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Comparing formulas for type $GL_n$, Macdonald polynomials


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Comparing formulas for type $GL_n$
Macdonald polynomials

Weiying Guo & Arun Ram

Abstract The paper compares (and reproves) the alcove walk and the nonattacking fillings formulas for type $GL_n$ Macdonald polynomials which were given in [10], [1] and [18]. The “compression” relating the two formulas in this paper is the same as that of Lenart [13]. We have reformulated it so that it holds without conditions and so that the proofs of the alcove walks formula and the nonattacking fillings formula are parallel. This reformulation highlights the role of the double affine Hecke algebra and Cherednik’s intertwiners. An exposition of the type $GL_n$ double affine braid group, double affine Hecke algebra, and all definitions and proofs regarding Macdonald polynomials are provided to make this paper self-contained.

0. Introduction

The Macdonald polynomials are an incredible family of orthogonal polynomials which simultaneously generalize Schur functions, Weyl characters, Demazure characters, Askey-Wilson polynomials, Koornwinder polynomials, Hall-Littlewood polynomials, Jack polynomials and spherical functions on $p$-adic groups. They are eigenfunctions of a family of difference operators which generalize the classical Laplacian and, in this sense, the Macdonald polynomials $E_\mu$ are generalizations of spherical harmonics.

This paper is a study of the relationship between combinatorial formulas for $GL_n$-type Macdonald polynomials:

(a) The nonattacking fillings formulas from [10, Theorem 3.5.1] and [1], and
(b) The alcove walks formula from [18, Theorem 3.1].

Except for Section 4, which contains the recursions and the calculations for the proofs, we have made an effort to try to make the different sections of this paper readable independent of each other. The reader should not hesitate to go directly to Section 5 for an introduction to the double affine Hecke algebra, to Section 2 for an entrée to $n$-periodic permutations and the affine Weyl group, and to Section 3 for the basics of Macdonald polynomials and some explicit examples of them.

The first half of Section 3 defines the various kinds of Macdonald polynomials, the $E_\mu$, the $P_\lambda$ and the $E_\mu^\alpha$; the second half of Section 3 computes some examples. In [1], the relative Macdonald polynomials $E_\mu^\alpha = (\text{const}) T_\mu E_\mu$ of this paper are called “permuted basement Macdonald polynomials”. These “$T_\mu$ shifted Macdonald polynomials” are useful for all root systems and have an alcove walks formula [18, Theorem 2.2]. In the general root system setting, the notion of a “basement” has a different flavor...
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As explained in Macdonald’s book [16], the $n$-periodic permutation $u_\mu$ defined in Section 2 is a critical ingredient for the understanding of the combinatorics of Macdonald polynomials and their construction by intertwiners $\tau^\gamma_i$. Proposition 2.2 provides a favorite reduced word for $u_\mu$ and determines its inversions. The inversions of $u_\mu$ provide the “arms” and “legs” that appear in [10] (denoted $N\text{arm}_\mu$ and $N\text{leg}_\mu$ in this paper), and this observation connects those statistics with the roots of the affine root system for type $GL_n$. Proposition 4.3 derives a box-by-box recursion for computing Macdonald polynomials and Remark 4.4 shows that the statistic that falls out of this derivation (in terms of a comparison of lengths of permutations) counts the coinversion triples that are used in [10]. This observation completes the interpretation of the statistics in the nonattacking fillings formula in terms of the Weyl group and the root system. Let us highlight that using the box-by-box recursion to compute Macdonald polynomials is equivalent to using a special reduced word for the $n$-periodic permutation $u_\mu$, the box greedy reduced word for $u_\mu$.

The proof of the alcove walks formula is obtained by iterating the step-by-step recursion for the relative Macdonald polynomials $E_z^\mu$. The proof of the nonattacking fillings formula is obtained by iterating the box-by-box recursion for the relative Macdonald polynomials $E_z^\mu$. Except for the effort to normalize the $E_z^\mu$ so that $x^\mu$ has coefficient 1, the proof of the step-by-step recursion does not differ from the proof of [18, Theorem 2.2]. The proof of the box-by-box recursion is, at its core, the same as [13, Proposition 4.1] (Lenart’s main results are stated for symmetric Macdonald polynomials $P_\lambda$ where $\lambda$ has distinct parts, we treat the general relative case $E_z^\mu$). Our reformulation and proof highlights the role of the intertwiners and the connection to the affine root system and pinpoints exactly which intertwiners get “compressed”.

Section 5 provides a Type $GL_n$ specific exposition, from scratch, of the double affine Hecke algebra and its use for defining and studying Macdonald polynomials. In [9], a supplement to this paper, we provide examples and further observations.

A small warning: Even though they all have a Type A root system, type $SL_n$ Macdonald polynomials, type $PGL_n$ Macdonald polynomials and type $GL_n$ Macdonald polynomials are all different (though the relationship is well known and not difficult). We should stress that this paper is specific to the $GL_n$-case and some results of this paper do not hold for Type $SL_n$ or type $PGL_n$ unless properly modified.

1. BOXES, ALCOVE WALKS AND NONATTACKING FILLINGS

The goal of this section is to state the main results: the alcove walks formula and the compression map $\psi$ which relates them. We begin by setting up the combinatorics of boxes, diagrams, alcove walks and nonattacking fillings. Then, after specifying the weights attached to alcove walks and to nonattacking fillings we state the alcove walks formula and the nonattacking fillings formula for Macdonald polynomials as weighted sums of alcove walks and nonattacking fillings, respectively.

1.1. BOXES. Fix $n \in \mathbb{Z}_{>0}$. A box is an element of $\{1, \ldots, n\} \times \mathbb{Z}_{\geq 0}$ so that

$$\{\text{boxes}\} = \{(i, j) \mid i \in \{1, \ldots, n\}, j \in \mathbb{Z}_{\geq 0}\}.$$  

To conform to [14, p.2], we draw the box $(i, j)$ as a square in row $i$ and column $j$ using the same coordinates as are usually used for matrices.

(1) The cylindrical coordinate of the box $(i, j)$ is the number $i + nj$.  

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The *basement* is the set \( \{(i,0) \mid i \in \{1, \ldots, n\} \} \), so that the basement is the collection of boxes in the 0th column. Pictorially,

\[
\begin{array}{cccccccc}
1 (1,0) & 6 (1,1) & 11 (1,2) & 16 (1,3) & 21 (1,4) & \cdots \\
2 (2,0) & 7 (2,1) & 12 (2,2) & 17 (2,3) & 22 (2,4) & \cdots \\
3 (3,0) & 8 (3,1) & 13 (3,2) & 18 (3,3) & 23 (3,4) & \cdots \\
4 (4,0) & 9 (4,1) & 14 (4,2) & 19 (4,3) & 24 (4,4) & \cdots \\
5 (5,0) & 10 (5,1) & 15 (5,2) & 20 (5,3) & 25 (5,4) & \cdots \\
\end{array}
\]

with box \((i,j)\) numbered \(i+nj\).

Let \( \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n \) be an \( n \)-tuple of nonnegative integers. The *diagram of \( \mu \)* is the set \( \text{dg}(\mu) \) of boxes with \( \mu_i \) boxes in row \( i \) and the *diagram of \( \mu \) with basement \( \text{dg}(\mu) \) includes the extra boxes \((i,0)\) for \( i \in \{1, \ldots, n\} \):

\[
dg(\mu) = \{(i,j) \mid i \in \{1, \ldots, n\} \text{ and } j \in \{1, \ldots, \mu_i\}\} \quad \text{and} \quad \text{dg}(\mu) = \{(i,j) \mid i \in \{1, \ldots, n\} \text{ and } j \in \{0,1, \ldots, \mu_i\}\}.
\]

It is often convenient to abuse notation and identify \( \mu, \text{dg}(\mu) \) and \( \text{dg}(\mu) \) (because these are just different ways of viewing the sequence \((\mu_1, \ldots, \mu_n)\)). For example, if \( \mu = (0,4,1,5,4) \) then

\[
dg(\mu) = \begin{array}{cccc}
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\end{array}
\quad \text{and} \quad \text{dg}(\mu) = \begin{array}{cccc}
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\end{array}
\]

1.2. Alcove walks and nonattacking fillings. Let \( \mu \in \mathbb{Z}_{\geq 0}^n \). Using cylindrical coordinates for boxes as specified (1), define, for a box \( b \in \text{dg}(\mu) \),

\[
\begin{align*}
\text{(2)} \quad \text{attack}_\mu(b) &= \{b-1, \ldots, b-n+1\} \cap \text{dg}(\mu), \\
\text{(3)} \quad \text{Nleg}_\mu(b) &= (b+n\mathbb{Z}_{>0}) \cap \text{dg}(\mu) \quad \text{and} \\
\text{(4)} \quad \text{Narm}_\mu(b) &= \{a \in \text{attack}_\mu(b) \mid \#\text{Nleg}_\mu(a) \leq \#\text{Nleg}_\mu(b)\},
\end{align*}
\]

where \( \#\text{Nleg}_\mu(a) \) denotes the number of elements of \( \text{Nleg}_\mu(a) \). For example, with \( \mu = (3,0,5,1,4,3,4) \) and \( b = (5,2) \), which has cylindrical coordinate \( b = 5+7 \cdot 2 = 19 \) the sets \( \text{attack}_\mu(b) \), \( \text{Narm}_\mu(b) \) and \( \text{Nleg}_\mu(b) \) are pictured as

\[
\begin{align*}
\text{attack}_\mu(b) &= \begin{array}{cccc}
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\end{array} \\
\text{Narm}_\mu(b) &= \begin{array}{cccc}
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\text{x} & \text{x} & \text{x} & \text{x} \\
\text{x} & \text{x} & \text{x} & \text{x} \\
\end{array} \\
\text{Nleg}_\mu(b) &= \begin{array}{cccc}
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\text{b} & \text{f} & \text{f} & \text{f} \\
\text{f} & \text{f} & \text{f} & \text{f} \\
\text{f} & \text{f} & \text{f} & \text{f} \\
\text{f} & \text{f} & \text{f} & \text{f} \\
\end{array}
\end{align*}
\]
Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{>0}$ and let $u_\mu$ be the $n$-periodic permutation defined in (33). Letting $u_\mu(i, j) = n - 1 - \#\text{attack}_\mu(i, j)$, the box-greedy reduced word for $u_\mu$ is

$$u_\mu^\square = \prod_{\text{boxes } (i, j) \text{ in } \text{diag}(\mu)} (s_{u_\mu(i, j)} \cdots s_2 s_1 \pi).$$

For the purposes of this section it is only necessary to recognize $u_\mu^\square$ as an abstract word in symbols $s_1, \ldots, s_{n-1}, \pi$. For an example, if $\mu = (0, 4, 1, 5, 4)$ then the box-greedy reduced word for $u_\mu$ is

$$u_\mu^\square = (s_1 \pi)^3(s_2 s_1 \pi)^3(s_3 s_2 s_1 \pi).$$

(The reduced word is a product of the boxes read in increasing order by cylindrical coordinate).

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{>0}$ and $z \in S_n$. Let $\vec{u}_\mu = w_1 w_2 \cdots w_\ell$ be a reduced word for $u_\mu$ so that $w_1, \ldots, w_\ell$ are the factors of $\vec{u}_\mu$ (a good choice is to let $\vec{u}_\mu = u_\mu^\square$).

An alcove walk of type $(z, \vec{u}_\mu)$ is a sequence $p = (p_0, p_1, \ldots, p_\ell)$ of elements of $W$ (see (19), but, as for $u_\mu^\square$, it is sensible just to view the $p_k$ as words in the symbols $s_1, \ldots, s_{n-1}, \pi$) such that

$$p_0 = z, \quad p_k = p_{k-1} \pi \text{ if } w_k = \pi, \quad \text{and} \quad p_k \in \{p_{k-1}, p_{k-1} w_k\} \text{ if } w_k \neq \pi.$$ 

In other words, an alcove walk of type $(z, \vec{u}_\mu)$ is equivalent to choosing a subset of the $s_i$ factors in $\vec{u}_\mu$ to cross out. For example,

$$p = (p_0, p_1, \ldots, p_\ell) = (z, z, z s_\pi, z s_1, z s_1 s_\pi, z s_1 s_\pi s_1, z s_1 s_\pi s_1 s_\pi, z s_1 s_\pi s_1 s_\pi s_1, \ldots)$$

(there is a repeat entry in $p$ each time there is an $s_i$ crossed out in $P$). In this example, there are $5 + 2 \cdot 8 + 3 = 24$ factors of the form $s_i$ in $u_\mu^\square$ and so there are a total of $2^{24}$ alcove walks of type $(z, u_\mu^\square)$ (for any fixed permutation $z \in S_n$).

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{>0}$ and $z \in S_n$. A nonattacking filling for $(z, \mu)$ is $T$: $\text{diag}(\mu) \rightarrow \{1, \ldots, n\}$ such that
(a) $T(i,0) = z(i)$ for $i \in \{1, \ldots, n\}$ and
(b) if $b \in dg(\mu)$ and $a \in \text{attack}_\mu(b)$ then $T(a) \neq T(b)$.

For example,

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 1 & 1 & 2 \\
3 & 4 & 4 & 4 \\
5 & 2 & 3 & 3 \\
\end{array}
\]

is a nonattacking filling for $(z, \mu)$

with $z = \text{id} \in S_5$ and $\mu = (0, 4, 1, 5, 4)$.

Let $b$ be a box in $\mu$. Starting at $b$ read, in succession, in reverse order by cylindrical coordinate, the entries from $T$ in (earlier) boxes, skipping values that have already been encountered. This process produces, for each box $b \in dg(\mu)$, a permutation $z_T(b)$ in $S_n$. For example, with $T$ as in (8), box $(4, 3)$ in $T$ (row 4, column 3) produces the permutation (in one line notation)

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 1 & 1 & 2 \\
3 & 3 & 4 & 4 \\
5 & 2 & 3 & 3 \\
\end{array}
\]

and doing this for all boxes in $T$ produces

\[
\begin{array}{cccc}
(23451) & (23451) & (35421) & (41532) \\
(24513) & (25134) & (23514) & (34215) & (15324) & (15234) \end{array}
\]

The sequence

\[
z_T = (z_T(b) \mid b \in dg(\mu)) \text{ is the permutation sequence of } T.
\]

Let $c_n = s_1 \cdots s_{n-1}$, an $n$-cycle in $S_n$. If $b = (i,j)$ is a box in $dg(\mu)$, the permutation $z_T(b')$ in the next box of $z_T$ (by cylindrical coordinate) is

\[
z_T(b') = z_T(b)s_r \cdots s_2s_1c_n, \quad \text{where } r \in \{0, 1, \ldots, u_\mu(b')\},
\]

and $s_r \cdots s_2s_1\pi$ is the entry in box $b'$ of the alcove walk $\varphi(T)$ corresponding to the nonattacking filling $T$. (If this construction of the permutation sequence feels ad hoc, the sentence before Lemma 4.2 may help to provide some insight into its source.)

For example, for $z_T$ as in (9), and with $z = (12345)$ the permutation in the basement of $T$, then

\[
\begin{array}{ccc}
(23451) &=& z_T(2, 1) = zc_n, \\
(24513) &=& z_T(3, 1) = z_T(2,1)s_1c_n, \\
(25134) &=& z_T(4, 1) = z_T(3,1)s_1c_n, \\
(21345) &=& z_T(5, 1) = z_T(4,1)s_1c_n,
\end{array}
\begin{array}{ccc}
(23451) &=& z_T(2, 2) = z_T(5,1)s_1c_n, \\
(25134) &=& z_T(4, 2) = z_T(2,2)s_2s_1c_n, \\
(21345) &=& z_T(5, 2) = z_T(4,2)c_n,
\end{array}
\]

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and so forth all the way to the last box of \( \mu \). Keeping track only of the factor which is the difference between successive boxes produces the alcove walk

\[
\varphi(T) = \begin{array}{cccc}
\times \pi & s_1 \pi & s_2 s_1 \pi & s_3 s_1 \pi \\
 s_1 \pi & s_2 s_1 \pi & s_3 s_1 \pi & s_3 s_2 s_1 \pi \\
 s_1 \pi & s_2 s_1 \pi & s_3 s_1 \pi & s_3 s_1 \pi \\
 s_1 \pi & s_2 s_1 \pi & s_3 s_1 \pi & s_3 s_1 \pi \\
\end{array}
\]

In summary, letting

\[\begin{align*}
\text{AW}_\mu^z &= \{ \text{alcove walks of type } (z, u_\square^\mu) \} \\
\text{NAF}_\mu^z &= \{ \text{nonattacking fillings for } (z, \mu) \}
\end{align*}\]

we have produced an injective map

\[\varphi: \text{NAF}_\mu^z \rightarrow \text{AW}_\mu^z.\]

By (11), the image of \( \varphi \) consists exactly of the alcove walks such that, each box \( b = (i, j) \in d g(\mu) \) contains a suffix \( s_r \cdots s_1 \pi \) of the entry \( s_{u_\mu(i,j)} \cdots s_1 \pi \) in box \( b \) in \( u_\mu^\square \). The compression map is the function

\[\psi: \text{AW}_\mu^z \rightarrow \text{NAF}_\mu^z\]

which, in each box, forces every \( s_i \) factor before the last crossed out factor in that box also to be crossed out. For example, if \( P \) is the alcove walk in (7) then

\[
\psi(P) = \begin{array}{cccc}
\times \pi & s_1 \pi & s_2 s_1 \pi & s_3 s_1 \pi \\
 s_1 \pi & s_2 s_1 \pi & s_3 s_1 \pi & s_3 s_1 \pi \\
 s_1 \pi & s_2 s_1 \pi & s_3 s_1 \pi & s_3 s_1 \pi \\
 s_1 \pi & s_2 s_1 \pi & s_3 s_1 \pi & s_3 s_1 \pi \\
\end{array}
\]

Identifying \( \text{NAF}_\mu^z \) with \( \text{im } \varphi \), then \( \psi \circ \varphi: \text{NAF}_\mu^z \rightarrow \text{NAF}_\mu^z \) is the identity map on nonattacking fillings.

