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Generating series of non-oriented constellations and marginal sums in the Matching-Jack conjecture

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Abstract Using the description of hypermaps with matchings, Goulden and Jackson have given an expression of the generating series of rooted bipartite maps in terms of the zonal polynomials. We generalize this approach to the case of constellations on non-oriented surfaces that have recently been introduced by Chapuy and Dołęga. A key step in the proof is an encoding of constellations with tuples of matchings.

We consider a one parameter deformation of the generating series of constellations using Jack polynomials and we introduce the coefficients $c^{\lambda}_{\mu^0,\dots,\mu^k}(b)$ obtained by the expansion of these functions in the power-sum basis. These coefficients are indexed by $k+2$ integer partitions and the deformation parameter b, and can be considered as a generalization for $k>1$ of the connection coefficients introduced by Goulden and Jackson. We prove that when we take some marginal sums, these coefficients enumerate b-weighted k-tuples of matchings. This can be seen as an “disconnected” version of a recent result of Chapuy and Dołęga for constellations. For $k=1$, this gives a partial answer to Goulden and Jackson Matching-Jack conjecture.

Lassalle has formulated a positivity conjecture for the coefficients $\theta^{(\alpha)}_{\mu}(\lambda)$, defined as the coefficient of the Jack polynomial $J^{(\alpha)}_{\lambda}$ in the power-sum basis. We use the second main result of this paper to give a proof of this conjecture in the case of partitions λ with rectangular shape.

1. Introduction

1.1. Jack polynomials and maps. Jack polynomials $J^{(\alpha)}_{\xi}$ are symmetric functions indexed by an integer partition ξ and a deformation parameter α that were introduced in [22]. Jack polynomials can be considered as one parameter deformation of Schur functions, which are obtained by evaluating the Jack polynomials at $\alpha=1$ and rescaling. For $\alpha=2$ we recover the zonal polynomials. This family of symmetric functions is related to various combinatorial problems [13, 17, 20]. Some properties of Jack polynomials have been investigated in [32] and [31, Chapter VI].

In this paper, we will be interested in relationships between Jack polynomial series and generating series of maps. A connected map is a 2-cell embedding of a connected graph into a closed surface without boundary, orientable or not. A map[(1)] is an unordered collection of connected maps. In this paper, we will use the word *orientable* for maps on orientable surfaces and the word *non-oriented* for maps on general surfaces, orientable or not. Maps appear in various branches of algebraic combinatorics, probability and physics. The study of maps involves various methods such as generating

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(1) This is not the usual definition of maps; what is usually called a map is called here a connected map.

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series, matrix integral techniques and bijective methods, see e.g. [1, 2, 6, 14, 29]. In this paper we will consider a class of vertex-colored maps that generalize bipartite maps, called k-constellations. Constellations on orientable surfaces were introduced in [29] and were generalized to the case of non-orientable surfaces in [8], see Section 1.3.

Let $j_\xi^{(\alpha)} := \langle J_\xi, J_\xi \rangle_\alpha$ be the squared norm of the Jack polynomial associated to ξ with respect to the α-deformation of the Hall scalar product, see Section 2.3. We consider $k+2$ different alphabets $\mathbf{x}^{(i)} := (x_1^{(i)}, x_2^{(i)}, \ldots)$, for $-1 \leqslant i \leqslant k$, and we set the power-sum variables associated respectively to these alphabets $\mathbf{p} := (p_1, p_2 \ldots)$ and $\mathbf{q}^{(i)} := (q_1^{(i)}, q_2^{(i)} \ldots)$ for $0 \leqslant i \leqslant k$. Chapuy and Dołęga have introduced(2) in [8] for every $k \geqslant 1$ a function $\tau_b^{(k)}$ with $k+2$ sets of variables, defined as follows:

$$(1) \qquad \tau_b^{(k)}(t, \mathbf{p}, \mathbf{q}^{(0)}, \ldots, \mathbf{q}^{(k)}) := \sum_{n \geqslant 0} t^n \sum_{\xi \vdash n} \frac{1}{j_\xi^{(\alpha)}} J_\xi^{(\alpha)}(\mathbf{p}) J_\xi^{(\alpha)}(\mathbf{q}^{(0)}) \ldots J_\xi^{(\alpha)}(\mathbf{q}^{(k)}),$$

where $J_\xi^{(\alpha)}$ are the Jack polynomials of parameter $\alpha = b+1$, expressed in the power-sum variables $J_\xi^{(\alpha)}(\mathbf{p}) := J_\xi^{(\alpha)}(\mathbf{x}^{(-1)})$ and $J_\xi^{(\alpha)}(\mathbf{q}^{(i)}) := J_\xi^{(\alpha)}(\mathbf{x}^{(i)})$ for $0 \leqslant i \leqslant k$. This function can be seen as an extension to $k+2$ sets of variables of the Cauchy sum for Jack symmetric functions. We also define

$$(2) \qquad \Psi_b^{(k)}(t, \mathbf{p}, \mathbf{q}^{(0)}, \mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(k)}) := (1+b) t \frac{\partial}{\partial t} \log \tau_b^{(k)}(t, \mathbf{p}, \mathbf{q}^{(0)}, \ldots, \mathbf{q}^{(k)}).$$

In the case $k = 1$, these functions were first introduced by Goulden and Jackson [17]. They suggested that the function $\tau_b^{(1)}$ is related to the generating series of matchings and $\Psi_b^{(1)}$ is related to the generating series of connected hypermaps (or by duality connected bipartite maps). The exponent of the shifted parameter $b := \alpha - 1$ is claimed to be correlated to the bipartiteness of the matchings in the first case and to the orientability of the maps in the second one. This was formulated in two conjectures that are still open, namely the b-conjecture and the Matching-Jack conjecture. These conjectures imply that the coefficients of the functions $\tau_b^{(1)}$ and $\Psi_b^{(1)}$ in the power-sum basis denoted respectively $c^\lambda_{\mu,\xi}(b)$ and $h^\lambda_{\mu,\xi}(b)$ are non-negative integer polynomials in b. In this paper, we consider a generalization of these quantities $c^\lambda_{\mu^0,\ldots,\mu^k}$ and $h^\lambda_{\mu^0,\ldots,\mu^k}$ indexed by $k+2$ partitions and defined by the expansion of $\tau_b^{(k)}$ and $\Psi_b^{(k)}$ in power-sum basis (see Equations (5) and (6)). We investigate their relationship with the enumeration of non-oriented constellations and tuples of matchings.

1.2. Main contributions.

We now say a word about the main contributions of the paper. The first four points will be discussed in more details in the next subsections of the introduction:

- We describe an encoding of non-oriented constellations with tuples of matchings; two versions of this correspondence are given, see Proposition 3.4 and Theorem 4.13.
- This correspondence is used to obtain Theorem 1.4 which relates the generating series of non-oriented k-constellations to the function $\Psi_b^{(k)}$ in the special case $b = 1$. The case $k = 1$ of this result was proved by Goulden and Jackson in [18].
- In the second part of this paper, we consider some marginal sums of the coefficients $c^\lambda_{\mu^0,\ldots,\mu^k}$ and $h^\lambda_{\mu^0,\ldots,\mu^k}$, where we control two partitions λ and μ and the number of parts of the other partitions, denoted respectively $c^\lambda_{\mu,l_1,\ldots l_k}(b)$

(2) The function introduced in [8] is a specialization of this function.

and $h^{\lambda}_{\mu,l_1,\dots,l_k}(b)$. Theorem 1.9 (see also Theorem 4.4) states that the coefficients $c^{\lambda}_{\mu,l_1,\dots l_k}(b)$ are non-negative integer polynomials in b and that they enumerate b-weighted k-tuples of matchings. The proof is based on the work of Chapuy and Dołęga [8] that gives an analogous result for the coefficients $h^{\lambda}_{\mu,l_1,\dots,l_k}(b)$. The fact that the coefficients $c^{\lambda}_{\mu,l_1,\dots l_k}(b)$ are polynomials in b with positive coefficients can directly be obtained from the result of [8], but not the integrality because of the derivative taken in Equation (2). In the proof of Theorem 1.9 we use symmetry properties to eliminate factors appearing in the denominator. When $k = 1$, Theorem 1.9 gives the marginal sum case in the Matching-Jack conjecture, and covers other partial results established for this conjecture [24, 25].

- Theorem 5.5 gives a combinatorial expression for the coefficients of the development of Jack polynomials J_λ in the power-sum basis, for rectangular partitions λ. In particular, this completes the proof of Lassalle's conjecture in the rectangular case. The proof is based on Theorem 1.9.
- Theorems 6.11 and 6.12 give a combinatorial interpretation of the top degree part in coefficients $c^{\lambda}_{\mu^0,\dots,\mu^k}$. In the case $k = 1$, this was investigated in [5] using Jack characters. We give here a different proof.

1.3. Constellations and matchings.

We consider the definition of constellations on general surfaces, orientable or not, given in [8]. The link with the usual definition of constellations in the orientable case is explained in Section 2.5.

Definition 1.1. *Let $k > 1$. A (non-oriented) k-constellation is a map, connected or not, whose vertices are colored with colors $\{0, 1, \dots, k\}$ such that*[(3)]*:*

(1) *Each vertex of color 0 (respectively k) has only neighbors of colors 1 (respectively $k - 1$).*
(2) *For $0 < i < k$, a vertex of color i has only neighbors of color $i - 1$ and $i + 1$, and each corner of such vertex separates two vertices of colors $i - 1$ and $i + 1$ (see Section 2.4 for the definition of corners).*

Constellations come with a natural notion of rooting; a connected constellation $\mathbf{M}$ is *rooted* by distinguishing an oriented corner c of color 0. The rooted constellation obtained is denoted $(\mathbf{M}, c)$. We can define the *size* of a constellation as the sum of the degrees of all vertices of color 0. An example of a rooted non-oriented 3-constellation of size 3 is illustrated in Figure 1.

Given a k-constellation $\mathbf{M}$, we can define its *profile* as the $(k + 2)$-tuple of integer partitions $(\lambda, \mu^0, \dots, \mu^k)$, where λ is the distribution of the face degrees, and μ^i is the distribution of the vertices of color i for $0 \leqslant i \leqslant k$, see Section 2.5 for a precise definition.

Constellations in the orientable case have been studied for a long time and have a well-known combinatorial description given by rotation systems; an orientable k-constellation of size n can be encoded by $(k + 1)$-tuple of permutations of $\mathfrak{S}_n$. Moreover the profile of the constellation is related to the cyclic type of the permutations, see [4, 7, 15, 29]. In Section 2.5, we introduce a notion of labelling for non-oriented constellations using *right-paths*. This leads to a correspondence between k-constellations and $(k+2)$-tuples of matchings, see Proposition 3.4. In fact, this completes the table 1.

(3) We use here the convention of [8], what we call k-constellation is often called $(k + 1)$-constellation in the orientable case.

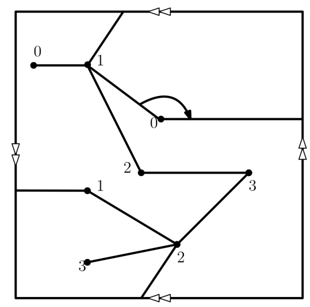


Figure 1. An example of a rooted 3-constellation drawn on the Klein bottle. The left-hand side of the square should be glued to the right-hand one (with a twist) and the top side should be glued to the bottom one (without a twist), as indicated by the white arrows. The root corner is indicated by a black arrow.

Table 1

	labelled bipartite maps	labelled k-constellations
orientable	pairs of permutations [classical]	$(k+1)$-tuples of permutations [23]
non-oriented	triples of matchings [18]	$(k+2)$-tuples of matchings, this paper

For $n \geq 1$, we will consider matchings on the set $A_n := \{1, \widehat{1}, \dots, n, \widehat{n}\}$, that is a set partition of A_n into pairs. A matching δ on A_n is *bipartite* if each one of its pairs is of the form $(i, \widehat{j})$. Given two matchings δ_1 and δ_2, we consider the partition $\Lambda(\delta_1, \delta_2)$ of n obtained by reordering the half-sizes of the connected components of the graph formed by δ_1 and δ_2. We consider the bipartite matching on A_n given by $\varepsilon := \Big\{\{1, \widehat{1}\}, \{2, \widehat{2}\}, \dots, \{n, \widehat{n}\}\Big\}$, and for each λ partition of n, we set the matching

$$\delta_\lambda := \Big\{\{1, \widehat{2}\}, \{2, \widehat{3}\} \dots \{\lambda_1 - 1, \widehat{\lambda_1}\}, \{\lambda_1, \widehat{1}\}, \{\lambda_1 + 1, \widehat{\lambda_1 + 2}\}, \dots\Big\}. \tag{3}$$

The matchings ε and δ_λ will have a specific role in this article as an example of bipartite matchings satisfying $\Lambda(\varepsilon, \delta_\lambda) = \lambda$.

Example 1.2. In Figure 2, the graph formed by ε and δ_λ is illustrated for $n = 8$ and $\lambda = [3, 3, 2]$.

Note that the symmetric group $\mathfrak{S}_n$ is in natural bijection with the set of bipartite matchings of A_n via the map $\sigma \longrightarrow \delta_\sigma$ where δ_σ is the bipartite matching whose pairs are $(i, \widehat{\sigma(i)})_{1 \leqslant i \leqslant n}$. Given a k-tuple of matchings, $\lambda, \mu^0, \dots, \mu^k \vdash n$, we consider the two sets

$$\mathfrak{F}^\lambda_{\mu^0,\dots,\mu^k} := \Big\{(\delta_0, \dots, \delta_{k-1}) \text{ matchings of } A_n \text{ such that } \Lambda(\varepsilon, \delta_0) = \mu^0, \tag{4}$$
$$\Lambda(\delta_{k-1}, \delta_\lambda) = \mu^k \text{ and } \Lambda(\delta_{i-1}, \delta_i) = \mu^i \quad \forall i \in \{1, \dots, k-1\}\Big\},$$

and

$$\widetilde{\mathfrak{F}}^\lambda_{\mu^0,\dots,\mu^k} := \Big\{(\delta_0, \dots, \delta_{k-1}) \in \mathfrak{F}^\lambda_{\mu^0,\dots,\mu^k} \text{ such that } \delta_i \text{ is bipartite for } 1 \leqslant i \leqslant k-1\Big\}.$$

In Theorem 4.13, we prove that k-constellations with profile $(\lambda, \mu^0, \dots, \mu^k)$ with a specific labelling are in bijection with $\mathfrak{F}^\lambda_{\mu^0,\dots,\mu^k}$.

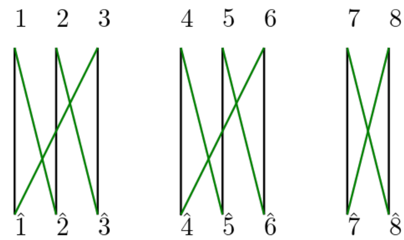


FIGURE 2. $G(\varepsilon, \delta_\lambda)$ for $\lambda = [3,3,2]$, ε in black and δ_λ in green.

1.4. GENERATING SERIES OF CONSTELLATIONS. As described in Section 1.3, constellations on the orientable case can be encoded with tuples of permutations. The multivariate generating series of permutations with respect to their cycle type, can be related via representation theory tools to the function $\tau_0^{(k)}$, see e.g. [19, Proposition 1.1] (in the following, we formulate this result using bipartite matchings rather than permutations). This gives a formula for the generating series of orientable constellations. To state this result we need to introduce the following notation; for a partition λ and a non-negative integer n we write $\lambda \vdash n$ if λ is a partition of n and if $\lambda = [\lambda_1, \lambda_2, \dots]$ we set as in [31]

$$z_\lambda \coloneqq \prod_{i > 1} m_i(\lambda)! i^{m_i(\lambda)},$$

where $m_i(\lambda)$ is the number of parts of λ equal to i. For a given set of variables $\mathbf{u} \coloneqq (u_1, u_2, \dots)$, we denote $u_\lambda \coloneqq u_{\lambda_1} u_{\lambda_2} \dots$.

THEOREM 1.3 ([19]). *For $k > 1$, we have*

(i) $\tau_0^{(k)}(t, \mathbf{p}, \mathbf{q}^{(0)}, \dots, \mathbf{q}^{(k)}) = 1 + \sum_{n > 1} t^n \sum_{\lambda, \mu^0, \dots, \mu^k \vdash n} \frac{|\widetilde{\mathfrak{F}}^\lambda_{\mu^0, \dots, \mu^k}|}{z_\lambda} p_\lambda q^{(0)}_{\mu^0} \dots q^{(k)}_{\mu^k}$,

(ii) $\Psi_0^{(k)}(t, \mathbf{p}, \mathbf{q}^{(0)}, \dots, \mathbf{q}^{(k)}) = \sum_{n > 1} t^n \sum_{\lambda, \mu^0, \dots, \mu^k \vdash n} \widetilde{C}^\lambda_{\mu^0, \dots, \mu^k} p_\lambda q^{(0)}_{\mu^0} \dots q^{(k)}_{\mu^k}$,

where $\widetilde{C}^\lambda_{\mu^0, \dots, \mu^k}$ is the number of rooted connected orientable k-constellations, with profile $(\lambda, \mu^0, \dots, \mu^k)$.

In this paper, we give an analogous result in the non-oriented case:

THEOREM 1.4. *For every $k > 1$, we have*

(i) $\tau_1^{(k)}(t, \mathbf{p}, \mathbf{q}^{(0)}, \dots, \mathbf{q}^{(k)}) = 1 + \sum_{n > 1} t^n \sum_{\lambda, \mu^0, \dots, \mu^k \vdash n} \frac{|\mathfrak{F}^\lambda_{\mu^0, \dots, \mu^k}|}{2^{\ell(\lambda)} z_\lambda} p_\lambda q^{(0)}_{\mu^0} \dots q^{(k)}_{\mu^k}$,

(ii) $\Psi_1^{(k)}(t, \mathbf{p}, \mathbf{q}^{(0)}, \dots, \mathbf{q}^{(k)}) = \sum_{n > 1} t^n \sum_{\lambda, \mu^0, \dots, \mu^k \vdash n} C^\lambda_{\mu^0, \dots, \mu^k} p_\lambda q^{(0)}_{\mu^0} \dots q^{(k)}_{\mu^k}$,

where $\ell(\lambda)$ is the number of parts of λ and $C^\lambda_{\mu^0, \dots, \mu^k}$ is the number of non-oriented rooted connected k-constellations with profile $(\lambda, \mu^0, \dots, \mu^k)$.

For $k = 1$, these results have been proved in [18]. In this case, constellations are bipartite maps. The generating series of non-oriented bipartite maps have been[4] investigated through the correspondence between bipartite maps and matchings on the one hand, and a relationship between matching enumeration and the structure coefficients of the double coset algebra of the Gelfand pair $(\mathfrak{S}_{2n}, \mathfrak{B}_n)$ on the other hand. In this paper, we extend this development to constellations.

[4] Actually, the maps considered in [18] are face colored maps called hypermaps. These maps are obtained from bipartite maps by duality.

