

ALGEBRAIC COMBINATORICS

Martin Bachratý, Marston Conder & Gabriel Verret

Skew product groups for monolithic groups

Volume 5, issue 5 (2022), p. 785-802.

<https://doi.org/10.5802/alco.206>

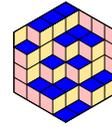
© The author(s), 2022.

 This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



*Algebraic Combinatorics is published by The Combinatorics Consortium
and is a member of the Centre Mersenne for Open Scientific Publishing*
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486





Skew product groups for monolithic groups

Martin Bachratý, Marston Conder & Gabriel Verret

ABSTRACT Skew morphisms, which generalise automorphisms for groups, provide a fundamental tool for the study of regular Cayley maps and, more generally, for finite groups with a complementary factorisation $G = BY$, where Y is cyclic and core-free in G . In this paper, we classify all examples in which B is monolithic (meaning that it has a unique minimal normal subgroup, and that subgroup is not abelian) and core-free in G . As a consequence, we obtain a classification of all proper skew morphisms of finite non-abelian simple groups.

1. INTRODUCTION

Let G be a finite group expressible as a product $G = BC$ of subgroups B and C such that $B \cap C = \{1\}$, sometimes called a *complementary factorisation* of G . If C is cyclic, generated by y , say, then for every $b \in B$ there exists a unique $b' \in B$ and a unique $j \in \{1, 2, \dots, |C| - 1\}$ such that $yb = b'y^j$. This induces a bijection $\varphi: B \rightarrow B$ and a function $\pi: B \rightarrow \mathbb{N}$, defined by $\varphi(b) = b'$ and $\pi(b) = j$, having the properties that $\varphi(1_B) = 1_B$ and $\varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b)$ for all $a, b \in B$. Any bijection $\varphi: B \rightarrow B$ with these two properties is called a *skew morphism* of B , and π is its associated *power function*. This is clearly a generalisation of the concept of a group automorphism, which occurs in the special case where $\pi(b) = 1$ for all b .

Conversely, let φ be a skew morphism of a group B . Then $\varphi \in \text{Sym}(B)$ and, identifying B with its left regular representation in $\text{Sym}(B)$, it can be easily verified that $B\langle\varphi\rangle$ is a group (see [15]). Moreover, one can show that $B\langle\varphi\rangle$ is a complementary factorisation and that $\langle\varphi\rangle$ is core-free in $B\langle\varphi\rangle$ (see [5, Lemma 4.1]). To find all skew morphisms of a group B , it therefore suffices to find all complementary factorisations $B\langle y \rangle$ with $\langle y \rangle$ core-free in $B\langle y \rangle$. A group containing B which has a cyclic core-free complement for B is called a *skew product group* for B , and the skew product group for B described in this paragraph is said to be *induced* by φ .

Skew morphisms were introduced and found to be important in the study of *regular Cayley maps*, which are embeddings of Cayley graphs on surfaces with an arc-transitive group of automorphisms (of the resulting map) that contains a vertex-regular subgroup; see [10]. In particular, they arose as an essentially algebraic concept

Manuscript received 20th September 2020, revised 21st August 2021, accepted 15th November 2021.

KEYWORDS. skew morphisms, regular Cayley maps, group factorisation.

ACKNOWLEDGEMENTS. The authors acknowledge the use of the MAGMA system [3] to investigate possibilities and check examples of groups and skew morphisms relevant to this paper. The first author acknowledges support from the APVV Research Grants 17-0428 and 19-0308, and the VEGA Research Grants 1/0206/20 and 1/0567/22. The second and third authors are grateful to the N.Z. Marsden Fund (via the projects UOA2030 and UOA1824 respectively) for its support.

with applications in topological graph theory. The theory of skew morphisms has been developed over the last 16 years, following the publication of [10]. For example, it is now known that the order of a skew morphism φ of a non-trivial group B is always less than $|B|$; see [5, Theorem 4.2].

It is easy to find all skew morphism of groups of very small order (see [1] for example). On the other hand, it has proved challenging to determine the skew morphisms of infinite classes of finite groups. In particular, this problem remains open for finite cyclic groups, despite positive progress recently (as in [15, 16] for example) and the determination of the corresponding regular Cayley maps for cyclic groups (see [6]). Indeed the skew morphisms of cyclic groups of order 162 are still not known. On the other hand, Kovács and Kwon recently obtained a full classification of regular Cayley maps for dihedral groups [14] (building on [13, 22]) and we understand that a full classification of skew morphisms for dihedral groups is imminent.

We refer to skew morphisms that are not automorphisms as *proper* skew morphisms. In the cases that have been worked out, there are usually many proper skew morphisms. In contrast, we show that non-abelian simple groups rarely have any proper skew morphisms. In fact we take this even further, to monolithic groups. A group B is *monolithic* if it has a unique minimal normal subgroup A , and this subgroup A is non-abelian. In that case, A is called the *monolith* of B . Also a subgroup H of a group G is said to be *core-free* in G if it contains no non-trivial normal subgroup of G .

Our main theorems are the following:

THEOREM 1.1. *Let G be a finite group with a complementary factorisation $G = BY$, where B is monolithic, and Y is cyclic and core-free in G . If B is core-free in G , then one of the following occurs:*

- (1) $G \cong \text{Sym}(n)$, $B \cong \text{Sym}(n - 1)$ and $Y \cong C_n$ for some $n \geq 6$,
- (2) $G \cong \text{Alt}(n)$, $B \cong \text{Alt}(n - 1)$ and $Y \cong C_n$ for some odd $n \geq 7$,
- (3) $G \cong \text{PSL}(2, 11)$, $B \cong \text{Alt}(5)$ and $Y \cong C_{11}$,
- (4) $G \cong M_{11}$, $B \cong M_{10}$ and $Y \cong C_{11}$,
- (5) $G \cong M_{23}$, $B \cong M_{22}$ and $Y \cong C_{23}$.

On the other hand, if B is not core-free in G , then the monolith A of B is normal in G , and either the centraliser $Z = C_Y(A)$ is trivial, or its order $|Z|$ is less than $|B/A|$.

COROLLARY 1.2. *Let G be a finite group with a complementary factorisation $G = BY$, where B is non-abelian and simple, and Y is cyclic and core-free in G . Then either B is normal in G , or one of the cases (2), (3) or (5) from Theorem 1.1 occurs. Hence in particular, every skew morphism of a non-abelian simple group B is an automorphism, unless B is $\text{Alt}(5)$, M_{22} or $\text{Alt}(k)$ for some even $k \geq 6$.*

Here we note that the upper bound on the order of the centraliser $Z = C_Y(A)$ given in Theorem 1.1 is important, for a reason we can explain as follows. First, since $G = BY$ and $B \cap Y = \{1\}$ we have

$$|G| = |B||Y| = |B||Y/Z||Z|.$$

Now for a given non-abelian group A , there are only finitely many possibilities for a monolithic group B with monolith A , because B must be isomorphic to a subgroup of $\text{Aut}(A)$, and in particular, $|B| \leq |\text{Aut}(A)|$. Moreover, if B is not core-free in G , then A is normal in G (see Corollary 3.3) and so $Y/Z = Y/C_Y(A)$ acts faithfully on A . It follows that $|Y/Z|$ is bounded above by the maximum order of an element in $\text{Aut}(A)$. Thus any upper bound on $|Z|$ gives also an upper bound on $|G|$ in terms of $\text{Aut}(A)$, and hence in terms of A . See for example Section 5.5, where we apply this to determine all skew morphisms of many monolithic groups.

Also we note that using Corollary 1.2, we can find all proper skew morphisms of all simple groups (see Section 5.3). On the other hand, we cannot say much about the regular Cayley maps for these groups. This contrasts with the situation for cyclic groups, for which the regular Cayley maps were completely classified in [6], but the skew morphisms are still not all known.

Very recently, Chen, Du and Li [4] have extended some of our work to obtain the skew product groups and regular Cayley maps on the nonabelian characteristically simple groups (equivalently, groups which are direct powers of a nonabelian simple groups).

There is a superficial resemblance between Theorem 1.1 and theorems which classify various factorisations of almost simple groups, such as in [18]. The main difference is that in [18] the overgroup G is assumed to be almost simple, whereas we make no assumption on the structure of G , but rather on the factors B and Y . In the end, there turns out to be a deeper connection, as we show that in the most important case, G is indeed almost simple (see Theorem 3.9), and then this allows us to apply previous results from the literature.

The proof of Theorem 1.1 is presented in Section 3, after some more background in Section 2. Then in Section 4 we give a variety of interesting examples regarding Theorem 1.1, including a family of examples that show that our upper bound on the order of $Z = C_Y(A)$ is sharp. Finally, in Section 5 we explain how to determine (and count) the skew morphisms of non-abelian simple groups and more general monolithic groups, using Theorem 1.1.