### 1.3. Formulas for the relative Macdonald polynomials \( E_\mu^z \)

In this subsection we state the alcove walks formula and the nonattacking fillings formula for \( E_\mu^z \). The proofs are by the step-by-step recursion (Proposition 4.1) and the box-by-box recursion (Proposition 4.3), respectively. The statistics \( \text{sh}(-\beta^\vee_k), \text{ht}(-\beta^\vee_k), \text{norm}(p_k) \) on alcove walks which are introduced below are read off of the step-by-step recursion, Proposition 4.1. Similarly, the statistics \( \# \text{Nleg}_\mu^\mu(b)+1, \# \text{Narm}_\mu^\mu(b)+1, \) and \( \# \text{bwn}_T(b) \) on nonattacking fillings which are introduced below are read off the box-by-box recursion, Proposition 4.3, and Remark 4.4.

Equations (13)–(15) use the notations of Section 2.3 so that \( W \) is the group of \( n \)-periodic permutations defined in (19) the root sequence for \( \vec{u}_\mu \) corresponds to the inversions of \( u_\mu \) as in (31) and the shift and height of an affine coroot are as given in (27).

Let \( \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n \) and \( z \in S_n \). Let \( s_x = \pi \) and let \( \vec{u}_\mu = s_i \cdots s_t \) be a reduced word for \( u_\mu \) (a good choice is to let \( \vec{u}_\mu = u_\square^\mu \)). An alcove walk of type \((z, \vec{u}_\mu)\) is

\[\text{a sequence } p = (p_0, p_1, \ldots, p_r) \text{ of elements of } W \text{ such that}\]
$p_0 = z$; if $s_{i_k} = \pi$ then $p_k = p_{k-1}\pi$; and if $s_{i_k} \neq \pi$ then $p_k \in \{p_{k-1}, p_{k-1}s_{i_k}\}$. The permutation sequence of $p$ is the sequence of elements of $S_n$,

\begin{equation}
\bar{z}_p = (z_0, \ldots, z_r), \text{ given by } z_j = \overline{b}_j,
\end{equation}

where $\overline{-} : W \to S_n$ is the homomorphism given by $\overline{t_v} = v$ (see (24)). The root sequence for $\mu$ is

the sequence $\{\beta_j^\vee | i_k \neq \pi\}$ given by $\beta_j^\vee = s_{i_r}^{-1}s_{i_{r-1}}^{-1}\cdots s_{i_{k+1}}^{-1}\alpha_j^\vee$.

Define

\begin{equation}
\text{ht}(\varepsilon_j^\vee - \varepsilon_j^\vee - \ell K) = j - i, \quad \text{and} \quad \text{sh}(\varepsilon_j^\vee - \varepsilon_j^\vee - \ell K) = \ell.
\end{equation}

For $k \in \{1, \ldots, r\}$ with $p_{k-1} = p_k$ define

\begin{equation}
\text{wt}(p_k) = \left\{ \begin{array}{ll}
1 - t - t^{\text{norm}(p_k)} & \text{if } p_k = p_{k-1} \text{ and } p_{k-1}s_{i_k} < p_{k-1}, \\
(1 - t)q^{\text{ht}(\beta_j^\vee)} & \text{if } p_k = p_{k-1} \text{ and } p_{k-1}s_{i_k} > p_{k-1}, \\
1 & \text{if } p_k = p_{k-1}s_{i_k}, \\
T_{x_{k-1}}(1) & \text{if } p_k = p_{k-1}\pi,
\end{array} \right.
\end{equation}

and define the weight of $p$ by

\begin{equation}
\text{wt}(p) = \prod_{k=1}^t \text{wt}_p(k), \quad \text{a product over the steps of } p.
\end{equation}

Let $\mu \in \mathbb{Z}_{\geq 0}$ and $z \in S_n$ and let $T$ be a nonattacking filling of shape $(z, \mu)$. For $b \in \text{dg}(\mu)$ let

\begin{equation}
\text{bwn}_{\text{mT}}(b) = \left\{ a \in \text{Narm}_\mu(b) \mid T(b-n) > T(a) > T(b) \right\}.
\end{equation}

The weight of $b$ in $T$ is

\begin{equation}
\text{wt}_T(b) = \left\{ \begin{array}{ll}
1 - t - t^{\text{bwn}_T(b)} & \text{if } T(b-n) > T(b), \\
(1 - t)q^{\text{Narm}_\mu(b)+1} & \text{if } T(b-n) < T(a) < T(b), \\
x_T(b) & \text{if } T(b-n) = T(b),
\end{array} \right.
\end{equation}

and the weight of $T$ is

\begin{equation}
\text{wt}(T) = \prod_{b \in \text{dg}(\mu)} \text{wt}_T(b), \quad \text{a product over the boxes of } T.
\end{equation}

The following theorem summarizes (and slightly generalizes) [18, Theorem 3.1], [1, Def. 5 and Prop. 6] and [10, Theorem 3.5.1].

**Theorem 1.1.** Let $\mu \in \mathbb{Z}_{\geq 0}$ and $z \in S_n$. Let $E^z_\mu$ be the relative (or permuted basement) nonsymmetric Macdonald polynomial defined in (43). Let $u_\mu$ be a reduced word for $u_\mu$ and let

\begin{align*}
\text{AW}^z_\mu &= \{ \text{alcove walks of type } (z, \bar{u}_\mu) \} \quad \text{and} \\
\text{NAF}^z_\mu &= \{ \text{nonattacking fillings for } (z, \mu) \}
\end{align*}
(a) Alcove walks formula: $E^z_\mu = \sum_{p \in AW^z_\mu} \text{wt}(p)$.

(b) Nonattacking fillings formula: $E^z_\mu = \sum_{T \in NAF^z_\mu} \text{wt}(T)$.

Proof. (a) is obtained by successive applications of the step-by-step recursion (Proposition 4.1), and (b) is obtained by successive applications of the box-by-box recursion (Proposition 4.3). The weight of each box $\text{wt}_T(b)$ comes from the coefficient of the corresponding term in Proposition 4.3 and, by Remark 4.4 and Remark 2.3, these weights can be stated in the form (16) and (17).

The following is a corollary of Lemma 4.2 (specifically, the step in line (52)). Lemma 4.2 is a version of [13, Proposition 4.1], which forms the core of the proof of the box-by-box recursion Proposition 4.3.

**Corollary 1.2.** Let $\mu \in \mathbb{Z}_{\geq 0}^n$ and $z \in S_n$. Let $\psi: AW^z_\mu \rightarrow NAF^z_\mu$ be the compression function defined in (12) and let $T \in NAF^z_\mu$. Then

$$\text{wt}(T) = \sum_{p \in \psi^{-1}(T)} \text{wt}(p).$$

The following example illustrates the proof of the nonattacking fillings formula by iterating the box-by-box recursion to produce the nonattacking fillings expansion of the Macdonald polynomial $E_{(2,2,1,1,0,0)} = E^{(123456)}_{(2,2,1,1,0,0)}$. The first four applications of Proposition 4.3 give

$$E^{(123456)}_{(2,2,1,1,0,0)} = x_1E^{(234561)}_{(2,1,1,0,0,1)} = x_1x_2E^{(345612)}_{(1,1,0,0,0,1,1)} = x_1x_2x_3E^{(456123)}_{(1,0,0,1,1,0)} = x_1x_2x_3x_4E^{(561234)}_{(0,0,1,1,0,0)}.$$

The fifth box is produced by applying Proposition 4.3 to $E^{(561234)}_{(0,0,1,1,0,0)}$ to obtain

$$E^{(561234)}_{(0,0,1,1,0,0)} = x_1E^{(562341)}_{(0,0,1,0,0,0)} + \left(\frac{1-t}{1-qt^{6-2}}\right)(qx_6E^{(512346)}_{(0,0,1,0,0,0)} + qx_5E^{(5612345)}_{(0,0,1,0,0,0)}).$$

The last box is obtained by applying Proposition 4.3 to each of the terms $E^{(562341)}_{(0,0,1,0,0,0)}$, $E^{(512346)}_{(0,0,1,0,0,0)}$, and $E^{(612345)}_{(0,0,1,0,0,0)}$ which have been generated in the previous step:

$$E^{(562341)}_{(0,0,1,0,0,0)} = x_2 + \left(\frac{1-t}{1-qt^{6-2}}\right)(qt x_6 + qx_5),$$

$$E^{(512346)}_{(0,0,1,0,0,0)} = x_2 + \left(\frac{1-t}{1-qt^{6-2}}\right)(x_1 + qx_5),$$

$$E^{(612345)}_{(0,0,1,0,0,0)} = x_2 + \left(\frac{1-t}{1-qt^{6-2}}\right)(x_1 + qx_6), \quad \text{since } E^z_{(0,0,0,0,0,0)} = 1 \text{ for } z \in S_n.$$

Compiling these produces an expansion of $E_{(2,2,1,1,0,0)}$ with 9 terms,

$$E^{(123456)}_{(2,2,1,1,0,0)} = x_1x_2x_3x_4E^{(561234)}_{(0,0,1,1,0,0)} + \left(\frac{1-t}{1-qt^{6-2}}\right)(qx_6E^{(512346)}_{(0,0,1,0,0,0)} + qx_5E^{(612345)}_{(0,0,1,0,0,0)}).$$
These 9 terms are exactly the 9 nonattacking fillings of $\mu = (2, 2, 1, 1, 0, 0)$ as follows

$$
\begin{align*}
&1\,1\,1 & 1\,1\,1 \\
&2\,2\,2 & 2\,2\,6 \\
&3\,3 & 3\,3 \\
&4\,4 & 4\,4 \\
&5 & 5 \\
&6 & 6 \\
\end{align*}
$$

$$
\begin{align*}
x_1x_2x_3x_4x_1x_2 & \quad x_1x_2x_3x_1\left(\frac{1-t}{1-qt^2}\right)qx_6 \\
&1\,1\,1 & 1\,6 \\
&2\,2\,5 & 2\,2\,6 \\
&3\,3 & 3\,3 \\
&4\,4 & 4\,4 \\
&5 & 5 \\
&6 & 6 \\
\end{align*}
$$

$$
\begin{align*}
x_1x_2x_3x_4x_1\left(\frac{1-t}{1-qt^2}\right)qx_5 & \quad x_1x_2x_3x_4\left(\frac{1-t}{1-qt^2}\right)qx_6x_2 \\
&1\,6 & 1\,6 \\
&2\,1 & 2\,5 \\
&3\,3 & 3\,3 \\
&4\,4 & 4\,4 \\
&5 & 5 \\
&6 & 6 \\
\end{align*}
$$

$$
\begin{align*}
x_1x_2x_3x_4\left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3x_4\left(\frac{1-t}{1-qt^2}\right)x_1 & \quad x_1x_2x_3x_4\left(\frac{1-t}{1-qt^2}\right)x_1 \\
&1\,5 & 1\,5 \\
&2\,2\,2 & 2\,2\,1 \\
&3\,3 & 3\,3 \\
&4\,4 & 4\,4 \\
&5 & 5 \\
&6 & 6 \\
\end{align*}
$$

$$
\begin{align*}
x_1x_2x_3x_4\left(\frac{1-t}{1-qt^2}\right)x_1x_2x_3x_4\left(\frac{1-t}{1-qt^2}\right)x_1 \\
&1\,5 & 1\,5 \\
&2\,6 & 2\,6 \\
&3\,3 & 3\,3 \\
&4\,4 & 4\,4 \\
&5 & 5 \\
&6 & 6 \\
\end{align*}
$$

In this table, the weight $\text{wt}(T)$ of the nonattacking filling is shown directly below the filling. These are exactly the weights produced by iterating the box-by-box recursion.
2. The affine Weyl group and the element $u_\mu$

The underlying permutation combinatorics that controls Macdonald polynomials is that of $n$-periodic permutations. In this section we define the group of $n$-periodic permutations (the affine Weyl group), and establish notations and facts about inversions and lengths of $n$-periodic permutations. At the end of this section we introduce the special $n$-periodic permutation $u_\mu$, which is used for the construction of the Macdonald polynomial $E_\mu$. Proposition 2.2 provides a favorite reduced word for $u_\mu$ (the box-greedy reduced word) and determines the inversions of $u_\mu$.

2.1. The finite Weyl group $W_{\text{fin}}$ and the affine Weyl group $W$. Let $n \in \mathbb{Z}_{>1}$. The finite Weyl group is

$$W_{\text{fin}} = S_n,$$

the symmetric group of bijections $v : \{1, \ldots, n\} \to \{1, \ldots, n\}$ with operation of composition of functions. The type $GL_n$ affine Weyl group $W$ is the group of $n$-periodic permutations $w : \mathbb{Z} \to \mathbb{Z}$ i.e.,

$$w(i + n) = w(i) + n.$$

Any $n$-periodic permutation $w$ is determined by its values $w(1), \ldots, w(n)$. Using $w(i + n) = w(i) + n$, any permutation $v : \{1, \ldots, n\} \to \{1, \ldots, n\}$ in $S_n$ extends to an $n$-periodic permutation in $W$, and so $S_n \subseteq W$.

Define $\pi \in W$ by

$$\pi(i) = i + 1, \quad \text{for } i \in \mathbb{Z}.$$

Define $s_0, s_1, \ldots, s_{n-1} \in W$ by

$$s_i(i) = i + 1, \quad s_i(i + 1) = i, \quad \text{and } s_i(j) = j \quad \text{for } j \in \{0, 1, \ldots, i - 1, i + 2, \ldots, n - 1\}.$$

The finite Weyl group $S_n$ is the subgroup of $W$ generated by $s_1, \ldots, s_{n-1}$.

For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ define $t_\mu \in W$ by

$$t_\mu(1) = 1 + n\mu_1, \quad t_\mu(2) = 2 + n\mu_2, \quad \ldots, \quad t_\mu(n) = n + n\mu_n.$$

Then

$$W = \{t_\mu v \mid \mu \in \mathbb{Z}^n, v \in S_n\} \quad \text{with} \quad vt_\mu = t_{\mu v} \text{ for } v \in S_n \text{ and } \mu \in \mathbb{Z}^n.$$

The map

$$\overline{\cdot} : W \to S_n \quad \text{given by} \quad \overline{t_\mu v} = v, \quad \text{for } \mu \in \mathbb{Z}^n \text{ and } v \in S_n,$$

is a surjective group homomorphism. If $v \in S_n$ and $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ then $(t_\mu v)(i) = v(i) + n\mu_{v(i)}$ for $i \in \{1, \ldots, n\}$. The two-line notation for $w = t_\mu v$ is

$$t_\mu v = \begin{pmatrix} 1 & 2 & \cdots & n \\ v(1) + n\mu_{v(1)} & v(2) + n\mu_{v(2)} & \cdots & v(n) + n\mu_{v(n)} \end{pmatrix}.$$

Another useful notation for $n$-periodic permutations is an extended one-line notation: If $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ and $v \in S_n$ write

$$t_\mu v = ((\mu_1)_{v^{-1}(1)}, (\mu_2)_{v^{-1}(2)}, \ldots, (\mu_n)_{v^{-1}(n)}).$$

For example, if $\mu = (0, 4, 5, 1, 4)$ with $n = 5$ and $v = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix}$ then

$$t_\mu v = (0_1, 4_3, 5_5, 1_2, 4_4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 + n & 2 + 4n & 5 + 4n & 3 + 5n \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 9 & 22 & 25 & 28 \end{pmatrix}.$$
2.2. Inversions of \( n \)-periodic permutations. Let \( w \in W \) be an \( n \)-periodic permutation. An inversion of \( w \) is

\[(j, k) \quad \text{with} \quad j < k \text{ and } w(j) > w(k).
\]

If \((j, k)\) is an inversion of \( w \) then \((j + \ell n, k + \ell n)\) is an inversion of \( w \) for \( \ell \in \mathbb{Z} \) and so it is sensible to assume \( j \in \{1, \ldots, n\} \) and define

\[
\text{Inv}(w) = \{(j, k) \mid j \in \{1, \ldots, n\}, k \in \mathbb{Z}, j < k \text{ and } w(j) > w(k)\}.
\]

The number of elements of Inv\((w)\),

\[
\ell(w) = \#\text{Inv}(w),
\]

is the length of \( w \).

For notational convenience when working with reduced words, let \( s_\pi = \pi \). Then

\[
\ell(s_\pi) = \ell(\pi) = 0 \quad \text{and} \quad \ell(s_i) = 1 \quad \text{for} \quad i \in \{1, \ldots, n-1\}.
\]

Let \( w \in W \). A reduced word for \( w \) is an expression of \( w \) as a product of \( s_1, \ldots, s_{n-1} \) and \( s_\pi \),

\[
w = s_{i_1} \ldots s_{i_\ell} \quad \text{such that} \quad \ell(w) = \ell(s_{i_1}) + \cdots + \ell(s_{i_\ell}),
\]

with \( i_1, \ldots, i_\ell \in \{1, \ldots, n-1, \pi\} \).