1.5. Generalized Goulden and Jackson conjectures. We define the coefficients $c^{\lambda}_{\mu^0,\ldots,\mu^k}$ and $h^{\lambda}_{\mu^0,\ldots,\mu^k}$ for partitions $\lambda,\mu^0,\ldots,\mu^k \vdash n \geq 1$ such that

$$\tau_b^{(k)}(t,\mathbf{p},\mathbf{q}^{(0)},\ldots,\mathbf{q}^{(k)}) = 1+\sum_{n\geq 1} t^n \sum_{\lambda,\mu_0,\ldots,\mu_k \vdash n} \frac{c^{\lambda}_{\mu_0,\ldots,\mu_k}(b)}{z_\lambda(1+b)^{\ell(\lambda)}} p_\lambda q^{(0)}_{\mu_0}\cdots q^{(k)}_{\mu_k}, \tag{5}$$

and

$$\Psi_b^{(k)}(t,\mathbf{p},\mathbf{q}^{(0)},\ldots,\mathbf{q}^{(k)}) = \sum_{n\geq 1} t^n \sum_{\lambda,\mu_0,\ldots,\mu_k \vdash n} h^{\lambda}_{\mu_0,\ldots,\mu_k}(b) p_\lambda q^{(0)}_{\mu_0}\cdots q^{(k)}_{\mu_k}. \tag{6}$$

For $k=1$, these coefficients were introduced by Goulden and Jackson. They have conjectured that these coefficients are non-negative integer polynomials in b and that they enumerate respectively matchings and bipartite maps. Based on the particular cases of Theorems 1.3 and 1.4, and on computational explorations we formulate a generalization of these conjectures for $k \geq 1$. These generalized conjectures are somehow implicit in [8]. First, we introduce a positivity conjecture.

Conjecture 1.5 (Generalized positivity conjecture.). *For every $k \geq 1$ and partitions $\lambda,\mu^0,\ldots,\mu^k$ of size $n \geq 1$, the coefficients $c^{\lambda}_{\mu^0,\ldots\mu^k}(b)$ and $h^{\lambda}_{\mu^0,\ldots\mu^k}(b)$ are polynomials in b with non-negative integer coefficients.*

It turns out that Goulden and Jackson's positivity conjecture for the coefficients c (the case $k=1$) implies the conjecture for any $k \geq 1$, since these coefficients satisfy a multiplicativity property (see Proposition 6.1). Such property does not a priori exist for the coefficients $h^{\lambda}_{\mu^0,\ldots,\mu^k}$, so the positivity for these coefficients is more general than the conjecture for $k=1$. Using computer explorations, this conjecture has been tested when $k \leq 5$ and $n \leq 12-k$. We attach[5] the values of the coefficients c and h for $k \leq 5$ and $n \leq 9-k$. Given Theorems 1.3 and 1.4, Conjecture 1.5 is equivalent to the two following combinatorial conjectures:

Conjecture 1.6 (Generalized Matching-Jack conjecture). *For every $k \geq 1$ and partitions $\lambda,\mu^0,\ldots,\mu^k$ of size $n \geq 1$, there exists a function ϑ_λ on $\mathfrak{F}^{\lambda}_{\mu^0,\ldots,\mu^k}$ with non-negative integer values such that $\vartheta_\lambda(\delta_0,\ldots,\delta_{k-1})$ is zero if and only if each one of the matchings $\delta_0,\ldots,\delta_{k-1}$ is bipartite, and*

$$c^{\lambda}_{\mu^0,\ldots,\mu^k}(b) = \sum_{(\delta_0,\ldots,\delta_{k-1}) \in \mathfrak{F}^{\lambda}_{\mu^0,\ldots,\mu^k}} b^{\vartheta_\lambda(\delta_0,\ldots,\delta_{k-1})}.$$

Conjecture 1.7 (Generalized b-conjecture). *For every $k \geq 1$ and partitions $\lambda,\mu^0,\ldots,\mu^k \vdash n \geq 1$, there exists a function ν on connected rooted constellations with profile $(\lambda,\mu^0,\ldots,\mu^k)$ with non-negative integer values such that $\nu(\mathbf{M},c)$ is zero if and only if $(\mathbf{M},c)$ is orientable, and*

$$h^{\lambda}_{\mu^0,\ldots,\mu^k}(b) = \sum_{(\mathbf{M},c)} b^{\nu(\mathbf{M},c)},$$

where the sum runs over rooted connected k-constellations with profile $(\lambda,\mu^0,\ldots,\mu^k)$.

The conjectures introduced by Goulden and Jackson in [17] correspond to the case $k=1$ and are still open, despite many partial results [8, 10, 11, 12, 24, 25, 28]. Using basic facts about Jack symmetric functions, we can see that the quantities $c^{\lambda}_{\mu^0,\ldots,\mu^k}(b)$ and $h^{\lambda}_{\mu^0,\ldots,\mu^k}(b)$ are rational functions in b with rational coefficients. The polynomiality of the quantities $c^{\lambda}_{\mu^0,\ldots,\mu^k}$ and $h^{\lambda}_{\mu^0,\ldots,\mu^k}$ for $k=1$ is not direct from

(5) To access the attached files with acrobat reader, click on view/navigation panels/attachments. Alternatively, you can download the source files from arXiv.org

the construction and has been open for twenty years. Dołęga and Féray have proved this polynomiality for coefficients $c^{\lambda}_{\mu^0,\mu^1}(b)$, see [11]. Shortly after, they deduced the polynomiality of $h^{\lambda}_{\mu^0,\mu^1}(b)$, see [12]. The proofs extend directly to any $k > 1$.

Conjecture 1.6 and Conjecture 1.7 are closely related, since the functions $\tau^{(k)}$ and $\Psi^{(k)}$ are related by Equation (2), and k-constellations can be encoded by matchings, see Proposition 3.4 and Theorem 4.13. However, we are not aware of any implication between them in the general case (even for $k = 1$). This difficulty to pass from one conjecture to another is due to the fact that we divide by $z_\lambda(1+b)^{\ell(\lambda)}$ in the definition of $c^{\lambda}_{\mu^0,\dots,\mu^k}$ and we should take the logarithm and the derivative to pass from $\Psi^{(k)}$ to $\tau^{(k)}$. Combinatorially, this can be explained by the fact that constellations appearing in the sum of the b-conjecture are rooted while there is no natural way to "root" the elements of the sets $\mathfrak{F}^{\lambda}_{\mu^0,\dots,\mu^k}$ appearing in the Matching-Jack conjecture. This notion of rooting will be discussed in more detail in Section 4. Nevertheless, we were able to overcome these difficulties in the case of marginal sums and deduce a result for $c^{\lambda}_{\mu^0,\dots,\mu^k}$ from the analog result of [8] on $h^{\lambda}_{\mu^0,\dots,\mu^k}$.

1.6. The case of marginal sums. We consider the marginal sums of coefficients $c^{\lambda}_{\mu,\mu^1,\dots,\mu^k}$ and $h^{\lambda}_{\mu,\mu^1,\dots,\mu^k}$, defined as follows: for all $\lambda, \mu \quad n > 1$, and for all $l_1, \dots, l_k > 1$ set

$$h^{\lambda}_{\mu,l_1,\dots,l_k} := \sum_{\mu^i \quad n,\ell(\mu^i)=l_i} h^{\lambda}_{\mu,\mu^1,\dots,\mu^k} \quad \text{and} \quad c^{\lambda}_{\mu,l_1,\dots,l_k} := \sum_{\mu^i \quad n,\ell(\mu^i)=l_i} c^{\lambda}_{\mu,\mu^1,\dots,\mu^k}, \tag{7}$$

where $\ell(\mu^i)$ denotes the number of parts of the partitions μ^i. The main motivation for formulating the generalized versions of Goulden and Jackson conjectures introduced above is the following theorem due to Chapuy and Dołęga [8] that establishes the generalized version of b-conjecture for the marginal sums $h^{\lambda}_{\mu,l_1,\dots,l_k}$;

Theorem 1.8 ([8]). *For every $k > 1$, partitions $\lambda, \mu \quad n > 1$ and $l_1, \dots l_k > 1$, there exists a function ν on connected rooted constellations with profile $(\lambda, \mu^0, \dots, \mu^k)$ with non-negative integer values such that $\nu(\mathbf{M}, c)$ is zero if and only if $(\mathbf{M}, c)$ is orientable, and*

$$h^{\lambda}_{\mu,l_1,\dots,l_k} = \sum_{(\mathbf{M},c)} b^{\nu(\mathbf{M},c)},$$

where the sum runs over rooted connected k-constellations with profile $(\lambda, \mu, \mu^1 \dots, \mu^k)$ for some partitions $\mu^1, \dots, \mu^k$ satisfying $\ell(\mu^i) = l_i$.

The second main result of this paper is an analog for the marginal sums $c^{\lambda}_{\mu,l_1,\dots,l_k}$:

Theorem 1.9. *For every $k > 1$, partitions $\lambda, \mu \quad n > 1$, and $l_1, \dots l_k > 1$, there exists a function ϑ_λ on $\mathfrak{F}^{\lambda}_{\mu^0,\dots,\mu^k}$ with non-negative integer values such that $\vartheta_\lambda(\delta_0, \dots, \delta_{k-1})$ is zero if and only if each one of the matchings $\delta_0, \dots, \delta_{k-1}$ is bipartite, and*

$$c^{\lambda}_{\mu,l_1,\dots,l_k} = \sum_{\ell(\mu^i)=l_i} \sum_{(\delta_0,\dots,\delta_{k-1}) \quad \mathfrak{F}^{\lambda}_{\mu,\mu^1\dots,\mu^k}} b^{\vartheta_\lambda(\delta_0,\dots,\delta_{k-1})}.$$

As explained above, the implications between the b-conjecture and the Matching-Jack conjecture are still open problems. The proof that we give here to deduce Theorem 1.9 from Theorem 1.8 cannot be applied in the general case of the conjectures, since it uses a property of symmetry of the statistic ν that appears in Theorem 1.8, see Equation (21). Note that for the other partial results established for these conjectures (the cases $b = 0$, $b = 1$, and polynomiality), we start by proving the result for the Matching-Jack conjecture and then we deduce it for the b-conjecture, this approach is reversed in the current case.

1.7. Lassalle's conjecture. For partitions $\lambda \vdash n \geqslant 1$ and $\mu \vdash m \geqslant 1$ we define $\theta^{(\alpha)}_{\mu}(\lambda)$ as follows:

$$\theta^{(\alpha)}_{\mu}(\lambda) := \begin{cases} 0, & \text{if } n < m. \\ [p_\mu]\, J^{(\alpha)}_{\lambda}, & \text{if } n = m. \\ \theta^{(\alpha)}_{\mu}(\lambda) = \binom{|\lambda|-|\mu|+m_1(\mu)}{m_1(\mu)}\theta^{(\alpha)}_{\mu,1^{n-m}}(\lambda), & \text{if } m < n. \end{cases}$$

where $[p_\mu]$ is the extraction symbol with respect to the variable $\mathbf{p}$ and $m_1(\mu)$ is the number of parts equal to 1 in the partition μ.

It is known that the coefficients $\theta^{(\alpha)}_{\mu}(\lambda)$ are polynomials in α with integer coefficients (see [27] and the discussion of [11, Section 3.6]). Actually, it has been proved that these quantities are also polynomials in the multirectangular coordinates of the partition λ (see [30] for a precise definition). Lassalle has conjectured that these polynomials satisfy some positivity properties:

Conjecture 1.10. *For every partition μ of size m such that $m_1(\mu) = 0$, the quantities $(-1)^{|\mu|} z_\mu \theta^{(\alpha)}_{\mu}(\lambda)$ are polynomials in the parameters $(b, q_1, q_2, \dots, -r_1, -r_2, \dots)$ with non-negative integers coefficients, where $b := \alpha - 1$, and $(q_1, q_2, \dots, r_1, r_2 \dots)$ are the multirectangular coordinates of λ.*

In this paper, we consider the case where the partition λ has a rectangular shape, i.e. a partition with q parts of size r, where $r, q \geqslant 1$. In this case, the multirectangular coordinates of λ are given by (q, r) and we write $\lambda = (q \times r)$, and $\theta^{(\alpha)}_{\mu}(q, r) := \theta^{(\alpha)}_{\mu}(\lambda)$.

Using a recurrence formula for the coefficients $\theta^{(\alpha)}_{\mu}(q, r)$, Lassalle has established in the same paper the positivity in Conjecture 1.10 for the rectangular case but not the integrality. In [13], a combinatorial expression of these coefficients in terms of weighted bipartite maps was given. However the weights considered are not integral.

In Theorem 5.5, we give a complete answer to the rectangular case in Conjecture 1.10 by proving the integrality of the coefficients. We obtain an expression of $(-1)^{|\mu|} z_\mu \theta^{(\alpha)}_{\mu}(q, -r)$ as a sum of bipartite maps with monomial weights in q, $-\alpha r$ and b. The approach we use here is different from the one used in [30] and [13]. It is based on the marginal case in the Matching-Jack conjecture (Theorem 1.9) and Corollary 5.2 that relates the Jack polynomials indexed by rectangular partitions to some specializations of the function $\tau^{(1)}_b$.

1.8. Outline of the paper. In Section 2 we introduce some necessary definitions and notation. In Section 3, we introduce a notion of labelling for non-oriented constellations, in order to construct a correspondence between k-constellations and matchings. Building on that, we prove Theorem 1.4. Section 4 is devoted to the proof of Theorem 1.9. In Section 5, we prove the rectangular case in Conjecture 1.10. In Section 6, we discuss some general properties of the generalized connection coefficients $c^{\lambda}_{\mu^0,\dots,\mu^k}$ and we give a new proof for the positivity of the top degree part of these coefficients.

2. Preliminaries

2.1. Partitions. A *partition* $\lambda = [\lambda_1, \dots, \lambda_\ell]$ is a weakly decreasing sequence of integers $\lambda_1 \geqslant \dots \geqslant \lambda_\ell > 0$. The quantity ℓ is called the *length* of λ and is denoted $\ell(\lambda)$. The size of λ is the integer $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. If n is the *size* of λ, we say that λ is a partition of n and we write $\lambda \vdash n$. The integers $\lambda_1, \dots, \lambda_\ell$ are called the *parts* of λ. For every $i \geqslant 1$, we denote by $m_i(\lambda)$ the number of parts of λ which are equal to i. The partition 2λ is given by $2\lambda := [2\lambda_1, 2\lambda_2, \dots]$.

We denote by $\mathbb{P}$ the set of all partitions, including the empty partition. For every partition λ and $i \geq 1$, we set $\lambda_i = 0$ if $i > \ell(\lambda)$. The dominance partial ordering $\leq$ on $\mathbb{P}$ is given by

$$\mu \leq \lambda \iff |\mu| = |\lambda| \text{ and } \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \text{ for } i \geq 1.$$

We identify a partition λ with its Young diagram defined by

$$\lambda \coloneqq \{(i,j), 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}.$$

Fix a box $\square \coloneqq (i,j) \in \lambda$. Its *arm-length* is given by $a_\lambda(\square) \coloneqq |\{(i,r) \in \lambda, r > j\}|$ and its *leg-length* is given by $\ell_\lambda(\square) \coloneqq |\{(r,j) \in \lambda, r > i\}|$. Two α-deformations of the hook-length product were introduced in [32]:

$$\text{hook}_\lambda^{(\alpha)} \coloneqq \prod_{\square \in \lambda} \left(\alpha a_\lambda(\square) + \ell_\lambda(\square) + 1\right), \quad \text{hook'}_\lambda^{(\alpha)} \coloneqq \prod_{\square \in \lambda} \left(\alpha(a_\lambda(\square) + 1) + \ell_\lambda(\square)\right).$$

With these notations, the classical hook-length product is given by (see e.g. [32])

$$H_\lambda \coloneqq \text{hook}_\lambda^{(1)} = \text{hook'}_\lambda^{(1)}.$$

Finally, we define the α-*content* of a box $\square \coloneqq (i,j)$ by $c_\alpha(\square) \coloneqq \alpha(j-1) - (i-1)$.

2.2. Matchings. We introduce some notation related to matchings as defined in [17]. We recall that for every $n \geq 1$, we set $\mathcal{A}_n \coloneqq \{1, \widehat{1}, \dots, n, \widehat{n}\}$. We also denote by $\mathfrak{F}_n$ the set of matchings on $\mathcal{A}_n$. For $\delta_1, \dots, \delta_r \in \mathfrak{F}_n$ we denote by $G(\delta_1, \dots, \delta_r)$ the multigraph with vertex-set $\mathcal{A}_n$, and edges all the pairs of $\delta_1 \cup \dots \cup \delta_k$. In the case $r = 2$, we note that all connected components of $G(\delta_1, \delta_2)$ are cycles of even size, so we can define $\Lambda(\delta_1, \delta_2)$ as the partition of n obtained by taking half of the size of each connected component of $G(\delta_1, \delta_2)$.

2.3. Symmetric functions and Jack polynomials. For the definitions and notation introduced in this subsection we refer to [31]. We denote by $\mathcal{S}$ the algebra of symmetric functions with coefficients in $\mathbb{Q}$. For every partition λ, we denote m_λ the monomial function, p_λ the power-sum function and s_λ the Schur function associated to λ. If α is an indeterminate, let $\mathcal{S}_\alpha \coloneqq \mathbb{Q}[\alpha] \otimes \mathcal{S}$ denote the algebra of symmetric functions with rational coefficients in α. We recall the following notation introduced in Section 1.4;

$$z_\lambda \coloneqq \prod_{i \geq 1} m_i(\lambda)! i^{m_i(\lambda)}.$$

We denote by $\langle \cdot, \cdot \rangle_\alpha$ the α-deformation of the Hall scalar product defined by

$$\langle p_\lambda, p_\mu \rangle_\alpha = z_\lambda \alpha^{\ell(\lambda)} \delta_{\lambda,\mu}, \text{ for } \lambda, \mu \in \mathbb{P}.$$

Macdonald [31, Chapter VI.10] has proved that there exists a unique family of symmetric functions (J_λ) in $\mathcal{S}_\alpha$ indexed by partitions, satisfying the following properties, called Jack polynomials;

$$\begin{cases} \text{Orthogonality:} & \langle J_\lambda, J_\mu \rangle_\alpha = 0, \text{ for } \lambda \neq \mu, \\ \text{Triangularity:} & [m_\mu] J_\lambda = 0, \text{ unless } \mu \leq \lambda, \\ \text{Normalization:} & [m_{1^n}] J_\lambda = n!, \text{ for } \lambda \vdash n, \end{cases}$$

where $[m_\mu]J_\lambda$ denotes the coefficient of m_μ in J_λ, and 1^n is the partition with n parts equal to 1. For $\alpha = 1$ and $\alpha = 2$, the Jack polynomials are given by

$$J_\lambda^{(1)} = H_\lambda s_\lambda, \quad J_\lambda^{(2)} = Z_\lambda, \tag{8}$$

where Z_λ denotes the zonal polynomial associated to λ, see [31, Chapters VI and VII]. The squared norm of Jack polynomials can be expressed in terms of the deformed hook-length products, (see [32, Theorem 5.8]):

$$j_\lambda^{(\alpha)} := \langle J_\lambda, J_\lambda \rangle_\alpha = \mathrm{hook}_\lambda^{(\alpha)} \, \mathrm{hook'}_\lambda^{(\alpha)} . \tag{9}$$

In particular, we have

$$j_\lambda^{(1)} = H_\lambda^2 \quad \text{and} \quad j_\lambda^{(2)} = H_{2\lambda}. \tag{10}$$

We conclude this subsection with the following theorem (see [31, Equation 10.25]).

Theorem 2.1 ([31]). *For every $\lambda \in \mathbb{P}$, we have*

$$J_\lambda^{(\alpha)}(\underline{u}) = \prod_{\square \in \lambda} (u + c_\alpha(\square)) ,$$

where $\underline{u} := (u, u, \ldots)$, and $J_\lambda^{(\alpha)}$ is expressed in the power-sum basis.

2.4. Maps.

We start by giving the definition of a map (see [29, Definition 1.3.6]).