2. FURTHER BACKGROUND

In this section we give some more background from group theory, and from the theory of skew morphisms.

All groups in this paper are assumed to be finite. We let C_n denote the cyclic group of order n , and $\text{Alt}(n)$ and $\text{Sym}(n)$ the alternating and symmetric groups of degree n , and M_n the Mathieu group of degree n .

The *core* of a subgroup H in a group G is the largest normal subgroup of G contained in H , and so H is core-free in G if and only if the core of H in G is trivial.

Every minimal normal subgroup N of a finite group G is a direct product of isomorphic simple groups (that is, $N \cong T^k = T \times \dots \times T$ for some simple group T and some positive integer k); see [21, Item 3.3.15]. Note that T is cyclic of some prime order p whenever N is soluble, and in that case N is an elementary abelian p -group, while otherwise N is non-abelian. The *socle* of a finite group G is the subgroup generated by all of the minimal normal subgroups of G , and denoted by $\text{soc}(G)$. In particular, if G is monolithic with monolith A , then $\text{soc}(G) = A$. A group B is *almost simple* if there is a non-abelian simple group A such that B can be embedded between A and its automorphism group, that is, with $A \leq B \leq \text{Aut}(A)$. Equivalently, a group is almost simple if its socle is non-abelian simple. The *soluble radical* of G is the largest soluble normal subgroup of G .

Next, recall that a skew morphism φ of a group B is a permutation $\varphi : B \rightarrow B$ fixing the identity element of B and having the property that $\varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b)$ for all $a, b \in B$, where $\pi : B \rightarrow \mathbb{N}$ is its associated power function.

The *order* $|\varphi|$ of φ is the smallest positive integer k for which φ^k is the identity permutation on B . We can therefore view π as a function from B to $\mathbb{Z}_{|\varphi|}$.

The set of all the elements $a \in B$ for which $\pi(a) = 1$ forms a subgroup of B , denoted by $\ker \varphi$ and called the *kernel* of φ . It is easy to see that two elements $a, b \in B$ belong to the same right coset of $\ker \varphi$ in B if and only if $\pi(a) = \pi(b)$, and that φ is an automorphism of B if and only if $\ker \varphi = B$. Also the following lemma

provides a helpful method for determining whether an element $y \in Y$ from a skew product $G = BY$ determines an automorphism or a proper skew morphism of B . This was already observed in [5], but we also prove it here.

LEMMA 2.1. *Let $G = BY$ be a skew product group for the group B , and let φ be the skew morphism of B given by left multiplication of B by a generator y of Y . Then $\ker \varphi$ is the largest subgroup K of B for which $yKy^{-1} \subseteq B$. In particular, φ is an automorphism of G if and only if B is normal in G .*

Proof. For $b \in B$, clearly $b \in \ker \varphi$ if and only if $yb = b'y$ for some $b' \in B$, which happens if and only if $yby^{-1} = b' \in B$. This proves the first assertion. The second one follows easily from the facts that φ is an automorphism of B if and only if $\ker \varphi = B$, and that B is normal in $G = BY$ if and only if it is normalised by Y . \square

More generally, the subgroup $\ker \varphi$ is not always normal in B , but it is non-trivial whenever B is non-trivial (see [5, Theorem 4.3]). Also the restriction of φ gives an isomorphism from $\ker \varphi$ to $\varphi(\ker \varphi)$, and hence φ restricts to an automorphism of $\ker \varphi$ if the latter is preserved by φ . In the case where B is abelian, every skew morphism φ of B induces an automorphism of $\ker \varphi$ (see [5, Lemma 5.1]).

An immediate consequence of non-triviality of the kernel is that every skew morphism of a cyclic group of prime order is an automorphism. Hence to classify skew morphisms of simple groups, it suffices to consider non-abelian simple groups. Moreover, by our comments in the Introduction (explained in more detail in [5, Lemma 4.1]), it is sufficient for us to consider skew product groups $G = BY$ with Y cyclic and core-free in G , because every skew morphism of B arises in this way.

3. PROOF OF THEOREM 1.1

3.1. PRELIMINARIES. We begin with a number of preliminary observations that are straightforward exercises, but we include their proofs for completeness.

LEMMA 3.1. *Let G be a group with subgroups B and C such that $G = BC$, and B is core-free and C is abelian. Then $B \cap C = \{1\}$.*

Proof. Consider the natural action of G on the right coset space $(G : B)$, with point-stabiliser B . Because B is core-free, this action is faithful. But also $G = BC$, so C acts transitively on $(G : B)$, and then since C is abelian, it must act regularly (because the stabiliser of every point is the same), and therefore $B \cap C = \{1\}$. \square

LEMMA 3.2. *If G has trivial soluble radical, and $\text{soc}(G) \leq H \leq G$, then $\text{soc}(H) = \text{soc}(G)$, and therefore H has trivial soluble radical.*

Proof. We first show that $\text{soc}(G) \leq \text{soc}(H)$. Let N be a minimal normal subgroup of G . Since G has trivial soluble radical, N is a direct product of non-abelian simple groups, and then since $\text{soc}(G) \leq H$ we see that N is a normal subgroup of H , which must then permute the direct factors of N . Thus N is a product of minimal normal subgroups of H , and so $N \leq \text{soc}(H)$, as required.

We now show that $\text{soc}(H) \leq \text{soc}(G)$. Suppose to the contrary that there is a minimal normal subgroup N of H not contained in $\text{soc}(G)$. Let M be a minimal normal subgroup of G . Then $M \leq \text{soc}(G) \leq H$ and so M is a normal subgroup of H . Also N is a minimal normal subgroup of H but $N \not\leq \text{soc}(G)$, and so $N \cap M = \{1\}$, and therefore N centralises M . Since M was an arbitrary minimal normal subgroup of G , it follows that N centralises $\text{soc}(G)$. But in a group with trivial soluble radical, the socle has trivial centraliser (see [8, Corollary 4.3A] for example), and so N cannot centralise $\text{soc}(G)$, contradiction. Thus $\text{soc}(H) = \text{soc}(G)$, and the rest follows easily. \square

COROLLARY 3.3. *Let G be a group with a monolithic subgroup B , and let A be the monolith of B . If B is not core-free in G , then A is normal in G .*

Proof. Let K be the core of B in G . Then $\text{soc}(B) = A \leq K \leq B$, and by Lemma 3.2 we find that $\text{soc}(K) = A$, making A characteristic in K and hence normal in G . \square

LEMMA 3.4. *Let G be a group with subgroups B and C such that $G = BC$ and $B \cap C = \{1\}$. If N is a normal subgroup of G , then $|N| = |BN \cap C| |B \cap N| = |B \cap CN| |C \cap N|$.*

Proof. First $G = (BN)(CN)$, and $|BN| = |B||N|/|B \cap N|$ and $|CN| = |C||N|/|C \cap N|$, so

$$|B||C| = |G| = |(BN)(CN)| = \frac{|BN||CN|}{|BN \cap CN|} = \frac{|B||C||N|^2}{|BN \cap CN| |B \cap N| |C \cap N|}$$

and it follows that

$$(1) \quad |N|^2 = |BN \cap CN| |B \cap N| |C \cap N|.$$

Next by Dedekind's modular law [21, Item 1.3.14] we have $(BN \cap C)N = BN \cap CN$, and so

$$|BN \cap CN| = |(BN \cap C)N| = \frac{|BN \cap C||N|}{|BN \cap C \cap N|} = \frac{|BN \cap C||N|}{|C \cap N|}.$$

Combining this with (1) gives the first equality $|N| = |BN \cap C| |B \cap N|$, and then exchanging the roles of B and C gives the second equality. \square

Next, the following theorem proved by Lucchini in [19] will be helpful. (It was also used to prove Theorems 4.2 and 4.3 on orders and kernels of skew morphisms in [5].)

THEOREM 3.5 ([19]). *If C is a core-free cyclic proper subgroup of a group G , then $|C| < |G : C|$.*

Finally, our proof of Theorem 1.1 depends heavily on results of a 2012 paper by Li and Praeger [17], on finite permutation groups containing a regular cyclic subgroup. In particular, we will frequently refer to the following theorem which we reproduce for the benefit of readers. Note that the classification for the primitive case (dealt with in item (a) and Table 1 and included in [17]) was first achieved by G.A. Jones in [11]. Also we have amended the third row of Table 1 (from [17]) by replacing $n \geq 4$ by $n \geq 5$, in order to make item (b) correct.

