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Abstract We prove that semi-infinite Bruhat order on an affine Weyl group is completely determined from those on the quotients by affine Weyl subgroups associated with various maximal (standard) parabolic subgroups of finite type. Furthermore, for an affine Weyl group of classical type, we give a complete classification of all cover relations of semi-infinite Bruhat order (or equivalently, all edges of the quantum Bruhat graphs) on the quotients in terms of tableaux. Combining these we obtain a tableau criterion for semi-infinite Bruhat order on an affine Weyl group of classical type. As an application, we give new tableau models for the crystal bases of a level-zero fundamental representation and a level-zero extremal weight module over a quantum affine algebra of classical untwisted type, which we call quantum Kashiwara–Nakashima columns and semi-infinite Kashiwara–Nakashima tableaux. We give an explicit description of the crystal isomorphisms among three different realizations of the crystal basis of a level-zero fundamental representation by quantum Lakshmibai–Seshadri paths, quantum Kashiwara–Nakashima columns, and (ordinary) Kashiwara–Nakashima columns.

Contents

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Keywords. Affine Weyl group, quantum affine algebra, semi-infinite Bruhat order, quantum Bruhat graph, level-zero fundamental representation, level-zero extremal weight module.

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1. Introduction

The aim of this paper is to give an explicit description of semi-infinite Bruhat order in terms of tableaux and its application to level-zero representation theory of quantum affine algebras.

Semi-infinite Bruhat order on an affine Weyl group is a variant of Bruhat order on a Coxeter group, and it is also an important tool to study representation theory of algebraic groups, quantum groups, and affine Kac–Moody Lie algebras, quantum and affine Schubert calculi, and so forth (see [2, 5, 10, 11, 19, 22, 24, 25, 20, 28, 29, 32] and the references given there). In fact, analogously to (ordinary) Bruhat order, each of the following structures is closely related (or equivalent) to semi-infinite Bruhat order: Lusztig's generic Bruhat order ([28, §1.5]) and Peterson's stable Bruhat order ([32, Lecture 12]) on an affine Weyl group, Littelmann's order on the affine Weyl group orbit through a level-zero integral weight of an affine Kac–Moody Lie algebra ([27, §4]), the quantum Bruhat graphs ([5, Definition 6.1] and [24, §4]), homomorphisms among Wakimoto modules over an affine Kac–Moody Lie algebra ([2, Proposition 4.10]), the containment relation among (opposite) Demazure subcrystals of the crystal basis of a level-zero extremal weight module over a quantum affine algebra ([29, Corollary 5.2.5]), and the containment relation among semi-infinite Schubert varieties ([19, §4.2]); see also (i)–(ii) below. Also, it is worth pointing out that semi-infinite Bruhat order has the lifting property (or "diamond lemma") with respect to the semi-infinite length function ([11, §4.1]; see also Lemma 2.6); note that Bruhat order on a Coxeter group is characterized uniquely by the lifting property with respect to the length function ([8, Theorem 1.1]; see also [4, Exercise 14 in §2]). However there indeed exist some differences between semi-infinite and ordinary Bruhat orders. For example, there are no minimal elements in semi-infinite Bruhat order, and this order depends on not only the Coxeter system but also the root data. Therefore the standard method to study Bruhat order (see for instance [4]) is not well adapted to the study of semi-infinite Bruhat order.

In [10], we intended to develop tableau combinatorics on semi-infinite Bruhat order. We introduced semi-infinite Young tableaux, and showed that these tableaux give a combinatorial realization of the crystal basis of a level-zero extremal weight module ([14, 16]) over the quantum affine algebra of type $A_{n-1}^{(1)}$. Our proof of this result was based on a tableau criterion for semi-infinite Bruhat order and standard monomial theory for semi-infinite Lakshmibai–Seshadri paths ([11, §3.1]). Therefore we can think of these tableaux as a natural generalization of (ordinary) Young tableaux. However we restricted the discussion in [10, §4] only to the affine Weyl group of type $A_{n-1}^{(1)}$, and the tableau criterion in [10, Theorem 4.7] is applicable only to semi-infinite Bruhat order on the quotient by an affine Weyl subgroup associated with a maximal parabolic subgroup of finite type.

In this paper, we wish to investigate semi-infinite Bruhat order on an affine Weyl group of all classical untwisted type via tableaux, and aim to give an application to level-zero representation theory (see for instance [1, 3, 16, 17]) of a quantum affine algebra of classical untwisted type. The main results of this paper are the following (see Theorems 3.1, 4.9, 4.14, 4.22, 4.38, 5.13, 5.16, 5.18, 5.21, and 5.23):

(I) a Deodhar-type criterion for semi-infinite Bruhat order on an affine Weyl group of arbitrary untwisted type,

(II) a tableau criterion for semi-infinite Bruhat order on an affine Weyl group of type $A_{n-1}^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, and $D_n^{(1)}$ in full generality,

(III) a tableau model for the crystal basis of a level-zero fundamental representation (resp. a level-zero extremal weight module) over a quantum affine algebra of type $B_n^{(1)}$ and $D_n^{(1)}$ (resp. $B_n^{(1)}, C_n^{(1)}$, and $D_n^{(1)}$), which we call quantum Kashiwara–Nakashima columns (resp. semi-infinite Kashiwara–Nakashima tableaux), and

(IV) an explicit description of the isomorphisms among three different realizations of the crystal basis of a level-zero fundamental representation by quantum Lakshmibai–Seshadri paths ([25]), quantum Kashiwara–Nakashima columns, and (ordinary) Kashiwara–Nakashima columns ([18]).

Let us give more precise explanation of our results. Let $\mathbf{U}$ be a quantum affine algebra of untwisted type, and let $\mathbf{U}'$ be its derived subalgebra. Let $W_{\mathrm{af}} = \langle r_i \mid i \in I_{\mathrm{af}} \rangle$ be the affine Weyl group associated with a finite Weyl group $W = \langle r_i \mid i \in I \rangle$, where $I_{\mathrm{af}} = \{0\} \sqcup I$ and r_i, $i \in I_{\mathrm{af}}$, is a simple reflection. For $J \subset I$, let $W_J = \langle r_j \mid j \in J \rangle \subset W$ be a (standard) parabolic subgroup, and let $W^J \subset W$ be the set of minimal(-length) coset representatives for W/W_J. Let $(W_J)_{\mathrm{af}} \subset W_{\mathrm{af}}$ be the affine Weyl subgroup associated with W_J (see (19)). Note that $(W_J)_{\mathrm{af}}$ is not a parabolic subgroup of W_{af}, but it is generated by reflections. Therefore there exists the subset $(W^J)_{\mathrm{af}} \subset W_{\mathrm{af}}$ of minimal coset representatives for $W_{\mathrm{af}}/(W_J)_{\mathrm{af}}$ (see (20)); note that if $J = \emptyset$, then $(W_J)_{\mathrm{af}}$ is trivial and $(W^J)_{\mathrm{af}} = W_{\mathrm{af}}$. Let $\Pi^J : W_{\mathrm{af}} \to (W^J)_{\mathrm{af}}$ be the canonical surjection. In this paper, following [11, §2.4] (see also [32, Lecture 12]), we define semi-infinite Bruhat order $\preceq$ on $(W^J)_{\mathrm{af}}$ by using the semi-infinite length function $\ell^{\frac{\infty}{2}} : W_{\mathrm{af}} \to \mathbb{Z}$ (see (21)). But we mainly use the following two realizations of semi-infinite Bruhat order (see Lemmas 3.7 (3) and 4.3 for precise formulation):

(i) the containment relation among the path model $\mathbb{B}^{\frac{\infty}{2}}_{x}(\lambda)$, $x \in W_{\mathrm{af}}$, of the (opposite) Demazure subcrystals of the crystal basis of the extremal weight $\mathbf{U}$-module of level-zero extremal weight λ ([29, §5]),

(ii) the "affinization" of the (parabolic) quantum Bruhat graph QB^J for W^J ([11, Appendix A]).

In §3, we prove a Deodhar-type criterion for semi-infinite Bruhat order (see (I) above), which states that for $x, y \in (W^J)_{\mathrm{af}}$ we have $x \preceq y$ in $(W^J)_{\mathrm{af}}$ if and only if

$$(1) \qquad \Pi^{I \setminus \{i\}}(x) \preceq \Pi^{I \setminus \{i\}}(y) \text{ in } (W^{I \setminus \{i\}})_{\mathrm{af}} \text{ for all } i \in I \setminus J$$

(cf. [4, Theorem 2.6.1]; see also [8, Lemma 3.6]). We prove this for W_{af} of arbitrary untwisted type (see Theorem 3.1 and Proposition 3.2). This is shown by investigating the path model $\mathbb{B}^{\frac{\infty}{2}}_{x}(\lambda)$, $x \in W_{\mathrm{af}}$, of Demazure subcrystals. More precisely, let ϖ_i, $i \in I$, be a level-zero fundamental weight, and let $\lambda = \sum_{i \in I} m_i \varpi_i$, $m_i \in \mathbb{Z}_{\geq 0}$, $i \in I$. Set $J_\lambda =$

$\{i \in I \mid m_i = 0\}$. The path model $\mathbb{B}^{\frac{\infty}{2}}_x(\lambda)$, $x \in W_{\mathrm{af}}$, is defined as a subset of the $\mathbf{U}$-crystal $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ of semi-infinite Lakshmibai–Seshadri paths of shape λ. We prove that any extremal element in the tensor product $\bigotimes_{i \in I \setminus J_\lambda} \mathbb{B}^{\frac{\infty}{2}}_{\Pi^{I \setminus \{i\}}(x)}(m_i \varpi_i)$ (see (46)) is in the image of $\mathbb{B}^{\frac{\infty}{2}}_x(\lambda)$ under the isomorphism $\psi_\lambda : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \bigotimes_{i \in I \setminus J_\lambda} \mathbb{B}^{\frac{\infty}{2}}(m_i \varpi_i)$ of $\mathbf{U}$-crystals (see Lemma 3.3 and Proposition 3.4 (2)). In §3.3, we see that this immediately yields the Deodhar-type criterion.

In §4, for W_{af} of type $A^{(1)}_{n-1}$, $B^{(1)}_n$, $C^{(1)}_n$, and $D^{(1)}_n$, we give a complete classification of the cover relations of semi-infinite Bruhat order on $(W^{I \setminus \{i\}})_{\mathrm{af}}$, $i \in I$, in terms of tableaux (Propositions 4.19, 4.27, and 4.43; see also [10, Proposition 4.11]). Moreover, we prove a tableau criterion for semi-infinite Bruhat order on $(W^{I \setminus \{i\}})_{\mathrm{af}}$, $i \in I$ (Definitions 4.7, 4.12, 4.20, and 4.36, and Propositions 4.8, 4.13, 4.21, and 4.37; see also [10, Theorem 4.7]). Combining this with the Deodhar-type criterion we obtain (II). We emphasize that (II) can be thought of as a generalization of the tableau criterion for Bruhat order on the symmetric group ([4, Theorem 2.6.3 (Tableau Criterion)]). Our main tool in §4 is the quantum Bruhat graph. In fact, we classify all (quantum) edges in the quantum Bruhat graph $\mathrm{QB}^{I \setminus \{i\}}$ for $W^{I \setminus \{i\}}$, $i \in I$ (Propositions 4.18, 4.26, and 4.42; see also [10, Lemma 4.15]). Combining this with the realization (ii) yields the classification results above. We should remark that, for W_{af} of type $A^{(1)}_{n-1}$ and $C^{(1)}_n$, Lenart's criterion ([22, Propositions 3.6 and 5.7]) for the edges of the quantum Bruhat graph $\mathrm{QB}^{?}$ for W is a necessary condition for our classification results. Indeed, the existence of an edge in $\mathrm{QB}^{I \setminus \{i\}}$ implies that in $\mathrm{QB}^{?}$ (see Lemma 4.4 for a precise statement).

We now briefly sketch the tableau criterion for semi-infinite Bruhat order on $(W^{I \setminus \{i\}})_{\mathrm{af}}$ when W_{af} is of type $A^{(1)}_{n-1}$ (§4.3 and [10, §4]; see also Example 4.10). First, we associate each element of W_{af} with a pair (T, c) of a column T of length i and an integer c (see (65)). Let $\mathrm{T}(u) \in \{1, 2, \dots, n\}$ denote the u-th entry of T. Let (T, c), (T', c') be such pairs for $x, y \in W_{\mathrm{af}}$, respectively. Then we have $\Pi^{I \setminus \{i\}}(x) \preceq \Pi^{I \setminus \{i\}}(y)$ in $(W^{I \setminus \{i\}})_{\mathrm{af}}$ if and only if

$$(2) \qquad (c \leqslant c') \text{ and } (\mathrm{T}(u) \leqslant \mathrm{T}'(u + c' - c) \text{ for } 1 \leqslant u \leqslant i - c' + c).$$

In §5, we prove (III)–(IV). For this purpose, we first investigate the subgraph $\mathrm{QB}(\varpi_i; 1/2)$ (see §5.1–§5.2) of the quantum Bruhat graph $\mathrm{QB}^{I \setminus \{i\}}$, where the vertex sets of these graphs are both $W^{I \setminus \{i\}}$. We see that $\mathrm{QB}(\varpi_i; 1/2)$ defines a partial order $\trianglelefteq$ on $W^{I \setminus \{i\}}$. Moreover, by the classification results in §4, we obtain an explicit description of the order $\trianglelefteq$ in terms of Maya diagrams (see Definitions 5.4 and 5.6). Then the $\mathbf{U}'$-crystal $\mathrm{QLS}(\varpi_i)$ of quantum Lakshmibai–Seshadri paths (QLS paths for short) of shape ϖ_i is given by

$$(3) \qquad \mathrm{QLS}(\varpi_i) = \{(v, w) \mid w, v \in W^{I \setminus \{i\}},\ w \trianglelefteq v\}.$$

It is well-known that $W^{I \setminus \{i\}}$ can be realized as a set of columns (see Lemmas 4.6, 4.11, and 4.35). Thus we can think of each QLS path $(v, w) \in \mathrm{QLS}(\varpi_i)$ as a pair of columns. We know from [25] that $\mathrm{QLS}(\varpi_i)$ is isomorphic, as a $\mathbf{U}'$-crystal, to the crystal basis of the level-zero fundamental representation $W(\varpi_i)$ ([16, 17]; see also §2.4 and §5.1).

Assume that $\mathbf{U}$ is of type $B^{(1)}_n$. In §5.4, we introduce quantum Kashiwara–Nakashima B_n-columns (QKN B_n-columns for short) and a $\mathbf{U}'$-crystal structure on them. In particular, we define Kashiwara operators e_j, f_j, $j \in I_{\mathrm{af}}$, acting on them. A QKN B_n-column $\tilde{\mathrm{C}}$ of shape ϖ_i is consisting of an (ordinary) Kashiwara–Nakashima

B_n-column (KN B_n-column for short) C of shape ϖ_{i-2m} and a multiset $\{\underbrace{\overline{0},\overline{0},\dots,\overline{0}}_{2m \text{ times}}\}$ for some integer $0 \leqslant m \leqslant \left\lfloor \frac{i}{2} \right\rfloor = \max\{k \in \mathbb{Z} \mid k \leqslant i/2\}$; we write $\tilde{\mathrm{C}} = \mathrm{C} \sqcup \{\underbrace{\overline{0},\overline{0},\dots,\overline{0}}_{2m \text{ times}}\}$ for brevity. Let $\mathrm{QKN}_{B_n}(\varpi_i)$ (resp. $\mathrm{KN}_{B_n}(\varpi_i)$) denote the set of QKN B_n-columns (resp. KN B_n-columns) of shape ϖ_i. Let us give an example of a QKN B_n-column. We have

$$(4) \qquad \tilde{\mathrm{C}} = \begin{array}{|c|}\hline 2\\\hline 3\\\hline 0\\\hline \overline{9}\\\hline \overline{3}\\\hline \overline{0}\\\hline \overline{0}\\\hline\end{array} = \mathrm{C} \sqcup \{\overline{0},\overline{0}\} \in \mathrm{QKN}_{B_9}(\varpi_7) \quad \text{and} \quad \mathrm{C} = \begin{array}{|c|}\hline 2\\\hline 3\\\hline 0\\\hline \overline{9}\\\hline \overline{3}\\\hline\end{array} \in \mathrm{KN}_{B_9}(\varpi_5).$$

For each QKN B_n-column $\tilde{\mathrm{C}}$ of shape ϖ_i, we construct a pair $(r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}})$ of columns, and show that it is a QLS path of shape ϖ_i. The construction of $(r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}})$ was motivated by [34, §4] (see also [21, §3]), and has previously been used by Briggs ([6]; see also [26, Algorithm 4.1]). For the QKN B_n-column $\tilde{\mathrm{C}}$ in (4), we have

$$(5) \qquad r\tilde{\mathrm{C}} = \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline \overline{9}\\\hline \overline{8}\\\hline \overline{1}\\\hline\end{array} \quad \text{and} \quad l\tilde{\mathrm{C}} = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 8\\\hline \overline{9}\\\hline \overline{5}\\\hline \overline{4}\\\hline \overline{3}\\\hline\end{array};$$

by using notation in §5.2, §5.4, and §5.6, we have

(1) $I_{\mathrm{C}} = \{0 \in 3\}$, $J_{\mathrm{C}} = \{8 > 1\}$, $K_{\tilde{\mathrm{C}}} = \{4 < 5\}$,
(2) $J(r\tilde{\mathrm{C}}) = J(l\tilde{\mathrm{C}}) = (\{1,2,3,4,5\} < \{8,9\}) \in S_7$, and
(3) $M(r\tilde{\mathrm{C}}) = (\{1\},\{8,9\})$, $M(l\tilde{\mathrm{C}}) = (\{3,4,5\},\{9\}) \in 2^{\{1,2,3,4,5\}} \times 2^{\{8,9\}}$.

Then we prove that the map $\mathrm{QKN}_{B_n}(\varpi_i) \to \mathrm{QLS}(\varpi_i)$, $\tilde{\mathrm{C}} \mapsto (r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}})$, is bijective. The important point to note here is that the inverse of this map can be described explicitly in terms of Maya diagrams (see Theorem 5.16 (2)); similar results have been obtained independently by Lenart–Schulze ([26, §4]), where they used the quantum alcove model ([23]). Thus we obtain the following crystal isomorphisms among the sets of QLS paths, QKN B_n-columns, and KN B_n-columns (cf. [7, Lemma 2.7]).

$$(6) \qquad \begin{gathered} \mathrm{QLS}(\varpi_i) \leftrightarrow \mathrm{QKN}_{B_n}(\varpi_i) \leftrightarrow \bigoplus_{m=0}^{\lfloor \frac{i}{2} \rfloor} \mathrm{KN}_{B_n}(\varpi_{i-2m}), \\ (r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) \leftrightarrow \tilde{\mathrm{C}} = \mathrm{C} \sqcup \{\underbrace{\overline{0},\overline{0},\dots,\overline{0}}_{2m \text{ times}}\} \leftrightarrow \mathrm{C}. \end{gathered}$$

Consequently, the $\mathbf{U}$-crystal $\mathrm{QKN}_{B_n}(\varpi_i)$ is isomorphic to the crystal basis of the level-zero fundamental representation $W(\varpi_i)$. Similar formulation and results also hold for $\mathbf{U}$ of type $D_n^{(1)}$ (see §5.5). We should remark that the crystal basis of $W(\varpi_i)$ is isomorphic to the Kirillov–Reshetikhin crystal $B^{i,1}$, and that there is another tableau model, called Kirillov–Reshetikhin tableaux ([30, 33]), more generally for the Kirillov–Reshetikhin crystals $B^{r,s}$ (see [9, §2.3]), $r \in I$, $s > 0$, of non-exceptional type. The advantage of using QKN columns lies in the explicit description of the isomorphisms (6). Likewise, the Kirillov–Reshetikhin tableaux model is an important ingredient to describe the rigged configuration bijections (see for instance [31]). It would be desirable to relate these two tableau models but we will not develop this point here.

Assume that $\mathbf{U}$ is of type $B_n^{(1)}$, $C_n^{(1)}$, or $D_n^{(1)}$. In §5.3–§5.5, we introduce semi-infinite Kashiwara–Nakashima tableaux (semi-infinite KN tableaux for short), and show that the set of these tableaux is isomorphic, as a $\mathbf{U}$-crystal, to the crystal basis of a level-zero extremal weight $\mathbf{U}$-module. This is achieved by applying standard monomial theory for semi-infinite Lakshmibai–Seshadri paths ([10, Theorem 3.4]) to the affinizations of $\mathbf{U}'$-crystals of (Q)KN columns. The definition of semi-infinite KN tableaux is based on the tableau criterion for semi-infinite Bruhat order obtained in §4.

This paper is organized as follows. In §2, we set up notation and terminology on untwisted affine root data, crystals, semi-infinite Bruhat order, extremal weight modules, and semi-infinite Lakshmibai–Seshadri paths. Also, we have compiled some basic facts on these. In §3, we prove a Deodhar-type criterion for semi-infinite Bruhat order on W_{af} of arbitrary untwisted type. In §4, we prove a tableau criterion for semi-infinite Bruhat order on W_{af} of type $A_{n-1}^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, and $D_n^{(1)}$, by classifying all cover relations of semi-infinite Bruhat order on $(W^{I \setminus \{i\}})_{\mathrm{af}}$ in terms of tableaux. In §5, we introduce the $\mathbf{U}'$-crystal of QKN columns and the $\mathbf{U}$-crystal of semi-infinite KN tableaux. We show that these tableaux give combinatorial models for crystal bases of level-zero fundamental representations and level-zero extremal weight modules. We give an explicit description of the crystal isomorphisms among QLS paths, QKN columns, and KN columns.

Notation. Let $\mathbb{Z}_{>0}$ (resp. $\mathbb{Z}_{\geq 0}$) denote the set of positive integers (resp. non-negative integers). For $k \in \mathbb{Z}$, set $[k] = \{1, 2, \dots, k\}$ if $k \geq 1$, and set $[k] = \emptyset$ if $k \leq 0$. For integers $k \leq l$, set $[k, l] = \{k, k+1, \dots, l\}$; we understand that $[k, l] = \emptyset$ if $k > l$. The disjoint union of two sets A, B will be denoted by $A \sqcup B$. For a (non-empty) set A, let $\mathfrak{S}(A)$ be the permutation group of A. For a finite set A, let $\#A$ denote the number of elements in A.

