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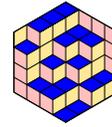
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On the action of the toggle group of the Dynkin diagram of type A

Yasuhide Numata & Yuiko Yamanouchi

ABSTRACT In this article, we consider involutions, called *toggings*, on the set of independent sets of the Dynkin diagram of type A , or a path graph. We are interested in the action of the subgroup of the symmetric group of the set of independent sets generated by toggings. We show that the subgroup coincides with the symmetric group.

1. INTRODUCTION

In this article, we are interested in the group generated by operations called *toggling*. The operation, *toggling*, is originally introduced for the set of order ideals of a poset in Cameron–Fon-Der-Flaass [1]. Let P be a finite poset, $J(P)$ the set of order ideals of P . For an element $p \in P$, we define the map $\tau_p: J(P) \rightarrow J(P)$ by

$$\tau_p(I) = \begin{cases} I \cup \{p\} & (p \text{ is a minimal element of } P \setminus I), \\ I \setminus \{p\} & (p \text{ is a maximal element of } I), \\ I & (\text{otherwise}), \end{cases}$$

for $I \in J(P)$. The toggle group is a subgroup of the symmetric group of $J(P)$ generated by toggings. Since we have a bijection from $J(P)$ to the set of antichains of P which maps an order ideal to the maximal elements of the order ideal, we can regard toggling as an involution on the set of antichains. Moreover we can define an analogue of the toggling τ_p as involutions not only on antichains of a poset, but also on important classes of subsets of a poset, e.g. order ideals of a poset, chains of a poset, antichains of a poset, interval-closed sets of a certain family of posets. Moreover we can consider a toggle group on nice subsets of a finite set with a combinatorial structure, e.g. independent sets of a graph, which is the main topic in this article. The toggle groups and their orbit structures are studied from the viewpoint of dynamical algebraic combinatorics, e.g. Striker–Williams [8], Striker [7], Joseph [2], and Joseph–Roby [4].

In this article, we write A_n to denote the Dynkin diagram of type A_n , i.e. the path graph with n vertices, and consider toggings on the set of the independent sets, i.e. subsets of vertices such that no pair are adjacent in the graph, of the graph A_n . In [3], Joseph and Roby study the orbit structure and the phenomenon called

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homomesy and related properties for the togglings. They consider the action of the product of togglings of each vertex of the graph A_n from left to right, and a map χ_i defined as $\chi_i(I) = 1$ if the independent set I contains i -th vertices, and otherwise $\chi_i(I) = 0$. They give a proof for Propp’s conjecture that the statistics $\chi_j - \chi_{n+1-j}$ are 0-mesic under the action for all $1 \leq j \leq n$. Moreover they also show that the statistics $2\chi_1 + \chi_2$ and $\chi_{n-1} + 2\chi_n$ are 1-mesic under the action.

In this article, we are interested in the transitivity of the action of the toggle group on the set of independent sets. For a nonempty independent set I , the number of vertices in the independent set I is greater than the number of vertices in the resulting independent set $\tau_v(I)$ by the toggling with respect to a vertex v in I . Hence we can obtain the empty independent set from I by applying the product of the togglings with respect to all vertices in the independent set. It follows from this observation that the action is 1-transitive. Multiple transitivity, however, does not seem trivial. In this paper, we consider the toggle group of the independent sets of A_n , and show that the action of the toggle group is f_{n+2} -transitive, where f_{n+2} is the $n + 2$ -th Fibonacci number, i.e. the number of independent sets of A_n . In other words, we show that the toggle group coincides with the symmetric group on the set of independent sets of A_n .

REMARK 1.1. In [5], Macauley, McCammond and Mortveit studied asynchronous cellular automata and their “dynamics groups.” Consider the binary words of length n with cyclic index, i.e. the sequence $x_0x_1x_2 \cdots x_{n-1}$ for $x_i \in \{0, 1\}$ with indices $i \in \mathbb{Z}/n\mathbb{Z}$. For a given map $F: \{0, 1\}^3 \rightarrow \{0, 1\}$, we can consider the operation to replace the i -th alphabet x_i of the word $x_0x_1x_2 \cdots x_{n-1}$ with $F(x_{i-1}, x_i, x_{i+1})$ for some index i . Let us consider the case where the local rule F is Rule 201, i.e. a map defined by

$$F(a, b, c) = \begin{cases} 1 - b & (a = c = 0), \\ b & (\text{otherwise}). \end{cases}$$

In this case, the operation for each index i is invertible. Hence the operations for all indices i generate a group. Naturally the binary word $x_0x_1x_2 \cdots x_{n-1}$ can be identified with the set of indices i such that $x_i = 1$. Since indices are in $\mathbb{Z}/n\mathbb{Z}$, the set can be regarded as the subset of the vertex set of the cycle with n vertices. The orbit containing the binary word $0 \cdots 0$ consists of binary words $x_0x_1x_2 \cdots x_{n-1}$ without indices i such that $x_i = x_{i+1} = 1$, corresponding to independent sets of the cycle. The group acts on the orbit as the toggle group for the cycle. Next consider the orbit containing the binary word $0 \cdots 00110$. A binary word $x_0x_1x_2 \cdots x_{n-1}$ is in the orbit if and only if $(x_{n-4}, x_{n-3}, x_{n-2}, x_{n-1}) = (0, 1, 1, 0)$ and $(x_i, x_{i+1}) \neq (1, 1)$ for $i \in \{0, 1, \dots, n - 6\}$. For the binary word in the orbit, the subword $x_0x_1x_2 \cdots x_{n-5}$ corresponds to an independent set of A_{n-4} . The group acts on the orbit as the toggle group for A_{n-4} . In [5], Macauley, McCammond and Mortveit conjectured that the group acts on the orbit with $0 \cdots 0$ as the symmetric group or the alternating group on the orbit, and that the group acts on the orbit with $0 \cdots 00110$ as the symmetric group on the orbit. In [6], Salo proved the conjecture by automaton-theoretical or language-theoretical methods. Our results give another proof of the conjecture.

This article is organized as follows: In Section 2, we consider the family of the symmetric groups indexed by the Fibonacci sequence. We give systems of generators for them as Theorem 2.6. In Section 3, we recall the definition of toggling on independent sets of a graph, and we consider the toggle group for independent sets of the graph A_n . As Theorem 3.7, we state that the toggle groups for independent sets of the graph A_n are isomorphic to the symmetric groups indexed by the Fibonacci sequence. We give proofs of main theorems in Section 4.

