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Abstract In this paper we unify several existing regularity conditions for graphs, including strong regularity, k-isoregularity, and the t-vertex condition. We develop an algebraic composition/decomposition theory of regularity conditions. Using our theoretical results we show that a family of non rank 3 graphs known to satisfy the 7-vertex condition fulfills an even stronger condition, $(3,7)$-regularity (the notion is defined in the text). Derived from this family we obtain a new infinite family of non rank 3 strongly regular graphs satisfying the 6-vertex condition. This strengthens and generalizes previous results by Reichard.

1. Introduction

Strongly regular graphs (srgs) are simple regular graphs with the property that the number of common neighbors of a pair of distinct vertices depends only on whether the two vertices are connected by an edge or not. Originally introduced by R. C. Bose in [4], they are one of the central notions of modern algebraic graph theory. Small examples include the pentagon, the Petersen graph, triangular graphs, the Clebsch graph, … (A. E. Brouwer maintains a list of known small examples at [6]). Srgs arise, e.g. as orbital graphs of permutation groups of rank three (such srgs are usually called *rank* 3 *graphs* or 2-*homogeneous graphs*). Thanks to the classification of finite simple groups, all rank 3 graphs are known by now (cf. [2, 32, 39]). However, by no means, all srgs arise in this way. Srgs exist in such an abundance that nowadays a complete classification up to isomorphism seems hopeless (cf. [17, 43, 58]). To single out the more interesting specimen it is necessary to impose stronger regularity conditions. One possible such regularity condition is the so-called *t-vertex condition* that was introduced by D. G. Higman in [25] (cf. also [24]). A graph is said to fulfill the t-vertex condition if the number of subgraphs with at most t vertices of a given isomorphism type over a fixed pair of vertices depends only on whether or not the vertices are connected by an edge or whether they are equal. Thus the t-vertex condition is, in fact, a class of regularity conditions parameterized by t which generalizes the regularity conditions of strongly regular graphs. In particular, the srgs are precisely the graphs that fulfill the 3-vertex condition. Clearly, all rank 3 graphs satisfy the t-vertex condition for arbitrary t. Of special interest are non-rank 3 graphs that satisfy the t-vertex condition for some $t > 3$. The smallest examples for $t = 4$ have order 36 (cf. [35]). As non-rank 3 srgs with the t-vertex condition for $t > 3$ appear to be very rare, there has been

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an ongoing research effort to discover new examples and to understand their nature (cf. [28, 29, 33, 35, 49, 51]).

Another class of regularity conditions strengthening strong regularity is k-isoregularity. A graph is said to be k-isoregular if for every set S of at most k vertices the number of common neighbors of the elements of S depends only on the isomorphism type of the subgraph induced by S. The srgs are precisely the 2-isoregular graphs. In the same way that the t-vertex condition is a combinatorial approximation of 2-homogeneity, k-isoregularity is a combinatorial approximation of k-homogeneity. The notion of k-isoregularity has its origins in works by J. M. J. Buczak, Ja. Ju. Gol'fand, and M. Klin ([8, 22]).

For a comprehensive overview of the history and the literature related to the t-vertex condition and k-isoregularity, we refer to Section 9 of Reichard's [51].

Every 5-isoregular finite graph is homogeneous (cf. [10]), i.e. every isomorphism between subgraphs extends to an automorphism. Similarly, it was conjectured by M. Klin (cf. [16]) that there is a number t_0 such that an srg is 2-homogeneous if and only if it satisfies the t_0-vertex condition. To prove or refute this conjecture, it is necessary to have good methods for observing whether or not a given graph fulfills the t-vertex condition. Already in [24] Hestenes and Higman noticed that to verify the 4-vertex condition it is enough to test it just for two types of subgraphs. More results on how to simplify the testing of the t-vertex condition were given by A. V. Ivanov and S. Reichard [29, 49].

In this paper, we develop a theory of regularity conditions applicable, in principle, to many categories of combinatorial objects. This leads us to new criteria for the t-vertex condition and for (k,t)-regularity (a regularity condition that strengthens the concept of k-isoregularity in the same way as the t-vertex condition strengthens the concept of 2-isoregularity).

Using our techniques, we show that the point graphs of partial quadrangles (in the sense of [9]) fulfill the 5-vertex condition (see Theorem 5.7). Moreover, we show that if the point graph of a partial quadrangle is 3-isoregular, then it is $(3,7)$-regular (see Theorem 5.17). In particular, the point graphs of generalized quadrangles of order (q,q^2) are $(3,7)$-regular (this strengthens a recent result by S. Reichard [51] stating that the point graphs of $\mathrm{GQ}(q,q^2)$ satisfy the 7-vertex condition). As a consequence we obtain that the point graphs of partial quadrangles of order $(s,t,\mu)=(q-1,q^2,q^2-q)$ satisfy the 6-vertex condition (see Corollary 5.19).

The paper is structured into two main parts, a theoretical one (consisting of Sections 3 and 4) and a more applied one (consisting of Section 5).

Section 3 is the technical backbone of the paper. Here graph types and the related regularity conditions (like $\mathbb{T}$-regularity and (m,n)-regularity) are defined and compared to classical regularity conditions (like k-isoregularity and the t-vertex condition). The main result of this section is the *type counting lemma* (Lemma 3.28). Roughly speaking it states that graphs that are regular for some graph types are also regular for some other, bigger graph types. Its proof hinges on an elementary notion from category theory, namely, the universal property of colimits, that provides a bijection between compatible cocones of a diagram with the morphisms starting from a given fixed colimit of this diagram. In the rest of Section 3, the type counting lemma is used to derive those criteria for (m,n)-regularity that are used in the applied second part of the paper. Of particular interest for the reader may be Corollary 3.42, a criterion for the (m,n)-regularity formulated purely in graph-theoretical language.

In Section 4 the results from Section 3 are used to improve known criteria for the t-vertex condition.

At some places of the paper we are faced with the problem of enumerating unlabeled 3- and 4-connected graphs of small orders ($\leqslant 8$). While these tasks can certainly be completed "by hand" using the known inductive methods for their construction from literature (notably Tutte's characterization of 3-connected graphs [56], and Slater's characterization of 4-connected graphs [53]), it is safer to trust in computers for such calculations. We used the `geng`-utility from the package *nauty and traces* (cf. [42]) in conjunction with *GAP* (cf. [18]) and *GRAPE* (cf. [54]) for the automatic enumeration of small 3- and 4-connected graphs.

In this paper, problems from algebraic graph theory are treated using methods from category theory. The results are then applied to graphs constructed out of finite incidence geometries (notably partial quadrangles and generalized quadrangles). While the paper is written in a mostly self-contained manner, it may be helpful to have some standard literature from these fields at hand. A modern source for algebraic graph theory is [21]. For notions from category theory, we refer to the classics [3, 40]. For incidence geometries we recommend [14] as a starting point (see also [47]). Finally, for a recent survey on homogeneous structures, we refer to [41].

2. Preliminaries about the category of graphs

Let us start by fixing some notations: A graph is a pair (V,E) where V is a finite set of vertices and $E \subseteq \binom{V}{2}$ is a set of undirected edges.[1] If Γ is a graph, then by $V(\Gamma)$ we denote the vertex set and by $E(\Gamma)$ we denote the edge set of Γ. If $M \subseteq V(\Gamma)$, then by $\Gamma(M)$ we denote the subgraph of Γ induced by M. As usual, the *order* of a graph is the number of its vertices and the *valency* of a vertex is the number of its neighbors. A graph Γ for which $E(\Gamma) = \binom{V(\Gamma)}{2}$ is called a *complete graph.* A complete graph of order n is denoted by K_n. The *complement* of a graph Γ is $(V(\Gamma), \binom{V(\Gamma)}{2} \setminus E(\Gamma))$. It is denoted by $\overline{\Gamma}$.

The class of all graphs can be naturally equipped with a concept of homomorphisms: A *graph homomorphism* (or short: *homomorphism*) from a graph Γ_1 to a graph Γ_2 is a function $f\colon V(\Gamma_1) \to V(\Gamma_2)$ with the property that for each $\{v,w\} \in E(\Gamma_1)$ we have that $\{f(v), f(w)\} \in E(\Gamma_2)$. A one-to-one homomorphism $f\colon \Gamma_1 \to \Gamma_2$ is called an *embedding* if for all $\{v,w\} \in \binom{V(\Gamma_1)}{2} : \{v,w\} \in E(\Gamma_1) \iff \{f(v), f(w)\} \in E(\Gamma_2)$.

Following the tradition of category theory (and somewhat conflicting with the tradition of algebraic graph theory), whenever $f\colon A \to B$, and $g\colon B \to C$, then the composition of f and g is a morphism from A to C that is denoted by $g \circ f$. That is, we use the convention that morphisms are applied to elements of their domain from the left so that $(f \circ g)(x) = f(g(x))$.

Next, we introduce the main construction principle of graphs relevant to this paper. It has a combinatorial and a category-theoretic dimension. Let us start with the category-theoretic one. In what follows we will use capital greek letters to denote suitable (local) subgraphs of a considered global graph. As a rule, the global graph itself is denoted by the letter Γ.

Definition 2.1. *Let Δ, Θ_1, Θ_2 be graphs and let $f_1\colon \Delta \to \Theta_1$, $f_2\colon \Delta \to \Theta_2$ be homomorphisms. A* compatible cocone *of (f_1, f_2) is a pair (g_1, g_2) where $g_1\colon \Theta_1 \to \Theta$,*

[1] Here and below, for a set M and a non-negative integer k, by $\binom{M}{k}$ we denote the set of k-element subsets of M.

$g_2\colon \Theta_2 \to \Theta$ *for some graph* Θ*, such that the following diagram commutes:*

$$\begin{tikzcd} \Theta_1 \arrow[r, "g_1"] & \Theta \\ \Delta \arrow[u, "f_1"] \arrow[r, "f_2"] & \Theta_2 \arrow[u, "g_2"'] \end{tikzcd} . \tag{1}$$

The cocone (g_1, g_2) *is called a* limiting cocone *of* (f_1, f_2) *if for any other compatible cocone* (h_1, h_2) *of* (f_1, f_2) *where* $h_1\colon \Theta_1 \to \Gamma$*,* $h_2\colon \Theta_2 \to \Gamma$ *there exists a unique homomorphism* $k\colon \Theta \to \Gamma$ *such that the following diagram commutes:*

$$\begin{tikzcd} & & \Gamma \\ \Theta_1 \arrow[r, "g_1"] \arrow[urr, bend left, "h_1"] & \Theta \arrow[ur, dashed, "k"] & \\ \Delta \arrow[u, "f_1"] \arrow[r, "f_2"] & \Theta_2 \arrow[u, "g_2"'] \arrow[uur, bend right, "h_2"'] & \end{tikzcd} .$$

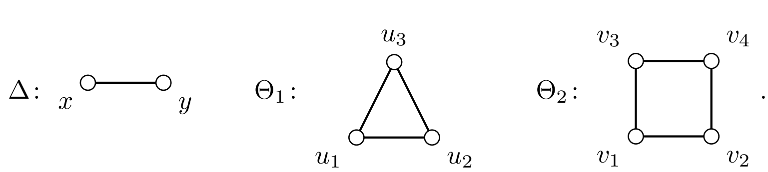
In that case the diagram (1) *is called a* pushout square.

For us, only the special case when (f_1, f_2) is a pair of embeddings is of interest. In this case, for every limiting cocone (g_1, g_2) of (f_1, f_2) we have that g_1 and g_2 are embeddings, too. A concrete construction of limiting cocones of pairs of embeddings in the category of graphs goes as follows:

Construction. Let $f_1\colon \Delta \hookrightarrow \Theta_1$, $f_2\colon \Delta \hookrightarrow \Theta_2$ be embeddings. Let $\widetilde{\Theta}$ be the disjoint union of Θ_1 and Θ_2. Let $\theta \subseteq V(\widetilde{\Theta})^2$ be the smallest equivalence relation that contains $\{(f_1(v), f_2(v)) \mid v \in V(\Delta)\}$. Let $\Theta \coloneqq \widetilde{\Theta}/\theta$ (vertices of Θ are equivalence classes of θ and two classes are connected by an edge if some representatives of the classes are connected by an edge in $\widetilde{\Theta}$). Finally, let $g_1\colon \Theta_1 \hookrightarrow \Theta$ and $g_2\colon \Theta_2 \hookrightarrow \Theta$ be given by $g_1\colon v \mapsto [v]_\theta$, $g_2\colon w \mapsto [w]_\theta$. Then (g_1, g_2) is a limiting cocone for (f_1, f_2).

Note that θ has equivalence classes of size $\leqslant 2$. One can imagine that Θ is obtained by glueing Θ_1 and Θ_2 together at a copy of Δ, which is marked in Θ_1 and Θ_2 through f_1 and f_2, respectively. This construction is also known under the name *graph amalgamation*, *fibered sum* or *amalgamated free sum* (cf. [40, 44]).

Example 2.2. Consider the following three graphs:



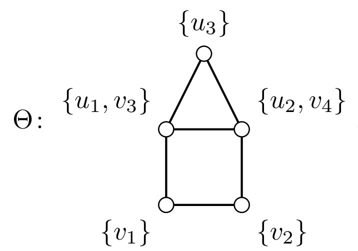
Define $f_1\colon \Delta \hookrightarrow \Theta_1$ and $f_2\colon \Delta \hookrightarrow \Theta_2$ according to

$$f_1\colon x \mapsto u_1, y \mapsto u_2; \qquad f_2\colon x \mapsto v_3, y \mapsto v_4.$$

According to the construction of amalgamated free sums we have that $V(\Theta) = (V(\Theta_1)\dot\cup V(\Theta_2))/\theta$, where θ is the equivalence relation on $V(\Theta_1)\dot\cup V(\Theta_2)$ generated by

$$\{(f_1(x), f_2(x)), (f_1(y), f_2(y))\} = \{(u_1, v_3), (u_2, v_4)\},$$

and where the operation $\dot\cup$ denotes the disjoint union of sets. In other words, $V(\Theta) = \{\{u_1, v_3\}, \{u_2, v_4\}, \{u_3\}, \{v_1\}, \{v_2\}\}$, and Θ is given by



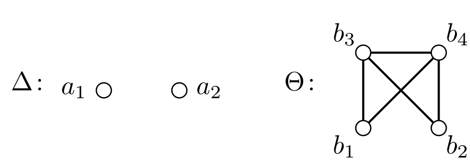
3. Graph types and regularity conditions

The t-vertex condition arises from a local invariant of pairs of vertices of a graph. Let $\Gamma = (V, E)$ be a graph and let $(x, y) \in V^2$. We consider all induced subgraphs of Γ that contain x and y and that have order $\leqslant t$. Two such subgraphs are said to be of the same type if they are isomorphic by an isomorphism that fixes x and y. The possible types of subgraphs correspond to all isomorphism classes of graphs of order $\leqslant t$ with a pair of distinguished vertices. To the pair $(x, y) \in V^2$ we may associate a function $\varphi_{x,y}$ from the types to the natural numbers that maps every type to the number of induced subgraphs of Γ that contain x and y and that are of this type. Graphs Γ where the function $\varphi_{x,y}$ does not depend directly on the pair (x, y) but only on whether $x = y$ or $\{x, y\} \in E$ or $\{x, y\} \in \binom{V}{2} \setminus E$, are said to fulfill the *t-vertex condition*. In the following, we give an equivalent definition of the t-vertex condition using the language of category theory.

3.1. Basic definitions.

Definition 3.1. *A* graph type $\mathbb{T}$ *is a triple* (Δ, ι, Θ) *where* Δ *and* Θ *are graphs and* $\iota\colon \Delta \hookrightarrow \Theta$ *is an embedding. The* order *of* $\mathbb{T}$ *is the pair* (m, n) *where* m *is the order of* Δ *and* n *is the order of* Θ*. The graphs* Δ *and* Θ *are called* base graph *and* underlying graph *of* $\mathbb{T}$*, respectively.*

Example 3.2. Consider the following graphs:



$\iota\colon \Delta \hookrightarrow \Theta$ shall be given by $\iota\colon a_1 \mapsto b_1, a_2 \mapsto b_2$. Then $\mathbb{T} = (\Delta, \iota, \Theta)$ is a graph type of order $(2, 4)$.

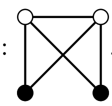
For given graph types $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$ and $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$ a *morphism* from $\mathbb{T}_1$ to $\mathbb{T}_2$ is pair (f, g) of graph homomorphisms such that $f\colon \Delta_1 \to \Delta_2$, $g\colon \Theta_1 \to \Theta_2$ and such that the following diagram commutes.

$$\begin{tikzcd} \Delta_2 \arrow[r, hook, "\iota_2"] & \Theta_2 \\ \Delta_1 \arrow[u, "f"] \arrow[r, hook, "\iota_1"] & \Theta_1 \arrow[u, "g"'] \end{tikzcd}$$

With this choice of morphisms graph types form a category. In particular, there is a natural concept of isomorphism between graph types.