THEOREM 3.6 ([17]). *Let G be a finite permutation group containing a regular cyclic subgroup. Then:*

- (a) G is quasiprimitive if and only if G is primitive and appears in Table 1;
- (b) G is almost simple if and only if G appears in rows (3) to (6) of Table 1; and
- (c) G is imprimitive but has a transitive minimal normal subgroup N if and only if G and N appear in Table 2.

Moreover, if N is a non-abelian minimal normal subgroup of G , and Δ is an orbit of N , then N^Δ is simple, and the permutation group G_Δ^Δ induced by the setwise stabiliser G_Δ in G of Δ appears in rows (3) to (6) of Table 1, or in Table 2.

3.2. THE NON-CORE-FREE CASE. Here we deal with the case when B is not core-free in $G = BY$.

PROPOSITION 3.7. *Let G be a group with subgroups B and Y such that $G = BY$, where B is monolithic, and Y is cyclic and core-free in G . Also let A be the monolith of B , and let Z be the centraliser of A in Y . If B is not core-free in G , then A is normal in G , and either $G = AY$ and $Z = \{1\}$, or $|Z| < |B/A|$.*

	Degree	Group G	Comments
(1)	$p \geq 2$	$C_p \leq G \leq \text{AGL}(1, p)$	p prime
(2)	4	$\text{Sym}(4)$	
(3)	$n \geq 5$	$\text{Alt}(n)$ or $\text{Sym}(n)$	n odd if $G = \text{Alt}(n)$
(4)	$(q^d - 1)/(q - 1)$	$\text{PGL}(d, q) \leq G \leq \text{PTL}(d, q)$	$d \geq 2$, and q a prime-power
(5)	11	$\text{PSL}(2, 11)$ or M_{11}	
(6)	23	M_{23}	

TABLE 1. Primitive groups with a regular cyclic subgroup [11]

Degree	Group G	Subgroup N
15	$\text{Alt}(5) \times C_3$ or $(\text{Alt}(5) \times C_3).C_2$	$\text{Alt}(5)$
22	$M_{11} \times C_2$	M_{11}
$r(q^d - 1)/(q - 1)$	$(\text{PSL}(d, q) \times C_r).C_k$ for certain k	$\text{PSL}(d, q)$

TABLE 2. Imprimitve groups with a regular cyclic subgroup and a transitive minimal normal subgroup

Proof. First, by Corollary 3.3 we find that A is normal in G . Let \bar{X} denote XA/A ($\cong X/(X \cap A)$) for every $X \leq G$. If $\bar{Y} = \bar{G}$, then $G = AY$. It follows that Z is central in G and, as Y is core-free in G , we have $Z = \{1\}$. We therefore assume that $\bar{Y} < \bar{G}$.

Since $G = BY$, we have $\bar{G} = \bar{B}\bar{Y}$ and thus $|\bar{G}| \leq |\bar{B}||\bar{Y}|$. Also because the centre of A must be trivial, we have $A \cap Z = \{1\}$ and therefore $\bar{Z} \cong Z$.

We now show that \bar{Z} is core-free in \bar{G} . Let X be a subgroup of Z such that \bar{X} is normal in \bar{G} . Then AX is normal in G , and also $AX \cong A \times X$, where X is cyclic while A is a direct product of non-abelian simple groups, so X is characteristic in AX and therefore normal in G . But Y is core-free, and so it follows that $X = \{1\}$, and hence $\bar{X} = \{1\}$.

Next let \bar{K} be the core of \bar{Y} in \bar{G} . Since \bar{K} is cyclic, all its subgroups are characteristic and hence normal in \bar{G} . In particular, $\bar{Z} \cap \bar{K}$ is normal in \bar{G} , but then since \bar{Z} is core-free in \bar{G} , we find that $\bar{Z} \cap \bar{K} = \{1\}$, and therefore $|\bar{Z}||\bar{K}| \leq |\bar{Y}|$.

Since $\bar{Y} < \bar{G}$, it follows that \bar{Y}/\bar{K} is a proper cyclic core-free subgroup of \bar{G}/\bar{K} , and so by Theorem 3.5 we have $|\bar{Y}/\bar{K}| < |(\bar{G}/\bar{K})/(\bar{Y}/\bar{K})| = |\bar{G}|/|\bar{Y}|$, and therefore

$$|Z| = |\bar{Z}| \leq |\bar{Y}|/|\bar{K}| < |\bar{G}|/|\bar{Y}| \leq |\bar{B}| = |B/A|.$$

□

3.3. THE CORE-FREE CASE. Here we deal with the other case, where B is core-free in $G = BY$.

PROPOSITION 3.8. *Let G be a group with core-free subgroups B and Y such that $G = BY$, where B is monolithic with monolith A , and Y is cyclic. Then G has a unique minimal normal subgroup N , and this normal subgroup N contains A .*

Proof. First, by Lemma 3.1 we find that $B \cap Y = \{1\}$. Now let N be any minimal normal subgroup of G and let \bar{X} denote XN/N ($\cong X/(X \cap N)$), for every $X \leq G$. We will show that N intersects A non-trivially, and thus N contains A and is the unique minimal normal subgroup of G .

Suppose that $A \cap N = \{1\}$. Since A is the unique minimal normal subgroup of B , this implies that $B \cap N = \{1\}$ and hence that $\bar{B} \cong B$. Moreover, as $B \cap N = B \cap YN \cap N$ this gives

$$B \cap YN \cong (B \cap YN)/(B \cap N) \cong (B \cap YN)N/N \leq YN/N \cong Y/(Y \cap N),$$

and so $B \cap YN$ is cyclic. Then by Itô's Theorem [9] on products of abelian groups, $(B \cap YN)Y$ is metabelian, and therefore so is its subgroup $BY \cap YNY = BY \cap YN = G \cap YN = YN$. This implies that the minimal normal subgroup N of G is abelian, and hence is isomorphic to C_p^n for some prime p . Also $C_B(N)$ is a normal subgroup of B , and because A is the monolith of B , it follows that either $C_B(N) = \{1\}$ or $A \leq C_B(N)$. We eliminate these two cases separately.

Case (a) $C_B(N) = \{1\}$.

In this case, B acts faithfully on N and so is isomorphic to a subgroup of $\text{Aut}(N) \cong \text{GL}(n, p)$. Also $B \cap YN$ and $Y \cap N$ are cyclic, and so using Lemma 3.4 we find that $B \cap YN$ has order $|B \cap YN| = |N|/|Y \cap N| \geq |N|/p = p^{n-1}$. Next, a Sylow p -subgroup of $\text{GL}(n, p)$ has exponent $p^{\lceil \log_p n \rceil}$ (as shown in [20, Theorem 1], for example), and again since $B \cap YN$ is cyclic, it follows that $p^{n-1} \leq |B \cap YN| \leq p^{\lceil \log_p n \rceil}$ and so $n - 1 \leq \lceil \log_p n \rceil$. This is a strong restriction, which holds only when $n \leq 2$ or $(n, p) = (3, 2)$.

But if $n \leq 2$ then $n = 2$ and p is odd since B is insoluble, and then $[B, B]$ is isomorphic to an insoluble subgroup of $\text{SL}(2, p)$, and hence must contain the unique involution in $\text{SL}(2, p)$, namely $-I$, which contradicts the fact that B has trivial centre.

Thus $(n, p) = (3, 2)$, so B is isomorphic to an insoluble subgroup of $\text{GL}(3, 2)$. This is actually the second smallest simple group, of order 168, and hence $B \cong \text{GL}(3, 2)$. Also $|N| = p^n = 8$, and then since $B \cap N = \{1\}$ we find that $BN \cong N \rtimes B$ is isomorphic to the semi-direct product $C_2^3 \rtimes \text{GL}(3, 2) \cong \text{AGL}(3, 2)$. Moreover, by Lemma 3.4 we find that $|N| = |BN \cap Y||B \cap N| = |BN \cap Y|$, and so $BN \cap Y$ is cyclic of order 8. But $\text{AGL}(3, 2)$ has no element of order 8, contradiction.

Case (b) $A \leq C_B(N)$.

In this case, AN is isomorphic to $A \times N$, and since A is a direct product of non-abelian simple groups while N is abelian, we see that A is characteristic in AN . But A is not normal in G (since B is core-free in G), and so AN cannot be normal in G , and therefore \bar{A} is not normal in \bar{G} . Also \bar{B} is monolithic with monolith \bar{A} , hence using Corollary 3.3 we find that \bar{B} is core-free in \bar{G} . Now by Lemma 3.1, it follows that $\bar{B} \cap \bar{Y} = \{1\}$, so $|\bar{G}| = |\bar{B}||\bar{Y}|$, and therefore

$$|B||Y| = |G| = |\bar{G}||N| = |\bar{B}||\bar{Y}||N| = |B||YN/N||N| = |B||YN|.$$

This gives $|Y| = |NY|$, and so $N \leq Y$, which contradicts the fact that Y is core-free in G .