2. SYSTEMS OF GENERATORS FOR SYMMETRIC GROUPS INDEXED BY THE FIBONACCI SEQUENCE

Let $\{f_n\}_{n=0,1,\dots}$ be the Fibonacci sequence, i.e. the sequence of numbers defined by $f_0 = 0, f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$. We define F_{n+2} to be the set $\{1, 2, \dots, f_{n+2}\}$. We decompose F_{n+2} into F_{n+1} and \hat{F}_n , where

$$\begin{aligned} \hat{F}_n &= F_{n+2} \setminus F_{n+1} \\ &= \{1 + f_{n+1}, 2 + f_{n+1}, \dots, f_n + f_{n+1} = f_{n+2}\} \\ &= \{i + f_{n+1} \mid i \in F_n\}. \end{aligned}$$

Here we consider the family $\{\mathfrak{S}_{f_n}\}_{n=3,4,\dots}$ of symmetric groups indexed by the Fibonacci sequence. For $n \leq m$, we regard \mathfrak{S}_n as a subset of \mathfrak{S}_m in the usual manner. In this article, for a set X , \mathfrak{S}_X stands for the symmetric group on X . Under this notation, $\mathfrak{S}_{f_n} = \mathfrak{S}_{F_n}$.

For $n = 1, 2, \dots$, we define $\hat{t}_n \in \mathfrak{S}_{f_{n+2}}$ to be the product

$$(1, f_{n+1} + 1)(2, f_{n+1} + 2) \cdots (f_n, f_{n+1} + f_n)$$

of f_n transpositions. For $n \geq 1$, the inner automorphism by \hat{t}_n induces the isomorphism

$$\begin{aligned} \mathfrak{S}_{F_n} &\rightarrow \mathfrak{S}_{\hat{F}_n} \\ t &\mapsto \hat{t}_n t \hat{t}_n^{-1}. \end{aligned}$$

Since $F_n = \{1, 2, \dots, f_n\}$ and $\check{F}_n = \{i + f_{n+1} \mid i \in F_n\}$, we have $F_n \cap \check{F}_n = \emptyset$, which implies that every element of \mathfrak{S}_{F_n} commutes with every element of $\mathfrak{S}_{\check{F}_n}$. For t and $t' \in \mathfrak{S}_{f_n}$, it follows that $t' \in \mathfrak{S}_{f_n}$ commutes with $\hat{t}_n t \hat{t}_n^{-1} \in \mathfrak{S}_{\check{F}_n}$, which implies

$$t \cdot \hat{t}_n t \hat{t}_n^{-1} \cdot t' \cdot \hat{t}_n t' \hat{t}_n^{-1} = t \cdot t' \cdot \hat{t}_n t \hat{t}_n^{-1} \cdot \hat{t}_n t' \hat{t}_n^{-1} = t t' \cdot \hat{t}_n t t' \hat{t}_n^{-1}$$

Hence the inner automorphism by \hat{t}_n induces the isomorphism

$$\begin{aligned} \mathfrak{S}_{f_n} &\rightarrow \tilde{\mathfrak{S}}_{f_n} \\ t &\mapsto t \cdot \hat{t}_n t \hat{t}_n^{-1}, \end{aligned}$$

where

$$\begin{aligned} \tilde{\mathfrak{S}}_{f_n} &= \left\{ g \in \mathfrak{S}_{f_{n+2}} \mid \begin{array}{l} \forall i \leq f_n, g(i + f_{n+1}) = g(i) + f_{n+1} \\ f_n < \forall i \leq f_{n+1}, g(i) = i \end{array} \right\} \\ &\subset \mathfrak{S}_{F_n} \times \mathfrak{S}_{\hat{F}_n} \subset \mathfrak{S}_{F_{n+1}} \times \mathfrak{S}_{\hat{F}_n} \subset \mathfrak{S}_{F_{n+2}}. \end{aligned}$$

For $1 \leq k \leq n$, we define $t_{k,n} \in \mathfrak{S}_{F_{n+2}}$ by

$$t_{k,n} = \begin{cases} t_{k,n-1} \cdot \hat{t}_n t_{k,n-2} \hat{t}_n^{-1} & (k \leq n-2), \\ t_{n-1,n-1} & (k = n-1), \\ \hat{t}_n & (i = n), \end{cases}$$

recursively. By definition, for $k < n$, $t_{k,n}$ is an element of $\mathfrak{S}_{F_{n+1}} \times \mathfrak{S}_{\hat{F}_n}$, and $t_{n-1,n}$ is an element of $\mathfrak{S}_{F_{n+1}}$. We define subsets G_n and G'_n of $\mathfrak{S}_{F_{n+2}}$ by

$$\begin{aligned} G'_n &= \{t_{k,n} \mid k \leq n-2\}, \\ G_n &= \{t_{k,n} \mid k \leq n\}. \end{aligned}$$

EXAMPLE 2.1. Consider the case where $n = 1$. Since \hat{t}_1 is the transposition $(1, 2) \in \mathfrak{S}_{f_3} = \mathfrak{S}_2$, we have $t_{1,1} = \hat{t}_1 = (1, 2)$ and $G_1 = \{(1, 2)\}$. Hence G_1 generates $\mathfrak{S}_2 = \mathfrak{S}_{f_3}$.

EXAMPLE 2.2. In the case where $n = 2$, \hat{t}_2 is the transposition $(1, 3) \in \mathfrak{S}_{f_4} = \mathfrak{S}_3$, we have $t_{1,2} = t_{1,1} = (1, 2)$ and $t_{2,2} = \hat{t}_2 = (1, 3)$. Hence $G_2 = \{ (1, 2), (1, 3) \}$. The set G_2 generates $\mathfrak{S}_3 = \mathfrak{S}_{f_4}$.

EXAMPLE 2.3. In the case where $n = 3$, \hat{t}_4 is the product $(1, 4)(2, 5) \in \mathfrak{S}_{f_5} = \mathfrak{S}_5$ of the transpositions, we have

$$\begin{aligned} t_{1,3} &= (1, 2) \cdot (4, 5), \\ t_{2,3} &= t_{2,2} = (1, 3), \\ t_{3,3} &= \hat{t}_3 = (1, 4)(2, 5). \end{aligned}$$

Hence

$$\begin{aligned} G_3 &= \{ (1, 2)(4, 5), (1, 3), (1, 4)(2, 5) \}, \\ G'_3 &= \{ (1, 2)(4, 5) \}. \end{aligned}$$

Since

$$\begin{aligned} \tilde{\mathfrak{S}}_{f_3} &= \left\{ g \in \mathfrak{S}_{f_5} \mid \begin{array}{l} \forall i \leq f_3, g(i + f_4) = g(i) + f_4 \\ f_3 < \forall i \leq f_4, g(i) = i \end{array} \right\} \\ &= \{ g \in \mathfrak{S}_{f_5} \mid g(1 + 3) = g(1) + 3, g(2 + 3) = g(2) + 3, g(3) = 3 \}, \end{aligned}$$

the group $\tilde{\mathfrak{S}}_{f_3}$ is generated by $\{ (1, 2)(4, 5) \}$. It follows from direct calculation that

$$\begin{aligned} (1, 3) &= t_{2,3}, \\ (2, 3) &= t_{1,3}(1, 3)t_{1,3}^{-1}, \\ (4, 3) &= t_{3,3}(1, 3)t_{3,3}^{-1}, \\ (5, 3) &= t_{1,3}(4, 3)t_{1,3}^{-1}. \end{aligned}$$

Hence G_3 generates $\mathfrak{S}_5 = \mathfrak{S}_{f_5}$.