Definition 2.2. *A* connected map *is a connected graph embedded into a surface such that all the connected components of the complement of the graph are simply connected. These connected components are called the* faces *of the map. We consider maps up to homeomorphisms of the surface (see [29, Definition 1.3.7]). A connected map is* orientable *if it is the case for the underlying surface. A* map *is an unordered collection of connected maps. A map is orientable if each one of its connected components is orientable. We will use the word* non-oriented maps *for maps which are orientable or not.*

Another description of orientable maps is the following: a map is orientable if each one of its faces can be endowed with an orientation such that for every edge e of the map the two edge-sides of e are oriented in opposite ways. In Figure 3 we have an edge e whose sides are incident to two faces F_1 and F_2 (not necessarily distinct), and that are oriented in opposite ways. In this case we say that the orientation of the faces is *consistent*. A pair of edge-sides that appear consecutively while travelling along a face F is called a *corner* of F. An *oriented corner* is a corner endowed with an order on its pair of edge-sides. A corner of a vertex v is a corner whose edge-sides are incident to v. In this case, we say that the corner is incident to v. In this paper, we will consider rooted maps, i.e. maps with a distinguished oriented corner. We call *canonical orientation* of a rooted connected orientable map the unique orientation on the faces of the map which is consistent and such that the face containing the root is oriented by the root corner.

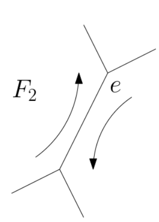


Figure 3. Consistent orientation from either side of an edge e.

2.5. k-Constellations. In this subsection, we introduce the same notation related to constellations as in [8].

We say that a corner of a constellation has color i if it is incident to a vertex of color i. We call a ***right-path*** of a k-constellation $\mathbf{M}$, a path of length k along the boundary of a face of $\mathbf{M}$ that separates a corner of color 0 with a consecutive corner of color k incident to this face. We recall that a connected k-constellation $\mathbf{M}$ is ***rooted*** if it is equipped with a distinguished oriented corner c of color 0. This oriented corner c is called the ***root*** of the constellation. This is equivalent to distinguishing in $\mathbf{M}$, that will be the right-path following the root corner, see Figure 4. We will use the term root to design the root corner or the root right-path. The rooted constellation will be denoted $(\mathbf{M}, c)$. We say that an edge is of color $\{i, i+1\}$ if its end points are of color i and $i+1$. When $k = 1$, 1-constellations are bipartite maps and right-paths are edge-sides.

Since the number of right-paths contained in each face is even, we can define the degree of a face as half the number of its right-paths. Similarly, we define the degree of a vertex as half the number of right-paths that passes by this vertex (we can see that this is the number of edges incident to this vertex if it has color 0 or k, and half the number of edges incident to this vertex if it has color in $\{1, \dots, k-1\}$). We also define the size of a k-constellation $\mathbf{M}$ as half the number of its right-paths, it will be denoted $|\mathbf{M}|$. Therefore, for every k-constellation $\mathbf{M}$ and for every color $0 \leq i \leq k$, we have $|\mathbf{M}| = \sum_v \deg(v)$, where the sum runs over vertices of color i. We also have $|\mathbf{M}| = \sum_f \deg(f)$, where the sum runs over the faces of $\mathbf{M}$. We define the ***face-type*** of a k-constellation as the partition obtained by reordering the degrees of the faces of $\mathbf{M}$. Similarly, for $i \in \{0, \dots, k\}$, the type of the vertices of color i, i.e. the partition obtained by reordering the degrees of the vertices of color i. We define the ***profile*** of a k-constellation $\mathbf{M}$ as the $(k+2)$-tuple of partitions $(\lambda, \mu^0, \dots, \mu^k)$ such that λ is the face-type of $\mathbf{M}$, and for $i \in \{0, \dots, k\}$, μ^i the type of the vertices of color i. If $\mathbf{M}$ is a k-constellation of size n, then $\lambda, \mu^0, \dots, \mu^k \vdash n$.

Example 2.3. The 2-constellation of Figure 4 has size 4 and profile $([2,1,1],[4],[2,1,1],[2,2])$.

Finally, we say that a k-constellation of size n is labelled if it is equipped with a bijection between its right-paths and the set $A_n = \{1, \widehat{1}, \dots, n, \widehat{n}\}$. Labelled k-constellations will be decorated with a check, as in $\check{\mathbf{M}}$.

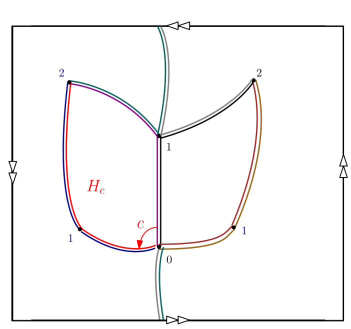


Figure 4. An example of a rooted 2-constellation on the projective plane. Here c is the root corner of the constellation, and H_c is the root right-path. The 8 right-paths are represented in different colors. Note that the two sides of one edge are always in different right-paths.

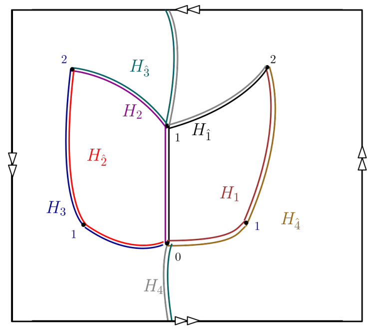


FIGURE 5. An example of a labelling of the 2-constellation illustrated in Figure 4. The right-path labelled by i is denoted H_i.

We now compare the definition of constellations that we use here (given in Definition 1.1) to the usual description of orientable constellations given by hypermaps, (see e.g. [4, 7, 15]). This correspondence between the two descriptions is mentioned in [8] without details.

Link with the usual definition of orientable constellations. We define a *hypermap* as a map with faces colored in two colors such that each edge separates two faces of different colors. The faces of one color are called the *hyperedges* of the hypermap, and the faces of the other color are called the *faces* of the hypermap. The usual definition of orientable constellations is the following:

DEFINITION 2.4 (Usual definition of orientable constellations.). *Let $k > 1$. An orientable k-constellation is an orientable hypermap with vertices colored in the colors $\{0, 1, 2 \dots, k\}$ with the following properties:*

- *The degree of each* hyperedge *is equal to $k + 1$.*
- *The degree of each* face *is a multiple of $k + 1$.*
- *There exits a consistent orientation of the faces such that when we travel along a face in this orientation we read the colors $\{0, 1, 2 \dots, k, 0, \dots\}$.*

A connected orientable k-constellation is rooted if it has a distinguished hyperedge.

Let us prove that this definition is equivalent to the definition of constellations that we use in this paper (see Definition 1.1).

To each connected rooted orientable k-constellation (in the sense of Definition 1.1), we can associate a hypermap as follows: we travel along each face with respect to the canonical orientation (see Section 2.4), and we add an edge between each corner of color 0 and the following corner of color k. In other terms, we close each right-path traversed from its corner of color 0 to its corner of color k by adding an edge of color $(0, k)$, thus forming a face of degree k. Such face will be considered as a *hyperedge* of the hypermap. The other faces of the map will be considered as *faces*. In Figure 6, we have an example of this transformation illustrated on a planar 2-constellation. Let us prove that the map obtained is a hypermap; since the constellation is orientable, the orientations from either side of a given edge e of the constellation are consistent. This implies that one of the two right-paths that contain e is traversed form the corner of color 0 to the corner of color k and the other right-path will be traversed in the opposite way. Only the first right-path will be transformed to a hyperedge. This proves that e separates a *hyperedge* and a *face*. Hence the map obtained is a hypermap that satisfies the properties of Definition 2.4. Moreover, this constellation can be rooted by distinguishing the hyperedge associated to the root right-path. We thus recover the usual definition of orientable constellations. Conversely, if we have a hypermap with the properties of Definition 2.4, we can delete the edges of color $(0, k)$ to obtain a constellation as described in Definition 1.1.

Remark 2.5. Note that the orientability of the constellation is necessary to obtain a hypermap by the transformation described above. The description of orientable constellations with hypermaps has the advantage of being symmetric in the $k + 1$ colors, while in the definition with right-paths the colors 0 and k have a particular role. This lack of symmetry seems inevitable in the case of non-oriented constellations.

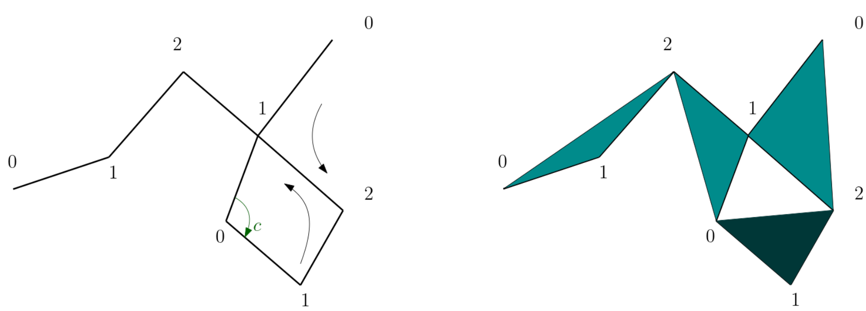


Figure 6. An example of an orientable 2-constellation in the plane, illustrated on the left with the description of Definition 1.1, and on the right with the description of hypermaps. The hyperedges are colored and the root hyperedge is represented with a darker color.

3. The case $b = 1$

3.1. Correspondence between constellations and tuples of matchings.

The purpose of this subsection is to give a bijection between labelled k-constellations of size n and $(k + 2)$-tuples of matchings on $\mathcal{A}_n$. This is a generalization of the construction given in [18] which corresponds to the case $k = 1$.

Definition 3.1. *If $\mathbf{M}$ is a labelled k-constellation of size n, we define $\mathcal{M}(\mathbf{M})$ as the $(k + 2)$-tuple $(\delta_{-1}, \delta_0, \dots, \delta_k)$ of matchings on $\mathcal{A}_n$ defined as follows:*

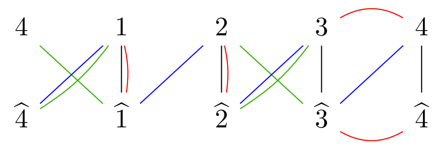
- *δ_{-1} (respectively δ_k) is the matching whose pairs are the labels of right-paths of the same face, that have a corner of color 0 (respectively k) in common.*
- *For $i \in \{0, \dots, k - 1\}$, δ_i is the matching whose pairs are the labels of right-paths having an edge of color $(i, i + 1)$ in common.*

It is easy to see that the profile of a k-constellation $\mathbf{M}$ can be determined by the associated matchings $\mathcal{M}(\mathbf{M})$:

- $\Lambda(\delta_{-1}, \delta_k)$ is the face-type.
- For $i \in \{0, \dots, k\}$, $\Lambda(\delta_{i-1}, \delta_i)$ is the type of vertices of color i.

Given a $(k+2)$-tuple of matchings $(\delta_{-1}, \dots, \delta_k)$, we define its *profile* as the $(k+2)$-tuple of partitions $\Big(\Lambda(\delta_{-1}, \delta_k), \Lambda(\delta_{-1}, \delta_0) \dots, \Lambda(\delta_{k-1}, \delta_k)\Big)$. We get from the previous remark that $\mathbf{M}$ and $\mathcal{M}(\mathbf{M})$ have the same profile.

Example 3.2. The labelled 2-constellation of Figure 5 is associated to the matchings $(\delta_{-1}, \delta_0, \delta_1, \delta_2)$ below, δ_{-1} in red, δ_0 in blue, δ_1 in green and δ_2 in black.



Conversely, if $(\delta_{-1}, \dots, \delta_k)$ is a $(k+2)$-tuple of matchings of A_n we can construct a labelled k-constellation $\vec{\mathbf{M}}$ such that $M(\vec{\mathbf{M}}) := (\delta_{-1}, \dots, \delta_k)$ (this construction is described with more details in [13, Section 3.1] in the case $k = 1$):

- For each connected component C of the graph $G(\delta_{-1}, \delta_k)$ of size $2r$ we consider a polygon consisting of $2r$ right-paths labelled by vertices in C, as follows: two right-paths have a vertex of color 0 in common (respectively of color k) if and only if their labels in $G(\delta_{-1}, \delta_k)$ are connected by δ_{-1} (respectively δ_k).
- For each $0 \leqslant i \leqslant k-1$, and for each edge $e = (j,k)$ of the matching δ_i (where $j, k \in A_n$), we glue the two edge-sides of color $(i, i+1)$ of the right-paths labelled by j and k.

Example 3.3. The construction of the 2-constellation of Figure 5 by gluing polygons with respect to the matchings of Example 3.2 is illustrated in Figure 7.

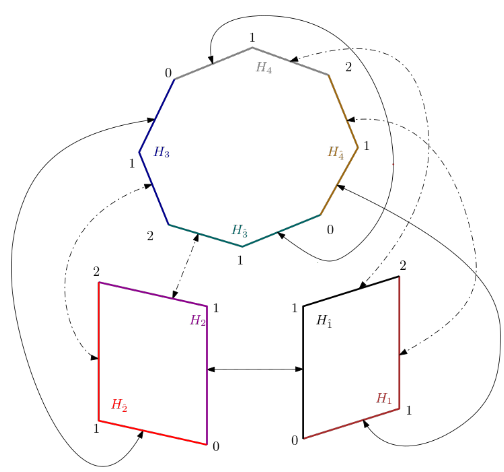


Figure 7. Polygons obtained from matchings δ_{-1} and δ_2 of Example 3.2. Continuous arrows illustrate how to glue edge-sides of color $(0,1)$ with respect to matchings δ_0 and dotted arrows illustrate how to glue edge-sides of color $(1,2)$ with respect to δ_1.

From the definition of a map, we know that the faces of a constellation are isomorphic to open polygons. This implies every map can be obtained by gluing polygons as above (see [29, Construction 1.3.20] for a complete proof in the orientable case). We deduce the following proposition:

Proposition 3.4. *For $\lambda, \mu^0, \dots, \mu^k \vdash n$, the map $\vec{\mathbf{M}} \longmapsto M(\vec{\mathbf{M}})$ is a bijection between labelled k-constellation with profile $(\lambda, \mu^0, \dots, \mu^k)$ and $(k+2)$-tuples of matchings on A_n with the same profile.*

Finally, we use the previous correspondence between constellations and matchings to introduce the notion of duality that will be useful in Section 4.

Definition 3.5. *Let $(\mathbf{M}, c)$ be a rooted k-constellation. We define the dual constellation $(\widetilde{\mathbf{M}}, \widetilde{c})$ as follows. First, we choose a labelling of $\mathbf{M}$ such that the root right-path* *is labelled by 1, we obtain a labelled constellation $\overset{\downarrow}{\mathbf{M}}$. Let $(\delta_{-1}, \delta_0, \ldots, \delta_k) := M(\overset{\downarrow}{\mathbf{M}})$. Then, we define the labelled constellation $\overset{\downarrow}{\mathbf{M}}$ such that $(\delta_{-1}, \delta_k, \ldots, \delta_0) = M(\overset{\downarrow}{\mathbf{M}})$ (i.e. we exchange the matchings $\delta_i \leftrightarrow \delta_{k-i}$ for $0 \leq i \leq k$). Finally we forget the labels of $\overset{\downarrow}{\mathbf{M}}$ except for the label 1. We thus obtain a rooted constellation $(\widetilde{\mathbf{M}}, \widetilde{c})$. It is clear that $(\widetilde{\mathbf{M}}, \widetilde{c})$ does not depend on the labelling chosen for $(\mathbf{M}, c)$.*

One can check that this definition is consistent with the definition of duality given in [8, Definition 2.4].

Remark 3.6. It is straightforward from the definition that duality is an involution that exchanges faces with vertices of color 0, and vertices of color i with vertices of color $k+1-i$, for $1 \leq i \leq k$. More precisely, given partitions $\lambda, \mu^0, \ldots, \mu^k$ duality is a bijection between k-constellations with profile $(\lambda, \mu^0 \ldots, \mu^k)$ and constellations with profile $(\mu^0, \lambda, \mu^k, \ldots, \mu^1)$.

Remark 3.7. It is also possible, using matchings, to generalize this notion of duality in order to exchange colors in all possible ways, while controlling the profile as in the previous remark. However, these generalizations do not have a simple description in terms of maps.

3.2. The Gelfand pair $(\mathfrak{S}_{2n}, \mathfrak{B}_n)$.

In this subsection, we give some results that will be useful in the proof of Theorem 1.4(i). We follow the computations given in [18] when $k = 1$, we recall the most important steps of this proof and give a generalized version for the key lemmas. For this purpose we need to recall some results on the Gelfand pair $(\mathfrak{S}_{2n}, \mathfrak{B}_n)$ (see [31, Section VII.2]). We consider $\mathfrak{S}_{2n}$ as the permutation group of the set $A_n := \{1, \widehat{1}, \ldots, n, \widehat{n}\}$. We define the following action of $\mathfrak{S}_{2n}$ on $\mathfrak{F}_n$, the set of matchings on A_n.

Definition 3.8. *Let $\sigma \in \mathfrak{S}_{2n}$ and $\delta \in \mathfrak{F}_n$. We define $\sigma.\delta$ as the matching of $\mathfrak{F}_n$ such that $\{i, j\}$ is a pair of $\sigma.\delta$ if and only if $\left\{\sigma^{-1}(i), \sigma^{-1}(j)\right\}$ is a pair of δ.*

This action is both transitive and faithful. We define the hyperoctahedral group $\mathfrak{B}_n$ as the stabilizer subgroup of the matching ε. One has that $|\mathfrak{B}_n| = n!2^n$.

Definition 3.9. *Let $\sigma \in \mathfrak{S}_{2n}$. We define the coset-type of σ as the partition of n given by $\Lambda(\varepsilon, \sigma.\varepsilon)$.*

The double cosets $\mathfrak{B}_n \backslash \mathfrak{S}_{2n} / \mathfrak{B}_n$ can be indexed by the partitions of n. In fact, for all $\sigma, \tau \in \mathfrak{S}_{2n}$, one has $\mathfrak{B}_n \sigma \mathfrak{B}_n = \mathfrak{B}_n \tau \mathfrak{B}_n$ if and only if σ and τ have the same coset-type (see [21, Lemma 3.1]). We denote K_λ the class of $\mathfrak{B}_n \backslash \mathfrak{S}_{2n} / \mathfrak{B}_n$ indexed by the partition λ, i.e. the class of permutations of coset-type λ and $K_\lambda \in \mathbb{C}\mathfrak{S}_{2n}$ defined by

$$K_\lambda := \sum_{\sigma \in K_\lambda} \sigma.$$

These sums are the basis of a commutative subalgebra of $\mathbb{C}\mathfrak{S}_{2n}$, the Hecke algebra of the Gelfand pair $(\mathfrak{S}_{2n}, \mathfrak{B}_n)$ (see [31, Section VII.2]). Hence for $\lambda, \mu^0, \ldots, \mu^k \vdash n$, we define $a^\lambda_{\mu^0, \mu^1, \ldots \mu^k}$ such that

$$\prod_{0 \leq i \leq k} K_{\mu^i} = \sum_{\lambda \vdash n} a^\lambda_{\mu^0, \mu^1, \ldots \mu^k} K_\lambda.$$

For $\sigma \in K_\lambda$, the coefficient $a^\lambda_{\mu^0, \ldots, \mu^k}$ can be interpreted as the number of factorizations $\sigma = \sigma_0 \ldots \sigma_k$ where $(\sigma_0, \ldots, \sigma_k) \in K_{\mu^0} \times \cdots \times K_{\mu^k}$. We deduce that

$$a^\lambda_{\mu^0, \mu^1, \ldots \mu^k} |K_\lambda| = \left| \left\{ (\sigma_0, \ldots, \sigma_k) \in K_{\mu^0} \times \cdots \times K_{\mu^k} \text{ such that } \sigma_0 \ldots \sigma_k \in K_\lambda \right\} \right|. \tag{11}$$

For every $\lambda \quad n$, there exist $\frac{n!}{z_\lambda}2^{n-\ell(\lambda)}$ matchings δ such that $\Lambda(\varepsilon,\delta)=\lambda$, see [17, Proposition 5.2]. On the other hand, using the fact that the action of $\mathfrak{S}_{2n}$ on $\mathfrak{F}_n$ is transitive, we can see that

$$|\{\sigma \quad \mathfrak{S}_{2n} \text{ such that } \sigma.\varepsilon=\delta_\lambda\}|=|\mathfrak{B}_n|, \tag{12}$$

where δ_λ is the matching defined in Section 1.3. We deduce that

$$|K_\lambda|=|\mathfrak{B}_n|\frac{n!}{z_\lambda}2^{n-\ell(\lambda)}=\frac{|\mathfrak{B}_n|^2}{z_\lambda 2^{\ell(\lambda)}}. \tag{13}$$

The coefficients $a^\lambda_{\mu^0,\mu^1,\dots\mu^k}$ are related to the size of the sets $\mathfrak{F}^\lambda_{\mu^0,\dots,\mu^k}$, defined in Equation (4). The following lemma is a generalization of [21, Lemma 3.2].