2.3. Affine coroots and the root sequence of a reduced word. Let \( \mathfrak{a}_\mathbb{Z} \) be the set of \( \mathbb{Z} \)-linear combinations of symbols \( \varepsilon_1^\vee, \ldots, \varepsilon_n^\vee, K \). The affine coroots are

\[
\varepsilon_i^\vee - \varepsilon_j^\vee + \ell K \quad \text{with} \quad i, j \in \{1, \ldots, n\} \text{ and } i \neq j \text{ and } \ell \in \mathbb{Z}
\]

(in the context of the corresponding affine Lie algebra the symbol \( K \) is the central element). The shift and height of an affine coroot are given by

\[
\text{sh}(\varepsilon_i^\vee - \varepsilon_j^\vee + \ell K) = -\ell \quad \text{and} \quad \text{ht}(\varepsilon_i^\vee - \varepsilon_j^\vee + \ell K) = j - i.
\]

The affine coroot corresponding to an inversion

\[(i, k) = (i, j + \ell n) \quad \text{with} \quad i, j \in \{1, \ldots, n\} \text{ and } \ell \in \mathbb{Z}, \quad \text{is} \quad \beta_i^\vee = \varepsilon_i^\vee - \varepsilon_j^\vee + \ell K.
\]

Define a \( \mathbb{Z} \)-linear action of the affine Weyl group \( W \) on \( \mathfrak{a}_\mathbb{Z} \) by

\[
\pi^{-1} \varepsilon_i^\vee = \varepsilon_i^\vee + K, \quad \pi^{-1} \varepsilon_i^\vee = \varepsilon_{i-1}^\vee \quad \text{for} \quad i \in \{2, \ldots, n\},
\]

\[
s_i \varepsilon_i^\vee = \varepsilon_{i+1}^\vee, \quad s_i \varepsilon_i^\vee = \varepsilon_i^\vee, \quad s_i \varepsilon_j = \varepsilon_j^\vee \quad \text{if} \quad j \in \{1, \ldots, n\} \text{ and } j \notin \{i, i+1\}.
\]

If \( \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \) then \( t_\mu \varepsilon_i^\vee = \varepsilon_i^\vee - \mu_i K \). This action matches the action from the double affine Hecke algebra results in Proposition 5.5 and equation (75).

Let

\[
\alpha_0^\vee = \varepsilon_n^\vee - \varepsilon_1^\vee + K, \quad \text{and} \quad \alpha_i^\vee = \varepsilon_i^\vee - \varepsilon_{i+1}^\vee \quad \text{for} \quad i \in \{1, \ldots, n-1\}.
\]

Let \( w \in W \) and let \( w = s_{i_1} \cdots s_{i_\ell} \) be a reduced word for \( w \). The root sequence of the reduced word \( w = s_{i_1} \cdots s_{i_\ell} \) (recall that \( s_\pi = \pi \)) is

\[(30) \quad \text{the sequence } (\beta_k^\vee \mid k \in \{1, \ldots, \ell\} \text{ and } i_k \neq \pi) \text{ given by } \beta_k^\vee = s_k^{-1} \cdots s_{i_k+1}^{-1} \alpha_{i_k}^\vee.
\]

Then, identifying inversions with affine coroots as in (28),

\[(31) \quad \text{Inv}(w) = \{\beta_k^\vee \mid k \in \{1, \ldots, \ell\} \text{ and } k \neq \pi\}
\]

(see [16, (2.2.9)] or [3, Ch. VI §1 no. 6 Cor. 2]).
2.4. The element $u_\mu$ in the affine Weyl group. Define an action of $W$ on $\mathbb{Z}^n$ by

\begin{equation}
\pi(\mu_1, \ldots, \mu_n) = (\mu_n + 1, \mu_1, \ldots, \mu_{n-1}) \quad \text{and} \quad s_i(\mu_1, \ldots, \mu_n) = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \ldots, \mu_n),
\end{equation}

for $i \in \{1, \ldots, n-1\}$. Let $u_\mu$ be the minimal length element of $W$ such that $u_\mu(0, 0, \ldots, 0) = (\mu_1, \ldots, \mu_n)$ and define $v_\mu \in S_n$ by $u_\mu = t_\mu v_\mu^{-1}$, where $t_\mu \in W$ is as defined in (22). As noted in [16, (2.4.3)], $u_\mu$ is the minimal length element of the coset $t_\mu S_n$ in $W$ and the choice of the notation $u_\mu$ and $v_\mu$ for these elements follows that lead. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be the decreasing rearrangement of $\mu$ and let

\begin{equation}
z_\mu \in S_n \quad \text{be minimal length such that } \mu = z_\mu \lambda.
\end{equation}

The following result is the translation of [16, (2.4.1)-(2.4.5) and (2.4.14)(i) and (2.4.12)] to our current setting.

**Proposition 2.1.** Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$. Let $u_\mu$, $v_\mu$, $\lambda$ and $z_\mu$ be as defined in (33) and (34).

(a) $v_\mu$ is the minimal length element of $S_n$ such that $v_\mu \mu$ is (weakly) increasing.

(b) The permutation $v_\mu: \{1, \ldots, n\} \to \{1, \ldots, n\}$ is given by

\[ v_\mu(i) = 1 + \#\{ j \in \{1, \ldots, i - 1\} \mid \mu_j \leq \mu_i \} + \#\{ j \in \{i + 1, \ldots, n\} \mid \mu_j < \mu_i \}.
\]

(c) The $n$-periodic permutations $u_\mu: \mathbb{Z} \to \mathbb{Z}$ and $u_\mu^{-1}: \mathbb{Z} \to \mathbb{Z}$ are given by

\[ u_\mu(i) = v_\mu^{-1}(i) + n\mu_i \quad \text{and} \quad u_\mu^{-1}(i) = v_\mu(i) - n\mu_{v_\mu(i)} \quad \text{for } i \in \{1, \ldots, n\}.
\]

(d) Let $\lambda$ be the decreasing rearrangement of $\mu$. The lengths of $t_\mu$, $u_\mu$ and $v_\mu$ are given by

\[ \ell(t_\mu) = \ell(t_\lambda) = \sum_{i<j} \lambda_i - \lambda_j,
\]

\[ \ell(v_\mu) = \#\{ i < j \mid \mu_i > \mu_j \} \quad \text{and} \quad \ell(u_\mu) = \ell(t_\mu) - \ell(v_\mu).
\]

(e) Let $i \in \{1, \ldots, n-1\}$. If $\mu_i \neq \mu_{i+1}$ so that $s_i u_\mu \neq u_\mu$ then $s_i u_\mu = s_i u_\mu v_\mu$ and

\[ \ell(s_i u_\mu) = \begin{cases} \ell(u_\mu) + 1, & \text{if } \mu_i > \mu_{i+1}, \\ \ell(u_\mu) - 1, & \text{if } \mu_i < \mu_{i+1}, \end{cases}
\]

and $\ell(s_i v_\mu) = \begin{cases} \ell(v_\mu) + 1, & \text{if } \mu_i > \mu_{i+1}, \\ \ell(v_\mu) - 1, & \text{if } \mu_i < \mu_{i+1}, \end{cases}$

(f) With $\pi$ as in (20), then $u_\pi = \pi u_\mu$ and $\ell(u_\pi) = \ell(u_\mu)$ and

\[ \ell(v_\mu) - \ell(v_\pi) = (n-1) - 2(v_\mu(n) - 1).
\]

**Proof.** (c) The first formula follows from $u_\mu = t_\mu v_\mu^{-1}$ and (22). To verify the second formula:

\[ u_\mu^{-1} u_\mu(i) = u_\mu^{-1}(v_\mu^{-1}(i) + n\mu_i) = u_\mu^{-1}(v_\mu^{-1}(i)) + n\mu_i = v_\mu(v_\mu^{-1}(i)) - n\mu_{v_\mu v_\mu^{-1}(i)} + n\mu_i = i.
\]

(d) From the definition of $t_\mu$ and $\text{Inv}(w)$,

\[ \text{Inv}(t_\mu) = \left( \bigcup_{\mu_i \neq \mu_j} \{(i, j), (i, j + n), \ldots, (i, j + n(\mu_j - \mu_i - 1))\} \right) \cup \left( \bigcup_{\mu_i \leq \mu_j} \{(j, i + n), \ldots, (j, i + n(\mu_i - \mu_j))\} \right)
\]

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and so $\ell(t_\mu) = \#\Inv(t_\mu) = \sum_{i<j} |\mu_i - \mu_j|$, which gives the first statement. More generally,

$$\Inv(t_\mu v) = \left( \bigcup_{i<j, v(i) < v(j) \atop \mu(v) < \mu(j)} \{(i,j), (i,j+n), \ldots, (i,j+n(\mu(j) - \mu(i) - 1))\} \right) \cup \left( \bigcup_{i<j, v(i) > v(j) \atop \mu(v) < \mu(j)} \{(i,j), (i,j+n), \ldots, (i,j+n(\mu(j) - \mu(i)))\} \right) \cup \left( \bigcup_{i<j, v(i) > v(j) \atop \mu(v) < \mu(j)} \{(j,i+n), \ldots, (j,i+n(\mu(j) - \mu(i) - 1))\} \right).$$

The length of $t_\mu v$ is $\ell(t_\mu v) = \#\Inv(t_\mu v)$. Thus the minimal length element of the coset $t_\mu S_n$ is the element $t_\mu v_\mu^{-1}$ where, if $i < j$ then $v_\mu^{-1}(i) < v_\mu^{-1}(j)$ if $\mu(v_\mu^{-1}(i)) < \mu(v_\mu^{-1}(j))$ and $v_\mu^{-1}(i) < v_\mu^{-1}(j)$ if $\mu(v_\mu^{-1}(i)) > \mu(v_\mu^{-1}(j))$. Thus $v_\mu = v_\mu(\mu_1, \ldots, \mu_n) = (\mu^{-1}(1), \ldots, \mu^{-1}(n))$ is in weakly increasing order and $\ell(t_\mu) = \ell(u_\mu) + \ell(v_\mu)$.

(a) These now follow from the last line of the proof of (d).

(b) In order for $v_\mu$ to rearrange $\mu$ into increasing order $v_\mu$ must move the $i$th part of $\mu$ to the right of the number of parts of $\mu$ which are less than $\mu_i$, or equal to $\mu_i$ and to the left of $\mu_i$.

(f) Write $\gamma = (\mu_1, \ldots, \mu_{n-1})$. Then

$$\ell(v_\mu) = \ell(v_\gamma) + \#\{i \in \{1, \ldots, n-1\} \mid \mu_i > \mu_n\} \quad \text{and} \quad \ell(v_\mu) = \ell(v_\gamma) + \#\{i \in \{1, \ldots, n-1\} \mid \mu_i < \mu_n + 1\}, \quad \text{giving}$$

$$\ell(v_\mu) - \ell(v_\gamma) = \#\{i \in \{1, \ldots, n-1\} \mid \mu_i > \mu_n\} - \#\{i \in \{1, \ldots, n-1\} \mid \mu_i < \mu_n + 1\} = (n-1) - \#\{i \in \{1, \ldots, n-1\} \mid \mu_i \leq \mu_n\} - \#\{i \in \{1, \ldots, n-1\} \mid \mu_i \leq \mu_n\} = (n-1) - 2\mu_n(n-1),$$

where the third equality follows from the description of $v_\mu(n)$ in (b). \(\square\)

2.5. The box-greedy reduced word for $\mu$. Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ and let $u_\mu$ be as defined in (33). The box-greedy reduced word for the element $u_\mu$ is the sequence $u_\mu^{\Box}$ defined inductively by the conditions $u_\mu^{\Box}(0, \ldots, 0) = 1$ and, when $\mu_k \neq 0$,

$$u_\mu^{\Box}(0, \ldots, 0, \mu_k, \mu_{k+1}, \ldots, \mu_n) = sk^{-1} \cdots s2s1\pi u_\mu^{\Box}(0, \ldots, 0, \mu_{k+1}, \ldots, \mu_n, \mu_k-1).$$

This is the reduced word for $u_\mu$ that is used implicitly in [11, 12, 19]. Under the action in (32), the factor $sk^{-1} \cdots s2s1\pi$ which appears in (35) is an element of $W$ of minimal length which moves $(0, \ldots, 0, 0, \mu_k, \mu_{k+1}, \ldots, \mu_n)$ to a composition with one less box.

**Proposition 2.2.** For a box $(i, j) \in dg(\mu)$ (i.e. $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, \mu_i\}$) define

$$u_\mu(i, j) = \#\{i' \in \{1, \ldots, i-1\} \mid \mu_{i'} < j \leq \mu_i\}$$

$$\quad + \#\{i' \in \{i+1, \ldots, n\} \mid \mu_{i'} < j-1 < \mu_{i}\}, \quad \text{and}$$

$$R_\mu(i, j) = \begin{cases} \varepsilon_{v_\mu(i)}^\Box - \varepsilon_1^\Box + (\mu_i - j + 1)K, \ldots, \varepsilon_{v_\mu(i)}^\Box - \varepsilon_{v_\mu(i+1)}^\Box + (\mu_i - j + 1)K \end{cases},$$

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(a) The box greedy reduced word for $u_\mu$ is

$$u_\mu^{\square} = \prod_{(i,j) \in dg(\mu)} (s_{u_\mu(i,j)} \cdots s_{1\pi}),$$

where the product is over the boxes of $\mu$ in increasing cylindrical wrapping order.

(b) The inversion set of $u_\mu$ is

$$\text{Inv}(u_\mu) = \bigcup_{(i,j) \in dg(\mu)} R_\mu(i,j), \quad \text{and} \quad \ell(u_\mu) = \sum_{(i,j) \in \mu} u_\mu(i,j).$$

Proof. Let $\mu = (0, \ldots, 0, \mu_k, \ldots, \mu_n)$ and let

$$\nu = \pi^{-1} s_1 s_2 \cdots s_{k-1} \mu = (0, \ldots, 0, \mu_{k+1}, \ldots, \mu_n, \mu_k - 1).$$

From the definition of $u_\mu(i,j)$ in (36),

$$u_\mu(k,1) = k - 1,$$

$$u_\mu(i,j) = u_\nu(i-1,j) \quad \text{for} \quad i \in \{k + 1, \ldots, n\}, \quad \text{and}$$

$$u_\mu(k,j) = u_\nu(n, j - 1), \quad \text{if} \quad j \in \{2, \ldots, \mu_k\},$$

which already establishes (a). Then, using Proposition 2.1 gives $v_\mu(i) = i$ for $i \in \{1, \ldots, k - 1\}$, $v_\mu(i) = v_\nu(i-1)$ for $i \in \{k + 1, \ldots, n\}$ and $v_\mu(k) = v_\nu(n)$.

These expressions for $u_\mu(i,j)$ and $v_\mu(i)$ in terms of $u_\nu(i,j)$ and $v_\nu(i)$ establish that

$$R_\mu(i,j) = R_\nu(i-1,j), \quad \text{if} \quad i \neq k, \quad \text{and}$$

$$R_\mu(k,j) = R_\nu(n, j - 1), \quad \text{if} \quad j \in \{2, \ldots, \mu_k\}.$$

It remains to compute $R_\mu(k,1)$. Since $u_\nu^{-1} \varepsilon_1^\nu = v_\nu^{-1} t_\nu^{-1} \varepsilon_1^\nu = \varepsilon_1^\nu + v_\nu K$, then

$$R_\mu(k,1) = \{u_\nu^{-1} \varepsilon_1^\nu, \ldots, u_\nu^{-1} \varepsilon_1^\nu s_1 s_2 \cdots s_{k-2} \varepsilon_1^\nu \}$$

$$= \{u_\nu^{-1} \varepsilon_1^\nu - \varepsilon_2^\nu, \ldots, u_\nu^{-1} \varepsilon_1^\nu \varepsilon_2^\nu \varepsilon_3^\nu \}$$

$$= \{u_\nu^{-1} (v_\nu K + K), v_\nu K + K, \ldots \}$$

$$= \{v_\nu \varepsilon_1^\nu, v_\nu \varepsilon_1^\nu + v_\nu K, v_\nu \varepsilon_1^\nu + v_\nu K + K, \ldots \}$$

where the next to last equality uses $v_1 = \cdots = v_{k-1} = 0$ and $v_n = \mu_k - 1$.

\[\square\]

Remark 2.3. Relating affine roots to $\#N_\mu(b)$ and $\#N_\mu(b)$. In the derivation of box-by-box recursion for relative Macdonald polynomials (Proposition 4.3), the last root in each box in the expression of $\text{Inv}(u_\mu)$ in Proposition 2.2(b) gets picked out (this is the $d_{-\beta_1}$ and $f_{-\beta_1}$ in the proof of Lemma 4.2). More precisely, for $(i,j) \in dg(\mu)$, let $\beta_\mu(i,j)$ be the last element of $R_\mu(i,j)$ in (37):

$$\beta_\mu(i,j) = \varepsilon_1^\nu + (\mu_i - j + 1) K.$$

With the shift and height of an affine root as defined in (27), then

$$\text{sh}(-\beta_\mu(i,j)) = \#N_\mu(i,j) + 1, \quad \text{and} \quad \text{ht}(-\beta_\mu(i,j)) = \#N_\mu(i,j) + 1,$$
3. Type $GL_n$ Macdonald polynomials

In this section we define the Macdonald polynomials $E_\mu$ and provide explicit formulas for all $E_\mu$ for $\mu$ with less than 3 boxes. These examples are helpful for getting a feel for what Macdonald polynomials actually look like. Although we have hidden the double affine Hecke algebra (DAHA) from our exposition in this section, Section 5 derives, from scratch, all the formulas for the operators $T_i$ and $Y_i$ and the Macdonald polynomials $E_\mu$ which are efficiently pulled out of a hat in this section.