Remark 3.3. When we depict a graph type $\mathbb{T} = (\Delta, \iota, \Theta)$, we prefer a more compact representation than in Example 3.2. We draw a picture of Θ. Then we mark $\iota(v)$ in black, for all $v \in V(\Delta)$. Clearly, this determines the graph type up to isomorphism. For instance, the graph type from Example 3.2 is depicted as follows:

$\mathbb{T}$:



.

In case it is not implied otherwise by the context, we always assume that the base graph Δ is an induced subgraph of Θ and that the embedding ι is the identical embedding.

A first observation about the category of graph types is:

Lemma 3.4. *Given natural numbers m and n such that $m \leqslant n$, there are just finitely many isomorphism classes of graph types of order (m, n).*

Proof. There are just finitely many (say, l) unlabeled graphs of order n. Moreover, every graph of order n accounts for at most $\binom{n}{m}$ graph types of order (m, n), up to isomorphism. Hence, there are at most $l \cdot \binom{n}{m}$ isomorphism classes of graph types of order (m, n). □

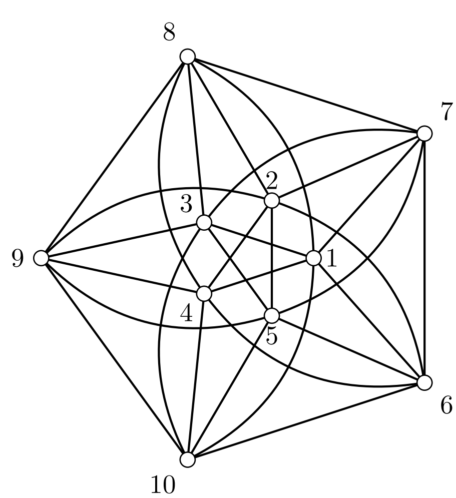
Definition 3.5. *Let $\mathbb{T} = (\Delta, \iota, \Theta)$ be a graph type, let Γ be a graph, and let $\kappa\colon \Delta \hookrightarrow \Gamma$ be an embedding. An embedding $\hat{\kappa}\colon \Theta \hookrightarrow \Gamma$ is called an* extension of κ along ι *if the following diagram commutes:*

$$\begin{tikzcd} \Theta \arrow[r, hook, "\hat{\kappa}"] & \Gamma \\ \Delta \arrow[u, hook, "\iota"] \arrow[ur, hook, "\kappa"'] & \end{tikzcd} .$$

The number of all extensions of κ along ι is denoted by $\#(\Gamma, \mathbb{T}, \kappa)$. If Δ embeds into Γ and if for every pair of embeddings $\kappa, \kappa'\colon \Delta \hookrightarrow \Gamma$ we have $\#(\Gamma, \mathbb{T}, \kappa) = \#(\Gamma, \mathbb{T}, \kappa')$, then this number is denoted by $\#(\Gamma, \mathbb{T})$. In case that Δ does not embed into Γ, we define $\#(\Gamma, \mathbb{T}) \coloneqq 0$. In both cases Γ is called $\mathbb{T}$-regular.

Example 3.6. Let us consider the complement graph Γ_1 of the Petersen graph:

Γ_1:



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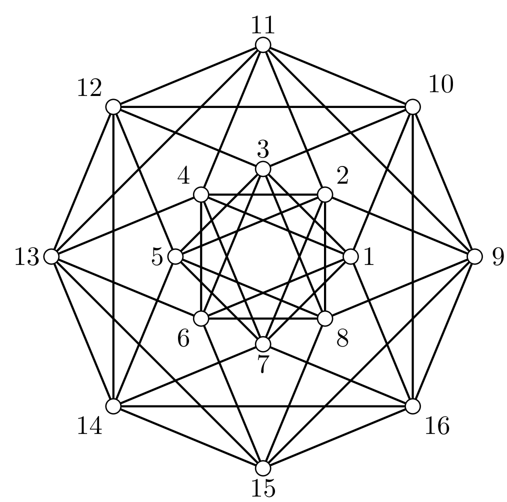
Take the graph type from Example 3.2. Let us fix an embedding $\kappa\colon \Delta \hookrightarrow \Gamma_1$, say, $\kappa\colon a_1 \mapsto 1, a_2 \mapsto 2$. The joint neighbors of 1 and 2 in Γ_1 are 4, 6, 7, and 8. These vertices induce a 4-cycle. Thus, there are exactly eight extensions of κ along ι, namely

$$\begin{aligned}
&\hat{\kappa}_1\colon b_1 \mapsto 1, b_2 \mapsto 2, b_3 \mapsto 4, b_4 \mapsto 6, && \hat{\kappa}_2\colon b_1 \mapsto 1, b_2 \mapsto 2, b_3 \mapsto 6, b_4 \mapsto 4,\\
&\hat{\kappa}_3\colon b_1 \mapsto 1, b_2 \mapsto 2, b_3 \mapsto 4, b_4 \mapsto 8, && \hat{\kappa}_4\colon b_1 \mapsto 1, b_2 \mapsto 2, b_3 \mapsto 8, b_4 \mapsto 4,\\
&\hat{\kappa}_5\colon b_1 \mapsto 1, b_2 \mapsto 2, b_3 \mapsto 6, b_4 \mapsto 7, && \hat{\kappa}_6\colon b_1 \mapsto 1, b_2 \mapsto 2, b_3 \mapsto 7, b_4 \mapsto 6,\\
&\hat{\kappa}_7\colon b_1 \mapsto 1, b_2 \mapsto 2, b_3 \mapsto 7, b_4 \mapsto 8, && \hat{\kappa}_8\colon b_1 \mapsto 1, b_2 \mapsto 2, b_3 \mapsto 8, b_4 \mapsto 7.
\end{aligned}$$

In particular, we observe that $\#(\Gamma_1, \mathbb{T}, \kappa) = 8$. Since the automorphism group of Γ_1 is a rank 3 group, this number does not depend on the particular choice of κ. In other words, Γ_1 is $\mathbb{T}$-regular with $\#(\Gamma_1, \mathbb{T}) = 8$.

Let us now consider the Shrikhande graph Γ_2:

$\Gamma_2\colon$



consider the embedding $\kappa\colon \Delta \hookrightarrow \Gamma_2$ given by $\kappa\colon a_1 \mapsto 1, a_2 \mapsto 9$. The joint neighbors of 1 and 9 are 10 and 16, respectively. These two vertices are connected by an edge. Hence there are exactly two extensions of κ along ι, namely

$$\hat{\kappa}_1\colon b_1 \mapsto 1, b_2 \mapsto 9, b_3 \mapsto 10, b_4 \mapsto 16, \qquad \hat{\kappa}_2\colon b_1 \mapsto 1, b_2 \mapsto 9, b_3 \mapsto 16, b_4 \mapsto 10.$$

Thus, we have that $\#(\Gamma_2, \mathbb{T}, \kappa) = 2$. However, if we consider $\kappa'\colon \Delta \hookrightarrow \Gamma_2$ given by $\kappa'\colon a_1 \mapsto 1, a_2 \mapsto 5$, then the two joint neighbors 3 and 7 of 1 and 5 are not connected by an edge in Γ_2. Consequently, there is no extension of κ' along ι in Γ_2. In other words, $\#(\Gamma_2, \mathbb{T}, \kappa') = 0$. It follows that the Shrikhande graph is not $\mathbb{T}$-regular.

Remarks 3.7.

- If $\mathbb{T} = (\Delta, \iota, \Theta)$ is a graph type of order $(0, n)$, and if Γ is an arbitrary graph, then Γ is $\mathbb{T}$-regular. In this case $\#(\Gamma, \mathbb{T})$ is equal to the number of embeddings of Θ into Γ.
- If $\mathbb{T} = (\Delta, \iota, \Theta)$ is a graph type of order (n, n), and if Γ is an arbitrary graph, then Γ is $\mathbb{T}$-regular. In this case $\#(\Gamma, \mathbb{T}) \in \{0, 1\}$. It is 1 if Γ has a subgraph isomorphic to Θ and 0 otherwise.
- If $\mathbb{T}_1$ and $\mathbb{T}_2$ are isomorphic graph types, then every graph Γ that is $\mathbb{T}_1$-regular, is also $\mathbb{T}_2$-regular. Moreover, in this case we have $\#(\Gamma, \mathbb{T}_1) = \#(\Gamma, \mathbb{T}_2)$.
- A concept equivalent to $\mathbb{T}$-regularity, but in the category of complete colored graphs, was introduced and studied by S. Evdokimov and I. Ponomarenko in [15] in relation with the t-vertex condition for association schemes.

A simple but important observation is:

Lemma 3.8. *Let Γ be $\mathbb{T}$-regular for $\mathbb{T} = (\Delta, \iota, \Theta)$. Then $\overline{\Gamma}$ is $\overline{\mathbb{T}}$-regular, where $\overline{\mathbb{T}} \coloneqq (\overline{\Delta}, \iota, \overline{\Theta})$.*

Proof. Clear. $\square$

Definition 3.9. *Let $m \leqslant n$ be two natural numbers. We say that a graph Γ is*

- *$({}_{=}m, {}_{=}n)$-regular if it is $\mathbb{T}$-regular for all graph types $\mathbb{T}$ of order (m, n).*
- *$({}_{=}m, n)$-regular if it is $({}_{=}m, {}_{=}l)$-regular for all $m \leqslant l \leqslant n$,*
- *$(m, {}_{=}n)$-regular if it is $({}_{=}k, {}_{=}n)$-regular for all $k \leqslant m$,*
- *(m, n)-regular if it is $({}_{=}k, n)$-regular for all $k \leqslant m$.*

The concept of (m, n)-regularity is a combinatorial approximation of the notion of m-homogeneity. Recall:

Definition 3.10. *A graph Γ is called m-*homogeneous *if every isomorphism between induced subgraphs of order at most m extends to an automorphism of Γ. It is called* homogeneous *if every isomorphism between finite induced subgraphs extends to an automorphism.*

It is not hard to see that for every graph Γ of order n we have that m-homogeneity is equivalent to (m, n)-regularity.

Lemma 3.11. *A graph Γ satisfies the t-vertex condition if and only if it is $(2, t)$-regular.*

Proof. Clear. $\square$

3.2. Composition of graph types.

Definition 3.12. *Let $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$ and $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$ be graph types, and let $e\colon \Delta_2 \hookrightarrow \Theta_1$.*

Let Λ be a graph, $\lambda_1\colon \Theta_1 \hookrightarrow \Lambda$, $\lambda_2\colon \Theta_2 \hookrightarrow \Lambda$ such that the following is a pushout square (see Definition 2.1):

$$\begin{tikzcd} \Theta_2 \arrow[r, hook, "\lambda_2"] & \Lambda \\ \Delta_2 \arrow[u, hook, "\iota_2"] \arrow[r, hook, "e"] & \Theta_1 \arrow[u, hook, "\lambda_1"'] \end{tikzcd} .$$

Then the graph type $(\Delta_1, \lambda_1 \circ \iota_1, \Lambda)$ is called the free sum of $\mathbb{T}_1$ and $\mathbb{T}_2$ with respect to e. *It is denoted by $\mathbb{T}_1 \oplus_e \mathbb{T}_2$.*

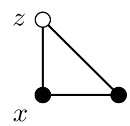
Remark 3.13. The following picture illustrates the construction of a free sum of types:

$\mathbb{T}_1$ $\quad$ $\mathbb{T}_2$ $\quad$ $\mathbb{T}_1 \oplus_e \mathbb{T}_2$

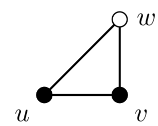
In the picture on the left we see $\mathbb{T}_1$. In the picture in the middle we see $\mathbb{T}_2$, and how Δ_2 is embedded by e into Θ_1. In the picture on the right we see how Θ_1 and Θ_2 are glued together along Δ_2, to obtain Λ. Now, Δ_1 still naturally embeds into Λ and we obtain the free sum of the types with respect to e.

Example 3.14. Let us consider the graph types $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$ and $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$ given by the following pictures:

$\mathbb{T}_1\colon$



$\mathbb{T}_2\colon$



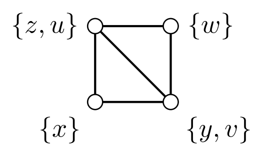
.

Let $e\colon \Delta_2 \hookrightarrow \Theta_1$ be given by

$$e\colon u \mapsto z, \quad v \mapsto y.$$

To obtain the free sum of $\mathbb{T}_1$ and $\mathbb{T}_2$ with respect to e, we have to take the disjoint union of Θ_1 and Θ_2, and to identify u with z and v with y. We end up with the graph Λ in the following picture:

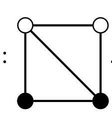
$\Lambda\colon$



.

Finally, we have $\mathbb{T}_1 \oplus_e \mathbb{T}_2 = (\Delta_1, \iota, \Lambda)$, where $\iota\colon \Delta_1 \hookrightarrow \Lambda$ is given by $\iota\colon x \mapsto \{x\}, y \mapsto \{y, v\}$. If we forget about the labelling, then we obtain:

$\mathbb{T}_1 \oplus_e \mathbb{T}_2\colon$



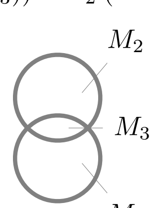
.

3.3. Decomposition of graph types.

Definition 3.15. *Let $\mathbb{T}, \mathbb{T}_2$ be graph types. We say that $\mathbb{T}$ is* $\mathbb{T}_2$-reducible *if $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \mathbb{T}_2$ for some $\mathbb{T}_1 \ncong \mathbb{T}$ and for some e.*

Remark 3.16. With the notions from above $\mathbb{T} = (\Delta, \iota, \Theta)$ is $\mathbb{T}_2$-reducible if and only if the set $V(\Theta)$ can be decomposed as a disjoint union of subsets M_1, M_2, and M_3, such that

(1) $\mathrm{im}(\iota) \subseteq M_1 \cup M_3$,
(2) $M_2 \neq \varnothing$
(3) there are no edges in Θ between vertices from M_1 and vertices from M_2,
(4) $\mathbb{T}_2' := (\Theta(M_3), \iota', \Theta(M_2 \cup M_3)) \cong \mathbb{T}_2$ (here ι' denotes the identical embedding).



In this case, we have $\mathbb{T}_1 = (\Delta, \iota, \Theta(M_1 \cup M_3))$ and $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \mathbb{T}_2'$, where e is the identical embedding of M_3 into $M_1 \cup M_3$.

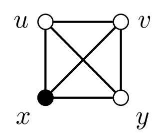
Remark 3.17. In Example 3.14, $\mathbb{T}_1 \oplus_e \mathbb{T}_2$ is $\mathbb{T}_2$-reducible. Beware that there are some degenerate forms of reducibility that we need to take care of: Every graph type $\mathbb{T} = (\Delta, \iota, \Theta)$ is $\mathbb{T}$-reducible, since $\mathbb{T} \cong \mathbb{T}_\Delta \oplus_{1_\Delta} \mathbb{T}$, where $\mathbb{T}_\Delta = (\Delta, 1_\Delta, \Delta)$ (here 1_Δ denotes the identity on $V(\Delta)$). In general, whenever $\mathbb{T} \cong \mathbb{T}' \oplus_e \mathbb{T}$ for some $\mathbb{T}'$ and some e, then $\mathbb{T}' \cong \mathbb{T}_\Delta$ and e is an isomorphism of Δ to the base-graph of $\mathbb{T}'$.

Definition 3.18. *A graph type* $\mathbb{T}$ *is called* (m,n)-irreducible *if whenever* $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \mathbb{T}_2$ *for a graph type* $\mathbb{T}_1$ *and a graph type* $\mathbb{T}_2$*, where* $\mathbb{T}_2$ *is of order* (k,l) *with* $k \leqslant m$ *and* $l \leqslant n$*, then we already have* $\mathbb{T} \cong \mathbb{T}_1$ *or* $\mathbb{T} \cong \mathbb{T}_2$*. Otherwise, we call* $\mathbb{T}$ (m,n)-reducible.

