 3.$$

By the Chinese Remainder Theorem we obtain

$$y_0\equiv 5 \pmod{12}.$$

Now we can use Equation (4.6) to compute the values for y_1 and y_2, namely

$$y_1=\frac{c_1y_0+c_{0,1}}{c_0}=\frac{2\cdot 5+2}{12}=1 \quad\text{and}\quad y_2=\frac{c_2y_0+c_{0,2}}{c_0}=\frac{2\cdot 5+26}{12}=3.$$

Thus we obtain the frieze with coefficients as in Figure 9, where we draw thick lines for diagonals with label 1, that is, for those diagonals which will appear in the final triangulation.

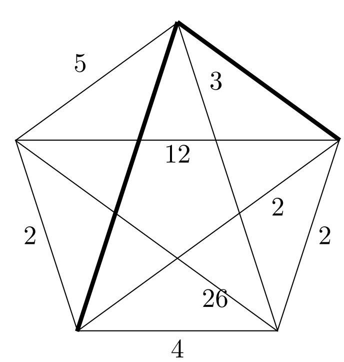

FIGURE 9. First step of the extension of $\mathcal{C}$.

Now we extend further at the boundary edge with label 5. We then have $c_0=5$, $c_1=1$, $c_2=3$ and $c_4=1$. For the relevant prime number $p=5=c_0$, we have $m=\min\{\nu_5(c_i)\mid 0\leqslant i\leqslant 3\}=0$ and we choose the vertex $i_5=3$.

Since $m=0$, the only condition in Lemma 4.3 is that y_{i_5} is not divisible by $p=5$. So we can choose $y_{i_5}=1$.

With Lemma 4.4 we then compute the value for the new edge as

$$y_0\equiv (c'_{i_5})^{-1}\left(\frac{c_0}{1}y_{i_5}-\frac{c_{3,0}}{1}\right)\equiv 5-12\equiv 3 \pmod 5.$$

From Equation (4.6) we then determine the values for the other diagonals

$$y_1=\frac{c_1y_0+c_{0,1}}{c_0}=\frac{1\cdot 3+2}{5}=1,$$

$$y_2=\frac{c_2y_0+c_{0,2}}{c_0}=\frac{3\cdot 3+26}{5}=7,$$

and

$$y_3=\frac{c_3y_0+c_{0,3}}{c_0}=\frac{1\cdot 3+12}{5}=3.$$

This leads to the frieze with coefficients given in Figure 10, where for clarity we only include those labels which we just computed. Note that the original square $\mathcal{C}$ forms the bottom half of the hexagon.

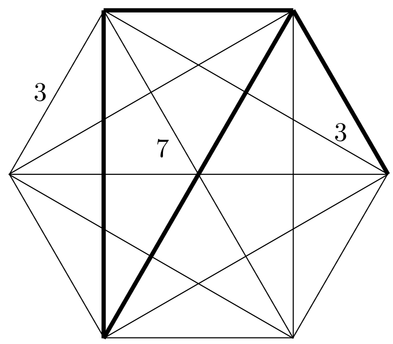

Figure 10. Second step of the extension of $\mathcal{C}$.

The third step in the extension procedure for the edge with label 12 in $\mathcal{C}$ is to extend the new boundary edge with label 3. Hence we set $c_0 = 3$, $c_1 = 1$, $c_2 = 7$, $c_3 = 3$ and $c_4 = 1$. For the only relevant prime number $p = 3 = c_0$ we have $m = 0$ and we choose $i_3 = 4$.

Since $m = 0$, by Lemma 4.3 we can choose as y_{i_3} any positive integer not divisible by $p = 3$. We choose $y_{i_3} = 2$. Then from Lemma 4.4 we obtain

$$y_0 \equiv (c'_{i_3})^{-1}\left(\frac{c_0}{1}y_{i_3} - \frac{c_{4,0}}{1}\right) \equiv 1\cdot(3\cdot 2 - 5) \equiv 1 \pmod 3.$$

Equation (4.6) gives the following values for the diagonals y_j,

$$y_1 = \frac{c_1y_0 + c_{0,1}}{c_0} = \frac{1\cdot 1+2}{3} = 1 \qquad \text{and} \qquad y_2 = \frac{c_2y_0 + c_{0,2}}{c_0} = \frac{7\cdot 1+26}{3} = 11,$$
$$y_3 = \frac{c_3y_0 + c_{0,3}}{c_0} = \frac{3\cdot 1+12}{3} = 5 \qquad \text{and} \qquad y_4 = \frac{c_4y_0 + c_{0,4}}{c_0} = \frac{1\cdot 1+5}{3} = 2.$$

This leads to the frieze with coefficients given in Figure 11, where again for clarity we only show a few of the diagonals and only the labels we just computed and the remaining boundary labels not equal to 1.