3.1. The polynomial representation and Cherednik-Dunkl operators. For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ let

$$x^\mu = x_{1}^{\mu_1} \cdots x_{n}^{\mu_n}.$$  

The Laurent polynomial ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ has basis $\{x^{\mu} \mid \mu \in \mathbb{Z}^n\}$ and the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ has basis $\{x^{\mu} \mid \mu \in \mathbb{Z}_{\geq 0}^n\}$, indexed by the set $\mathbb{Z}_0^n$ of compositions. The symmetric group $S_n$ acts on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $\mathbb{C}[x_1, \ldots, x_n]$ by permuting the variables $x_1, \ldots, x_n$.

The symmetric group $S_n$ acts on $\mathbb{Z}^n$ by permuting the positions of the entries so that $wx^\mu = x^\nu$ for $w \in S_n$ and $\mu \in \mathbb{Z}^n$.

Let $q, t \in \mathbb{C}^\times$. Following the notation of [14, Ch. VI (3.1)], let $T_{q^{-1}, x_n}$ be the operator on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ given by

$$(T_{q^{-1}, x_n} h)(x_1, \ldots, x_n) = h(x_1, \ldots, x_{n-1}, q^{-1} x_n).$$

For $i \in \{1, \ldots, n-1\}$ let $s_i$ be the transposition which switches $i$ and $i+1$. Define operators $T_1, T_2, \ldots, T_{n-1}, g$ and $g^\vee$ on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by

$$T_i = t^{-\frac{1}{2}}(t - \frac{t x_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i)), \quad g = s_1 s_2 \cdots s_{n-1} T_{q^{-1}, x_n} \quad \text{and} \quad g^\vee = x_1 T_1 \cdots T_{n-1}.$$  

In §5.6 we give the derivation of these operators from the type $GL_n$ double affine Hecke algebra (DAHA). Except for the factor of $t^{-\frac{1}{2}}$, $T_i$ is the operator in [2, (2.3)], which appears in the form (79) in [10, (7)]). The Cherednik-Dunkl operators are

$$Y_1 = g T_{n-1} \cdots T_1, \quad Y_2 = T_{n-1}^{-1} Y_1 T_1^{-1}, \quad \ldots, \quad Y_n = T_{n-1}^{-1} Y_{n-1} T_{n-1}^{-1}.$$  

3.2. Macdonald polynomials. Let $g^\vee$, $T_i$ and $Y_i$ be as in (39) and (40) and define

$$\tau_\pi^\vee = g^\vee, \quad \text{and} \quad \tau_i^\vee = T_i + \frac{t^{-\frac{1}{2}}(1 - t)}{1 - Y_i^{-1} Y_{i+1}} t^{-\frac{1}{2}} (1 - t)$$

for $i \in \{1, \ldots, n-1\}$.

Using the action of $s_1, \ldots, s_n, \pi$ on $\mathbb{Z}^n$ given in (32), the (nonsymmetric) Macdonald polynomials $E_\mu$, for $\mu \in \mathbb{Z}^n$, are determined by $E_0 = 1$,

$$E_{\pi \mu} = t^{|\{i \in \{1, \ldots, n\} \mid \mu_i > \mu_{i+1}\}|} \tau_\mu^\vee E_\mu, \quad \text{and} \quad E_{s_i \mu} = t^\frac{1}{2} \tau_i^\vee E_\mu \quad \text{if} \ i \in \{1, \ldots, n-1\} \text{ and } \mu_i > \mu_{i+1}.$$
Remark 3.1. The source of the strange coefficients in (42) is Proposition 2.1(e) and (f) which gives that $\frac{1}{2}(\ell(v_\mu) - \ell(v_{\nu})) = \frac{1}{2}$ and $-\frac{1}{2}(\ell(v_\mu) - \ell(v_{\nu})) = \frac{1}{2}(n-1) - (v_\mu(n) - 1) = \frac{1}{2}(n-1) - \#\{i \in \{1, \ldots, n-1\} \mid \mu_i \leq \mu_n\}$. The role of these coefficients is to force the coefficient of $x^\mu$ in $E_\mu$ to be 1.

The following theorem, the type $GL_n$ case of [4, Theorem 4.1 and Proposition 4.2], shows that the $E_\mu$ are simultaneous eigenvectors of the Cherednik-Dunkl operators. We provide a proof in Theorem 5.7 of this paper.

Theorem 3.2. Let $\mu \in \mathbb{Z}^n$ and let $v_\mu \in S_n$ be the minimal length permutation that rearranges $\mu$ into weakly increasing order. Then $E_\mu$ is the unique element of $C[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that

$$\text{if } i \in \{1, \ldots, n\} \quad \text{then} \quad Y_i E_\mu = q^{-\mu_i} t^{-(v_\mu(i) - 1) + \frac{1}{2}(n-1)} E_\mu,$$

and the coefficient of $x^\mu$ in $E_\mu$ is 1.

Let $\mu = (\mu_1, \ldots, \mu_n)$ and let $z \in S_n$. Define $T_z = T_{i_1} \cdots T_{i_r}$ if $z = s_{i_1} \cdots s_{i_r}$ is a reduced word for $z$.

(43) The relative Macdonald polynomial $E_\mu^z$ is

$$E_\mu^z = t^{-\frac{1}{2}(\ell(v_{\mu^{-1}}) - \ell(v_{\mu}))} T_z E_\mu.$$

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$.

(44) The symmetric Macdonald polynomial $P_\lambda$ is

$$P_\lambda = \sum_{\nu \in S_n \lambda} t^{\frac{1}{2}(\ell(z_{\nu}) - \ell(\nu))} T_z E_\lambda,$$

where the sum is over rearrangements $\nu$ of $\lambda$ and $z_\nu \in S_n$ is minimal length such that $\nu = z_\nu \lambda$. These definitions follow [15, Remarks after (6.8)], [16, (5.7.6), (5.7.7)], [8, Definition 4.4.2], [1, Definition 5] and [7, (2.8)] (Ferreira references private communication with Haglund). In [1], the $E_\mu^z$ are called permuted basement Macdonald polynomials.

Remark 3.3. The following properties of the $E_\mu$ are proved in Proposition 5.8:

$$E_{(\mu_n+1, \mu_1, \ldots, \mu_{n-1})} = q^{\mu_n} x_1 E_\mu(x_2, \ldots, x_{n}, q^{-1} x_1),$$

$$E_{(\mu_1+1, \ldots, \mu_{n+1})} = x_1 \cdots x_n E_{(\mu_1, \ldots, \mu_n)},$$

$$E_{(-\mu_n, \ldots, -\mu_1)}(x_1, \ldots, x_n; q, t) = E_\mu(x_1^{-1}, \ldots, x_n^{-1}; q, t).$$

Remark 3.4. In generalization of (43), one could, for any $\mu \in \mathbb{Z}^n$ and any $n$-periodic permutation $z \in W$, define $E_\mu^z = (\text{const}) T_z E_\mu$, where (const) is a constant determined by requiring the coefficient of $x^{\mu}$ in $E_\mu^z$ to be 1. A more useful alternative might be to define $E_\mu^z = X^z E_\mu = X^z T_\mu 1$ in the notation of [18, (2.26) and Theorem 2.2].

3.3. Explicit $E_\mu$ with less than three boxes. The following explicit formulas for $E_\mu$ with 1 and 2 boxes already provide enough data that one might have a chance at guessing the nonattacking fillings formula.

Proposition 3.5. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the sequence with 1 in the $i$th component.

(a) If $i \in \{1, \ldots, n\}$ then

$$E_{e_i} = x_i + \frac{(1 - t)}{(1 - q t^{n-i+1})} (x_{i-1} + \cdots + x_1).$$
(b) If $i \in \{1, \ldots, n\}$ then
\[
E_{2x_i} = x_i^2 + \left(\frac{1-t}{1-q^{2n-1}}\right) \sum_{k \in \{1, \ldots, i-1\}} x_k^2 + \left(\sum_{t \in \{i+1, \ldots, n\}} x_t x_k\right)
\]
\[
+ \left(\frac{1-t}{1-q^t}\right) \left(1 + \left(\frac{1-t}{1-q^{2n-(i-1)}}\right) q \sum_{k \in \{1, \ldots, i-1\}} x_k x_i\right)
\]
\[
+ \left(\frac{1-t}{1-q^t}\right) \left(1 - \left(\frac{1-t}{1-q^{2n-(i-1)}}\right) (1 + q) \sum_{k \in \{1, \ldots, i-1\}} x_k x_{t+1}\right)
\]
\[
+ \left(\frac{1-t}{1-q^{2n-(i-1)}}\right) q \sum_{k \in \{1, \ldots, i-1\}} x_k x_{t+1}.
\]
(c) If $j_1, j_2 \in \{1, \ldots, n\}$ with $j_1 < j_2$ then
\[
E_{x_{j_1} + x_{j_2}} = x_{j_1} x_{j_2} + \left(\frac{1-t}{1-q^{n-j_1}}\right) \sum_{k=1}^{j_1-1} x_k x_{j_1} + \left(\frac{1-t}{1-q^{n-j_2}}\right) \sum_{t=j_1+1}^{j_2-1} x_{j_1} x_t
\]
\[
+ \left(\frac{1-t}{1-q^{n-j_2}}\right) \left(1 - \frac{1-t}{1-q^{n-j_2}} \right) q \sum_{k=1}^{j_1-1} x_k x_{j_1} + \sum_{t=j_1+1}^{j_2-1} x_{j_1} x_t
\]
\[
+ \left(\frac{1-t}{1-q^{n-j_2}}\right) \left(1 - \frac{1-t}{1-q^{n-j_2}} \right) (1 + t) \sum_{k \in \{1, \ldots, j_1-1\}} x_k x_{t+1}.
\]

Proof. Using the first identity in (39), if $r \in \{1, \ldots, n\}$ then
\[
t^2T_i(x_r) = \begin{cases} x_{i+1}, & \text{if } r = i, \\ tx_i + (1-t)x_{i+1}, & \text{if } r = i + 1, \\ tx_r, & \text{otherwise.} \end{cases}
\]

Assuming $r, s \in \{1, \ldots, n\}$ with $r < s$ then
\[
t^2T_i(x_r x_s) = \begin{cases} x_r x_{i+1}, & \text{if } s = i, \\ tx_r + (1-t)x_r x_{i+1}, & \text{if } s = i + 1 \text{ and } r < i, \\ tx_r x_{i+1}, & \text{if } r = i \text{ and } s = i + 1, \\ x_{i+1} x_s, & \text{if } r = i \text{ and } s > i + 1, \\ (1-t)x_{i+1} x_s + tx_r x_s, & \text{if } r = i + 1 \text{ and } s > i + 1, \\ tx_r x_s, & \text{otherwise.} \end{cases}
\]

If $r \in \{1, \ldots, n\}$ then
\[
t^2T_i(x_r^2) = \begin{cases} x_{i+1}^2 + (1-t)x_r x_{i+1}, & \text{if } r = i, \\ tx_r^2 + (1-t)x_r^2 + (1-t)x_r x_{i+1}, & \text{if } r = i + 1, \\ tx_r^2, & \text{otherwise.} \end{cases}
\]

(a) The proof is by induction on $i$. The base case $i = 1$ is
\[
E_{c_{1}} = t^{-\frac{1}{2}(n-1)}x_{1}x_{1}^{n-1} = t^{-\frac{1}{2}(n-1)}X_{1}T_{1} \cdots T_{n-1} \cdot 1 = x_{1}.
\]

For the induction step (note that $Y_{i}^{-1} Y_{i+1} E_{c_{i}} = q^{1-q^{2n-i}} E_{c_{i}}$) and use
\[
E_{c_{i+1}} = t^2x_{i+1}Y_{i}E_{c_{i}} = \left(t^2 T_{i} + \frac{1-t}{1-q^{2n-i}}\right) E_{c_{i}}.
\]
(b) The proof is by induction on \(i\). Using part (a) and the first identity in Remark 3.3 applied to \(E_{\varepsilon_i}\),

\[
E_{2\varepsilon_i} = x_1^2 + \frac{1-t}{1-q}q(x_1x_n + \cdots + x_1x_2),
\]

and this provides the base of the induction. Then use \(Y_i^{-1}Y_{i+1}E_{2\varepsilon_i} = q^{2-\ell t^{n-1}}E_{2\varepsilon_i}\), and

\[
E_{2\varepsilon_i+1} = \left(t^2 T_i + \frac{1-t}{1-Y_i^{-1}Y_{i+1}}\right)E_{2\varepsilon_i} = \left(t^2 T_i + \frac{1-t}{1-q^{2t^{n-1}}}\right)E_{2\varepsilon_i},
\]

(c) The proof is by induction on \(j_1\). From part (a) and the first identity in Remark 3.3 applied to \(E_{\varepsilon_{j_2}-1}\),

\[
E_{\varepsilon_{j_1}+\varepsilon_{j_2}} = x_1x_{j_2} + \left(\frac{1-t}{1-q^{t^{n-(j_2-1)}}}\right)(x_1x_{j_2-1} + \cdots + x_1x_3 + x_1x_2),
\]

and this provides the base of the induction. Then use

\[
E_{\varepsilon_{j_1}+\varepsilon_{j_2}} = \left(t^2 T_{j_1-1} + \frac{1-t}{1-q^{t^{n-j_1}}}\right)E_{\varepsilon_{j_1-1}+\varepsilon_{j_2}}.
\]

4. Recursions for computing \(E_{\mu}^z\)

In this section we derive the recursions which are used to produce expansions of Macdonald polynomials in terms of monomials. These computations are extensions of the defining recursions given in (42). It will be helpful to record carefully the action of \(t^2 \tau_i^\vee\) and \(t^2 T_i\) on the Macdonald polynomials \(E_{\mu}\) as follows.

Let \(\mu \in \mathbb{Z}^n\) and, with notations as in Theorem 3.2, let

\[
a_{\mu} = q^{\mu_i-\mu_{i+1}}t^{v_i(i)-v_i(i+1)}, \quad a_{s_i \mu} = q^{\mu_i-\mu_{i+1}}t^{v_i(i+1)-v_i(i)}, \quad \text{and} \quad D_{\mu} = \frac{(1-ta_{\mu})(1-ta_{s_i \mu})}{(1-a_{\mu})(1-a_{s_i \mu})}.
\]

Assume that \(\mu_i > \mu_{i+1}\). Using the identity \(E_{s_i \mu} = t^2 \tau_i^\vee E_{\mu}\) if \(\mu_i > \mu_{i+1}\) from (42), the eigenvalue from Theorem 3.2, and (74) gives

\[
Y_i^{-1}Y_{i+1}E_{\mu} = a_{\mu}E_{\mu}, \quad E_{s_i \mu} = a_{s_i \mu}E_{\mu}, \quad t^2 \tau_i^\vee E_{s_i \mu} = D_{\mu}E_{s_i \mu}, \quad t^2 \tau_i^\vee E_{s_i \mu} = D_{\mu}E_{s_i \mu},
\]

(47) \(t^2 T_iE_{\mu} = -\frac{1-t}{1-a_{\mu}}E_{\mu} + E_{s_i \mu}\) and \(t^2 T_iE_{s_i \mu} = D_{\mu}E_{\mu} + \frac{1-t}{1-a_{s_i \mu}}E_{s_i \mu}\).

Now assume that \(\mu_i = \mu_{i+1}\). Then \(v_i(i + 1) = v_i(i) + 1\) and \(a_{\mu} = t^{-1}\) so that

(48) \(Y_i^{-1}Y_{i+1}E_{\mu} = t^{-1}E_{\mu}, \quad (t^2 \tau_i^\vee)E_{\mu} = 0, \quad \text{and} \quad (t^2 T_i)E_{\mu} = tE_{\mu}..\)

4.1. Step-by-step recursion for computing \(E_{\mu}^z\). Proposition 4.1(a) is used to reduce the number of boxes in \(\mu\) and part (b) is used to reduce the computation to decreasing \(\mu\). Iterating these steps delivers a monomial expansion of \(E_{\mu}^z\) as a weighted sum of alcove walks \(p\). The permutation sequence \(z_p\) of the alcove walk which appears in (14) is the sequence \(z_{n}, z_{n-1}, \ldots\) of permutations which arise as superscripts of the \(E_{\mu}^z\) which occur in the intermediate applications of the step-by-step recursion to obtain the monomial expansion.

**Proposition 4.1.** Let \(\mu \in \mathbb{Z}^n\) and let \(z \in S_n\). Let \(v_{\mu}\) be the minimal length element of \(S_n\) such that \(v_{\mu}\) rearranges \(\mu\) to be weakly increasing.