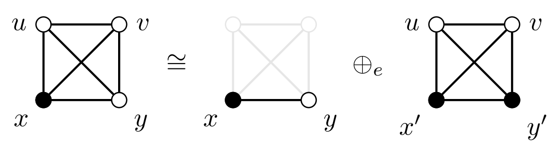
Lemma 3.19. *A graph type* $\mathbb{T}$ *is* (m,n)*-reducible if and only if it is* $\mathbb{T}'$*-reducible, for some graph type* $\mathbb{T}'$ *of order* (k,l)*, where* $k \leqslant m$ *and* $l \leqslant n$*, such that* $\mathbb{T}' \ncong \mathbb{T}$*.*

Proof. Clear. □

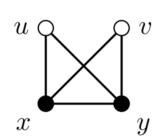
Example 3.20. Consider the following graph type of order $(1,4)$:



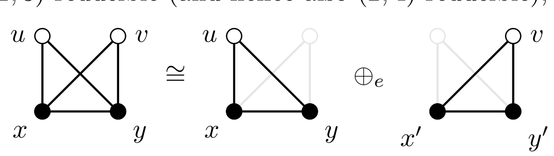
It is $(2,4)$-reducible, since



Moreover, the graph type



of order $(2,4)$ is $(2,3)$-reducible (and hence also $(2,4)$-reducible), because

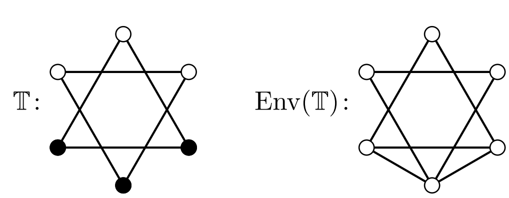


In both examples, $e\colon x' \mapsto x, y' \mapsto y$.

In the following it is our goal to link the concept of (m,n)-reducibility to classical graph-theoretical terms.

Definition 3.21. *Let* $\mathbb{T} = (\Delta, \iota, \Theta)$ *be a graph type. Let* $S \subseteq V(\Theta)$ *be the image of* ι*. Then we define the enveloping graph of* $\mathbb{T}$ *to be the graph with vertex set* $V(\Theta)$ *and with edge set* $E(\Theta) \cup \binom{S}{2}$*. The enveloping graph of* $\mathbb{T}$ *will be denoted by* $\operatorname{Env}(\mathbb{T})$*.*

Example 3.22.



Recall that a graph Γ is called *l-decomposable* if there exists an l-element set of vertices whose deletion makes the graph disconnected. Moreover, Γ is called $(n+1)$-*connected* if it is l-indecomposable, for all $l \in \{0,\dots,n\}$. Note that our definition of $(n+1)$-connectedness slightly deviates from the classical one (cf. e.g. [23, p. 45]). In particular, the usual definition allows a graph of order n to be $n-1$-connected, at most. Of course, such highly connected graphs are exactly the complete graphs. For technical convenience, in this paper the complete graphs are k-connected for every $k \in \mathbb{N}$.

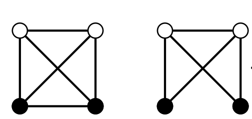
Lemma 3.23. *A graph type $\mathbb{T} = (\Delta, \iota, \Theta)$ of order $(m_1, n+1)$ is (m_2, n)-irreducible if and only if* $\mathrm{Env}(\mathbb{T})$ *is (m_2+1)-connected.*

Proof. "$\Leftarrow$:" Suppose that $\mathbb{T} = (\Delta, \iota, \Theta)$ is (m_2, n)-reducible. That is, $\mathbb{T}$ is $\mathbb{T}'$-reducible for some graph type $\mathbb{T}'$ of order (k, l), where $k \leqslant m_2$ and $l \leqslant n$, such that $\mathbb{T}' \ncong \mathbb{T}$ (see Lemma 3.19). Let us fix such a graph type $\mathbb{T}'$. Then, as described in Remark 3.16, we may decompose $V(\Theta)$ into a disjoint union of subsets M_1, M_2, M_3, such that $\mathrm{im}(\iota) \subseteq M_1 \cup M_3$, $M_2 \neq \varnothing$, there are no edges in Θ between vertices from M_1 and M_2, and such that $\mathbb{T}' \cong \mathbb{T}'' := (\Theta(M_3), \iota'', \Theta(M_2 \cup M_3))$, where ι'' is the identical embedding. Since $|M_1|+|M_2|+|M_3| = |V(\Theta)| = n+1$ and since $|M_2|+|M_3| = l < n+1$, we conclude that M_1 is non-empty.

Now we observe that in $\mathrm{Env}(\mathbb{T})$ there are still no edges between vertices from M_1 and vertices from M_2, since only edges between vertices in $\mathrm{im}(\iota) \subseteq M_1 \cup M_3$ are added in the course of the construction of $\mathrm{Env}(\mathbb{T})$. Thus, removing the k vertices of M_3 from $\mathrm{Env}(\mathbb{T})$ makes the remainder disconnected. It follows that $\mathrm{Env}(\mathbb{T})$ is k-decomposable. Consequently, $\mathrm{Env}(\mathbb{T})$ is not (m_2+1)-connected.

"$\Rightarrow$:" Suppose that $\hat{\Theta} := \mathrm{Env}(\mathbb{T})$ is not (m_2+1)-connected. Then there exists some $k \leqslant m_2$ such that $\hat{\Theta}$ is k-decomposable. Thus, there exists pairwise disjoint subsets M_1, M_2, M_3 of $V(\hat{\Theta})$, such that $M_1 \cup M_2 \cup M_3 = V(\hat{\Theta})$, $M_1, M_2 \neq \varnothing$, $|M_3| = k$, and such that there are no edges in $\hat{\Theta}$ between vertices from M_1 and vertices from M_2. Thus, if $M \subseteq V(\Theta)$ denotes the image of ι, then we have $M \subseteq M_1 \cup M_3$ or $M \subseteq M_2 \cup M_3$. Without loss of generality assume that $M \subseteq M_1 \cup M_3$. Then, with $\mathbb{T}' = (\Theta(M_3), \iota', \Theta(M_2 \cup M_3))$ (where ι' is the identical embedding), we obtain that $\mathbb{T}$ is $\mathbb{T}'$-reducible (see Remark 3.16). By construction we have that $\mathbb{T}'$ is of order (k, l), where $l = |M_2 \cup M_3| \leqslant n$. Thus $\mathbb{T}' \ncong \mathbb{T}$. Consequently, $\mathbb{T}$ is (m_2, n)-reducible (see Lemma 3.19). □

Example 3.24. The only 3-connected graph of order 4 is the complete graph K_4. Thus, the only $(2,3)$-irreducible graph types of order $(2,4)$ are:



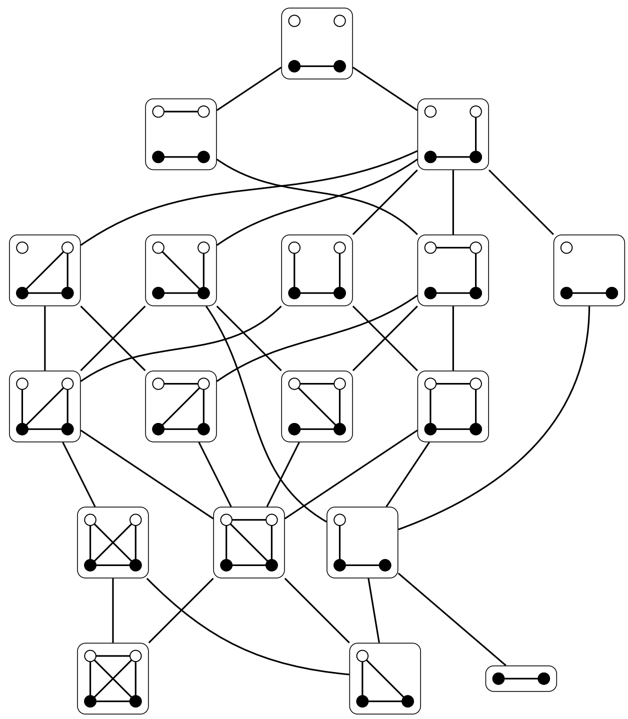
3.4. The dominance quasiorder of graph types.

Definition 3.25. *Let $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$, $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$ be graph types. Then we define $\mathbb{T}_1 \preceq \mathbb{T}_2$ ($\mathbb{T}_2$* dominates *$\mathbb{T}_1$) if there exists a morphism $(f, g)\colon \mathbb{T}_2 \to \mathbb{T}_1$ such that $f\colon \Delta_2 \to \Delta_1$ is an isomorphism and such that $g\colon \Theta_2 \to \Theta_1$ is surjective on vertices. If, in addition, g is not an isomorphism, then we write $\mathbb{T}_1 \prec \mathbb{T}_2$.*

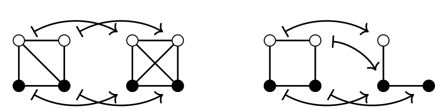
Lemma 3.26. *The relation $\preceq$ defines a quasiorder on graph types. For finite graph types $\mathbb{T}_1$, $\mathbb{T}_2$ we have $\mathbb{T}_1 \cong \mathbb{T}_2$ if and only if $\mathbb{T}_1 \preceq \mathbb{T}_2$ and $\mathbb{T}_2 \preceq \mathbb{T}_1$.*

Proof. Clear. □

Example 3.27. In the picture below the order diagram of the domination quasiorder of all graph types of order $(2, t)$ for $2 \leqslant t \leqslant 4$ with base graph Δ isomorphic to K_2 can be found (in this diagram, a graph type $\mathbb{T}_2$ dominates a graph type $\mathbb{T}_1$ iff $\mathbb{T}_2$ can be reached by an upwards-sloped path starting from $\mathbb{T}_1$).



Two typical examples of covering pairs in this diagram are given below together with the morphisms mapping the dominating types to the dominated ones (indicated by arrows $\mapsto$). Each time, the two arrows between the black vertices determine the isomorphism f between the base graphs and all four arrows together determine the surjective homomorphism g between the underlying graphs of the types.



3.5. The type counting lemma. Now all preparations are made so that we can come to the central auxiliary result of this paper from which all other results depend crucially. It is the place where algebraic graph theory meets category theory. Its proof critically depends on the universal property of amalgamated free sums.

Lemma 3.28 (Type counting lemma). *Given a graph* Γ *and graph types* $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$ *and* $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$*. Let* $e\colon \Delta_2 \hookrightarrow \Theta_1$ *be an embedding. Then* Γ *is* $\mathbb{T}_1 \oplus_e \mathbb{T}_2$*-regular if*

(1) Γ *is* $\mathbb{T}_1$*-regular,*
(2) Γ *is* $\mathbb{T}_2$*-regular, and*
(3) Γ *is* $\mathbb{T}$*-regular for every* $\mathbb{T} \prec \mathbb{T}_1 \oplus_e \mathbb{T}_2$*.*

Before coming to the proof of the type counting lemma, we need to prepare a few tools:

Definition 3.29. *Let* Θ *and* Γ *be graphs, and let* $h\colon \Theta \to \Gamma$ *be a graph homomorphism. By* Θ/h *we denote the graph whose vertex set is* $V(\Theta)/\ker h$ *and whose edge set is* *given by*

$$E(\Theta/h) \coloneqq \{\{M_1, M_2\} \mid M_1, M_2 \in V(\Theta)/h,$$
$$\{h(m_1), h(m_2)\} \in E(\Gamma), \textit{for some } m_1 \in M_1, \textit{ and } m_2 \in M_2\}.$$

Lemma 3.30. *Let $h\colon \Theta \to \Gamma$ be a graph homomorphism. Then the natural mapping $\chi_h\colon V(\Theta) \to V(\Theta/h)$ defined by $\chi_h\colon v \mapsto [v]_{\ker h}$ is a surjective graph homomorphism to Θ/h. Moreover, there is a unique graph embedding $\tilde{h}$ from Θ/h to Γ such that $h = \tilde{h} \circ \chi_h$.*

Proof. Straightforward. □

Now we are ready to prove the type counting lemma. The reader is invited to study Example 3.31 in parallel.

Proof of Lemma 3.28. Let us start by fixing some notations. Suppose $\mathbb{T}_1 \oplus_e \mathbb{T}_2 = (\Delta_1, \iota, \Theta)$. Let λ_1, λ_2 be given such that the following is a pushout square:

$$\begin{tikzcd} \Theta_2 \arrow[r, hook, "\lambda_2"] & \Theta \\ \Delta_2 \arrow[u, hook, "\iota_2"] \arrow[r, hook, "e"] & \Theta_1 \arrow[u, hook, "\lambda_1"'] \end{tikzcd}$$

and such that $\iota = \lambda_1 \circ \iota_1$.

For every compatible cocone (μ_1, μ_2) of (e, ι_2), let us denote by $h_{\mu_1,\mu_2}\colon \Theta \to \Upsilon$ the unique homomorphism that makes the following diagram commutative:

$$\begin{tikzcd} & & \Upsilon \\ \Theta_2 \arrow[r, hook, "\lambda_2"] \arrow[urr, bend left, "\mu_2"] & \Theta \arrow[ur, dashed, "h_{\mu_1,\mu_2}"] & \\ \Delta_2 \arrow[u, hook, "\iota_2"] \arrow[r, hook, "e"] & \Theta_1 . \arrow[u, hook, "\lambda_1"'] \arrow[uur, bend right, "\mu_1"'] & \end{tikzcd}$$

By Lemma 3.30 we have that every h_{μ_1,μ_2} decomposes uniquely into the natural homomorphism $\chi_{\mu_1,\mu_2}\colon \Theta \to \Theta/h_{\mu_1,\mu_2}$ and an embedding $\tilde{h}_{\mu_1,\mu_2}\colon \Theta/h_{\mu_1,\mu_2} \hookrightarrow \Upsilon$.

Let us define $\mathbb{T}_{\mu_1,\mu_2} \coloneqq (\Delta_1, \chi_{\mu_1,\mu_2} \circ \iota, \Theta/h_{\mu_1,\mu_2})$. We claim that if μ_1 and μ_2 are embeddings, then $\mathbb{T}_{\mu_1,\mu_2}$ is a graph type, that is, $\chi_{\mu_1,\mu_2} \circ \iota$ is an embedding. To see this, observe that

$$\tilde{h}_{\mu_1,\mu_2} \circ \chi_{\mu_1,\mu_2} \circ \iota = h_{\mu_1,\mu_2} \circ \iota = h_{\mu_1,\mu_2} \circ \lambda_1 \circ \iota_1 = \mu_1 \circ \iota_1.$$

Thus, since $\mu_1 \circ \iota_1$ and $\tilde{h}_{\mu_1,\mu_2}$ are embeddings, it follows that so is $\chi_{\mu_1,\mu_2} \circ \iota$. Note that $\mathbb{T}_1 \oplus_e \mathbb{T}_2$ dominates $\mathbb{T}_{\mu_1,\mu_2}$, since $(1_{\Delta_1}, \chi_{\mu_1,\mu_2})\colon \mathbb{T}_1 \oplus_e \mathbb{T}_2 \to \mathbb{T}_{\mu_1,\mu_2}$, and since χ_{μ_1,μ_2} is surjective:

$$\begin{tikzcd} \Delta_1 \arrow[r, hook, "\chi_{\mu_1,\mu_2} \circ \iota"] & \Theta/h_{\mu_1,\mu_2} \\ \Delta_1 \arrow[u, "1_{\Delta_1}"] \arrow[r, hook, "\iota"] & \Theta . \arrow[u, two heads, "\chi_{\mu_1,\mu_2}"'] \end{tikzcd}$$

Let us collect the graph types obtained in this way in a set $\mathcal{T}$:

$$\mathcal{T} \coloneqq \{\mathbb{T}_{\mu_1,\mu_2} \mid (\mu_1, \mu_2) \text{ is a compatible cocone of } (e, \iota_2),\ \mu_1, \mu_2 \text{ are embeddings}\}.$$

Note that in the definition of $\mathcal{T}$ the compatible cocones (μ_1, μ_2) of (e, ι_2) are *not* restricted to a fixed codomain Υ. In particular they form a proper class. So we need to show that $\mathcal{T}$ is well-defined. Next we will prove the following claims:

(A) $\mathcal{T}$ is a finite set.

(B) Exactly one element of $\mathcal{T}$, namely $\mathbb{T}_{\lambda_1,\lambda_2}$, is isomorphic to $\mathbb{T}_1 \oplus_e \mathbb{T}_2$. In particular, all other elements of $\mathcal{T}$ are strictly dominated by $\mathbb{T}_1 \oplus_e \mathbb{T}_2$.

About (A): Recall that for every compatible cocone (μ_1,μ_2) of (e,ι_2) we have $\mathbb{T}_{\mu_1,\mu_2} = (\Delta_1, \chi_{\mu_1,\mu_2} \circ \iota, \Theta/h_{\mu_1,\mu_2})$. Let us analyze $\Theta/h_{\mu_1,\mu_2}$. According to Definition 3.29 its vertex set is $V(\Theta)/\ker h_{\mu_1,\mu_2}$. Thus, the number of possible quotients $\Theta/h_{\mu_1,\mu_2}$ is bounded from above by $B_n \cdot 2^{\binom{n}{2}}$, where $n = |V(\Theta)|$ and where B_n denotes the n-th Bell number. Since $\chi_{\mu_1,\mu_2} \circ \iota \colon \Delta_1 \to \Theta/h_{\mu_1,\mu_2}$ is an embedding, it is in particular a function. Thus, the cardinality of $\mathcal{T}$ can be estimated from above by $B_n \cdot 2^{\binom{n}{2}} \cdot n^m$, where $m = |V(\Delta_1)|$.