Lemma 3.10. *For* $\lambda,\mu^0,\dots,\mu^k \quad n$, *we have*

$$|\mathfrak{F}^\lambda_{\mu^0,\dots,\mu^k}|=\frac{a^\lambda_{\mu^0,\mu^1,\dots\mu^k}}{|\mathfrak{B}_n|^k}.$$

Proof. We define E as the subset of $\mathfrak{S}_{2n}^{k+1}$, consisting of the elements $(\sigma_0,\dots,\sigma_k)$ of $K_{\mu^0}\times\dots\times K_{\mu^k}$ such that $\sigma_0\dots\sigma_k.\varepsilon=\delta_\lambda$. For every permutation σ_λ such that $\sigma_\lambda.\varepsilon=\delta_\lambda$, one has that $\sigma_\lambda \quad K_\lambda$, and that σ_λ has $a^\lambda_{\mu^0,\dots,\mu^k}$ factorizations of the form $\sigma_\lambda=\sigma_0\dots\sigma_k$ where $(\sigma_0,\dots,\sigma_k) \quad K_{\mu^0}\times\dots\times K_{\mu^k}$. Using Equation (12), we get

$$|E|=|\mathfrak{B}_n|a^\lambda_{\mu^0,\mu^1,\dots\mu^k}. \tag{14}$$

We now consider the map

$$\begin{aligned}\psi: \qquad\qquad E &\longrightarrow \mathfrak{F}_n^k\\ (\sigma_0,\dots,\sigma_k) &\longrightarrow (\sigma_0.\varepsilon,\sigma_0\sigma_1.\varepsilon,\dots,\sigma_0\dots\sigma_{k-1}.\varepsilon).\end{aligned}$$

For all $i \quad \{0,\dots,k\}$, we have

$$\Lambda(\sigma_0\sigma_1\dots\sigma_i.\varepsilon,\sigma_0\dots\sigma_{i-1}.\varepsilon)=\Lambda(\sigma_i.\varepsilon,\varepsilon)=\mu^i,$$

since $\sigma_i \quad K_{\mu^i}$. Hence, $\psi(E) \quad \mathfrak{F}^\lambda_{\mu^0,\dots,\mu^k}$. Let $(\delta_0,\dots,\delta_{k-1}) \quad \mathfrak{F}^\lambda_{\mu^0,\dots,\mu^k}$. There exists $(\sigma_0,\dots,\sigma_k)$ such that $\sigma_0.\varepsilon=\delta_0$, $\sigma_k.\delta_{k-1}=\delta_\lambda$ and for $i \quad \{1,\dots,k-1\}$, $\sigma_i.\delta_{i-1}=\delta_i$. Then $(\sigma_0,\dots,\sigma_k) \quad E$ and $\psi(\sigma_0,\dots,\sigma_k)=(\delta_0,\dots,\delta_{k-1})$, proving that $\psi(E)=\mathfrak{F}^\lambda_{\mu^0,\dots,\mu^k}$. Moreover, $\psi(\sigma_0,\dots,\sigma_k)=\psi(\underline{\sigma}_0,\dots,\underline{\sigma}_k)$ if and only if there exist $\tau_0,\dots,\tau_k \quad \mathfrak{B}_n$ such that for $i \quad \{0,\dots,k\}$

$$\underline{\sigma}_0\dots\underline{\sigma}_i=\sigma_0\sigma_1\dots\sigma_i\tau_i.$$

We deduce that

$$|\psi^{-1}(\psi(\sigma_0,\dots,\sigma_k))|=|\mathfrak{B}_n|^{k+1}. \tag{15}$$

Equations (14) and (15) conclude the proof.

We shall now establish the connection between the coefficients $a^\lambda_{\mu^0,\dots,\mu^k}$ and the Jack polynomials for $\alpha=2$. To this purpose we define for all $\lambda,\xi \quad n$:

$$\phi^\xi(\lambda):=\sum_{\sigma \quad K_\lambda}\chi^{2\xi}(\sigma),$$

where $\chi^{2\xi}$ is the irreducible character of the symmetric group indexed by the partition 2ξ. We also introduce the orthogonal idempotents of the Hecke algebra of $(\mathfrak{S}_{2n},\mathfrak{B}_n)$ that can be defined as follows (see [21, Equation (3.5)]);

$$E_\xi=\frac{1}{H_{2\xi}}\sum_{\lambda \quad n}\frac{1}{|K_\lambda|}\phi^\xi(\lambda)K_\lambda, \tag{16}$$

where $H_{2\xi}$ is the hook-length product associated to the partition 2ξ (see Section 2.1 for the definitions). The orthogonal idempotents satisfy the property $E_\xi E_\eta = \delta_{\xi\eta} E_\xi$ for each $\xi, \eta \vdash n$, where δ is the Kronecker delta. Equation (16) can be inverted as follows (see [21, Eq. (3.3)]):

$$K_\lambda = \sum_{\xi \vdash n} \phi^\xi(\lambda) E_\xi. \tag{17}$$

The following lemma is a generalization of [21, Lemma 3.3].

Lemma 3.11. *For each partitions $\lambda, \mu^0, \dots, \mu^k \vdash n \geq 1$, we have*

$$a^\lambda_{\mu^0,\mu^1,\dots\mu^k} = \frac{1}{|K_\lambda|} \sum_{\xi \vdash n} \frac{1}{H_{2\xi}} \phi^\xi(\lambda)\phi^\xi(\mu^0)\dots\phi^\xi(\mu^k).$$

Proof. Using Equation (17) we can write

$$\prod_{0\leq i\leq k} K_{\mu^i} = \sum_{\xi \vdash n} \phi^\xi(\mu^0)\dots\phi^\xi(\mu^k) E_\xi.$$

We use Equation (16) to extract the coefficient of K_λ from the last equality to obtain $a^\lambda_{\mu^0,\mu^1,\dots\mu^k}$.

When $\alpha = 2$, the Jack polynomials are called zonal polynomials and denoted by Z_ξ, see [31, Chapter VII]. They can be expressed in the basis of power-sum functions as follows; for every $\xi \vdash n$ one has

$$Z_\xi = \frac{1}{|\mathfrak{B}_n|} \sum_{\mu \vdash n} \phi^\xi(\mu) p_\mu. \tag{18}$$

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4(i). For $\alpha = 2$, the function $\tau_1^{(k)}$ has the following expression; see Equations (1) and (10).

$$\tau_1^{(k)}(t, \mathbf{p}, \mathbf{q}^{(0)}, \dots, \mathbf{q}^{(k)}) = 1 + \sum_{n\geq 1} t^n \sum_{\xi \vdash n} \frac{1}{H_{2\xi}} Z_\xi(\mathbf{p}) Z_\xi(\mathbf{q}^{(0)})\dots Z_\xi(\mathbf{q}^{(k)}).$$

Using Equation (18) and Lemma 3.11, this can be rewritten as

$$\begin{aligned}
&\tau_1^{(k)}(t, \mathbf{p},\mathbf{q}^{(0)}, \dots, \mathbf{q}^{(k)}) \\
&\quad = 1 + \sum_{n\geq 1} t^n \sum_{\xi \vdash n} \frac{1}{H_{2\xi}|\mathfrak{B}_n|^{k+2}} \sum_{\lambda,\mu^0,\dots,\mu^k \vdash n} \phi^\xi(\lambda) p_\lambda \phi^\xi(\mu^0) q^{(0)}_{\mu^0} \dots \phi^\xi(\mu^k) q^{(k)}_{\mu^k} \\
&\quad = 1 + \sum_{n\geq 1} t^n \sum_{\lambda,\mu^0,\dots\mu^k \vdash n} a^\lambda_{\mu^0,\mu^1,\dots\mu^k} \frac{|K_\lambda|}{|\mathfrak{B}_n|^{k+2}} p_\lambda q^{(0)}_{\mu^0} \dots q^{(k)}_{\mu^k}.
\end{aligned}$$

Finally, we use Lemma 3.10 and Equation (13) to conclude.

Before deducing Theorem 1.4(ii), we introduce the following notation; if $\mathbf{M}$ is a k-constellation with profile $(\lambda, \mu^0, \dots, \mu^k)$, we define the *marking*[(6)] of $\mathbf{M}$ as the monomial

$$\widetilde{\kappa}(\mathbf{M}) := p_\lambda q^{(0)}_{\mu^0} q^{(1)}_{\mu^1} \dots q^{(k)}_{\mu^k}.$$

We define the marking of a labelled constellation as the marking of the underlying constellation. Theorem 1.4(ii) can be reformulated as follows:

$$\Psi_1^{(k)}(t, \mathbf{p}, \mathbf{q}^{(0)}, \dots, \mathbf{q}^{(k)}) = \sum_{(\mathbf{M},c)} t^{|\mathbf{M}|} \widetilde{\kappa}(\mathbf{M}),$$

(6) What is called marking in [8] will be called marginal marking in this paper, see Section 4.1.

where the sum runs over non-oriented rooted connected constellations.

Proof of Theorem 1.4(ii). Theorem 1.4(i) can be rewritten as follows;

$$\tau_1^{(k)}(t,\mathbf{p},\mathbf{q}^{(0)},\dots,\mathbf{q}^{(k)}) = 1+\sum_{n\geqslant 1}\frac{t^n}{(2n)!}\sum_{\lambda,\mu^0,\dots\mu^k \vdash n}|\mathfrak{F}^{\lambda}_{\mu^0,\dots,\mu^k}|\frac{(2n)!}{n!2^n}2^{n-\ell(\lambda)}\frac{n!}{z_\lambda}p_\lambda q^{(0)}_{\mu^0}\dots q^{(k)}_{\mu^k},$$

On the other hand, the number of $(k+2)$-tuple of matchings $(\delta_{-1},\dots,\delta_k)$ with profile $(\lambda,\mu^0\dots,\mu^k)$ is given by $\frac{(2n)!}{n!2^n}2^{n-\ell(\lambda)}\frac{n!}{z_\lambda}|\mathfrak{F}^{\lambda}_{\mu^0,\dots,\mu^k}|$; we have $\frac{(2n)!}{n!2^n}$ choices for δ_{-1}, $\frac{n!}{z_\lambda}2^{n-\ell(\lambda)}$ choices for δ_k and $|\mathfrak{F}^{\lambda}_{\mu^0,\dots,\mu^k}|$ choices for the other matchings. Using the description of labelled k-constellations with matchings (see Proposition 3.4) we obtain

$$\tau_1^{(k)}(t,\mathbf{p},\mathbf{q}^{(0)},\dots,\mathbf{q}^{(k)}) = 1+\sum_{\mathbb{M}}\frac{t^{|\mathbb{M}|}}{(2|\mathbb{M}|)!}\widetilde{\kappa}(\mathbb{M}),$$

where the sum is taken over labelled k-constellations, connected or not. Since the marking $\widetilde{\kappa}(\mathbf{M})$ is multiplicative on the connected components of $\mathbf{M}$, we can apply the logarithm on the last equality in order to obtain the exponential generating series of connected labelled constellations (we use here the exponential formula for labelled combinatorial classes see e.g. [16, Chapter II]). When we forget all the labels in a connected rooted constellation except for the label "1", we obtain a constellation with a marked right-path that we can consider as a rooted constellation, see Definition 1.1. As each rooted constellation of size n can be labelled in $(2n-1)!$ ways, we obtain

$$\log\left(\tau_1^{(k)}(t,\mathbf{p},\mathbf{q}^{(0)},\dots,\mathbf{q}^{(k)})\right) = \sum_{(\mathbf{M},c)}\frac{t^{|\mathbf{M}|}}{2|\mathbf{M}|}\widetilde{\kappa}(\mathbf{M}),$$

where the sum runs over connected rooted constellations. We conclude the proof by applying $2t\frac{\partial}{\partial t}$ on the last equality.

4. Matching-Jack conjecture for marginal sums

The purpose of this section is to give a proof for Theorem 1.9.

4.1. Notation. We fix $k\geqslant 1$. We consider two sequences of variables $\mathbf{p}=(p_1,p_2,\dots)$, $\mathbf{q}=(q_1,q_2,\dots)$ and k variables $u_1,\dots u_k$. For a variable u we denote $\underline{u}:=(u,u,\dots)$. From the definition of the marginal sums $c^{\lambda}_{\mu,l_1\dots,l_k}$ and $h^{\lambda}_{\mu,l_1\dots,l_k}$ (see Equation (7)), we have

$$\tau_b^{(k)}(t,\mathbf{p},\mathbf{q},\underline{u_1},\underline{u_2},\dots,\underline{u_k}) = 1+\sum_{n\geqslant 1}t^n\sum_{\lambda,\mu\vdash n}\sum_{l_1,\dots,l_k\geqslant 1}\frac{c^{\lambda}_{\mu,l_1\dots,l_k}(b)}{z_\lambda(1+b)^{\ell(\lambda)}}p_\lambda q_\mu u_1^{l_1}\dots u_k^{l_k},$$

$$\Psi_b^{(k)}(t,\mathbf{p},\mathbf{q},\underline{u_1},\underline{u_2},\dots,\underline{u_k}) = \sum_{n\geqslant 1}t^n\sum_{\lambda,\mu\vdash n}\sum_{l_1,\dots,l_k\geqslant 1}h^{\lambda}_{\mu,l_1\dots,l_k}(b)p_\lambda q_\mu u_1^{l_1}\dots u_k^{l_k}.$$

If $\mathbf{M}$ is a k-constellation with profile $(\lambda,\mu^0,\mu^1,\dots,\mu^k)$, we define the *marginal marking* of $\mathbf{M}$ by

$$\kappa(\mathbf{M}):=p_\lambda q_{\mu^0}u_1^{\ell(\mu^1)}\dots u_k^{\ell(\mu^k)},$$

and we say that $\left(\lambda,\mu^0,\ell(\mu^1),\dots,\ell(\mu^k)\right)$ is *the marginal profile* of $\mathbf{M}$. We can formulate Theorem 1.8 as follows.

Theorem 4.1 ([8]). *For every $k \geqslant 1$, we have*

$$\Psi_b^{(k)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u_1}, \underline{u_2}, \dots, \underline{u_k}) = \sum_{(\boldsymbol{M},c)} \kappa(\boldsymbol{M}) t^{|\boldsymbol{M}|} b^{\nu(\boldsymbol{M},c)},$$

where the sum is taken over rooted connected k-constellations and $\nu(\mathbf{M}, \mathbf{c})$ is a non-negative integer which is zero if and only if $(\mathbf{M}, c)$ is orientable.

Definition 4.2. *For a class of vertex-colored maps, we call a b-weight a function ρ that has values in $\mathbb{Q}[b]$ which has the two following properties:*

- *$\rho(\mathbf{M}) = 1$ if and only if $\mathbf{M}$ is orientable.*
- *When we take $b = 1$ we have $\rho(\mathbf{M}) = 1$.*

Moreover, we say that a b-weight ρ is integral if for every map $\mathbf{M}$ one has that $\rho(\mathbf{M})$ is a monomial in b.

With the definition above, the quantity $b^{\nu(\mathbf{M},c)}$ that appears in Theorem 4.1 is an integral b-weight on connected rooted constellations. In Section 4.3, we will consider b-weights on face-labelled constellations.

Remark 4.3. There is not a unique b-weight satisfying Theorem 1.8, see [8, Theorem 5.10]. In particular there exist non integral b-weights with this property. In this section, we fix once and for all an integral b-weight $b^{\nu(\mathbf{M},c)}$.

For every $\lambda, \mu \vdash n$ and $l_1, \dots, l_k \geqslant 1$ we define

$$\mathfrak{F}^{\lambda}_{\mu, l_1, \dots, l_k} := \bigcup_{\mu^i \vdash n, \ell(\mu^i) = l_i} \mathfrak{F}^{\lambda}_{\mu, \mu^1, \dots, \mu^k},$$

where $\mathfrak{F}^{\lambda}_{\mu,\mu^1,\dots,\mu^k}$ is defined in Equation (4). Theorem 1.9 can be reformulated as follows:

Theorem 4.4. *For every $k \geqslant 1$, we have*

$$\tau_b^{(k)}(t, \boldsymbol{p}, \boldsymbol{q}, \underline{u_1}, \dots, \underline{u_k}) = 1 + \sum_{n \geqslant 1} \sum_{\substack{\lambda, \mu \vdash n \\ l_1, \dots, l_k \geqslant 1}} \frac{p_\lambda q_\mu u_1^{l_1} \dots u_k^{l_k}}{z_\lambda (1+b)^{\ell(\lambda)}} \sum_{(\delta_0, \dots, \delta_{k-1}) \in \mathfrak{F}^{\lambda}_{\mu, l_1, \dots, l_k}} b^{\vartheta_\lambda(\delta_0, \dots, \delta_{k-1})},$$

where $\vartheta_\lambda(\delta_0, \dots, \delta_{k-1})$ is a non-negative integer which is zero if and only if each one of the matchings $\delta_0, \dots, \delta_{k-1}$ is bipartite.

The purpose of this section is to use the b-weight of rooted-constellations given in Theorem 4.1 in order to define a statistic ϑ on k-tuples of matchings that satisfies Theorem 4.4. We recall that in Proposition 3.4 we have established a bijection between $(k+2)$-tuples of matchings and labelled k-constellations. The difficulty here is that the sums run over k-tuples of matchings (we recall that in definition of $\mathfrak{F}^{\lambda}_{\mu^0,\dots,\mu^k}$ we fix the matchings δ_{-1} to be ε and the matching δ_k to be δ_λ; see Equation (4)). It turns out that the convenient objects to consider are the *face-labelled constellations*. The purpose of Sections 4.2, 4.3, and 4.4 is to introduce face-labelled constellations and define b-weights on them. In Section 4.6 we will establish a bijection between $\mathfrak{F}^{\lambda}_{\mu^0,\dots,\mu^k}$ and face-labelled constellations.

4.2. Face-labelled constellations.

Face-labelled maps were introduced in [5] in the case of bipartite maps, we give here an analog definition for constellations. We say that a k-constellation $\mathbf{M}$ is *face-labelled* if each face is rooted (with a marked oriented corner of color 0 or equivalently with a marked right-path), and for every $j > 0$, the faces of degree j are labelled i.e. if $\mathbf{M}$ contains $m_j > 0$ faces of degree j, these faces are labelled by $\{1,2,\dots,m_j\}$. Face-labelled constellations will be denoted with a hat: $\widehat{\mathbf{M}}$. In each face, the marked corner or right-path is called the *face-root*. We say that a connected face-labelled k-constellation is *oriented* if the underlying rooted constellation is orientable, and the orientations given by the face roots are consistent, see Figure 3. Finally, we say that a connected face-labelled k-constellation $\widehat{\mathbf{M}}$ is *rooted* if the underlying constellation has a root c such that the orientation of the root face (given by the definition of a face-labelled constellation above) is the same as the orientation induced by the root c. This constellation will be denoted $(\widehat{\mathbf{M}},c)$. Note that the root c of the constellation is not necessarily a face-root.