(a) If \(\mu_1 \neq 0\) then

\[
E_{\mu}^z = x_{z(1)}E_{(\mu_2, \ldots, \mu_n, \mu_1-1)},
\]

where \(c_n = s_1 \cdots s_{n-1}\) (an \(n\)-cycle in \(S_n\)).
(b) Let \( i \in \{1, \ldots, n-1\} \) such that \( \mu_i < \mu_{i+1} \) and let
\[
\beta^\nu = \gamma^\nu_{\nu_{i}(i+1)} - \gamma^\nu_{\nu_i(i)} + (\mu_{i+1} - \mu_i)K
\]
so that \( q^{sh(-\beta^\nu)}q^{ht(-\beta^\nu)} = q^{\mu_{i+1} - \mu_i}q_{\nu_{i}(i+1)-\nu_i}. \)
Let \( \text{norm}^z_{\mu}(i) = \frac{1}{2}(\ell(z_{\nu_{\mu}^{1-1}}) - \ell(z_{v_{\mu}^{1-1}}) - \ell(s_i)) \). Then
\[
E^z_{s_{\mu}} = \begin{cases} 
E^z_{s_{\mu}} + \frac{(1 - t)q^{sh(-\beta^\nu)}q^{ht(-\beta^\nu)}}{1 - q^{sh(-\beta^\nu)}q^{ht(-\beta^\nu)}} \text{norm}^z_{\mu}(i) E^z_{s_{\mu}}, & \text{if } z(i) < z(i + 1), \\
E^z_{s_{\mu}} + \frac{(1 - t)q^{sh(-\beta^\nu)}q^{ht(-\beta^\nu)}}{1 - q^{sh(-\beta^\nu)}q^{ht(-\beta^\nu)}} \text{norm}^z_{\mu}(i) E^z_{s_{\mu}}, & \text{if } z(i) > z(i + 1).
\end{cases}
\]

**Proof.** (a) By the second identity in Proposition 5.3, \( T_\nu g^\nu = x_{\nu(1)}T_{\nu_{\mu}^1} \) giving
\[
T_\nu \tau_{u_\mu} = T_\nu T_{\nu_{\mu}^1} \tau_{\nu_{\mu}^1} = T_\nu T_{\nu_{\mu}^1 \nu_{\mu}^1} = x_{\nu(1)}T_{\nu_{\mu}^1 \nu_{\mu}^1} = 1.
\]
Then (a) follows by using \( E^z_{\mu} = t^{-\frac{1}{2}}T_{z_{\mu}^{1-1}}T_\nu T_{\nu_{\mu}^1} \) to rewrite each side, and computing
\[
\ell(z_{\nu_{\mu}^{1-1}}) - \ell(z_{\nu_{\mu}^{1-1}}) = \ell(z_{\nu_{\mu}^{1-1}}) - \ell(z_{\nu_{\mu}^{1-1}})
= \ell(z_{\nu_{\mu}^{1-1}}) - \ell(z_{\nu_{\mu}^{1-1}}),
\]
where \( W \rightarrow S_n \) is the homomorphism defined in (24).

(b) Let \( \nu = s_{\mu} \) and let \( a_{\nu} = q^{\mu_{i+1} - \mu_i}q_{\nu_{i}(i+1)-\nu_i} = q^{sh(-\beta^\nu)}q^{ht(-\beta^\nu)} \). Using (42), (76) and the eigenvalue formula from (3.2), then
\[
T_\nu T_{\nu_{\mu}^1} = T_\nu T_{\nu_{\mu}^1} T_{\nu_{\mu}^1} = \begin{cases} 
T_\nu (T_\nu + \left(\frac{1}{1-a_{\nu}}\right) T_{\nu_{\mu}^1} 1, & \text{if } z_{s_{\mu}} > z, \\
T_\nu (T_{\nu_{\mu}^1}^{-1} + \left(\frac{1}{1-a_{\nu}}\right) a_{\nu} T_{\nu_{\mu}^1} 1, & \text{if } z_{s_{\mu}} < z,
\end{cases}
\]

Then, using \( E^z_{\mu} = t^{-\frac{1}{2}}T_{z_{\mu}^{1-1}}T_\nu T_{\nu_{\mu}^1} \) to obtain
\[
E^z_{\mu} = \begin{cases} 
\ell^{\frac{1}{2}}(\ell(z_{\nu_{\mu}^{1-1}}) + \ell(z_{\nu_{\mu}^{1-1}})) E^z_{s_{\mu}}, & \text{if } z_{s_{\mu}} > z, \\
\ell^{\frac{1}{2}}(\ell(z_{\nu_{\mu}^{1-1}}) + \ell(z_{\nu_{\mu}^{1-1}}) E^z_{s_{\mu}}, & \text{if } z_{s_{\mu}} < z,
\end{cases}
\]

By Proposition 2.1(e), \( v_{\mu}^{1-1} = s_{\nu}^{1-1} \) and so
\[
\ell^{\frac{1}{2}}(\ell(z_{\nu_{\mu}^{1-1}}) + \ell(z_{\nu_{\mu}^{1-1}})) = \ell^{\frac{1}{2}}(\ell(z_{s_{\mu}^{1-1}}) + \ell(z_{s_{\mu}^{1-1}})) = 1.
\]

4.2. **Box-by-box recursion for computing** \( E^z_{\mu} \). Proposition 4.3 executes several steps of Proposition 4.1 at once to provide a recursion for computing \( E^z_{\mu} \) which removes a box at each application of the recursion. Iterating this recursion delivers a monomial expansion of \( E^z_{\mu} \) as a weighted sum of nonattacking fillings \( T \). The permutation sequence \( z_{T} \) of the nonattacking filling \( T \) (see (10)) is the sequence of permutations \( z_0, z_1, \ldots \) which arise as superscripts of the \( E^z_{\mu} \) which occur in the intermediate applications of the box-by-box recursion.
Lemma 4.2. (Compressing 2j−1 terms to j terms) Let \( j \in \{1, \ldots, n\} \) and let 
\[
\mu = (0, \ldots, 0, \mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n \quad \text{with} \quad \mu_1 = 0, \ldots, \mu_{j-1} = 0 \text{ and } \mu_j \neq 0.
\]
and let \( \gamma = \pi \nu = (\mu_2, 0, \ldots, 0, \mu_{j+1}, \ldots, \mu_n) \). Let \( \tau_{\gamma}^1, \ldots, \tau_{\gamma}^{n-1} \) be the intertwiners of (41) acting on \( \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) by (39) and (40).

(a) \( \tau_{\gamma}^{j-1} \cdots \tau_1 \gamma E_\gamma = T_{j-1}^{1} \cdots T_1 E_\gamma + \frac{1-t}{1 - q^{\mu_1} t^{\nu_1(1)} - (j-1)} \sum_{a=1}^{j-1} T_{a-1}^{1} \cdots T_1 t^{-\frac{j}{2}(j-a)} E_\gamma. \)

(b) Let \( i \in \{1, \ldots, j-1\} \). Then
\[
\tau_{\gamma}^{j-1} \cdots \tau_i \gamma E_\gamma = T_{j-1}^{1} T_{j-2}^{1} \cdots T_i^{1} T_{i-1}^{1} \cdots T_1 E_\gamma
+ \frac{1-t}{1 - q^{\mu_1} t^{\nu_1(1)} - (j-1)} q^{\mu_i} t^{\nu_i(1)} (j-1) \sum_{a=1}^{j-1} T_{a-1}^{1} \cdots T_1 t^{-\frac{j}{2}(j-a)} E_\gamma
+ \frac{1-t}{1 - q^{\mu_1} t^{\nu_1(1)} - (j-1)} \sum_{a=1}^{i-1} T_{a-1}^{1} \cdots T_1 t^{-\frac{j}{2}(j-a)} E_\gamma.
\]

Proof. Let \( ev_\gamma : \mathbb{C}(Y) \to \mathbb{C}(q,t) \) be the homomorphism given by
\[
ev_\gamma(Y_i) = q^{-\gamma_i} (q^{\nu_i(1)} + t^{\gamma_i(1)-1} + 1) \quad \text{so that} \quad f E_\gamma = ev_\gamma(f) E_\gamma, \quad \text{for } f \in \mathbb{C}(Y).
\]
(we shall only apply this to rational expressions in \( Y_1, \ldots, Y_n \) where the denominator does not evaluate to 0.) For \( i \in \{1, \ldots, n-1\} \) set \( \alpha_i = \epsilon_i - \epsilon_{i+1} \) and let \( \beta_i^1 = \alpha_i, \beta_i^2 = \alpha_i, \beta_i^j = \beta_i \), \( \beta_{j-1} = s_{j-1} \alpha_{j-1} = \epsilon_{j-1} - \epsilon_j \).

For \( i \in \{1, \ldots, j-1\} \) let
\[
Y^{-\beta_i} = Y_1^{-1} Y_{i+1}, \quad F_{-\beta_i} = \frac{t^{-\frac{j}{2}(1-t)} - 1}{1 - Y^{-\beta_i}}, \quad C_{-\beta_i} = T_i + F_{-\beta_i}, \quad \text{and}
\]
\[
d_{-\beta_i} = ev_\gamma(Y^{-\beta_i}), \quad f_{-\beta_i} = ev_\gamma(F_{-\beta_i}) = c_{-\beta_i} = ev_\gamma(t^{-\frac{j}{2}} + F_{-\beta_i})
\]
If \( i \in \{1, \ldots, j-1\} \) then
\[
c_{-\beta_i} = ev_\gamma \left( \frac{t^{-\frac{j}{2}(1-t)Y^{-\beta_i}}}{1 - Y^{-\beta_i}} \right) = ev_\gamma(t^{-\frac{j}{2}} + F_{-\beta_i}) = t^{-\frac{j}{2}} + F_{-\beta_i},
\]
\[
d_{-\beta_i} = q^{-\gamma_i} q^{\nu_i(1) - \nu_i(1)+1} t^{\nu_i(1)-1} = td_{-\beta_i},
\]
\[
c_1 \cdots c_{j-1} f_{\beta_j} = \left( t^{-\frac{j}{2} \left( 1 - \frac{t d_{-\beta_j}}{1 - d_{-\beta_j}} \right) } \right) \cdots \left( t^{-\frac{j}{2} \left( 1 - \frac{t d_{-\beta_j}}{1 - d_{-\beta_j}} \right) } \right)
= t^{-\frac{j}{2}(j-1)} \left( t^{-\frac{j}{2} (1-t)} \right) \left( \frac{t^{-\frac{j}{2}(1-t)} - 1}{1 - d_{-\beta_j}} \right) = t^{-\frac{j}{2}(j-1)} t^{-\frac{j}{2}} f_{\beta_{j-1}},
\]
where the last equality follows from
\[
(t^{-\frac{j}{2}} - t^{-\frac{j}{2}}) + F_{-\beta_i} = (t^{-\frac{j}{2}} - t^{-\frac{j}{2}}) + \left( t^{-\frac{j}{2} (1-t)} \right) = Y^{-\beta_i} F_{-\beta_i}.
\]
Since
\[
\tau_{\gamma}^1 \tau_{\gamma}^{j-1} \cdots \tau_1 \gamma E_\gamma = (T_i + F_{-\alpha_i}) \tau_{\gamma}^{j-1} \cdots \tau_1 \gamma E_\gamma = (T_i + ev_\gamma(F_{-\alpha_i} \alpha_i) \tau_{\gamma}^{j-1} \cdots \tau_1 \gamma E_\gamma
= (T_i + f_{-\beta_i}) \tau_{\gamma}^{j-1} \cdots \tau_1 \gamma E_\gamma = C_{-\beta_i} \tau_{\gamma}^{j-1} \cdots \tau_1 \gamma E_\gamma
\]
then \( \tau_{\gamma}^{j-1} \cdots \tau_1 \gamma E_\gamma = C_{-\beta_j} \cdots C_{-\beta_1} \gamma E_\gamma. \)

For \( i \in \{2, \ldots, j-1\} \), \( C_{-\beta_i} E_\gamma = (T_i + f_{-\beta_i}) E_\gamma = (t^{-\frac{j}{2}} + f_{-\beta_i}) E_\gamma = c_{-\beta_i} E_\gamma. \)
Thus,
\[
\tau_{\gamma_1}^{\gamma_1} \cdots \tau_{\gamma_j}^{\gamma_j} E_\gamma = C_{-\beta_{\gamma_1}} \cdots C_{-\beta_{\gamma_j}} E_\gamma = C_{-\beta_{\gamma_1}} \cdots C_{-\beta_{\gamma_j}} (T_1 + f_{-\beta_{\gamma_1}}) E_\gamma \\
= C_{-\beta_{\gamma_1}} \cdots C_{-\beta_{\gamma_j}} T_1 E_\gamma + f_{-\beta_{\gamma_1}} C_{-\beta_{\gamma_1}} \cdots C_{-\beta_{\gamma_j}} E_\gamma \\
= C_{-\beta_{\gamma_1}} \cdots C_{-\beta_{\gamma_j}} T_1 E_\gamma + c_{-\beta_{\gamma_1}} \cdots c_{-\beta_{\gamma_j}} f_{-\beta_{\gamma_1}} E_\gamma \\
= C_{-\beta_{\gamma_1}} \cdots C_{-\beta_{\gamma_j}} (T_2 + f_{-\beta_{\gamma_1}}) T_1 E_\gamma + c_{-\beta_{\gamma_1}} \cdots c_{-\beta_{\gamma_j}} f_{-\beta_{\gamma_1}} E_\gamma \\
= C_{-\beta_{\gamma_1}} \cdots C_{-\beta_{\gamma_j}} T_2 T_1 E_\gamma + c_{-\beta_{\gamma_1}} \cdots c_{-\beta_{\gamma_j}} f_{-\beta_{\gamma_1}} T_1 E_\gamma \\
+ c_{-\beta_{\gamma_1}} \cdots c_{-\beta_{\gamma_j}} f_{-\beta_{\gamma_1}} E_\gamma,
\]
and continuing this process and using (50) gives (a).

Let \( R_i \) be the right hand side of the expression in statement of (b), so that the identity in (a) can be considered as \( \tau_{\gamma_1}^{\gamma_1} \cdots \tau_{\gamma_j}^{\gamma_j} E_\gamma = R_j \). Then, canceling the common terms in \( R_{i+1} \) and \( R_i \) gives
\[
R_{i+1} - R_i = T_{j-1}^{T_j \cdots T_1} E_\gamma + T_{j-1}^{T_j \cdots T_1} E_\gamma + T_{j-1}^{T_j \cdots T_1} E_\gamma - T_{j-1}^{T_j \cdots T_1} E_\gamma \\
= T_{j-1}^{T_j \cdots T_1} (T_1 + f_{-\beta_{\gamma_1}}) E_\gamma - T_{j-1}^{T_j \cdots T_1} E_\gamma \\
= t^{1/2} f_{-\beta_{\gamma_1}} + t^{1/2} (1 - d_{-\beta_{\gamma_1}}) T_1 E_\gamma.
\]

Using Proposition 4.1(a) and adjusting for the normalization in the definition of \( E_\mu^2 \) in (43) produces the following box-by-box recursion for the relative Macdonald polynomials \( E_\mu^2 \):

**Proposition 4.3.** Let \( z \in S_n \). Let
\[
\mu = (0, \ldots, 0, \mu_j, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n \quad \text{with} \quad \mu_1 = 0, \ldots, \mu_{j-1} = 0 \quad \text{and} \quad \mu_j \neq 0.
\]

and let \( \nu = (0, \ldots, 0, \mu_{j+1}, \ldots, \mu_n, \mu_j - 1) \). For \( m \in \{0, \ldots, n\} \) let \( c_m = s_{m-1} \cdots s_{2} s_1 \) (which is an m-cycle in \( S_n \)). Let \( y = (y(1), \ldots, y(n)) \) be the permutation which has \( y(k) = z(k) \) for \( k \in \{j, \ldots, n\} \)

and \( \{y(1), \ldots, y(j-1)\} = \{z(1), \ldots, z(j-1)\} \) and \( y(1) < \cdots < y(j-1) \).

Then
\[
E_\mu^2 = x_{y(j)} E_{\mu}^{y(j-1)c_n} x_{y(j)}^{1 - t} q^{\text{covid}_\mu(y)} x_{y(a)} E_{\mu}^{y(j-1)c_n},
\]
where, for \( a \in \{1, \ldots, j-1\} \),
\[
\text{covid}_\mu(y) = \begin{cases} 
0, & \text{if } y(j) = y(a), \\
\mu_j, & \text{if } y(j) < y(a), 
\end{cases}
\]
and
\[
\text{covid}_\mu^2(y) = \begin{cases} 
\frac{1}{2}(\ell(y c^{-1}_{\mu} c_j v_{\mu}^{-1}) - \ell(y c^{-1}_{\mu} c_j)), & \text{if } y(j) = y(a), \\
\ell(y c^{-1}_{\mu} c_j v_{\mu}^{-1}) - \ell(y c^{-1}_{\mu} c_j), & \text{if } y(j) < y(a).
\end{cases}
\]
Proof. Write $z = y\sigma$ with $\sigma \in S_{j-1}$ and $y$ minimal length in the coset $zS_{j-1}$. Then $y(j) = z(j)$ and, by the last identity in (48),

$$T_z E_\mu = T_y T_\sigma E_\mu = T_y t^{\frac{1}{2}(\sigma)} E_\mu, \text{ so that } E^z_\mu = E^y_\mu.$$  

To control the spacing let $c_0 = s_1 \cdots s_{a-1}$ (which is an a-cycle in $S_a$) and let

$$d_{-\beta^\vee_{j-1}} = q^\mu v^\vee_r (1) - (j-1) = q^\mu v^\vee_r (j-1) \quad \text{and} \quad t^\frac{1}{2} f_{-\beta^\vee_{j-1}} = \frac{1 - t}{1 - q^\mu v^\vee_r (j-1)}.$$  

Let $\gamma = \mu \nu = (\mu_1, 0, \ldots, 0, \mu_j, \ldots, \mu_n)$ as in Lemma 4.2 and note that $v_\gamma = v_{y(j) - 1} \cdots s_1$.  