2. Preliminaries

2.1. Untwisted affine root data. Let $\mathfrak{g}_{\mathrm{af}}$ be an untwisted affine Kac–Moody Lie algebra over $\mathbb{C}$ with a Cartan subalgebra $\mathfrak{h}_{\mathrm{af}}$. Let $\{\alpha_i\}_{i \in I_{\mathrm{af}}} \subset \mathfrak{h}_{\mathrm{af}}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathrm{af}}, \mathbb{C})$ and $\{\alpha_i^\vee\}_{i \in I_{\mathrm{af}}} \subset \mathfrak{h}_{\mathrm{af}}$ be the sets of simple roots and simple coroots, respectively. Here I_{af} denotes the vertex set of the (affine) Dynkin diagram of $\mathfrak{g}_{\mathrm{af}}$. Let $\langle \cdot, \cdot \rangle : \mathfrak{h}_{\mathrm{af}}^* \times \mathfrak{h}_{\mathrm{af}} \to \mathbb{C}$ be the canonical pairing. We take and fix an integral weight lattice $P_{\mathrm{af}} \subset \mathfrak{h}_{\mathrm{af}}^*$ satisfying the conditions that $\alpha_i \in P_{\mathrm{af}}$ and $\alpha_i^\vee \in \mathrm{Hom}_{\mathbb{Z}}(P_{\mathrm{af}}, \mathbb{Z})$ for all $i \in I_{\mathrm{af}}$, and for each $i \in I_{\mathrm{af}}$ there exists $\Lambda_i \in P_{\mathrm{af}}$ such that $\langle \alpha_j^\vee, \Lambda_i \rangle = \delta_{ij}$ for all $j \in I_{\mathrm{af}}$. Similarly, let $P_{\mathrm{af}}^\vee \subset \mathfrak{h}_{\mathrm{af}}$ be an integral coweight lattice such that $\alpha_i^\vee \in P_{\mathrm{af}}^\vee$ and $\alpha_i \in \mathrm{Hom}_{\mathbb{Z}}(P_{\mathrm{af}}^\vee, \mathbb{Z})$ for all $i \in I_{\mathrm{af}}$, and for each $i \in I_{\mathrm{af}}$ there exists $\Lambda_i^\vee \in P_{\mathrm{af}}^\vee$ such that $\langle \Lambda_i^\vee, \alpha_j \rangle = \delta_{ij}$ for all $j \in I_{\mathrm{af}}$. Let $\delta = \sum_{i \in I_{\mathrm{af}}} a_i \alpha_i \in \mathfrak{h}_{\mathrm{af}}^*$ and $c = \sum_{i \in I_{\mathrm{af}}} a_i^\vee \alpha_i^\vee \in \mathfrak{h}_{\mathrm{af}}$ be the null root and the canonical central element, respectively. For $\lambda \in P_{\mathrm{af}}$, the integer $\langle c, \lambda \rangle$ is called the level of λ. We take and fix $0 \in I_{\mathrm{af}}$ such that $a_0 = a_0^\vee = 1$. Set $I = I_{\mathrm{af}} \setminus \{0\}$; note that the subset I of I_{af} corresponds to the vertex set of the Dynkin diagram of a complex finite-dimensional simple Lie subalgebra $\mathfrak{g}$ of $\mathfrak{g}_{\mathrm{af}}$. Fix a section $\iota : P_{\mathrm{af}}/(P_{\mathrm{af}} \cap \mathbb{C}\delta) \to P_{\mathrm{af}}$ (resp. $\iota : P_{\mathrm{af}}^\vee/(P_{\mathrm{af}}^\vee \cap \mathbb{C}c) \to P_{\mathrm{af}}^\vee$) of the canonical surjection cl : $P_{\mathrm{af}} \to P_{\mathrm{af}}/(P_{\mathrm{af}} \cap \mathbb{C}\delta)$ (resp. cl : $P^\vee_{\mathrm{af}} \to P^\vee_{\mathrm{af}}/(P^\vee_{\mathrm{af}} \cap \mathbb{C}c)$) such that $(\iota \circ \mathrm{cl})(\alpha_i) = \alpha_i$ (resp. $(\iota \circ \mathrm{cl})(\alpha_i^\vee) = \alpha_i^\vee$) for $i \in I$. For each $i \in I_{\mathrm{af}}$, define $\varpi_i = (\iota \circ \mathrm{cl})(\Lambda_i - \langle c, \Lambda_i \rangle \Lambda_0)$ and $\varpi_i^\vee = (\iota \circ \mathrm{cl})(\Lambda_i^\vee - \langle \Lambda_i^\vee, \delta \rangle \Lambda_0^\vee)$; note that $\varpi_0 = 0$, $\varpi_0^\vee = 0$, $\langle c, \varpi_i \rangle = \langle \varpi_i^\vee, \delta \rangle = 0$, and $\langle \alpha_j^\vee, \varpi_i \rangle = \langle \varpi_i^\vee, \alpha_j \rangle = \delta_{ij}$ for all $i, j \in I$. We call ϖ_i the i-th level-zero fundamental weight of $\mathfrak{g}_{\mathrm{af}}$. Set

(7) $$Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i, \qquad Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee, \qquad P = \bigoplus_{i \in I} \mathbb{Z}\varpi_i, \qquad P^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\varpi_i;$$

note that $Q \subset P$ and $Q^\vee \cong \bigoplus_{i \in I} \mathbb{Z}\varpi_i^\vee$. We think of P (resp. Q) as a weight lattice (resp. a root lattice) of $\mathfrak{g}$.

Let $W_{\mathrm{af}} = \langle r_i \mid i \in I_{\mathrm{af}} \rangle$ be the (affine) Weyl group of $\mathfrak{g}_{\mathrm{af}}$, where r_i denotes the simple reflection with respect to α_i. The subgroup $W = \langle r_i \mid i \in I \rangle \subset W_{\mathrm{af}}$ is the (finite) Weyl group of $\mathfrak{g}$. Let $\ell : W_{\mathrm{af}} \to \mathbb{Z}_{\geq 0}$ be the length function. Let $e \in W_{\mathrm{af}}$ be the unit element. The action of W_{af} on $\mathfrak{h}^*_{\mathrm{af}}$ (resp. $\mathfrak{h}_{\mathrm{af}}$) is given by $r_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$ (resp. $r_i(h) = h - \langle h, \alpha_i \rangle \alpha_i^\vee$) for $i \in I_{\mathrm{af}}$ and $\lambda \in \mathfrak{h}^*_{\mathrm{af}}$ (resp. $h \in \mathfrak{h}_{\mathrm{af}}$). For $\xi \in Q^\vee$, we denote by $t_\xi \in W_{\mathrm{af}}$ the translation by ξ (see [13, §6.5]). We know from [13, Proposition 6.5] that $\{t_\xi \mid \xi \in Q^\vee\}$ forms an abelian normal subgroup of W_{af}, for which $t_\xi t_\zeta = t_{\xi+\zeta}$, $\xi, \zeta \in Q^\vee$, and $W_{\mathrm{af}} = W \ltimes \{t_\xi \mid \xi \in Q^\vee\}$. For $w \in W$ and $\xi \in Q^\vee$, we have $wt_\xi\lambda = w\lambda - \langle \xi, \lambda \rangle \delta$ if $\lambda \in \mathfrak{h}^*_{\mathrm{af}}$ satisfies $\langle c, \lambda \rangle = 0$.

Let Δ (resp. $\Delta^\vee$) be the root system (resp. the coroot system) of $\mathfrak{g}$ with a simple root system $\Pi = \{\alpha_i \mid i \in I\}$ (resp. a simple coroot system $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$). Set $\Delta^+ = \Delta \cap \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ and $\Delta^{\vee,+} = \Delta^\vee \cap \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i^\vee$. For a subset $J \subset I$, set

(8) $$Q_J = \bigoplus_{j \in J} \mathbb{Z}\alpha_j, \qquad \Delta_J = \Delta \cap Q_J, \qquad \Delta_J^+ = \Delta^+ \cap Q_J,$$

(9) $$Q_J^\vee = \bigoplus_{j \in J} \mathbb{Z}\alpha_j^\vee, \qquad \Delta_J^\vee = \Delta^\vee \cap Q_J^\vee, \qquad \Delta_J^{\vee,+} = \Delta^{\vee,+} \cap Q_J^\vee.$$

Denote by Δ_{af} the set of real roots of $\mathfrak{g}_{\mathrm{af}}$, and by Δ^+_{af} the set of positive real roots of $\mathfrak{g}_{\mathrm{af}}$; we know from [13, Proposition 6.3] that

(10) $$\Delta_{\mathrm{af}} = \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\}, \qquad \Delta^+_{\mathrm{af}} = \Delta^+ \sqcup \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}_{>0}\}.$$

Let $\beta^\vee \in \mathfrak{h}_{\mathrm{af}}$ denote the coroot of $\beta \in \Delta_{\mathrm{af}}$. Let $r_\beta \in W_{\mathrm{af}}$ be the reflection with respect to $\beta \in \Delta_{\mathrm{af}}$; if $\beta = \alpha + n\delta$, $\alpha \in \Delta$ and $n \in \mathbb{Z}$, then $r_\beta = r_\alpha t_{n\alpha^\vee}$.

2.2. Crystals. In this subsection, we set up notation and terminology on crystals. For a fuller treatment, we refer the reader to [14, 16].

Let $\mathbf{U}$ be the quantized universal enveloping algebra associated with $\mathfrak{g}_{\mathrm{af}}$. Let $\mathbf{U}'$ be the subalgebra of $\mathbf{U}$ corresponding to the derived subalgebra $[\mathfrak{g}_{\mathrm{af}}, \mathfrak{g}_{\mathrm{af}}]$ of $\mathfrak{g}_{\mathrm{af}}$ (see for instance [3, §2.2]).

A set B together with the maps $\mathrm{wt} : B \to P_{\mathrm{af}}$ (resp. $\mathrm{wt} : B \to P_{\mathrm{af}}/(P_{\mathrm{af}} \cap \mathbb{C}\delta)$), $e_i, f_i : B \to B \sqcup \{\mathbf{0}\}$, and $\varepsilon_i, \phi_i : B \to \mathbb{Z} \sqcup \{-\infty\}$, $i \in I_{\mathrm{af}}$, is called a $\mathbf{U}$-crystal (resp. a $\mathbf{U}'$-crystal) if the following conditions are satisfied:

- (C1) $\phi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle$ for $b \in B$ and $i \in I_{\mathrm{af}}$,
- (C2) $\mathrm{wt}(e_ib) = \mathrm{wt}(b) + \alpha_i$ if $e_ib \in B$,
- (C3) $\mathrm{wt}(f_ib) = \mathrm{wt}(b) - \alpha_i$ if $f_ib \in B$,
- (C4) $\varepsilon_i(e_ib) = \varepsilon_i(b) - 1$ and $\phi_i(e_ib) = \phi_i(b) + 1$ if $e_ib \in B$,
- (C5) $\varepsilon_i(f_ib) = \varepsilon_i(b) + 1$ and $\phi_i(f_ib) = \phi_i(b) - 1$ if $f_ib \in B$,
- (C6) $f_ib = b'$ if and only if $b = e_ib'$ for $b, b' \in B$ and $i \in I_{\mathrm{af}}$,
- (C7) if $\phi_i(b) = -\infty$, then $e_ib = f_ib = \mathbf{0}$.

A set B together with the maps $\mathrm{wt} : B \to P$ and $e_i, f_i, \varepsilon_i, \phi_i$ for $i \in I$ as above is called a $\mathfrak{g}$-crystal if these maps satisfy (C1)–(C7). We can think of a $\mathbf{U}'$-crystal B such that $\mathrm{wt}(B) \subset P/(P \cap \mathbb{C}\delta) = P$ as a $\mathfrak{g}$-crystal by forgetting the maps $e_0, f_0, \varepsilon_0, \phi_0$.

Following [16, §4.2], define the affinization $B_{\mathrm{af}} = B \times \mathbb{Z}$ of a $\mathbf{U}'$-crystal B to be the $\mathbf{U}$-crystal such that for $b \in B$, $c \in \mathbb{Z}$, and $i \in I_{\mathrm{af}}$, $\mathrm{wt}(b, c) = \iota(\mathrm{wt}(b)) - c\delta \in P_{\mathrm{af}}$, $e_i(b, c) = (e_i b, c - \delta_{i,0})$, $f_i(b, c) = (f_i b, c + \delta_{i,0})$, $\varepsilon_i(b, c) = \varepsilon_i(b)$, and $\phi_i(b, c) = \phi_i(b)$; we understand that $(\mathbf{0}, c) = \mathbf{0}$.

Let B_1 and B_2 be $\mathbf{U}$-crystals or $\mathbf{U}'$-crystals. A morphism $\Psi : B_1 \to B_2$ is, by definition, a map $B_1 \sqcup \{\mathbf{0}\} \to B_2 \sqcup \{\mathbf{0}\}$ such that

(CM1) $\Psi(\mathbf{0}) = \mathbf{0}$,
(CM2) if $b \in B_1$ and $\Psi(b) \in B_2$, then $\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$, and $\phi_i(\Psi(b)) = \phi_i(b)$ for all $i \in I_{\mathrm{af}}$,
(CM3) if $b, b' \in B_1$, $\Psi(b), \Psi(b') \in B_2$ and $f_i b = b'$, then $f_i \Psi(b) = \Psi(b')$ for all $i \in I_{\mathrm{af}}$.

A morphism $\Psi : B_1 \to B_2$ is called strict if $\Psi(f_i b) = f_i \Psi(b)$ and $\Psi(e_i b) = e_i \Psi(b)$ for all $b \in B_1$ and $i \in I_{\mathrm{af}}$. A morphism $\Psi : B_1 \to B_2$ is called a strict embedding if it is a strict morphism and the associated map $B_1 \sqcup \{\mathbf{0}\} \to B_2 \sqcup \{\mathbf{0}\}$ is injective. A morphism $\Psi : B_1 \to B_2$ is called an isomorphism if the associated map $B_1 \sqcup \{\mathbf{0}\} \to B_2 \sqcup \{\mathbf{0}\}$ is bijective. In the same manner we define morphisms of $\mathfrak{g}$-crystals.

The tensor product $B_1 \otimes B_2$ of crystals B_1 and B_2 is defined to be the set $\{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$ whose crystal structure is as follows:

(T1) $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$,
(T2) $\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \alpha_i^\vee, \mathrm{wt}(b_1) \rangle\}$,
(T3) $\phi_i(b_1 \otimes b_2) = \max\{\phi_i(b_2), \phi_i(b_1) + \langle \alpha_i^\vee, \mathrm{wt}(b_2) \rangle\}$,
(T4) $e_i(b_1 \otimes b_2) = \begin{cases} (e_i b_1) \otimes b_2 & \text{if } \phi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes (e_i b_2) & \text{if } \phi_i(b_1) < \varepsilon_i(b_2), \end{cases}$
(T5) $f_i(b_1 \otimes b_2) = \begin{cases} (f_i b_1) \otimes b_2 & \text{if } \phi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes (f_i b_2) & \text{if } \phi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$

Here, we understand that $b_1 \otimes \mathbf{0} = \mathbf{0} \otimes b_2 = \mathbf{0}$.

Let B be a regular $\mathbf{U}$-crystal in the sense of [16, §2.2]. It follows that

$$\phi_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid f_i^k b = \mathbf{0}\}, \quad \varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid e_i^k b = \mathbf{0}\} \tag{11}$$

for $b \in B$ and $i \in I_{\mathrm{af}}$. Define $f_i^{\max} b = f_i^{\phi_i(b)} b \in B$ and $e_i^{\max} b = e_i^{\varepsilon_i(b)} b \in B$ for $b \in B$ and $i \in I_{\mathrm{af}}$. By [14, §7], we have a W_{af}-action $S : W_{\mathrm{af}} \to \mathfrak{S}(B)$, $x \mapsto S_x$, on (the underlying set) B given by

$$S_{r_i} b = \begin{cases} f_i^{\langle \alpha_i^\vee, \mathrm{wt}(b) \rangle} b & \text{if } \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle \geq 0, \\ e_i^{-\langle \alpha_i^\vee, \mathrm{wt}(b) \rangle} b & \text{if } \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle \leq 0 \end{cases} \tag{12}$$

for $b \in B$ and $i \in I_{\mathrm{af}}$. Note that $\mathrm{wt}(S_x b) = x\mathrm{wt}(b)$ holds for all $x \in W_{\mathrm{af}}$ and $b \in B$. An element $b \in B$ of weight $\mathrm{wt}(b) = \lambda \in P_{\mathrm{af}}$ is called an extremal element if there exist $b_x \in B$, $x \in W_{\mathrm{af}}$, such that

(E1) $b_e = b$,
(E2) if $\langle \alpha_i^\vee, x\lambda \rangle \geq 0$, then $e_i b_x = \mathbf{0}$ and $f_i^{\langle \alpha_i^\vee, x\lambda \rangle} b_x = f_i^{\max} b_x = b_{r_i x}$,
(E3) if $\langle \alpha_i^\vee, x\lambda \rangle \leq 0$, then $f_i b_x = \mathbf{0}$ and $e_i^{-\langle \alpha_i^\vee, x\lambda \rangle} b_x = e_i^{\max} b_x = b_{r_i x}$.

Then $b_x = S_x b$ holds for all $x \in W_{\mathrm{af}}$, and each b_x is an extremal element of weight $x\lambda$. The proof of the next lemma is straightforward.

Lemma 2.1. (1) *Let B be a regular $\mathbf{U}$-crystal, and let $b \in B$ be an extremal element. If there exist $i_1, i_2, \ldots, i_N \in I_{\mathrm{af}}$ such that*

$$\langle \alpha_{i_n}^\vee, r_{i_{n-1}} \cdots r_{i_2} r_{i_1} \mathrm{wt}(b) \rangle \geq 0 \text{ for all } n \in [N], \tag{13}$$

then

$$f_{i_N}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max} b = S_{r_{i_N} \cdots r_{i_2} r_{i_1}} b. \tag{14}$$

(2) *Let* $B_1, B_2, \dots, B_M$ *be regular* $\mathbf{U}$*-crystals, and let* $b_\nu \in B_\nu$, $\nu \in [M]$, *be extremal elements such that, for every* $\beta \in \Delta_{\mathrm{af}}$, $\langle \beta^\vee, \mathrm{wt}(b_\nu) \rangle$, $\nu \in [M]$, *are all nonnegative or all nonpositive. Then the equalities*

$$f_i^{\max}(b_1 \otimes b_2 \otimes \cdots \otimes b_M) = f_i^{\max} b_1 \otimes f_i^{\max} b_2 \otimes \cdots \otimes f_i^{\max} b_M, \tag{15}$$

$$S_x(b_1 \otimes b_2 \otimes \cdots \otimes b_M) = S_x b_1 \otimes S_x b_2 \otimes \cdots \otimes S_x b_M \tag{16}$$

hold and (15)–(16) *are extremal elements for all* $i \in I_{\mathrm{af}}$ *and* $x \in W_{\mathrm{af}}$.

2.3. Semi-infinite Bruhat order. In this subsection, we collect some basic facts on semi-infinite Bruhat order on an affine Weyl group (see [11, 20, 32] for more details). Throughout this subsection, we take and fix $J \subset I$.

Let $W_J = \langle r_j \mid j \in J \rangle$, and let W^J be the set of minimal(-length) coset representatives for W/W_J. For $w \in W$, we denote by $\lfloor w \rfloor = \lfloor w \rfloor^J \in W^J$ the minimal coset representative for the coset $wW_J \in W/W_J$. Define

$$(\Delta_J)_{\mathrm{af}} = \{\alpha + n\delta \mid \alpha \in \Delta_J, n \in \mathbb{Z}\} \subset \Delta_{\mathrm{af}}, \tag{17}$$

$$(\Delta_J)^+_{\mathrm{af}} = (\Delta_J)_{\mathrm{af}} \cap \Delta^+_{\mathrm{af}} = \Delta^+_J \sqcup \{\alpha + n\delta \mid \alpha \in \Delta_J, n \in \mathbb{Z}_{>0}\}, \tag{18}$$

$$(W_J)_{\mathrm{af}} = W_J \ltimes \{t_\xi \mid \xi \in Q_J^\vee\} = \langle r_\beta \mid \beta \in (\Delta_J)^+_{\mathrm{af}} \rangle, \tag{19}$$

$$(W^J)_{\mathrm{af}} = \{x \in W_{\mathrm{af}} \mid x\beta \in \Delta^+_{\mathrm{af}} \text{ for all } \beta \in (\Delta_J)^+_{\mathrm{af}}\}; \tag{20}$$

note that $(W_\emptyset)_{\mathrm{af}} = \{e\}$ and $(W^\emptyset)_{\mathrm{af}} = W_{\mathrm{af}}$.

We see from [32] (see also [20, Lemma 10.5]) that, for each $x \in W_{\mathrm{af}}$, there exist a unique $x_1 \in (W^J)_{\mathrm{af}}$ and a unique $x_2 \in (W_J)_{\mathrm{af}}$ such that $x = x_1 x_2$. Define $\Pi^J : W_{\mathrm{af}} \to (W^J)_{\mathrm{af}}$ by $\Pi^J(x) = x_1$ if $x = x_1 x_2$ with $x_1 \in (W^J)_{\mathrm{af}}$ and $x_2 \in (W_J)_{\mathrm{af}}$. It follows immediately from (17)–(20) that $\Pi^J = \Pi^J \circ \Pi^K$ if $K \subset J$.

Set $\rho_J = (1/2) \sum_{\alpha \in \Delta_J^+} \alpha$; we abbreviate ρ_J to ρ if $J = I$. Define the semi-infinite length function $\ell^{\frac{\infty}{2}} : W_{\mathrm{af}} \to \mathbb{Z}$ by

$$\ell^{\frac{\infty}{2}}(x) = \ell(w) + 2\langle \xi, \rho \rangle \tag{21}$$

for $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$.

Define the (parabolic) semi-infinite Bruhat graph SiB^J to be the Δ^+_{af}-colored directed graph with vertex set $(W^J)_{\mathrm{af}}$ and edges of the form $x \xrightarrow{\beta} r_\beta x$ for $x \in (W^J)_{\mathrm{af}}$ and $\beta \in \Delta^+_{\mathrm{af}}$, where $r_\beta x \in (W^J)_{\mathrm{af}}$ and $\ell^{\frac{\infty}{2}}(r_\beta x) = \ell^{\frac{\infty}{2}}(x) + 1$. We know from [11, Appendix A] that the existence of the edge $x \xrightarrow{\beta} \Pi^J(r_\beta x)$ in SiB^J implies $r_\beta x = \Pi^J(r_\beta x) \in (W^J)_{\mathrm{af}}$. The semi-infinite Bruhat order is a partial order $\preceq$ on $(W^J)_{\mathrm{af}}$ defined as follows: for $x, y \in (W^J)_{\mathrm{af}}$, write $x \preceq y$ if there exists a directed path from y to x in SiB^J. Write $x \prec y$ if $x \preceq y$ and $x \neq y$.

Lemma 2.2. *Let* $\lambda \in P^+$ *be such that* $J = \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\}$. *We have* $x\lambda = \Pi^J(x)\lambda$ *for all* $x \in W_{\mathrm{af}}$.

Proof. The assertion follows from $\langle \beta^\vee, \lambda \rangle = 0$ for all $\beta \in (\Delta_J)^+_{\mathrm{af}}$.

Lemma 2.3 ([32]; see also [20, Lemma 10.7 and Proposition 10.10]).

(1) $\Pi^J(w) = \lfloor w \rfloor^J$ *for* $w \in W$.
(2) $\Pi^J(xt_\xi) = \Pi^J(x)\Pi^J(t_\xi)$ *for* $x \in W_{\mathrm{af}}$ *and* $\xi \in Q^\vee$.

For simplicity of notation, we let T_ξ^J stand for $\Pi^J(t_\xi) \in (W^J)_{\mathrm{af}}$ for $\xi \in Q^\vee$. By Lemma 2.3, we have

$$(W^J)_{\mathrm{af}} = \left\{ wT_\xi^J \;\middle|\; w \in W^J,\ \xi \in Q^\vee \right\}. \tag{22}$$

Let

$$[\,\cdot\,]_{I\setminus J} : Q^\vee = Q_J^\vee \oplus Q_{I\setminus J}^\vee \to Q_{I\setminus J}^\vee \tag{23}$$

be the projection. For $\xi_1, \xi_2 \in Q^\vee$, write $\xi_1 \geq \xi_2$ if $\xi_1 - \xi_2 \in \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i^\vee$.

Lemma 2.4 ([11, Lemmas 6.1.1 and 6.2.1]).