4.3. b-weights for connected face-labelled constellations.

Once and for all, and for every connected rooted k-constellation $(\mathbf{M},c)$, we fix an orientation $O_{(\mathbf{M},c)}$ of the faces of $\mathbf{M}$ that satisfies the two following properties (see [10, Section 5.1]):

- The orientation of the root face is given by the root c.
- If $\mathbf{M}$ is orientable, then $O_{(\mathbf{M},c)}$ is the canonical orientation of the constellation, see Section 2.4.

Definition 4.5. *Let $(\widehat{\mathbf{M}},c)$ be a connected rooted face-labelled constellation, and let $(\mathbf{M},c)$ be the underlying rooted constellation. We define the b-weight ϑ of $(\widehat{\mathbf{M}},c)$ by*

$$\vartheta(\widehat{\mathbf{M}},c) := \nu(\widetilde{\mathbf{M}},\widetilde{c}) + r,$$

where $(\widetilde{\mathbf{M}},\widetilde{c})$ is the dual constellation of $(\mathbf{M},c)$ as defined in Definition 3.5, $\nu(\widetilde{\mathbf{M}},\widetilde{c})$ is the non-negative integer of Theorem 4.1, and r is the number of faces of $\widehat{\mathbf{M}}$ whose orientation is di erent from the orientation given by $O_{(\mathbf{M},c)}$.

Remark 4.6. We note that $\vartheta(\widehat{\mathbf{M}},c)=0$ if and only if $\widehat{\mathbf{M}}$ is oriented. Moreover, for every connected rooted constellation $\mathbf{M}$ with face-type λ, we have

$$\sum_{(\widehat{\mathbf{M}},c)} b^{\vartheta(\widehat{\mathbf{M}},c)} = z_\lambda(1+b)^{\ell(\lambda)-1} b^{\nu(\widetilde{\mathbf{M}},\widetilde{c})}, \tag{19}$$

where the sum is taken over all possible face-labellings of $(\mathbf{M},c)$. Indeed, we have z_λ choices to label the faces of $(\mathbf{M},c)$ which have the same size and choose a corner of color 0 (which is not yet oriented) in each face. Besides, for each face other than the root face, we have to choose an orientation for the root corner (the orientation in the root face being fixed by the root c). The orientation consistent with $O_{(\mathbf{M},c)}$ contributes 1 to the b-weight and the other orientation contributes b, which gives us $1+b$ for each face different from the root face.

We now define b-weights for unrooted connected face-labelled constellations. These b-weights are given by different ways to root a face-labelled constellation.

Definition 4.7. *Let λ be a partition of n and let $\widehat{\mathbf{M}}$ be a connected face-labelled k-constellation of face-type λ. We define three b-weights on $\widehat{\mathbf{M}}$:*

(1) *We root $\widehat{\mathbf{M}}$ with c_0, the root of the face of maximal degree and which is labelled by 1. We define:*

$$\vec{\rho}(\widehat{\mathbf{M}}) := b^{\vartheta(\widehat{\mathbf{M}},c_0)}.$$

(2) *We take the average over all possible roots c that lie in a face of maximal degree (we recall that the orientation of this root should be consistent with the orientation given by the face-root):*

$$\widehat{\rho}(\widehat{\mathbf{M}}) \coloneqq \frac{1}{m\lambda_1} \sum_{c, \deg(f_c) = \lambda_1} b^{\vartheta(\widehat{\mathbf{M}},c)},$$

where $m \coloneqq m_{\lambda_1}(\lambda)$ is the number of faces of maximal degree.

(3) *We take the average over all possible roots c:*

$$\widetilde{\rho}(\widehat{\mathbf{M}}) \coloneqq \frac{1}{n} \sum_{c} b^{\vartheta(\widehat{\mathbf{M}},c)}.$$

Note that the b-weight $\vec{\rho}$ has the advantage of being integral, however it is a priori less symmetric than $\widetilde{\rho}$. The purpose of the next subsection is to show that the b-weights $\vec{\rho}$, $\widehat{\rho}$ and $\widetilde{\rho}$ are equivalent when we sum over connected face-labelled k-constellations of a given marginal profile $(\lambda, \mu, l_1, \dots, l_k)$.

4.4. Equivalence between the three b-weights. We start by the equivalence between $\vec{\rho}$ and $\widehat{\rho}$.

Lemma 4.8. *For every $k, n > 1$ and $\lambda, \mu^0, \dots, \mu^k \quad n$ we have*

$$\sum_{\widehat{\mathbf{M}}} \vec{\rho}(\widehat{\mathbf{M}}) = \sum_{\widehat{\mathbf{M}}} \widehat{\rho}(\widehat{\mathbf{M}}), \tag{20}$$

where the sums are taken over connected face-labelled k-constellation with profile $(\lambda, \mu^0, \dots, \mu^k)$.

Proof. We denote $m \coloneqq m_{\lambda_1}(\lambda)$, the number of parts in λ of maximal size. From Definitions 4.5 and 4.7, we know that $\vec{\rho}(\widehat{\mathbf{M}})$ is of the form $b^r b^{\nu(\widetilde{\mathbf{M}},\widetilde{c})}$. We rewrite the left-hand side of Equation (20) by putting together the terms having the same underlying rooted connected constellation $(\mathbf{M}, c)$. With the same argument as in the proof of Equation (19), for every rooted constellation $(\mathbf{M}, c)$ with a root c in a face of maximal degree we have

$$\sum_{\widehat{\mathbf{M}}} \vec{\rho}(\widehat{\mathbf{M}}) = (1+b)^{\ell(\lambda)-1} \frac{z_\lambda}{m\lambda_1} b^{\nu(\widetilde{\mathbf{M}},\widetilde{c})},$$

where the sum is taken over face-labelled constellations that can be obtained from $(\mathbf{M}, c)$ by labelling its faces, with the condition that the root face is always labelled by 1 and rooted by c (see Definition 4.7 item (1)). We deduce that the left-hand side of Equation (20) side is equal to

$$(1+b)^{\ell(\lambda)-1} \frac{z_\lambda}{m\lambda_1} \sum_{(\mathbf{M},c)} b^{\nu(\widetilde{\mathbf{M}},\widetilde{c})},$$

where the sum is taken over rooted connected k-constellations with profile $(\lambda, \mu^0, \dots, \mu^k)$ such that the root face has maximal degree λ_1.

On the other hand, we can rewrite the right-hand side of Equation (20) (using Definition 4.7 item (2)) as follows

$$\sum_{(\widehat{\mathbf{M}},c)} \frac{1}{m\lambda_1} b^{\vartheta(\widehat{\mathbf{M}},c)},$$

where the sum is taken over face-labelled rooted constellation, for which the root is in a face of maximal degree.

We use Equation (19) to conclude.

The link between the two b-weights $\widehat{\rho}$ and $\widetilde{\rho}$ is less obvious. We need a property of symmetry of the b-weight defined in [8] on rooted connected constellations. We start by defining for every $s \geq 1$ the series

$$U_s := (1+b)s\frac{\partial}{\partial q_s}\log(\tau_b^{(k)}) \quad , \quad V_s := (1+b)s\frac{\partial}{\partial p_s}\log(\tau_b^{(k)}).$$

We also define the operator π that switches the variables $\mathbf{p} \leftrightarrow \mathbf{q}$ and $u_i \leftrightarrow u_{k+1-i}$ for $1 \leq i \leq k$. Since $\pi\tau_b^{(k)} = \tau_b^{(k)}$, we get $\pi U_s = V_s$. On the other hand, one has (see [8, Corollary 5.9])

$$U_s = q_s^{-1} \sum_{\substack{(\mathbf{M},c) \\ \deg(v_c)=s}} t^{|\mathbf{M}|}\kappa(\mathbf{M})b^{\nu(\mathbf{M},c)}, \tag{21}$$

where the sum is taken over rooted connected k-constellation whose root vertex has degree s. Moreover, it is straightforward from Remark 3.6 that for every k-constellation $\mathbf{M}$ we have

$$\pi\big(\kappa(\mathbf{M})\big) = \kappa(\widetilde{\mathbf{M}}), \tag{22}$$

where $\widetilde{\mathbf{M}}$ denotes the dual constellation of $\mathbf{M}$. Applying π to Equation (21), we get

$$V_s = p_s^{-1} \sum_{\substack{(\mathbf{M},c) \\ \deg(f_c)=s}} t^{|\mathbf{M}|}\kappa(\mathbf{M})b^{\nu(\widetilde{\mathbf{M}},\widetilde{c})}. \tag{23}$$

We deduce the following lemma.

Lemma 4.9. *Let $\lambda, \mu \vdash n$ and $l_1, \dots l_k \geq 1$, and let $s \geq 1$ such that $m := m_s(\lambda) \geq 1$. Then*

$$\frac{1}{n}\sum_{(\mathbf{M},c)} b^{\nu(\widetilde{\mathbf{M}},\widetilde{c})} = \frac{1}{ms}\sum_{\substack{(\mathbf{M},c) \\ \deg(f_c)=s}} b^{\nu(\widetilde{\mathbf{M}},\widetilde{c})},$$

where the sums are taken over connected rooted k-constellations of marginal profile $(\lambda, \mu, l_1, \dots, l_k)$, with the condition that the root face has degree s in the sum of the right-hand side.

Proof. From Theorem 4.1 we have

$$(1+b)\log(\tau_b^{(k)}) = \sum_{(\mathbf{M},c)} \frac{t^{|\mathbf{M}|}}{|\mathbf{M}|}\kappa(\mathbf{M})b^{\nu(\mathbf{M},c)}.$$

Applying π on the last equality, we get

$$(1+b)\log(\tau_b^{(k)}) = \sum_{(\mathbf{M},c)} \frac{t^{|\mathbf{M}|}}{|\mathbf{M}|}\kappa(\mathbf{M})b^{\nu(\widetilde{\mathbf{M}},\widetilde{c})}.$$

We deduce then that the coefficient of the monomial $t^n p_\lambda q_{\mu^0} u_1^{l_1} \dots u_k^{l_k}$ in $p_s V_s$, is given by

$$\frac{ms}{n}\sum_{(\mathbf{M},c)} b^{\nu(\widetilde{\mathbf{M}},\widetilde{c})},$$

where the sum is taken over connected rooted k-constellations of marginal profile $(\lambda, \mu, l_1, \dots, l_k)$. On the other hand, using Equation (23) we get that this coefficient is also equal to

$$\sum_{\substack{(\mathbf{M},c) \\ \deg(f_c)=s}} b^{\nu(\widetilde{\mathbf{M}},\widetilde{c})},$$

where the sum is taken over connected rooted k-constellations of marginal profile $(\lambda, \mu, l_1, \ldots, l_k)$ with the condition that the root face has degree s, which finishes the proof.

This lemma has the following interpretation: conditioning the root to be in a face of a given degree does not affect the b-weight obtained when summing over constellations of a given marginal profile. We deduce the following corollary that establishes the equivalence claimed between $\widehat{\rho}$ and $\widetilde{\rho}$:

Corollary 4.10. *Let $\lambda, \mu \quad n$, and $l_1, \ldots, l_k > 1$. Then we have*

$$\sum_{\widehat{\mathbf{M}}} \widehat{\rho}(\widehat{\mathbf{M}}) = \sum_{\widehat{\mathbf{M}}} \widetilde{\rho}(\widehat{\mathbf{M}}),$$

where the sums run over connected face-labelled k-constellation of marginal profile $(\lambda, \mu, l_1, \ldots, l_k)$.

Proof. We apply Lemma 4.9 for $s = \lambda_1$ and multiply both sides of the equation by $z_\lambda (1+b)^{\ell(\lambda)-1}$. Using Equation (19) we obtain:

$$\frac{1}{n} \sum_{(\widehat{\mathbf{M}},c)} b^{\vartheta(\widehat{\mathbf{M}},c)} = \frac{1}{m\lambda_1} \sum_{\substack{(\widehat{\mathbf{M}},c) \\ \deg(f_c) = \lambda_1}} b^{\vartheta(\widehat{\mathbf{M}},c)},$$

where $m \coloneqq m_{\lambda_1}(\lambda)$, which finishes the proof.

4.5. Extension to disconnected face-labelled constellations. We extend multiplicatively the b-weight $\vec{\rho}$ to disconnected constellations. More precisely, if $\widehat{\mathbf{M}}$ is a disconnected face-labelled constellation and $\widehat{\mathbf{M}}_i$ is a connected component of $\widehat{\mathbf{M}}$, then it can be considered as a face-labelled constellation where the labelling of the faces having the same degree in $\widehat{\mathbf{M}}_i$ is inherited from their labelling in $\widehat{\mathbf{M}}$. This allow us to define $\vec{\rho}(\widehat{\mathbf{M}})$ as the product over all its connected components of $\vec{\rho}(\widehat{\mathbf{M}}_i)$, where $\vec{\rho}(\widehat{\mathbf{M}}_i)$ is given by Definition 4.7 item (1).

Remark 4.11. By definition $\widehat{\mathbf{M}}$ is oriented if and only if each one of its connected components is oriented. Hence, we can deduce from Remark 4.6 that $\vec{\rho}(\widehat{\mathbf{M}})$ is a monomial, and it equals 1 if and only if $\widehat{\mathbf{M}}$ is oriented. Hence $\vec{\rho}$ is an integral b-weight on face-labelled constellations.

The following lemma establishes the connection between the generating functions of connected and disconnected constellations. It is a variant of the exponential formula in the combinatorial class theory. However, one has to take care of the multiplicities since we have a separate labelling for each size of faces. We give here the proof in completeness.

Lemma 4.12. *For every $k > 1$, we have*

$$1 + \sum_{n>1} t^n \sum_{\lambda,\mu \quad n} \sum_{l_1,\ldots,l_k>1} \frac{p_\lambda q_\mu u_1^{l_1} \ldots u_k^{l_k}}{z_\lambda (1+b)^{\ell(\lambda)}} \sum_{\widehat{\mathbf{M}}} \vec{\rho}(\widehat{\mathbf{M}})$$
$$= \exp\Big(\sum_{n>1} t^n \sum_{\lambda,\mu \quad n} \sum_{l_1,\ldots,l_k>1} \frac{p_\lambda q_\mu u_1^{l_1} \ldots u_k^{l_k}}{z_\lambda (1+b)^{\ell(\lambda)}} \sum_{\widehat{\mathbf{M}} \text{ connected}} \vec{\rho}(\widehat{\mathbf{M}}) \Big),$$

where the $\widehat{M}$ in the final sums range over face-labelled constellations of marginal profile $(\lambda, \mu, l_1, \ldots, l_k)$.

Proof. When we develop the exponential of the right-hand side, we obtain a sum over tuples of connected face-labelled constellations. Let $\widehat{\mathbf{M}}_1,\dots,\widehat{\mathbf{M}}_r$ be a list of r connected face-labelled constellations, with face-types $\lambda^{(1)},\dots,\lambda^{(r)}$. We define $\lambda := \bigcup_{i=1}^{r} \lambda^{(i)}$. Taking the disjoint union of the constellations $\widehat{\mathbf{M}}_1,\dots,\widehat{\mathbf{M}}_r$, we obtain a constellation of face-type λ. In such operations, we deal with the labellings as in the theory of labelled combinatorial classes [16, Chapter II]; namely for every j such that $m_j(\lambda) > 0$, we consider all the ways to relabel the faces of degree j of $\widehat{\mathbf{M}}_1,\dots,\widehat{\mathbf{M}}_r$ in an increasing way such that their label sets become disjoint and the union of their label sets is $[\![m_j(\lambda)]\!]$. So we have $\binom{m_j(\lambda)}{m_j(\lambda^1),\dots,m_j(\lambda^r)}$ choices to relabel the faces of degree j, and

$$\frac{z_\lambda}{z_{\lambda^1}\dots z_{\lambda^r}} = \prod_j \binom{m_j}{m_j(\lambda^1),,\dots,m_j(\lambda^r)}$$

choices to relabel all the faces of $\bigcup_{i=1}^{r} \widehat{\mathbf{M}}_i$ to obtain a face-labelled constellation $\widehat{\mathbf{M}}$. Finally, we notice that the marking and the quantity $(1+b)^{\ell(\lambda)}$ are multiplicative which concludes the proof.

4.6. Face-labelled constellations and Matchings. Let $\widehat{\mathbf{M}}$ be a face-labelled constellation of face-type λ. We describe a canonical way to obtain a labelled constellation $\vec{\mathbf{M}}$ from $\widehat{\mathbf{M}}$ that will be useful in the next proposition. We start by defining the following order on $\mathcal{A}_n$: $\widehat{1} < 1 < 2 \dots < \widehat{n} < n$. We label the right-paths starting from faces of highest degree and smallest label: We start from the face of degree λ_1 and label 1. We travel along this face starting from the right-path preceding the root corner, and we attribute to each right-path the smallest label not yet used. We restart with the face of highest degree and smallest label whose right-paths are not yet labelled. This will be called the *canonical labelling* of the face-labelled constellation $\widehat{\mathbf{M}}$. Note that for every face-labelled constellation $\widehat{\mathbf{M}}$ of face-type λ, the matchings δ_{-1} and δ_k associated to the canonical labelling $\vec{\mathbf{M}}$ (as in Definition 3.1) satisfy $\delta_{-1} = \varepsilon$ and $\delta_k = \delta_\lambda$, where δ_λ is the matching defined in Equation (3).

We now prove the following proposition that establishes a correspondence between face-labelled k-constellations and k-tuples of matchings.

Theorem 4.13. *For $\lambda, \mu^0, \dots, \mu^k \vdash n$, there exists a bijection $\widehat{\mathcal{M}}$ between face-labelled k-constellations with profile $(\lambda,\mu^0,\dots,\mu^k)$ and $\mathfrak{F}^{\lambda}_{\mu^0,\dots,\mu^k}$. Moreover, a face-labelled constellation $\widehat{\mathbf{M}}$ is oriented if and only if $\widehat{\mathcal{M}}(\widehat{\mathbf{M}})$ is a k-tuple of bipartite matchings.*

Proof. Let $\widehat{\mathbf{M}}$ be a face-labelled constellation, and let $\vec{\mathbf{M}}$ be its canonical labelling. By construction of the canonical labelling, the matchings associated to $\vec{\mathbf{M}}$ by the bijection of Proposition 3.4 is of the form $\mathcal{M}(\vec{\mathbf{M}}) = (\varepsilon,\delta_0,\dots,\delta_{k-1},\delta_\lambda)$. Then we define $\widehat{\mathcal{M}}(\widehat{\mathbf{M}}) := (\delta_0,\dots,\delta_{k-1})$.

Conversely, let $(\delta_0,\dots,\delta_{k-1}) \in \mathfrak{F}^{\lambda}_{\mu^0,\dots,\mu^k}$. The bijection of Proposition 3.4 gives us a labelled k-constellation $\vec{\mathbf{M}} := \mathcal{M}^{-1}(\varepsilon,\delta_0,\dots,\delta_{k-1},\delta_\lambda)$. Then $\widehat{\mathcal{M}}^{-1}(\delta_0,\dots,\delta_{k-1})$ is the face-labelled constellation having $\vec{\mathbf{M}}$ as a canonical labelling.