About (B): First we note that $\mathbb{T}_{\lambda_1,\lambda_2} \in \mathcal{T}$, since λ_1 and λ_2 are embeddings and since (λ_1,λ_2) is a limiting cocone for (e,ι_2). Clearly, $h_{\lambda_1,\lambda_2} = 1_\Theta$. So $\ker h_{\lambda_1,\lambda_2}$ is the equality relation and $\Theta/h_{\lambda_1,\lambda_2}$ is obtained from Θ by renaming each vertex v to the singleton class $\{v\} = [v]_{\ker h_{\lambda_1,\lambda_2}}$. In particular, $\chi_{\lambda_1,\lambda_2} \colon \Theta \to \Theta/h_{\lambda_1,\lambda_2}$ is an isomorphism. Thus $(1_{\Delta_1}, \chi_{\lambda_1,\lambda_2}) \colon \mathbb{T}_1 \oplus_e \mathbb{T}_2 \to \mathbb{T}_{\lambda_1,\lambda_2}$ is an isomorphism, too. It remains to show that $\mathbb{T}_{\lambda_1,\lambda_2}$ is the only element of $\mathcal{T}$ that is isomorphic to $\mathbb{T}_1 \oplus_e \mathbb{T}_2$: Suppose that $\mathbb{T}_{\mu_1,\mu_2} = (\Delta_1, \chi_{\mu_1,\mu_2} \circ \iota, \Theta/h_{\mu_1,\mu_2})$ is an element of $\mathcal{T}$ isomorphic to $\mathbb{T}_1 \oplus_e \mathbb{T}_2$. Then in particular, $\Theta/h_{\mu_1,\mu_2}$ is isomorphic to Θ. Since $|V(\Theta/h_{\mu_1,\mu_2})| = |V(\Theta)|$, we have that $\ker h_{\mu_1,\mu_2}$ is the equality relation. Thus $V(\Theta/h_{\lambda_1,\lambda_2}) = V(\Theta/h_{\mu_1,\mu_2})$, and $\chi_{\lambda_1,\lambda_2}$ and χ_{μ_1,μ_2} coincide as functions. Moreover, since $|E(\Theta)| = |E(\Theta/h_{\mu_1,\mu_2})|$, we obtain, that χ_{μ_1,μ_2} is an isomorphism. Consequently, $\mathbb{T}_{\mu_1,\mu_2} = \mathbb{T}_{\lambda_1,\lambda_2}$, which proves Claim (B).

At this point it is essential to notice that $\mathcal{T}$ only depends on $\mathbb{T}_1$, $\mathbb{T}_2$, and e, but not on Γ. Let us fix an embedding $\kappa \colon \Delta_1 \to \Gamma$. Our goal is to determine $\#(\Gamma, \mathbb{T}_1 \oplus_e \mathbb{T}_2, \kappa)$. However, we are not able to do so directly. Instead we are going to prove the following identity:

$$\#(\Gamma,\mathbb{T}_1)\cdot\#(\Gamma,\mathbb{T}_2) = \sum_{\mathbb{T}\in\mathcal{T}} \#(\Gamma,\mathbb{T},\kappa). \tag{2}$$

Note now that by the assumption and by (B), we have that Γ is $\mathbb{T}$-regular for all graph types $\mathbb{T} \in \mathcal{T} \setminus \{\mathbb{T}_{\lambda_1,\lambda_2}\}$. Thus, from (2) we obtain that

$$\#(\Gamma,\mathbb{T}_1\oplus_e\mathbb{T}_2,\kappa) = \#(\Gamma,\mathbb{T}_{\lambda_1,\lambda_2},\kappa) = \#(\Gamma,\mathbb{T}_1)\cdot\#(\Gamma,\mathbb{T}_2) - \sum_{\mathbb{T}\in\mathcal{T}\setminus\{\mathbb{T}_{\lambda_1,\lambda_2}\}} \#(\Gamma,\mathbb{T}),$$

which obviously does not depend on κ. Thus, once we show identity (2), then we are done. The rest of the proof is dedicated to the task of showing (2).

Let $\mu_1 \colon \Theta_1 \hookrightarrow \Gamma$, $\mu_2 \colon \Theta_2 \hookrightarrow \Gamma$. Then (μ_1,μ_2) is called a *κ-compatible pair* if

(a) μ_1 extends κ along ι_1 (i.e. $\kappa = \mu_1 \circ \iota_1$),

(b) μ_2 extends $\mu_1 \circ e$ along ι_2 (i.e. $\mu_1 \circ e = \mu_2 \circ \iota_2$).

Clearly, every κ-compatible pair is a compatible cocone for (e,ι_2). Thus, to every κ-compatible pair (μ_1,μ_2) we can associate the graph type $\mathbb{T}_{\mu_1,\mu_2}$ from $\mathcal{T}$. Let us define

$$\begin{aligned} P_\kappa &\coloneqq \{(\mu_1,\mu_2) \mid (\mu_1,\mu_2) \text{ is a } \kappa\text{-compatible pair}\},\\ P_{\kappa,\mathbb{T}} &\coloneqq \{(\mu_1,\mu_2) \mid (\mu_1,\mu_2) \in P_\kappa, \mathbb{T}_{\mu_1,\mu_2} = \mathbb{T}\}. \end{aligned}$$

Then, by definition we have

$$\#(\Gamma,\mathbb{T}_1)\cdot\#(\Gamma,\mathbb{T}_2) = |P_\kappa| = \sum_{\mathbb{T}\in\mathcal{T}} |P_{\kappa,\mathbb{T}}|. \tag{3}$$

In the following we are going to show:

$$\forall \mathbb{T}\in\mathcal{T} : |P_{\kappa,\mathbb{T}}| = \#(\Gamma,\mathbb{T},\kappa). \tag{4}$$

Let $\mathbb{T} \in \mathcal{T}$. Then there exists a compatible cocone (ν_1, ν_2) of (e, ι_2), such that $\mathbb{T} = \mathbb{T}_{\nu_1,\nu_2} = (\Delta_1, \chi_{\nu_1,\nu_2} \circ \iota, \Theta/h_{\nu_1,\nu_2})$ and such that both, ν_1 and ν_2 are embeddings.

Let $\hat{\kappa}\colon \Theta/h_{\nu_1,\nu_2} \hookrightarrow \Gamma$ be an extension of κ along $\chi_{\nu_1,\nu_2} \circ \iota$ (i.e. $\kappa = \hat{\kappa} \circ \chi_{\nu_1,\nu_2} \circ \iota$). Define $\mu_1^{[\hat{\kappa}]}\colon \Theta_1 \hookrightarrow \Gamma$ by $\mu_1^{[\hat{\kappa}]} \coloneqq \hat{\kappa}\circ\chi_{\nu_1,\nu_2}\circ\lambda_1$ and $\mu_2^{[\hat{\kappa}]}\colon \Theta_2 \hookrightarrow \Gamma$ by $\mu_2^{[\hat{\kappa}]} \coloneqq \hat{\kappa}\circ\chi_{\nu_1,\nu_2}\circ\lambda_2$.

$$\begin{tikzcd}
\Theta_2 \arrow[r, hook, "\lambda_2"] \arrow[rrr, hook, dashed, bend left, "\mu_2^{[\hat{\kappa}]}"] & \Theta \arrow[r, two heads, "\chi_{\nu_1,\nu_2}"] & \Theta/h_{\nu_1,\nu_2} \arrow[r, hook, "\hat{\kappa}"] & \Gamma \\
\Delta_2 \arrow[u, hook, "\iota_2"] \arrow[r, hook, "e"] & \Theta_1 \arrow[u, hook, "\lambda_1"] \arrow[urr, hook, dashed, "\mu_1^{[\hat{\kappa}]}"'] & & \Delta_1. \arrow[u, hook, "\kappa"'] \arrow[ll, hook', "\iota_1"]
\end{tikzcd}$$

We claim that $(\mu_1^{[\hat{\kappa}]}, \mu_2^{[\hat{\kappa}]})$ is a κ-compatible pair. First we note that $\mu_1^{[\hat{\kappa}]}$ is an embedding, since $\tilde{h}_{\nu_1,\nu_2} \circ (\chi_{\nu_1,\nu_2} \circ \lambda_1) = \nu_1$ is an embedding and $\mu_2^{[\hat{\kappa}]}$ is an embedding, since $\tilde{h}_{\nu_1,\nu_2} \circ (\chi_{\nu_1,\nu_2} \circ \lambda_2) = \nu_2$ is an embedding. Next we compute that

$$\mu_1^{[\hat{\kappa}]} \circ \iota_1 = \hat{\kappa} \circ \chi_{\nu_1,\nu_2} \circ \lambda_1 \circ \iota_1 = \hat{\kappa} \circ \chi_{\nu_1,\nu_2} \circ \iota = \kappa,$$

thus $\mu_1^{[\hat{\kappa}]}$ extends κ along ι_1, and

$$\mu_1^{[\hat{\kappa}]} \circ e = \hat{\kappa} \circ \chi_{\nu_1,\nu_2} \circ \lambda_1 \circ e = \hat{\kappa} \circ \chi_{\nu_1,\nu_2} \circ \lambda_2 \circ \iota_2 = \mu_2^{[\hat{\kappa}]} \circ \iota_2,$$

thus $\mu_2^{[\hat{\kappa}]}$ extends $\mu_1^{[\hat{\kappa}]} \circ e$ along ι_2 and the claim is proved.

The next step is to show that the assignment $\hat{\kappa} \mapsto (\mu_1^{[\hat{\kappa}]}, \mu_2^{[\hat{\kappa}]})$ is a bijection:

"injectivity": Let $\hat{\kappa}_1$ and $\hat{\kappa}_2$ be extensions of κ along $\chi_{\nu_1,\nu_2} \circ \iota$ and suppose that $(\mu_1^{[\hat{\kappa}_1]}, \mu_2^{[\hat{\kappa}_1]}) = (\mu_1^{[\hat{\kappa}_2]}, \mu_2^{[\hat{\kappa}_2]})$. Note that $\hat{\kappa}_1 \circ \chi_{\nu_1,\nu_2}$ is the unique mediating morphism from the limiting cocone (λ_1, λ_2) to $(\mu_1^{[\hat{\kappa}_1]}, \mu_2^{[\hat{\kappa}_1]})$, and that $\hat{\kappa}_2 \circ \chi_{\nu_1,\nu_2}$ is the unique mediating morphism from (λ_1, λ_2) to $(\mu_1^{[\hat{\kappa}_2]}, \mu_2^{[\hat{\kappa}_2]})$. Since $(\mu_1^{[\hat{\kappa}_1]}, \mu_2^{[\hat{\kappa}_1]}) = (\mu_1^{[\hat{\kappa}_2]}, \mu_2^{[\hat{\kappa}_2]})$, we have $\hat{\kappa}_1 \circ \chi_{\nu_1,\nu_2} = \hat{\kappa}_2 \circ \chi_{\nu_1,\nu_2}$. Since χ_{ν_1,ν_2} is surjective, we conclude $\hat{\kappa}_1 = \hat{\kappa}_2$.

"surjectivity": Let (μ_1, μ_2) be any κ-compatible pair such that $\mathbb{T}_{\mu_1,\mu_2} = \mathbb{T} = \mathbb{T}_{\nu_1,\nu_2}$. In particular, $\Theta/h_{\mu_1,\mu_2} = \Theta/h_{\nu_1,\nu_2}$, and thus also $\chi_{\mu_1,\mu_2} = \chi_{\nu_1,\nu_2}$. We claim that $\tilde{h}_{\mu_1,\mu_2}$ is an extension of κ along $\chi_{\nu_1,\nu_2} \circ \iota$. Indeed, we may compute

$$\tilde{h}_{\mu_1,\mu_2} \circ \chi_{\nu_1,\nu_2} \circ \iota = \tilde{h}_{\mu_1,\mu_2} \circ \chi_{\mu_1,\mu_2} \circ \iota = h_{\mu_1,\mu_2} \circ \iota = h_{\mu_1,\mu_2} \circ \lambda_1 \circ \iota_1 = \mu_1 \circ \iota_1 = \kappa.$$

It remains to show that $\tilde{h}_{\mu_1,\mu_2}$ is really a preimage of (μ_1, μ_2) under our correspondence. For this we compute

$$\tilde{h}_{\mu_1,\mu_2} \circ \chi_{\nu_1,\nu_2} \circ \lambda_1 = \tilde{h}_{\mu_1,\mu_2} \circ \chi_{\mu_1,\mu_2} \circ \lambda_1 = h_{\mu_1,\mu_2} \circ \lambda_1 = \mu_1,$$

and

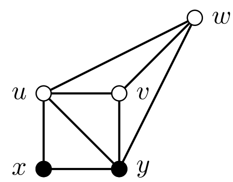
$$\tilde{h}_{\mu_1,\mu_2} \circ \chi_{\nu_1,\nu_2} \circ \lambda_2 = \tilde{h}_{\mu_1,\mu_2} \circ \chi_{\mu_1,\mu_2} \circ \lambda_2 = h_{\mu_1,\mu_2} \circ \lambda_2 = \mu_2.$$

This finishes the proof of (4). Now, identity (2) is a direct consequence of (3) and (4). □

The type counting lemma is the technical backbone of all further results in this paper. Alas, while the language of category theory used in the proof is convenient for assuring correctness, it is not ideal to illustrate the combinatorial intuitions behind the proof. To amend this situation, we elaborate on an extended example:

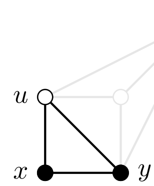
Example 3.31. Suppose, we are given a $(2,4)$-regular graph Γ. In other words, Γ is strongly regular and satisfies the 4-vertex condition. Let us illustrate the idea behind the proof of the type counting lemma by analyzing the graph type $\mathbb{T} = (\Delta, \iota, \Theta)$

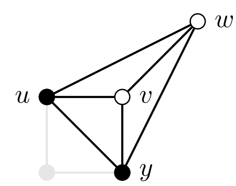
given by the following picture (here $\Delta = \Theta(\{x,y\})$, and $\iota\colon \Delta \hookrightarrow \Theta$ is the identical embedding):

$\mathbb{T}\colon$ 

.

Our first observation is that $\mathbb{T}$ is $(2,4)$-reducible. In particular we have $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \mathbb{T}_2$, where $\mathbb{T}_1 = (\Delta, \iota_1, \Theta_1)$ and $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$ are given by:

$\mathbb{T}_1\colon$ 

$\mathbb{T}_2\colon$ 

and where $e\colon \Delta_2 \hookrightarrow \Theta_1$ is the identical embedding. Since Γ is $(2,4)$-regular, it is $\mathbb{T}_1$- and $\mathbb{T}_2$-regular.

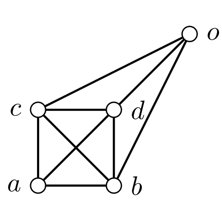
Let (μ_1, μ_2) be an arbitrary compatible cocone of (e, ι_2), where $\mu_i\colon \Theta_i \hookrightarrow \Upsilon$ ($i \in \{1,2\}$), say

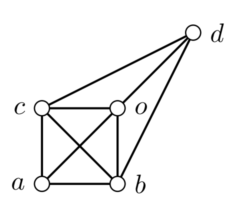
$$\begin{aligned}&\mu_1\colon x \mapsto a, y \mapsto b, u \mapsto c,\\&\mu_2\colon y \mapsto b, u \mapsto c, v \mapsto d, w \mapsto o, \text{ where } a,b,c,d,o \in V(\Upsilon).\end{aligned}$$

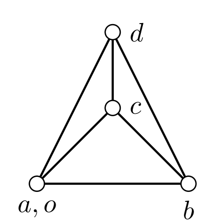
Then the unique mediating morphism h_{μ_1,μ_2} is given by

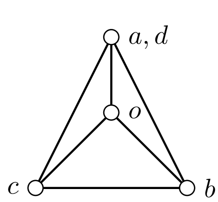
$$h_{\mu_1,\mu_2}\colon x \mapsto a, y \mapsto b, u \mapsto c, v \mapsto d, w \mapsto o.$$

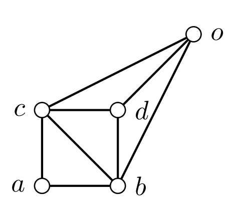
In the following we list all possibilities what the subgraph of Υ induced by $\{a,b,c,d,o\}$ might look like (depending on Υ and on (μ_1,μ_2)). This list is obtained by constructing all graphs vertex labeled by $\{a,b,c,d,o\}$ in such a way that every vertex has at least one label (though, it may have more than one label) and such that every label is used exactly once, subject to the condition that the above given functions μ_1 and μ_2 define graph-embeddings. In our case this means that the vertices labeled by elements of $\{a,b,c\}$ induce K_3 and those labeled by elements from $\{b,c,d,o\}$ induce K_4:

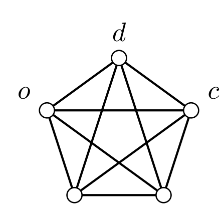
(1): 

(2): 

(3): 

(4): 

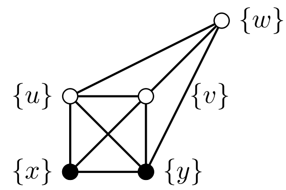
(5): 

(6): 

.