(1) *Let $K \subset J$ and $x, y \in (W^K)_{\mathrm{af}}$. If $x \preceq y$ in $(W^K)_{\mathrm{af}}$, then $\Pi^J(x) \preceq \Pi^J(y)$ in $(W^J)_{\mathrm{af}}$.*

(2) *Let $\xi_1, \xi_2 \in Q^\vee$. We have $\Pi^J(t_{\xi_1}) \preceq \Pi^J(t_{\xi_2})$ if and only if $[\xi_1]_{I\setminus J} \geq [\xi_2]_{I\setminus J}$. In particular, we have $\Pi^J(t_{\xi_1}) \preceq \Pi^J(t_{\xi_2})$ if and only if $\Pi^{I\setminus\{i\}}(t_{\xi_1}) \preceq \Pi^{I\setminus\{i\}}(t_{\xi_2})$ for all $i \in I \setminus J$.*

Lemma 2.5 ([11, Remark 4.1.3]). *Let $x \in (W^J)_{\mathrm{af}}$, $i \in I_{\mathrm{af}}$, and let $\lambda \in P^+$ be such that $J = \{j \in I \mid \langle \alpha_j^\vee, \lambda \rangle = 0\}$.*

(1) *$r_i x = \Pi^J(r_i x) \in (W^J)_{\mathrm{af}}$ if and only if $\langle \alpha_i^\vee, x\lambda \rangle \neq 0$.*
(2) *$x = \Pi^J(r_i x)$ if and only if $\langle \alpha_i^\vee, x\lambda \rangle = 0$.*
(3) *$r_i x \xleftarrow{\alpha_i} x$ (resp. $x \xleftarrow{\alpha_i} r_i x$) if and only if $\langle \alpha_i^\vee, x\lambda \rangle > 0$ (resp. $\langle \alpha_i^\vee, x\lambda \rangle < 0$).*

The next lemma is a reformulation of the "diamond lemma" for semi-infinite Bruhat order obtained in [11, §4.1].

Lemma 2.6. *Let $x, y \in (W^J)_{\mathrm{af}}$ and $i \in I_{\mathrm{af}}$ be such that $\Pi^J(r_i x) \prec x$ and $\Pi^J(r_i y) \prec y$.*

(1) *If $\Pi^J(r_i x) \preceq y$, then $x \preceq y$ and $\Pi^J(r_i x) \preceq \Pi^J(r_i y)$.*
(2) *$x \preceq y$ if and only if $\Pi^J(r_i x) \preceq \Pi^J(r_i y)$.*

Lemma 2.7. *Let $w, v \in W$ and $\xi, \zeta \in Q^\vee$. If $\Pi^J(wt_\xi) \preceq \Pi^J(wt_\zeta)$, then $\Pi^J(vt_\xi) \preceq \Pi^J(vt_\zeta)$.*

Proof. It suffices to prove that $\Pi^J(wt_\xi) \preceq \Pi^J(wt_\zeta)$ if and only if $\Pi^J(t_\xi) \preceq \Pi^J(t_\zeta)$. The proof is by induction on $\ell(w)$. If $\ell(w) = 0$, then the assertion is obvious. Assume that $\ell(w) > 0$. Let $i \in I$ be such that $\ell(r_i w) < \ell(w)$. By induction hypothesis, $\Pi^J(r_i wt_\xi) \preceq \Pi^J(r_i wt_\zeta)$ if and only if $\Pi^J(t_\xi) \preceq \Pi^J(t_\zeta)$. The proof is completed by showing that $\Pi^J(r_i wt_\xi) \preceq \Pi^J(r_i wt_\zeta)$ if and only if $\Pi^J(wt_\xi) \preceq \Pi^J(wt_\zeta)$. Let $\lambda \in P^+$ be such that $J = \{j \in I \mid \langle \alpha_j^\vee, \lambda \rangle = 0\}$; note that $0 > \langle \alpha_i^\vee, w\lambda \rangle = \langle \alpha_i^\vee, \Pi^J(wt_\xi)\lambda \rangle = \langle \alpha_i^\vee, \Pi^J(wt_\zeta)\lambda \rangle$. We see from Lemma 2.5 (2)–(3) that $\Pi^J(wt_\xi) \xleftarrow{\alpha_i} \Pi^J(r_i wt_\xi)$ and $\Pi^J(wt_\zeta) \xleftarrow{\alpha_i} \Pi^J(r_i wt_\zeta)$. By Lemma 2.6 (2), $\Pi^J(r_i wt_\xi) \preceq \Pi^J(r_i wt_\zeta)$ if and only if $\Pi^J(wt_\xi) \preceq \Pi^J(wt_\zeta)$.

2.4. Extremal weight modules and their crystal bases. In this subsection, following [3, 14, 16], we review some of the standard facts on extremal weight modules and their crystal bases.

For $\lambda \in P^+$, let $V(\lambda)$ be the extremal weight $\mathbf{U}$-module generated by an extremal weight vector u_λ of extremal weight λ, and let $B(\lambda)$ be the crystal basis of $V(\lambda)$ ([14, Proposition 8.2.2]; see also [16, §3.2]). Note that $B(\lambda)$ is a regular $\mathbf{U}$-crystal in the sense of [16, §2.2] (see §2.2). Let z_i, $i \in I$, be the $\mathbf{U}'$-linear automorphism of $V(\varpi_i)$ of weight δ introduced in [16, §5.2]; z_i sends a (unique) global basis element of weight ϖ_i to a (unique) global basis element of weight $\varpi_i + \delta$. Then z_i induces an automorphism of $B(\varpi_i)$ as a $\mathbf{U}'$-crystal; by abuse of notation, we use the same letter z_i for the automorphism of $B(\varpi_i)$. The $\mathbf{U}'$-module $W(\varpi_i) = V(\varpi_i)/(z_i - 1)V(\varpi_i)$ is called a level-zero fundamental representation. We know from [16, Theorem 5.17] that $W(\varpi_i)$ is a finite-dimensional irreducible $\mathbf{U}'$-module and has a (simple) crystal basis.

For $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$, with $m_i \in \mathbb{Z}_{>0}$, $i \in I$, set $\breve{B}(\lambda) = \bigotimes_{i \in I} B(\varpi_i)^{\otimes m_i}$. For each $i \in I$ and $\nu \in [m_i]$, let $z_{i,\nu}$ be the automorphism of the $\mathbf{U}'$-crystal $\breve{B}(\lambda)$ obtained by the action of z_i on the ν-th factor $B(\varpi_i)$ of $B(\varpi_i)^{\otimes m_i}$ in $\breve{B}(\lambda)$. Set

$$\text{(24)}\quad \operatorname{Par}(\lambda) = \left\{ \rho = (\rho^{(i)})_{i \in I} \;\middle|\; \rho^{(i)} \text{ is a partition of length less than } m_i \text{ for } i \in I \right\};$$

we understand that a partition of length less than 1 is an empty partition $\varnothing$. Let $\rho = (\rho^{(i)})_{i \in I} \in \operatorname{Par}(\lambda)$ and $\rho^{(i)} = (\rho^{(i)}_1 \geq \rho^{(i)}_2 \geq \cdots \geq \rho^{(i)}_{m_i - 1} \geq 0)$, $i \in I$. Define the automorphism $z^{-\rho}$ of the $\mathbf{U}'$-crystal $\breve{B}(\lambda)$ by

$$\text{(25)}\qquad z^{-\rho} = \prod_{i \in I} z_{i,1}^{-\rho^{(i)}_1} z_{i,2}^{-\rho^{(i)}_2} \cdots z_{i,m_i-1}^{-\rho^{(i)}_{m_i-1}}.$$

Let S^-_ρ be the (PBW-type) basis element of weight

$$\text{(26)}\qquad \operatorname{wt}(\rho) = -\sum_{i \in I} \sum_{\nu=1}^{m_i - 1} \rho^{(i)}_\nu \delta$$

of the negative imaginary part of $\mathbf{U}$ constructed in [3, the element $S^-_{\mathbf{c}_0}$ in §3.1; see also Remark 4.1]. We know from [3, §4.2] that

$$\text{(27)}\qquad \begin{aligned} B(\lambda) = \Big\{ & g_1 g_2 \cdots g_l S^-_\rho u_\lambda \\ & \;\Big|\; g_k \in \{e_i, f_i \mid i \in I_{\mathrm{af}}\},\ k \in [l],\ l \in \mathbb{Z}_{>0},\ \rho \in \operatorname{Par}(\lambda) \Big\} \setminus \{0\}. \end{aligned}$$

Define

$$\text{(28)}\qquad \Phi^{\mathrm{LT}}_{\lambda|q=0} : B(\lambda) \to \breve{B}(\lambda),\ g_1 g_2 \cdots g_l S^-_\rho u_\lambda \mapsto g_1 g_2 \cdots g_l z^{-\rho} \left(\bigotimes_{i \in I} u_{\varpi_i}^{\otimes m_i} \right),$$

where $g_k \in \{e_i, f_i \mid i \in I_{\mathrm{af}}\}$, $k \in [l]$, $l \in \mathbb{Z}_{>0}$, and $\rho \in \operatorname{Par}(\lambda)$. We know from [10, Lemma 3.1] that the map $\Phi^{\mathrm{LT}}_{\lambda|q=0}$ is a strict embedding of $\mathbf{U}$-crystals.

The next theorem will be needed in § 5.

Theorem 2.8 ([3, Remark 4.17]; see also [16, Conjecture 13.1 (iii)]). *Let $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$. We have an isomorphism $B(\lambda) \cong \bigotimes_{i \in I} B(m_i \varpi_i)$ of $\mathbf{U}$-crystals.*

2.5. Path model for Demazure crystals. In this subsection, we give a brief exposition of the path model for the crystal bases of level-zero extremal weight $\mathbf{U}$-modules and their Demazure submodules. For a fuller treatment we refer the reader to [3, 11, 14, 16, 17, 29].

For $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$, set

$$\text{(29)}\qquad J_\lambda = \{ j \in I \mid \langle \alpha_j^\vee, \lambda \rangle = 0 \}, \qquad J^c_\lambda = I \setminus J_\lambda.$$

For a rational number $0 < a \leq 1$, define $\operatorname{SiB}(\lambda; a)$ to be the subgraph of $\operatorname{SiB}^{J_\lambda}$ with the same vertex set but having only the edges of the form

$$\text{(30)}\qquad x \xrightarrow{\beta} y \ \text{ with } a\langle \beta^\vee, x\lambda \rangle \in \mathbb{Z};$$

note that $\operatorname{SiB}(\lambda; 1) = \operatorname{SiB}^{J_\lambda}$. A semi-infinite Lakshmibai–Seshadri path of shape λ is, by definition, a pair $(x; a)$ of a decreasing sequence $x : x_1 \succ x_2 \succ \cdots \succ x_l$ of elements in $(W^{J_\lambda})_{\mathrm{af}}$ and an increasing sequence $a : 0 = a_0 < a_1 < \cdots < a_l = 1$ of rational numbers such that there exists a directed path from x_{u+1} to x_u in $\operatorname{SiB}(\lambda; a_u)$ for each $u \in [l-1]$. Let $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ denote the set of semi-infinite Lakshmibai–Seshadri paths of shape λ.

Following [11, §3.1], we equip the set $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ with a $\mathbf{U}$-crystal structure. For $\eta = (x_1, \dots, x_l; a_0, \dots, a_l) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, define the map $\overline{\eta} : \{t \in \mathbb{R} \mid 0 \leqslant t \leqslant 1\} \to \mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{af}}$ by

$$\overline{\eta}(t) = \sum_{p=1}^{u-1} (a_p - a_{p-1}) x_p \lambda + (t - a_{u-1}) x_u \lambda \quad \text{for } a_{u-1} \leqslant t \leqslant a_u \text{ and } u \in [l]. \tag{31}$$

Define $\mathrm{wt} : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to P_{\mathrm{af}}$ by $\mathrm{wt}(\eta) = \overline{\eta}(1) \in P_{\mathrm{af}}$. Set

$$h_i^\eta(t) = \langle \alpha_i^\vee, \overline{\eta}(t) \rangle \quad \text{for } 0 \leqslant t \leqslant 1, \qquad m_i^\eta = \min\{h_i^\eta(t) \mid 0 \leqslant t \leqslant 1\}. \tag{32}$$

We define $e_i \eta$, $i \in I_{\mathrm{af}}$, as follows: if $m_i^\eta = 0$, then we set $e_i \eta = \mathbf{0}$. If $m_i^\eta \leqslant -1$, then we set

$$\begin{cases} t_1 = \min\{t \mid 0 \leqslant t \leqslant 1,\ h_i^\eta(t) = m_i^\eta\}, \\ t_0 = \max\{t \mid 0 \leqslant t \leqslant t_1,\ h_i^\eta(t) = m_i^\eta + 1\}. \end{cases} \tag{33}$$

Let $1 \leqslant p \leqslant q \leqslant l$ be such that $a_{p-1} \leqslant t_0 < a_p$ and $t_1 = a_q$. Then we define

$$\begin{aligned} e_i \eta = (&x_1, \dots, x_p, r_i x_p, \dots, r_i x_q, x_{q+1}, \dots, x_l; \\ &a_0, \dots, a_{p-1}, t_0, a_p, \dots, a_q = t_1, \dots, a_l); \end{aligned} \tag{34}$$

if $t_0 = a_{p-1}$, then we drop x_p and a_{p-1}, and if $r_j x_q = x_{q+1}$, then we drop x_{q+1} and $a_q = t_1$.

Next, we define $f_i \eta$, $i \in I_{\mathrm{af}}$, as follows: if $m_i^\eta = h_i^\eta(1)$, then we set $f_i \eta = \mathbf{0}$. If $h_i^\eta(1) - m_i^\eta \geqslant 1$, then we set

$$\begin{cases} t_0 = \max\{t \mid 0 \leqslant t \leqslant 1,\ h_i^\eta(t) = m_i^\eta\}, \\ t_1 = \min\{t \mid t_0 \leqslant t \leqslant 1,\ h_i^\eta(t) = m_i^\eta + 1\}. \end{cases} \tag{35}$$

Let $1 \leqslant p \leqslant q \leqslant l - 1$ be such that $t_0 = a_p$ and $a_q < t_1 \leqslant a_{q+1}$. Then we define

$$\begin{aligned} f_i \eta = (&x_1, \dots, x_p, r_i x_{p+1}, \dots, r_i x_{q+1}, x_{q+1}, \dots, x_l; \\ &a_0, \dots, a_p = t_0, \dots, a_q, t_1, a_{q+1}, \dots, a_l); \end{aligned} \tag{36}$$

if $t_1 = a_{q+1}$, then we drop x_{q+1} and a_{q+1}, and if $x_p = r_i x_{p+1}$, then we drop x_p and $a_p = t_0$.

For $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ and $i \in I_{\mathrm{af}}$, define

$$\begin{cases} \varepsilon_i(\eta) = -m_i^\eta, \\ \phi_i(\eta) = h_i^\eta(1) - m_i^\eta. \end{cases} \tag{37}$$

For $\eta = (x_1, x_2, \dots, x_l; a) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, set $\kappa(\eta) = x_l$. Following [29, Equation (4.2.1)], for each $x \in (W^{J_\lambda})_{\mathrm{af}}$, set

$$\mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) = \left\{ \eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda) \,\middle|\, \kappa(\eta) \succeq x \right\}. \tag{38}$$

Following [11, Equation (7.2.2)], we define an extremal element $\eta_\rho \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ of weight $\lambda + \mathrm{wt}(\rho)$ for each $\rho = \left(\rho^{(i)}\right)_{i \in I} \in \mathrm{Par}(\lambda)$, with $\rho^{(i)} = (\rho_1^{(i)} \geqslant \cdots \geqslant \rho_{m_i}^{(i)} = 0)$. Let s be the least common multiple of $\{m_i \mid i \in J_\lambda^c\}$. Let $c_i(\xi) \in \mathbb{Z}$ denote the coefficient of $\alpha_i^\vee$ in $\xi \in Q^\vee$. For $\xi, \zeta \in Q^\vee$, write $\xi > \zeta$ if $\xi \geqslant \zeta$ and $\xi \neq \zeta$. Let $\zeta_1, \dots, \zeta_s \in Q^\vee$ be such that

(i) $c_i(\zeta_t) = \rho_u^{(i)}$ if $i \in J_\lambda^c$ and $\dfrac{s(u-1)}{m_i} < t \leqslant \dfrac{su}{m_i}$, and
(ii) $c_j(\zeta_t) = 0$ for all $j \in J_\lambda$ and $t \in [s]$;

note that $\zeta_1 \geq \cdots \geq \zeta_s$ and $\zeta_s = 0$. Assume that

$$\zeta_1 = \cdots = \zeta_{s_1} > \zeta_{s_1+1} = \cdots = \zeta_{s_2} > \cdots\cdots > \zeta_{s_{k-1}+1} = \cdots = \zeta_{s_k}, \tag{39}$$

where $1 \leq s_1 < \cdots < s_{k-1} < s_k = s$. Set

$$\eta_\rho = \left(T^{J_\lambda}_{\zeta_{s_1}}, T^{J_\lambda}_{\zeta_{s_2}}, \ldots, T^{J_\lambda}_{\zeta_{s_{k-1}}}, e;\ 0, \frac{s_1}{s}, \frac{s_2}{s}, \ldots, \frac{s_{k-1}}{s}, 1 \right). \tag{40}$$

Theorem 2.9 ([11, Theorem 3.2.1 and Proposition 7.2.1]). *Let $\lambda \in P^+$.*

1. *For each connected component C of (the crystal graph of) $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$, there exists a unique $\rho \in \mathrm{Par}(\lambda)$ such that $\eta_\rho \in C$.*
2. *There exists a unique isomorphism $\phi_\lambda : B(\lambda) \to \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ of $\mathbf{U}$-crystals sending $S^-_\rho u_\lambda$ to η_ρ for every $\rho \in \mathrm{Par}(\lambda)$.*

Remark 2.10. Let $\lambda \in P^+$.

1. Since $B(\lambda)$ is a regular $\mathbf{U}$-crystal, it follows from Theorem 2.9 (2) that $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is also a regular $\mathbf{U}$-crystal. Hence W_{af} acts on $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ as (12). By (40) and [29, Remark 3.5.2 (2)],

$$S_x\eta_\rho = \left(\Pi^{J_\lambda}\left(xT^{J_\lambda}_{\zeta_{s_1}} \right), \ldots, \Pi^{J_\lambda}\left(xT^{J_\lambda}_{\zeta_{s_{k-1}}} \right), \Pi^{J_\lambda}(x);\ 0, \frac{s_1}{s}, \ldots, \frac{s_{k-1}}{s}, 1 \right) \tag{41}$$

   for $\rho \in \mathrm{Par}(\lambda)$ and $x \in W_{\mathrm{af}}$.
2. Let $B^-_x(\lambda)$ be the (opposite) Demazure subcrystal of $B(\lambda)$ associated with $x \in (W^{J_\lambda})_{\mathrm{af}}$ ([29, §4.1]; see also [17, §2.8]). We know from [29, Theorem 4.2.1] that $\phi_\lambda(B^-_x(\lambda)) = \mathbb{B}^{\frac{\infty}{2}}_x(\lambda)$. However, we will not use this fact in any essential way in the remainder of this paper.

3. Deodhar-type criterion for semi-infinite Bruhat order

This section is devoted to the proof of the next theorem.

Theorem 3.1. *Let $J, K_1, K_2, \ldots, K_s \subset I$ be such that $J = \bigcap_{\nu=1}^{s} K_\nu$, and let $x, y \in (W^J)_{\mathrm{af}}$. We have $x \preceq y$ in $(W^J)_{\mathrm{af}}$ if and only if $\Pi^{K_\nu}(x) \preceq \Pi^{K_\nu}(y)$ in $(W^{K_\nu})_{\mathrm{af}}$ for all $\nu \in [s]$.*

This theorem is an analogue of Deodhar's criterion for Bruhat order on Coxeter groups ([4, Theorem 2.6.1]; see also [8, Lemma 3.6]). It is easily seen that Theorem 3.1 is equivalent to the next proposition.

Proposition 3.2. *Let $J \subset I$ and $x, y \in (W^J)_{\mathrm{af}}$. We have $x \preceq y$ in $(W^J)_{\mathrm{af}}$ if and only if $\Pi^{I \setminus \{i\}}(x) \preceq \Pi^{I \setminus \{i\}}(y)$ in $(W^{I \setminus \{i\}})_{\mathrm{af}}$ for all $i \in I \setminus J$.*

We give a proof of Proposition 3.2 in §3.3. For this purpose, we first introduce an isomorphism $\psi_\lambda : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \bigotimes_{i \in J^c_\lambda} \mathbb{B}^{\frac{\infty}{2}}(m_i\varpi_i)$ of $\mathbf{U}$-crystals in §3.1. Next, in Proposition 3.4, we give a partial characterization of the image of $\mathbb{B}^{\frac{\infty}{2}}_x(\lambda)$, $x \in (W^{J_\lambda})_{\mathrm{af}}$, under the map ψ_λ. Finally, we show that Proposition 3.4 (2) implies Proposition 3.2.

3.1. The map ψ_λ and Demazure crystals.

In this subsection, we give a partial characterization of the image of $\mathbb{B}^{\frac{\infty}{2}}_x(\lambda)$ under the isomorphism $\psi_\lambda : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \bigotimes_{i \in J^c_\lambda} \mathbb{B}^{\frac{\infty}{2}}(m_i\varpi_i)$ of $\mathbf{U}$-crystals obtained in the next lemma.

Lemma 3.3 (cf. Theorem 2.8). *Let $\lambda = \sum_{i \in I} m_i\varpi_i \in P^+$. There exists a unique isomorphism $\psi_\lambda : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \bigotimes_{i \in J^c_\lambda} \mathbb{B}^{\frac{\infty}{2}}(m_i\varpi_i)$ of $\mathbf{U}$-crystals such that*

$$\psi_\lambda(\eta_\rho) = \bigotimes_{i \in J^c_\lambda} \eta_{\rho^{(i)}} \quad \text{for all } \rho = \left(\rho^{(i)}\right) \in \mathrm{Par}(\lambda), \tag{42}$$

where, for each $i \in J_\lambda^c$, $\eta_{\rho^{(i)}} \in \mathbb{B}^{\frac{\infty}{2}}(m_i\varpi_i)$ *denotes the element* (40) *associated with* $\rho^{(i)} \in \mathrm{Par}(m_i\varpi_i)$.

Proof. We first claim that there exists a unique strict embedding $\breve{\psi}_\lambda : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \bigotimes_{i \in J_\lambda^c} \mathbb{B}^{\frac{\infty}{2}}(\varpi_i)^{\otimes m_i}$ of $\mathbf{U}$-crystals such that

$$\breve{\psi}_\lambda(\eta_\rho) = \bigotimes_{i \in J_\lambda^c} \bigotimes_{\nu=1}^{m_i} \left(T^{I \setminus \{i\}}_{\rho_\nu^{(i)}\alpha_i}; 0, 1 \right) \tag{43}$$

for all $\rho = \left(\rho_1^{(i)} > \rho_2^{(i)} > \cdots > \rho_{m_i-1}^{(i)} > \rho_{m_i}^{(i)} = 0\right)_{i \in I} \in \mathrm{Par}(\lambda)$; the uniqueness follows from Theorem 2.9 (1). Indeed, we see from (28) and [10, Lemma 3.8 (1)] that the map

$$\breve{\psi}_\lambda := \left(\bigotimes_{i \in J_\lambda^c} \phi_{\varpi_i}^{\otimes m_i} \right) \circ \Phi^{\mathrm{LT}}_{\lambda|q=0} \circ \phi_\lambda^{-1} : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \bigotimes_{i \in J_\lambda^c} \mathbb{B}^{\frac{\infty}{2}}(\varpi_i)^{\otimes m_i} \tag{44}$$

is a strict embedding satisfying (43). We can now construct the map ψ_λ as follows. By (43) and Theorem 2.9 (1), the image of $\breve{\psi}_\lambda$ equals that of $\bigotimes_{i \in J_\lambda^c} \breve{\psi}_{m_i\varpi_i}$. Hence the map

$$\psi_\lambda := \left(\bigotimes_{i \in J_\lambda^c} \breve{\psi}_{m_i\varpi_i} \right)^{-1} \circ \breve{\psi}_\lambda : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \bigotimes_{i \in J_\lambda^c} \mathbb{B}^{\frac{\infty}{2}}(m_i\varpi_i) \tag{45}$$

is well-defined and satisfies (42); the uniqueness follows from Theorem 2.9 (1).