Thus $A \cap N \neq \{1\}$. But A and N are both normalised by B , so $A \cap N$ is normal in B , and then since A is the unique minimal normal subgroup of B , we find that $A \leq N$. Hence every minimal normal subgroup of G contains A .

Finally, any two minimal normal subgroups intersect trivially, and so there cannot be more than one, and hence G has a unique minimal normal subgroup, as required. \square

THEOREM 3.9. *Let G be a group with core-free subgroups B and C such that $G = BC$. If B is monolithic and C is cyclic, then G is almost simple.*

Proof. We use induction on $|G|$. By Lemma 3.1, once again we have $B \cap C = \{1\}$. Now let A be the socle of B . Then A is not normal in G because B is core-free in G ,

and by Proposition 3.8 we know that G has a unique minimal normal subgroup N , and $A \leq N$. In particular, $N = \text{soc}(G)$, and because A is not soluble, $N \cong T^n$ for some non-abelian simple group T and some positive integer $n \geq 1$. Also by Lemma 3.2 with $H = BN$ we have $\text{soc}(BN) = \text{soc}(G) = N$.

Next, since B is core-free in G , we may view G as a permutation group on the coset space $(G : B)$ with point-stabiliser B , and a regular cyclic subgroup C . We proceed by considering three different cases, where $BN = G$, or $BN < G$ and B is core-free in BN , or $BN < G$ and B is not core-free in BN .

Case (a) $BN = G$.

In this case, N is a transitive subgroup of G , and then since N is the only minimal normal subgroup of G , it follows that every non-trivial normal subgroup is transitive, and so by definition G is *quasiprimitive* on $(G : B)$. By Theorem 3.6, we know that G appears in Table 1, but also G is not soluble, and hence we can rule out rows (1) and (2) of the table, and conclude that G is almost simple.

Case (b) $BN < G$ and B is core-free in BN .

In this case, first we note that BN has trivial soluble radical, since $\text{soc}(BN) = N \cong T^n$ where T is non-abelian simple, and hence the cyclic subgroup $BN \cap C$ is core-free in BN . Next, by Dedekind’s Modular Law, $B(BN \cap C) = BN \cap BC = BN \cap G = BN$. We may now apply the inductive hypothesis to BN (with $BN \cap C$ playing the role of C) to conclude that BN is almost simple, and then since $\text{soc}(BN) = \text{soc}(G) = N$, it follows that also G is almost simple.

Case (c) $BN < G$ and B is not core-free in BN .

In this case, let K be the core of B in BN . Then by hypothesis K is a non-trivial normal subgroup of B , so must contain A , and then by Lemma 3.2 applied to $A = \text{soc}(B) \leq K \leq B$, we find that $\text{soc}(K) = \text{soc}(B) = A$. In particular, A is characteristic in K and hence normal in BN , so A is also normal in N . Hence A must be a product of some of the (simple) direct factors of N . It then follows that every subgroup of N containing A must be of the form $A \times H$ for some $H \leq N$, and in particular, $B \cap N$ has this form. But also A is the unique minimal subgroup of B , so the centraliser of A in B is trivial, and so H is trivial, and $B \cap N = A$.

Again viewing G as a permutation group on the coset space $(G : B)$ with point-stabiliser $G_v = B$, with regular cyclic subgroup C , we now have $A = B \cap N = N_v$. Since this is normal in N , it follows that N induces a regular subgroup on each of its orbits.

Let Δ be an N -orbit on $(G : B)$, and let G_Δ denote the set-wise stabiliser of Δ in G . Then since N is normal in G and preserves Δ , we know that also N is normal in G_Δ , and hence N^Δ is normal in G_Δ^Δ . By Theorem 3.6 it now follows that G_Δ^Δ appears either in rows (3) to (6) of Table 1 or in Table 2, and that N^Δ is a simple subgroup of G_Δ^Δ acting regularly on Δ . This is not possible, however, because for each choice of G_Δ^Δ from those two tables, the degree of the permutation group is strictly smaller than the order of its smallest non-abelian simple normal subgroup. Hence this third case is eliminated. □

We can now complete the proof of Theorem 1.1, below.

Proof. Let G be a group with core-free subgroups B and Y such that $G = BY$, where B is monolithic and Y is cyclic. By Lemma 3.1, we know that $B \cap Y = \{1\}$, and Theorem 3.9 implies that G is almost simple. Once again, we may view G as a permutation group on the coset space $(G : B)$ with point-stabiliser B , and a regular cyclic subgroup Y . It follows by Theorem 3.6 that G appears in rows (3) to (6) of Table 1. If G appears in row (4), then $\text{PGL}(d, q) \leq G \leq \text{PTL}(d, q)$ and hence

$\text{AGL}(d-1, q) \leq G_v \leq \text{AFL}(d-1, q)$, but $B \cong G_v$, contradicting the fact that B is monolithic. We may therefore exclude this case. The other possibilities are genuine examples, and give the cases listed in the statement of Theorem 1.1. \square

4. EXAMPLES AND REMARKS CONCERNING THEOREM 1.1

Our first example in this section gives an infinite family for which the upper bound on the order of $Z = C_Y(A)$ in Theorem 1.1 is sharp.

EXAMPLE 4.1. Let p be a prime, let $G = \text{P}\Sigma\text{L}(2, 2^p) \times \text{AGL}(1, p)$ and write $\text{P}\Sigma\text{L}(2, 2^p) = A \rtimes C \cong \text{PSL}(2, 2^p) \rtimes C_p$ and $\text{AGL}(1, p) = T \rtimes M \cong C_p \rtimes C_{p-1}$. Let D be the diagonal subgroup of $C \times T \cong C_p \times C_p$, let $B = \langle A, D \rangle$ and let $Y = C \times M \cong C_p \times C_{p-1}$. Then B is isomorphic to $\text{P}\Sigma\text{L}(2, 2^p)$, and hence is almost simple with socle A , which is normal in G . Also Y is cyclic, $B \cap Y = \{1\}$ and, by order considerations, $G = BY$. Note also that Y is core-free in G . Thus B , Y and G satisfy the hypotheses of Theorem 1.1. Finally, we see that the centraliser of A in Y is $M \cong C_{p-1}$, while $|B/A| = |C| = p$.

Taking $p \geq 3$ in Example 4.1 also provides examples for which B is neither core-free nor normal in G . (Note that B is normal in G when $p = 2$.)

The next example shows that although the order of $Z = C_Y(A)$ is always less than $|B/A|$, the order of the centraliser of A in G can be arbitrarily large, even when $B = A$.

EXAMPLE 4.2. For $n \geq 5$, take $A = B = \text{Alt}(n)$, let g be an element of odd order m in A , and let G be the direct product $A \times C_m$. Also let Y be the cyclic subgroup of G of order m generated by gc , where c is a generator of the direct factor C_m . Then clearly $G = BY$ with $B \cap Y = \{1\}$, and B is simple and normal in G , while Y is cyclic and core-free in G , so again B , Y and G satisfy the hypotheses of Theorem 1.1. Here $Z = C_Y(A)$ is trivial, while $C_G(A)$ has order m , which can be arbitrarily large (depending on the choice of n and g).

Finally, we give an example where B is neither core-free nor normal in G , and A is not simple.

EXAMPLE 4.3. Let H be a permutation group of degree n with a transitive non-normal subgroup X and a core-free cyclic subgroup Y such that $H = XY$ with $X \cap Y = \{1\}$.

Note that there are many possibilities for (H, X, Y) . The smallest example in terms of $|H|$ is obtained by taking $H = \text{Sym}(3) \times C_3 \cong (C_3 \rtimes C_2) \times C_3$ with $X \cong C_2 \times C_3 \cong C_6$ and Y the diagonal subgroup of $C_3 \times C_3$, and $n = 6$. The smallest example in terms of the degree n is obtained by taking $H = \text{Sym}(4)$, with $X = D_4$ and $Y \cong C_3$, and $n = 4$.

Next, let T be a non-abelian simple group, let G be the wreath product $T \wr H$ (which is isomorphic to $T^n \rtimes H$), let $A = T^n$, let $B = T^n \rtimes X$, and consider Y as a subgroup of $\{1\} \wr H$ in G . Since X conjugates transitively the n copies of T in A among themselves, the subgroup A is minimal normal in B . Furthermore, A is the only minimal normal subgroup of B , because any other minimal normal subgroup of B would be contained in the centraliser of A in B , which is trivial. It follows that B is monolithic, with monolith A . Also $G = BY$ with $B \cap Y = \{1\}$, and B not normal in G , while Y is cyclic and core-free in G .