If $y(j) > y(j-1)$ then $T_{y(j) \nu} = T_y T_{j-1} \cdots T_1$, and using Lemma 4.2(a) gives

$$T_y \tau^\vee_{j-1} \cdots \tau^\vee_1 E_\gamma = T_{y(j) \nu} E_\gamma + t^\frac{1}{2} f_{-\beta^\vee_{j-1}} \sum_{a=1}^{j-1} t^{-\frac{1}{2}(j-a)} T_{y(a) \nu} E_\gamma.$$  

If $y(j) < y(j-1)$ then $T_{y(j) \nu} = T_y T_{j-1}^{-1} \cdots T_1^{-1} T_1 \cdots T_1$ with $i = \min\{r \in \{1, \ldots, j-1\} \mid y(r) > y(j)\}$,  

and using Lemma 4.2(b) gives

$$T_y \tau^\vee_{j-1} \cdots \tau^\vee_1 E_\gamma = T_{y(j) \nu} E_\gamma + t^\frac{1}{2} f_{-\beta^\vee_{j-1}} \sum_{a=1}^{i-1} t^{-\frac{1}{2}(j-a)} T_{y(a) \nu} E_\gamma.$$  

For $a \in \{1, \ldots, j\}$, let

\[ \text{norm}_{a}^{\nu}(a) = (l(y_a^{-1} c_j v^{-1}_a) - l(c_j v^{-1}_a)) - (l(y_a^{-1}) - l(v^{-1}_a) - (j-1)) \]

\[ = (l(y_a^{-1} c_j v^{-1}_a) - l(v^{-1}_a) + j - 1) - (l(y_a^{-1}) + (j-1)) \]

\[ = (l(y_a^{-1} c_j v^{-1}_a) - l(v^{-1}_a)). \]

With this notation, the identities

\[ E^\nu_\mu = t^{-\frac{1}{2}((\ell(y^{-1}_a c_j v^{-1}_a) - \ell(v^{-1}_a)))} T_{y(j) \nu} E_\gamma = t^{-\frac{1}{2}((\ell(y^{-1}_a) - \ell(v^{-1}_a) - (j-1)))} T_y \tau^\vee_{j-1} \cdots \tau^\vee_1 E_\gamma, \]

and

\[ E^{\nu_{a-1}}_\gamma \tau^\vee_{j-1} \cdots \tau^\vee_1 E_\gamma = t^{-\frac{1}{2}((\ell(y_{a-1}^{-1} c_j v^{-1}_{a-1}) - \ell(v^{-1}_{a-1})))} T_y T_{a-1} \cdots T_1 E_\gamma \]

then give

\[ E^\nu_\mu = t^\frac{1}{2} \text{norm}^{\nu}_{a}(a) E^{\nu_{a-1}}_\gamma + t^\frac{1}{2} f_{-\beta^\vee_{j-1}} \sum_{a=1}^{j-1} t^{-\frac{1}{2}(j-a)} t^\frac{1}{2} \text{norm}^{\nu}_{a}(a) E^{\nu_{a-1}}_\gamma \]

for $y(j) > y(j-1)$; and

\[ E^\nu_\mu = t^\frac{1}{2} \text{norm}^{\nu}_{a}(a) E^{\nu_{a-1}}_\gamma + t^\frac{1}{2} f_{-\beta^\vee_{j-1}} \sum_{a=1}^{i-1} d_{-\beta^\vee_{j-1}} t^{-\frac{1}{2}(j-a)} t^\frac{1}{2} \text{norm}^{\nu}_{a}(a) E^{\nu_{a-1}}_\gamma + t^\frac{1}{2} f_{-\beta^\vee_{j-1}} \sum_{a=1}^{i-1} t^{-\frac{1}{2}(j-a)} t^\frac{1}{2} \text{norm}^{\nu}_{a}(a) E^{\nu_{a-1}}_\gamma, \]

for $y(j) < y(j-1)$ and $i = \min\{r \in \{1, \ldots, j-1\} \mid y(r) > y(j)\}$.
If \( a = j \) then \( \text{norm}_\mu^\pm (j) = 0 \). Since
\[
\ell(c_a^{-1}c_j) = \ell(s_{a-1} \cdots s_1 s_j \cdots s_{j-1}) = \ell(s_a \cdots s_{j-1}) = (j - a)
\]
then
\[
\text{norm}_\mu^\pm (a) - (j - a) = \ell(y c_a^{-1}c_j v_\mu^{-1}) - \ell(y v_\mu^{-1}) - \ell(c_a^{-1}c_j).
\]
Applying Proposition 4.1(a) to the right hand side of the expressions that have been obtained for \( E_\mu^y \) and substituting
\[
d_{a}^{\nu} = q^{\nu_{a}^{1}}v_{\nu_{a}^{1}}^{(1)}(j-1) = q^{\nu_{a}^{1}}v_{\nu_{a}^{1}}(j)(j-1) = q^{\nu_{a}^{1}}(j-1)n_{j}(j-1)
\]
gives
\[
E_\mu^z = E_\mu^y = y(j)E_\mu^{y c_a^{-1}c_j} + \frac{(1 - t)}{1 - q^{\nu_{a}^{1}}v_{\nu_{a}^{1}}(j)(j-1)} \sum_{a=1}^{j-1} y^{\text{max}_{\mu}^\pm (a)}c_{\text{covid}_{\mu}^\pm (a)}x_{a}(a)E_{\mu_{a}}^{y c_a^{-1}c_j},
\]
where, if \( i = \min\{r \in \{1, \ldots, j\} \mid y(r) \geq y(j)\} \) then
\[
\text{covid}_{\mu}^\pm (a) = \begin{cases} \frac{1}{2}\text{norm}_\mu^\pm (a) - \frac{1}{2}(j - a), & \text{if } a < i, \\ v_{\mu}(j) - (j - 1) + \frac{1}{2}\text{norm}_\mu^\pm (a) - \frac{1}{2}(j - a), & \text{if } a \geq i, \\ \frac{1}{2}(\ell(y c_a^{-1}c_j v_\mu^{-1}) - \ell(y v_\mu^{-1}) - \ell(c_a^{-1}c_j)), & \text{if } y(j) > y(a), \\ v_{\mu}(j) - (j - 1) + \frac{1}{2}(\ell(y c_a^{-1}c_j v_\mu^{-1}) - \ell(y v_\mu^{-1}) - \ell(c_a^{-1}c_j)), & \text{if } y(j) < y(a). \end{cases}
\]

Remark 4.4. Relating \( \text{covid}_{\mu}^\pm (a) \) to coinvversion triples. To give an alternate point of view on the statistic \( \text{covid}_{\mu}^\pm (a) \) which fell out of the computation in the proof of Proposition 4.3, let us analyze how the inversions of \( y v_\mu^{-1} \) change when the factor \( c_a^{-1}c_j \) is inserted to form \( y c_a^{-1}c_j v_\mu^{-1} \). To do this note that
\[
y c_a^{-1}c_j v_\mu^{-1} = y v_\mu^{-1}(s_{v_{\nu}(j)-1}s_{v_{\nu}(j)-2} \cdots s_{j})(s_a \cdots s_{j-2})s_{j-1}(s_j \cdots s_{v_{\nu}(j)-1}).
\]
and analyze the effect of each of the factors on the right hand side.

(a) Since \( y(a) < y(a+1) < \cdots < y(j-1) \) then \((s_a \cdots s_{j-2}) \) creates \((j-1-a)\) inversions in \( y c_a^{-1}c_j v_\mu^{-1} \) which do not occur in \( x v_\mu^{-1} \).

(b) The factor \( s_{j-1} \) creates an inversion if \( y(j) > y(a) \) and removes an inversion if \( y(j) < y(a) \).

(c) The factor \((s_j \cdots s_{v_{\nu}(j)-1})\)
undoes inversions \( y v_\mu^{-1}(k) < y v_\mu^{-1}(a) \) for \( k \in \{j, \ldots, v_{\nu}(j) - 1\} \),
adds inversions \( y v_\mu^{-1}(k) > y v_\mu^{-1}(a) \) for \( k \in \{j, \ldots, v_{\nu}(j) - 1\} \).

(d) The factor \((s_{v_{\nu}(j)-1}s_{v_{\nu}(j)-2} \cdots s_{j})\)
undoes inversions \( y v_\mu^{-1}(k) > y v_\mu^{-1}(v_{\nu}(j)) \) for \( k \in \{j, \ldots, v_{\nu}(j) - 1\} \),
adds inversions \( y v_\mu^{-1}(k) < y v_\mu^{-1}(v_{\nu}(j)) \) for \( k \in \{j, \ldots, v_{\nu}(j) - 1\} \).
Thus, if $yv_{\mu}^{-1}(v_\mu(j)) > yv_{\mu}^{-1}(a)$ (so that $y(j) > y(a)$) then

$$\ell(yv_{\mu}^{-1}c_jv_{\mu}^{-1}) - \ell(yv_{\mu}^{-1}) - \ell(c_{\alpha}^{-1}c_j) = \ell(yv_{\alpha}^{-1}c_jv_{\mu}^{-1}) - \ell(yv_{\mu}^{-1}) - (j-a)$$

$$= (j-1-a) + 1$$

$$- \#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(k) < yv_{\mu}^{-1}(a)\}$$

$$+ \#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(k) > yv_{\mu}^{-1}(a)\}$$

$$- \#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(v_\mu(j)) < yv_{\mu}^{-1}(k)\}$$

$$+ \#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(v_\mu(j)) > yv_{\mu}^{-1}(k)\}$$

$$= (j-a)$$

$$= -\#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(k) < yv_{\mu}^{-1}(a)\}$$

$$+ \#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(a) < yv_{\mu}^{-1}(k)\}$$

$$- \#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(v_\mu(j)) < yv_{\mu}^{-1}(k)\}$$

$$+ \#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(v_\mu(j)) > yv_{\mu}^{-1}(k)\}$$

$$= 2 \cdot (\#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(a) < yv_{\mu}^{-1}(k) < yv_{\mu}^{-1}(v_\mu(j))\}) - 1$$

Then, if $yv_{\mu}^{-1}(v_\mu(j)) < yv_{\mu}^{-1}(a)$ (so that $y(j) < y(a)$) then

$$\ell(yv_{\mu}^{-1}c_jv_{\mu}^{-1}) - \ell(yv_{\mu}^{-1}) - \ell(c_{\alpha}^{-1}c_j) = \ell(yv_{\alpha}^{-1}c_jv_{\mu}^{-1}) - \ell(yv_{\mu}^{-1}) - (j-a)$$

$$= (j-1-a) - 1$$

$$- \#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(k) < yv_{\mu}^{-1}(a)\}$$

$$+ \#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(a) < yv_{\mu}^{-1}(k)\}$$

$$- \#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(v_\mu(j)) < yv_{\mu}^{-1}(k)\}$$

$$+ \#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(v_\mu(j)) > yv_{\mu}^{-1}(k)\}$$

$$= 2 \cdot (\#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(a) < yv_{\mu}^{-1}(k) < yv_{\mu}^{-1}(v_\mu(j))\}) - 2$$

If $y(j) > y(a)$ then

$$\frac{1}{2}(\ell(yv_{\alpha}^{-1}c_jv_{\mu}^{-1}) - \ell(yv_{\mu}^{-1}) - \ell(c_{\alpha}^{-1}c_j))$$

$$= \#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid yv_{\mu}^{-1}(a) < yv_{\mu}^{-1}(k) < yv_{\mu}^{-1}(v_\mu(j))\}$$

$$= \#\{k \in \{j, \ldots, v_\mu(j)-1\} \mid y(a) < yv_{\mu}^{-1}(k) < y(j)\}$$

$$= \#\{v_\mu(b) \in \{j, \ldots, v_\mu(j)-1\} \mid y(a) < yv_{\mu}^{-1}(v_\mu(b)) < y(j)\}$$

$$= \#\left\{b \in \{v_{\mu}^{-1}(j), \ldots, v_{\mu}^{-1}(v_\mu(j)-1)\} \mid y(a) < y(b) < y(j)\right\}$$
and, if \( y(j) < y(a) \) then
\[
(v_\mu(j) - (j - 1)) + \frac{1}{2}(\ell(yv_\mu^{-1}c_jv_\mu^{-1}) - \ell(yv_\mu^{-1} - \ell(c_\mu^{-1}c_j))
\]
\[
= (v_\mu(j) - j + 1) - 1
\]
\[
- \#\{k \in \{j, \ldots, v_\mu(j) - 1\} \mid yv_\mu^{-1}(a) < yv_\mu^{-1}(k) < yv_\mu^{-1}(v_\mu(j))\}
\]
\[
= ((v_\mu(j) - 1) - (j - 1))
\]
\[
- \#\{k \in \{j, \ldots, v_\mu(j) - 1\} \mid yv_\mu^{-1}(a) < yv_\mu^{-1}(k) < yv_\mu^{-1}(v_\mu(j))\}
\]
\[
= \#\{k \in \{j, \ldots, v_\mu(j) - 1\} \mid yv_\mu^{-1}(k) < y(v_\mu^{-1}(a) < yv_\mu^{-1}(v_\mu(j)) \}
\]
\[
= \#\{b \in \{v_\mu^{-1}(j), \ldots, v_\mu^{-1}(v_\mu(j) - 1)\} \mid y(b) < y(a) < y(j) \}
\]
These last two expressions are exactly the numbers of coinversion triples that appear in [10, Lemma 3.6.3] for the box \((j, 1)\) filled with \(y(a)\) in a filling of shape \(\mu\) with basement \((y(1), \ldots, y(n))\).

5. **Type GL\( n\) DAArt, DAHA and the Polynomial Representation**

The power tools that enable us to construct and manipulate Macdonald polynomials with ease are the polynomial generators \(X_1, \ldots, X_n\), the Cheveri-Dunkl operators \(Y_1, \ldots, Y_n\) and the intertwiners \(\tau^\vee, \tau^\vee_{n-1}, \tau^\vee_n\) which all live inside the double affine Hecke algebra \(\tilde{H}_{GL_n}\). In this section we will build the Macdonald polynomials \(E_\mu\) by first constructing the double affine Artin group \(\tilde{B}_{GL_n}\), then the elements \(X_1, \ldots, X_n\) and \(Y_1, \ldots, Y_n\), then the DAHA \(\tilde{H}_{GL_n}\) and the intertwiners \(\tau^\vee, \tau^\vee_{n-1}, \tau^\vee_n\). Let us begin by defining the DAArt \(\tilde{B}_{GL_n}\) and establishing its primary dualities. The definition is by generators and relations and the dualities are automorphisms of \(\tilde{B}_{GL_n}\).

The double affine Hecke algebra \(\tilde{H}_{GL_n}\) is constructed as a quotient of the group algebra of \(\tilde{B}_{GL_n}\) by the Hecke relations \(T_i^2 = (t_i^\vee - t_i^{-\frac{1}{2}})T_i + 1\). Alternative expositions of the material in this section are found in [5] and [16].

Use Coxeter diagram shorthand for relations so that

\[
\begin{array}{c}
\circ & \overset{a}{\longrightarrow} & \circ \quad \text{indicates} \quad aba = bab, \quad \text{and} \quad \overset{a}{\circ} \overset{b}{\circ} \quad \text{indicates} \quad ab = ba, \\
\end{array}
\]

5.1. **The Type GL\( n\) Double Affine Artin Group (DAArt).** The element \(q\) will be a parameter in the Macdonald polynomials. In the definition of the DAArt by generators and relations the element \(q\) appears as a central element of the group, but in Section 5.6 the element \(q\) will get specialized to be a complex parameter.

The type GL\( n\) double affine Artin group (DAArt) \(\tilde{B}_{GL_n}\) is generated by \(q, g^\vee, g, S_0^\vee, S_0, T_1, \ldots, T_{n-1}\) with the relations

\[(53)\]

\[
gS_0g^{-1} = T_1, \quad gT_i g^{-1} = T_{i+1}, \quad gT_{n-1} g^{-1} = S_0, \quad g^\vee S_0^\vee (g^\vee)^{-1} = T_1, \quad g^\vee T_i (g^\vee)^{-1} = T_{i+1}, \quad g^\vee T_{n-1} (g^\vee)^{-1} = S_0^\vee, \quad q \in Z(\tilde{B}_{GL_n}) \quad \text{and} \]
for $i \in \{1, \ldots, n-2\}$.

The two visible symmetries in this definition, switching the Coxeter diagram containing $S_0$ and the Coxeter diagram containing $S_0'$, and flipping the Coxeter diagrams about the middle, form two important dualities. These dualities are expressed as involutive automorphisms of the DAArt $\tilde{B}_{GL_n}$.

**Theorem 5.1**. (a) (Duality) There is an involution $\iota: \tilde{B}_{GL_n} \rightarrow \tilde{B}_{GL_n}$ with

$$\iota(q) = q^{-1}, \quad \iota(T_i) = T_i^{-1}, \quad \iota(S_0^\vee) = S_0^{-1}, \quad \iota(g) = g^\vee.$$ 

(b) (Duality) There is an involution $\eta: \tilde{B}_{GL_n} \rightarrow \tilde{B}_{GL_n}$ with

$$\eta(q) = q, \quad \eta(T_i) = T_{n-1}, \quad \eta(g) = g^{-1}, \quad \eta(g^\vee) = (g^\vee)^{-1}.$$ 

**Proof.** (a) Applying $\iota$ to the relations in (53) switches the upper (nonchecked) relations. Applying $\iota$ to the relations in (54) produces the relations $q^{-1} \in Z(\tilde{B}_{GL_n})$.

$$T_i^{-1}gg^\vee = g^\vee gT_{n-1} \quad \text{and} \quad T_{n-1} \cdots T_1 g^\vee g^{-1} = g^{-1}g^{-1}g^\vee T_{n-1} \cdots T_1,$$

respectively. Thus the relations in (54) are preserved under $\iota$.