For $\lambda = \sum_{i \in I} m_i\varpi_i \in P^+$ and $x \in (W^{J_\lambda})_{\mathrm{af}}$, set

$$\begin{aligned} &\bigotimes_{i \in J_\lambda^c} \mathbb{B}^{\frac{\infty}{2}}_{\Pi^{I \setminus \{i\}}(x)}(m_i\varpi_i) \\ &= \left\{ \bigotimes_{i \in J_\lambda^c} \eta^{(i)} \in \bigotimes_{i \in J_\lambda^c} \mathbb{B}^{\frac{\infty}{2}}(m_i\varpi_i) \;\middle|\; \kappa(\eta^{(i)}) \succeq \Pi^{I \setminus \{i\}}(x) \text{ for all } i \in J_\lambda^c \right\}. \end{aligned} \tag{46}$$

Proposition 3.4. *Let* $\lambda = \sum_{i \in I} m_i\varpi_i \in P^+$ *and* $x \in (W^{J_\lambda})_{\mathrm{af}}$.

(1) $\psi_\lambda\left(\mathbb{B}^{\frac{\infty}{2}}_x(\lambda)\right) \subset \bigotimes_{i \in J_\lambda^c} \mathbb{B}^{\frac{\infty}{2}}_{\Pi^{I \setminus \{i\}}(x)}(m_i\varpi_i)$.

(2) *If* $\pi \in \bigotimes_{i \in J_\lambda^c} \mathbb{B}^{\frac{\infty}{2}}_{\Pi^{I \setminus \{i\}}(x)}(m_i\varpi_i)$ *is of the form*

$$\pi = \bigotimes_{i \in J_\lambda^c} S_y\eta_{\rho^{(i)}} \quad \text{for some } y \in W_{\mathrm{af}} \text{ and } (\rho^{(i)}) \in \mathrm{Par}(\lambda), \tag{47}$$

then $\pi \in \psi_\lambda\left(\mathbb{B}^{\frac{\infty}{2}}_x(\lambda)\right)$.

Remark 3.5. It follows from (43), [1, Lemma 1.6 (2)] and [16, Proposition 5.4 (i)] that an element $\pi \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is extremal if and only if π is of the form (47).

3.2. Proof of Proposition 3.4.

This subsection is devoted to the proof of Proposition 3.4. We begin by recalling some fundamental properties of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$.

Lemma 3.6. *Let* $\lambda \in P^+$ *and* $\rho \in \mathrm{Par}(\lambda)$.

(1) $\kappa(S_x\eta_\rho) = \Pi^{J_\lambda}(x)$ *for* $x \in W_{\mathrm{af}}$.

(2) $(W_{J_\lambda})_{\mathrm{af}} = \{x \in W_{\mathrm{af}} \mid S_x\eta_\rho = \eta_\rho\}$.

Proof. (1) follows from (41).

By (40)–(41), $S_x\eta_\rho = \eta_\rho$ if and only if $\Pi^{J_\lambda}\big(xT^{J_\lambda}_{\zeta_{s_p}}\big) = T^{J_\lambda}_{\zeta_{s_p}}$ for $p \in [k]$. By Lemma 2.3 (2),

$$\Pi^{J_\lambda}\big(xT^{J_\lambda}_{\zeta_{s_p}}\big) = \Pi^{J_\lambda}\big(x\Pi^{J_\lambda}(t_{\zeta_{s_p}})\big) = \Pi^{J_\lambda}\big(xt_{\zeta_{s_p}}\big)$$
$$= \Pi^{J_\lambda}(x)\Pi^{J_\lambda}(t_{\zeta_{s_p}}) = \Pi^{J_\lambda}(x)T^{J_\lambda}_{\zeta_{s_p}}$$

for $p \in [k]$. This implies that $S_x\eta_\rho = \eta_\rho$ if and only if $\Pi^{J_\lambda}(x) = e$, or equivalently, $x \in (W_{J_\lambda})_{\mathrm{af}}$.

Lemma 3.7 ([29, §5]). *Let $\lambda \in P^+$ and $x, y \in (W^{J_\lambda})_{\mathrm{af}}$.*

(1) *$\mathbb{B}^{\frac{\infty}{2}}_{x}(\lambda) \cup \{\mathbf{0}\}$ is stable under the action of f_j for all $j \in I_{\mathrm{af}}$.*
(2) *$\mathbb{B}^{\frac{\infty}{2}}_{x}(\lambda) \cup \{\mathbf{0}\}$ is stable under the action of e_j for $j \in I_{\mathrm{af}}$ such that $\langle \alpha_j^{\vee}, x\lambda\rangle > 0$.*
(3) *$x \succeq y$ if and only if $\mathbb{B}^{\frac{\infty}{2}}_{x}(\lambda) \subset \mathbb{B}^{\frac{\infty}{2}}_{y}(\lambda)$.*
(4) *For every $\eta \in \mathbb{B}^{\frac{\infty}{2}}_{x}(\lambda)$, we have $f_j^{\max}\eta \in \mathbb{B}^{\frac{\infty}{2}}_{\Pi^{J_\lambda}(r_jx)}(\lambda)$ for all $j \in I_{\mathrm{af}}$.*

The next lemma is a slight refinement of [29, Lemma 5.4.1].

Lemma 3.8. *For $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ $\big($resp. $\eta \in \bigotimes_{i \in J^c_\lambda} \mathbb{B}^{\frac{\infty}{2}}(m_i\varpi_i)\big)$, $x \in W_{\mathrm{af}}$ and $w \in W$, there exist $i_1, i_2, \ldots, i_N \in I_{\mathrm{af}}$ such that*

(i) *$\langle \alpha^{\vee}_{i_n}, r_{i_{n-1}}\cdots r_{i_2}r_{i_1}x\lambda\rangle > 0$ for all $n \in [N]$, and*
(ii) *$f^{\max}_{i_N}\cdots f^{\max}_{i_2}f^{\max}_{i_1}\eta = S_{wt_\xi}\eta_\rho$ $\big($resp. $f^{\max}_{i_N}\cdots f^{\max}_{i_2}f^{\max}_{i_1}\eta = \bigotimes_{i\in J^c_\lambda} S_{wt_\xi}\eta_{\rho^{(i)}}\big)$ for some $\xi \in Q^{\vee}$ and $\rho = (\rho^{(i)})_{i\in J^c_\lambda} \in \mathrm{Par}(\lambda)$.*

Proof. By Lemma 3.3, it suffices to prove the assertion only for $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$. By [29, Lemma 5.4.1], there exist $j_1, j_2, \ldots, j_p \in I_{\mathrm{af}}$ such that

(i) $\langle \alpha^{\vee}_{j_m}, r_{j_{m-1}}\cdots r_{j_2}r_{j_1}x\lambda\rangle > 0$ for all $m \in [p]$, and
(ii) $f^{\max}_{j_p}\cdots f^{\max}_{j_2}f^{\max}_{j_1}\eta = S_{t_\xi}\eta_\rho$ for some $\xi \in Q^{\vee}$ and $\rho \in \mathrm{Par}(\lambda)$.

We see from (the proof of) [29, Lemma 5.4.1] that

$$r_{j_p}\cdots r_{j_2}r_{j_1}x\lambda \equiv \lambda \mod \mathbb{C}\delta.$$

For a reduced expression $w = r_{k_q}r_{k_{q-1}}\cdots r_{k_1}$, $k_1, k_2, \ldots, k_q \in I$, we have

$$\langle \alpha^{\vee}_{k_m}, r_{k_{m-1}}\cdots r_{k_2}r_{k_1}r_{j_p}\cdots r_{j_2}r_{j_1}x\lambda\rangle = \langle \alpha^{\vee}_{k_m}, r_{k_{m-1}}\cdots r_{k_2}r_{k_1}\lambda\rangle > 0$$

for all $m \in [q]$. It follows from $\mathrm{wt}(S_{t_\xi}\eta_\rho) \equiv \lambda \mod \mathbb{C}\delta$ and Lemma 2.1 (1) that

$$f^{\max}_{k_q}\cdots f^{\max}_{k_2}f^{\max}_{k_1}\underbrace{f^{\max}_{j_p}\cdots f^{\max}_{j_2}f^{\max}_{j_1}\eta}_{=S_{t_\xi}\eta_\rho} = S_wS_{t_\xi}\eta_\rho = S_{wt_\xi}\eta_\rho.$$

This completes the proof.

We are now in a position to prove Proposition 3.4. In what follows, we write

$$\mathbb{B}_x = \mathbb{B}^{\frac{\infty}{2}}_{x}(\lambda) \text{ and } \breve{\mathbb{B}}_x = \bigotimes_{i\in J^c_\lambda} \mathbb{B}^{\frac{\infty}{2}}_{\Pi^{I\setminus\{i\}}(x)}(m_i\varpi_i) \text{ for } x \in (W^{J_\lambda})_{\mathrm{af}}. \tag{48}$$

Proof of Proposition 3.4 (1). To see $\psi_\lambda(\mathbb{B}_x) \subset \breve{\mathbb{B}}_x$, let $\eta \in \mathbb{B}_x$, and show that $\psi_\lambda(\eta) \in \breve{\mathbb{B}}_x$. By Lemma 3.8, there exist $i_1, i_2, \ldots, i_N \in I_{\mathrm{af}}$ such that

(i) $\langle \alpha^{\vee}_{i_n}, r_{i_{n-1}}r_{i_{n-2}}\cdots r_{i_1}x\lambda\rangle > 0$ for all $n \in [N]$, and
(ii) $f^{\max}_{i_N}\cdots f^{\max}_{i_2}f^{\max}_{i_1}\eta = S_{t_\xi}\eta_\rho$ for some $\xi \in Q^{\vee}$ and $\rho \in \mathrm{Par}(\lambda)$.

The proof is by induction on N.

If $N = 0$, then $\eta = S_{t_\xi}\eta_\rho$ and

$$\psi_\lambda(\eta) = \psi_\lambda(S_{t_\xi}\eta_\rho) = S_{t_\xi}\psi_\lambda(\eta_\rho) = S_{t_\xi}\bigotimes_{i\in J_\lambda^c}\eta_{\rho^{(i)}} = \bigotimes_{i\in J_\lambda^c} S_{t_\xi}\eta_{\rho^{(i)}}$$

by Lemmas 2.1 (2) and 3.3. By Lemma 3.6 (1), $\kappa(S_{t_\xi}\eta_{\rho^{(i)}}) = \Pi^{I\setminus\{i\}}(t_\xi)$ for all $i \in J_\lambda^c$. Since $\Pi^{J_\lambda}(t_\xi) = \kappa(\eta) \succeq x$, it follows from Lemma 2.4 (1) that

$$\Pi^{I\setminus\{i\}}(t_\xi) = \Pi^{I\setminus\{i\}}\left(\Pi^{J_\lambda}(t_\xi)\right) \succeq \Pi^{I\setminus\{i\}}(x) \ \text{ for all } i \in J_\lambda^c. \tag{49}$$

Hence $\kappa(S_{t_\xi}\eta_{\rho^{(i)}}) \succeq \Pi^{I\setminus\{i\}}(x)$ for all $i \in J_\lambda^c$, which implies $\psi_\lambda(\eta) \in \breve{\mathbb{B}}_{\succeq x}$.

If $N > 0$, then $f_{i_1}^{\max}\eta \in \mathbb{B}_{\succeq \Pi^{J_\lambda}(r_{i_1}x)}$ by Lemma 3.7 (4). By induction hypothesis, $\psi_\lambda(f_{i_1}^{\max}\eta) \in \breve{\mathbb{B}}_{\succeq \Pi^{J_\lambda}(r_{i_1}x)}$. We claim that $\psi_\lambda(f_{i_1}^{\max}\eta) \in \breve{\mathbb{B}}_{\succeq x}$. Indeed, since $\langle \alpha_{i_1}^\vee, x\lambda\rangle > 0$ (see (i) above), Lemma 2.5 (2)–(3) gives $\Pi^{J_\lambda}(r_{i_1}x) \succeq x$. By Lemma 2.4 (1),

$$\Pi^{I\setminus\{i\}}(r_{i_1}x) = \Pi^{I\setminus\{i\}}(\Pi^{J_\lambda}(r_{i_1}x)) \succeq \Pi^{I\setminus\{i\}}(x) \text{ for all } i \in J_\lambda^c. \tag{50}$$

Lemma 3.7 (3) now shows that $\breve{\mathbb{B}}_{\succeq \Pi^{J_\lambda}(r_{i_1}x)} \subset \breve{\mathbb{B}}_{\succeq x}$, and hence $\psi_\lambda(f_{i_1}^{\max}\eta) \in \breve{\mathbb{B}}_{\succeq x}$. Since ψ_λ is an isomorphism of $\mathbf{U}$-crystals, there exists $k \in \mathbb{Z}_{>0}$ such that $\psi_\lambda(\eta) = e_{i_1}^k\psi_\lambda(f_{i_1}^{\max}\eta)$. We see from $\Pi^{I\setminus\{i\}}(r_{i_1}x) \succeq \Pi^{I\setminus\{i\}}(x)$ and Lemma 2.5 (2)–(3) that

$$\langle \alpha_{i_1}^\vee, \Pi^{I\setminus\{i\}}(x)m_i\varpi_i\rangle = m_i\langle \alpha_{i_1}^\vee, \Pi^{I\setminus\{i\}}(x)\varpi_i\rangle > 0 \text{ for all } i \in J_\lambda^c. \tag{51}$$

By Lemma 3.7 (2) and tensor product rule, we conclude that $\psi_\lambda(\eta) = e_{i_1}^k\psi_\lambda(f_{i_1}^{\max}\eta) \in \breve{\mathbb{B}}_{\succeq x}$.

Proof of Proposition 3.4 (2). Assume that $\pi = \bigotimes_{i\in J_\lambda^c} S_y\eta_{\rho^{(i)}} \in \breve{\mathbb{B}}_{\succeq x}$, $y \in W_{\mathrm{af}}$ and $\rho = (\rho^{(i)}) \in \mathrm{Par}(\lambda)$. By Lemmas 2.1 (2) and 3.3,

$$\pi = \bigotimes_{i\in J_\lambda^c} S_y\eta_{\rho^{(i)}} = S_y\bigotimes_{i\in J_\lambda^c}\eta_{\rho^{(i)}} = S_y\psi_\lambda(\eta_\rho) = \psi_\lambda(S_y\eta_\rho). \tag{52}$$

Let $w_0 \in W$ be the longest element. By Lemma 3.8, there exist $i_1, i_2, \dots, i_N \in I_{\mathrm{af}}$ such that

(i) $\langle \alpha_{i_n}^\vee, r_{i_{n-1}}r_{i_{n-2}}\cdots r_{i_1}x\lambda\rangle > 0$ for all $n \in [N]$, and
(ii) $f_{i_N}^{\max}\cdots f_{i_2}^{\max}f_{i_1}^{\max}\pi = \bigotimes_{i\in J_\lambda^c} S_{w_0t_\xi}\eta_{\rho^{(i)}}$ for some $\xi \in Q^\vee$.

We show by induction on N that $\pi \in \psi_\lambda(\mathbb{B}_{\succeq x})$, or equivalently, $\kappa(S_y\eta_\rho) \succeq x$; note that $\kappa(S_y\eta_\rho) = \Pi^{J_\lambda}(y)$ by Lemma 3.6 (1).

We first assume that $N = 0$. In the same manner as in (52) we see that $\pi = \psi_\lambda(S_{w_0t_\xi}\eta_\rho)$. Since ψ_λ is an isomorphism of $\mathbf{U}$-crystals, we have $S_y\eta_\rho = S_{w_0t_\xi}\eta_\rho$, and hence $\Pi^{J_\lambda}(y) = \Pi^{J_\lambda}(w_0t_\xi)$ by Lemma 3.6 (1). Write $x = wt_\zeta$, $w \in W$, $\zeta \in Q^\vee$. We proceed by induction on $\ell(w_0) - \ell(w)$.

If $\ell(w_0) - \ell(w) = 0$, then $w = w_0$. By Lemma 3.6 (1),

$$\Pi^{I\setminus\{i\}}(w_0t_\xi) = \Pi^{I\setminus\{i\}}(y) = \kappa(S_y\eta_{\rho^{(i)}}) \text{ for all } i \in J_\lambda^c. \tag{53}$$

Since $\pi \in \breve{\mathbb{B}}_{\succeq x}$, we have

$$\kappa(S_y\eta_{\rho^{(i)}}) \succeq \Pi^{I\setminus\{i\}}(x) = \Pi^{I\setminus\{i\}}(wt_\zeta) = \Pi^{I\setminus\{i\}}(w_0t_\zeta) \text{ for all } i \in J_\lambda^c. \tag{54}$$

Hence $\Pi^{I\setminus\{i\}}(w_0t_\xi) \succeq \Pi^{I\setminus\{i\}}(w_0t_\zeta)$ for all $i \in J_\lambda^c$. By Lemma 2.7, we have $\Pi^{I\setminus\{i\}}(t_\xi) \succeq \Pi^{I\setminus\{i\}}(t_\zeta)$ for all $i \in J_\lambda^c$, which implies $\Pi^{J_\lambda}(t_\xi) \succeq \Pi^{J_\lambda}(t_\zeta)$, by Lemma 2.4 (2). Again, by Lemma 2.7, we have $\Pi^{J_\lambda}(w_0t_\xi) \succeq \Pi^{J_\lambda}(w_0t_\zeta)$. Since $\kappa(S_y\eta_\rho) = \Pi^{J_\lambda}(y) = \Pi^{J_\lambda}(w_0t_\xi)$ and $x = \Pi^{J_\lambda}(w_0t_\zeta)$, we conclude that $\kappa(S_y\eta_\rho) \succeq x$.

If $\ell(w_0) - \ell(w) > 0$, then $w \neq w_0$. There exists $j \in I$ such that $r_jx \succ x$; note that $r_jx = r_jwt_\zeta$, $r_jw \in W$, and $\ell(w_0) - \ell(r_jw) < \ell(w_0) - \ell(w)$. Since $\langle \alpha_j^\vee, \Pi^{J_\lambda}(y)\lambda\rangle =$

$\langle \alpha_j^\vee, w_0 t_\xi \lambda \rangle \leqslant 0$ by Lemma 2.2, we have $\Pi^{J_\lambda}(y) \succeq \Pi^{J_\lambda}(r_j y)$ by Lemma 2.5. By Lemma 2.4 (1),

$$\text{(55)} \qquad \Pi^{I \setminus \{i\}}(r_j x) \succeq \Pi^{I \setminus \{i\}}(x) \text{ and } \Pi^{I \setminus \{i\}}(y) \succeq \Pi^{I \setminus \{i\}}(r_j y) \text{ for all } i \in J_\lambda^c.$$

Since $\pi \in \breve{\mathbb{B}}_{\succeq x}$, we have $\Pi^{I \setminus \{i\}}(y) = \kappa(S_y \eta_{\rho^{(i)}}) \succeq \Pi^{I \setminus \{i\}}(x)$ for all $i \in J_\lambda^c$. It follows from Lemma 2.6 (1) that $\Pi^{I \setminus \{i\}}(y) \succeq \Pi^{I \setminus \{i\}}(r_j x)$ for all $i \in J_\lambda^c$. By induction hypothesis, we have $\Pi^{J_\lambda}(y) \succeq \Pi^{J_\lambda}(r_j x)$. We see from $\Pi^{I \setminus \{i\}}(r_j x) \succeq \Pi^{I \setminus \{i\}}(x)$, $i \in J_\lambda^c$, Lemmas 2.2 and 2.5 (2)–(3) that $\langle \alpha_j^\vee, x\varpi_i \rangle = \langle \alpha_j^\vee, \Pi^{I \setminus \{i\}}(x)\varpi_i \rangle > 0$ for all $i \in J_\lambda^c$, which gives $\langle \alpha_j^\vee, x\lambda \rangle = \sum_{i \in J_\lambda^c} m_i \langle \alpha_j^\vee, x\varpi_i \rangle > 0$. Again, by Lemma 2.5 (2)–(3), we have $\Pi^{J_\lambda}(r_j x) \succeq x$. Thus $\kappa(S_y \eta_\rho) = \Pi^{J_\lambda}(y) \succeq x$.

We next assume that $N > 0$. It follows from Lemmas 2.1 and 3.7 (4) that

$$\text{(56)} \qquad f_{i_1}^{\max} \pi = f_{i_1}^{\max} \left(\bigotimes_{i \in J_\lambda^c} S_y \eta_{\rho^{(i)}} \right) = \bigotimes_{i \in J_\lambda^c} \underbrace{f_{i_1}^{\max} S_y \eta_{\rho^{(i)}}}_{= S_{r_{i_1} y} \eta_{\rho^{(i)}}} \in \breve{\mathbb{B}}_{\succeq \Pi^{J_\lambda}(r_{i_1} x)}$$

is of the form (47). By induction hypothesis, there exists $\eta \in \mathbb{B}_{\succeq \Pi^{J_\lambda}(r_{i_1} x)}$ such that $f_{i_1}^{\max} \pi = \psi_\lambda(\eta)$. Since ψ_λ is an isomorphism of U-crystals, there exists $k \in \mathbb{Z}_{>0}$ such that $\pi = \psi_\lambda(e_{i_1}^k \eta)$. We see from $\langle \alpha_{i_1}^\vee, x\lambda \rangle > 0$ (see (i) above) and Lemma 2.5 (2)–(3) that $\Pi^{J_\lambda}(r_{i_1} x) \succeq x$, which gives $\mathbb{B}_{\succeq \Pi^{J_\lambda}(r_{i_1} x)} \subset \mathbb{B}_{\succeq x}$, by Lemma 3.7 (3). It follows from $\langle \alpha_{i_1}^\vee, x\lambda \rangle > 0$ (see (i) above) and Lemma 3.7 (2) that

$$\text{(57)} \qquad \begin{aligned} e_{i_1}^k \eta &\in \left\{ e_{i_1}^k \eta \;\middle|\; \eta \in \mathbb{B}_{\succeq \Pi^{J_\lambda}(r_{i_1} x)} \right\} \setminus \{\mathbf{0}\} \\ &\subset \left\{ e_{i_1}^k \eta \;\middle|\; \eta \in \mathbb{B}_{\succeq x} \right\} \setminus \{\mathbf{0}\} \subset \mathbb{B}_{\succeq x}. \end{aligned}$$

Thus $\pi = \psi_\lambda(e_{i_1}^k \eta) \in \psi_\lambda(\mathbb{B}_{\succeq x})$.