EXAMPLE 2.4. Consider the case where $n = 4$. In this case, $\hat{t}_4 = (1, 6)(2, 7)(3, 8) \in \mathfrak{S}_{f_6} = \mathfrak{S}_8$ of the transpositions. Hence

$$\begin{aligned} t_{1,4} &= (1, 2)(4, 5) \cdot (6, 7), \\ t_{2,4} &= (1, 3) \cdot (6, 8), \\ t_{3,4} &= t_{3,3} = (1, 4)(2, 5), \\ t_{4,4} &= \hat{t}_4 = (1, 6)(2, 7)(3, 8). \end{aligned}$$

Hence

$$\begin{aligned} G_4 &= \{ (1, 2)(4, 5)(6, 7), (1, 3)(6, 8), (1, 4)(2, 5), (1, 6)(2, 7)(3, 8) \}, \\ G'_4 &= \{ (1, 2)(4, 5)(6, 7), (1, 3)(6, 8) \}. \end{aligned}$$

We show the following theorems by induction on n in Subsection 4.1.

THEOREM 2.5. For $n = 3, 4, \dots$, the set G'_n generates the group $\tilde{\mathfrak{S}}_{f_n}$. Hence the group $\langle G'_n \rangle$ generated by G'_n is isomorphic to \mathfrak{S}_{f_n} .

THEOREM 2.6. For $n = 1, 2, \dots$, the set G_n generates the f_{n+2} -th symmetric group $\mathfrak{S}_{f_{n+2}}$.

3. THE TOGGLE GROUP FOR INDEPENDENT SETS OF THE DYNKIN DIAGRAM OF TYPE A

In this article, we consider simple graphs. We regard a graph as the pair (V, E) of the set V of vertices and the set E of edges. We also regard an edge as a subset of V of size two. An independent set of a graph is a subset I of the vertex set V of the graph satisfying

$$u, v \in I \implies \{u, v\} \notin E.$$

Roughly speaking, an independent set of a graph is a subset of vertices such that no pair is adjacent in (V, E) .

Let \mathcal{I} be the set of independent sets of a graph (V, E) . For a vertex $v \in V$, we define the map τ_v by

$$\tau_v : \mathcal{I} \rightarrow \mathcal{I}$$

$$I \mapsto \begin{cases} I \setminus \{v\} & (v \in I), \\ I \cup \{v\} & (v \notin I \text{ and } I \cup \{v\} \in \mathcal{I}), \\ I & (\text{otherwise}). \end{cases}$$

By definition, the map τ_v is an involution on \mathcal{I} . Hence τ_v is an element of the symmetric group $\mathfrak{S}_{\mathcal{I}}$ on \mathcal{I} . We call the map τ_v the *toggling* on \mathcal{I} with respect to v . We also call the subgroup generated by all togglings the *toggle group* of the graph (V, E) .

REMARK 3.1. Let \mathcal{I} be the set of independent sets of a graph (V, E) . For a vertex $v \in V$, we can also describe the toggling τ_v as follows: We define $N_v(V, E)$ to be the set of neighbours of v , i.e. $\{u \mid \{u, v\} \in E\}$. Let $\mathcal{I}' = \{I \in \mathcal{I} \mid I \cap N_v(V, E) \neq \emptyset\}$. The toggling τ_v acts on \mathcal{I}' trivially. Let

$$V' = V \setminus (\{v\} \cup N_v(V, E)),$$

$$E' = \{\{u, w\} \in E \mid u, w \in V'\}.$$

The graph (V', E') is the induced subgraph of the graph (V, E) with respect to all vertices except v and its neighbours. Let \mathcal{I}'' be the set of independent sets of (V', E') . For $I \in \mathcal{I}''$, I and $I \cup \{v\}$ are independent sets of (V, E) . Then we have $\tau_v(I) = I \cup \{v\}$ and $\tau_v(I \cup \{v\}) = I$ for $I \in \mathcal{I}''$. Since $\mathcal{I} = \mathcal{I}' \cup \mathcal{I}'' \cup \mathcal{I}'''$ for $\mathcal{I}''' = \{I \cup \{v\} \mid I \in \mathcal{I}''\}$, the toggling τ_v is the involution on \mathcal{I} defined by

$$\begin{cases} \tau(I) = \tau(I) & (I \in \mathcal{I}'), \\ \tau(I) = \tau(I \cup \{v\}) & (I \in \mathcal{I}''), \\ \tau(I \cup \{v\}) = \tau(I) & (I \in \mathcal{I}'''). \end{cases}$$

The toggling $\tau_v \in \mathfrak{S}_{\mathcal{I}}$ is the product of transpositions swapping I and $I \cup \{v\}$. Hence, the toggling $\tau_v \in \mathfrak{S}_{\mathcal{I}}$ is an even permutation if and only if $\#\mathcal{I}''$ is even.

REMARK 3.2. The toggle group acts on the set \mathcal{I} of independent sets naturally. Moreover, for an independent set $I = \{v_1, v_2, \dots, v_l\}$ of size l , we have

$$\tau_{v_1} \circ \tau_{v_2} \circ \dots \circ \tau_{v_l}(I) = \emptyset.$$

We can obtain the empty set from any independent set by applying some togglings. Hence the group generated by all togglings of a graph acts transitively on the set of independent sets of the graph. In other words, the action of the toggle group is 1-transitive.

We give some small examples. First we consider cycles. The toggle group of some cycle is a proper subgroup of the symmetric group on the set of independent sets.

EXAMPLE 3.3 (\tilde{A}_3). Consider the cycle (V, E) such that

$$V = \{0, 1, 2, 3\}, \quad E = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 0\}\}.$$

and the set \mathcal{I} of its independent sets. Consider the toggling τ_0 for the vertex 0. The induced subgraph with respect to all vertices except 0 and neighbours of 0 is $(\{2\}, \emptyset)$. The set of its independent sets is $\{\emptyset, \{2\}\}$, consisting of two elements. Hence $\tau_0 \in \mathfrak{S}_{\mathcal{I}}$ is an even permutation. Similarly, we can show that $\tau_v \in \mathfrak{S}_{\mathcal{I}}$ is an even permutation for each vertex $v \in V$. Hence we have the toggle group is a subgroup of the alternating group on \mathcal{I} , which is a proper subgroup of the symmetric group $\mathfrak{S}_{\mathcal{I}}$.

Next we consider trees. For the tree in Example 3.4, the toggle group equals the symmetric group on the set of independent sets. For the tree in Example 3.5, the toggle group is a proper subgroup of the symmetric group on the set of independent sets.