(b) The involution $\eta$ preserves the relations in (53). Applying $\eta$ to the relations in (54) produces the relations $q \in Z(\tilde{B}_{GL_n})$, 

$$T_{n-1}(g^\vee)^{-1}g^{-1} = g^{-1}(g^\vee)^{-1}T_1 \quad \text{and} \quad T_{n-1}^{-1} \cdots T_1^{-1}g^\vee = g^\vee g^{-1}T_{n-1} \cdots T_1,$$

which are equivalent to the original relations in (54) by taking inverses. \qed

**5.2. The Elements $X^{x_1}, \ldots, X^{x_n}$ and $Y^{x_1}, \ldots, Y^{x_n}$**. The elements $X^{x_1}, \ldots, X^{x_n}$ will be used as the generators for a polynomial ring (inside the group algebra of $\tilde{B}_{GL_n}$), and the Macdonald polynomials are polynomials in these variables. Inside the DAArt, these elements form a large commutative subgroups and, because of duality, there is another large commutative subgroup generated by elements $Y^{x_1}, \ldots, Y^{x_n}$. In this section we define these elements and give alternate presentations of $\tilde{B}_n$ in terms of these elements.

Define $Y^{x_1}, \ldots, Y^{x_n}$ and $X^{x_1}, \ldots, X^{x_n}$ in $\tilde{B}_{GL_n}$ by

$$Y^{x_j} = g_{n-1} \cdots T_1 \quad \text{and} \quad Y^{x_{j+1}} = T_j^{-1}Y^{x_j}T_j^{-1},$$

$$X^{x_j} = g^\vee T_{n-1}^{-1} \cdots T_1^{-1} \quad \text{and} \quad X^{x_{j+1}} = T_j X^{x_j} T_j,$$

for $j \in \{1, \ldots, n-1\}$. If $\iota: \tilde{B}_{GL_n} \rightarrow \tilde{B}_{GL_n}$ and $\eta: \tilde{B}_{GL_n} \rightarrow \tilde{B}_{GL_n}$ are the involutions in Theorem 5.1 then

$$\iota(X^{x_i}) = Y^{x_i} \quad \text{and} \quad \eta(X^{x_i}) = X^{-x_{i+1}} \quad \text{and} \quad \eta(Y^{x_i}) = Y^{-x_{i+1}},$$

for $i \in \{1, \ldots, n-1\}$.

The subgroup generated by $g^\vee, T_1, \ldots, T_{n-1}$ has a pictorial representation given by

$$g^\vee = \quad \text{and} \quad T_i = \quad \text{for } i = 1, \ldots, n-1,$$

\[ \quad \text{for } i = 1, \ldots, n-1, \]

\[ \quad \text{for } i = 1, \ldots, n-1, \]
so that

\[ X^{\varepsilon_i} = T_{i-1} \cdots T_1 g^T T_{n-1}^{-1} \cdots T_i^{-1} \]

for \( i \in \{1, \ldots, n\} \).

Use the notation \( X_i = X^{\varepsilon_i} \), and let \( X^\mu = X_1^{\mu_1} \cdots X_n^{\mu_n} = X^{\mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n} \), for \( \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \).

Similarly use the notation \( Y_i = Y^{\varepsilon_i} \), and let \( Y^\lambda = Y_1^{\lambda_1} \cdots Y_n^{\lambda_n} = Y^{\lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n} \), for \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \).

The pictorial representation provides an easy check of the relations

\[
(g^\varepsilon)^n = X^{\varepsilon_1 + \cdots + \varepsilon_n} \quad \text{and} \quad X^{\varepsilon_i} X^{\varepsilon_j} = X^{\varepsilon_j} X^{\varepsilon_i},
\]

\[
g = Y^{\varepsilon_1 + \cdots + \varepsilon_n} \quad \text{and} \quad Y^{\varepsilon_i} Y^{\varepsilon_j} = Y^{\varepsilon_j} Y^{\varepsilon_i},
\]

for \( i, j \in \{1, \ldots, n\} \). The pictorial perspective also verifies the relations

\[
S_0 T_{-1} = Y^{\varepsilon_1 - \varepsilon_n} \quad \text{and} \quad (S_0^\varepsilon)^{-1} T_{n-1}^{-1} = X^{\varepsilon_1 - \varepsilon_n},
\]

where \( T_{n-1} = T_{n-1} \cdots T_1 \cdots T_{n-1} \).

Theorem 5.2.

(a) The group \( \mathcal{B}_{GL_n} \) is presented by generators \( q, g^\varepsilon, S_0^\varepsilon, T_1, \ldots, T_{n-1}, Y^{\varepsilon_1}, \ldots, Y^{\varepsilon_n} \), and relations

\[
g^T T_i (g^\varepsilon)^{-1} = T_{i+1} \quad \text{and} \quad g^T T_{n-1} (g^\varepsilon)^{-1} = S_0^\varepsilon,
\]

where \( T_{n-1} = T_{n-1} \cdots T_1 \cdots T_{n-1} \).

(b) The group \( \mathcal{B}_{GL_n} \) is presented by generators \( q, g, T_0, T_1, \ldots, T_{n-1} \) and \( X^{\varepsilon_1}, \ldots, X^{\varepsilon_n} \) and relations

\[
g T_i g^{-1} = T_{i+1} \quad \text{and} \quad g T_{n-1} g^{-1} = S_0,
\]

where \( T_{n-1} = T_{n-1} \cdots T_1 \cdots T_{n-1} \).

Conclusion.

\[
X^{\varepsilon_{i+1}} = T_i X^{\varepsilon_i} T_i \quad \text{and} \quad T_i X^{\varepsilon_j} = X^{\varepsilon_j} T_i.
\]
for \( i \in \{1, \ldots, n-1\} \) and \( j \neq i, i+1 \), and

\[ gX^i g^{-1} = X^{i+1}, \quad \text{for} \ i \in \{1, 2, \ldots, n-1\} \quad \text{and} \quad gX^n g^{-1} = g^{-1}X^1. \]

**Proof.** The proof is by showing that the relations in (61), (62) and (63) follow from the defining relations of \( \mathcal{B}G_{\mathcal{L}^0} \), and vice versa.

(53) \& (54) \implies (a): Using (55) to define \( Y^\circ i \), the pictorial perspective establishes the relations in (61) and (62). The proof of the relations in (63) is completed by

\[ gY^{\circ i} = gY gT_{n-1} \cdots T_1 = T_{i+1}^{-1} gT_{n-1} \cdots T_1 gT_{n-1} \cdots T_2 gY = T_{i+1}^{-1} Y T_{i+1}^{-1} Y \cdots Y T_{i+1}^{-1} Y = Y^{\circ i} g, \]

\[ gY^{\circ i} = gY T_{n-1} \cdots T_1 = T_{i+1}^{-1} Y T_{i+1}^{-1} Y \cdots Y T_{i+1}^{-1} Y = Y^{\circ i} g. \]

and

\[ gY^{\circ i} = gY T_{n-1} \cdots T_1 = T_{i+1}^{-1} Y T_{i+1}^{-1} Y \cdots Y T_{i+1}^{-1} Y = Y^{\circ i} g. \]

and

\[ gS_0 g^{-1} = Y^{\circ i} T_{n-1}^{\circ i} \cdots T_1^{\circ i} \cdots T_1^{\circ i} Y^{\circ i} T_{n-1}^{\circ i} \cdots T_1^{\circ i} = T_{i+1}^{-1} Y T_{i+1}^{-1} Y \cdots Y T_{i+1}^{-1} Y = Y^{\circ i} T_{i+1}, \]

which establishes the relations in (53).

To prove the first relation in (54):

\[ T_i gY = T_i gY Y^{\circ i} T_i^{\circ i} \cdots T_{n-1}^{\circ i} = T_i Y^{\circ i} gY T_i^{\circ i} \cdots T_{n-1}^{\circ i}, \]

and to prove the second relation in (54):

\[ T_i^{-1} Y gT_{n-1} \cdots T_1 gT_{n-1} \cdots T_1 Y^{-1} = gT_{n-1} \cdots T_1 gT_{n-1} \cdots T_1 Y^{-1} = gY T_{n-1} \cdots T_1 T_{i+1}^{-1} T_{i+1}^{-1} = gY T_{n-1} \cdots T_1 T_{i+1}^{-1} T_{i+1}^{-1}, \]

Part (b) follows from part (a) by applying the duality involution \( \iota \).

\[ \square \]
5.3. The elements $X^{s_i}$. Use the notation $X_i = X^{e_i}$, 
and let $X^μ = X_1^{μ_1} \cdots X_n^{μ_n} = X^{μ_1 ε_1 + \cdots + μ_n ε_n}$, for $μ = (μ_1, \ldots, μ_n) ∈ ℤ^n$.

Using the notation of the affine Weyl group $W = \{ t_μ w | μ ∈ ℤ^n, w ∈ S_n \}$ from Section 2.1, for $μ ∈ ℤ^n$ and $w ∈ S_n$ define

\[(65) \quad X^μT_w = X^μT_w, \quad \text{where} \quad T_w = T_{i_1} \cdots T_{i_t} \]

if $w = s_{i_1} \cdots s_{i_t}$ is a reduced word. The following proposition establishes how these elements are affected by right multiplication by the generators $T_1, \ldots, T_n$ and $g^\vee$ of $B_{GL_n}$.

**Proposition 5.3.** Let $μ ∈ ℤ^n$ and $w ∈ S_n$. Then

\[X^μT_w = \begin{cases} X^μT_wT_{i_1}, & \text{if } ℓ(ws_i) > ℓ(w), \\ X^μT_wT_{i_1}^{-1}, & \text{if } ℓ(ws_i) < ℓ(w), \end{cases} \]

\[X^μT_wg^\vee = X^μX_w(1)T_{ws_1 \cdots s_{n-1}}.\]

**Proof.** The first equality follows from the fact that if $z ∈ S_n$ and $ℓ(zs_i) > ℓ(z)$ then $T_zT_i = T_{zs_i}$. For the second equality: Let $k = w(1)$ and write $w = s_{k-1} \cdots s_{1}z$ with $z$ in the subgroup of $S_n$ that is generated by $s_2, \ldots, s_{n-1}$. Letting $c_n = s_1 \cdots s_{n-1}$ and using $g^\vee T_1(g^\vee)^{-1} = T_{i_1}$ then $(g^\vee)^{-1}T_zg^\vee = T_{c_n^{-1}zc_n}$ and

\[T_wg^\vee = T_{k-1}T_1T_zg^\vee = T_{k-1}T_1g^\vee((g^\vee)^{-1}T_zg^\vee)\]

\[= T_{k-1}T_1g^\veeT_{c_n^{-1}zc_n}((T_{k-1}T_1g^\veeT_{c_n^{-1}zc_n})^{-1})T_k \cdots T_{n-1}T_{c_n^{-1}zc_n} = X_kT_{c_n^{-1}zc_n}X_kT_w,c_n.\]

\[\square\]

5.4. The type $GL_n$ double affine Hecke algebra (DAHA). The type $GL_n$ double affine Hecke algebra $\tilde{H}_{GL_n}$ is the quotient of the group algebra of $B_{GL_n}$ by the relations

\[(66) \quad (T_i - t_{i-\frac{1}{2}})(T_i + t_{-i-\frac{1}{2}}) = 0, \quad \text{for } i ∈ \{1, \ldots, n-1\}.\]

The involutions $ι: \tilde{B}_{GL_n} → \tilde{B}_{GL_n}$ and $η: \tilde{B}_{GL_n} → \tilde{B}_{GL_n}$ from Theorem 5.1 preserve the relations in (66) to provide involutions

\[(67) \quad ι: \tilde{H}_{GL_n} → \tilde{H}_{GL_n} \quad \text{and} \quad η: \tilde{H}_{GL_n} → \tilde{H}_{GL_n}.\]

The following proposition explains how $g, T_1, \ldots, T_n$ move past the $X_1, \ldots, X_n$ inside the affine Hecke algebra.

**Proposition 5.4.** Let $μ = (μ_1, \ldots, μ_n) ∈ ℤ^n$ and define $X^μ = X^{μ_1 ε_1 + \cdots + μ_n ε_n}$. The symmetric group $S_n$ acts on $ℤ^n$ by permuting coordinates. Let $s_1, \ldots, s_{n-1}$ be the simple reflections in $S_n$. Then, as elements of $\tilde{H}_{GL_n}$,

\[gX^μ = q^{-μ_n}X^{s_1s_2 \cdots s_{n-1}μ_n}g = q^{-μ_n}X^{(μ_n, -μ_1, \ldots, -μ_{n-1})}g,\]

and

\[T_iX^μ = (s_iX^μ)T_i + \frac{t_i^2 - t_i^{-2}}{1 - X_iX_{i+1}^{-1}}(1 - s_i)X^μ, \quad \text{for } i ∈ \{1, \ldots, n-1\}.\]

**Proof.** Start with $X_{i+1} = T_iX_iT_i$ and use $T_i^{-1} = T_i - (t_i^2 - t_i^{-2})$ to get

\[T_iX_i = X_{i+1}T_i^{-1} = X_{i+1}(T_i - (t_i^2 - t_i^{-2}))X^{s_ε_i}T_i + (t_i^2 - t_i^{-2})\frac{X_i - X_{i+1}}{1 - X_iX_{i+1}}X_i = X^{s_ε_i}T_i + \frac{(t_i^2 - t_i^{-2})}{1 - X_iX_{i+1}}(1 - s_i)X_i.\]
and
\[ T_iX_{i+1} = T_i^2X_iX_i = ((t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1)X_iX_i = X_iT_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})X_{i+1} \]
\[ = X_iT_{i+1} + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}}X_{i+1} \]
\[ = (s_iX_{i+1})T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}}(1 - s_i)X_{i+1} \]

and, for \( j \not\in \{i, i+1\} \),
\[ T_iX_j = X_jT_i = (s_iX_j)T_i + 0 = (s_iX_j)T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}}(1 - s_i)X_j. \]

If
\[ T_iX_\mu = (s_iX_\mu)T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}}(1 - s_i)X_\mu \quad \text{and} \]
\[ T_iX_\nu = (s_iX_\nu)T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}}(1 - s_i)X_\nu \]

then
\[ T_iX^{\mu+\nu} = T_iX_\mu X_\nu = \left( (s_iX_\mu)T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}}(1 - s_i)X_\mu \right) X_\nu \]
\[ = (s_iX_\mu) \left( (s_iX_\nu)T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}}(1 - s_i)X_\nu \right) + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}}(X_\mu - (s_iX_\mu))X_\nu \]
\[ = (s_iX^{\mu+\nu})T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}}X_\mu X_\nu - X^{(i+1)}_\mu X^{\nu} + X^{\mu+\nu} - X^{\mu+\nu} \]
\[ = (s_iX^{\mu+\nu})T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}}(1 - s_i)X^{\mu+\nu}. \]

Since \( gX_n g^{-1} = q^{-1}X_1 \) and \( gX_1 g^{-1} = X_{i+1} \) then
\[ gX^{(\mu_1, \ldots, \mu_n)} = gX_1^{\mu_1} \cdots X_n^{\mu_n} = X_1^{\mu_1} \cdots X_n^{\mu_n-1}X_{n+1}^{\mu_{n+1}}g. \]

5.5. Intertwiners. Structurally, the elements \( X^{\varepsilon_1}, \ldots, X^{\varepsilon_n} \) are playing the role of generators of a polynomial ring inside of the double affine Hecke algebra \( \tilde{H}_{GL_n} \). The next key point is that we can produce elements \( \tau_1^\varepsilon, \ldots, \tau_n^\varepsilon \) which are “replacements” for the generators \( T_1, \ldots, T_n \) and \( g^\varepsilon \), and which move past the elements \( Y^{\varepsilon_1}, \ldots, Y^{\varepsilon_n} \) in the best possible way, by permuting the \( Y_i \), as seen in (75).

Define \( Y_i \) for \( i \in \mathbb{Z} \) by setting
\[ (70) \quad Y_i = Y^{\varepsilon_i} \quad \text{for } i \in \{1, \ldots, n\} \quad \text{and} \quad Y_{j+n} = qY_j \quad \text{for } j \in \mathbb{Z}. \]

Letting \( Y^K = q^{-1} \) and \( e_0^\varepsilon = e_0^\varepsilon + K \) then
\[ (71) \quad Y_0 = Y^{\varepsilon_0} = Y^{\varepsilon_0 + K} = Y^K Y^{\varepsilon_0} = q^{-1} Y_n. \]

Let
\[ (72) \quad \tau_i^\varepsilon = g^\varepsilon \quad \text{and} \quad \tau_i^\varepsilon = T_i + \frac{t^{\frac{1}{2}}(1 - t)}{1 - Y_i^{-1} Y_{i+1}} \quad \text{for } i \in \{1, \ldots, n-1\} \]

(the \( \tau_i^\varepsilon \) lie in a localization of the double affine Hecke algebra \( \tilde{H}_{GL_n} \) which allows the denominators \( 1 - Y_i Y_{j-1}^{-1} \) for \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \)). For \( w \in W \) define
\[ \tau_w^\varepsilon = \tau_{i_1}^\varepsilon \cdots \tau_{i_t}^\varepsilon \quad \text{for a reduced word } w = s_{i_1} \cdots s_{i_t}. \]
The following proposition establishes that the $\tau_i^\vee$ satisfy the braid relations so that the element $\tau^\vee_w$ does not depend on the choice of reduced word for $w$. The $\tau_i^\vee$ do not quite generate a symmetric group, because $(\tau_i^\vee)^2$ is not the identity.