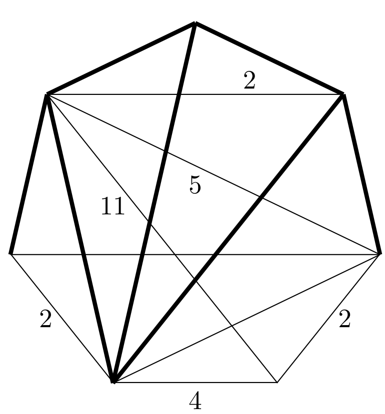

Figure 11. Third step of the extension of $\mathcal{C}$.

Note that we have now completed the extension for the boundary edge with label 12 in the original frieze with coefficients $\mathcal{C}$. It now remains to apply the same procedure to the other boundary edges with labels 2, 4 and 2. We leave the computations to the reader. Eventually, one can find the triangulation of a decagon given in Figure 12, containing the original frieze with coefficients $\mathcal{C}$ as a subpolygon.

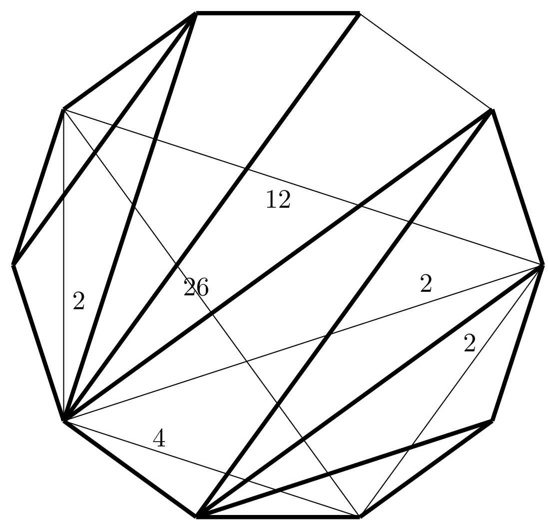

FIGURE 12. The frieze with coefficients $\mathcal{C}$ as a subpolygon of a Conway–Coxeter frieze.

REFERENCES

[1] John H. Conway and Harold S. M. Coxeter, *Triangulated polygons and frieze patterns*, Math. Gaz. **57** (1973), 87–94 and 175–183.
[2] Harold S. M. Coxeter, *Frieze patterns*, Acta Arith. **18** (1971), 297–310.
[3] ———, *Regular complex polytopes*, second ed., Cambridge University Press, Cambridge, 1991.
[4] Michael Cuntz, *On wild frieze patterns*, Experimental Mathematics **26** (2017), no. 3, 342–348.
[5] Michael Cuntz, Thorsten Holm, and Peter Jørgensen, *Frieze patterns with coefficients*, Forum Math. Sigma **8** (2020), Paper no. e17 (36 pages).
[6] J. Grobe, *Frieze patterns with coefficients*, Master's thesis, Leibniz Universität Hannover, 2020.
[7] Sophie Morier-Genoud, *Coxeter's frieze patterns at the crossroads of algebra, geometry and combinatorics*, Bull. Lond. Math. Soc. **47** (2015), no. 6, 895–938.
[8] Tadao Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 15, Springer-Verlag, Berlin, 1988, An introduction to the theory of toric varieties, Translated from the Japanese.
[9] Oskar Perron, *Die Lehre von den Kettenbrüchen*, Teubner Verlag, Leipzig, 1929.
[10] James Propp, *The combinatorics of frieze patterns and Markoff numbers*, Integers **20** (2020), Paper no. A12 (38 pages).

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