Now we are ready to construct the set $\mathcal{T}$ mentioned in the proof of the type counting Lemma. In cases (1) and (2) we obtain

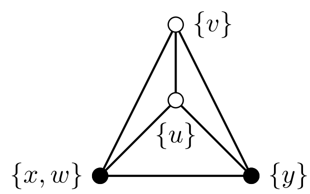
$$\mathbb{T}^{(1)} := \mathbb{T}_{\mu_1,\mu_2}\colon$$



$$\tilde{h}_{\mu_1,\mu_2}\colon \begin{cases} \{x\} \mapsto a, \\ \{y\} \mapsto b, \\ \{u\} \mapsto c, \\ \{v\} \mapsto d, \\ \{w\} \mapsto o. \end{cases}$$

In case (3) we obtain

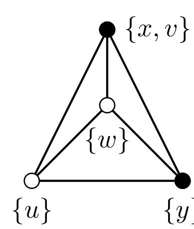
$$\mathbb{T}^{(2)} := \mathbb{T}_{\mu_1,\mu_2}\colon$$



$$\tilde{h}_{\mu_1,\mu_2}\colon \begin{cases} \{x, w\} \mapsto a(= o), \\ \{y\} \mapsto b, \\ \{u\} \mapsto c, \\ \{v\} \mapsto d. \end{cases}$$

In case (4) we obtain

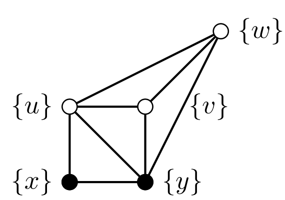
$$\mathbb{T}^{(3)} := \mathbb{T}_{\mu_1,\mu_2} \cong \mathbb{T}^{(2)}\colon$$



$$\tilde{h}_{\mu_1,\mu_2}\colon \begin{cases} \{x, v\} \mapsto a(= d), \\ \{y\} \mapsto b, \\ \{u\} \mapsto c, \\ \{w\} \mapsto o. \end{cases}$$

In case (5) we obtain

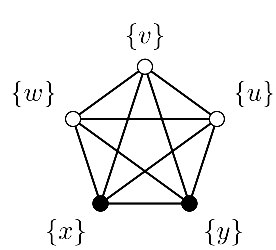
$$\mathbb{T}^{(4)} := \mathbb{T}_{\mu_1,\mu_2} \cong \mathbb{T}\colon$$



$$\tilde{h}_{\mu_1,\mu_2}\colon \begin{cases} \{x\} \mapsto a, \\ \{y\} \mapsto b, \\ \{u\} \mapsto c, \\ \{v\} \mapsto d, \\ \{w\} \mapsto o. \end{cases}$$

In case (6) we obtain

$$\mathbb{T}^{(5)} := \mathbb{T}_{\mu_1,\mu_2}\colon$$



$$\tilde{h}_{\mu_1,\mu_2}\colon \begin{cases} \{x\} \mapsto a, \\ \{y\} \mapsto b, \\ \{u\} \mapsto c, \\ \{v\} \mapsto d, \\ \{w\} \mapsto o. \end{cases}$$

To sum up, we have

$$\mathcal{T} = \{\mathbb{T}^{(1)}, \mathbb{T}^{(2)}, \mathbb{T}^{(3)}, \mathbb{T}^{(4)}, \mathbb{T}^{(5)}\}.$$

Let us fix an embedding $\kappa\colon \Delta \hookrightarrow \Gamma$. Then the set P_κ of all κ-compatible pairs is given by

$$P_\kappa = \{(\mu_1, \mu_2) \mid \mu_1 \text{ extends } \kappa \text{ along } \iota_1 \text{ and } \mu_2 \text{ extends } \mu_1 \circ e \text{ along } \iota_2\}.$$

Thus, we have

$$\#(\Gamma, \mathbb{T}_1) \cdot \#(\Gamma, \mathbb{T}_2) = |P_\kappa| = \sum_{i=1}^{5} \#(\Gamma, \mathbb{T}^{(i)}, \kappa).$$

If we suppose that Γ is $\mathbb{T}^{(1)}$-, $\mathbb{T}^{(2)}$-, and $\mathbb{T}^{(5)}$-regular, then, taking into account that $\mathbb{T}^{(2)} \cong \mathbb{T}^{(3)}$, we obtain

$$\#(\Gamma, \mathbb{T}^{(4)}, \kappa) = \#(\Gamma, \mathbb{T}_1) \cdot \#(\Gamma, \mathbb{T}_2) - \#(\Gamma, \mathbb{T}^{(1)}) - 2 \cdot \#(\Gamma, \mathbb{T}^{(2)}) - \#(\Gamma, \mathbb{T}^{(5)}).$$

Finally, observing that $\#(\Gamma, \mathbb{T}, \kappa) = \#(\Gamma, \mathbb{T}^{(4)}, \kappa)$. we arrive at

$$\#(\Gamma, \mathbb{T}, \kappa) = \#(\Gamma, \mathbb{T}_1) \cdot \#(\Gamma, \mathbb{T}_2) - \#(\Gamma, \mathbb{T}^{(1)}) - 2 \cdot \#(\Gamma, \mathbb{T}^{(2)}) - \#(\Gamma, \mathbb{T}^{(5)}).$$

As this does not depend on κ, we conclude that Γ is $\mathbb{T}$-regular.

Remark 3.32. The formulation of the type counting Lemma is not as strong as it could be. In particular, when analyzing the proof it becomes clear that the third condition can be weakened. It is not necessary that Γ is $\mathbb{T}$-regular for *all* graph types $\mathbb{T}$ strictly dominated by $\mathbb{T}_1 \oplus_e \mathbb{T}_2$. Instead it is sufficient to claim that Γ is $\mathbb{T}$-regular for all those graph types $\mathbb{T}$ for which there exists a morphism $(f, g)\colon \mathbb{T}_1 \oplus_e \mathbb{T}_2 \twoheadrightarrow \mathbb{T}$ such that

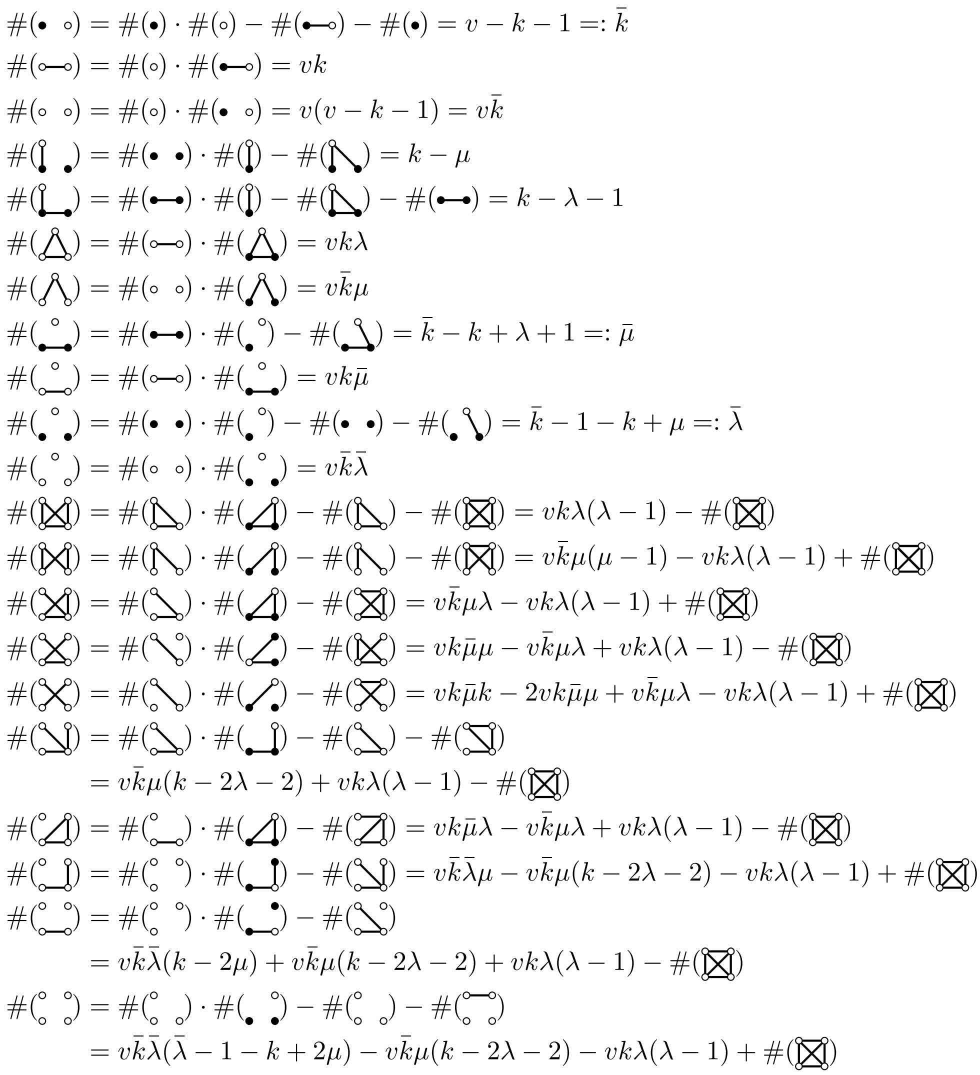
(1) f is an isomorphism,
(2) g is surjective and not an isomorphism,
(3) $g \circ \lambda_1$ and $g \circ \lambda_2$ are embeddings,

where (λ_1, λ_2) is a limiting cocone for (e, ι_2).

Example 3.33. The type counting lemma is a qualitative statement about regularities. It makes no claim about $\#(\Gamma, \mathbb{T}_1 \oplus_e \mathbb{T}_2, \kappa)$, only that it is independent of κ. However, when studying its proof, it becomes clear that there is also a quantitative dimension. While it is not the topic of this paper, let us have a little look into this aspect, just to get a taste. We consider the problem of counting subgraphs in strongly regular graphs. In N. Kriger's D.Phil thesis [38], following the spirit of the paper [24] by M.D. Hestenes and D.G. Higman, formulae for counting four-vertex subgraphs in strongly regular graphs are given and proved. Following Kriger's notation, by $F(\Theta)$ the number of induced subgraphs of Γ isomorphic to Θ is denoted. In general, if we define $\mathbb{T}_\Theta \coloneqq (\varnothing, \iota, \Theta)$, then $\#(\Gamma, \mathbb{T}_\Theta)$ is equal to the number of embeddings of Θ into Γ. Thus we have $F(\Theta) = \#(\Gamma, \mathbb{T}_\Theta)/|\operatorname{Aut}(\Theta)|$. Let Γ be a strongly regular graph with parameters (v, k, λ, μ). That is, we know a priori that

$$\#(\Gamma, \circ) = v, \qquad \#(\Gamma, \bullet\!\!-\!\!\circ) = k \qquad \#(\Gamma, \triangle) = \lambda \qquad \#(\Gamma, \wedge) = \mu.$$

In order to save some space, in the following, instead of $\#(\Gamma, \mathbb{T})$ we will write just $\#(\mathbb{T})$.



Note above how the counting of embeddings of 4-vertex graphs into Γ may be reduced to counting $\#(K_4)$.

3.6. Criteria for (m,n)-regularity. The proofs of the following propositions make use of a very basic induction principle for finite posets:

Lemma 3.34. *Let $(P, \leqslant)$ be a finite partially ordered set and let $B \subseteq P$. If*

$$\forall p \in P : \big(\{q \in P \mid q < p\} \subseteq B \Rightarrow p \in B\big), \tag{5}$$

then we already have that B is equal to P.

Proof. Suppose that (5) holds for B, but that $B \neq P$. Let x be a minimal element of $P \setminus B$ in $(P, \leqslant)$ (this exists because P is finite). Then for all $y < x$ we have $y \in B$. Thus, by (5), we also have $x \in B$, a contradiction. $\square$

Proposition 3.35. *Let* Γ *be an* (m,m)*-regular graph. Then,* Γ *is* (m,n)*-regular if and only if it is* $({}_{=}m,n)$*-regular.*

Proof. By definition, from (m,n)-regularity follows $({}_{=}m,n)$-regularity.

Suppose that Γ is $({}_{=}m,n)$-regular and (m,m)-regular. Let $\mathcal{M}$ be a transversal of the isomorphism classes of graph types of order (k,l) for $k \leqslant m$ and for $l \leqslant n$. Then, by Lemma 3.4, $(\mathcal{M},\preceq)$ is a finite poset. Moreover, whenever $\mathbb{T} \in \mathcal{M}$ and $\mathbb{T}' \preceq \mathbb{T}$, then $\mathbb{T}'$ is isomorphic to an element of $\mathcal{M}$.

Let $\mathbb{T} = (\Delta,\iota,\Theta) \in \mathcal{M}$ be of order (k,l). Suppose that for all $\mathbb{T}' \prec \mathbb{T}$ the graph Γ is $\mathbb{T}'$-regular. If $l \leqslant m$, then Γ is $\mathbb{T}$-regular, by assumption. So suppose that $m < l \leqslant n$. Let $\hat{\Delta}$ be an induced subgraph of order m of Θ that contains the image of ι, and let $\hat{\iota}$ be the identical embedding of $\hat{\Delta}$ into Θ. Then $\mathbb{T}_1 \coloneqq (\Delta,\iota,\hat{\Delta})$ is a graph type of order (k,m), and $\mathbb{T}_2 \coloneqq (\hat{\Delta},\hat{\iota},\Theta)$ is a graph type of order (m,l). Moreover, $\mathbb{T} \cong \mathbb{T}_1 \oplus_{\hat{\iota}} \mathbb{T}_2$. By the assumptions, we have that Γ is $\mathbb{T}_1$- and $\mathbb{T}_2$-regular. Hence, by the type counting lemma, we conclude that Γ is $\mathbb{T}$-regular.

By the arguments above and by Lemma 3.34, Γ is $\mathbb{T}$-regular for all graph types $\mathbb{T}$ from $\mathcal{M}$. In other words, Γ is (m,n)-regular. □

Note that a graph is $(2,2)$-regular if and only if it is regular. Thus, the previous proposition generalizes a classic result by A.V. Ivanov:

Theorem 3.36 (A.V. Ivanov [29, Proposition 2.1]). *Let* Γ *be a regular graph. Then* Γ *satisfies the* t*-vertex condition if and only if it is* $({}_{=}2,t)$*-regular.*

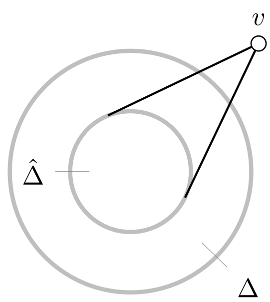
Definition 3.37. *A graph* $\Gamma = (V,E)$ *is called* k-isoregular *if for every subset* $X \subseteq V$ *with* $|X| \leqslant k$ *the number of vertices* $v \notin X$ *that are adjacent to all elements of* X *does not depend on* X *but only on the isomorphism type of the subgraph of* Γ *induced by* X*.*

Proposition 3.38. *Let* Γ *be a graph and let* $k > 0$ *be a natural number. Then the following are equivalent:*

(1) Γ *is* k*-isoregular,*
(2) Γ *is* $({}_{=}l,{}_{=}l+1)$*-regular for every* $1 \leqslant l \leqslant k$*,*
(3) Γ *is* $(k,k+1)$*-regular.*

Proof. "(1)⇒(2):" Let $l \in \{1,\dots,k\}$, and let $\mathcal{M}$ be a transversal of the isomorphism classes of graph types of order (l,m) where $m \in \{l,l+1\}$. Without loss of generality we may assume for every graph type $\mathbb{T} = (\Delta,\iota,\Theta) \in \mathcal{M}$ that ι is the identical embedding (i.e. Δ is an induced subgraph of Θ). By Lemma 3.4, $(\mathcal{M},\preceq)$ is a finite poset. Moreover, for every $\mathbb{T} \in \mathcal{M}$ of order (l,m) and for every $\mathbb{T}' \prec \mathbb{T}$ we have that the order of $\mathbb{T}'$ is (l,n) for some $l \leqslant n \leqslant m$; hence there exists a unique $\mathbb{T}'' \in \mathcal{M}$ such that $\mathbb{T}' \cong \mathbb{T}''$.