EXAMPLE 3.4 (D_4). Consider the tree (V, E) such that

$$V = \{1, 2, 3, 4\}, \quad E = \{\{1, 3\}, \{2, 3\}, \{3, 4\}\}.$$

The set \mathcal{I} of independent sets of the tree consists of the following:

$$\begin{aligned} I_1 = \emptyset & & I_2 = \{1\} & & I_3 = \{2\} & & I_4 = \{1, 2\} \\ I_5 = \{4\} & & I_6 = \{1, 4\} & & I_7 = \{2, 4\} & & I_8 = \{1, 2, 4\} & & I_9 = \{3\}. \end{aligned}$$

We regard \mathcal{I} as $\{1, 2, \dots, 9\}$ by identifying I_k with k . Then togglings are the product of the transpositions as follows:

$$\begin{aligned} \tau_1 &= (1, 2)(3, 4)(5, 6)(7, 8) & \tau_2 &= (1, 3)(2, 4)(5, 7)(6, 8) \\ \tau_4 &= (1, 5)(2, 6)(3, 7)(4, 8) & \tau_3 &= (1, 9). \end{aligned}$$

Let

$$\begin{aligned} \sigma_2 &= \tau_1, & \sigma_3 &= \tau_2, & \sigma_4 &= \tau_2\tau_1, & \sigma_5 &= \tau_4, \\ \sigma_6 &= \tau_4\tau_1, & \sigma_7 &= \tau_4\tau_2, & \sigma_8 &= \tau_4\tau_2\tau_1. \end{aligned}$$

Then we have $\sigma_k(1) = k$ and $\sigma_k(9)$. Hence we have

$$\sigma_k\tau_3\sigma_k^{-1} = (\sigma_k(1), \sigma_k(9)) = (k, 9).$$

Since we have $(1, 9), (2, 9), \dots, (8, 9)$ in the toggle group, the toggle group is the symmetric group $\mathfrak{S}_{\mathcal{I}}$.

EXAMPLE 3.5 (\tilde{D}_5). Consider the tree (V, E) such that

$$V = \{0, 1, 2, 3, 4, 5\}, \quad E = \{\{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{4, 0\}\},$$

and the set \mathcal{I} of its independent sets. Consider the toggling τ_3 for the vertex 3. The induced subgraph with respect to all vertices except 3 and neighbours of 3 is the disjoint union of the graphs $(\{0\}, \emptyset)$ and $(\{5\}, \emptyset)$. The number of the independent sets of the disjoint union of graphs is the product of the number of the independent sets of each graph. Since the number of the independent sets of $(\{0\}, \emptyset)$ is two, the number of the independent sets of the induced graph is even. Hence $\tau_3 \in \mathfrak{S}_{\mathcal{I}}$ is even. Similarly, we have τ_4 is even. Next consider the toggling τ_1 for the vertex 1. The induced subgraph with respect to all vertices except 1 and neighbours of 1 is the disjoint union of the graphs $(\{2\}, \emptyset)$ and $(\{0, 4, 5\}, \{\{0, 4\}, \{4, 5\}\})$. Hence we have the number of independent sets of the induced subgraph is even, which implies τ_1 is even. Similarly, it follows that τ_v is even for $v \in \{0, 2, 5\}$. Hence the toggle group is a subgroup of the alternating group on \mathcal{I} , which is a proper subgroup of $\mathfrak{S}_{\mathcal{I}}$.

Let $A_n = (V_n, E_n)$ be the graph such that

$$V_n = \{ 1, \dots, n \},$$

$$E_n = \{ \{1, 2\}, \{2, 3\} \dots, \{n - 1, n\} \},$$

i.e. the Dynkin diagram of type A_n . Let \mathcal{I}_n be the set of independent sets of A_n . In this article, we are interested in the togglings of the graph A_n .

For $1 \leq k \leq n$, we define $\tau_{k,n}$ to be the toggling τ_k on \mathcal{I}_n with respect to $k \in V_n$, i.e.

$$\tau_{k,n}: \mathcal{I}_n \rightarrow \mathcal{I}_n$$

$$I \mapsto \begin{cases} I \setminus \{ k \} & (k \in I), \\ I \cup \{ k \} & (k - 1, k, k + 1 \notin I), \\ I & (\text{otherwise}). \end{cases}$$

We also define Γ_n to be the group generated by togglings

$$\{ \tau_{1,n}, \tau_{2,n}, \dots, \tau_{n,n} \}.$$

We call this group the *toggle group of A_n* . By definition the toggle group Γ_n of A_n acts on the set \mathcal{I}_n of independent sets on A_{n+1} , naturally. In the other words, the toggle group Γ_n is a subgroup of the symmetric group $\mathfrak{S}_{\mathcal{I}_n}$ of \mathcal{I}_n .

REMARK 3.6. The togglings $\tau_{k,n}$ for $1 \leq k \leq n$ satisfy the following relations:

$$\tau_{k,n}^2 = \text{id}.$$

$$\tau_{k,n} \tau_{k',n} = \tau_{k',n} \tau_{k,n} \quad (|k - k'| > 1).$$

$$(\tau_{k,n} \tau_{k+1,n})^6 = \text{id}.$$

Hence the toggle group Γ_n is a finite quotient group of the Coxeter group with respect to

$$\underbrace{\bullet \overset{6}{-} \bullet \overset{6}{-} \bullet \overset{6}{-} \dots \overset{6}{-} \bullet}_n,$$

which is infinite if $n > 2$.

We will show the following theorem in Subsection 4.2.

THEOREM 3.7. *For $n = 1, 2, \dots$, the toggle group Γ_n of A_n is the symmetric group $\mathfrak{S}_{\mathcal{I}_n}$ of the set \mathcal{I}_n of independent sets of A_n . Hence the toggle group Γ_n of A_n is isomorphic to the f_{n+2} -th symmetric group $\mathfrak{S}_{f_{n+2}}$.*

COROLLARY 3.8. *For $n = 1, 2, \dots$, the action of the toggle group Γ_n of A_{n+1} on \mathcal{I}_n is f_{n+2} -transitive.*

4. PROOF OF MAIN THEOREMS

4.1. THEOREMS 2.5 AND 2.6. Here we show these Theorems 2.5 and 2.6 by induction on n . To show theorems, we show some lemmas.

LEMMA 4.1. *If $n \geq 3$ and G_{n-2} generates \mathfrak{S}_{f_n} , then G'_n generates $\tilde{\mathfrak{S}}_{f_n}$.*

Proof. Consider the isomorphism

$$\varphi: \mathfrak{S}_{f_n} \rightarrow \tilde{\mathfrak{S}}_{f_n}$$

$$t \mapsto t \cdot \hat{t}_n \hat{t}_n^{-1}.$$

By definition, for $k \leq n - 2$, $t_{k,n}$ is the image $\varphi(t_{k,n-2})$ of $t_{k,n-2}$. Hence we have

$$\begin{aligned} \langle G'_n \rangle &= \langle \varphi(t_{1,n-2}), \varphi(t_{2,n-2}), \dots, \varphi(t_{n-2,n-2}) \rangle \\ &= \varphi(\langle t_{1,n-2}, t_{2,n-2}, \dots, t_{n-2,n-2} \rangle) \\ &= \varphi(\langle G_{n-2} \rangle). \end{aligned}$$

Since G_{n-2} generates \mathfrak{S}_{f_n} , we have $\langle G'_n \rangle = \varphi(\mathfrak{S}_{f_n}) = \tilde{\mathfrak{S}}_{f_n}$. □