We complete this section with a remark that includes an alternative proof of Theorem 3.9 in the case where B is simple.

REMARK 4.4. The proofs of the preliminaries in Section 3.1 and Propositions 3.7 and 3.8 are elementary, in the sense of not relying on the CFSG (the classification

of finite simple groups). On the other hand, the proof of Theorem 3.9 relies on the CFSG many times, when we cite Theorem 3.6. This, however, can be avoided in the case when B is simple. We present a short proof of this fact, as we believe it could be of some interest.

Proof. Let G be a group with core-free subgroups B and Y such that $G = BY$, where B is non-abelian simple, and Y is cyclic. Then $\text{soc}(B) = B$ since B is simple, and by Proposition 3.8 we know that G has a unique minimal normal subgroup, say N , and this subgroup contains B . Also $N \cong T^k$ for some simple group T , and since $B \leq N$, we have $YN = G$, and therefore Y conjugates transitively the k direct factors of N among themselves. Also because Y is abelian, it is an easy exercise to show that $Y \cap N$ is isomorphic to a subgroup of T . Hence by Theorem 3.5, we find that $|Y \cap N| < |T : (Y \cap N)|$ and therefore $|Y \cap N| < |T|^{1/2}$. Next, Lemma 3.4 gives $|N| = |B \cap YN| |Y \cap N| = |B| |Y \cap N|$, and since $B \leq N \cong T^k$ and B is simple, it can be easily verified that B is isomorphic to a subgroup of T , and therefore $|B| \leq |T|$. Consequently $|T|^k = |N| = |B| |Y \cap N| < |B| |T|^{1/2} \leq |T|^{3/2}$, which gives $k < \frac{3}{2}$. It follows that $k = 1$, and therefore N is simple, and G is almost simple. \square

Finally, note that the last part of our proof of Theorem 1.1 (after Theorem 3.9) again relies heavily on the CFSG through the use of Theorem 3.6, even in the case where B is simple.

5. SKEW MORPHISMS OF NON-ABELIAN SIMPLE AND OTHER MONOLITHIC GROUPS

In this section, we first describe a method for determining the skew morphisms of a finite group B using skew product groups. Using Theorem 1.1, we then apply this method to the case where B is monolithic and core-free in its skew product group. We complete the paper by giving a summary of information about the case where B is simple, and making some observations in the case where B is not core-free.

5.1. DETERMINING SKEW MORPHISMS USING SKEW PRODUCT GROUPS. We first define a few notions that will be very useful. If φ is a skew morphism of a group B and ψ is a group automorphism of B , then $\psi \circ \varphi \circ \psi^{-1}$ is also a skew morphism of B (see [2] for example) that is said to be *equivalent* to φ . Now let G be a skew product group for B . By definition, G contains a copy of B and an element y such that $G = B\langle y \rangle$ is a complementary factorisation and $\langle y \rangle$ is core-free in G . We call such a pair (B, y) a *skew generating pair* for G , and we say that two skew generating pairs (B, y) and (B, y') for groups G and G' are *equivalent* if there is an isomorphism $\alpha: G \rightarrow G'$ such that $B^\alpha = B$ and $y^\alpha = y'$. It is easy to see that equivalent skew generating pairs induce equivalent skew morphisms.

To enumerate all skew morphisms of B , we first determine all equivalence classes and then determine the size of each class. To determine the classes, we first determine (up to isomorphism) all skew product groups G for B . Note that if we want only proper skew morphisms of B , then by Lemma 2.1 we can restrict ourselves to the skew products groups in which B is not normal. Then for each such G , we determine the equivalence classes of generating pairs, choose a representative from each class, and take the corresponding skew morphisms. By the previous paragraph, we now have at least one representative of each equivalence class of skew morphisms. It remains to check if any of these actually represent the same class, which can be done by a direct check of conjugacy under $\text{Aut}(B)$ in $\text{Sym}(B)$. Finally, the size of the equivalence class of a given skew morphism φ is given by the index $|\text{Aut}(B) : C_{\text{Aut}(B)}(\varphi)|$.

The method above is designed to minimise the amount of calculations that have to be undertaken in $\text{Sym}(B)$, noting that this is often a very large group compared to $\text{Aut}(B)$ or even $\text{Aut}(G)$.

REMARK 5.1. If (B, y) is a generating pair for a skew product group G , then no two different elements of $\langle y \rangle$ induce the same skew morphism of B . For if y and y' are elements of Y that induce the same skew morphism φ of B , with power function π , then for every $b \in B$ we have $yb = \varphi(b)y^{\pi(b)}$ and $y'b = \varphi(b)(y')^{\pi(b)}$, and so $b^{-1}y^{-1}y'b = (yb)^{-1}y'b = (\varphi(b)y^{\pi(b)})^{-1}(\varphi(b)(y')^{\pi(b)}) = y^{-\pi(b)}(y')^{\pi(b)} = (y^{-1}y')^{\pi(b)}$. This implies that the cyclic subgroup of $\langle y \rangle$ generated by $y^{-1}y'$ is normalised by b , for all $b \in B$, and hence is normal in $B\langle y \rangle = G$. But $\langle y \rangle$ is core-free in G , so $y^{-1}y'$ must be trivial, and therefore $y' = y$. Note also that some elements of $\langle y \rangle$ might not induce a skew morphism of B at all (as in [2] for example).

Next, we note the following, obtainable from the proof of [10, Theorem 2]:

PROPOSITION 5.2. *A skew morphism φ of a finite group B gives rise to a regular Cayley map for B if and only if the set of elements in some cycle of φ (when regarded as a permutation of B) is closed under taking inverses and generates B .*

Finally we note that every regular Cayley map for the group B that is *balanced* (in the sense defined in [10]) comes from an automorphism of B , and so proper skew morphisms give rise only to non-balanced regular Cayley maps.

5.2. SKEW MORPHISMS OF MONOLITHIC GROUPS: THE CORE-FREE CASE. Here we apply the method from the previous subsection to determine all skew morphisms of a monolithic group B when B is core-free in the skew product group $G = BY$. All such skew morphisms will be proper, since the hypothesis requires that B itself is not normal in G .

By Theorem 1.1, the only possibilities for B are $\text{Alt}(5)$, M_{10} , M_{22} , $\text{Alt}(n)$ for even $n \geq 6$, and $\text{Sym}(n)$ for $n \geq 5$. We enumerate both the number of (proper) skew morphisms, and the number of equivalence classes of these. This enumeration splits into seven cases, coming from the five cases in Theorem 1.1, with the two additional cases $\text{Alt}(6)$ and $\text{Sym}(6)$ treated separately because of their exceptional outer automorphisms. In particular, we have five sporadic cases (including some for which we relied on MAGMA [3] to verify), and two infinite families.

Case (1) $B = \text{Alt}(5)$.

Here the skew product group G is $\text{PSL}(2, 11)$, with $B = \text{Alt}(5)$ core-free in G , and Y cyclic of order 11. There is just one conjugacy class of possibilities for Y (since Y is a Sylow 11-subgroup), and there are two conjugacy classes of subgroups isomorphic to B in G , but the latter form a single class within $\text{Aut}(G)$, so we may take B as any representative of just one of them. Then the subgroup H of $\text{Aut}(G)$ preserving B has two orbits on elements of order 11 in G , with y and y^{-1} always lying in different orbits, for every such y . Hence we have two equivalence classes of skew generating pairs for G .

Next, take y of order 11 in G , and let φ be the skew morphism of B induced by y . Then every skew morphism of B associated with the skew product group G is equivalent to φ or φ^{-1} . Moreover, the centraliser in $\text{Aut}(B) \cong \text{Sym}(5)$ of φ is trivial, and φ^{-1} is not a conjugate of φ under an element of $\text{Aut}(B)$, so these two skew morphisms are inequivalent. Hence there are two equivalence classes of skew morphisms of $\text{Alt}(5)$ associated with $\text{PSL}(2, 11)$, each of size 120.

Moreover, it is easy to check that φ induces at least one 11-cycle on $\text{Alt}(5)$ that contains three involutions, four 3-cycles and four 5-cycles, forming a set that is closed under inverses and clearly generates $\text{Alt}(5)$. It follows that the same holds

for φ^{-1} , and hence for all skew morphisms arising in this case (each of which is equivalent to φ or φ^{-1}). By Proposition 5.2, it follows that each of these skew morphisms gives rise to a (non-balanced) regular Cayley map for $\text{Alt}(5)$.