In the following we show that Γ is $\mathbb{T}$-regular, for all $\mathbb{T} \in \mathcal{M}$. Let $\mathbb{T} = (\Delta,\iota,\Theta)$ be an element of $\mathcal{M}$. Moreover, suppose that for all $\mathbb{T}' \prec \mathbb{T}$ from $\mathcal{M}$ the graph Γ is $\mathbb{T}'$-regular. If the order of $\mathbb{T}$ is (l,l), then Γ is $\mathbb{T}$-regular. So suppose that $\mathbb{T}$ has order $(l,l+1)$. Let v be the unique vertex of Θ that is not in $V(\Delta)$. If v has valency l in Θ, then Γ is $\mathbb{T}$-regular, because Γ is k-isoregular. So, suppose that the valency of v in Θ is equal to $m < l$. Let $\hat{\Delta}$ be the subgraph of Δ induced by the neighbors of v, let $\hat{\Theta}$ be the subgraph of Θ induced by the vertices of $\hat{\Delta}$ together with v itself, and let $\hat{\iota}\colon \hat{\Delta} \hookrightarrow \hat{\Theta}$ be the identical embedding. Then $\mathbb{T}_1 \coloneqq (\Delta,1_\Delta,\Delta)$ and $\mathbb{T}_2 \coloneqq (\hat{\Delta},\hat{\iota},\hat{\Theta})$ are graph types. Moreover, $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \mathbb{T}_2$, where e denotes the identical embedding of $\hat{\Delta}$ into Δ.



$$\begin{tikzcd} \Delta \arrow[r, hook, "\iota", "="'] & \Theta \\ \hat{\Delta} \arrow[u, hook, "e", "="'] \arrow[r, hook, "\hat{\iota}", "="'] & \hat{\Theta} \arrow[u, hook, "="'] \end{tikzcd} \qquad \begin{array}{l} V(\Theta) = V(\Delta) \cup \{v\} \\ \\ V(\hat{\Theta}) = V(\hat{\Delta}) \cup \{v\}. \end{array}$$

Then $\mathbb{T}_1$ is of order (l, l) thus, Γ is $\mathbb{T}_1$-regular. Moreover, $\mathbb{T}_2$ is of order $(m, m+1)$ and the $\mathbb{T}_2$-regularity of Γ follows from the k-isoregularity of Γ. Now, from the type counting lemma it follows that Γ is $\mathbb{T}$-regular. Finally, from Lemma 3.34 it follows that Γ is regular for all types from $\mathcal{M}$. In particular, Γ is $({}_{=}l, {}_{=}l+1)$-regular.

"(2)⇒(3):" We show that Γ is $(l, l+1)$-regular for all $l \in \{1, \dots, k\}$. We proceed by induction on l. For the induction base we note that Γ is $(1, 2)$-regular if and only if it is $({}_{=}1, {}_{=}1)$, and $({}_{=}1, {}_{=}2)$-regular. The first regularity condition is trivially fulfilled and the $({}_{=}1, {}_{=}2)$-regularity is given by assumption. Suppose, we know that Γ is $(l, l+1)$-regular and $({}_{=}l+1, {}_{=}l+2)$-regular, for some $1 \leqslant l \leqslant k-1$. Then from the $(l, l+1)$-regularity follows immediately the $(l+1, l+1)$-regularity (indeed, a graph is $(l+1, l+1)$-regular iff it is $(l, l+1)$-regular and $({}_{=}l+1, {}_{=}l+1)$-regular; however, trivially, every graph is $({}_{=}l+1, {}_{=}l+1)$-regular). Moreover, we have that Γ is $({}_{=}l+1, l+2)$-regular, because Γ is $({}_{=}l+1, {}_{=}l+1)$-regular and Γ is $({}_{=}l+1, {}_{=}l+2)$-regular. Hence, from Proposition 3.35, it follows that Γ is $(l+1, l+2)$-regular.

"(3)⇒(1):" k-isoregularity of Γ follows immediately from the $(k, k+1)$-regularity. □

The following criterion by S. Reichard characterizes, when a k-isoregular graph with the $(t-1)$-vertex condition satisfies the t-vertex condition:

Theorem 3.39 ([49, Theorem 3]). *Let Γ be a k-isoregular graph that satisfies the $(t-1)$-vertex condition for $t > 3$. Then, in order to verify the t-vertex condition, it suffices to test the $\mathbb{T}$-regularity for graph types $\mathbb{T} = (\Delta, \iota, \Theta)$ of order $(2, t)$ with the property that all vertices of Θ that are not in the image of ι have valency $\geqslant k+1$ in Θ.*

Our next goal is to generalize this result:

Proposition 3.40. *Let Γ be an (m, t)-regular graph. Let $\mathcal{M}$ be a set of graph types and suppose that Γ is $\mathbb{T}$-regular, for all $\mathbb{T} \in \mathcal{M}$. Then, in order to verify the $(m, t+1)$-regularity of Γ it suffices to test the $\mathbb{T}$-regularity for graph types of order $(m, t+1)$ that are $\widehat{\mathbb{T}}$-irreducible for all $\widehat{\mathbb{T}} \in \mathcal{M}$.*

Proof. Let $\mathcal{T}$ be a transversal of the isomorphism classes of graph types of order $(m, t+1)$. Then, by Lemma 3.4, $(\mathcal{T}, \preceq)$ is a finite poset. Moreover, whenever $\mathbb{T} \in \mathcal{T}$ and $\mathbb{T}' \preceq \mathbb{T}$ is a graph type of order $(m, t+1)$, then $\mathbb{T}'$ is isomorphic to an element of $\mathcal{T}$.

We will use the induction principle from Lemma 3.34 on $(\mathcal{T}, \preceq)$: Let $\mathbb{T} = (\Delta, \iota, \Theta) \in \mathcal{T}$ and suppose that Γ is $\mathbb{T}'$-regular for all $\mathbb{T}' \in \mathcal{T}$ with $\mathbb{T}' \prec \mathbb{T}$. Note that for every graph type $\mathbb{T}'' \prec \mathbb{T}$ we either have that $\mathbb{T}''$ is isomorphic to an element of $\mathcal{T}$ or it has order (m, l) for some $l < t+1$. In both cases we conclude that Γ is $\mathbb{T}''$-regular.

If $\mathbb{T}$ is $\widehat{\mathbb{T}}$-irreducible for all $\widehat{\mathbb{T}} \in \mathcal{M}$, then Γ is $\mathbb{T}$-regular, by assumption. So suppose that there exists a $\widehat{\mathbb{T}} \in \mathcal{M}$, such that $\mathbb{T}$ is $\widehat{\mathbb{T}}$-reducible. Then $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \widehat{\mathbb{T}}$ for some graph type $\mathbb{T}_1 \ncong \mathbb{T}$. But then the order of $\mathbb{T}_1$ is (m, l), for some $l < t+1$. Hence, by assumption Γ is $\mathbb{T}_1$-regular and $\widehat{\mathbb{T}}$-regular. By the type counting lemma we obtain that Γ is $\mathbb{T}$-regular.

Now, it remains to invoke Lemma 3.34, to obtain that Γ is regular for all types from $\mathcal{T}$. Consequently, Γ is $({}_{=}m, {}_{=}t+1)$-regular. By assumption, Γ is (m,t)- and in particular (m,m)-regular. Hence, by Proposition 3.35, we have that Γ is $(m,t+1)$-regular. ☐

Proposition 3.41. *Let Γ be a graph. Then Γ is $(m,n+1)$-regular if and only if Γ is (m,n)-regular and it is $\mathbb{T}$-regular for every (m,n)-irreducible graph type $\mathbb{T}$ of order $(m,n+1)$.*

Proof. "$\Rightarrow$:" This is clear.

"$\Leftarrow$:" Let $\mathcal{M}$ be a transversal of the isomorphism classes of graph types of order (k,l), where $k \leqslant m$ and where $l \leqslant n$. By assumption, Γ is regular for all graph types from $\mathcal{M}$. By Proposition 3.40, in order to show that Γ is $(m,n+1)$-regular it suffices to show that Γ is $\mathbb{T}$-regular, for all graph types $\mathbb{T}$ of order $(m,n+1)$ that are $\widehat{\mathbb{T}}$-irreducible, for all $\widehat{\mathbb{T}} \in \mathcal{M}$.

By Lemma 3.19 we have that a graph type $\mathbb{T}$ of order $(m,n+1)$ is (m,n)-reducible if and only if it is $\widehat{\mathbb{T}}$-reducible for some $\widehat{\mathbb{T}} \in \mathcal{M}$. In particular, if $\mathbb{T}$ is (m,n)-irreducible, then it is $\widehat{\mathbb{T}}$-irreducible for all $\widehat{\mathbb{T}} \in \mathcal{M}$. This finishes the proof. ☐

Corollary 3.42. *A graph Γ is $(m,n+1)$-regular if and only if it is (m,n)-regular and it is $\mathbb{T}$-regular for all graph types $\mathbb{T}$ of order $(m,n+1)$ for which* $\operatorname{Env}(\mathbb{T})$ *is $(m+1)$-connected.*

Proof. This follows immediatelyfrom Proposition 3.41 together with Lemma 3.23. ☐

Definition 3.43. *Let Γ be a graph and let $u \in V(\Gamma)$. Then with $\Gamma_1(u)$ we denote the subgraph of Γ induced by the neighbors of u. Moreover, with $\Gamma_2(u)$ we denote the subgraph of Γ induced by the non-neighbors of u (except u itself). $\Gamma_1(u)$ and $\Gamma_2(u)$ are called the* first *and the* second subconstituent *of Γ with respect to u, respectively.*

The following proposition relates the regularities of a graph with the regularities of its subconstituents. This is used later on to identify a new class of graphs satisfying the 6-vertex condition:

Proposition 3.44. *Let Γ be an (m,n)-regular graph where $m \geqslant 1$, and let $u \in V(\Gamma)$. Then $\Gamma_1(u)$ and $\Gamma_2(u)$ are both $(m-1,n-1)$-regular.*

Proof. About $\Gamma_1(u)$: Let $\mathbb{T} = (\Delta, \iota, \Theta)$ be a graph type of order (r,s) where $r \leqslant m-1$ and $s \leqslant n-1$. Let $\Delta' := \Delta + \{x\}$ and $\Theta' := \Theta + \{y\}$ be graphs obtained from Δ and Θ by adjoining a single new vertex that is connected to vertices of Δ and of Θ, respectively. Let $\iota' \colon \Delta' \hookrightarrow \Theta'$ be defined according to

$$\iota' \colon w \mapsto \begin{cases} \iota(w) & w \in V(\Delta), \\ y & w = x. \end{cases}$$

Then $\mathbb{T}' := (\Delta', \iota', \Theta')$ is a graph type of order $(r+1, s+1)$. As $r+1 \leqslant m$ and $s+1 \leqslant n$, we have that Γ is $\mathbb{T}'$-regular. Let $\kappa \colon \Delta \hookrightarrow \Gamma_1(u)$. Define $\kappa' \colon \Delta' \hookrightarrow \Gamma$ according to

$$\kappa' \colon w \mapsto \begin{cases} \kappa(w) & w \in V(\Delta), \\ u & w = x. \end{cases}$$

We claim that there is a bijection between set of extensions of κ along ι in $\Gamma_1(u)$ and the set of extensions of κ' along ι' in Γ:

Let $\hat{\kappa}$ be any extension of κ along ι in $\Gamma_1(u)$. We define $\hat{\kappa}'\colon \Theta' \hookrightarrow \Gamma$ according to

$$\hat{\kappa}'\colon w \mapsto \begin{cases} \hat{\kappa}(w) & w \in V(\Theta), \\ u & w = y. \end{cases}$$

Clearly, $\hat{\kappa}'$ is an extension of κ' along ι' in Γ.

Let on the other hand $\hat{\kappa}'$ be any extension of κ' along ι' in Γ. Then $\hat{\kappa} \coloneqq \hat{\kappa}'\restriction_{V(\Theta)}$ is an extension of κ along ι in $\Gamma_1(u)$. This establishes the desired bijection between extensions of κ along ι in $\Gamma_1(u)$ and extensions of κ' along ι' in Γ. In particular, we have $\#(\Gamma_1(u), \mathbb{T}, \kappa) = \#(\Gamma, \mathbb{T}', \kappa') = \#(\Gamma, \mathbb{T}')$. Thus, $\Gamma_1(u)$ is $\mathbb{T}$-regular. As $\mathbb{T}$ was chosen arbitrarily, we conclude that $\Gamma_1(u)$ is $(m-1, n-1)$-regular.

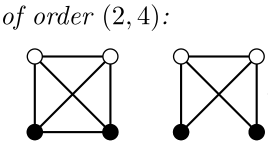
About $\Gamma_2(u)$: From Lemma 3.8 it follows that $\overline{\Gamma}$ is (m,n)-regular. Clearly, we have $\overline{\Gamma}_1(u) = \overline{\Gamma_2(u)}$, as the neighbours of u in $\overline{\Gamma}$ are exactly the non-neighbors of u in Γ, and the edges in $\overline{\Gamma}_1(u)$ are exactly the non-edges in $\Gamma_2(u)$. From the first part of the proof it follows that $\overline{\Gamma}_1(u)$ is $(m-1, n-1)$-regular. Again using Lemma 3.8 we conclude that $\Gamma_2(u)$ is $(m-1, n-1)$-regular. $\square$

Remark 3.45. The previous proposition generalizes a result from the folklore of algebraic graph theory to the case of (m,n)-regular graphs. Namely, if a graph is $(k+1)$-isoregular, then all its first and second subconstituents are k-isoregular. This observation, together with spectral methods, stands at the center of Gol'fand's lost proof that 5-isoregular graphs are homogeneous (see [51, Section 9.2] for a historical account and for further references).

4. Checking the t-vertex condition

Every graph satisfies the 1-vertex condition. A graph satisfies the 2-vertex condition if and only if it is regular. A bit less obvious but rather straightforward is the observation that a graph satisfies the 3-vertex condition if and only if it is strongly regular, i.e. it is regular and the number of common neighbors of every edge is equal to a constant λ and the number of common neighbors of every non-edge is equal to a constant μ (the first half of Example 3.33 contains the calculations necessary for a proof that strong regularity implies the 3-vertex condition). A criterion for the 4-vertex condition is given by:

Theorem 4.1 (M.D. Hestenes, D.G. Higman [24]). *Let Γ be a strongly regular graph. Then, in order to verify the 4-vertex condition it suffices to test the $\mathbb{T}$-regularity for the following two graph types of order $(2,4)$:*



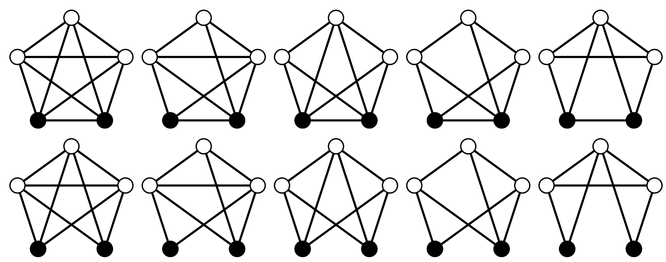
In our terminology, this is a special case of Corollary 3.42 ($m = 2$ and $n = 3$, see Example 3.24). More generally, we have:

Proposition 4.2. *Let Γ be a graph that satisfies the t-vertex condition for $t \geqslant 3$. Then, in order to verify the $(t+1)$-vertex condition it suffices to test the $\mathbb{T}$-regularity for all those graph types $\mathbb{T}$ of order $(2, t+1)$ for which $\mathrm{Env}(\mathbb{T})$ is 3-connected.*

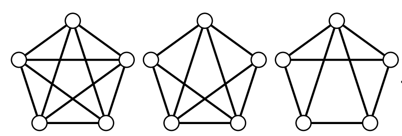
Proof. This is a special case of Corollary 3.42 ($m = 2$, $n = t$). $\square$

In [50, Theorem 4.9] S. Reichard proved that a graph satisfying the 4-vertex condition satisfies the 5-vertex condition if and only if it is regular for a list of 16 graph types. The following proposition reduces the number of graph types to be tested to 10:

Proposition 4.3. *Given a graph Γ that fulfills the 4-vertex condition. Then in order to test whether Γ satisfies also the 5-vertex condition it suffices to count the graph types in the table below.*



Proof. According to Proposition 4.2, Γ satisfies the 5-vertex condition if and only if it is $\mathbb{T}$-regular for all $\mathbb{T}$ such that $\mathrm{Env}(\mathbb{T})$ is 3-connected. So we start by constructing all 3-connected graphs of order 5. This gives us the following three graphs:



Next, for each graph Θ from this list we computed the orbits of $\mathrm{Aut}(\Theta)$ in its action on edges. Each orbit representative corresponds to a two-vertex subgraph $\Delta \cong K_2$, producing a graph type (Δ, ι, Θ) (as usual, ι is the identical embedding). This produces the upper row of graph types. The lower row is obtained by removing the distinguished edge in each case. Clearly, this produces a transversal of the isomorphism classes of graph types $\mathbb{T}$ of order $(2, 5)$ for which $\mathrm{Env}(\mathbb{T})$ is 3-connected. □

5. Point graphs of partial quadrangles

An incidence structure is a triple $(\mathscr{P}, \mathscr{L}, I)$, where $\mathscr{P}$ is a set of points (denoted by capital Latin letters $P, Q, \ldots$), $\mathscr{L}$ is a set of lines (denoted by small Latin letters $l, s, t, \ldots$), and $I \subseteq \mathscr{P} \times \mathscr{L}$ is an incidence relation. The elements of I are called *flags* and the elements of $(\mathscr{P} \times \mathscr{L}) \smallsetminus I$ are called *antiflags*. A point P is called incident with a line l if (P, l) is a flag. Slightly abusing the notation we write in this case $P \in l$. Two distinct flags (P, p) and (Q, q) are called *collinear* if $p = q$, and *concurrent* if $P = Q$. Two distinct points P and Q are called collinear if there exists a line l such that (P, l) and (Q, l) are flags. In this case we say that l goes through P and Q. Dually, we say that two lines p and q are intersecting each other if there is a point P such that $P \in p$ and $P \in q$.