3.3. Proof of Proposition 3.2.

Proof of Proposition 3.2. The "only if" part follows immediately from Lemma 2.4 (1). Let us show the "if" part. Let $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$ be such that $J_\lambda = J$. Let $\rho = (\rho^{(i)}) \in \mathrm{Par}(\lambda)$. Since $\kappa(S_x \eta_{\rho^{(i)}}) = \Pi^{I \setminus \{i\}}(x)$, by Lemma 3.6 (1), and $\Pi^{I \setminus \{i\}}(x) \succeq \Pi^{I \setminus \{i\}}(y)$ for all $i \in I \setminus J = J_\lambda^c$, it follows that $\bigotimes_{i \in J_\lambda^c} S_x \eta_{\rho^{(i)}} \in \bigotimes_{i \in J_\lambda^c} \mathbb{B}^{\frac{\infty}{2}}_{\succeq \Pi^{I \setminus \{i\}}(y)}(m_i \varpi_i)$. Similarly to (52), we have $\bigotimes_{i \in J_\lambda^c} S_x \eta_{\rho^{(i)}} = \psi_\lambda(S_x \eta_\rho)$. By Proposition 3.4 (2), $S_x \eta_\rho \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq y}(\lambda)$, which implies $\kappa(S_x \eta_\rho) \succeq y$. Since $x = \kappa(S_x \eta_\rho)$, by Lemma 3.6 (1), we conclude that $x \succeq y$.

4. Tableau criterion for semi-infinite Bruhat order

Proposition 3.2 shows that the study of semi-infinite Bruhat order on W_{af} is reduced to those on the sets $(W^{I \setminus \{i\}})_{\mathrm{af}}$, $i \in I$, of "semi-infinite Grassmannian elements." In this section, we proceed with the study of semi-infinite Bruhat order on $(W^{I \setminus \{i\}})_{\mathrm{af}}$, $i \in I$, for W_{af} of type $A_{n-1}^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, and $D_n^{(1)}$. The main results of this section are Theorems 4.9, 4.14, 4.22, and 4.38 (see also Definitions 4.7, 4.12, 4.20, and 4.36), which give combinatorial criteria for semi-infinite Bruhat order in terms of tableaux. In order to get these results, we give a complete classification of the edges in the quantum Bruhat graph $\mathrm{QB}^{I \setminus \{i\}}$, $i \in I$ (see §4.2 and Propositions 4.18, 4.26, and 4.42). For combinatorial descriptions of Bruhat order on finite Weyl groups of classical type, we refer the reader to [4, §8].

4.1. Explicit description of $(W^J)_{\mathrm{af}}$. In this subsection, following [24, §3], we give an explicit description of $(W^J)_{\mathrm{af}}$ for later use.

We take and fix $J = \bigsqcup_{m=1}^{k} I_m \subset I$, where $I_1, I_2, \dots, I_k$ are the sets of vertices of the connected components of the Dynkin diagram of Δ_J; note that $\Delta_J = \bigsqcup_{m=1}^{k} \Delta_{I_m}$. Set $(I_m)_{\mathrm{af}} = \{0\} \sqcup I_m \subset I_{\mathrm{af}}$, $m \in [k]$. Set

$$Q^{\vee,J} = \{ \xi \in Q^{\vee} \mid \langle \xi, \alpha \rangle \in \{-1, 0\} \text{ for all } \alpha \in \Delta_J^{+} \}. \tag{58}$$

Lemma 4.1 ([24, Equation (3.6)]). *For each $\xi \in Q^{\vee}$ there exist a unique $\varphi_J(\xi) \in Q_J^{\vee}$ and a unique $(j_1, j_2, \dots, j_k) \in \prod_{m=1}^{k} (I_m)_{\mathrm{af}}$ such that*

$$\xi + \varphi_J(\xi) + \sum_{m=1}^{k} \varpi_{j_m}^{\vee} \in \sum_{i \in I \setminus J} \mathbb{Z} \alpha_i^{\vee}. \tag{59}$$

In particular, $\xi + \varphi_J(\xi) \in Q^{\vee,J}$ for any $\xi \in Q^{\vee}$, and hence $Q^{\vee,J}$ is a complete system of coset representatives for $Q^{\vee}/Q_J^{\vee}$.

For a subset $K \subset I$, let w_0^K be the longest element of W_K. For $j_m \in (I_m)_{\mathrm{af}}$, set

$$v_{j_m}^{I_m} = w_0^{I_m} w_0^{I_m \setminus \{j_m\}} \in W_{I_m} \subset W_J; \tag{60}$$

note that $v_0^{I_m} = e$. For $\xi \in Q^{\vee}$, define

$$z_\xi = z_\xi^J = v_{j_1}^{I_1} v_{j_2}^{I_2} \cdots v_{j_k}^{I_k} \in W_J, \tag{61}$$

where $(j_1, j_2, \dots, j_k) \in \prod_{m=1}^{k} (I_m)_{\mathrm{af}}$, satisfying (59) for ξ, is determined uniquely by Lemma 4.1; note that $z_\xi = z_\zeta$ if $\xi \equiv \zeta \mod Q_J^{\vee}$.

Lemma 4.2 ([24, Lemma 3.7]). *We have $T_\xi = \Pi^J(t_\xi) = z_\xi t_{\xi + \varphi_J(\xi)}$ for every $\xi \in Q^{\vee}$. Therefore, by Lemma 2.3, $\Pi^J(w t_\xi) = \lfloor w \rfloor z_\xi t_{\xi + \varphi_J(\xi)}$ for every $w \in W$ and $\xi \in Q^{\vee}$, and we have a bijection $W^J \times Q^{\vee,J} \to (W^J)_{\mathrm{af}}$, $(w, \xi) \mapsto w T_\xi$. In particular,*

$$(W^J)_{\mathrm{af}} = \{ w T_\xi = w z_\xi t_\xi \mid w \in W^J,\ \xi \in Q^{\vee,J} \}. \tag{62}$$

4.2. Quantum Bruhat graphs. Following [24, §4] (see also [5]), define the (parabolic) quantum Bruhat graph QB^J to be the $(\Delta^+ \setminus \Delta_J^+)$-colored directed graph with vertex set W^J and edges of the form $w \xrightarrow{\gamma} \lfloor w r_\gamma \rfloor$ for $w \in W^J$ and $\gamma \in \Delta^+ \setminus \Delta_J^+$, where $\ell(\lfloor w r_\gamma \rfloor) - \ell(w) = 1 - 2\chi \langle \gamma^{\vee}, \rho - \rho_J \rangle$ and $\chi \in \{0, 1\}$. We say that an edge $w \xrightarrow{\gamma} \lfloor w r_\gamma \rfloor$ in QB^J is Bruhat (resp. quantum) if $\ell(\lfloor w r_\gamma \rfloor) - \ell(w) = 1$ (resp. $\ell(\lfloor w r_\gamma \rfloor) - \ell(w) = 1 - 2\langle \gamma^{\vee}, \rho - \rho_J \rangle$). We see that if there exists a Bruhat edge $w \xrightarrow{\gamma} \lfloor w r_\gamma \rfloor$ in QB^J, then $w r_\gamma = \lfloor w r_\gamma \rfloor \in W^J$. Note that QB^J does not define a partial order on W^J.

Lemma 4.3 ([11, Proposition A.1.2]). *Let $w \in W^J$, $\xi \in Q^{\vee,J}$ and $\beta \in \Delta_{\mathrm{af}}^{+}$. Write $\beta = w\gamma + \chi\delta$ with $\gamma \in \Delta$ and $\chi \in \mathbb{Z}_{\geq 0}$. Then $r_\beta w T_\xi \in (W^J)_{\mathrm{af}}$ and there exists an edge $w T_\xi \xrightarrow{\beta} r_\beta w T_\xi$ in SiB^J if and only if $\gamma \in \Delta^+ \setminus \Delta_J^+$ and one of the following conditions holds:*

1. *$\chi = 0$ and $w \xrightarrow{\gamma} w r_\gamma$ is a Bruhat edge in QB^J,*
2. *$\chi = 1$ and $w \xrightarrow{\gamma} \lfloor w r_\gamma \rfloor$ is a quantum edge in QB^J;*

in these cases, we have $r_\beta w T_\xi = \lfloor w r_\gamma \rfloor T_{\xi + \chi\gamma^{\vee}} = \lfloor w r_\gamma \rfloor z_{\xi + \chi\gamma^{\vee}}^J t_{\xi + \chi\gamma^{\vee} + \varphi_J(\xi + \chi\gamma^{\vee})}$ and $c_i(\xi + \chi\gamma^{\vee} + \varphi_J(\xi + \chi\gamma^{\vee})) = c_i(\xi) + \chi c_i(\gamma^{\vee})$ for all $i \in I \setminus J$.

Lemma 4.4 ([20, Proof of Theorem 10.16]). *Let $w \in W^J$ and $\gamma \in \Delta^+ \setminus \Delta_J^+$. There exists a quantum edge $w \xrightarrow{\gamma} \lfloor w r_\gamma \rfloor$ in QB^J if and only if $\ell(w r_\gamma) - \ell(w) = 1 - 2\langle \gamma^{\vee}, \rho \rangle$ and $w r_\gamma t_{\gamma^{\vee}} \in (W^J)_{\mathrm{af}}$; note that $w r_\gamma t_{\gamma^{\vee}} \in (W^J)_{\mathrm{af}}$ and Lemma 4.2 imply $\gamma^{\vee} \in Q^{\vee,J} \cap (\Delta^{\vee,+} \setminus \Delta_J^{\vee,+})$ and $\lfloor w r_\gamma \rfloor = w r_\gamma z_{\gamma^{\vee}}^{-1}$.*

Lemma 4.5. *Let $i \in I$ and $\gamma \in Q^{\vee, I\setminus\{i\}} \cap (\Delta^{\vee,+} \setminus \Delta^{\vee,+}_{I\setminus\{i\}})$. We have $\langle \gamma, \rho - \rho_{I\setminus\{i\}} \rangle = c_i(\gamma) \langle \alpha_i^\vee, \rho - \rho_{I\setminus\{i\}} \rangle = c_i(\gamma)(1 - \langle \alpha_i^\vee, \rho_{I\setminus\{i\}} \rangle)$.*

Proof. The assertion follows from $\langle \xi, \rho - \rho_{I\setminus\{i\}} \rangle = 0$ for $\xi \in Q^\vee_{I\setminus\{i\}}$, $\gamma \equiv c_i(\gamma)\alpha_i^\vee \mod Q^\vee_{I\setminus\{i\}}$, and $\langle \alpha_i^\vee, \rho \rangle = 1$.

4.3. Type $A_{n-1}^{(1)}$. Fix an integer $n > 2$. Set $I = [n-1]$. We assume that the labeling of the vertices of the Dynkin diagram of type A_{n-1} is as follows.

$$\underset{1}{\bullet} - \underset{2}{\bullet} - \cdots - \underset{n-2}{\bullet} - \underset{n-1}{\bullet}$$

Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be an orthonormal basis of an n-dimensional Euclidean space $\mathbb{R}^n$. Let $\Delta = \{\pm(\varepsilon_s - \varepsilon_t) \mid s, t \in [n],\ s < t\}$ be a root system of type A_{n-1}, and let $\Pi = \{\alpha_s = \varepsilon_s - \varepsilon_{s+1} \mid s \in I\}$ be a simple root system of Δ.

Let W be the Weyl group of Δ; note that W can be described by $W = \mathfrak{S}([n])$ as the permutation group of $\{\varepsilon_s \mid s \in [n]\} \subset \mathbb{R}^n$. The longest element of W is given by $u \mapsto n - u + 1$, $u \in [n]$. Let $\mathrm{CST}_{A_{n-1}}(\varpi_i)$ be the family of i-element subsets of $[n]$. We identify $\mathrm{T} = \{\mathrm{T}(1) < \mathrm{T}(2) < \cdots < \mathrm{T}(i)\} \in \mathrm{CST}_{A_{n-1}}(\varpi_i)$ with the column-strict tableau

(63)
$$\begin{array}{|c|}\hline \mathrm{T}(1) \\ \hline \mathrm{T}(2) \\ \hline \vdots \\ \hline \mathrm{T}(i) \\ \hline \end{array}\ .$$

For $w \in W$, let $\mathrm{T}^{(i)}_w \in \mathrm{CST}_{A_{n-1}}(\varpi_i)$ be such that

(64)
$$\mathrm{T}^{(i)}_w = \left(\mathrm{T}^{(i)}_w(1) < \mathrm{T}^{(i)}_w(2) < \cdots < \mathrm{T}^{(i)}_w(i) \right) = \{w(1), w(2), \ldots, w(i)\}.$$

The proof of the next lemma is standard (cf. [4, §2.4]).

Lemma 4.6. *Let $i \in I$. We have*

$$W^{I\setminus\{i\}} = \{w \in W \mid w(1) < w(2) < \cdots < w(i), \text{ and } w(i+1) < w(i+2) < \cdots < w(n)\}.$$

The map $W^{I\setminus\{i\}} \to \mathrm{CST}_{A_{n-1}}(\varpi_i)$, $w \mapsto \mathrm{T}^{(i)}_w$, is bijective.

We see from Lemmas 4.1–4.2 and 4.6 that the map

(65)
$$Y_i^{A_{n-1}} : W_{\mathrm{af}} \to \mathrm{CST}_{A_{n-1}}(\varpi_i) \times \mathbb{Z},\ wt_\xi \mapsto \left(\mathrm{T}^{(i)}_w, c_i(\xi) \right),$$

induces a bijection from the subset $(W^{I\setminus\{i\}})_{\mathrm{af}} \subset W_{\mathrm{af}}$ to $\mathrm{CST}_{A_{n-1}}(\varpi_i) \times \mathbb{Z}$.

Definition 4.7 ([10, Definition 4.2 (1)]). *Define a partial order $\preceq$ on $\mathrm{CST}_{A_{n-1}}(\varpi_i) \times \mathbb{Z}$ as follows: for $(\mathrm{T}, c), (\mathrm{T}', c') \in \mathrm{CST}_{A_{n-1}}(\varpi_i) \times \mathbb{Z}$, set $(\mathrm{T}, c) \preceq (\mathrm{T}', c')$ if*

(66)
$$(c \le c') \text{ and } (\mathrm{T}(u) \le \mathrm{T}'(u + c' - c) \text{ for } u \in [i - c' + c]).$$

Proposition 4.8. *Let $i \in I$.*

1. $Y_i^{A_{n-1}} \circ \Pi^{I\setminus\{i\}} = Y_i^{A_{n-1}}$.
2. *For $x, y \in W_{\mathrm{af}}$, we have $\Pi^{I\setminus\{i\}}(x) \preceq \Pi^{I\setminus\{i\}}(y)$ in $(W^{I\setminus\{i\}})_{\mathrm{af}}$ if and only if $Y_i^{A_{n-1}}(x) \preceq Y_i^{A_{n-1}}(y)$ in $\mathrm{CST}_{A_{n-1}}(\varpi_i) \times \mathbb{Z}$.*

(3) *Let* $(\mathrm{T}, c), (\mathrm{T}', c') \in \mathrm{CST}_{A_{n-1}}(\varpi_i) \times \mathbb{Z}$*. If* $c' - c > \min\{i, n-i\}$*, then* $(\mathrm{T}, c) \preceq (\mathrm{T}', c')$.

Proof. (1): Let $x = wt_\xi \in W_{\mathrm{af}}$ and $y = \Pi^{I \setminus \{i\}}(x) = vt_\zeta$, where $w, v \in W$ and $\xi, \zeta \in Q^\vee$. We see from Lemma 2.3 that $\lfloor w \rfloor^{I \setminus \{i\}} = \lfloor v \rfloor^{I \setminus \{i\}}$ and $c_i(\xi) = c_i(\zeta)$. Since $\{1, 2, \ldots, i\}$ is stable under the action of $W_{I \setminus \{i\}}$, $\lfloor w \rfloor^{I \setminus \{i\}} = \lfloor v \rfloor^{I \setminus \{i\}}$ implies $\mathrm{T}_w^{(i)} = \mathrm{T}_v^{(i)}$. Thus $Y_i^{A_{n-1}}(x) = Y_i^{A_{n-1}}(y)$.

(2): The assertion follows from (1) and [10, Theorem 4.7 and Equation (67)].

(3): We first assume that $c' - c > i$. Obviously, $c \leq c'$ holds. Since $i - c' + c \leq 0$, the latter condition in (66) is trivial. Thus $(\mathrm{T}, c) \preceq (\mathrm{T}', c')$.

We next assume that $i > c' - c > n - i$. Obviously, $c \leq c'$ holds. Since $\mathrm{T}(u) \in [u, u+n-i]$ holds for all $u \in [i]$, we have

$$\mathrm{T}(u) \leq u + n - i \leq u + c' - c \leq \mathrm{T}'(u + c' - c) \text{ for } u \in [i - c' + c]. \tag{67}$$

This implies $(\mathrm{T}, c) \preceq (\mathrm{T}', c')$.

By combining Propositions 3.2 and 4.8 (2), we obtain the following tableau criterion for the semi-infinite Bruhat order on W_{af} of type $A_{n-1}^{(1)}$.

Theorem 4.9. *Let* $J \subset I$ *and* $x, y \in (W^J)_{\mathrm{af}}$*. We have* $x \preceq y$ *in* $(W^J)_{\mathrm{af}}$ *if and only if* $Y_i^{A_{n-1}}(x) \preceq Y_i^{A_{n-1}}(y)$ *in* $\mathrm{CST}_{A_{n-1}}(\varpi_i) \times \mathbb{Z}$ *for all* $i \in I \setminus J$.

For $w \in W$, the notation $w = i_1 i_2 \cdots i_n$ means that $w(u) = i_u$ for $u \in [n]$. A column-strict tableau (of skew shape) is called semi-standard if its entries are weakly increasing from right to left in each row.

Example 4.10. Assume that $n = 6$. Let $w = 564213$, $v = 412635 \in W$, and

$$\xi = \alpha_1^\vee - \alpha_3^\vee + \alpha_4^\vee + 2\alpha_5^\vee, \ \zeta = 2\alpha_1^\vee + 3\alpha_2^\vee + \alpha_3^\vee + 2\alpha_4^\vee + 5\alpha_5^\vee \in Q^\vee.$$

Let us compare $x = wt_\xi$ and $y = vt_\zeta$ in semi-infinite Bruhat order on W_{af}. We have

$$\mathrm{T}_w^{(1)}\mathrm{T}_w^{(2)}\mathrm{T}_w^{(3)}\mathrm{T}_w^{(4)}\mathrm{T}_w^{(5)} = \begin{array}{|c|c|c|c|c|} \hline 5 & 5 & 4 & 2 & 1 \\ \hline & 6 & 5 & 4 & 2 \\ \hline & & 6 & 5 & 4 \\ \hline & & & 6 & 5 \\ \hline & & & & 6 \\ \hline \end{array}$$

and

$$\mathrm{T}_v^{(1)}\mathrm{T}_v^{(2)}\mathrm{T}_v^{(3)}\mathrm{T}_v^{(4)}\mathrm{T}_v^{(5)} = \begin{array}{|c|c|c|c|c|} \hline 4 & 1 & 1 & 1 & 1 \\ \hline & 4 & 2 & 2 & 2 \\ \hline & & 4 & 4 & 3 \\ \hline & & & 6 & 4 \\ \hline & & & & 6 \\ \hline \end{array}.$$

(1) $Y_1^{A_{n-1}}(x) \preceq Y_1^{A_{n-1}}(y)$ since $c_1(\zeta) - c_1(\xi) = 1 = \min\{1, 6-1\}$ (see Proposition 4.8 (3)).

(2) $Y_2^{A_{n-1}}(x) \preceq Y_2^{A_{n-1}}(y)$ since $c_2(\zeta) - c_2(\xi) = 3 > 2 = \min\{2, 6-2\}$ (see Proposition 4.8 (3)).

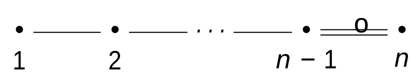
(3) $Y_3^{A_{n-1}}(x) \preceq Y_3^{A_{n-1}}(y)$ since $c_3(\zeta) - c_3(\xi) = 2$ and $\begin{array}{|c|c|} \hline 1 & \\ \hline 2 & \\ \hline 4 & 4 \\ \hline & 5 \\ \hline & 6 \\ \hline \end{array}$ is semi-standard (see (66)), where $\mathrm{T}_w^{(3)} = \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline 6 \\ \hline \end{array}$ and $\mathrm{T}_v^{(3)} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array}$.

(4) $Y_4^{A_{n-1}}(x) \quad Y_4^{A_{n-1}}(y)$ since $c_4(\zeta) - c_4(\xi) = 1$ and $\begin{array}{|c|c|} \hline 1 & \\ \hline 2 & 2 \\ \hline 4 & 4 \\ \hline 6 & 5 \\ \hline & 6 \\ \hline \end{array}$ is semi-standard (see (66)), where $\mathrm{T}_w^{(4)} = \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline 5 \\ \hline 6 \\ \hline \end{array}$ and $\mathrm{T}_v^{(4)} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 6 \\ \hline \end{array}$.

(5) $Y_5^{A_{n-1}}(x) \quad Y_5^{A_{n-1}}(y)$ since $c_5(\zeta) - c_5(\xi) = 3 > 1 = \min\{5, 6-5\}$ (see Proposition 4.8 (3)).

By Theorem 4.9, we conclude that $x \quad y$.

4.4. Type $C_n^{(1)}$. Fix an integer $n > 2$. Set $I = [n]$. We assume that the labeling of the vertices of the Dynkin diagram of type C_n is as follows.



Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be an orthonormal basis of an n-dimensional Euclidean space $\mathbb{R}^n$. Let $\Delta = \{\pm(\varepsilon_s \pm \varepsilon_t) \mid s, t \quad [n],\ s < t\} \quad \{\pm 2\varepsilon_s \mid s \quad [n]\}$ be a root system of type C_n, and let $\Pi = \{\alpha_s = \varepsilon_s - \varepsilon_{s+1} \mid s \quad [n-1]\} \quad \{\alpha_n = 2\varepsilon_n\}$ be a simple root system of Δ.

Let W be the Weyl group of Δ. Note that W acts faithfully on $\{\pm\varepsilon_s \mid s \quad [n]\} \quad \mathbb{R}^n$. Define a totally ordered set $\mathcal{C}_n$ by

(68) $$\mathcal{C}_n = \{1 \quad 2 \quad \cdots \quad n-1 \quad n \quad \overline{n} \quad \overline{n-1} \quad \cdots \quad \overline{2} \quad \overline{1}\}.$$

Let $\sigma : \mathcal{C}_n \to \mathcal{C}_n$ be the bijection defined by $s \leftrightarrow \overline{s}$ for $s \quad [n]$. If we identify $\mathcal{C}_n$ with $\{\pm\varepsilon_s \mid s \quad [n]\}$ by $s = \varepsilon_s$ and $\overline{s} = -\varepsilon_s$ for $s \quad [n]$, then W can be described as follows:

(69) $$W = \{w \quad \mathfrak{S}(\mathcal{C}_n) \mid w(\sigma(s)) = \sigma(w(s)) \text{ for } s \quad [n]\}.$$

Let $(s_1\ s_2\ \cdots\ s_l) \quad \mathfrak{S}(\mathcal{C}_n)$ denote the cyclic permutation $s_1 \to s_2 \to \cdots \to s_l \to s_1$, where $l > 1$ and $s_1, s_2, \dots, s_l \quad \mathcal{C}_n$ are all distinct. For $s, t \quad [n]$, $s < t$, we have $r_{\pm(\varepsilon_s - \varepsilon_t)} = (s\ t)(\overline{s}\ \overline{t})$, $r_{\pm 2\varepsilon_s} = (s\ \overline{s})$, and $r_{\pm(\varepsilon_s + \varepsilon_t)} = (s\ \overline{t})(\overline{s}\ t)$ in $\mathfrak{S}(\mathcal{C}_n)$.