**Proposition 5.5.** For $i \in \{1, \ldots, n-2\}$ and $j, k \in \{1, \ldots, n-1\}$ with $k \not\in \{j+1, j-1\}$,

$$\tau^\vee_i \tau^\vee_i = \tau^\vee_{i+1} \tau^\vee_i, \quad \tau^\vee_i \tau^\vee_{i+1} \tau^\vee_i = \tau^\vee_{i+1} \tau^\vee_i \tau^\vee_{i+1} \quad \text{and} \quad \tau^\vee_j \tau^\vee_k = \tau^\vee_k \tau^\vee_j;$$

$$\left(\tau^\vee_i \tau^\vee_i\right)^2 = \frac{(1-tY_{i+1}^{-1}Y_i^{-1})(1-tY_{i+1}^{-1})}{(1-Y_i^{-1}Y_{i+1})(1-Y_i^{-1}Y_{i+1}^{-1})}, \quad \text{for } i \in \{1, \ldots, n-1\};$$

$$Y_i \tau^\vee_w = \tau^\vee_w Y_{w^{-1}(i)}, \quad \text{for } w \in W \text{ and } i \in \mathbb{Z}.$$ 

**Proof.** Using $T_i = T_i^{-1} + (t^{1/2} - t^{-1/2})$,

$$\tau^\vee_i = T_i + \frac{t^{1/2}(1-t)}{1-Y_i^{-1}Y_{i+1}} = (T_i^{-1} + (t^{1/2} - t^{-1/2}) + \frac{t^{1/2}(1-t)}{1-Y_i^{-1}Y_{i+1}}$$

$$= T_i^{-1} + \frac{(Y_i^{-1}Y_{i+1}^{-1}+1)t^{1/2}(1-t)}{1-Y_i^{-1}Y_{i+1}} = T_i^{-1} + \frac{t^{1/2}(1-t)Y_{i+1}^{-1}Y_i^{-1}}{1-Y_i^{-1}Y_{i+1}}.$$

To prove (75), prove that

$$Y_i \tau^\vee_w = q^{-1} \tau^\vee_w Y_i \quad \text{and} \quad Y_i \tau^\vee_w = \tau^\vee_w Y_{w^{-1}(i)}, \quad \text{for } i \in \{2, \ldots, n\}, \quad \text{and}$$

$$Y_i \tau^\vee_w = \tau^\vee_w Y_i, \quad Y_{i+1} \tau^\vee_i = \tau^\vee_i Y_{i+1}, \quad Y_{i+1} \tau^\vee_i = \tau^\vee_i Y_{i+1} \quad \text{and} \quad Y_k \tau^\vee_i = \tau^\vee_i Y_k,$$

for $i \in \{1, \ldots, n-1\}$ and $k \in \{1, \ldots, n\}$ with $k \not\in \{i, i+1\}$. By (63) and (20),

$$\tau^\vee_i \tau^\vee_k = q^{\epsilon_{i+1}-\epsilon_i} \tau^\vee_k \quad \text{gives} \quad Y_i \tau^\vee_w = \tau^\vee_w Y_i = \tau^\vee_w Y_{w^{-1}(i)} = \tau^\vee_w Y_{w^{-1}(i)} = \tau^\vee_w Y_{w^{-1}(i)},$$

and $Y^{\epsilon_{i+1}} g^\vee = g^\vee Y^{\epsilon_i}$ for $i \in \{1, \ldots, n-1\}$. Using $Y_{i+1} = T_{i+1}^{-1} Y_i T_{i+1}^{-1}$,

$$\tau^\vee_i Y_i = \left(T_{i+1}^{-1} + \frac{t^{1/2}(1-t)Y_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}}\right) Y_i = \left(Y_{i+1} T_i + \frac{t^{1/2}(1-t)Y_{i+1}^{-1}}{1-Y_{i+1}^{-1}}\right) = Y_{i+1} \tau^\vee_i,$$

and

$$\tau^\vee_i Y_{i+1} = \left(T_i Y_{i+1} + \frac{t^{1/2}(1-t)Y_{i+1}}{1-Y_{i+1}^{-1}}\right) = Y_i \left(T_{i+1}^{-1} + \frac{t^{1/2}(1-t)Y_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}}\right) = Y_i \tau^\vee_i.$$

If $k \not\in \{i, i+1\}$ then $T_i Y_k = Y_k T_i$ and $Y_i Y_k = Y_k Y_i$ and $Y_{i+1} Y_k = Y_k Y_{i+1}$ and so

$$\tau^\vee_i Y_k = \left(T_i + \frac{t^{1/2}(1-t)}{1-Y_i^{-1}Y_{i+1}}\right) Y_k = Y_k \left(T_i + \frac{t^{1/2}(1-t)}{1-Y_i^{-1}Y_{i+1}}\right) = Y_k \tau^\vee_i.$$
Using (76),

\[
(t_i^+)^2 = \left( T_i + \frac{t_i^{-\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}} \right) t_i^+ T_i t_i^+ + t_i^+ \frac{t_i^{-\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}} T_i^{\text{span}} t_i^+ 
\]

\[
= T_i \left( T_i^{-1} + \frac{t_i^{-\frac{1}{2}}(1-t)Y_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}} \right) + \left( T_i + \frac{t_i^{-\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}} \right) \frac{t_i^{-\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}} 
\]

\[
= 1 + T_i \frac{t_i^{-\frac{1}{2}}(1-t)Y_i^{-1}Y_{i+1} + T_i t_i^{-\frac{1}{2}}(1-t) + \left( \frac{t_i^{-\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}} \right) \frac{t_i^{-\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}}}{(1-Y_i^{-1}Y_{i+1})(1-Y_i^{-1}Y_{i+1})} 
\]

\[
= \frac{(1-Y_i^{-1}Y_{i+1})Y_i^{-1}Y_{i+1} + t^{-1} - 2 + t}{(1-Y_i^{-1}Y_{i+1})(1-Y_i^{-1}Y_{i+1})} 
\]

The proof of the relations in (73) can be done by comparing the brute force expansion of each side using the relations in (62) and (63). An alternative, often used, argument is to note that the action of each side on the polynomial representation (which is a faithful representation of \( \hat{H} \)) produces the same output (see Proposition 2.14(e) of [17]).

5.6. The Polynomial Representation. In this section we build the action of the double affine Hecke algebra on Laurent polynomials in \( X_1, \ldots, X_n \). The elements \( Y_1, \ldots, Y_n \) are then a large family of commuting elements acting on the polynomial ring \( \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \). The Macdonald polynomials are the simultaneous eigenvectors for the family of commuting elements \( Y_1, \ldots, Y_n \).

Let \( q, t \in \mathbb{C}^\times \) such that \( 1 \not\in \{q^a t^b \mid a, b \in \mathbb{Z} \) and \( a \) and \( b \) not both \( 0 \) \) (alternatively, one may let \( q \) and \( t \) be formal parameters). The polynomial representation is

\[
\mathbb{C}[X] = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] = \mathbb{C}\text{-span}\{X^\mu \mid \mu \in \mathbb{Z}^n\}
\]

with the action of DAHA determined by \( T_i 1 = t_i^+ 1 \) and \( g 1 = 1 \) so that, by (55),

\[
Y_i^\mu 1 = t_i^{\mu(n-1)} 1 \quad \text{and} \quad Y_i^{-\mu} 1 = t_i^{(n-1)2 - (i-1)^2} 1 = t_i^{-i(i-1)+\frac{1}{2}(n-1)} 1.
\]

Following the notation of [14, Ch. VI (3.1)], let \( T_{q^{-1},X_n} \) be the operator on \( \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \) given by

\[
(T_{q^{-1},X_n} h)(X_1, \ldots, X_n) = h(X_1, \ldots, X_{n-1}, q^{-1} X_n).
\]

**Proposition 5.6.** As operators on the polynomial representation

\[
g = s_1 s_2 \cdots s_{n-1} T_{q^{-1},X_n} \quad \text{and} \quad T_i = t_i^{-\frac{1}{2}}(t - \frac{t X_i - X_{i+1}(1 - s_i)}{X_i - X_{i+1}(1 - s_i)}),
\]

for \( i \in \{1, \ldots, n-1\} \).

**Proof.** The first statement in Proposition 5.4 gives

\[
g X^\mu 1 = (s_1 s_2 \cdots s_{n-1} T_{q^{-1},X_n} X^\mu) 1,
\]

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since \( T_{q^{-1},X_n}^{-1}X^\mu = q^{-\mu_n}X^\mu \). Using the second statement in Proposition 5.4, (79) \[ T_iX^\mu 1 = \left( (s_iX^\mu)t^\frac{1}{2} + \frac{t^\frac{1}{2} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}}(1 - s_i)X^\mu \right) 1 \]

\[ = \left( t^\frac{1}{2} - t^\frac{1}{2}(1 - s_i) + \frac{t^\frac{1}{2} - t^{-\frac{1}{2}}}{1 - X_iX_{i+1}}(1 - s_i) \right) X^\mu 1 \]

\[ = \left( t^\frac{1}{2} + \frac{1}{1 - X_iX_{i+1}}(X_iX_{i+1}^{-1}t^\frac{1}{2} - t^{-\frac{1}{2}}) (1 - s_i) \right) X^\mu 1 \]

\[ = \left( t^\frac{1}{2} + \frac{1}{X_i - X_{i+1}}(-X_it^\frac{1}{2} + X_{i+1}t^{-\frac{1}{2}}) (1 - s_i) \right) X^\mu 1 \]

\[ = t^{-\frac{1}{2}} \left( t - \frac{tX_i - X_{i+1}}{X_i - X_{i+1}}(1 - s_i) \right) X^\mu 1. \]

□

5.7. Constructing the nonsymmetric Macdonald polynomials \( E_\mu \). The Macdonald polynomials are the simultaneous eigenvectors for the action of \( Y_1, \ldots, Y_n \) on the polynomial ring \( \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \). Because the intertwiners \( \tau_1^\vee, \ldots, \tau_{n-1}^\vee, \tau_n^\vee \) move past \( Y_1, \ldots, Y_n \) in the best possible way, they are the perfect tools for explicitly computing the Macdonald polynomials \( E_\mu \).

**Proposition 5.7.** Let \( \mu \in \mathbb{Z}_{\geq 0}^n \) and let \( u_\mu \) and \( v_\mu \) be as in (33) and let \( \ell(v_\mu) \) be the number of inversions of \( v_\mu \). Choose a reduced word \( \bar{u}_\mu = s_{i_1} \cdots s_{i_\ell} \) (where \( i_1, \ldots, i_\ell \in \{1, \ldots, n-1\} \)) and let \( \tau_{u_\mu} = \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \). Define

\[ E_\mu = t^{-\frac{1}{2}\ell(v_\mu)}\tau_{u_\mu}^\vee 1. \]

Then \( Y_i E_\mu = q^{-\mu_i}t^{-(\nu_\mu(i) - 1) + \frac{1}{2}(n-1)} E_\mu, \)

for \( i \in \{1, \ldots, n\} \), and the coefficient of \( x^\mu \) in \( E_\mu \) is 1.

**Proof.** Compute the eigenvalue as follows:

\[ Y_i E_\mu = Y_i t^{-\frac{1}{2}\ell(v_\mu)}\tau_{u_\mu}^\vee 1 \quad \text{(by definition of } E_\mu) \]

\[ = t^{-\frac{1}{2}\ell(v_\mu)}\tau_{u_\mu}^\vee Y_{\nu(i)}^{-1} v_\mu^{-1} 1 \quad \text{(by (75))} \]

\[ = t^{-\frac{1}{2}\ell(v_\mu)}\tau_{u_\mu}^\vee Y_{\nu(i)}^{-1} v_\mu^{-1} 1 \quad \text{(by (33))} \]

\[ = t^{-\frac{1}{2}\ell(v_\mu)}\tau_{u_\mu}^\vee Y_{\nu(i)-\mu_i}^{-1} 1 \quad \text{(by (22))} \]

\[ = t^{-\frac{1}{2}\ell(v_\mu)}\tau_{u_\mu}^\vee Y_{\nu(i)-\mu_i}^{-1} 1 \quad \text{(by (19))} \]

\[ = t^{-\frac{1}{2}\ell(v_\mu)}\tau_{u_\mu}^\vee q^{-\mu_i} Y_{\nu(i)}^{-1} 1 \quad \text{(by (70))} \]

\[ = q^{-\mu_i} t^{-(\nu_\mu(i) - 1) + \frac{1}{2}(n-1)} \left( t^{-\frac{1}{2}\ell(v_\mu)}\tau_{u_\mu}^\vee 1 \right) \quad \text{(by (78))} \]

\[ = q^{-\mu_i} t^{-(\nu_\mu(i) - 1) + \frac{1}{2}(n-1)} E_\mu \quad \text{(by definition of } E_\mu) \]

Using (67) and (5.3), the top term of the expansion of in \( t^{-\frac{1}{2}\ell(v_\mu)}\tau_{u_\mu}^\vee 1 \) is

\[ t^{-\frac{1}{2}\ell(v_\mu)} X^\mu 1 = t^{-\frac{1}{2}\ell(v_\mu)} T_{\nu(i)}^{-1} 1 = t^{-\frac{1}{2}\ell(v_\mu)} X^\mu 1 = X^\mu 1 = x^\mu. \]

□
5.8. Steps and Symmetries of $E_{\mu}$. The following Proposition establishes the inductive construction of the $E_{\mu}$ and the symmetries in (3.3). For examples of the $E_{\mu}$ see Proposition 3.5, which provides explicit formulas for the cases when $\mu$ has 1 or 2 boxes.

**Proposition 5.8.** Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$.

(a) If $i \in \{1, \ldots, n - 1\}$ and $\mu_i > \mu_{i+1}$ then $E_{\mu_{i+1}} = t^{\frac{1}{2}}x_i^\vee E_{\mu}$.

(b) $E_{(\mu_1, \mu_2, \ldots, \mu_{n-1})} = t^{\# \{i \in \{1, \ldots, n\} | \mu_i > \mu_{i+1} \}} \tau_\pi E_{(\mu_1, \ldots, \mu_n)}$.

(c) $E_{(\mu_1, \mu_2, \ldots, \mu_{n-1})} = q^{\mu_1 x_1 E_{\mu}(x_2, \ldots, x_n, q^{-1} x_1)}$.

(d) $E_{(\mu_1, \ldots, \mu_n)} = x_1^{\mu_1} \cdots x_n E_{(\mu_1, \ldots, \mu_n)}$.

(e) $E_{(-\mu_n, \ldots, -\mu_1)}(x_1, \ldots, x_n; q, t) = E_{\mu}(x_1^{-1}, \ldots, x_n^{-1}; q, t)$.

**Proof.** (a) Let $\mu = (\mu_1, \ldots, \mu_n)$ and let $i$ be such that $\mu_i > \mu_{i+1}$.

By Proposition 2.1(e), $\ell(v_{n, \mu}) - \ell(v_{n, \mu}) = -1$, giving $E_{\mu_{i+1}} = t^{\frac{1}{2}}x_i^\vee E_{\mu}$.

(b) The left hand side is $E_{\mu}$ and

$$E_{\mu_{i+1}} = t^{\frac{1}{2}}(\ell(v_{n, \mu}) - \ell(v_{n, \mu})) E_{\mu} = t^{\frac{1}{2}}(\tau_\pi) E_{\mu} = (t^{\frac{1}{2}}) E_{\mu}.$$ 

The result then follows from Proposition 2.1(b).

(c) The second relation in (54) and the second relation in (63) give $X_1 g = g^Y Y_n$. Beginning with the right hand side of (b) and using $g^Y Y_n = X_1 g$ gives

$$E_{\mu_{i+1}} = t^{\frac{1}{2}}(\ell(v_{n, \mu}) - \ell(v_{n, \mu})) E_{\mu} = t^{\frac{1}{2}}(\tau_\pi) E_{\mu} = (t^{\frac{1}{2}}) E_{\mu}.$$ 

(d) By (58), $(\tau_\pi)^n = X_1 \cdots X_n$ and so

$$E_{(\mu_1, \ldots, \mu_n)} = (\tau_\pi)^n E_{(\mu_1, \ldots, \mu_n)} = x_1^{\mu_1} \cdots x_n E_{(\mu_1, \ldots, \mu_n)}.$$

(e) Let $\eta: \tilde{HGL}_n \rightarrow \tilde{HGL}_n$ be the involution in (69). Let $w_0$ be the longest element of $S_n$ so that $w_0(i) = n - i + 1$ for $i \in \{1, \ldots, n\}$. Using the last relation in (56),

$$Y_i \eta(E_{\mu}(X_1, \ldots, X_n)) = \eta(Y_{\mu - i} E_{\mu}(X_1, \ldots, X_n)) = q^{\mu_n-i} E_{\mu}(X_1, \ldots, X_n),$$

so that $\eta(E_{\mu}(X_1, \ldots, X_n)) = E_{\mu}(x_1^{-1}, \ldots, x_n^{-1})$ satisfies the conditions (from Theorem 5.7) determining $E_{-\mu}(x_1, \ldots, x_n)$. 

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References


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