For every incidence structure, we may define its *point graph*. This is a simple graph which has as vertices the points of the incidence structure such that between two points there is an edge if and only if the points are collinear.

In the following we restrict our attention to so-called partial linear spaces of order (s, t) (in the sense of [14, p. 3]):

Definition 5.1. *Let $s, t \in \mathbb{N} \smallsetminus \{0\}$. A* partial linear space of order (s, t) *(short* $\mathrm{PLS}(s, t)$*) is an incidence structure $(\mathscr{P}, \mathscr{L}, I)$ with the following properties:*

PLS1. *Every line is incident with the same number $s + 1$ of points.*
PLS2. *Every point is incident with the same number $t + 1$ of lines.*
PLS3. *Through any two distinct points goes at most one line.*

If two lines p and q of a partial linear space intersect each other, then we denote the unique point of intersection by $p \cap q$.

Remark 5.2. Note that in a partial linear space two lines are equal if and only if they are incident with exactly the same points. Below we will implicitly identify a line in a partial linear space with the set of points it is incident with. Moreover, a partial linear space $(\mathscr{P}, \mathscr{L}, I)$ will be denoted just like $(\mathscr{P}, \mathscr{L})$.

We are interested in partial linear spaces because there are significant classes of them whose point graphs are strongly regular. Two such classes are defined below. The first class of interest consists of the generalized quadrangles as introduced by J. Tits in [55]:

Definition 5.3. *A* generalized quadrangle *of order (s,t) (abbreviated to* $\mathrm{GQ}(s,t)$*) is a partial linear space of order (s,t) with the the following additional property:*

GQ1. *For every antiflag (P,q) there is a unique point Q such that P and Q are collinear and such that $Q \in q$.*

It is well-known that the point graph of a generalized quadrangle of order (s,t) is strongly regular with parameters (v,k,λ,μ) where

$$v = (s+1)(st+1), \qquad k = s(t+1), \qquad \lambda = s-1, \qquad \mu = t+1.$$

Axiom GQ1 ensures that a generalized quadrangle does not contain triples of pairwise collinear points that are not all three on one line. From this follows that every set of points that induces a clique in the point graph is a subset of some line. In particular, the generalized quadrangle can be reconstructed from its point graph up to isomorphism by taking as points the vertices of the point graph, as lines the maximal cliques and as incidence relation the $\in$-relation. Moreover, the point graph of a generalized quadrangle cannot contain $K_4 - e$ as an induced subgraph because this would imply the existence of two distinct maximal cliques that intersect in at least two points which cannot happen because of axiom PLS3.

In [9] P. J. Cameron examined point graphs of generalized quadrangles and made the above observations. These observations lead him to study strongly regular graphs that do not contain $K_4 - e$ as an induced subgraph. It turns out that such graphs always arise as point graph of certain partial linear spaces. The class of partial linear spaces that have as a point graph an srg without $K_4 - e$ as an induced subgraph, are called *partial quadrangles*. Below we give an axiomatization:

Definition 5.4. *A* partial quadrangle *with parameters (s,t,μ) (short* $\mathrm{PQ}(s,t,\mu)$*) is a partial linear space $(\mathscr{P}, \mathscr{L})$ of order (s,t) with the following properties:*

PQ1. *If three points are pairwise collinear, then they are all three on one line.*

PQ2. *For every pair (P,Q) of non-collinear points there exist μ points X that are collinear with both points P and Q.*

The point graphs of partial quadrangles have an elegant characterization:

Theorem 5.5 (P. J. Cameron [9, Theorem 2]). *Let $\Gamma = (V,E)$ be a strongly regular graph with parameters (v,k,λ,μ). Then Γ is isomorphic to the point graph of a partial quadrangle if and only if $\mu > 0$ and it does not contain any induced subgraph isomorphic to $K_4 - e$.*

Let us recall that starting from a strongly regular graph Γ with parameters (v,k,λ,μ) that has no induced subgraph isomorphic to $K_4 - e$, we can construct a partial quadrangle by taking as points the vertices of Γ and as lines the maximal cliques. The resulting partial quadrangle has parameters $(\lambda+1, \frac{k}{\lambda+1} - 1, \mu)$. On the other hand, the parameters of the point graph of a $\mathrm{PQ}(s,t,\tilde{\mu})$ are

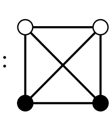
$$v = \frac{s(t+1)(\tilde{\mu} + st)}{\tilde{\mu}} + 1, \qquad k = s(t+1), \qquad \lambda = s-1, \qquad \mu = \tilde{\mu}.$$

Remark 5.6. Every $\mathrm{GQ}(s,t)$ is at the same time a $\mathrm{PQ}(s,t,t+1)$. While there are many known constructions for generalized quadrangles (cf. [47]), much fewer constructions are known for *proper* partial quadrangles, i.e. for partial quadrangles that are not generalized quadrangles. A first source of proper partial quadrangles is given by the triangle-free strongly regular graphs. They correspond to the $\mathrm{PQ}(1,t,\mu)$. The known triangle-free srgs are the pentagon ($\mathrm{PQ}(1,1,1)$), the Petersen graph ($\mathrm{PQ}(1,2,1)$, the Clebsch graph ($\mathrm{PQ}(1,4,2)$), the Hoffman–Singleton graph ($\mathrm{PQ}(1,6,1)$), the Gewirtz graph ($\mathrm{PQ}(1,9,2)$), the Mesner graph ($\mathrm{PQ}(1,15,4)$), and the Higman–Sims graph ($\mathrm{PQ}(1,21,6)$. Two more infinite sources of proper partial quadrangles are related to generalized quadrangles of order (q,q^2). For the first one we start with a $\mathrm{GQ}(q,q^2)$ and select a point P. Then we delete P, all lines through P, and all points that are collinear with P in this generalized quadrangle. When this is done, we end up with a $\mathrm{PQ}(q-1,q^2,q^2-q)$ (see [11, Theorem 7.9]). The second source is induced by so-called hemisystems (in the sense of Segre [52, p. 161]). Whenever a hemisystem exists in a $\mathrm{GQ}(q,q^2)$, it gives rise to a $\mathrm{PQ}((q-1)/2,q^2,(q-1)^2/2)$. Such partial quadrangles were constructed by Cossidente and Penttila (see [13]) for all odd prime powers q. Meanwhile a number of other constructions of hemisystems in generalized quadrangles were found. We refer to [57] for a relatively recent overview together with further links to topics from algebraic graph theory. Also, the papers [1, 12, 14] may be used as a starting point to get an overview of the known constructions of proper partial quadrangles.

Now we are ready to formulate the first result of this section:

Theorem 5.7. *Let Γ be the point graph of a partial quadrangle. Then Γ is $(2,5)$-regular, i.e. it satisfies the 5-vertex condition.*

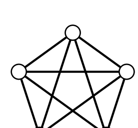
Proof. At first we note that by Theorem 4.1, in order to test the 4-vertex condition for Γ it is enough to test it for

$\mathbb{T}_1$:  ,

as Γ does not contain K_4-e as an induced subgraph. Clearly, we have

$$\#(\Gamma,\mathbb{T}_1)=(s-1)(s-2).$$

Secondly we note that from all the graph types given in Proposition 4.3 only the underlying graph of the first one does not contain K_4-e as an induced subgraph. Thus, in order to test the 5-vertex condition, we have only to consider

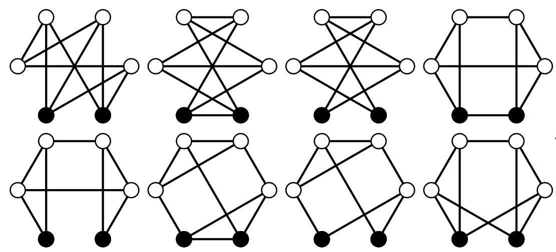
$\mathbb{T}_2$:  .

However, we easily compute

$$\#(\Gamma,\mathbb{T}_2)=(s-1)(s-2)(s-3). \qquad \square$$

Let us have a look at a criterion for the 6-vertex condition for partial quadrangles:

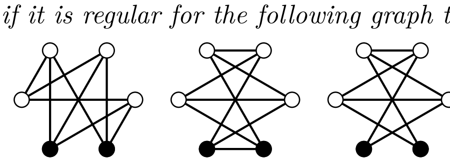
Proposition 5.8. *Let Γ be the point graph of a partial quadrangle. Then in order to test the 6-vertex condition for Γ it suffices to check it for the following 8 graph types:*



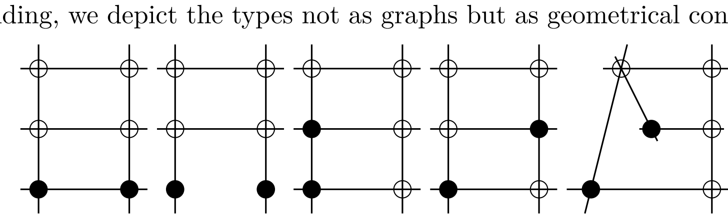
Proof. The above given 8 graph types form a transversal of the isomorphism classes of all those graph types $\mathbb{T} = (\Delta, \iota, \Theta)$ of order $(2, 6)$ for which $\mathrm{Env}(\mathbb{T})$ is 3-connected and for which Θ does not contain an induced subgraph isomorphic to $K_4 - e$. Now the claim follows from Theorem 5.7 together with Proposition 4.2. $\square$

S. Reichard showed in [50] that among these 8 graph types there are 5 types $\mathbb{T}$ such that the point graph of every generalized quadrangle is $\mathbb{T}$-regular. Together with this observation we obtain:

Proposition 5.9. *The point graph of a generalized quadrangle satisfies the 6-vertex condition if and only if it is regular for the following graph types:*



Proof. Let Γ be the point graph of a $\mathrm{GQ}(s, t)$. By Proposition 5.8, in order to prove the claim, we need to show that Γ is regular for the following graph types (to get a better understanding, we depict the types not as graphs but as geometrical configurations):



$\mathbb{T}_1$ $\mathbb{T}_2$ $\mathbb{T}_3$ $\mathbb{T}_4$ $\mathbb{T}_5$.

However, it is not hard to see that:

$$\begin{aligned}
\#(\Gamma, \mathbb{T}_1) &= t^2 s(s-1),\\
\#(\Gamma, \mathbb{T}_2) &= (t+1)t(s-1)(s-2),\\
\#(\Gamma, \mathbb{T}_3) &= t^2 s(s-1),\\
\#(\Gamma, \mathbb{T}_4) &= (t+1)t(s-1),\\
\#(\Gamma, \mathbb{T}_5) &= (t+1)t(t-1)s.
\end{aligned}$$
$\square$

Recall, that in a partial linear space, three pairwise non-collinear points are called a *triad*. Moreover, a *center* of a triad is a point collinear to all three points of the triad.

Theorem 5.10 (P. J. Cameron [9, Theorem 2]). *Let $\Pi = (\mathscr{P}, \mathscr{L})$ be a partial quadrangle of order (s, t, μ). Then*

$$\big(s(t-1) + (\mu-1)(\mu-2)\big)\left(\frac{(t+1)ts^2}{\mu} - 1 - (t+1)s + \mu\right) \geqslant \mu(t-1)^2 s^2.$$

Moreover, equality holds if and only if every triad in Π *has the same number* c *of centers. In this case we have*

$$c = 1 + \frac{(\mu-1)(\mu-2)}{s(t-1)}.$$

For the special case of generalized quadrangles this simplifies to the following well-known result:

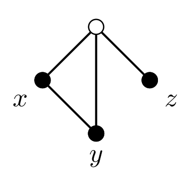
Theorem 5.11 ([26, Theorem 3.2], [5, Corollary 3.1], [9, Corollary to Theorem 1]). *Let* $\Pi = (\mathscr{P}, \mathscr{L})$ *be a generalized quadrangle of order* (s,t)*. Then* $s^2 \geqslant t$*. Moreover, equality holds if and only if every triad in* Π *has the same number* $(s+1)$ *of centers.*

Remark 5.12. According to P.J. Cameron (cf. [9, Abstract]), the first part of the above theorem was proved by D.G. Higman in 1971. The fact that in generalized quadrangles of order (s, s^2) every triad has exactly $s+1$ centers (which, in turn, are pairwise non-collinear) was proved by R.C. Bose and S.S. Shrikhande in 1971. In its full generality the theorem was proved by P.J. Cameron in 1973.

Corollary 5.13 ([51, Corollary 3]). *Let* Γ *be the point graph of a generalized quadrangle of order* (q, q^2)*. Then* Γ *is* 3*-isoregular.*

Proposition 5.14. *Let* Π *be a partial quadrangle of order* (s,t,μ)*, such that every triad in* Π *has the same number* c *of centers, and let* Γ *be its point graph. Then* Γ *is* 3*-isoregular if and only if either* Π *is a generalized quadrangle and* $t = s^2$*, or* Γ *is triangle-free (i.e.* $s = 1$*).*

Proof. "$\Rightarrow$:" Suppose that Γ is 3-isoregular. Consider the following graph type $\mathbb{T} = (\Delta, \iota, \Theta)$ of order $(3,4)$:



.

Then any embedding κ of Δ into Γ determines a line l of Π (spanned by $\kappa(x)$ and $\kappa(y)$) and a vertex $p = \kappa(z)$ not on this line such that neither $\kappa(x)$ nor $\kappa(y)$ is collinear with p. In any partial quadrangle there exists at most one vertex q on l that is collinear with p (otherwise Π would contain a triangle of lines). So we have $\#(\Gamma, \mathbb{T}) \in \{0,1\}$. If $\#(\Gamma, \mathbb{T}) = 0$, then Γ is triangle-free and if $\#(\Gamma, \mathbb{T}) = 1$, then Π is a generalized quadrangle. By Theorem 5.11, we obtain that $t = s^2$.

"$\Leftarrow$:" If Π is a generalized quadrangle of order (s, s^2), then Γ is 3-isoregular, by Corollary 5.13. So suppose that Γ is triangle-free. Let u, v, w be three mutually distinct vertices of Γ. If the subgraph of Γ induced by u, v, and w contains an edge, then none of the edges has a common neighbor (otherwise Γ would contain triangles). So u, v, w form a triad in Π. Hence, they have c common neighbors in Γ. Consequently, Γ is 3-isoregular. □

3-isoregular triangle-free graphs appear to be extremely rare. The following observation was made by R. Noda:

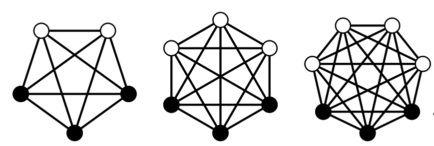
Proposition 5.15 (cf. [9, p. 70]). *Let* Γ *be a non-degenerate triangle-free* 3*-isoregular graph in which any three pairwise non-adjacent points are joint to exactly* n *vertices. Then* Γ *is the point graph of a* $\mathrm{PQ}(1, (n^2+2n-1)(n+1), n(n+1))$*.*

Remark 5.16. The first two members of this series are the Clebsch graph ($n=1$) and the Higman–Sims graph ($n=2$). For $n=3$ the putative graph would have parameters

$(v, k, \lambda, \mu) = (324, 57, 0, 12)$. It was shown by A. L. Gavrilyuk and A. A. Makhnev in [20] that such a graph does not exist. For a very interesting account of the history of the discovery of the Higman–Sims graph, we refer to [36].

Theorem 5.17. *Let Γ be the point graph of a partial quadrangle and suppose that Γ is 3-isoregular. Then Γ is $(3, 7)$-regular.*

Proof. As Γ is 3-isoregular, it is $(3, 4)$-regular. By Corollary 3.42, in order to prove $(3, 5)$-regularity of Γ it suffices to prove the $\mathbb{T}$-regularity for all graph types $\mathbb{T}$ of order $(3, 5)$ for which $\mathrm{Env}(\mathbb{T})$ is 4-connected. Since Γ does not have $K_4 - e$ as an induced subgraph, we can shorten this list by all $\mathbb{T}$ whose underlying graph contains $K_4 - e$. A computer search reveals that only the graph type $\mathbb{T}_a$ depicted below fulfills all these requirements:



$\mathbb{T}_a$ $\mathbb{T}_b$ $\mathbb{T}_c$

However, it is easy to see that

$$\#(\Gamma, \mathbb{T}_a) = \begin{cases} (s-2)(s-3) & s \geqslant 4, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, Γ is $(3, 5)$-regular.