LEMMA 4.2. *If $n \geq 4$ and G'_n generates $\tilde{\mathfrak{S}}_{f_n}$, then the group $\langle G_n \rangle$ generated by G_n contains cyclic permutations $(i, i + 1, i + 2)$ for $1 \leq i \leq f_{n+2} - 2$.*

Proof. We decompose F_{n+2} into three subsets

$$\begin{aligned} F_n &= \{ 1, 2, \dots, f_n \}, \\ \hat{F}_{n-1} &= \{ i + f_n \mid i \in F_{n-1} \}, \\ \hat{F}_n &= \{ i + f_{n+1} \mid i \in F_n \}. \end{aligned}$$

First we show that $(1, 2, 3) \in \langle G_n \rangle$. Since $n \geq 4$, $f_n > f_3 = 2$. Since $\tilde{\mathfrak{S}}_{f_n} = \langle G'_n \rangle \subset \langle G_n \rangle$, the group $\langle G_n \rangle$ contains the following elements:

$$\begin{aligned} (1, 2)\hat{t}_n(1, 2)\hat{t}_n^{-1} &= (1, 2)(f_{n+1} + 1, f_{n+1} + 2), \\ (2, 3)\hat{t}_n(2, 3)\hat{t}_n^{-1} &= (2, 3)(f_{n+1} + 2, f_{n+1} + 3). \end{aligned}$$

Since $t_{n-1,n} = \hat{t}_{n-1} \in G_n$, we have

$$\hat{t}_{n-1}(1, 2)(f_{n+1} + 1, f_{n+1} + 2)\hat{t}_{n-1}^{-1} \in \langle G_n \rangle.$$

Since $\hat{t}_{n-1} \in \mathfrak{S}_{F_{n+1}}$,

$$\begin{aligned} \hat{t}_{n-1}(1, 2)(f_{n+1} + 1, f_{n+1} + 2)\hat{t}_{n-1}^{-1} &= \hat{t}_{n-1}(1, 2)\hat{t}_{n-1}^{-1} \cdot (f_{n+1} + 1, f_{n+1} + 2) \\ &= (f_n + 1, f_n + 2)(f_{n+1} + 1, f_{n+1} + 2). \end{aligned}$$

Hence $\langle G_n \rangle$ contains

$$\begin{aligned} &(f_n + 1, f_n + 2)(f_{n+1} + 1, f_{n+1} + 2) \cdot (2, 3)(f_{n+1} + 2, f_{n+1} + 3) \\ &= (2, 3) \cdot (f_n + 1, f_n + 2) \cdot (f_{n+1} + 1, f_{n+1} + 2)(f_{n+1} + 2, f_{n+1} + 3) \\ &= (2, 3) \cdot (f_n + 1, f_n + 2) \cdot (f_{n+1} + 1, f_{n+1} + 2, f_{n+1} + 3). \end{aligned}$$

Hence $\langle G_n \rangle$ contains the square

$$\begin{aligned} &((2, 3) \cdot (f_n + 1, f_n + 2) \cdot (f_{n+1} + 1, f_{n+1} + 2, f_{n+1} + 3))^2 \\ &= (2, 3)^2 \cdot (f_n + 1, f_n + 2)^2 \cdot (f_{n+1} + 1, f_{n+1} + 2, f_{n+1} + 3)^2 \\ &= (f_{n+1} + 1, f_{n+1} + 3, f_{n+1} + 2) \end{aligned}$$

of the element. Since $\hat{t}_n(f_{n+1} + 1, f_{n+1} + 3, f_{n+1} + 2)\hat{t}_n^{-1} = (1, 3, 2)$, $\langle G_n \rangle$ contains cyclic permutations $(1, 3, 2)$ and $(1, 2, 3)$.

Next we show that $\langle G_n \rangle$ contains the cyclic permutation (i, j, k) for $i, j, k \in F_n$. Fix an element $\sigma \in \mathfrak{S}_{F_n}$ such that $\sigma(1) = i, \sigma(2) = j, \sigma(3) = k$. Since $\tilde{\mathfrak{S}}_{f_n} = \langle G'_n \rangle \subset \langle G_n \rangle$, $\langle G_n \rangle$ contains $\tilde{\sigma} = \sigma\hat{t}_n\sigma\hat{t}_n^{-1} \in \tilde{\mathfrak{S}}_{f_n}$. Hence $\langle G_n \rangle$ contains

$$\tilde{\sigma}(1, 2, 3)\tilde{\sigma}^{-1} = (\tilde{\sigma}(1), \tilde{\sigma}(2), \tilde{\sigma}(3)) = (\sigma(1), \sigma(2), \sigma(3)) = (i, j, k).$$

Next we show that $\langle G_n \rangle$ contains the cyclic permutation (i, j, k) for $i, j, k \in \hat{F}_{n-1} = \{ i' + f_n \mid i' \in F_{n-1} \}$. Since F_{n-1} contains $i - f_n, j - f_n$ and $k - f_n$ for $i, j, k \in \hat{F}_{n-1}$,

$\langle G_n \rangle$ contains $(i - f_n, j - f_n, k - f_n)$. Hence $\langle G_n \rangle$ contains

$$\begin{aligned} \hat{t}_{n-1}(i - f_n, j - f_n, k - f_n)\hat{t}_{n-1}^{-1} &= (\hat{t}_{n-1}(i - f_n), \hat{t}_{n-1}(j - f_n))\hat{t}_{n-1}(k - f_n) \\ &= (i, j, k). \end{aligned}$$

Next we show that $\langle G_n \rangle$ contains the cyclic permutation (i, j, k) for $i, j, k \in \hat{F}_n = \{i' + f_{n+1} \mid i' \in F_n\}$. Since F_n contains $i - f_{n+1}, j - f_{n+1}$ and $k - f_{n+1}$ for $i, j, k \in \hat{F}_n$, $\langle G_n \rangle$ contains $(i - f_{n+1}, j - f_{n+1}, k - f_{n+1})$. Hence $\langle G_n \rangle$ contains

$$\begin{aligned} \hat{t}_n(i - f_{n+1}, j - f_{n+1}, k - f_{n+1})\hat{t}_n^{-1} &= (\hat{t}_n(i - f_{n+1}), \hat{t}_{n-1}(j - f_{n+1}))\hat{t}_n(k - f_{n+1}) \\ &= (i, j, k). \end{aligned}$$

Next $\langle G_n \rangle$ contains the cyclic permutation (i, j, k) for $i \in F_n$ and $j, k \in \hat{F}_{n-1} = \{i' + f_n \mid i' \in F_{n-1}\}$. For $j, k \in \hat{F}_{n-1}$, we have $j - f_n$ and $k - f_n \in F_{n-1} \subset F_n$. Since $f_n \in F_n \setminus F_{n-1}$, $\langle G_n \rangle$ contains the cyclic permutation $(f_n, j - f_n, k - f_n)$. Hence $\langle G_n \rangle$ contains

$$\begin{aligned} \hat{t}_{n-1}(f_n, j - f_n, k - f_n)\hat{t}_{n-1}^{-1} &= (\hat{t}_{n-1}(f_n), \hat{t}_{n-1}(j - f_n), \hat{t}_{n-1}(k - f_n)) \\ &= (f_n, j, k). \end{aligned}$$