We now prove the second part of the proposition. Let $\widehat{\mathbf{M}}$ be a face-labelled constellation. We start by the following remark: when we travel along the boundary of each face of $\widehat{\mathbf{M}}$ in the orientation induced by its root, a right-path H in $\widehat{\mathbf{M}}$ is traversed from the corner of color 0 to the corner of color k (respectively from the corner of color k to the corner of color 0) if it has a label in the first class of $\mathcal{A}_n$ (respectively in second class) with respect to the canonical labelling of $\widehat{\mathbf{M}}$. Indeed, this property is clear for the root right-path of each face. Moreover, this can be extended to the other right-paths since when we travel along a face we alternate right-paths traversed form 0 to k and right-paths traversed from k to 0, and when we traverse a connected component of $G(\varepsilon, \delta_\lambda)$ we alternate labels of first and second class.

We recall that $\widehat{\mathbf{M}}$ is oriented if and only if the orientations induced by the faces roots are consistent as in Figure 3. Note that the orientations of the faces of $\widehat{\mathbf{M}}$ are consistent from either side of edges of color $(i, i+1)$ if and only if each two right-paths having an edge of color $(i, i+1)$ in common are traversed in opposite ways. By the previous remark, this is equivalent to saying that δ_i is bipartite. In particular $\widehat{\mathbf{M}}$ is oriented if and only if each one of the matchings δ_0,…,δ_{k-1} is bipartite.

4.7. A statistic ϑ for elements of $\mathfrak{F}^{\lambda}_{\mu,l_1,\dots,l_k}$ and proof of Theorem 1.9.

Definition 4.14. *Let* $\lambda, \mu \quad n$ *and* $l_1, \dots, l_k > 1$*. For each* $(\delta_1, \dots, \delta_k) \quad \mathfrak{F}^{\lambda}_{\mu,l_1,\dots,l_k}$*, we define the non-negative integer* $\vartheta_\lambda(\delta_0, \dots, \delta_{k-1})$ *such that*

$$\vec{\rho}(\widehat{\mathbf{M}}) = b^{\vartheta_\lambda(\delta_0,\dots,\delta_{k-1})},$$

where $\widehat{\mathbf{M}}$ *is the face-labelled constellation associated to* $(\delta_1, \dots, \delta_k)$ *by the bijection of Theorem* 4.13.

Since the bijection of Theorem 4.13 ensures that $\widehat{\mathbf{M}}$ is oriented if and only if the matchings $\delta_0, \dots, \delta_{k-1}$ are bipartite, we note that $\vartheta_\lambda(\delta_0, \dots, \delta_{k-1})$ is equal to zero if and only if each one of the matchings δ_1,…,δ_k is bipartite.

Proof of Theorem 4.4. From Theorem 4.1 we have

$$\Psi_b^{(k)}(t, \mathbf{p}, \mathbf{q}, \underline{u_1}, \dots, \underline{u_k}) = t\frac{\partial}{\partial t} \sum_{n>1} \frac{t^n}{n} \sum_{(\mathbf{M},c)} \kappa(\mathbf{M}) b^{\nu(\mathbf{M},c)},$$

where the second sum runs over connected rooted constellations of size n. Applying the operator π on the last equation and using Equation (22) we get

$$\Psi_b^{(k)}(t, \mathbf{p}, \mathbf{q}, \underline{u_1}, \dots, \underline{u_k}) = t\frac{\partial}{\partial t} \sum_{n>1} \frac{t^n}{n} \sum_{(\mathbf{M},c)} \kappa(\mathbf{M}) b^{\nu(\widetilde{\mathbf{M}},c)}.$$

Using Equation (19) and Definition 4.7(3), we obtain

$$\begin{aligned}\Psi_b^{(k)}(t, \mathbf{p}, \mathbf{q}, \underline{u_1}, \dots, \underline{u_k}) &= t\frac{\partial}{\partial t} \sum_{n>1} \frac{t^n}{n} \sum_{\lambda,\mu \quad n} \sum_{l_1,\dots,l_k>1} \frac{p_\lambda q_\mu u_1^{l_1} \dots u_k^{l_k}}{z_\lambda (1+b)^{\ell(\lambda)-1}} \sum_{(\widehat{\mathbf{M}},c)} b^{\vartheta(\widehat{\mathbf{M}},c)} \\ &= t\frac{\partial}{\partial t} \sum_{n>1} t^n \sum_{\lambda,\mu \quad n} \sum_{l_1,\dots,l_k>1} \frac{p_\lambda q_\mu u_1^{l_1} \dots u_k^{l_k}}{z_\lambda (1+b)^{\ell(\lambda)-1}} \sum_{\widehat{\mathbf{M}}} \widetilde{\rho}(\widehat{\mathbf{M}}),\end{aligned}$$

where the last sums are taken over connected face labelled constellations of marginal profile $(\lambda, \mu, l_1, \dots, l_k)$, rooted in the first equation and unrooted in the second one. This can be rewritten using Lemma 4.8 and Corollary 4.10 as follows

$$\begin{aligned}&\Psi_b^{(k)}(t, \mathbf{p}, \mathbf{q}, \underline{u_1}, \dots, \underline{u_k}) \\ &\quad = t\frac{\partial}{\partial t}(1+b) \sum_{n>1} t^n \sum_{\lambda,\mu \quad n} \sum_{l_1,\dots,l_k>1} \frac{p_\lambda q_\mu u_1^{l_1} \dots u_k^{l_k}}{z_\lambda (1+b)^{\ell(\lambda)}} \sum_{\widehat{\mathbf{M}} \text{ connected}} \vec{\rho}(\widehat{\mathbf{M}}).\end{aligned}$$

Applying Lemma 4.12, we obtain

$$\Psi_b^{(k)}(t,\mathbf{p},\mathbf{q},\underline{u_1},\dots,\underline{u_k}) = t\frac{\partial}{\partial t}(1+b)\log\Big(1+\sum_{n\geq 1}t^n\sum_{\lambda,\mu\vdash n}\sum_{l_1,\dots,l_k\geq 1}\frac{p_\lambda q_\mu u_1^{l_1}\dots u_k^{l_k}}{z_\lambda(1+b)^{\ell(\lambda)}}\sum_{\widehat{\mathbf{M}}}\vec{\rho}(\widehat{\mathbf{M}})\Big),$$

where the last sum runs over face-labelled constellations of marginal profile $(\lambda,\mu,l_1,\dots l_k)$, connected or not. Comparing the last equation with Equation (2), we deduce that

$$\tau_b^{(k)}(t,\mathbf{p},\mathbf{q},\underline{u_1},\underline{u_2},\dots,\underline{u_k}) = 1+\sum_{n\geq 1}t^n\sum_{\lambda,\mu\vdash n}\sum_{l_1,\dots,l_k\geq 1}\frac{p_\lambda q_\mu u_1^{l_1}\dots u_k^{l_k}}{z_\lambda(1+b)^{\ell(\lambda)}}\sum_{\widehat{\mathbf{M}}}\vec{\rho}(\widehat{\mathbf{M}}).$$

Using the bijection of Theorem 4.13 and Definition 4.14, the last equation can be rewritten as follows:

$$\tau_b^{(k)}(t,\mathbf{p},\mathbf{q},\underline{u_1},\dots,\underline{u_k}) = 1+\sum_{n\geq 1}\sum_{\substack{\lambda,\mu\vdash n\\ l_1,\dots,l_k\geq 1}}\frac{p_\lambda q_\mu u_1^{l_1}\dots u_k^{l_k}}{z_\lambda(1+b)^{\ell(\lambda)}}\sum_{(\delta_0,\dots,\delta_{k-1})\in\mathfrak{F}^{\lambda}_{\mu,l_1,\dots,l_k}} b^{\vartheta_\lambda(\delta_0,\dots,\delta_{k-1})}.$$

Since ϑ_λ has the properties required in Theorem 4.4, this concludes the proof.

5. Application: Lassalle's conjecture for rectangular partitions

In this section, we take $k=1$, and we consider the function $\tau_b^{(1)}$ with the following specializations: $\mathbf{q}^{(0)}=\underline{q}:=(q,q,\dots)$ and $\mathbf{q}^{(1)}=\underline{-r\alpha}:=(-r\alpha,-r\alpha,\dots)$. We also replace t by $-t$. We recall that for $k=1$, k-constellations correspond to bipartite maps; these are maps with vertices colored in white and black such that each edge separates two vertices of different colors. For every face-labelled bipartite map $\mathbf{M}$, we denote by $\mathrm{w}^{(\alpha)}(\widehat{\mathbf{M}},q,r)$ the quantity:

$$\mathrm{w}^{(\alpha)}(\widehat{\mathbf{M}},q,r):=(-1)^{|\mathbf{M}|}(-r\alpha)^{|V_\circ(\widehat{\mathbf{M}})|}q^{|V_\bullet(\widehat{\mathbf{M}})|},$$

where $|V_\circ(\widehat{\mathbf{M}})|$ (respectively $|V_\bullet(\widehat{\mathbf{M}})|$) denotes the number of white (respectively black) vertices of $\widehat{\mathbf{M}}$. The generating series of bipartite maps can be written as follows (see Theorem 4.4):

$$\tau_b^{(1)}(-t,\mathbf{p},\underline{q},\underline{-r\alpha})=\sum_{\mu\in\mathbb{P}}\frac{p_\mu t^{|\mu|}}{z_\mu\alpha^{\ell(\mu)}}\sum_{\widehat{\mathbf{M}}}\vec{\rho}(\widehat{\mathbf{M}})\,\mathrm{w}^{(\alpha)}(\widehat{\mathbf{M}},q,r), \tag{24}$$

where the second sum runs over face-labelled bipartite maps of face-type μ.

The key step of the proof is Corollary 5.2, that gives an expression of Jack polynomials associated to partitions with rectangular shape in terms of the function $\tau_b^{(1)}$. We state here a more general result related to Jack polynomials indexed by multirectangular partitions. Unfortunately, we were not able to use it to obtain polynomiality information on coefficients $\theta_\mu^{(\alpha)}(\lambda)$ for multirectangular partitions λ.

Lemma 5.1. *Fix $s\geq 1$, and two sequences of positive integers $1=q_1<q_2<\dots<q_s$ and $r_1>\dots>r_s=1$. We consider the boxes $\square_i:=(q_i+1,r_i+1)$ for $1\leq i\leq s$, and the respective opposite of their α-contents $u_i:=-c_\alpha(\square_i)$. Let λ be the partition* *of maximal size that does not contain any of the boxes* $\square_i$*, and let* n *be its size. Then, we have*

$$J_\lambda^{(\alpha)} = \frac{[t^n]\tau_b^{(s-1)}\left(t, \mathbf{p}, \underline{u_1}, \dots, \underline{u_s}\right)}{[t^n]\tau_b^{(s-2)}\left(t, \underline{u_1}, \dots, \underline{u_s}\right)}, \tag{25}$$

where $[.]$ *denotes the extraction symbol with respect to the variable* t*.*

Proof. Recall that

$$[t^n]\tau_b^{(s-1)}(-t, \mathbf{p}, \underline{u_1}, \dots, \underline{u_s}) = (-1)^n \sum_{\xi \vdash n} \frac{J_\xi^{(\alpha)}(\mathbf{p}) J_\xi^{(\alpha)}(\underline{u_1}) \dots J_\xi^{(\alpha)}(\underline{u_s})}{j_\xi^{(\alpha)}}. \tag{26}$$

Let $\square_0$ be fixed box, and let $u \coloneqq -c_\alpha(\square_0)$ be the opposite of its α-content, see Section 2.1. Using Theorem 2.1, we can see that $J_\xi(\underline{u}) = 0$ if and only if $\square_0 \in \xi$. In particular, the partitions that contribute to the sum of Equation (26) are the partitions that do not contain anyone of the boxes $\square_i$, for $1 \leqslant i \leqslant s$. By definition, the only partition of size n that fulfills this condition is the partition λ. Hence

$$[t^n]\tau_b^{(s-1)}(-t, \mathbf{p}, \underline{u_1}, \dots, \underline{u_s}) = (-1)^n \frac{J_\lambda^{(\alpha)}(\mathbf{p}) J_\lambda^{(\alpha)}(\underline{u_1}) \dots J_\lambda^{(\alpha)}(\underline{u_s})}{j_\lambda^{(\alpha)}}.$$

Similarly, we have

$$[t^n]\tau_b^{(s-2)}(-t, \underline{u_1}, \dots, \underline{u_s}) = (-1)^n \frac{J_\lambda^{(\alpha)}(\underline{u_1}) \dots J_\lambda^{(\alpha)}(\underline{u_s})}{j_\lambda^{(\alpha)}}.$$

This concludes the proof of the lemma.

In the case of rectangular partitions, Equation (25) has a simpler expression.

Corollary 5.2. *For every partition* $\lambda = (q \times r) \vdash n$*, we have*

$$J_\lambda^{(\alpha)} = [t^n]\tau_b^{(1)}(-t, \mathbf{p}, \underline{q}, \underline{-r\alpha}), \tag{27}$$

where $[.]$ *denotes the extraction symbol with respect to the variable* t*.*

Proof. It is enough to prove that

$$(-1)^n \frac{J_\lambda^{(\alpha)}(\underline{q}) J_\lambda^{(\alpha)}(\underline{-\alpha r})}{j_\lambda^{(\alpha)}} = 1.$$

The last equality can be checked directly from Theorem 2.1 and Equation (9).

The purpose of the following lemma is to explain how to add faces of degree 1 on b-weighted bipartite maps. We will need a variant of Equation (24) where we replace $\vec{\rho}$ by another b-weight on face-labelled maps $\vec{\rho}_{SYM}$ that we now define. As noticed in Remark 4.3, the b-weight $b^{\nu(\mathbf{M},c)}$ that we consider in Section 4 is not the only one that satisfies Theorem 4.1. We consider now the b-weight $\rho_{SYM}(\mathbf{M}, c)$ defined in [8, Remark 3], which is not integral but has more symmetry properties that will be useful in the proof of Lemma 5.3. We define $\vec{\rho}_{SYM}$ as the b-weight on face-labelled bipartite maps obtained in Section 4.5 when we replace $b^{\nu(\mathbf{M},c)}$ by $\rho_{SYM}(\mathbf{M}, c)$ in Section 4 (see also Definition 4.5 and Definition 4.7 (1)). With the same arguments used in Section 4, one can check that Equation (24) also holds for $\vec{\rho}_{SYM}$.

Lemma 5.3. *For every partition* $\mu \vdash m$ *such that* $m_1(\mu) = 0$*, and* $\lambda = (q \times r) \vdash n \geqslant m$*, we have*

$$[p_{\mu \cup 1^{n-m}} t^n]\tau_b^{(1)}(-t, \mathbf{p}, \underline{q}, \underline{-r\alpha}) = [p_\mu t^m]\tau_b^{(1)}(-t, \mathbf{p}, \underline{q}, \underline{-r\alpha}), \tag{28}$$

where $\mu \cup 1^\ell$ *denotes the partition obtained by adding* ℓ *parts equal to 1 to* μ*.*

Proof. We start by proving that for every partition $\xi \vdash \ell$ we have the following equation:

$$2(m_1(\xi)+1)[p_{\xi\cup 1}t^{\ell+1}]\tau_b^{(1)}(-t,\mathbf{p},\underline{q},\underline{-r\alpha}) = 2(n-\ell)[p_\xi t^\ell]\tau_b^{(1)}(-t,\mathbf{p},\underline{q},\underline{-r\alpha}). \tag{29}$$

Using Equation (24) for the b-weight $\vec{\rho}_{SYM}$ introduced above, we can see that the two terms of the previous equation are generating series of bipartite maps. Hence, the last equality can be rewritten as follows:

$$2\sum_{\widehat{\mathbf{M}}}\vec{\rho}_{SYM}(\widehat{\mathbf{M}})\,\mathrm{w}^{(\alpha)}(\widehat{\mathbf{M}},q,r) = 2\alpha(n-\ell)\sum_{\widehat{\mathbf{M}}}\vec{\rho}_{SYM}(\widehat{\mathbf{M}})\,\mathrm{w}^{(\alpha)}(\widehat{\mathbf{M}},q,r) \tag{30}$$

where the sums run over face-labelled bipartite maps, of face-type $\xi\cup 1$ in the left hand-side and ξ in the right hand-side. The factor 2 in the left hand side of the last equation will be interpreted as marking an edge-side on the face of degree 1 with the highest label of each face-labelled bipartite map $\widehat{\mathbf{M}}$ of face-type $\xi\cup 1$. Such a map can be obtained by adding an edge e with a marked side to a bipartite map $\widehat{\mathbf{M}}$ of face-type ξ so that the marked side is in a face of degree 1 in the map $\widehat{\mathbf{M}}\cup\{e\}$.

In the following, we show that this corresponds to the right-hand side of Equation (30). Let $\widehat{\mathbf{M}}$ be a map of face-type ξ. We have two ways to add such an edge e with a marked side to $\widehat{\mathbf{M}}$:

- We add an isolated edge with a marked side. We chose the highest label for the face of degree 1 that we form by adding e. We thus obtain a face-labelled map. In this case we have: $\mathrm{w}^{(\alpha)}(\widehat{\mathbf{M}}\cup\{e\},q,r) = 2n\alpha\cdot\mathrm{w}^{(\alpha)}(\widehat{\mathbf{M}},q,r)$; the black vertex has weight q, the white $-r\alpha$, and we multiply by -1 for adding an edge. Finally we have two choices for the marked edge-side.
- We choose a side of an edge s to which we add the marked side of the edge e in order to form a face of degree 1. Since the map is of size ℓ we have 2ℓ choices for the edge-side s. Once s is fixed, we chose the highest label for the face of degree 1 formed by adding e. Since we have two choices of the orientation of this face, we obtain two face-labelled maps of face-type $\xi\cup 1$, that we denote $\widehat{\mathbf{M}}_1$ and $\widehat{\mathbf{M}}_2$. They satisfy $\mathrm{w}^{(\alpha)}(\widehat{\mathbf{M}}_1,q,r) = \mathrm{w}^{(\alpha)}(\widehat{\mathbf{M}}_2,q,r) = -\mathrm{w}^{(\alpha)}(\widehat{\mathbf{M}},q,r)$.

On the other hand, we claim that the b-weight $\vec{\rho}_{SYM}$ defined above has the following property: if e is an edge that we add to a bipartite map $\widehat{\mathbf{M}}$ to form a face of degree 1 then we have:

- $\vec{\rho}_{SYM}(\widehat{\mathbf{M}}\cup\{e\}) = \vec{\rho}_{SYM}(\widehat{\mathbf{M}})$, if e is an isolated edge.
- $\vec{\rho}_{SYM}(\widehat{\mathbf{M}}_1)+\vec{\rho}_{SYM}(\widehat{\mathbf{M}}_2) = \alpha\vec{\rho}_{SYM}(\widehat{\mathbf{M}})$, if e is added on an edge side of $\widehat{\mathbf{M}}$, where $\widehat{\mathbf{M}}_1$ and $\widehat{\mathbf{M}}_2$ are as above.

Let us explain how to obtain this property. As explained above $\vec{\rho}_{SYM}$ is obtained from ρ_{SYM} by duality (see Definition 4.5 and Definition 4.7(1)). Notice that adding a face of degree 1 on a map is equivalent to adding a white vertex of degree 1 on the dual map. But such operation does not affect the b-weight ρ_{SYM} (this is clear from the combinatorial model used in [8] and the definition of ρ_{SYM} [8, Remark 3]). Finally, observe that when e is not an isolated edge, one of the possible orientations of the added face does not affect the b-weight $\rho_{SYM}(\widehat{\mathbf{M}})$, and for the second one $\rho_{SYM}(\widehat{\mathbf{M}})$ is multiplied by b (see Definition 4.5). This concludes the proof of the previous property and thus the proof of Equation (29).