With the same reasoning as before and again using a computer, we obtain that Γ is $(3, 6)$-regular if and only if it is $\mathbb{T}_b$-regular. However, it is easy to see that

$$\#(\Gamma, \mathbb{T}_b) = \begin{cases} (s-2)(s-3)(s-4) & s \geqslant 5, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, Γ is $(3, 6)$-regular.

Finally, once more using the same reasoning as above and using a computer, we obtain that Γ is $(3, 7)$-regular if and only if it is $\mathbb{T}_c$-regular. However, it is easy to see that

$$\#(\Gamma, \mathbb{T}_c) = \begin{cases} (s-2)(s-3)(s-4)(s-5) & s \geqslant 6, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, Γ is $(3, 7)$-regular. □

The previous theorem generalizes and strengthens a result by Reichard ([51, Theorem 2]) that states that the point graphs of generalized quadrangles of order (q, q^2) satisfy the 7-vertex condition.

Corollary 5.18. *Let Γ be the point graph of a partial quadrangle and suppose that Γ is 3-isoregular. Then, for every $u \in V(\Gamma)$, the second subconstituent $\Gamma_2(u)$ satisfies the 6-vertex condition.*

Proof. This follows from Proposition 3.44. □

Note that by Proposition 5.14, the previous corollary applies in particular to the point graphs of generalized quadrangles of order (q, q^2). This has the following consequence:

Corollary 5.19. *Let* Γ *be the point graph of a partial quadrangle of order* $(q-1,q^2,q^2-q)$*. Then* Γ *satisfies the* 6*-vertex condition.*

Proof. It was shown by A. A. Ivanov and S. V. Shpectorov in [27, Theorem A(i)] that whenever a graph Γ is strongly regular with parameters $(v,k,\lambda,\mu)=(q^4,(q^2+1)(q-1),q-2,q(q-1)))$ for some $q\geqslant 2$, such that in Γ every edge is contained in a complete subgraph of order q, then Γ is of the shape $\widehat{\Gamma}_2(u)$, where $\widehat{\Gamma}$ is the point graph of some generalized quadrangle of order (q,q^2), and where u is some vertex of $\widehat{\Gamma}$.

Since the given graph Γ is the point graph of a $\mathrm{PQ}(q-1,q^2,q^2+q)$, the result by Ivanov and Shpectorov applies to it. Let $\widehat{\Gamma}$ be the point graph of a $\mathrm{GQ}(q,q^2)$ and let u be a vertex of $\widehat{\Gamma}$, such that $\Gamma=\widehat{\Gamma}_2(u)$. By Proposition 5.14, $\widehat{\Gamma}$ is 3-isoregular. Finally, by Proposition 3.44, we have that $\widehat{\Gamma}_2(u)$ is $(2,6)$-regular. In other words, Γ satisfies the 6-vertex condition. □

Example 5.20. There exists an infinite family of generalized quadrangles of order (q,q^2) whose point graphs are non rank 3 graphs (cf. [30, 31, 46]). By Theorem 5.17, the point graph of any such generalized quadrangle is $(3,7)$-regular. The second subconstituents of these graphs give rise to a hitherto unknown family of non-rank 3 graphs satisfying the 6-vertex condition.

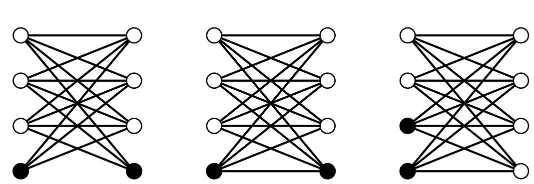
The smallest actual example is the point graph Γ of a non-classical generalized quadrangle of order $(5,25)$. Its parameters are given by

$$(v,k,\lambda,\mu)=(756,130,4,26).$$

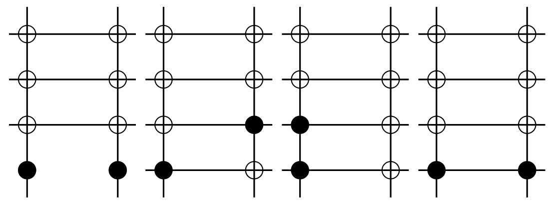
Its automorphism group is intransitive of rank 11.

Γ has two non-isomorphic second subconstituents Γ' and Γ''. Both satisfy the 6-vertex condition and both are in turn point graphs of partial quadrangles of order $(4,25,20)$. The automorphism group of Γ' is intransitive of rank 52 and the automorphism group of Γ'' is transitive of rank 5.

Proposition 5.21. *Let* Γ *be the point graph of a partial quadrangle, and suppose that* Γ *is* 3*-isoregular. Then* Γ *satisfies the* 8*-vertex condition if and only if it is regular for the following graph types of order* $(2,8)$*:*



Proof. By Theorem 5.17 we already know that Γ is $(3,7)$-regular. Let $\mathcal{M}$ be a transversal of all isomorphism classes of graph types of order (m,n), where $m\leqslant 3$ and $n\leqslant 7$. To show that Γ satisfies the 8-vertex condition means to show that it is $(2,8)$-regular. By Proposition 3.40 it suffices to show that Γ is $\mathbb{T}$-regular for all graph types of order $(2,8)$ that are $\widehat{\mathbb{T}}$-irreducible, for all $\widehat{\mathbb{T}}\in\mathcal{M}$. However, these are precisely the $(3,7)$-irreducible graphs types $\mathbb{T}$ of order $(2,8)$. In turn, by Lemma 3.23, these are the graph types $\mathbb{T}$ of order $(2,8)$ for which $\mathrm{Env}(\mathbb{T})$ is 4-connected. By the computer we may obtain a list of all such graph types. Since the point graph of a partial quadrangle does not contain K_4-e as an induced subgraph, we may decrease the list of graph types further to those whose underlying graph does not contain K_4-e. We end up with the above depicted graph types and the four graph types given below (for better visibility they are depicted as geometric configurations rather than graphs):



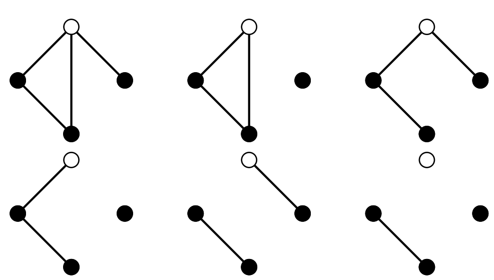
$\mathbb{T}_1$ $\mathbb{T}_2$ $\mathbb{T}_3$ $\mathbb{T}_4$

By Proposition 5.14, Γ is either the point graph of a generalized quadrangle of order (q, q^2) or it is triangle-free. Neither of the graph types $\mathbb{T}_1, \dots, \mathbb{T}_4$ is triangle-free. Thus, if Γ is triangle-free then we are done. Suppose therefore that Γ is the point graph of a generalized quadrangle $\Pi = (\mathscr{P}, \mathscr{L})$ of order $(s, t) = (q, q^2)$. Then we compute:

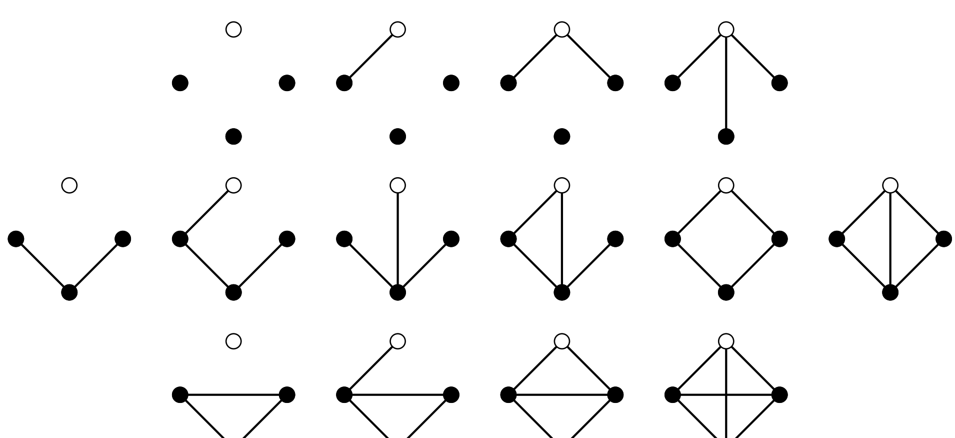
$$\begin{aligned}
\#(\Gamma, \mathbb{T}_1) &= (t+1)t(s-1)(s-2)(s-3), \\
\#(\Gamma, \mathbb{T}_2) &= (t+1)t(s-1)(s-2), \\
\#(\Gamma, \mathbb{T}_3) &= t^2 s(s-1)(s-2), \\
\#(\Gamma, \mathbb{T}_4) &= t^2 s(s-1)(s-2).
\end{aligned}$$ □

Let us at the end have a look on partial quadrangles Π in which every triad has the same number c of centers, but where the point graph Γ is not necessarily 3-isoregular.

Lemma 5.22. *Let Π be a partial quadrangle in which every triad has c centers, and let Γ be the point graph of Π. Then Γ is regular for all graph types of order $(3, 4)$, except possibly the following:*



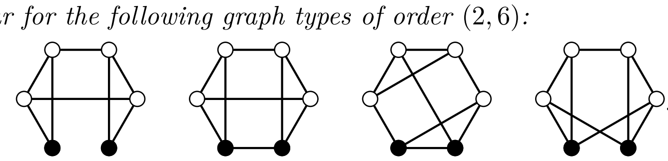
Proof. Let us first of all list all graph types of order $(3, 4)$ not mentioned above:



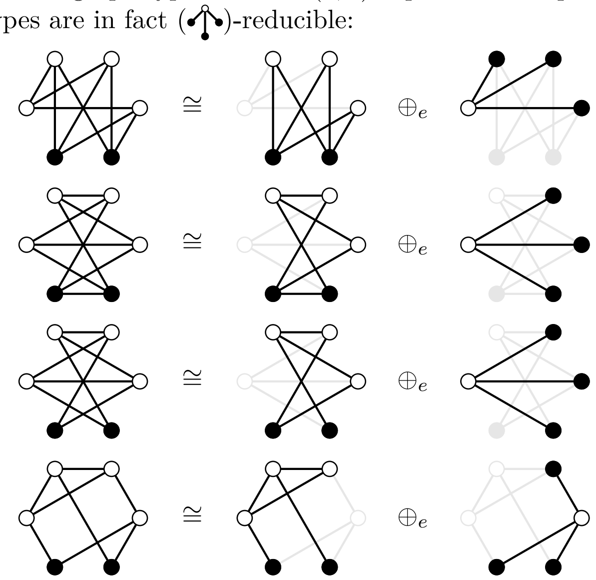
Suppose that the parameters of Γ as a strongly regular graph are (v,k,λ,μ). Then we count (using some of the previous calculations from Example 3.33):

$$\#(\ ) = c,$$
$$\#(\ ) = \#(\ )\cdot\#(\ ) - \#(\ ) = \mu - c,$$
$$\#(\ ) = \#(\ )\cdot\#(\ ) - \#(\ ) = (k-\mu) - (\mu - c) = k - 2\mu + c,$$
$$\#(\ ) = \#(\ )\cdot\#(\ ) - \#(\ ) - \#(\ ) = \bar{\lambda} - k + 2\mu - c - 1,$$
$$\#(\ ) = 0,$$
$$\#(\ ) = \#(\ )\cdot\#(\ ) - \#(\ ) - \#(\ ) = \mu - 1,$$
$$\#(\ ) = \#(\ )\cdot\#(\ ) - \#(\ ) = \lambda,$$
$$\#(\ ) = \#(\ )\cdot\#(\ ) - \#(\ ) - \#(\ ) = k - 2\lambda - 2,$$
$$\#(\ ) = \#(\ )\cdot\#(\ ) - \#(\ ) = k - \lambda - \mu,$$
$$\#(\ ) = \#(\ )\cdot\#(\ ) - \#(\ ) - \#(\ ) = \bar{\mu} - k + \lambda + \mu - 1,$$
$$\#(\ ) = \lambda - 1,$$
$$\#(\ ) = 0,$$
$$\#(\ ) = \#(\ )\cdot\#(\ ) - \#(\ ) = k - \lambda - 1,$$
$$\#(\ ) = \#(\ )\cdot\#(\ ) - \#(\ ) = \bar{\mu} - k + \lambda + 1.$$
□

Proposition 5.23. *Let Π be a partial quadrangle in which every triad has c centers, and let Γ be the point graph of Π. Then Γ satisfies the 6-vertex condition if and only if it is regular for the following graph types of order $(2,6)$:*



Proof. The four given graph types are exactly those from Proposition 5.8 that are irreducible for any of the graph types of order $(3,4)$ depicted in the proof of Lemma 5.22. All the other types are in fact ( )-reducible:



Now the claim follows from Proposition 3.40. □

6. Concluding remarks

The (m,n)-regularity introduced in Section 3 is a very strong condition. It is, in fact, interesting only for $m \leqslant 4$, because any 5-isoregular graph is 5-homogeneous and, in fact, homogeneous ([10, 19, 22]). At first sight, this appears to limit the use of the regularity conditions introduced in this paper. However, in principle, the definitions and results from Section 3 apply to other categories of combinatorial objects. Finite metric spaces (possibly with integer or with rational distances), directed graphs, or semilinear spaces come to mind.

For the category of finite graphs, the most interesting are (m,n)-regular graphs where $m \in \{2,3,4\}$. Here the goal is to find (m,n)-regular graphs that are not m-homogeneous and, if feasible, to classify such graphs completely, up to isomorphism.

As was noted above, every graph Γ is $(0,n)$-regular, because for every graph type $\mathbb{T} = (\varnothing, \iota, \Theta)$ the number $\#(\Gamma, \mathbb{T})$ is equal to the number of embeddings of Θ into Γ. Nevertheless, counting subgraphs of a graph has been used as a global invariant for distinguishing non-isomorphic graphs. For instance, in [34] subgraphs isomorphic to K_4 are counted in order to distinguish point-symmetric strongly regular graphs in three infinite families.

We would also like to mention K. Kováčiková's dissertation thesis [37] about counting subgraphs in strongly regular graphs, where she, among other things, counts the induced subgraphs of order $\leqslant 9$ in a putative Moore graph of valency 57. Her methods involve the solution of huge linear systems of equations. It will be interesting to compare her approach with the one given in this paper.

The $(2,t)$-regular graphs correspond exactly to the graphs that satisfy the t-vertex condition. There is a longstanding conjecture by M. Klin [16], that there exists a natural number t_0 such that for each $t \geqslant t_0$ all $(2,t)$-regular graphs are 2-homogeneous (i.e. they are rank 3 graphs). The largest t for which the existence of a non-rank 3, $(2,t)$-regular graph is settled is $t = 7$, due to Reichard [51, Theorem 2]. Thus in Klin's conjecture, we have $t_0 \geqslant 8$.

We should mention that the motivation to study graphs with the t-vertex condition comes not only from Klin's conjecture. The driving motivation to introduce the t-vertex condition was to distinguish the rank 3 graphs from other strongly regular graphs with the same parameters. In the times before the announcement of the classification of finite simple groups, there was the hope to uncover in this way new sporadic finite simple groups. A typical example of the use of the t-vertex condition as a distinguishing invariant is [45].

Up till now, $(3,t)$-regular graphs were known only for $t = 4$ (apart from the 3-homogeneous graphs). In this paper, the first cases of non-3-homogeneous $(3,7)$-regular graphs are observed. Among the examples, there are graphs whose automorphism group is intransitive. Given Klin's conjecture and because of the observation that $(3,t)$-regular graphs appear to be much rarer than $(2,t)$-regular graphs, it seems sensible to ask whether there exists a t_1 such that all $(3,t)$-regular graphs with $t \geqslant t_1$ are 3-homogeneous. This paper shows that if such a t_1 exists, then $t_1 \geqslant 8$. Note that every $(3,t)$-regular graph is $(2,t)$-regular. Thus, if Klin's conjecture turns out to be true, then this question can be answered using the classification of rank 3 graphs.

Recently, in [48] a classical family of strongly regular graphs originally constructed by Brouwer, Ivanov, and Klin (see [7]) was analyzed for regularities. It was shown there that these graphs are $(3,5)$-regular but not 2-homogeneous.

There is only one known $(4,5)$-regular graph that is not 4-homogeneous, the McLaughlin graph on 275 vertices. A computer experiment showed that this graph is not $(4,6)$-regular. Is every $(4,6)$-regular graph 4-homogeneous?

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Christian Pech, Radebeul, Germany
E-mail : `cpech@freenet.de`
Url : `https://www.researchgate.net/profile/Christian_Pech2`