In conclusion, $\text{Alt}(5)$ has 240 proper skew morphisms associated with $\text{PSL}(2, 11)$, which split into two equivalence classes of size 120, with one of the classes consisting of the inverses of the members of the other one, and also every skew morphism associated with $\text{PSL}(2, 11)$ gives rise to a (non-balanced) regular Cayley map for $\text{Alt}(5)$.

Case (2) $B = M_{10}$.

In this case the skew product group G is M_{11} , with $B = M_{10}$ and Y cyclic of order 11. Again here we have two classes of skew generating pairs, and the skew morphisms associated with their representatives are not equivalent. We find there are two equivalence classes of proper skew morphisms, each of size $|\text{Aut}(M_{10})| = 1440$, giving us a total of 2880 proper skew morphisms. Also one of the classes consists of the inverses of members of the other one. Finally, each of the two class representatives (and hence every skew morphism arising in this case) induces at least one 11-cycle on M_{10} , consisting of three involutions, six elements of order 4 and two of order 8, and forming a generating set that is closed under inverses, and hence every skew morphism of M_{10} associated with M_{11} gives rise to a (non-balanced) regular Cayley map for M_{10} .

Case (3) $B = M_{22}$.

The skew product group G in this case is M_{23} , with $B = M_{22}$ and Y cyclic of order 23. Again there are two classes of skew generating pairs, and the skew morphisms associated with representatives of different classes are not equivalent. Hence we have two equivalence classes of proper skew morphisms, each of size $|\text{Aut}(M_{12})| = 887040$, giving us a total of 1774080. Again in this case, each class consists of the inverses of members of the other one. Finally, each of the two class representatives (and hence every skew morphism arising in this case) induces at least one 23-cycle on M_{22} , consisting of seven involutions, eight elements of order 7 and eight of order 11, and forming a generating set that is closed under inverses. Thus every skew morphism of M_{22} associated with M_{23} gives rise to a (non-balanced) regular Cayley map for M_{22} .

Case (4) $B = \text{Alt}(6)$.

Here the skew product group G is $\text{Alt}(7)$, with $B = \text{Alt}(6)$ and Y cyclic of order 7. This time there is only one equivalence class of skew generating pairs, giving us a single class of proper skew morphisms, of size $|\text{Aut}(\text{Alt}(6))| = 1440$. (Each one comes from taking Y as the cyclic subgroup generated by a single 7-cycle, and B as the stabiliser in $\text{Alt}(7)$ of any point.) Moreover, every skew morphism (arising in this case) induces at least one 7-cycle on $\text{Alt}(6)$, consisting of three involutions and four elements of order 5, and forming a generating set that is closed under inverses. Thus every skew morphism of $\text{Alt}(6)$ associated with $\text{Alt}(7)$ gives rise to a (non-balanced) regular Cayley map for $\text{Alt}(6)$.

Case (5) $B = \text{Sym}(6)$.

The skew product group G in this case is $\text{Sym}(7)$, with $B = \text{Sym}(6)$ and Y cyclic of order 7. Just as in the previous case, there is only one equivalence class of skew generating pairs, giving a single class of proper skew morphisms, of size $|\text{Aut}(\text{Sym}(6))| = 1440$. (This time B can be taken as the stabiliser in $\text{Sym}(7)$ of any point.) Moreover, a representative of that class induces at least one 7-cycle on $\text{Sym}(6)$ that consists of five involutions and two elements of order 6, forming a generating set that is closed under inverses. Thus every skew morphism of $\text{Sym}(6)$

associated with $\text{Sym}(7)$ gives rise to a (non-balanced) regular Cayley map for $\text{Sym}(6)$.

Case (6) $B = \text{Alt}(n)$ for even $n \geq 8$.

Here the skew product group G is $\text{Alt}(n + 1)$, with $B = \text{Alt}(n)$ and Y cyclic of order $n + 1$. Up to equivalence, there is only one skew generating pair, and hence a single equivalence class of proper skew morphisms of $\text{Alt}(n)$, of size at most $|\text{Aut}(\text{Alt}(n))| = n!$.

Now take $B = \text{Alt}(n)$ as the stabiliser of the point $n + 1$ in G , let $y = (1, 2, 3, \dots, n + 1)$, and let φ be the skew morphism of $\text{Alt}(n)$ induced by y . If π is the power function of φ , then by considering the effect of $yb = \varphi(b)y^{\pi(b)} \in G = \text{Alt}(n + 1)$ on the point $n + 1$, we find that $1^b = ((n + 1)^y)^b = (n + 1)^{yb} = (n + 1)^{\varphi(b)y^{\pi(b)}} = (n + 1)^{y^{\pi(b)}}$ for all $b \in B$, because $\varphi(b) \in B$ fixes the point $n + 1$. Hence if $\pi(b) = k$ then $1^b = (n + 1)^{y^k} = k$, and therefore $\pi(b) = 1^b$ for all $b \in B$.

Next, suppose z is another $(n + 1)$ -cycle in $G = \text{Alt}(n + 1)$ that gives exactly the same skew morphism φ of $\text{Alt}(n)$, say $z = (p_1, p_2, p_3, \dots, p_n, n + 1)$. Then the same argument applies also to $zb = \varphi(b)z^{\pi(b)}$, giving $((n + 1)^z)^b = (n + 1)^{z^{\pi(b)}}$ for all $b \in B$, so $p_1^b = ((n + 1)^z)^b = (n + 1)^{z^k} = p_k$ whenever $\pi(b) = k$.

In particular, if we let a be the $(n - 1)$ -cycle $(1, 2, 3, \dots, n - 1)$ in $B = \text{Alt}(n)$, then taking $b = a^j$ we find that $\pi(a^j) = 1^{a^j} = j + 1$ and so $p_1^{a^j} = p_{j+1}$ for $1 \leq j \leq n - 1$. It follows that a takes p_i to p_{i+1} for $1 \leq i \leq n - 1$, and so has the same effect as z on $\{p_1, p_2, \dots, p_{n-1}\}$, and therefore $z = (1, 2, 3, \dots, n, n + 1) = y$.

Hence for a given copy of $B = \text{Alt}(n)$ in $G = \text{Alt}(n + 1)$, every two different $(n + 1)$ -cycles give two different skew morphisms of B . Since the number of $(n + 1)$ -cycles is $n!$, it follows that the number of skew morphisms of $\text{Alt}(n)$ associated with $\text{Alt}(n + 1)$ is exactly $n!$. In particular, the centraliser in $\text{Aut}(B) \cong \text{Sym}(n)$ of φ is trivial.

Also the set S of elements of the cycle of φ on $\text{Alt}(n)$ containing the double transposition $x = (1, 2)(3, 4)$ consists of the following:

$$\begin{aligned} x &= (1, 2)(3, 4), \\ yxy^{-2} &= (1, n, n - 1, n - 2, \dots, 5, 4, 3), \\ y^2xy^{-1} &= (1, 3, 4, 5, \dots, n - 2, n - 1, n), \\ y^3xy^{-4} &= (1, n, n - 2, n - 3, \dots, 4, 3, 2), \\ y^4xy^{-3} &= (1, 2, 3, 4, \dots, n - 3, n - 2, n), \\ y^5xy^{-5} &= (n - 3, n - 2)(n - 1, n), \\ y^6xy^{-6} &= (n - 4, n - 3)(n - 2, n - 1), \\ &\vdots \\ y^{n-1}xy^{-(n-1)} &= (3, 4)(5, 6), \\ y^nx y^{-n} &= (2, 3)(4, 5). \end{aligned}$$

It is easy to see that this set S generates a transitive subgroup of $\text{Alt}(n)$, and contains the $(n - 1)$ -cycle $y^2xy^{-1} = (1, 3, 4, 5, \dots, n - 2, n - 1, n)$. It follows that $\langle S \rangle$ is 2-transitive, and hence primitive. Then by Jordan's Theorem [8, Theorem 3.3E], the presence of the 5-cycle $xy^nx y^{-n} = (1, 3, 5, 4, 2)$ implies that $\langle S \rangle = \text{Alt}(n)$.

As also S is closed under inverses, it follows that φ , and hence every skew morphism of $\text{Alt}(n)$ associated with $\text{Alt}(n + 1)$, gives rise to a (non-balanced) regular Cayley map for $\text{Alt}(n)$.

Case (7) $B = \text{Sym}(n)$ for $n = 5$ and $n \geq 7$.

This case, where the skew product group G is $\text{Sym}(n + 1)$, and $B = \text{Sym}(n)$ and Y cyclic of order $n + 1$, is similar to the previous case, with only one equivalence class of skew generating pairs and a single equivalence class of proper skew morphisms. Again Y can be taken as the cyclic subgroup generated by a single $(n + 1)$ -cycle, and B as the stabiliser in $\text{Sym}(n + 1)$ of any point, and the size of the equivalence class is $|\text{Aut}(B)| = n!$.