Using Equation (29), we prove by induction on ℓ that

$$[p_{\mu\cup 1^{\ell-m}}t^\ell]\tau_b^{(1)}(-t,\mathbf{p},\underline{q},\underline{-r\alpha}) = \binom{n-m}{\ell-m}[p_\mu t^m]\tau_b^{(1)}(-t,\mathbf{p},\underline{q},\underline{-r\alpha}).$$

This gives Equation (28) when $\ell = n$.

Remark 5.4. Observe that Equation (29) can be rewritten as follows

$$\frac{\partial}{\partial p_1}\tau_b^{(1)}(-t,\mathbf{p},\underline{q},\underline{-r\alpha}) = -\left[\frac{(-r\alpha)q}{\alpha}+t\frac{\partial}{\partial t}\right]\tau_b^{(1)}(-t,\mathbf{p},\underline{q},\underline{-r\alpha}).$$

This equation can be seen as a b-deformation of the first Virasoro constraint for bipartite maps (see [26] and [8, Equation (17)]). After the first version of this article has been made public, Virasoro constraints have been proved in a greater generality by Bonzom, Chapuy and Dołęga (see [3, Proposition A.1]).

We now prove the main result of this section.

Theorem 5.5. *For every partition $\mu \vdash m > 1$ such that $m_1(\mu) = 0$, we have that $(-1)^m z_\mu \theta_\mu^{(\alpha)}(q,r)$ is a polynomial in $(q,-r,b)$ with non-negative integer coefficients. More precisely, we have*

$$z_\mu\theta_\mu^{(\alpha)}(q,r) = \sum_{\widehat{\mathbf{M}}} \vec{\rho}(\widehat{\mathbf{M}})\frac{\mathrm{w}^{(\alpha)}(\widehat{\mathbf{M}},q,r)}{\alpha^{\ell(\mu)}}, \tag{31}$$

where the sum is taken over face-labelled bipartite maps of face-type μ.

Proof. We start by proving Equation (31). Let $q,r > 1$, and let $\lambda := q \times r$. We denote by $n = q\cdot r$ the size of λ. Since $m_1(\mu) = 0$, we have that $\theta_\mu^{(\alpha)}(\lambda) = \theta_{\mu\cup 1^{n-m}}^{(\alpha)}(\lambda)$. Applying Corollary 5.2 and Lemma 5.3, we get

$$\theta_\mu^{(\alpha)}(\lambda) = [p_{\mu\cup 1^{n-m}}t^n]\tau_b^{(1)}(-t,\mathbf{p},q,-r\alpha) = [p_\mu t^m]\tau_b^{(1)}(-t,\mathbf{p},q,-r\alpha).$$

Using Equation (24), this leads to Equation (31).

Let us now prove that Equation (31) implies the positivity and the integrality of the coefficients of $(-1)^m z_\mu\theta_\mu^{(\alpha)}(q,r)$. Since $(-1)^m\,\mathrm{w}^{(\alpha)}(\widehat{\mathbf{M}},q,r)$ is a polynomial in $(q,-r,b)$ with non-negative integer coefficients, it suffices to eliminate the term $\alpha^{\ell(\mu)}$ that appears in the denominator of the right-hand side of Equation (31). We say that a bipartite map is *weakly face-labelled* if it is obtained from a face-labelled bipartite map for which we keep the labelling of faces, but we forget the orientation of all the faces except for the face of maximal degree and smallest label in each connected component. For such a map, we have a natural notion of rooting for every connected component given by the root of the face of maximal degree and minimal label. Using a variant of Equation (19), Equation (31) can be rewritten as follows

$$z_\mu\theta_\mu^{(\alpha)}(q,r) = \sum_{\mathbf{M}}\prod_{(\mathbf{M}_i,c_i)} b^{\nu(\widetilde{\mathbf{M}}_i,c_i)}\frac{\mathrm{w}^{(\alpha)}(\mathbf{M}_i,q,r)}{\alpha}, \tag{32}$$

where the sum is taken over weakly face-labelled bipartite maps $\mathbf{M}$ with face-type μ and the product runs over the connected components of $\mathbf{M}$ rooted as explained above. To conclude, notice that it is direct from the definition that $\mathrm{w}^{(\alpha)}(\mathbf{M}_i,q,r)$ is divisible by α.

Remark 5.6. As noticed by Lassalle [30, Conjecture 1, item (*iii*)], z_μ is the good normalization to obtain integrality in Theorem 5.5. Indeed, if $\mu \vdash m$

$$[q^m](-1)^m z_\mu\theta_\mu^{(\alpha)}(q,r) = (-r)^{\ell(\mu)},$$

where [.] denotes the extraction symbol with respect to the variable q. To see this, observe that the only face-labelled bipartite map that contributes to the monomial q^m in Equation (31) is the map of size m and face-type μ that contains m black vertices.

REMARK 5.7. Lassalle suggested that the coefficients $\theta_\mu^{(\alpha)}(q,r)$ have a natural expression as a positive polynomial in both variables α and b (see [30, Conjecture 2]). Such an expression can be obtained from Equation (32) by considering the two terms $\prod_{(\mathbf{M}_i,c_i)} b^{\nu(\widetilde{\mathbf{M}}_i,c_i)}$ and $\prod_{(\mathbf{M}_i,c_i)} \frac{\mathsf{w}^{(\alpha)}(\mathbf{M}_i,q,r)}{\alpha}$. This expression in α and b and the one given by Lassalle in [30] are related but not the same.

6. GENERALIZED COEFFICIENTS $c^\lambda_{\mu^0,\dots,\mu^k}$

In this section, we state some properties of the coefficients $c^\lambda_{\mu^0,\dots,\mu^k}$, and we give a new proof for a combinatorial interpretation of the top degree part in these coefficients. This part is also related to the evaluation of $c^\lambda_{\mu^0,\dots,\mu^k}$ at $b=-1$, see Corollary 6.5.

6.1. GENERAL PROPERTIES OF $c^\lambda_{\mu^0,\dots,\mu^k}$.

We start by a multiplicativity property of the coefficients $c^\lambda_{\mu^0,\dots,\mu^k}$ due to Chapuy and Dołęga (private communication).

PROPOSITION 6.1. *Let* $k > 2$ *and* $\lambda,\mu^0,\dots,\mu^k \quad n > 1$. *We have*

$$c^\lambda_{\mu^0,\dots,\mu^k}(b)=\sum_{\eta \quad n} c^\lambda_{\mu^0,\dots\mu^{k-2},\eta}(b)c^\eta_{\mu^{k-1},\mu^k}(b).$$

Proof. Let $\mathbf{r} := (r_1, r_2, \dots)$ be an additional sequence of power-sum variables. We consider the two functions $\tau_b^{(k-1)}(t,\mathbf{p},\mathbf{q}^{(0)},\dots,\mathbf{q}^{(k-2)},\mathbf{r})$ and $\tau_b^{(1)}(t,\mathbf{q}^{(k-1)},\mathbf{q}^{(k)},\mathbf{r})$, and we take their scalar product with respect to the variable $\mathbf{r}$. Since

$$J^{(\alpha)}_{\xi^1}(\mathbf{r}), J^{(\alpha)}_{\xi^2}(\mathbf{r}) \ _\alpha = \delta_{\xi^1,\xi^2} j^{(\alpha)}_{\xi^1},$$

this scalar product gives the function $\tau_b^{(k)}(t,\mathbf{p},\mathbf{q}^{(0)},\dots,\mathbf{q}^{(k-2)},\mathbf{q}^{(k-1)},\mathbf{q}^{(k)})$. On the other hand, the expansion of these functions in the power-sum basis can be written as:

$$\tau_b^{(k-1)}(t,\mathbf{p},\mathbf{q}^{(0)},\dots,\mathbf{q}^{(k-2)},\mathbf{r}) = \sum_{n>0} t^n \sum_{\lambda,\mu^0,\dots,\mu^{k-2},\eta \quad n} \frac{c^\lambda_{\mu^0,\dots,\mu^{k-2},\eta}}{z_\lambda(1+b)^{\ell(\lambda)}} p_\lambda q^{(0)}_{\mu^0}\dots q^{(k-2)}_{\mu^{k-2}} r_\eta,$$

and

$$\tau_b^{(1)}(t,\mathbf{q}^{(k-1)},\mathbf{q}^{(k)},\mathbf{r}) = \sum_{n>0} t^n \sum_{\eta,\mu^{k-1},\mu^k \quad n} \frac{c^\eta_{\mu^{k-1},\mu^k}}{z_\eta(1+b)^{\ell(\eta)}} q^{(k-1)}_{\mu^{k-1}} q^{(k)}_{\mu^k} r_\eta.$$

We conclude by taking the scalar product of the two last equations.

The previous property can be used to extend some results known for coefficients c with three parameters (the case $k=1$) to the general case. In particular, we can deduce the following corollary.

COROLLARY 6.2. *Conjecture* 1.5 *for the coefficients* $c^\lambda_{\mu^0,\dots,\mu^k}$ *when* $k=1$ *implies the conjecture for* $c^\lambda_{\mu^0,\dots,\mu^k}$ *for any* $k > 1$.

Proof. We use induction on k and Proposition 6.1.

As mentioned in the introduction, the polynomiality of the quantities $c^\lambda_{\mu^0,\dots,\mu^k}$ has been proved in [11] when $k=1$. This can be generalized for any $k > 1$.

THEOREM 6.3. *For all* $\lambda,\mu^0,\dots,\mu^k \quad n$, *the coefficient* $c^\lambda_{\mu^0,\dots,\mu^k}(b)$ *is a polynomial with rational coefficients, and we have the following bounds on the degree:*

$$\deg(c^\lambda_{\mu^0,\dots,\mu^k}) \ 6 \min_{-1 6 i 6 k} d_i(\lambda,\mu^0,\dots,\mu^k),$$

where

$$d_{-1}(\lambda,\mu^0,\dots,\mu^k) := kn + \ell(\lambda) - \big(\ell(\mu^0)+\dots+\ell(\mu^k)\big),$$

and

$$d_i(\lambda, \mu^0, \ldots, \mu^k) \coloneqq kn - \sum_{j=i} \ell(\mu^j),$$

for $0 \leqslant i \leqslant k$.

Proof. The polynomiality and the bound d_{-1} follow from [11, Proposition B.2] and Proposition 6.1. To deduce the other bounds, we use the symmetry of coefficients $c^{\lambda}_{\mu^0 \ldots \mu^k}$ in partitions μ^i for $0 \leqslant i \leqslant k$ and the following relation that exchanges λ and μ^0 (see Equation (5)):

$$\frac{c^{\lambda}_{\mu^0,\ldots,\mu^k}}{z_\lambda (1+b)^{\ell(\lambda)}} = \frac{c^{\mu^0}_{\lambda,\mu^1,\ldots,\mu^k}}{z_{\mu^0}(1+b)^{\ell(\mu^0)}}. \tag{33}$$

In Section 6.2, we give a combinatorial interpretation of the term associated to each one of these bounds. We now state some results that will be useful in Section 6.2.

Proposition 6.4. *For every* $k, n \geqslant 1$ *and* $\lambda, \mu^0, \ldots, \mu^k \vdash n$*, the coefficient* $c^{\lambda}_{\mu^0,\ldots,\mu^k}$ *has the following form;*

$$c^{\lambda}_{\mu^0,\ldots,\mu^k} = \sum_{0 \leqslant i \leqslant \left\lfloor \frac{d_{-1}}{2} \right\rfloor} a_i b^{d_{-1}-2i}(1+b)^i,$$

where $a_i \in \mathbb{Q}$ *and* $d_{-1} \coloneqq d_{-1}(\lambda, \mu^0, \ldots, \mu^k)$.

The proof is essentially the same as in the case $k = 1$ proved in [28]. For completeness, we give the key steps of the proof in Appendix A.

The previous proposition has the following implication:

Corollary 6.5. *For all* $\lambda, \mu^0, \ldots, \mu^k \vdash n \geqslant 1$

$$[b^{d_{-1}}] c^{\lambda}_{\mu^0,\ldots,\mu^k} = (-1)^{d_{-1}} c^{\lambda}_{\mu^0,\ldots,\mu^k}(-1).$$

The polynomiality of coefficients $h^{\lambda}_{\mu^0,\ldots,\mu^k}$ has been deduced from the polynomiality of $c^{\lambda}_{\mu^0,\ldots,\mu^k}$ when $k = 1$ in [11]. The proof works in a similar way for $k \geqslant 1$. We give the key steps of this proof in Appendix B. We obtain the following theorem.

Theorem 6.6. *For all* $\lambda, \mu^0, \ldots, \mu^k \vdash n \geqslant 1$*, the coefficient* $h^{\lambda}_{\mu^0,\ldots,\mu^k}$ *is a polynomial in* b *with rational coefficients, and we have the following bound on its degree:*

$$\deg(h^{\lambda}_{\mu^0,\ldots,\mu^k}) \leqslant kn + 2 - \big(\ell(\lambda) + \ell(\mu^0) + \cdots + \ell(\mu^k)\big).$$

The following property has been proved by Dołęga in the case $k = 1$ (see [10, Proposition 4.1]). We copy here the proof, adapting it to the case $k > 1$.

Lemma 6.7. *For all* $\lambda, \mu^0, \ldots, \mu^{k-1} \vdash n \geqslant 1$ *we have*

$$\sum_{\eta \vdash n} h^{\lambda}_{\mu^0,\ldots,\mu^{k-1},\eta}(b) = (1+b)^{kn+1-(\ell(\lambda)+\ell(\mu^0)+\cdots+\ell(\mu^{k-1}))} \sum_{\eta \vdash n} h^{\lambda}_{\mu^0,\ldots,\mu^{k-1},\eta}(0).$$

Proof. From Equation (6), we have

$$\sum_{\eta \vdash n} h^{\lambda}_{\mu^0,\ldots,\mu^{k-1},\eta}(b) = [t^n p_\lambda q^{(0)}_{\mu^0} q^{(1)}_{\mu^1} \ldots q^{(k-1)}_{\mu^{k-1}}] \Psi^{(k)}_b \left(t, \mathbf{1}, \mathbf{q}^{(0)}, \mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(k)}\right),$$

where the variable $\mathbf{p}$ is specialized to $\mathbf{p} = \mathbf{1} \coloneqq (1, 1, \ldots)$. With this specialization, Jack polynomials have the following expression

$$J^{(\alpha)}_\xi(\mathbf{1}) = \begin{cases} (1+\alpha)(1+2\alpha)\ldots(1+(n-1)\alpha) & \text{if } \xi = [n], \\ 0 & \text{if } \ell(\xi) > 1. \end{cases}$$

Moreover, we have the following properties related to partitions of one single part;

$$J_n^{(\alpha)}(\mathbf{r}) = \sum_{\mu \vdash n} \frac{n!\alpha^{n-\ell(\mu)}}{z_\mu} r_\mu,$$

for a fixed variable $\mathbf{r}$, and

$$j_{[n]}^{(\alpha)} = (1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)n!\alpha^n,$$

(see [32] for the proof of these properties). Hence

$$\begin{aligned}
&\sum_{\eta \vdash n} h^{\lambda}_{\mu^0,\dots,\mu^{k-1},\eta}(b) \\
&\quad = [t^n p_\lambda q^{(0)}_{\mu^0} q^{(1)}_{\mu^1} \dots q^{(k-1)}_{\mu^{k-1}}](1+b)t\frac{\partial}{\partial t}\log \sum_{n>0} t^n \frac{J_n^{(\alpha)}(\mathbf{p}) J_n^{(\alpha)}(\mathbf{q}^{(0)}) \dots J_n^{(\alpha)}(\mathbf{q}^{(k-1)})}{n!\alpha^n} \\
&\quad = [t^n p_\lambda q^{(0)}_{\mu^0} q^{(1)}_{\mu^1} \dots q^{(k-1)}_{\mu^{k-1}}](1+b)^d t\frac{\partial}{\partial t}\log \sum_{n>0} t^n \sum_{\widetilde{\lambda},\widetilde{\mu^0},\dots,\widetilde{\mu^{k-1}} \vdash n} \frac{(n!)^k p_{\widetilde{\lambda}} q^{(0)}_{\widetilde{\mu^0}} \dots q^{(k-1)}_{\widetilde{\mu^{k-1}}}}{z_{\widetilde{\lambda}} z_{\widetilde{\mu^0}} \dots z_{\widetilde{\mu^{k-1}}}},
\end{aligned}$$

where $d = kn + 1 - (\ell(\lambda) + \ell(\mu^0) + \dots + \ell(\mu^{k-1}))$. We conclude by observing that the last expression is equal to

$$(1+b)^d \sum_{\eta \vdash n} h^{\lambda}_{\mu^0,\dots,\mu^{k-1},\eta}(0).$$

We deduce the following corollary that will be useful in the proof of Theorem 6.11.

Corollary 6.8. *For* $\lambda, \mu^0, \dots, \mu^{k-1} \vdash n$ *we have*

$$[b^{kn+1-(\ell(\lambda)+\ell(\mu^0)+\dots+\ell(\mu^{k-1}))}] h^{\lambda}_{\mu^0,\dots,\mu^k} = \delta_{\mu^k,[n]} \sum_{\eta \vdash n} h^{\lambda}_{\mu^0,\dots,\mu^{k-1},\eta}(0),$$

where δ *is the Kronecker delta.*

Proof. If $\mu^k = [n]$, then from Theorem 6.6 we have

$$[b^{kn+1-(\ell(\lambda)+\ell(\mu^0)+\dots+\ell(\mu^{k-1}))}] h^{\lambda}_{\mu^0,\dots,\mu^{k-1},\mu^k}(b) = 0.$$

The previous lemma finishes the proof.

6.2. Top degree in coefficients $c^{\lambda}_{\mu^0,\dots,\mu^k}$. Theorem 6.3 gives $k+2$ upper bounds on the degrees of coefficients $c^{\lambda}_{\mu^0,\dots,\mu^k}$. Using the symmetry property, we can see that the bounds $d_i(\lambda, \mu^0, \dots, \mu^k)$ for $0 \leqslant i \leqslant k$ are equivalent. Theorem 6.11 gives a combinatorial interpretation for the coefficient in $c^{\lambda}_{\mu^0,\dots,\mu^k}$ associated to the bounds $d_i(\lambda, \mu^0, \dots, \mu^k)$ for $0 \leqslant i \leqslant k$ and Theorem 6.12 gives an interpretation for the bound $d_{-1}(\lambda, \mu^0, \dots, \mu^k)$. The bound $d_{-1}(\lambda, \mu^0, \dots, \mu^k)$ was investigated in [5] when $k = 1$ and the combinatorial interpretation was given in terms of unhandled maps, while we give here an interpretation with orientable maps with a different proof. In fact, there exists a bijection between the two objects, showing that Theorem 6.12 for $k = 1$ is equivalent to the result of [5] (see [9, Theorem 1.8]).

As explained in Section 1.4, the labelling of constellations is simpler in the orientable case. We introduce the following definition of labelling for orientable constellations that will be used to state the main results of this section.

Definition 6.9. *If $\mathbf{M}$ is an orientable k-constellation of size n. We say that:*

- $\mathbf{M}$ *is labelled if its hyperedges are labelled by $\{1,\dots,n\}$, when we consider $\mathbf{M}$ as a hypermap (see Section 2.5). In terms of right-paths, this is equivalent to label the right-paths of the constellation $\mathbf{M}$ traversed from the corner of color 0 to the corner of color k, when $\mathbf{M}$ is equipped with the canonical orientation.*
- $\mathbf{M}$ *has* labelled faces *if each face has a distinguished corner of color 0, and the faces of same size are labelled.*

Note that the definition of face-labelling that we give here for orientable constellations is slightly different from the definition given in Section 4.2; in each face we do not choose an orientation for the distinguished corner. The reason is that in the orientable case all faces have a canonical orientation (see Definition 1.1). We also introduce the following definition.