Since $\tilde{\mathfrak{S}}_{f_n} = \langle G'_n \rangle \subset \langle G_n \rangle$, $\langle G_n \rangle$ contains

$$\tilde{\sigma} = (i, f_n)\hat{t}_n(i, f_n)\hat{t}_n^{-1} = (i, f_n)(f_{n+1} + i, f_{n+1} + f_n)$$

for $i < f_n$. Hence $\langle G_n \rangle$ contains

$$\tilde{\sigma}(f_n, j, k)\tilde{\sigma}^{-1} = (\tilde{\sigma}(f_n), \tilde{\sigma}(j), \tilde{\sigma}(k)) = (i, j, k).$$

Next $\langle G_n \rangle$ contains the cyclic permutation (i, j, k) for

$$i, j \in \hat{F}_{n-1} = \{i' + f_n \mid i' \in F_{n-1}\} \text{ and } k \in \hat{F}_n = \{i + f_{n+1} \mid i \in F_n\}.$$

Since $k - f_{n+1} \in F_n$ for $k \in \hat{F}_n$, $\langle G_n \rangle$ contains the cyclic permutation $(k - f_{n+1}, i, j)$. Hence $\langle G_n \rangle$ contains

$$\begin{aligned} \hat{t}_n(k - f_{n+1}, i, j)\hat{t}_n^{-1} &= (\hat{t}_n(k - f_{n+1}), \hat{t}_n(i), \hat{t}_n(j)) \\ &= (k, i, j) = (i, j, k). \end{aligned}$$

Next $\langle G_n \rangle$ contains the cyclic permutation (i, j, k) for $i, j \in F_n$ and $k \in \hat{F}_{n-1} = \{i' + f_n \mid i' \in F_{n-1}\}$. Since F_{n-1} contains $k - f_n$ for $k \in \hat{F}_{n-1}$ and \hat{F}_{n-1} contains $f_n + 1, f_n + 2$, $\langle G_n \rangle$ contains $(k - f_n, f_n + 1, f_n + 2)$. Hence $\langle G_n \rangle$ contains

$$\begin{aligned} \hat{t}_{n-1}(k - f_n, f_n + 1, f_n + 2)\hat{t}_{n-1}^{-1} &= (\hat{t}_{n-1}(k - f_n), \hat{t}_{n-1}(f_n + 1), \hat{t}_{n-1}(f_n + 2)) \\ &= (k, 1, 2). \end{aligned}$$

Fix an element $\sigma \in \mathfrak{S}_{f_n}$ such that $\sigma(1) = i$ and $\sigma(2) = j$. Since $\tilde{\mathfrak{S}}_{f_n} = \langle G'_n \rangle \subset \langle G_n \rangle$, $\langle G_n \rangle$ contains $\tilde{\sigma} = \sigma\hat{t}_n\sigma\hat{t}_n^{-1}$. Hence $\langle G_n \rangle$ contains

$$\tilde{\sigma}(k, 1, 2)\tilde{\sigma}^{-1} = (\tilde{\sigma}(k), \tilde{\sigma}(1), \tilde{\sigma}(2)) = (k, i, j) = (i, j, k).$$

Next $\langle G_n \rangle$ contains the cyclic permutation (i, j, k) for

$$i \in \hat{F}_{n-1} = \{i' + f_n \mid i' \in F_{n-1}\} \text{ and } j, k \in \hat{F}_n = \{i + f_{n+1} \mid i \in F_n\}.$$

Since F_n contains $j - f_{n+1}, k - f_{n+1}$ for $j, k \in \hat{F}_n$, $\langle G_n \rangle$ contains the cyclic permutation $(j - f_{n+1}, k - f_{n+1}, i)$. Hence $\langle G_n \rangle$ contains

$$\begin{aligned} \hat{t}_n(j - f_{n+1}, i - f_{n+1}, i)\hat{t}_n^{-1} &= (\hat{t}_n(j - f_{n+1}), \hat{t}_n(k - f_{n+1}), \hat{t}_n(i)) \\ &= (j, k, i) = (i, j, k). \end{aligned}$$

Since $n \geq 4$, $\#\hat{F}_{n-1} = f_{n-1} \geq f_3 = 2$. Hence we have $(i, i + 1, i + 2) \in \langle G \rangle$ for $1 \leq i \leq f_{n+2} - 2$. \square

LEMMA 4.3. *If $n \geq 4$ and G'_n generates $\tilde{\mathfrak{S}}_{f_n}$, then G_n generates $\mathfrak{S}_{F_{n+2}}$.*

Proof. Since $\langle G_n \rangle$ contains cyclic permutations $(i, i + 1, i + 2)$ for $1 \leq i \leq f_{n+2} - 2$ by Lemma 4.2, $\langle G_n \rangle$ contains the f_{n+2} -th alternating group $\mathfrak{A}_{f_{n+2}}$. Since either f_n or f_{n-1} is an odd number, the alternating group $\mathfrak{A}_{f_{n+2}}$ does not contain either $t_{n,n} = \hat{t}_n$ or $t_{n,n-1} = \hat{t}_{n-1}$. Since the alternating group $\mathfrak{A}_{f_{n+2}}$ is a subgroup of $\mathfrak{S}_{f_{n+2}}$ of index 2, the subgroup $\langle G_n \rangle$ coincide with $\mathfrak{S}_{f_{n+2}}$. \square

By Examples 2.1, 2.2 and 2.3, we can show the base case, i.e. the case where $n = 1, 2, 3$. Moreover we have Lemmas 4.1 and 4.3 for induction step. Hence we have Theorems 2.5 and 2.6.

4.2. PROOF OF THEOREM 3.7. To show Theorem 3.7, we give a bijection between \mathcal{I}_n and F_{n+2} , and translate toggling to permutations in $\mathfrak{S}_{f_{n+2}}$.

We can decompose the set \mathcal{I}_n of independent set of the graph A_n into the following two subsets:

$$\begin{aligned} & \{ I \in \mathcal{I}_n \mid n \notin I \}, \\ & \{ I \in \mathcal{I}_n \mid n \in I \}. \end{aligned}$$

It is easy to see that $\{ I \in \mathcal{I}_n \mid n \notin I \} = \mathcal{I}_{n-1}$. We define $\hat{\mathcal{I}}_{n-2}$ to be the set $\{ I \in \mathcal{I}_n \mid n \in I \}$. For $I \in \mathcal{I}_n$, by definition, we have $n - 1 \notin I$ if $n \in I$. Hence we have the bijection φ_{n-2} from $\hat{\mathcal{I}}_{n-2}$ to \mathcal{I}_{n-2} by removing the vertex n , i.e. the map defined by

$$\begin{aligned} \varphi_{n-2}: \hat{\mathcal{I}}_{n-2} &\rightarrow \mathcal{I}_{n-2} \\ I &\mapsto I \setminus \{ n \}. \end{aligned}$$

Hence we have the recurrence relation $\#\mathcal{I}_n = \#\mathcal{I}_{n-1} + \#\mathcal{I}_{n-2}$. Since $\#\mathcal{I}_1 = 2$ and $\#\mathcal{I}_2 = 3$, the number $\#\mathcal{I}_n$ of independent sets of A_n is equal to the $(n + 2)$ -th Fibonacci number f_{n+2} .