Moreover, if we take y as the $(n + 1)$ -cycle $(1, 2, 3, \dots, n, n + 1)$, and φ as the skew morphism of $\text{Sym}(n)$ induced by y , then the set T of elements of the cycle of φ on $\text{Sym}(n)$ containing the single transposition $(1, 2)$ consists of the n -cycle $(1, 2, 3, 4, \dots, n - 2, n - 2, n)$ and its inverse, plus the $n - 1$ transpositions $(1, 2), (2, 3), \dots, (n - 2, n - 1), (n - 1, n)$. The first n -cycle and any one of these transpositions generate $\text{Sym}(n)$, and clearly T is closed under inverses, so again it follows that φ , and hence every skew morphism of $\text{Sym}(n)$ associated with $\text{Sym}(n + 1)$, gives rise to a (non-balanced) regular Cayley map for $\text{Sym}(n)$.

REMARK 5.3. Note that all skew morphisms considered in **Case (1)** through **Case (7)** induce a cycle which forms a generating set and is closed under inverses. Hence by Proposition 5.2 they all give rise to a regular Cayley map. Moreover, every such regular Cayley map is non-balanced, because the skew morphism is a proper skew morphism. Conversely, by what we have proved here, every non-balanced regular Cayley map for a finite monolithic group B arises in this way. We can summarise the situation even more strongly, in the following, which shows that almost every regular Cayley map for a monolithic group is balanced.

COROLLARY 5.4. *Every non-balanced regular Cayley map for a monolithic group B arising from a proper skew morphism of B for which B is core-free in the corresponding skew product group is one with:*

- (a) $B = \text{Alt}(5)$ and valency 11,
- (b) $B = M_{10}$ and valency 11,
- (c) $B = M_{22}$ and valency 23,
- (d) $B = \text{Alt}(n)$ and valency $n + 1$, where n is even and $n \geq 6$, or
- (e) $B = \text{Sym}(n)$ and valency $n + 1$, where $n \geq 5$.

To complete this subsection, we note that in all of the cases above, the centraliser in $\text{Aut}(B)$ of φ is trivial. Consequently, this is true for all proper skew morphisms of simple groups. On the other hand, there are many examples of groups which admit a skew morphism with non-trivial centraliser. The latter is true even for some almost simple groups (see Example 5.9).

5.3. SUMMARY OF SKEW MORPHISMS OF SIMPLE GROUPS. Let $G = BY$ be the skew product group induced by some skew morphism of a simple group B . If B is not core-free in G , then it is normal in G and, by Lemma 2.1, the skew morphism is an automorphisms of B . If B is core-free in G , then by Theorem 1.1, B is one of $\text{Alt}(5)$, M_{22} or $\text{Alt}(n)$ for even $n \geq 6$. In particular, this proves Corollary 1.2.

Moreover, as we have shown in Section 5.2, $\text{Alt}(5)$, $\text{Alt}(6)$ and M_{22} admit 240, 1440 and 1774080 proper skew morphisms, respectively while, for even $n \geq 8$, $\text{Alt}(n)$ admits $n!$ proper skew morphisms. Skew morphisms of $\text{Alt}(5)$ and M_{22} fall into two equivalence classes of equal size, while those of $\text{Alt}(n)$ form a single equivalence class, for even $n \geq 6$.

By Remark 5.3 and Corollary 5.4 we also have the following.

THEOREM 5.5. *Every proper skew morphism of a non-abelian finite simple group B gives rise to a non-balanced regular Cayley map for B . Moreover, every non-balanced*

regular Cayley map for a non-abelian finite simple group is one for either $\text{Alt}(5)$ with valency 11, or M_{22} with valency 23, or $\text{Alt}(n)$ with valency $n+1$ for some even $n \geq 6$.

Unfortunately we cannot say much about the (balanced) regular Cayley maps that arise from automorphisms of non-abelian finite simple groups, or of monolithic groups in general, because that requires much greater knowledge about generating sets for such groups.

5.4. SKEW MORPHISMS OF MONOLITHIC GROUPS: THE NON-CORE-FREE CASE. We now consider the case where the monolithic group B is not core-free in the skew product group $G = BY$. If B is simple, then it is normal in G , and hence all skew morphisms will be automorphisms of B . In contrast, if B is monolithic but not simple, it might not be normal in G (see Example 4.1) and hence the corresponding skew-morphism might be proper. Moreover, if B is almost simple, the corresponding skew morphism always restricts to an automorphism of $\text{soc}(B)$; see Corollary 5.7.

LEMMA 5.6. *If B is an almost simple group with socle A , then A is the only subgroup of B isomorphic to A .*

Proof. Let T be a subgroup of B isomorphic to A . Then $A \cap T$ is normal in T since A is normal in B , but T is simple, so either $A \cap T = T$ or $A \cap T = \{1\}$. In the latter case, T is isomorphic to a subgroup of B/A , which in turn is a subgroup of $\text{Out}(A)$ since A is the socle of the almost simple group B , but that is impossible since T is insoluble while $\text{Out}(A)$ is soluble, by the proof of Schreier's Conjecture (see [8, Section 4.7]). Thus $A \cap T = T$, and so $T = A$. \square

COROLLARY 5.7. *Let B be an almost simple group with socle A , and let φ be a skew morphism of B associated with the skew product group $G = BY$. If B is not core-free in G , then φ restricts to an automorphism of A .*

Proof. By Proposition 3.7, A is normal in G , and so $\ker \varphi$ contains A by Lemma 2.1. It follows that φ restricts to an isomorphism from A to the subgroup $T = \varphi(A)$ of B , but then $A = T$ by Lemma 5.6, and therefore φ restricts to an automorphism of A . \square

Note here that if B is almost simple with socle A , then A must be a relatively large subgroup of B , because $|B : A| \leq |\text{Out}(A)| < \log_2 |A|$; see [12] for example.

It would be interesting to obtain a generalisation of Lemma 5.6 to the monolithic case, with a corresponding generalisation of Corollary 5.7.

Note also that the remarks before Lemma 5.6 help to explain the fact that if G is the automorphism group of a balanced regular Cayley map for a non-abelian simple group A , then the customary factorisation $G = AY$ cannot be re-written to produce a non-balanced regular Cayley map for A (or more specifically, for a subgroup of G that is isomorphic to A). This contrasts strongly with the situation for cyclic groups, where the latter is often possible; see [6, Section 2].

5.5. SKEW MORPHISMS OF MONOLITHIC GROUPS WITH SOCLE OF INDEX TWO. We now show how Proposition 3.7 can be used to find all skew product groups $G = BY$ for a monolithic group B with monolith A in the case when $B \cong \text{Aut}(A)$ and is not core-free in G , and $|B : A| = 2$.

PROPOSITION 5.8. *Let B be a monolithic group with monolith A , and let $G = BY$ be a skew product group for B with B not core-free in G . If B is isomorphic to $\text{Aut}(A)$ and $|B : A| = 2$, then $G = C_G(A) \rtimes B$. Moreover, $|C_G(A)| = |Y|$ and is bounded above by the maximum order of an element in B , and $C_G(A)$ has a cyclic subgroup of index at most two.*

Proof. By Proposition 3.7, A is normal in G and $C_Y(A) = \{1\}$. Since $B \cong \text{Aut}(A)$, we have $G/C_G(A) \cong \text{Aut}(A)$, and then since B is monolithic, $C_B(A) = \{1\}$ and so $G = C_G(A) \rtimes B$. In particular, by order considerations, we find that $|C_G(A)| = |Y|$. But also Y acts faithfully by conjugation on A since $C_Y(A) = \{1\}$, and thus Y embeds in $\text{Aut}(A) \cong B$. As Y is cyclic, it follows that $|C_G(A)| = |Y|$ is bounded above by the maximum order of an element in B .

Next, write \overline{X} for XA/A when X is a subgroup of G . Since $A \cap Y = \{1\}$ and $|B : A| = 2$, we see that $\overline{Y} \cong Y$ and $|\overline{B}| = 2$. It follows that $|\overline{G}| = 2|\overline{Y}|$, and hence \overline{Y} is a cyclic subgroup of index two in \overline{G} . This implies that $\overline{C_G(A)} \cap \overline{Y}$ is a cyclic subgroup of $\overline{C_G(A)}$ of index at most 2. Finally, since A is the monolith of B , we find that $A \cap C_G(A) = \{1\}$, and thus $\overline{C_G(A)} \cong C_G(A)$. \square

There are many interesting examples of a monolithic group B with monolith A such that $B \cong \text{Aut}(A)$ and $|B : A| = 2$, such as the following:

- $\text{Sym}(n)$ for $n = 5$ and $n \geq 7$,
- $\text{PGL}(2, p)$ for an odd prime p ,
- $\text{PGL}(n, p)$ for $n \geq 3$ and a prime p such that $\gcd(n, p - 1) = 1$,
- $T \wr C_2$, where T is a non-abelian simple group such that $T \cong \text{Aut}(T)$.