For $w \quad \mathfrak{S}(\mathcal{C}_n)$ and $s \quad [n]$, set

(70) $$\mathrm{A}_s(w) = \{t \quad [s+1, n] \mid w(s) \quad w(t) \text{ in } \mathcal{C}_n\}, \quad \mathrm{a}_s(w) = \#\mathrm{A}_s(w),$$

(71) $$\mathrm{B}_s(w) = \{t \quad [s+1, n] \mid w(s) \quad \sigma(w(t)) \text{ in } \mathcal{C}_n\}, \quad \mathrm{b}_s(w) = \#\mathrm{B}_s(w),$$

(72) $$\mathrm{e}_s(w) = \begin{cases} 0 & \text{if } w(s) \quad n, \\ 1 & \text{if } w(s) \quad \overline{n}; \end{cases}$$

note that $\mathrm{a}_n(w) = \mathrm{b}_n(w) = 0$ for $w \quad \mathfrak{S}(\mathcal{C}_n)$. The length function $\ : W \to \mathbb{Z}_{>0}$ is given by

(73) $$(w) = \sum_{s=1}^{n} (\mathrm{a}_s(w) + \mathrm{b}_s(w) + \mathrm{e}_s(w))$$

for $w \quad W$. The longest element of W is given by $u \to \overline{u}$, $u \quad [n]$.

Let $\ \cdot\ : \mathcal{C}_n \to [n]$ be the map defined by $\ s\ = s$ and $\ \overline{s}\ = s$ for $s \quad [n]$. We identify an i-element subset $\mathrm{T} = \{\mathrm{T}(1) \quad \mathrm{T}(2) \quad \cdots \quad \mathrm{T}(i)\} \quad \mathcal{C}_n$ with the column-strict tableau of the form (63). Set

(74) $$\mathrm{CST}_{\mathcal{C}_n}(\ _i) = \{\mathrm{T} \mid \mathrm{T} \quad \mathcal{C}_n,\ \#\mathrm{T} = i, \text{ and } \ \mathrm{T}(u)\ ,\ u \quad [i], \text{ are all distinct}\}.$$

For $w \in W$, let $\mathrm{T}^{(i)}_w \in \mathrm{CST}_{C_n}(\varpi_i)$ be such that

(75) $$\mathrm{T}^{(i)}_w = \left(\mathrm{T}^{(i)}_w(1) \prec \mathrm{T}^{(i)}_w(2) \prec \cdots \prec \mathrm{T}^{(i)}_w(i)\right) = \{w(1), w(2), \ldots, w(i)\}.$$

The proof of the next lemma is standard (cf. [4, §8.1]).

Lemma 4.11. *Let $i \in I$. We have*

$$W^{I \setminus \{i\}} = \{w \in W \mid w(1) \prec w(2) \prec \cdots \prec w(i), \text{ and } w(i+1) \prec w(i+2) \prec \cdots \prec w(n) \preceq n\}.$$

If $w \in W^{I\setminus\{i\}}$, then $\ell(w) = \sum_{s=1}^{i}(\mathrm{a}_s(w) + \mathrm{b}_s(w) + \mathrm{e}_s(w))$ and $\mathrm{A}_s(w) \subset [i+1, n]$ for $s \in [i]$. The map $W^{I\setminus\{i\}} \to \mathrm{CST}_{C_n}(\varpi_i)$, $w \mapsto \mathrm{T}^{(i)}_w$, is bijective.

We see from Lemmas 4.1–4.2 and 4.11 that the map

(76) $$Y_i^{C_n} : W_{\mathrm{af}} \to \mathrm{CST}_{C_n}(\varpi_i) \times \mathbb{Z}, \ wt_\xi \mapsto \left(\mathrm{T}^{(i)}_w, c_i(\xi)\right),$$

induces a bijection from the subset $(W^{I\setminus\{i\}})_{\mathrm{af}} \subset W_{\mathrm{af}}$ to $\mathrm{CST}_{C_n}(\varpi_i) \times \mathbb{Z}$.

Definition 4.12. *Define a partial order $\preceq$ on $\mathrm{CST}_{C_n}(\varpi_i) \times \mathbb{Z}$ as follows: for $(\mathrm{T}, c), (\mathrm{T}', c') \in \mathrm{CST}_{C_n}(\varpi_i) \times \mathbb{Z}$, set $(\mathrm{T}, c) \preceq (\mathrm{T}', c')$ if*

(77) $$(c' \leq c) \text{ and } (\mathrm{T}(u) \preceq \mathrm{T}'(u + c - c') \text{ in } C_n \text{ for } u \in [i - c + c']).$$

Proposition 4.13. *Let $i \in I$.*

1. $Y_i^{C_n} \circ \Pi^{I\setminus\{i\}} = Y_i^{C_n}$.
2. *For $x, y \in W_{\mathrm{af}}$, we have $\Pi^{I\setminus\{i\}}(x) \preceq \Pi^{I\setminus\{i\}}(y)$ in $(W^{I\setminus\{i\}})_{\mathrm{af}}$ if and only if $Y_i^{C_n}(x) \preceq Y_i^{C_n}(y)$ in $\mathrm{CST}_{C_n}(\varpi_i) \times \mathbb{Z}$.*
3. *Let $(\mathrm{T}, c), (\mathrm{T}', c') \in \mathrm{CST}_{C_n}(\varpi_i) \times \mathbb{Z}$. If $c - c' > i$, then $(\mathrm{T}, c) \preceq (\mathrm{T}', c')$.*

By combining Propositions 3.2 and 4.13(2), we obtain the following tableau criterion for the semi-infinite Bruhat order on W_{af} of type $C_n^{(1)}$.

Theorem 4.14. *Let $J \subsetneq I$. For $x, y \in (W^J)_{\mathrm{af}}$, we have $x \preceq y$ in $(W^J)_{\mathrm{af}}$ if and only if $Y_i^{C_n}(x) \preceq Y_i^{C_n}(y)$ in $\mathrm{CST}_{C_n}(\varpi_i) \times \mathbb{Z}$ for all $i \in I \setminus J$.*

The remainder of this subsection is devoted to the proof of Proposition 4.13.

The proofs of Lemmas 4.15–4.16 below are straightforward.

Lemma 4.15. *Let $i \in I$ and $\gamma \in \Delta^+ \setminus \Delta^+_{I\setminus\{i\}}$. We have $\gamma^\vee \in Q^{\vee, I\setminus\{i\}}$ if and only if one of the following conditions holds:*

1. $\gamma^\vee = \varepsilon_i = \alpha_i^\vee + \alpha_{i+1}^\vee + \cdots + \alpha_n^\vee$.
2. $\gamma^\vee = \varepsilon_{i-1} + \varepsilon_i = \alpha_{i-1}^\vee + 2\alpha_i^\vee + \cdots + 2\alpha_n^\vee$.

Lemma 4.16. *Let $i \in I$. We have $2\langle \alpha_i^\vee, \rho - \rho_{I\setminus\{i\}}\rangle = 2n - i + 1$.*

Proposition 4.17 (cf. [4, §8.1]). *Let $i \in I$, $w \in W^{I\setminus\{i\}}$, and $\gamma \in \Delta^+$. There exists a Bruhat edge $w \xrightarrow{\gamma} \lfloor wr_\gamma \rfloor = wr_\gamma$ in $\mathrm{QB}^{I\setminus\{i\}}$ if and only if $\gamma \in \Delta^+ \setminus \Delta^+_{I\setminus\{i\}}$ and one of the following statements holds.*

(b-C1) *$i \in [n-1]$, $c_i(\gamma^\vee) = 1$, and there exists $s \in [i]$ such that $wr_\gamma(u) = w(u)$ for $u \in [i] \setminus \{s\}$, $1 \preceq w(s) \prec n$, and $wr_\gamma(s) = \min([w(s)+1, n] \setminus \{|w(u)| \mid u \in [i],\ w(u) \preceq \overline{n}\})$; in this case, we have $\gamma^\vee = \varepsilon_s - \varepsilon_t = \alpha_s^\vee + \alpha_{s+1}^\vee + \cdots + \alpha_t^\vee$ for some $t \in [i+1, n]$.*

(b-C2) *$i \in [n-1]$, $c_i(\gamma^\vee) = 1$, and there exists $s \in [i]$ such that $wr_\gamma(u) = w(u)$ for $u \in [i] \setminus \{s\}$, $\overline{n} \preceq w(s) \prec \overline{1}$, and $wr_\gamma(s) = \sigma(\max([|w(s)| - 1] \setminus \{w(u) \mid u \in [i],\ w(u) \preceq n\}))$; in this case, we have $\gamma^\vee = \varepsilon_s - \varepsilon_t = \alpha_s^\vee + \alpha_{s+1}^\vee + \cdots + \alpha_t^\vee$ for some $t \in [i+1, n]$.*

(b-C3) $i \in [2,n]$, $c_i(\gamma) = 2$, *and there exist* $s,t \in [i]$ *such that* $s < t$, $wr_\gamma(u) = w(u)$ *for* $u \in [i] \setminus \{s,t\}$, *and* $wr_\gamma(s) = w(s) + 1 = \sigma(w(t)) = \sigma(wr_\gamma(t)) + 1 \preceq n$*; in this case, we have* $\gamma = \varepsilon_s + \varepsilon_t = \alpha_s + \cdots + \alpha_{t-1} + 2\alpha_t + \cdots + 2\alpha_n$.

(b-C4) $c_i(\gamma) = 1$, *and there exists* $s \in [i]$ *such that* $wr_\gamma(u) = w(u)$ *for* $u \in [i] \setminus \{s\}$ *and* $\sigma(wr_\gamma(s)) = w(s) = n$*; in this case, we have* $\gamma = \varepsilon_s = \alpha_s + \alpha_{s+1} + \cdots + \alpha_n$.

Moreover, for $w, v \in W^{I \setminus \{i\}}$, *we have* $w \le v$ *if and only if* $w(u) \preceq v(u)$ *in* C_n *for* $u \in [i]$.

PROPOSITION 4.18. *Let* $i \in I$, $w \in W^{I \setminus \{i\}}$ *and* $\gamma \in \Delta^+$. *There exists a quantum edge* $w \xrightarrow{\gamma} \lfloor wr_\gamma \rfloor$ *in* $\mathrm{QB}^{I \setminus \{i\}}$ *if and only if* $\gamma \in \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$ *and the following statement holds:*

(q-C) $c_i(\gamma) = 1$, $\lfloor wr_\gamma \rfloor(1) = 1$, $\lfloor wr_\gamma \rfloor(u) = w(u-1)$ *for* $u \in [2,i]$, *and* $w(i) = \overline{1}$*; in this case, we have* $\gamma = \varepsilon_i = \alpha_i + \alpha_{i+1} + \cdots + \alpha_n$.

Before starting the proof of Proposition 4.18, we mention a consequence of Lemma 4.3 and Propositions 4.17–4.18.

PROPOSITION 4.19. *Let* $i \in I$, $x, y \in (W^{I \setminus \{i\}})_{\mathrm{af}}$, $Y_i^{C_n}(x) = (\mathsf{T}, c)$, *and* $Y_i^{C_n}(y) = (\mathsf{T}', c')$. *There exists an edge* $x \xrightarrow{\beta} y$ *in* $\mathrm{SiB}^{I \setminus \{i\}}$ *for some* $\beta \in \Delta^+_{\mathrm{af}}$ *if and only if one of the following conditions holds:*

($\frac{\infty}{2}$-C1) $i \in [n-1]$, $c' = c$, *and there exists* $s \in [i]$ *such that* $\mathsf{T}'(u) = \mathsf{T}(u)$ *for* $u \in [i] \setminus \{s\}$, $1 \preceq \mathsf{T}(s) \prec n$, *and* $\mathsf{T}'(s) = \min([\mathsf{T}(s)+1, n] \setminus \{\mathsf{T}(u) \mid u \in [i], \mathsf{T}(u) \preceq \overline{n}\})$.

($\frac{\infty}{2}$-C2) $i \in [n-1]$, $c' = c$, *and there exists* $s \in [i]$ *such that* $\mathsf{T}'(u) = \mathsf{T}(u)$ *for* $u \in [i] \setminus \{s\}$, $\overline{n} \preceq \mathsf{T}(s) \prec \overline{1}$, *and* $\mathsf{T}'(s) = \sigma(\max([\sigma\mathsf{T}(s) - 1] \setminus \{\mathsf{T}(u) \mid u \in [i], \mathsf{T}(u) \preceq n\}))$.

($\frac{\infty}{2}$-C3) $i \in [2,n]$, $c' = c$, *and there exist* $s,t \in [i]$ *such that* $s < t$, $\mathsf{T}'(u) = \mathsf{T}(u)$ *for* $u \in [i] \setminus \{s,t\}$, *and* $\mathsf{T}'(s) = \mathsf{T}(s) + 1 = \sigma(\mathsf{T}(t)) = \sigma(\mathsf{T}'(t)) + 1 \preceq n$.

($\frac{\infty}{2}$-C4) $c' = c$, *and there exists* $s \in [i]$ *such that* $\mathsf{T}'(u) = \mathsf{T}(u)$ *for* $u \in [i] \setminus \{s\}$ *and* $\sigma(\mathsf{T}'(s)) = \mathsf{T}(s) = n$.

($\frac{\infty}{2}$-C5) $c' = c + 1$, $\mathsf{T}'(1) = 1$, $\mathsf{T}'(u) = \mathsf{T}(u-1)$ *for* $u \in [2,i]$, *and* $\mathsf{T}(i) = \overline{1}$.

For $i \in I$, $w \in W^{I \setminus \{i\}}$ and $\gamma \in \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$, let $\mathrm{Q}(i,w,\gamma)$ denote the following statement.

$\mathrm{Q}(i,w,\gamma)$**:** There exists a quantum edge $w \xrightarrow{\gamma} \lfloor wr_\gamma \rfloor$ in $\mathrm{QB}^{I \setminus \{i\}}$.

Proof of Proposition 4.18. By Lemmas 4.4 and 4.15, we may assume that $\gamma \in Q^{\vee, I \setminus \{i\}}$, $\lfloor wr_\gamma \rfloor = wr_\gamma (z_\gamma^{I \setminus \{i\}})^{-1}$ and $c_i(\gamma) \in \{1,2\}$. Let $I \setminus \{i\} = I_1 \sqcup I_2$, where $I_1 = [i-1]$ is of type A_{i-1} and $I_2 = [i+1,n]$ is of type C_{n-i}. The proof will be divided into three steps.

Step 1. We show that $c_i(\gamma) = 1$ and $\mathrm{Q}(i,w,\gamma)$ imply (q-C). It follows from Lemmas 4.5 and 4.16 that $c_i(\gamma) = 1$ and $\mathrm{Q}(i,w,\gamma)$ are equivalent to $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = i - 2n$. By Lemma 4.15, $\gamma = \varepsilon_i$ and $r_\gamma = (i\ \overline{i})$. We see that $(i-1, 0) \in (I_1)_{\mathrm{af}} \times (I_2)_{\mathrm{af}}$ satisfies the condition for $\gamma^\vee \in Q^\vee$ in Lemma 4.1; note that $I_1 \setminus \{i-1\} = [i-2]$ is of type A_{i-2}. Hence $z_\gamma^{I \setminus \{i\}} = w_0^{I_1} w_0^{I_1 \setminus \{i-1\}} = (1\ 2\ \cdots\ i)(\overline{1}\ \overline{2}\ \cdots\ \overline{i})$ and $\lfloor wr_\gamma \rfloor = w(i\ \overline{i})(i\ \cdots\ 2\ 1)(\overline{i}\ \cdots\ \overline{2}\ \overline{1})$. We have $\lfloor wr_\gamma \rfloor(1) = w(\overline{i})$, $\lfloor wr_\gamma \rfloor(u) = w(u-1)$ for $u \in [2,i]$, and $\lfloor wr_\gamma \rfloor(u) = w(u)$ for $u \in [i+1,n]$. It follows from Lemma 4.11 that

$$(78) \qquad \underbrace{w(\overline{i})}_{\preceq n} \prec w(1) \prec w(2) \prec \cdots \prec w(i-1) \prec \underbrace{w(i)}_{\succeq \overline{n}}.$$

It remains to prove that $w(i) = \overline{1}$. If we prove that

(1) $e_1(wr_\gamma) = 0$,
(2) $a_1(wr_\gamma) + b_1(wr_\gamma) = w(\bar{i}) - 1$,
(3) $a_i(w) = n - i$, $b_i(w) = n - i - (w(\bar{i}) - 1)$, $e_i(w) = 1$,
(4) $e_s(wr_\gamma) = e_{s-1}(w)$ for $s \in [2, i]$,
(5) $a_s(wr_\gamma) = a_{s-1}(w)$ and $b_s(wr_\gamma) = b_{s-1}(w) - 1$ for $s \in [2, i]$,

then the assertion follows. Indeed, by Lemma 4.11, we have

$$\begin{aligned}
\ell(wr_\gamma) - \ell(w) &= \underbrace{a_1(wr_\gamma) + b_1(wr_\gamma)}_{= w(\bar{i}) - 1} + \underbrace{e_1(wr_\gamma)}_{=0} \\
&\quad - \underbrace{(a_i(w) + b_i(w) + e_i(w))}_{= 1 + 2(n-i) - (w(\bar{i}) - 1)} \\
&\quad + \sum_{s=2}^{i} \Big(\underbrace{a_s(wr_\gamma) - a_{s-1}(w)}_{=0} \Big) \\
&\quad + \sum_{s=2}^{i} \Big(\underbrace{b_s(wr_\gamma) - b_{s-1}(w)}_{=-1} \Big) \\
&\quad + \sum_{s=2}^{i} \Big(\underbrace{e_s(wr_\gamma) - e_{s-1}(w)}_{=0} \Big) \\
&= i - 2n + 2(w(\bar{i}) - 1).
\end{aligned}$$

Since $\ell(wr_\gamma) - \ell(w) = i - 2n$, we get $w(\bar{i}) = 1$ and $w(i) = \bar{1}$.

We prove (1)–(5) as follows.

(1) follows from $wr_\gamma(1) = w(\bar{i})$ and (78).

(2): We see from $wr_\gamma(1) = w(\bar{i})$, $wr_\gamma(u) = w(u-1)$ for $u \in [2, i]$, $wr_\gamma(u) = w(u)$ for $u \in [i+1, n]$, and (78) (see also Lemma 4.11) that $A_1(wr_\gamma), B_1(wr_\gamma) \subset [i+1, n]$ and

$$A_1(wr_\gamma) = \{t \in [i+1, n] \mid w(\bar{i}) > w(t)\},$$
$$B_1(wr_\gamma) = \{t \in [i+1, n] \mid w(i) > w(t)\}.$$

Hence $A_1(wr_\gamma) \cap B_1(wr_\gamma) = \emptyset$ and

$$A_1(wr_\gamma) \sqcup B_1(wr_\gamma) = \{t \in [n] \mid |w(t)| < w(\bar{i})\},$$

which implies $a_1(wr_\gamma) + b_1(wr_\gamma) = w(\bar{i}) - 1$.

(3): Since $\bar{n} \neq w(i)$, we have $e_i(w) = 1$, $A_i(w) = [i+1, n]$, and hence $a_i(w) = n - i$. We see from $w(\bar{i}) \neq w(1)$ that $B_i(w) = [i + w(\bar{i}), n]$ and $b_i(w) = n - i - (w(\bar{i}) - 1)$.

(4) follows from $wr_\gamma(s) = w(s-1)$ for $s \in [2, i]$.

(5): Let $s \in [2, i]$. We see from Lemma 4.11 that

$$A_s(wr_\gamma) = \{t \in [i+1, n] \mid \underbrace{wr_\gamma(s)}_{=w(s-1)} > \underbrace{wr_\gamma(t)}_{=w(t)}\} = A_{s-1}(w),$$

which implies $a_s(wr_\gamma) = a_{s-1}(w)$ for $s \in [2, i]$. Similarly, we deduce from $i \in B_{s-1}(w)$ that the map

$$B_s(wr_\gamma) \to B_{s-1}(w) \setminus \{i\}, \quad t \mapsto \begin{cases} t - 1 & \text{if } t \in [s+1, i], \\ t & \text{if } t \in [i+1, n], \end{cases}$$

is bijective, which implies $b_s(wr_\gamma) = b_{s-1}(w) - 1$ for $s \in [2, i]$.

Step 2. We show that $c_i(\gamma) = 2$ and $\mathrm{Q}(i, w, \gamma)$ lead to a contradiction; the computation below will be used again in the proofs of Lemmas 4.32–4.33 in §4.5. It follows from Lemmas 4.5 and 4.16 that $c_i(\gamma) = 2$ and $\mathrm{Q}(i, w, \gamma)$ are equivalent to $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = 2i-4n-1$. By Lemma 4.15, $\gamma = \varepsilon_{i-1}+\varepsilon_i$ and $r_\gamma = (i-1\ \bar{i})(\overline{i-1}\ i)$. We see that $(i-2, 0) \in (I_1)_{\mathrm{af}} \times (I_2)_{\mathrm{af}}$ satisfies the condition for $\gamma \in Q$ in Lemma 4.1; note that $I_1 \setminus \{i-2\} = [i-3] \sqcup \{i-1\}$ is of type $A_{i-3}\times A_1$. Hence $z_\gamma^{I \setminus \{i\}} = w_0^{I_1} w_0^{I_1 \setminus \{i-2\}}$ is given by $u \to u+2$ for $u \in [i-2]$, $i-1 \to 1$, and $i \to 2$. We have $\lfloor wr_\gamma \rfloor(1) = w(\bar{i})$, $\lfloor wr_\gamma \rfloor(2) = w(\overline{i-1})$, $\lfloor wr_\gamma \rfloor(u) = w(u-2)$ for $u \in [3, i]$, and $\lfloor wr_\gamma \rfloor(u) = w(u)$ for $u \in [i+1, n]$. It follows from Lemma 4.11 that

$$\underbrace{w(\bar{i}) \succ w(\overline{i-1})}_{\succeq n} \succ w(1) \succ w(2) \succ \cdots \succ w(i-2) \succ \underbrace{w(i-1) \succ w(i)}_{\preceq \bar{n}}; \tag{79}$$

note that $w(\bar{i}) - 1 > 0$ and $w(\overline{i-1}) - 2 > 0$. If we prove that

(1) $\mathrm{e}_1(\lfloor wr_\gamma \rfloor) = \mathrm{e}_2(\lfloor wr_\gamma \rfloor) = 0$, $\mathrm{e}_s(\lfloor wr_\gamma \rfloor) = \mathrm{e}_{s-2}(w)$ for $s \in [3, i]$, $\mathrm{e}_{i-1}(w) = \mathrm{e}_i(w) = 1$,
(2) $\mathrm{a}_1(\lfloor wr_\gamma \rfloor) = w(\bar{i}) - 1$, $\mathrm{a}_2(\lfloor wr_\gamma \rfloor) = w(\overline{i-1}) - 2$, $\mathrm{a}_{i-1}(w) = \mathrm{a}_i(w) = n - i$,
(3) $\mathrm{b}_1(\lfloor wr_\gamma \rfloor) = \mathrm{b}_2(\lfloor wr_\gamma \rfloor) = 0$, $\mathrm{b}_{i-1}(w) = n - i - (w(\overline{i-1}) - 2) + 1$, $\mathrm{b}_i(w) = n - i - (w(\bar{i}) - 1)$,
(4) $\mathrm{a}_s(\lfloor wr_\gamma \rfloor) = \mathrm{a}_{s-2}(w)$ and $\mathrm{b}_s(\lfloor wr_\gamma \rfloor) = \mathrm{b}_{s-2}(w) - 2$ for $s \in [3, i]$,

then the assertion follows. Indeed, (1)–(4) and Lemma 4.11 imply

$$\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = 2i - 4n + 1 + 2\underbrace{(w(\bar{i}) - 1)}_{>0} + 2\underbrace{(w(\overline{i-1}) - 2)}_{>0}$$
$$> 2i - 4n - 1,$$

contrary to $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = 2i - 4n - 1$.