Definition 6.10. *Let λ be a partition. We say that a k-constellation $\mathbf{M}$ is λ-*connected, *if λ is the partition obtained by reordering the sizes of the connected components of $\mathbf{M}$. We say that a k-constellation $\mathbf{M}$ has* labelled connected components, *if each connected component is rooted, and the connected components of the same size are labelled, i.e. for $r \geq 1$ if $\mathbf{M}$ has j connected components of size r, they are labelled with $\{1,\dots,j\}$. For all partitions $\lambda,\eta,\mu^0,\dots,\mu^k \vdash n \geq 1$, we denote $\widetilde{h}^{\lambda,\eta}_{\mu^0,\dots,\mu^k}$ the number of labelled orientable η-connected k-constellations with profile $(\lambda,\mu^0,\dots,\mu^k)$. Finally, we say that a k-constellation has* partial profile *$(\lambda,\mu^0,\dots,\mu^{k-1},\bullet)$ if its profile is given by $(\lambda,\mu^0,\dots,\mu^{k-1},\mu^k)$ for some partition μ^k.*

Theorem 6.11. *For all $\lambda,\mu^0,\dots,\mu^k \vdash n \geq 1$, the top degree $[b^{d_k}]c^{\lambda}_{\mu^0\mu^1\dots\mu^k}$ is equal to the number of μ^k-connected orientable k-constellations with labelled faces with partial profile $(\lambda,\mu^0,\dots\mu^{k-1},\bullet)$, and where $d_k := d_k(\lambda,\mu^0,\dots,\mu^k)$.*

Proof. From equations (5) and (6) and by developing the exponential in Equation (2), we obtain

$$\frac{c^{\lambda}_{\mu^0,\dots,\mu^k}}{z_\lambda(1+b)^{\ell(\lambda)}} = \sum_{r\geq 1}\frac{1}{r!}\sum_{(n_i)}\sum_{(\lambda_{(i)},\mu^0_{(i)},\dots,\mu^k_{(i)})}\prod_{1\leq i\leq r}\frac{h^{\lambda_{(i)}}_{\mu^0_{(i)},\dots,\mu^k_{(i)}}(b)}{n_i(1+b)}, \tag{34}$$

where the second sum is taken over r-tuples of positive integers which sum to r, and the third sum is taken over r-tuples $(\lambda_{(i)},\mu^1_{(i)},\dots,\mu^k_{(i)})_{1\leq i\leq r}$ such that $\bigcup_{1\leq i\leq r}\lambda_{(i)}=\lambda$, $\bigcup_{1\leq i\leq r}\mu^j_{(i)}=\mu^j$, for all $j \in [\![0,k]\!]$ and $n_i=|\lambda_{(i)}|=|\mu^j_{(i)}|$, for all $i \in [\![1,r]\!]$ and $j \in [\![0,k]\!]$. The last equality can be rewritten as follows

$$\frac{c^{\lambda}_{\mu^0,\dots,\mu^k}}{z_\lambda} = \sum_{r\geq 1}\frac{1}{r!}\sum_{(n_i)}\sum_{(\lambda_{(i)},\mu^0_{(i)},\dots,\mu^k_{(i)})}\prod_{1\leq i\leq r}\frac{(1+b)^{\ell(\lambda_{(i)})-1}h^{\lambda_{(i)}}_{\mu^0_{(i)},\dots,\mu^k_{(i)}}(b)}{n_i}.$$

But from Theorem 6.6 we know that for all i, $h^{\lambda_{(i)}}_{\mu^0_{(i)},\dots,\mu^k_{(i)}}(b)$ is a polynomial in b and

$$\deg\left(\frac{(1+b)^{\ell(\lambda_{(i)})-1}h^{\lambda_{(i)}}_{\mu^0_{(i)},\dots,\mu^k_{(i)}}(b)}{n_i}\right) \leq kn_i+1-\sum_{0\leq j\leq k}\ell(\mu^j_{(i)}) \leq kn_i-\sum_{0\leq j\leq k-1}\ell(\mu^j_{(i)}).$$

Taking the product over i, it gives us

$$\deg\left(\frac{1}{r!}\prod_{1\leq i\leq r}\frac{(1+b)^{\ell(\lambda_{(i)})-1}h^{\lambda_{(i)}}_{\mu^0_{(i)}\cdots\mu^k_{(i)}}(b)}{n_i}\right)\leq kn-\sum_{0\leq j\leq k-1}\ell(\mu^j)=d_k.$$

To have equality in the last line, we should have $\ell(\mu^k_{(i)})=1$ for all $i\in[\![r]\!]$. In other words, $\mu^k_{(i)}=[n_i]$ (and hence r should be equal to $\ell(\mu^k)$). Therefore, one has

$$(35)\quad [b^{d_k}]\frac{c^{\lambda}_{\mu^0,\dots,\mu^k}}{z_\lambda}$$
$$=\frac{1}{\ell(\mu^k)!}\sum_{(\lambda_{(i)},\mu^0_{(i)},\dots,n_i)}[b^{d_k}]\prod_{1\leq i\leq \ell(\mu^k)}\frac{(1+b)^{\ell(\lambda_{(i)})-1}h^{\lambda_{(i)}}_{\mu^0_{(i)},\dots,\mu^{k-1}_{(i)},[n_i]}}{n_i}$$
$$=\frac{1}{\ell(\mu^k)!}\sum_{(\lambda_{(i)},\mu^0_{(i)},\dots,n_i)}\prod_{1\leq i\leq \ell(\mu^k)}\left[b^{d_{(i)}}\right]\frac{(1+b)^{\ell(\lambda_{(i)})-1}h^{\lambda_{(i)}}_{\mu^0_{(i)},\dots,\mu^{k-1}_{(i)},[n_i]}}{n_i},$$

where the sums run over $\ell(\mu^k)$-tuples $(\lambda_{(i)},\mu^0_{(i)},\dots,n_i)_{1\leq i\leq \ell(\mu^k)}$ such that $(n_i)_{1\leq i\leq\ell(\mu^k)}$ is a reordering of μ^k, $\bigcup_{1\leq i\leq r}\lambda_{(i)}=\lambda$ and $\bigcup_{1\leq i\leq r}\mu^j_{(i)}=\mu^j$ for all $j\in[\![0,k-1]\!]$, and where $d_{(i)}\coloneqq kn_i-\sum_{0\leq j\leq k-1}\ell(\mu^j_{(i)})$. From Corollary 6.8, we know that

$$(36)\qquad \left[b^{d_{(i)}}\right](1+b)^{\ell(\lambda_{(i)})-1}h^{\lambda_{(i)}}_{\mu^0_{(i)},\dots,\mu^{k-1}_{(i)},[n_i]}(b)=\sum_{\eta_{(i)}\vdash n_i}h^{\lambda_{(i)}}_{\mu^0_{(i)},\dots,\mu^{k-1}_{(i)},\eta_{(i)}}(0).$$

Hence

$$[b^{d_k}]\frac{c^{\lambda}_{\mu^0,\dots,\mu^k}}{z_\lambda}=\frac{1}{\ell(\mu^k)!}\sum_{(\lambda_{(i)},\mu^0_{(i)},\dots,\mu^{k-1}_{(i)},\eta_{(i)})}\prod_{1\leq i\leq\ell(\mu^k)}\frac{h^{\lambda_{(i)}}_{\mu^0_{(i)},\dots,\mu^{k-1}_{(i)},\eta_{(i)}}(0)}{n_i}.$$

On the other hand, from Theorem 1.3 we know that $h^{\lambda_{(i)}}_{\mu^0_{(i)},\dots,\mu^{k-1}_{(i)},\eta_{(i)}}(0)$ is the number of rooted connected orientable k-constellations. Then for all partitions $\lambda_{(i)},\mu^0_{(i)},\dots,\mu^{k-1}_{(i)},\eta_{(i)}\vdash n_i$

$$\frac{h^{\lambda_{(i)}}_{\mu^0_{(i)},\dots\mu^{k-1}_{(i)},\eta_{(i)}}(0)}{n_i}=\frac{\widetilde{h}^{\lambda_{(i)},[n_i]}_{\mu^0_{(i)},\dots,\mu^{k-1}_{(i)},\eta_{(i)}}}{n_i!},$$

where $\widetilde{h}^{\lambda_{(i)},[n_i]}_{\mu^0_{(i)},\dots,\mu^{k-1}_{(i)},\eta_{(i)}}$ is the quantity defined in Definition 6.10. Hence, Equation (35) can be rewritten as follows

$$\frac{[b^{d_k}]c^{\lambda}_{\mu^0,\dots,\mu^k}}{z_\lambda}=\frac{\widetilde{h}^{\lambda,\mu^k}_{\mu^0,\dots\mu^{k-1},\bullet}}{n!}.$$

Finally, we multiply $\widetilde{h}^{\lambda,\mu^k}_{\mu^0,\dots\mu^{k-1},\bullet}$ by $\frac{z_\lambda}{n!}$ to pass from labelled constellations to face-labelled constellations.

We now deduce from Theorem 6.11 an analogous theorem for the bound $d_{-1}(\lambda,\mu^0,\dots,\mu^k)$.

Theorem 6.12. *For $\lambda, \mu^0, \dots, \mu^k \vdash n \geq 1$, the top degree term $[b^{d_{-1}}]c^{\lambda}_{\mu^0,\dots,\mu^k}$ is equal to the number of λ-connected orientable k-constellation with labelled connected components and partial profile $(\mu^k, \mu^0, \dots \mu^{k-1}, \bullet)$, where $d_{-1} \coloneqq d_{-1}(\lambda, \mu^0, \dots, \mu^k)$.*

Proof. From Equation (33), we know that

$$[b^{d_{-1}}]c^{\lambda}_{\mu^0,\dots,\mu^k} = \frac{z_\lambda}{z_{\mu^k}}[b^{d_k(\mu^k,\mu^0,\dots,\mu^{k-1},\lambda)}]c^{\mu^k}_{\mu^0,\dots,\lambda}.$$

We apply Theorem 6.11 and multiply by z_λ to choose the labels of the connected components and we divide by z_{μ^k} to forget the labels of the faces, which concludes the proof.

Appendix A. Sketch of the proof of Proposition 6.4

In this appendix, we give the key steps of the proof of Proposition 6.4. The same proof given in [28] for $k = 1$ still works in the general case. But we prefer here for completeness to use the multiplicativity property of Proposition 6.1 to extend some key steps of this proof to the case $k > 1$. We start by introducing some definitions and results due to La Croix [28].

Definition A.1 ([28]). *Let $g \geq 0$. We denote by Ξ_g the set of rational functions in b with coefficients in $\mathbb{Q}$, satisfying the following functional equation:*

$$f(b-1) = (-b)^g f\left(\frac{1}{b} - 1\right).$$

We have the following multiplicativity property (see [28, Lemma 5.6]).

Lemma A.2 ([28]). *Let $g_1, g_2 \geq 0$, and let $f_1 \in \Xi_{g_1}$ and $f_2 \in \Xi_{g_2}$, then $f_1 f_2 \in \Xi_{g_1+g_2}$.*

La Croix has proved the following lemma [28, Lemma 5.7].

Lemma A.3 ([28]). *Let f be a polynomial in b with coefficients in $\mathbb{Q}$. Then $f \in \Xi_g$ if and only if f is of the form*

$$f = \sum_{0 \leq i \leq \frac{g}{2}} a_i b^{g-2i}(1+b)^i,$$

where $a_i \in \mathbb{Q}$.

On the other hand, we have the following lemma.

Lemma A.4. *For every $k, n \geq 1$, and for every partitions $\lambda, \mu^0, \dots, \mu^k \vdash n$, we have that $c^{\lambda}_{\mu^0,\dots,\mu^k} \in \Xi_g$, where $g = kn + \ell(\lambda) - \ell(\mu^0) \dots - \ell(\mu^k)$.*

Proof. We start by the case $k = 1$. La Croix has proved that $\frac{c^{\lambda}_{\mu^0,\mu^1}}{z_\lambda(1+b)^{\ell(\lambda)}} \in \Xi_{g-2\ell(\lambda)}$ (see [28, Lemma 5.14]). On the other hand, it is easy to see that $z_\lambda(1+b)^{\ell(\lambda)} \in \Xi_{2\ell(\lambda)}$. Using Lemma A.2, we obtain that $c^{\lambda}_{\mu^0,\mu^1} \in \Xi_g$.

Using the multiplicativity property (see Proposition 6.1) and Lemma A.2, we deduce the lemma for $k > 1$.

We now deduce Proposition 6.4:

Proof of Proposition 6.4. The proposition is a straight consequence of the polynomiality of the coefficients $c^{\lambda}_{\mu^0,\dots,\mu^k}$ (see Theorem 6.3) and Lemmas A.4 and A.3.

Proposition A.5. *For partitions $\lambda, \mu^0, \dots, \mu^k$, we have $h^{\lambda}_{\mu^0,\dots,\mu^k} \in \Xi_g$, where $g = kn + 2 - \big(\ell(\lambda) + \ell(\mu^0) + \dots + \ell(\mu^k)\big)$.*

Proof. The proposition is obtained by induction on n, using Equation (34), and Lemmas A.4 and A.2.

Appendix B. Sketch of the proof of Theorem 6.6

In [12], Dołęga and Féray have deduced the polynomiality of the coefficients $h^{\lambda}_{\mu,\nu}$ in b using the polynomiality of $c^{\lambda}_{\mu,\nu}$ proved in [11]. The same proof can be used to obtain the polynomiality of $h^{\lambda}_{\mu^0,\dots,\mu^k}$. We give here the key steps of this proof.

Notation B.1. We recall that α is the Jack parameter related to the parameter b by $\alpha = b+1$. Let R be a field. We denote by $R(\alpha)$ the field of rational functions in α with coefficients in R. For $f \in R(\alpha)$ and an integer m, we write $f = O(\alpha^m)$ if the rational function $\alpha^{-m} \cdot f$ has no pole at 0.

Let $\lambda^1, \dots, \lambda^r$ be a family of partitions. We denote by $\bigoplus_{1\leqslant i\leqslant r} \lambda^i$ its entry-wise sum defined by $\left(\bigoplus_{1\leqslant i\leqslant r} \lambda^i\right)_j = \sum_{1\leqslant i\leqslant r} \lambda^i_j$, for every $j \geqslant 1$.

Definition B.2. *Let F be a function on Young diagrams, and let $\lambda^1, \lambda^2, \dots, \lambda^r$ be a family of r partitions. We define its* partial cumulants $\boldsymbol{\kappa}^F_H(\lambda^1, \dots, \lambda^r)$ *inductively by*

$$F\left(\bigoplus_{i\in H} \lambda^i\right) = \sum_{\pi\in P(H)} \prod_{B\in\pi} \boldsymbol{\kappa}^F_B(\lambda^1, \dots, \lambda^r),$$

for every subset H of $[\![r]\!]$, where $P(H)$ denotes the set of set partitions of H.

Definition B.3 ([12]). *We say that a function F on Young diagrams has* the small cumulant property *if for every Young diagrams $\lambda^1, \dots, \lambda^r$ and every subset H of $[\![r]\!]$ of size at least 2, we have*

$$\boldsymbol{\kappa}^F_H(\lambda^1, \dots, \lambda^r) = \left(\prod_{i\in[\![r]\!]} F(\lambda^i)\right) O(\alpha^{|H|-1}).$$

We now give some key results due to Dołęga and Féray.

Theorem B.4 ([12]). *The following functions have the small cumulant property:*

- *the function $\lambda \to J^{(\alpha)}_\lambda(\mathbf{p})$, where $\mathbf{p}$ is a fixed alphabet.*
- *the function $\lambda \to j^{(\alpha)}_\lambda / \left(\alpha^{\lambda_1} \prod_i m_i(\lambda')!\right)$, where λ' denotes the conjugate partition of λ and $m_i(\lambda')$ is the number of parts equal to i in λ'.*

Proposition B.5 ([12]). *If F_1 and F_2 have the small cumulant property and take non-zero values then this is also the case for $F_1 \cdot F_2$ and F_1/F_2.*

We deduce from the two previous propositions that the function

$$\lambda \to \frac{\alpha^{\lambda_1} \prod_i m_i(\lambda')!}{j^{(\alpha)}_\lambda} J_\lambda(\mathbf{p}) J_\lambda(\mathbf{q}^{(0)}) \dots J_\lambda(\mathbf{q}^{(k)})$$

has the small cumulant property.

We consider an alphabet of infinite variables $\boldsymbol{t} = (t_1, t_2, \dots)$. We recall the notation $\boldsymbol{t}_\lambda := t_{\lambda_1} \cdots t_{\lambda_{\ell(\lambda)}}$. In the following we denote for a family of partitions $\lambda^1, \dots, \lambda^r$, the cumulant $\boldsymbol{\kappa}^F_{[\![r]\!]}(\lambda^1, \dots, \lambda^r)$ by $\boldsymbol{\kappa}^f(\lambda^1, \dots, \lambda^r)$.

Lemma B.6 ([12]). *For every function F on Young diagrams,*

$$\log \sum_\lambda \frac{F(\lambda)}{\alpha^{\lambda_1} \prod_i m_i(\lambda')!} \boldsymbol{t}_{\lambda'} = \sum_{r\geqslant 1} \frac{1}{r!\alpha^r} \sum_{(j_1,\dots,j_r)} \boldsymbol{\kappa}^F(1^{j_1}, \dots, 1^{j_r}) t_{j_1} \cdots t_{j_r}.$$

We now prove Theorem 6.6.

Proof of Theorem 6.6. Using Equation (34) and Theorem 6.3 we inductively prove that for all partitions $\lambda, \mu^0, \dots, \mu^k$ $n > 1$ the coefficient $h^{\lambda}_{\mu^0,\dots,\mu^k}$ is a rational function in α with only possible pole at $\alpha = 0$. Hence, to obtain the polynomiality of these coefficients in α (or equivalently in b), it is enough to show that the function $\Psi_b^{(k)}$, defined in Equation (2), is $O(1)$. To this end, we write

$$\Psi_b^{(k)}(t, \mathbf{p}, \mathbf{q}^{(0)}, \dots, \mathbf{q}^{(k)}) = \alpha \cdot \log \sum_{\lambda} \frac{F(\lambda)}{\alpha^{\lambda_1} \prod_i m_i(\lambda)!} t^{|\lambda|},$$

where

$$F(\lambda) = \frac{\alpha^{\lambda_1} \prod_i m_i(\lambda)!}{j_\lambda^{(\alpha)}} J_\lambda(\mathbf{p}) J_\lambda(\mathbf{q}^{(0)}) \dots J_\lambda(\mathbf{q}^{(k)}).$$

We now use Lemma B.6 with the function F and $\boldsymbol{t} = (t, t, \dots)$. Since F has the small cumulant property, and for every partition λ the quantity

$$\frac{\alpha^{\lambda_1} \prod_i m_i(\lambda)!}{j_\lambda^{(\alpha)}} J_\lambda(\mathbf{p}) J_\lambda(\mathbf{q}^{(0)}) \dots J_\lambda(\mathbf{q}^{(k)})$$

has no pole at 0 (see [32, Proposition 7.6] and [32, Theorem 5.8]), we deduce that

$$\boldsymbol{\kappa}^F(1^{j_1}, \dots, 1^{j_r}) = O(\alpha^{r-1}),$$

for every positive integers $(j_1, \dots, j_r)$. This finishes the proof of the polynomiality. To obtain the bound on the degree we use Proposition A.5 and Lemma A.3.

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