We now explain how to find all skew product groups G for such a monolithic group B . By the results of Section 5.2, it suffices to consider the case where B is not core-free in G , and apply Proposition 5.8. This gives us a good upper bound on the order of $C_G(A)$. Also the groups having a cyclic subgroup of index at most two are very well-understood (see [7] for example), and hence we can easily identify all possible candidates for $C_G(A)$. For each candidate for $C_G(A)$, we find all semi-direct products of the form $C_G(A) \rtimes B$. As A acts trivially on $C_G(A)$ by conjugation and $|B : A| = 2$, conjugation by B induces a group of automorphisms of $C_G(A)$ of order at most two. This gives us a complete list of candidates for G . As a final step, we reject a candidate G if it does not admit a cyclic core-free complement of B .

Once we have all possible skew product groups for B , we can attempt to find all skew morphisms using the method outlined in Section 5.1. We now illustrate this method in the smallest relevant case, namely $B = \text{Sym}(5)$.

EXAMPLE 5.9. Let $B = \text{Sym}(5)$, and $A = \text{soc}(B) = \text{Alt}(5)$, and let $G = BY$ be a skew product group induced by some skew morphism of B . If B is normal in G , then by Lemma 2.1, the skew morphism is an automorphism of B . Also the case when B is core-free in G was dealt with in Section 5.2, and hence we may assume that the core of B in G is A .

By Proposition 5.8, we find that $G = C_G(A) \rtimes B$, and that $|C_G(A)|$ is bounded above by the maximum order of an element in B , namely 6. Then since $G = C_G(A) \rtimes B$ where B is not normal in G , it follows that $|C_G(A)| \geq 3$. In particular, $C_G(A)$ is isomorphic to $C_3, C_4, C_2^2, C_5, C_6$ or $\text{Sym}(3)$.

It remains to determine the conjugation action of B on $C_G(A)$. Since B is not normal in G , this action is non-trivial, but its index 2 subgroup A acts trivially by definition, and so B induces an automorphism of $C_G(A)$ of order 2. In each case, there is a unique conjugacy class of element of order 2 in $\text{Aut}(C_G(A))$, and hence a unique possibility for G .

Now that we know G , it remains to find Y , which must be a cyclic core-free complement for B in G . It turns out that G has no such subgroup when $C_G(A) \cong C_4$ or $C_G(A) \cong C_6$, and a unique class of such subgroups in all other cases, up to conjugacy under $N_G(B)$. (This is a slightly tedious computation to do by hand, but it is easy to do by computer.)

Using similar methods as in Section 5.2, we enumerate all skew morphisms for these skew product groups. When $C_G(A)$ is isomorphic to C_3 , C_2^2 , C_5 or $\text{Sym}(3)$, we get a single equivalence class of skew morphisms in each case, containing 20, 30, 24 and 20 proper skew morphisms, respectively. Indeed in all of these cases, the centraliser in $\text{Aut}(B)$ of the skew morphism is non-trivial.

Thus $\text{Sym}(5)$ admits $20 + 30 + 24 + 20 = 94$ proper skew morphisms (in four equivalence classes) when $\text{Alt}(5)$ is the core of $\text{Sym}(5)$ in its skew product group, plus another 120 proper skew morphisms in a single equivalence class when the core is trivial, and 120 automorphisms when the core is $\text{Sym}(5)$.

Using a computer to automate the approach used in Example 5.9, we are able to find all skew morphisms for all monolithic groups B with monolith A such that $B \cong \text{Aut}(A)$, $|B : A| = 2$, and $|B| < 200000$. This includes the cases where $B = \text{Sym}(8)$, $\text{Aut}(M_{12})$ or $\text{PGL}(2, 53)$, for example. By comparison, the best method we know for determining all skew morphisms of an arbitrary finite group A is computationally feasible only for $|A|$ up to 47 (by considering all transitive permutation groups of degree $|A|$ with cyclic point-stabiliser).

Finally, we note that the methods of this section, based on Proposition 3.7, can be generalised to other monolithic groups, but the difficulty increases quickly with respect to $|\text{Aut}(A) : A|$.

REFERENCES

- [1] Martin Bachratý, *List of skew-morphisms of cyclic groups up to order 161*, 2020.
- [2] Martin Bachratý and Robert Jajcay, *Powers of skew-morphisms*, in *Symmetries in graphs, maps, and polytopes*, Springer Proc. Math. Stat., vol. 159, Springer, 2016, pp. 1–25.
- [3] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, *J. Symbolic Comput.* **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993).
- [4] Jiyong Chen, Shaofei Du, and Cai Heng Li, *Skew-morphisms of nonabelian characteristically simple groups*, *J. Combin. Theory Ser. A* **185** (2022), article no. 105539 (17 pages).
- [5] Marston D. E. Conder, Robert Jajcay, and Thomas W. Tucker, *Cyclic complements and skew morphisms of groups*, *J. Algebra* **453** (2016), 68–100.
- [6] Marston D. E. Conder and Thomas W. Tucker, *Regular Cayley maps for cyclic groups*, *Trans. Amer. Math. Soc.* **366** (2014), no. 7, 3585–3609.
- [7] Kálmán Csiszter and Mátyás Domokos, *The Noether number for the groups with a cyclic subgroup of index two*, *J. Algebra* **399** (2014), 546–560.
- [8] John D. Dixon and Brian Mortimer, *Permutation groups*, Graduate Texts in Mathematics, vol. 163, Springer-Verlag, New York, 1996.
- [9] Noboru Itô, *Über das Produkt von zwei abelschen Gruppen*, *Math. Z.* **62** (1955), 400–401.
- [10] Robert Jajcay and Jozef Širáň, *Skew-morphisms of regular Cayley maps*, *Discrete Math.* **244** (2002), no. 1-3, 167–179, Algebraic and topological methods in graph theory (Lake Bled, 1999).
- [11] Gareth A. Jones, *Cyclic regular subgroups of primitive permutation groups*, *J. Group Theory* **5** (2002), no. 4, 403–407.
- [12] Stefan Kohl, *A bound on the order of the outer automorphism group of a finite simple group of given order*, <https://stefan-kohl.github.io/preprints/outbound.pdf>.
- [13] István Kovács and Young Soo Kwon, *Classification of reflexible Cayley maps for dihedral groups*, *J. Combin. Theory Ser. B* **127** (2017), 187–204.
- [14] ———, *Regular Cayley maps for dihedral groups*, *J. Combin. Theory Ser. B* **148** (2021), 84–124.
- [15] István Kovács and Roman Nedela, *Decomposition of skew-morphisms of cyclic groups*, *Ars Math. Contemp.* **4** (2011), no. 2, 329–349.
- [16] ———, *Skew-morphisms of cyclic p -groups*, *J. Group Theory* **20** (2017), no. 6, 1135–1154.
- [17] Cai Heng Li and Cheryl E. Praeger, *On finite permutation groups with a transitive cyclic subgroup*, *J. Algebra* **349** (2012), 117–127.
- [18] Martin W. Liebeck, Cheryl E. Praeger, and Jan Saxl, *The maximal factorizations of the finite simple groups and their automorphism groups*, *Mem. Amer. Math. Soc.* **86** (1990), no. 432, iv+151.

- [19] Andrea Lucchini, *On the order of transitive permutation groups with cyclic point-stabilizer*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **9** (1998), no. 4, 241–243 (1999).
- [20] Ivan Niven, *Fermat's theorem for matrices*, Duke Math. J. **15** (1948), 823–826.
- [21] Derek J. S. Robinson, *A course in the theory of groups*, second ed., Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1996.
- [22] Jun-Yang Zhang and Shaofei Du, *On the skew-morphisms of dihedral groups*, J. Group Theory **19** (2016), no. 6, 993–1016.

MARTIN BACHRATÝ, Mathematics Department, Slovak University of Technology, 81005 Bratislava, Slovakia
E-mail : `martin.bachraty@stuba.sk`

MARSTON CONDER, Mathematics Department, University of Auckland, PB 92019, Auckland, New Zealand
E-mail : `m.conder@auckland.ac.nz`

GABRIEL VERRET, Mathematics Department, University of Auckland, PB 92019, Auckland, New Zealand
E-mail : `g.verret@auckland.ac.nz`