We prove (1)–(4) as follows.

(1) follows from (79).

(2): We see from Lemma 4.11 and (79) that

$$\mathrm{A}_1(\lfloor wr_\gamma \rfloor) = \{t \in [i+1, n] \mid \underbrace{\lfloor wr_\gamma \rfloor(1)}_{=w(\bar{i})} \succ \underbrace{\lfloor wr_\gamma \rfloor(t)}_{=w(t)}\} = [i+1, i+w(\bar{i})-1],$$

which implies $\mathrm{a}_1(\lfloor wr_\gamma \rfloor) = w(\bar{i}) - 1$. Similarly, we have

$$\mathrm{A}_2(\lfloor wr_\gamma \rfloor) = [i+1, i+w(\overline{i-1})-2],$$

$$\mathrm{A}_{i-1}(w) = \mathrm{A}_i(w) = [i+1, n],$$

which imply $\mathrm{a}_2(\lfloor wr_\gamma \rfloor) = w(\overline{i-1}) - 2$ and $\mathrm{a}_{i-1}(w) = \mathrm{a}_i(w) = n - i$.

(3): We claim that $\mathrm{B}_1(\lfloor wr_\gamma \rfloor) = \emptyset$. Suppose that $2 \in \mathrm{B}_1(\lfloor wr_\gamma \rfloor)$. Then $w(\bar{i}) = \lfloor wr_\gamma \rfloor(1) \preceq \sigma(\lfloor wr_\gamma \rfloor(2)) = w(i-1)$, contrary to (79). Suppose that $t \in \mathrm{B}_1(\lfloor wr_\gamma \rfloor) \cap [3, i]$. Then $w(\bar{i}) = \lfloor wr_\gamma \rfloor(1) \preceq \sigma(\lfloor wr_\gamma \rfloor(t)) = \sigma(w(t-2))$ and hence $w(i) \succeq w(t-2)$, contrary to (79). Suppose that $t \in \mathrm{B}_1(\lfloor wr_\gamma \rfloor) \cap [i+1, n]$. Then $w(\bar{i}) = \lfloor wr_\gamma \rfloor(1) \preceq \sigma(\lfloor wr_\gamma \rfloor(t)) = \sigma(w(t))$ and hence $\bar{n} \succeq w(i) \succeq w(t)$, contrary to Lemma 4.11. Consequently, we have $\mathrm{B}_1(\lfloor wr_\gamma \rfloor) = \emptyset$ and $\mathrm{b}_1(\lfloor wr_\gamma \rfloor) = 0$ as claimed. Similarly, we have $\mathrm{B}_2(\lfloor wr_\gamma \rfloor) = \emptyset$ and $\mathrm{b}_2(\lfloor wr_\gamma \rfloor) = 0$. We next claim that $\mathrm{B}_i(w) = [i+1, n] \setminus \mathrm{A}_1(\lfloor wr_\gamma \rfloor)$. Indeed,

$$\begin{aligned}\mathrm{B}_i(w) &= \{t \in [i+1, n] \mid w(i) \succeq \sigma(w(t))\} \\ &= \{t \in [i+1, n] \mid w(\bar{i}) \preceq w(t)\} \\ &= [i+1, n] \setminus \mathrm{A}_1(\lfloor wr_\gamma \rfloor).\end{aligned}$$

This implies $b_i(w) = n - i - (w(\bar{i}) - 1)$. Similarly, we have $B_{i-1}(w) = [i, n] \smallsetminus A_2(wr_y)$ and $b_{i-1}(w) = n - i - (w(\overline{i-1}) - 2) + 1$.

(4): Let $s \in [3, i]$. We see from Lemma 4.11 that

$$A_s(wr_y) = \{t \in [i+1, n] \mid \underbrace{wr_y(s)}_{=w(s-2)} \le \underbrace{wr_y(t)}_{=w(t)}\} = A_{s-2}(w),$$

which implies $a_s(wr_y) = a_{s-2}(w)$ for $s \in [3, i]$. Similarly, we deduce from $i-1, i \in B_{s-2}(w)$ that the map

$$B_s(wr_y) \to B_{s-2}(w) \smallsetminus \{i-1, i\},\quad t \mapsto \begin{cases} t-2 & \text{if } t \in [s+1, i], \\ t & \text{if } t \in [i+1, n], \end{cases}$$

is bijective, which implies $b_s(wr_y) = b_{s-2}(w) - 2$ for $s \in [3, i]$.

Step 3. We show that (q-C) implies $Q(i, w, y)$. Assume that (q-C) is true. Since $c_i(y) = 1$, $Q(i, w, y)$ is equivalent to $\ell(wr_y) - \ell(w) = i - 2n$, by Lemmas 4.5 and 4.16. We see from (q-C) that w and wr_y satisfy (1)–(5) in Step 1. As in Step 1, this gives $\ell(wr_y) - \ell(w) = i - 2n + (w(\bar{i}) - 1)$. Since $w(\bar{i}) = 1$ by (q-C), we conclude that $\ell(wr_y) - \ell(w) = i - 2n$.

The proof of Proposition 4.18 is complete.

Proof of Proposition 4.13. (1) and (3) follow by the same method as in the proof of Proposition 4.8.

We prove (2). Let $x, y \in (W^{I \smallsetminus \{i\}})_{\mathrm{af}}$, $Y_i^{C_n}(x) = (T, c)$, and $Y_i^{C_n}(y) = (T', c')$. By Proposition 4.19, we may assume that $d := c' - c > 0$. The proof is by induction on d.

If $d = 0$, the assertion follows from Propositions 4.17 and 4.19.

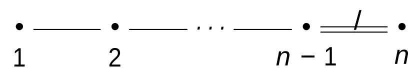
Assume that $d > 0$. It follows immediately from Proposition 4.19 that $x \le y$ implies $(T, c) \le (T', c')$. Conversely, we prove that $(T, c) \le (T', c')$ implies $x \le y$; assume that $T(u) \le T'(u + d)$ for $u \in [i - d]$. To this end, we construct $x_1, x_2 \in (W^{I \smallsetminus \{i\}})_{\mathrm{af}}$ and $T_1, T_2 \in \mathrm{CST}_{C_n}(\varpi_i)$ such that $Y_i^{C_n}(x_1) = (T_1, c')$, $Y_i^{C_n}(x_2) = (T_2, c' - 1)$, and $x \le x_2 \le x_1 \le y$ as follows. Let $s \in [0, i]$ be such that $T'(u) = \overline{i - u + 1}$ for $u \in [s + 1, i]$, and $T'(s) \ne \overline{i - s + 1}$ if $s \ne 0$. Define $T_1, T_2 \in \mathrm{CST}_{C_n}(\varpi_i)$ by

$$T_1(u) = \begin{cases} 1 & \text{if } u = 1, \\ T'(u) & \text{if } u \in [2, s], \\ \overline{i - u + 2} & \text{if } u \in [\max\{2, s+1\}, i], \end{cases}$$

$$T_2(u) = \begin{cases} T_1(u + 1) & \text{if } u \in [i - 1], \\ \bar{1} & \text{if } u = i. \end{cases}$$

To see that T_1 is well-defined, it suffices to show that $|T_1(u)|$, $u \in [i]$, are all distinct. This follows from $|T_1(1)| = 1 < |T_1(u)| \le i - s + 1 < |T_1(v)|$ for $u \in [\max\{2, s+1\}, i]$ and $v \in [2, s]$. Let $x_1, x_2 \in (W^{I \smallsetminus \{i\}})_{\mathrm{af}}$ be such that $Y_i^{C_n}(x_1) = (T_1, c')$ and $Y_i^{C_n}(x_2) = (T_2, c' - 1)$. Since $T_1(u) \le T'(u)$ for $u \in [i]$, we see from the assertion for $d = 0$ that $x_1 \le y$. By Proposition 4.19($\frac{\infty}{2}$-C5), we have $x_2 \le x_1$. By induction hypothesis, it remains to prove that $T(u) \le T_2(u + d - 1)$ for $u \in [i - d + 1]$. Let $u \in [i - d + 1]$. If $u + d - 1 \in [i - 1] \cap [1, s - 1]$, then $T_2(u + d - 1) = T_1(u + d) = T'(u + d) \ge T(u)$. If $u + d - 1 \in [i - 1] \cap [\max\{2, s + 1\} - 1, i - 1]$, then $T_2(u + d - 1) = T_1(u + d) = \overline{i - (u + d) + 2} \ge T(u + d - 1) \ge T(u)$. We have $T_2(i) = \bar{1} \ge T(i - d + 1)$.

4.5. Type $B_n^{(1)}$. Fix an integer $n > 3$. Set $I = [n]$. We assume that the labeling of the vertices of the Dynkin diagram of type B_n is as follows.



Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be an orthonormal basis of an n-dimensional Euclidean space $\mathbb{R}^n$. Let $\Delta = \{\pm(\varepsilon_s \pm \varepsilon_t) \mid s, t \quad [n],\ s < t\} \quad \{\pm\varepsilon_s \mid s \quad [n]\}$ be a root system of type B_n, and let $\Pi = \{\alpha_s = \varepsilon_s - \varepsilon_{s+1} \mid s \quad [n-1]\} \quad \{\alpha_n = \varepsilon_n\}$ be a simple root system of Δ.

Let W be the Weyl group of Δ; we see from §4.4 that

$$W = \{w \quad \mathfrak{S}(C_n) \mid w(\sigma(s)) = \sigma(w(s)) \text{ for } s \quad [n]\}. \tag{80}$$

Note that $r_{\pm(\varepsilon_s - \varepsilon_t)} = (s\ t)(\overline{s}\ \overline{t})$, $r_{\pm\varepsilon_s} = (s\ \overline{s})$, and $r_{\pm(\varepsilon_s + \varepsilon_t)} = (s\ \overline{t})(\overline{s}\ t)$ for $s, t \quad [n]$, $s < t$. Write $\mathrm{CST}_{B_n}(\ _i) = \mathrm{CST}_{C_n}(\ _i)$. We know from Lemmas 4.1–4.2 and 4.11 that the map

$$Y_i^{B_n} : W_{\mathrm{af}} \to \mathrm{CST}_{B_n}(\ _i) \times \mathbb{Z},\ wt_\xi \to \left(\mathrm{T}_w^{(i)}, c_i(\xi)\right), \tag{81}$$

induces a bijection from the subset $(W^{I \smallsetminus \{i\}})_{\mathrm{af}} \quad W_{\mathrm{af}}$ to $\mathrm{CST}_{B_n}(\ _i) \times \mathbb{Z}$.

Definition 4.20. *Let $i \quad I$, $(\mathrm{T}, c), (\mathrm{T}\ , c\ ) \quad \mathrm{CST}_{B_n}(\ _i) \times \mathbb{Z}$, and $d := c\ - c$. Define a partial order on $\mathrm{CST}_{B_n}(\ _i) \times \mathbb{Z}$ as follows.*

(1) *Assume that $i = 1$. Set $(\mathrm{T}, c) \quad (\mathrm{T}\ , c\ )$ if either of the following holds:*

$$\begin{gathered}(d > 2),\ (d = 1,\ \mathrm{T}(1) = \overline{1},\ \text{and } \mathrm{T}\ (1) = 1),\\ (d = 1 \text{ and } \mathrm{T}(1) = \overline{1}),\ \text{or } (d = 0 \text{ and } \mathrm{T}(1) \quad \mathrm{T}\ (1) \text{ in } C_n).\end{gathered} \tag{82}$$

(2) *Assume that $i \quad [2, n-1]$. Set $(\mathrm{T}, c) \quad (\mathrm{T}\ , c\ )$ if*

$$(d > 0),\ (\mathrm{T}(u) \quad \mathrm{T}\ (u + d) \text{ in } C_n \text{ for } u \quad [i - d]), \tag{83}$$

and one of the following conditions holds:
- (i) *d is even.*
- (ii) *d is odd. If $d \quad [i-1]$, then $\mathrm{T}(i-d) \quad n$. If $d \quad [i]$, $\mathrm{T}(i-d+1) \quad \overline{n}$, $\{a_1 < a_2 < \cdots < a_{n-i+d}\} = [n] \smallsetminus \{\mathrm{T}(u) \mid u \quad [i-d]\}$, and $\mathrm{T}(i-d+1) = \overline{a_d}$, then $1 \quad a_d \quad \mathrm{T}\ (a_d)$; note that $a_d \quad [d, i]$ since $a_u \quad [u, u+i-d]$ for $u \quad [n-i+d]$.*

(3) *Assume that $i = n$. Set $(\mathrm{T}, c) \quad (\mathrm{T}\ , c\ )$ if*

$$(d > 0) \text{ and } (\mathrm{T}(u) \quad \mathrm{T}\ (u + 2d) \text{ in } C_n \text{ for } u \quad [n - 2d]). \tag{84}$$

Proposition 4.21. *Let $i \quad I$.*

(1) *$Y_i^{B_n} \circ \Pi^{I \smallsetminus \{i\}} = Y_i^{B_n}$.*
(2) *For $x, y \quad W_{\mathrm{af}}$, we have $\Pi^{I \smallsetminus \{i\}}(x) \quad \Pi^{I \smallsetminus \{i\}}(y)$ in $(W^{I \smallsetminus \{i\}})_{\mathrm{af}}$ if and only if $Y_i^{B_n}(x) \quad Y_i^{B_n}(y)$ in $\mathrm{CST}_{B_n}(\ _i) \times \mathbb{Z}$.*
(3) *Let $i \quad [n-1]$ and $(\mathrm{T}, c), (\mathrm{T}\ , c\ ) \quad \mathrm{CST}_{B_n}(\ _i) \times \mathbb{Z}$. If $c\ - c > i$, then $(\mathrm{T}, c) \quad (\mathrm{T}\ , c\ )$.*
(4) *Let $i = n$ and $(\mathrm{T}, c), (\mathrm{T}\ , c\ ) \quad \mathrm{CST}_{B_n}(\ _n) \times \mathbb{Z}$. If $2(c\ - c) > i$, then $(\mathrm{T}, c) \quad (\mathrm{T}\ , c\ )$.*

By combining Propositions 3.2 and 4.21 (2), we obtain the following tableau criterion for the semi-infinite Bruhat order on W_{af} of type $B_n^{(1)}$.

Theorem 4.22. *Let $J \quad I$. For $x, y \quad (W^J)_{\mathrm{af}}$, we have $x \quad y$ in $(W^J)_{\mathrm{af}}$ if and only if $Y_i^{B_n}(x) \quad Y_i^{B_n}(y)$ in $\mathrm{CST}_{B_n}(\ _i) \times \mathbb{Z}$ for all $i \quad I \smallsetminus J$.*

The remainder of this subsection is devoted to the proof of Proposition 4.21.

The proofs of Lemmas 4.23–4.24 below are straightforward.

Lemma 4.23. *Let $i \in I$ and $\gamma \in \Delta^+ \setminus \Delta^+_{I\setminus\{i\}}$. We have $\gamma \in Q^{\vee, I\setminus\{i\}}$ if and only if one of the following conditions holds:*

(1) $i = 1$ *and* $\gamma^\vee = \varepsilon_1 - \varepsilon_2 = \alpha_1^\vee$.
(2) $i \in [2, n-1]$ *and* $\gamma^\vee = \varepsilon_i - \varepsilon_{i+1} = \alpha_i^\vee$.
(3) $i \in [2, n-1]$ *and* $\gamma^\vee = \varepsilon_{i-1} + \varepsilon_i = \alpha_{i-1}^\vee + 2\alpha_i^\vee + \cdots + 2\alpha_{n-1}^\vee + \alpha_n^\vee$.
(4) $i = n$ *and* $\gamma^\vee = \varepsilon_{n-1} + \varepsilon_n = \alpha_{n-1}^\vee + \alpha_n^\vee$.

Lemma 4.24. *Let $i \in I$. We have*

$$2\langle \alpha_i^\vee, \rho - \rho_{I\setminus\{i\}}\rangle = \begin{cases} 2n - i & \text{if } i \in [n-1], \\ 2n & \text{if } i = n. \end{cases} \tag{85}$$

Proposition 4.25 (cf. [4, §8.1]). *Let $i \in I$, $w \in W^{I\setminus\{i\}}$, and $\gamma \in \Delta^+$. There exists a Bruhat edge $w \xrightarrow{\gamma} \lfloor wr_\gamma \rfloor = wr_\gamma$ in $\mathrm{QB}^{I\setminus\{i\}}$ if and only if $\gamma \in \Delta^+ \setminus \Delta^+_{I\setminus\{i\}}$ and one of the following statements holds.*

(b-B1) *$i \in [n-1]$, $c_i(\gamma^\vee) = 1$, and there exists $s \in [i]$ such that $wr_\gamma(u) = w(u)$ for $u \in [i] \setminus \{s\}$, $1 \le w(s) \le n$, and $wr_\gamma(s) = \min([w(s)+1, n] \setminus \{|w(u)| \mid u \in [i], w(u) \le \overline{n}\})$; in this case, we have $\gamma^\vee = \varepsilon_s - \varepsilon_t = \alpha_s^\vee + \alpha_{s+1}^\vee + \cdots + \alpha_t^\vee$ for some $t \in [i+1, n]$.*

(b-B2) *$i \in [n-1]$, $c_i(\gamma^\vee) = 1$, and there exists $s \in [i]$ such that $wr_\gamma(u) = w(u)$ for $u \in [i] \setminus \{s\}$, $\overline{n} \le w(s) \le \overline{1}$, and $wr_\gamma(s) = \sigma(\max([|w(s)| - 1] \setminus \{|w(u)| \mid u \in [i], w(u) \le n\}))$; in this case, we have $\gamma^\vee = \varepsilon_s - \varepsilon_t = \alpha_s^\vee + \alpha_{s+1}^\vee + \cdots + \alpha_t^\vee$ for some $t \in [i+1, n]$.*

(b-B3) *$i \in [2, n-1]$, $c_i(\gamma^\vee) = 2$, and there exist $s, t \in [i]$ such that $s < t$, $wr_\gamma(u) = w(u)$ for $u \in [i] \setminus \{s, t\}$ and $wr_\gamma(s) = w(s) + 1 = \sigma(w(t)) = \sigma(wr_\gamma(t)) + 1 \le n$; in this case, we have $\gamma^\vee = \varepsilon_s + \varepsilon_t = \alpha_s^\vee + \cdots + \alpha_{t-1}^\vee + 2\alpha_t^\vee + \cdots + 2\alpha_{n-1}^\vee + \alpha_n^\vee$.*

(b-B4) *$i = n$, $c_n(\gamma^\vee) = 1$, and there exist $s, t \in [i]$ such that $s < t$, $wr_\gamma(u) = w(u)$ for $u \in [i] \setminus \{s, t\}$ and $wr_\gamma(s) = w(s) + 1 = \sigma(w(t)) = \sigma(wr_\gamma(t)) + 1 \le n$; in this case, we have $\gamma^\vee = \varepsilon_s + \varepsilon_t = \alpha_s^\vee + \cdots + \alpha_{t-1}^\vee + 2\alpha_t^\vee + \cdots + 2\alpha_{n-1}^\vee + \alpha_n^\vee$.*

(b-B5) *$i \in [n-1]$, $c_i(\gamma^\vee) = 2$, and there exists $s \in [i]$ such that $wr_\gamma(u) = w(u)$ for $u \in [i] \setminus \{s\}$ and $\sigma(wr_\gamma(s)) = w(s) = n$; in this case, we have $\gamma^\vee = \varepsilon_s = 2\alpha_s^\vee + 2\alpha_{s+1}^\vee + \cdots + 2\alpha_{n-1}^\vee + \alpha_n^\vee$.*

(b-B6) *$i = n$, $c_n(\gamma^\vee) = 1$, and there exists $s \in [i]$ such that $wr_\gamma(u) = w(u)$ for $u \in [i] \setminus \{s\}$ and $\sigma(wr_\gamma(s)) = w(s) = n$; in this case, we have $\gamma^\vee = \varepsilon_s = 2\alpha_s^\vee + 2\alpha_{s+1}^\vee + \cdots + 2\alpha_{n-1}^\vee + \alpha_n^\vee$.*

Moreover, for $w, v \in W^{I\setminus\{i\}}$, we have $w \le v$ if and only if $w(u) \le v(u)$ for $u \in [i]$.

For $i \in I$, $w \in W^{I\setminus\{i\}}$ and $\gamma \in \Delta^+ \setminus \Delta^+_{I\setminus\{i\}}$, let $\mathrm{Q}(i, w, \gamma)$ denote the following statement.

$\mathrm{Q}(i, w, \gamma)$**:** There exists a quantum edge $w \xrightarrow{\gamma} \lfloor wr_\gamma \rfloor$ in $\mathrm{QB}^{I\setminus\{i\}}$.

Proposition 4.26. *Let $i \in I$, $w \in W^{I\setminus\{i\}}$, and $\gamma \in \Delta^+ \setminus \Delta^+_{I\setminus\{i\}}$. Then $\mathrm{Q}(i, w, \gamma)$ is true if and only if one of the following statements holds.*

(q-B1) *$i = 1$, $c_1(\gamma^\vee) = 1$, and $(w(1), \lfloor wr_\gamma \rfloor(1)) \in \{(\overline{1}, 2), (\overline{2}, 1)\}$; in this case, we have $\gamma^\vee = \varepsilon_1 - \varepsilon_2 = \alpha_1^\vee$.*

(q-B2) *$i \in [2, n-1]$, $c_i(\gamma^\vee) = 1$, and $1 \le w(\overline{i}) \le i + 1$. If we set $k = w(\overline{i})$, then $w(u) = u + 1$ for $u \in [k-2]$, $k \le w(u) \le n$ for $u \in [k-1, i-1]$, $\lfloor wr_\gamma \rfloor(1) = 1$, and $\lfloor wr_\gamma \rfloor(u) = w(u-1)$ for $u \in [2, i]$; in this case, we have $\gamma^\vee = \varepsilon_i - \varepsilon_{i+1} = \alpha_i^\vee$.*

(q-B3) *$i \in [2, n-1]$, $c_i(\gamma^\vee) = 2$, $\lfloor wr_\gamma \rfloor(1) = w(\overline{i}) = 1$, $\lfloor wr_\gamma \rfloor(2) = w(\overline{i-1}) = 2$, and $\lfloor wr_\gamma \rfloor(u) = w(u-2)$ for $u \in [3, i]$; in this case, we have $\gamma^\vee = \varepsilon_{i-1} + \varepsilon_i = \alpha_{i-1}^\vee + 2\alpha_i^\vee + \cdots + 2\alpha_{n-1}^\vee + \alpha_n^\vee$.*

(q-B4) $i = n$, $c_n(\gamma^\vee) = 1$, $\lfloor wr_\gamma \rfloor(1) = w(\overline{n}) = 1$, $\lfloor wr_\gamma \rfloor(2) = w(\overline{n-1}) = 2$, *and* $\lfloor wr_\gamma \rfloor(u) = w(u-2)$ *for* $u \in [3,n]$; *in this case, we have* $\gamma^\vee = \varepsilon_{n-1} + \varepsilon_n = \alpha_{n-1}^\vee + \alpha_n^\vee$.

Before starting the proof of Proposition 4.26, we mention a consequence of Lemma 4.3 and Propositions 4.25–4.26.