Since the number of \mathcal{I}_n is $f_{n+2} = \#F_{n+2}$, the elements of \mathcal{I}_n can be indexed by F_{n+2} . We give the index $\iota_n(I)$ of an independent set I in \mathcal{I}_n in the following manner: In the case where $n = 1$, we define

$$\iota_1(I) = \begin{cases} 1 & (I = \emptyset), \\ 2 & (I = \{ 1 \}). \end{cases}$$

In the case where $n = 2$, we define

$$\iota_1(I) = \begin{cases} 1 & (I = \emptyset), \\ 2 & (I = \{ 1 \}), \\ 3 & (I = \{ 2 \}). \end{cases}$$

In the case where $n > 2$, we define

$$\iota_n(I) = \begin{cases} \iota_{n-1}(I) & (I \in \mathcal{I}_{n-1}), \\ \iota_{n-2}(\varphi_{n-2}(I)) + f_{n+1} & (I \in \hat{\mathcal{I}}_{n-2}), \end{cases}$$

recursively. By definition, the index of an independent set in \mathcal{I}_{n-1} is in F_{n+1} . The index of an independent set in $\hat{\mathcal{I}}_{n-2}$, corresponding to \mathcal{I}_{n-2} , is in \hat{F}_n . Moreover, if $I \in \mathcal{I}_n$ satisfies $\iota_n(I) \leq f_{k+2}$, then $I \in \mathcal{I}_k$. We also have

$$\iota_n(I \cup \{ n \}) = \iota_{n-2}(I) + f_{n+1} = \iota_{n-1}(I) + f_{n+1} = \iota_n(I) + f_{n+1}$$

for $I \in \mathcal{I}_{n-2}$.

EXAMPLE 4.4. For $n = 1, 2, 3, 4$, independent sets in I_n are indexed as in the Figure 1.

\mathcal{I}_1	F_3	\mathcal{I}_2	F_4	\mathcal{I}_3	F_5	\mathcal{I}_4	F_6
$\{ \}$	1	$\{ \}$	1	$\{ \}$	1	$\{ \}$	1
$\{ 1 \}$	2	$\{ 1 \}$	2	$\{ 1 \}$	2	$\{ 1 \}$	2
		$\{ 2 \}$	3	$\{ 2 \}$	3	$\{ 2 \}$	3
				$\{ 3 \}$	4	$\{ 3 \}$	4
				$\{ 1, 3 \}$	5	$\{ 1, 3 \}$	5
						$\{ 4 \}$	6
						$\{ 1, 4 \}$	7
						$\{ 2, 4 \}$	8

FIGURE 1. $\iota_n(I)$ for $n = 1, 2, 3, 4$

EXAMPLE 4.5. It follows from direct calculation that

$$\begin{aligned} \iota_1(\tau_{1,1}(\{ \})) &= \iota_1(\{ 1 \}) = 2, \\ t_{1,1}(\iota_1(\{ \})) &= t_{1,1}(1) = (1, 2)(1) = 2, \\ \iota_1(\tau_{1,1}(\{ 1 \})) &= \iota_1(\{ \}) = 1, \\ t_{1,1}(\iota_1(\{ 1 \})) &= t_{1,1}(2) = (1, 2)(2) = 1. \end{aligned}$$

Hence

$$\iota_1(\tau_{1,1}(I)) = t_{1,1}(\iota_n(I))$$

for $I \in \mathcal{I}_1$.

EXAMPLE 4.6. It follows from direct calculation that

$$\begin{aligned} \iota_2(\tau_{1,2}(\{ \})) &= \iota_2(\{ 1 \}) = 2, \\ t_{1,2}(\iota_2(\{ \})) &= t_{1,2}(1) = (1, 2)(1) = 2, \\ \iota_2(\tau_{1,2}(\{ 1 \})) &= \iota_2(\{ \}) = 1, \\ t_{1,2}(\iota_2(\{ 1 \})) &= t_{1,2}(2) = (1, 2)(2) = 1, \\ \iota_2(\tau_{1,2}(\{ 2 \})) &= \iota_2(\{ 2 \}) = 3, \\ t_{1,2}(\iota_2(\{ 2 \})) &= t_{1,2}(3) = (1, 2)(3) = 3. \end{aligned}$$

Hence $\iota_2(\tau_{1,2}(I)) = t_{1,2}(\iota_n(I))$. It also follows that

$$\begin{aligned} \iota_2(\tau_{2,2}(\{ \})) &= \iota_2(\{ 2 \}) = 3, \\ t_{2,2}(\iota_2(\{ \})) &= t_{2,2}(1) = (1, 3)(1) = 3, \\ \iota_2(\tau_{2,2}(\{ 1 \})) &= \iota_2(\{ 1 \}) = 2, \\ t_{2,2}(\iota_2(\{ 1 \})) &= t_{1,2}(2) = (1, 3)(2) = 2, \\ \iota_2(\tau_{2,2}(\{ 2 \})) &= \iota_2(\{ \}) = 1, \\ t_{2,2}(\iota_2(\{ 2 \})) &= t_{2,2}(3) = (1, 3)(3) = 1. \end{aligned}$$

Hence $\iota_2(\tau_{2,2}(I)) = t_{2,2}(\iota_n(I))$ for $I \in \mathcal{I}_2$.

By Examples 4.5 and 4.6, we have $\iota_n \circ \tau_{k,n} = t_{k,n} \circ \iota_n$ for $n = 1, 2$. For $n > 2$, we have the following lemmas:

LEMMA 4.7. *Let $n > 2$ and $k < n - 1$. If $\iota_{n-1} \circ \tau_{k,n-1} = t_{k,n-1} \circ \iota_{n-1}$, then $\iota_n \circ \tau_{k,n} = t_{k,n} \circ \iota_n$.*

Proof. First we consider the case where $\iota_n(I) \leq f_{n+1}$. In this case, \mathcal{I}_{n-1} contains I . Hence we have $\tau_{k,n}(I) = \tau_{k,n-1}(I)$, which implies $\iota_n(\tau_{k,n}(I)) = \iota_n(\tau_{k,n-1}(I))$. Since \mathcal{I}_{n-1} contains $\tau_{k,n-1}(I)$, we have $\iota_n(\tau_{k,n-1}(I)) = \iota_{n-1}(\tau_{k,n-1}(I))$. By assumption, we have

$$\iota_{n-1}(\tau_{k,n-1}(I)) = t_{k,n-1}(\iota_{n-1}(I)) = t_{k,n-1}(\iota_n(I)).$$

Since $t_{k,n} = t_{k,n-1}$, $\iota_n(\tau_{k,n}(I)) = t_{k,n}\iota_n(I)$.

Next consider the case where $f_{n+1} < \iota_n(I)$. We show that $\iota_n(\tau_{k,n}(I))$ equals

$$\begin{aligned} t_{k,n}(\iota_n(I)) &= t_{k,n-1}\hat{t}_n t_{k,n-1}\hat{t}_n^{-1}(\iota_n(I)) \\ &= \hat{t}_n t_{k,n-1}\hat{t}_n^{-1}(\iota_n(I)) \\ &= t_{k,n-1}(\iota_n(I) - f_{n+1}) + f_{n+1}. \end{aligned}$$

In this case we have $n \in I$. Hence $I' = I \setminus \{n\} \in \mathcal{I}_{n-1}$ and $\iota_n(I) = \iota(I') + f_{n+1}$. It is easy to show that

$$\tau_{k,n}(I) = \tau_{k,n-1}(I') \cup \{n\}.$$

Hence we have

$$\begin{aligned} \iota_n(\tau_{k,n}(I)) &= \iota_n(\tau_{k,n-1}(I') \cup \{n\}) \\ &= \iota_n(\tau_{k,n-1}(I')) + f_{n+1} \\ &= \iota_{n-1}(\tau_{k,n-1}(I')) + f_{n+1}. \end{aligned}$$

By assumption, we have $\iota_{n-1}(\tau_{k,n-1}(I')) = t_{k,n-1}(\iota_{n-1}(I'))$. Hence we have

$$\begin{aligned} \iota_n(\tau_{k,n}(I)) &= \iota_{n-1}(\tau_{k,n-1}(I')) + f_{n+1} \\ &= t_{k,n-1}(\iota_{n-1}(I')) + f_{n+1} \\ &= t_{k,n-1}(\iota_{n-1}(I) - f_{n+1}) + f_{n+1}. \quad \square \end{aligned}$$

LEMMA 4.8. *Let $n > 2$. If $\iota_{n-1} \circ \tau_{n-1,n-1} = t_{n-1,n-1} \circ \iota_{n-1}$, then $\iota_n \circ \tau_{n-1,n} = t_{n-1,n} \circ \iota_n$.*

Proof. First consider the case where $\iota_n(I) \leq f_{n+1}$. In this case, we have $I \in \mathcal{I}_{n-1}$. Hence we have $\tau_{n-1,n}(I) = \tau_{n-1,n-1}(I)$, which implies $\iota_n(\tau_{n-1,n}(I)) = \iota_n(\tau_{n-1,n-1}(I))$. Since $\tau_{n-1,n-1}(I) \in \mathcal{I}_{n-1}$, we have

$$\iota_n(\tau_{n-1,n-1}(I)) = \iota_{n-1}(\tau_{n-1,n-1}(I)).$$

Hence we have $\iota_{n-1}(\tau_{n-1,n-1}(I)) = t_{n-1,n-1}(\iota_{n-1}(I)) = t_{n-1,n-1}(\iota_n(I))$. Since $t_{n-1,n} = t_{n-1,n-1}$, we have $\iota_n(\tau_{n-1,n}(I)) = t_{n-1,n}(\iota_n(I))$.

Next we consider the case where $f_{n+1} < \iota_n(I)$. We show that $\iota_n(\tau_{n-1,n}(I))$ equals $\hat{t}_{n-1}(\iota_n(I)) = \iota_n(I)$. In this case, I contains n . Hence it follows that $\tau_{n-1,n}(I) = I$ and that $\iota_n(\tau_{n-1,n}(I)) = \iota_n(I)$. \square

LEMMA 4.9. *We have $\iota_n \circ \tau_{n,n} = t_{n,n} \circ \iota_n$ for $n > 2$.*

Proof. First we consider the case where $\iota_n(I) \leq f_n$. We show that $\iota_n(\tau_{n,n}(I))$ equals $\hat{t}_n(\iota_n(I)) = \iota_n(I) + f_{n+1}$. In this case, we have $I \in \mathcal{I}_{n-2}$, which implies I does not contain $n - 1, n$. Hence $\tau_{n,n}(I) = I \cup \{n\}$. Since $\iota_n(I \cup \{n\}) = \iota_n(I) + f_{n+1}$ for $I \in \mathcal{I}_{n-2}$, it follows that $\iota_n(\tau_{n,n}(I)) = \iota_n(I) + f_{n+1}$.

Next we consider the case where $f_n < \iota_n(I) \leq f_{n+1}$. We show that $\iota_n(\tau_{n,n}(I))$ equals $\hat{t}_n(\iota_n(I)) = \iota_n(I)$. In this case, we have $I \in \hat{\mathcal{I}}_{n-3}$, which implies I contains $n - 1$. Hence $\tau_{n,n}(I) = I$ and $\iota_n(\tau_{n,n}(I)) = \iota_n(I)$.

Finally we consider the case where $f_{n+1} < \iota_n(I) \leq f_{n+2}$. We show that $\iota_n(\tau_{n,n}(I))$ equals $\hat{t}_n(\iota_n(I)) = \iota_n(I) - f_{n+1}$. In this case, we have $I \in \hat{\mathcal{I}}_{n-2}$, which implies I

contains n . Hence $\tau_{n,n}(I) = I \setminus \{n\}$. Since $\iota_n(I \cup \{n\}) = \iota_n(I) + f_{n+1}$ for $I \in \mathcal{I}_{n-2}$, it follows that $\iota_n(\tau_{n,n}(I)) = \iota_n(I) - f_{n+1}$. \square

Lemmas 4.7, 4.8 and 4.9 imply the following lemma:

LEMMA 4.10. *If $n > 2$ and $\iota_{n-1} \circ \tau_{k,n-1} = t_{k,n-1} \circ \iota_{n-1}$, then $\iota_n \circ \tau_{k,n} = t_{k,n} \circ \iota_n$.*

Therefore, by induction on n , we have $\iota_n \circ \tau_{k,n} = t_{k,n} \circ \iota_n$ for $n \geq 1$.

Since ι_n induces a bijection between \mathcal{I}_n and F_{n+2} , the bijective map

$$\begin{aligned} \{ \tau_{1,n}, \dots, \tau_{n,n} \} &\rightarrow G_n \\ \tau_{k,n} &\mapsto t_{k,n} \end{aligned}$$

induces the homomorphism from Γ_n to $\langle G_n \rangle$ which preserves the actions. Since G_n generates the f_{n+2} -th symmetric group $\mathfrak{S}_{f_{n+2}}$, the group Γ_n is isomorphic to $\mathfrak{S}_{f_{n+2}}$. Hence $|\Gamma_n| = |\mathfrak{S}_{f_{n+2}}| = |\mathfrak{S}_{\mathcal{I}_n}|$. Since $\Gamma_n \subset \mathfrak{S}_{\mathcal{I}_n}$, we have Theorem 3.7.

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