Proposition 4.27. *Let* $i \in I$, $x, y \in (W^{I \setminus \{i\}})_{\mathrm{af}}$, $Y_i^{B_n}(x) = (\mathrm{T}, c)$, *and* $Y_i^{B_n}(y) = (\mathrm{T}', c')$. *There exists an edge* $x \xrightarrow{\beta} y$ *in* $\mathrm{SiB}^{I \setminus \{i\}}$ *for some* $\beta \in \Delta_{\mathrm{af}}^+$ *if and only if one of the following conditions holds:*

($\frac{\infty}{2}$-B1) $i \in [n-1]$, $c' = c$, *and there exists* $s \in [i]$ *such that* $\mathrm{T}'(u) = \mathrm{T}(u)$ *for* $u \in [i] \setminus \{s\}$, $1 \le \mathrm{T}(s) \le n$, *and* $\mathrm{T}'(s) = \min([\mathrm{T}(s)+1, n] \setminus \{\, |\mathrm{T}(u)| \mid u \in [i],\ \mathrm{T}(u) \le \overline{n}\})$.

($\frac{\infty}{2}$-B2) $i \in [n-1]$, $c' = c$, *and there exists* $s \in\,[i]$ *such that* $\mathrm{T}'(u) = \mathrm{T}(u)$ *for* $u \in [i] \setminus \{s\}$, $\overline{n}\, \le \mathrm{T}(s) \le \overline{1}$, *and* $\mathrm{T}'(s) = \sigma\,\max([\,|\mathrm{T}(s)| - 1] \setminus \{\mathrm{T}(u) \mid u \in [i],\ \mathrm{T}(u) \le n\})$.

($\frac{\infty}{2}$-B3) $i \in [2,n]$, $c' = c$, *and there exist* $s, t \in [i]$ *such that* $s < t$, $\mathrm{T}'(u) = \mathrm{T}(u)$ *for* $u \in [i] \setminus \{s,t\}$, *and* $\mathrm{T}'(s) = \mathrm{T}(s) + 1 = \sigma(\mathrm{T}(t)) = \sigma(\mathrm{T}'(t)) + 1 \le n$.

($\frac{\infty}{2}$-B4) $c' = c$, *and there exists* $s \in [i]$ *such that* $\mathrm{T}'(u) = \mathrm{T}(u)$ *for* $u \in [i] \setminus \{s\}$ *and* $\sigma(\mathrm{T}'(s)) = \mathrm{T}(s) = n$.

($\frac{\infty}{2}$-B5) $i = 1$, $c' = c + 1$, *and* $(\mathrm{T}(1), \mathrm{T}'(1)) \in \{(\overline{1}, 2), (\overline{2}, 1)\}$.

($\frac{\infty}{2}$-B6) $i \in [2, n-1]$, $c' = c + 1$, *and* $1 \le \sigma(\mathrm{T}(i)) \le i + 1$. *If we set* $k = \sigma(\mathrm{T}(i))$, *then* $\mathrm{T}(u) = u + 1$ *for* $u \in [k-2]$, $k \le \mathrm{T}(u) \le n$ *for* $u \in [k-1, i-1]$, $\mathrm{T}'(1) = 1$, *and* $\mathrm{T}'(u) = \mathrm{T}(u-1)$ *for* $u \in [2, i]$.

($\frac{\infty}{2}$-B7) $i \in [2, n-1]$, $c' = c + 2$, $\mathrm{T}'(1) = \sigma(\mathrm{T}(i)) = 1$, $\mathrm{T}'(2) = \sigma(\mathrm{T}(i-1)) = 2$, *and* $\mathrm{T}'(u) = \mathrm{T}(u-2)$ *for* $u \in [3, i]$.

($\frac{\infty}{2}$-B8) $i = n$, $c' = c + 1$, $\mathrm{T}'(1) = \sigma(\mathrm{T}(n)) = 1$, $\mathrm{T}'(2) = \sigma(\mathrm{T}(n-1)) = 2$, *and* $\mathrm{T}'(u) = \mathrm{T}(u-2)$ *for* $u \in [3, n]$.

We have divided the proof of Proposition 4.26 into a sequence of lemmas.

Lemma 4.28. $\mathrm{Q}(1, w, \gamma)$ *implies* (q-B1).

Proof. Assume that $\mathrm{Q}(1, w, \gamma)$ is true. By Lemmas 4.4 and 4.23, we have $\gamma^\vee = \varepsilon_1 - \varepsilon_2 \in Q^{\vee, I \setminus \{1\}}$, $r_\gamma = (1\ 2)(\overline{1}\ \overline{2})$, and $\lfloor wr_\gamma \rfloor = wr_\gamma (z_\gamma^{I \setminus \{1\}})^{-1}$. Set $J = I \setminus \{1\} = [2, n]$; note that J is of type B_{n-1}. We see that $2 \in J_{\mathrm{af}}$ satisfies the condition for $\gamma^\vee \in Q^\vee$ in Lemma 4.1; note that $J \setminus \{2\} = [3, n]$ is of type B_{n-2}. Hence $z_\gamma^J = w_0^J w_0^{J \setminus \{2\}} = (2\ \overline{2})$ and $\lfloor wr_\gamma \rfloor = w(1\ 2)(\overline{1}\ \overline{2})(2\ \overline{2}) = w(1\ 2\ \overline{1}\ \overline{2})$. From this we obtain $\lfloor wr_\gamma \rfloor(1) = w(2)$, $\lfloor wr_\gamma \rfloor(2) = w(\overline{1})$, and $\lfloor wr_\gamma \rfloor(u) = w(u)$ for $u \in [3, n]$. It follows from Lemma 4.11 that

$$\max\{w(\overline{1}), w(2)\} \le w(3) \le w(4) \le \cdots \le w(n) \le n. \tag{86}$$

Hence $\lfloor wr_\gamma \rfloor(u) = w(u) = u$ for $u \in [3, n]$. If $w(\overline{1}) \le w(2)$, then $(w(1), \lfloor wr_\gamma \rfloor(1)) = (\overline{1}, 2)$. If $w(2) \le w(\overline{1})$, then $(w(1), \lfloor wr_\gamma \rfloor(1)) = (\overline{2}, 1)$.

Lemma 4.29. (q-B1) *implies* $\mathrm{Q}(1, w, \gamma)$.

Proof. Assume that (q-B1) is true. We see from Lemmas 4.5 and 4.23–4.24 that $\mathrm{Q}(1, w, \gamma)$ is equivalent to $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = 2 - 2n$. Note that (q-B1) and Lemma 4.11 yield (86).

If $(w(1), \lfloor wr_\gamma \rfloor(1)) = (\overline{1}, 2)$, then $w = (1\ \overline{1})$ and $\lfloor wr_\gamma \rfloor = (1\ 2)(\overline{1}\ \overline{2}) = r_1$, by Lemma 4.11. Since $\ell(\lfloor wr_\gamma \rfloor) = 1$ and $\ell(w) = \mathrm{a}_1(w) + \mathrm{b}_1(w) + \mathrm{e}_1(w) = (n-1) + (n-1) + 1 = 2n - 1$, we have $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = 2 - 2n$.

If $(w(1), \ wr_y \ (1)) = (\overline{2}, 1)$, then $w = (1\ \overline{2}\ \overline{1}\ 2)$ and $\ wr_y \ = e$, by Lemma 4.11. Since $\ (\ wr_y \ ) = 0$ and $\ (w) = \mathrm{a}_1(w) + \mathrm{b}_1(w) + \mathrm{e}_1(w) = (n-1) + (n-2) + 1 = 2n - 2$, we have $\ (\ wr_y \ ) - \ (w) = 2 - 2n$.

Lemma 4.30. *$i \quad [2, n-1]$, $c_i(y \ ) = 1$, and* $\mathrm{Q}(i, w, y)$ *imply* (q-B2).

Proof. Assume that $i \quad [2, n-1]$ and $c_i(y \ ) = 1$, and that $\mathrm{Q}(i, w, y)$ is true. By Lemmas 4.4 and 4.23, we have $y \ = \varepsilon_i - \varepsilon_{i+1} \quad Q^{\ ,I \setminus \{i\}}$, $r_y = (i\ i+1)(\overline{i}\ \overline{i+1})$, and $\ wr_y \ = wr_y(z_y^{I \setminus \{i\}})^{-1}$. We see from Lemmas 4.5 and 4.24 that $\mathrm{Q}(i, w, y)$ is equivalent to $\ (\ wr_y \ ) - \ (w) = i - 2n + 1$. Let $I \setminus \{i\} = I_1 \quad I_2$, where $I_1 = [i-1]$ is of type A_{i-1} and $I_2 = [i+1, n]$ is of type B_{n-i}. We see that $(i-1, i+1) \quad (I_1)_{\mathrm{af}} \times (I_2)_{\mathrm{af}}$ satisfies the condition for $y \quad Q$ in Lemma 4.1; note that $I_1 \setminus \{i-1\} = [i-2]$ is of type A_{i-2} and $I_2 \setminus \{i+1\} = [i+2, n]$ is of type B_{n-i-1}. Hence $z_y^{I \setminus \{i\}} = w_0^{I_1} w_0^{I_1 \setminus \{i-1\}} w_0^{I_2} w_0^{I_2 \setminus \{i+1\}} = (1\ 2\ \cdots\ i)(\overline{1}\ \overline{2}\ \cdots\ \overline{i})(i+1\ \overline{i+1})$ and $\ wr_y \ = w(i\ i+1)(\overline{i}\ \overline{i+1})(i+1\ \overline{i+1})(i\ \cdots\ 2\ 1)(\overline{i}\ \cdots\ \overline{2}\ \overline{1}) = w(i\ i+1\ \overline{i}\ \overline{i+1})(i\ \cdots\ 2\ 1)(\overline{i}\ \cdots\ \overline{2}\ \overline{1})$. We have $\ wr_y \ (1) = w(i+1)$, $\ wr_y \ (u) = w(u-1)$ for $u \quad [2, i]$, $\ wr_y \ (i+1) = w(\overline{i})$, and $\ wr_y \ (u) = w(u)$ for $u \quad [i+2, n]$. It follows from Lemma 4.11 that

$$(87) \qquad \begin{aligned} & w(i+1) \quad w(1) \quad w(2) \quad \cdots \quad w(i-1) \quad w(i), \\ & \max\{w(\overline{i}), w(i+1)\} \quad w(i+2) \quad w(i+3) \quad \cdots \quad w(n) \quad n \quad w(i). \end{aligned}$$

Hence $w(\overline{i}) \quad i+1$. Set $k = w(\overline{i}) = \ wr_y \ (i+1)$. The rest of the proof will be divided into four steps.

Step 1. We claim that $k \quad 1$ and (1)–(5) below imply (q-B2). Let $l \quad [k-1, i]$ be such that $w(l-1) \quad n \quad w(l)$.

1. (1) $\mathrm{a}_1(\ wr_y \ ) = \mathrm{b}_1(\ wr_y \ ) = \mathrm{e}_1(\ wr_y \ ) = 0$,
2. (2) $\mathrm{a}_s(\ wr_y \ ) = \mathrm{a}_{s-1}(w) - 1$ for $s \quad [2, k-1]$, $\mathrm{a}_s(\ wr_y \ ) = \mathrm{a}_{s-1}(w)$ for $s \quad [k, i]$,
3. (3) $\mathrm{b}_s(\ wr_y \ ) = \mathrm{b}_{s-1}(w)$ for $s \quad [2, k-1] \quad [l+1, i]$, $\mathrm{b}_s(\ wr_y \ ) = \mathrm{b}_{s-1}(w) - 1$ for $s \quad [k, l]$,
4. (4) $\mathrm{e}_s(\ wr_y \ ) = \mathrm{e}_{s-1}(w)$ for $s \quad [2, i]$,
5. (5) $\mathrm{a}_i(w) = n - i$, $\mathrm{b}_i(w) = n - i - 1$, $\mathrm{e}_i(w) = 1$.

Indeed, if $k \quad 1$, then $\ wr_y \ (u) = u$ for $u \quad [k-1]$, by (87), Lemma 4.11, and $\ wr_y \ (i+1) = k$. Therefore $w(u) = \ wr_y \ (u+1) = u+1$ for $u \quad [k-2]$. It follows from $w(1) = 2$ and (87) that $\ wr_y \ (1) = w(i+1) = 1$. Also, by (87), we have $k - 1 = w(k-2) \quad w(u) \quad w(i) = \overline{k}$ for $u \quad [k-1, i-1]$, which implies $\ w(u) \ > k$ for $u \quad [k-1, i-1]$. It remains to prove that $w(u) \quad n$ for $u \quad [k-1, i-1]$; we only need to show that $l = i$. It follows from Lemma 4.11 and (1)–(5) above that $\ (\ wr_y \ ) - \ (w) = i - 2n + 1 + (i - l)$. Since $\ (\ wr_y \ ) - \ (w) = i - 2n + 1$, we get $l = i$.

Step 2. We prove (1)–(5) in Step 1 under the assumption that $k \quad 1$.

(1) follows from $\ wr_y \ (1) = 1$.

(2): If $s \quad [2, k-1]$, then $\mathrm{A}_s(\ wr_y \ ) = ?$ and $\mathrm{A}_{s-1}(w) = \{i+1\}$, which implies $\mathrm{a}_s(\ wr_y \ ) = \mathrm{a}_{s-1}(w) - 1$. If $s \quad [k, i]$, then $\mathrm{A}_s(\ wr_y \ ) = \mathrm{A}_s(w)$, which implies $\mathrm{a}_s(\ wr_y \ ) = \mathrm{a}_{s-1}(w)$.

(3): If $s \quad [2, k-1]$, then $\mathrm{B}_s(\ wr_y \ ) = \mathrm{B}_{s-1}(w) = ?$, which implies $\mathrm{b}_s(\ wr_y \ ) = \mathrm{b}_{s-1}(w)$. If $s \quad [k, l]$, then $k \quad \ wr_y \ (s) = w(s-1) \quad n$, $i \quad \mathrm{B}_{s-1}(w)$, and the map $\mathrm{B}_s(\ wr_y \ ) \to \mathrm{B}_{s-1}(w) \setminus \{i\}$, $t \to t - 1$, is bijective. This implies $\mathrm{b}_s(\ wr_y \ ) = \mathrm{b}_{s-1}(w) - 1$. If $s \quad [l+1, i]$, then $\overline{n} \quad \ wr_y \ (s) = w(s-1)$ and $\mathrm{B}_s(\ wr_y \ ) = \mathrm{B}_{s-1}(w)$, which implies $\mathrm{b}_s(\ wr_y \ ) = \mathrm{b}_{s-1}(w)$.

(4) follows from $\ wr_y \ (s) = w(s-1)$ for $s \quad [2, i]$.

(5): Lemma 4.11 and $w(i) = \overline{k} \succeq \overline{n}$ show that $\mathrm{a}_i(w) = n - i$ and $\mathrm{e}_i(w) = 1$. We see from $|w(s)| > k = |w(i)|$ for $s \in [k-1, i-1]$ that $\mathrm{b}_i(w) = n - i - 1$.

Step 3. It remains to prove that $k \neq 1$. On the contrary, suppose that $k = 1$. Then $w(\overline{i}) = k \preceq w(i+1) \preceq n$. If we prove that

(6) $\mathrm{e}_1(\lfloor wr_\gamma \rfloor) = 0$, $\mathrm{e}_s(\lfloor wr_\gamma \rfloor) = \mathrm{e}_{s-1}(w)$ for $s \in [2, i]$, and $\mathrm{e}_i(w) = 1$,
(7) $\mathrm{a}_i(w) = n - i$,
(8) $\mathrm{a}_1(\lfloor wr_\gamma \rfloor) = 1$, $\mathrm{b}_1(\lfloor wr_\gamma \rfloor) = w(i+1) - 2$,
(9) $\mathrm{a}_s(\lfloor wr_\gamma \rfloor) = \mathrm{a}_{s-1}(w)$ for $s \in [2, i]$,
(10) $\mathrm{b}_s(\lfloor wr_\gamma \rfloor) = \mathrm{b}_{s-1}(w) - 1$ for $s \in [2, i]$,
(11) $\mathrm{b}_i(w) = n - i$,

then $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = i - 2n + 1 + (w(i+1) + 2i - 4)$. Since $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = i - 2n + 1$ and $i > 2$, we have $w(i+1) = 4 - 2i \leq 0$, a contradiction.

Step 4. We prove (6)–(11) in Step 3 under the assumption that $k = 1$; in fact, we do not use $k = 1$ to prove (6)–(7).

(6) follows from (87) and $\lfloor wr_\gamma \rfloor(u) = w(u-1)$ for $u \in [2, i]$.

(7): We deduce from (87) that $\mathrm{A}_i(w) = [i+1, n]$, which gives $\mathrm{a}_i(w) = n - i$.

(8): By Lemma 4.11, $\mathrm{A}_1(\lfloor wr_\gamma \rfloor) \subset [i+1, n]$. Since $\lfloor wr_\gamma \rfloor(1) = w(i+1) \preceq w(\overline{i}) = \lfloor wr_\gamma \rfloor(i+1)$, we have $i+1 \in \mathrm{A}_1(\lfloor wr_\gamma \rfloor)$. Since $\lfloor wr_\gamma \rfloor(t) = w(t)$ for $t \in [i+2, n]$, we have $\mathrm{A}_1(\lfloor wr_\gamma \rfloor) \cap [i+2, n] = \emptyset$. Hence $\mathrm{A}_1(\lfloor wr_\gamma \rfloor) = \{i+1\}$ and $\mathrm{a}_1(\lfloor wr_\gamma \rfloor) = 1$ as claimed. Since $\lfloor wr_\gamma \rfloor(1) = w(i+1) \preceq \overline{1} = w(i) = \sigma(\lfloor wr_\gamma \rfloor(i+1))$, we have $i+1 \notin \mathrm{B}_1(\lfloor wr_\gamma \rfloor)$. Therefore

$$\mathrm{B}_1(\lfloor wr_\gamma \rfloor) = \{t \in [2, i] \mid \underbrace{\lfloor wr_\gamma \rfloor(1)}_{=w(i+1)} \succeq \underbrace{\sigma(\lfloor wr_\gamma \rfloor(t))}_{=\sigma(w(t-1))}\}$$
$$\sqcup \{t \in [i+2, n] \mid \underbrace{\lfloor wr_\gamma \rfloor(1)}_{=w(i+1)} \succeq \underbrace{\sigma(\lfloor wr_\gamma \rfloor(t))}_{=\sigma(w(t))}\}.$$

It follows that the map

$$[2, n] \to [n], \quad t \mapsto \begin{cases} t - 1 & \text{if } t \in [2, i], \\ t & \text{if } t \in [i+1, n], \end{cases}$$

induces a bijection from $\mathrm{B}_1(\lfloor wr_\gamma \rfloor)$ to $\{t \in [n] \mid w(i+1) \succeq \sigma(w(t))\} \setminus \{i\}$. Since $w(i+1) = \min\{w(u) \mid u \in [n]\}$, the latter set equals

$$\{w^{-1}(\overline{2}), w^{-1}(\overline{3}), \dots, w^{-1}(\overline{w(i+1) - 1})\}.$$

This proves $\mathrm{b}_1(\lfloor wr_\gamma \rfloor) = w(i+1) - 2$.

(9): Let $s \in [2, i]$. Since $\lfloor wr_\gamma \rfloor(i+1) = 1$ and $w(i+1) = \min\{w(u) \mid u \in [n]\}$, we have $i+1 \in \mathrm{A}_s(\lfloor wr_\gamma \rfloor)$ and $i+1 \in \mathrm{A}_{s-1}(w)$. It follows from Lemma 4.11 that

$$\mathrm{A}_s(\lfloor wr_\gamma \rfloor) = \{i+1\} \sqcup \{t \in [i+2, n] \mid \underbrace{\lfloor wr_\gamma \rfloor(s)}_{=w(s-1)} \succeq \underbrace{\sigma(\lfloor wr_\gamma \rfloor(t))}_{=w(t)}\} = \mathrm{A}_{s-1}(w),$$

which implies $\mathrm{a}_s(\lfloor wr_\gamma \rfloor) = \mathrm{a}_{s-1}(w)$.

(10): Let $s \in [2, i]$. Since $\lfloor wr_\gamma \rfloor(i+1) = w(\overline{i}) = \overline{1}$, we have $\sigma(\lfloor wr_\gamma \rfloor(i+1)) \succeq \lfloor wr_\gamma \rfloor(s)$. Therefore $i+1 \notin \mathrm{B}_s(\lfloor wr_\gamma \rfloor)$ and

$$\mathrm{B}_s(\lfloor wr_\gamma \rfloor) = (\mathrm{B}_s(\lfloor wr_\gamma \rfloor) \cap [s+1, i]) \sqcup (\mathrm{B}_s(\lfloor wr_\gamma \rfloor) \cap [i+2, n])$$
$$= \{t \in [s+1, i] \mid \underbrace{\lfloor wr_\gamma \rfloor(s)}_{=w(s-1)} \succeq \underbrace{\sigma(\lfloor wr_\gamma \rfloor(t))}_{=\sigma(w(t-1))}\}$$
$$\sqcup \{t \in [i+2, n] \mid \underbrace{\lfloor wr_\gamma \rfloor(s)}_{=w(s-1)} \succeq \underbrace{\sigma(\lfloor wr_\gamma \rfloor(t))}_{=\sigma(w(t))}\}.$$

Also, $w(\overline{1}) = 1$ implies $i \in \mathrm{B}_{s-1}(w)$. It follows that the map

$$[s+1, n] \to [s, n], \quad t \mapsto \begin{cases} t-1 & \text{if } t \in [s+1, i], \\ t & \text{if } t \in [i+1, n], \end{cases}$$

induces a bijection from $\mathrm{B}_s(wr_y)$ to $\mathrm{B}_{s-1}(w) \smallsetminus \{i\}$, which implies $\mathrm{b}_s(wr_y) = \mathrm{b}_{s-1}(w) - 1$.

(11): Since $w(i) = \overline{1}$, we have $\mathrm{B}_i(w) = \{t \in [i+1, n] \mid w(i) \quad \sigma(w(t))\} = [i+1, n]$. This proves $\mathrm{b}_i(w) = n - i$ as claimed.

The proof of Lemma 4.30 is complete.

Lemma 4.31. (q-B2) *implies* $\mathrm{Q}(i, w, y)$.

Proof. Assume that (q-B2) is true; note that $c_i(y) = 1$ and (87) in the proof of Lemma 4.30 holds. We see from Lemmas 4.5 and 4.24 that $\mathrm{Q}(i, w, y)$ is equivalent to $(wr_y) - (w) = i - 2n + 1$. If we prove that

(1) $\mathrm{e}_1(wr_y) = 0$, $\mathrm{e}_s(wr_y) = \mathrm{e}_{s-1}(w)$ for $s \in [2, i]$, and $\mathrm{e}_i(w) = 1$,

(2) $\mathrm{a}_i(w) = n - i$,

(3) $\mathrm{a}_1(wr_y) = \mathrm{b}_1(wr_y) = 0$,

(4) $\mathrm{b}_s(wr_y) = \begin{cases} \mathrm{b}_{s-1}(w) & \text{if } w(s-1) \quad w(\overline{i}), \\ \mathrm{b}_{s-1}(w) - 1 & \text{if } w(s-1) \quad w(\overline{i}) \end{cases}$ for $s \in [2, i]$,

(5) $\mathrm{a}_s(wr_y) = \begin{cases} \mathrm{a}_{s-1}(w) - 1 & \text{if } w(s-1) \quad w(\overline{i}), \\ \mathrm{a}_{s-1}(w) & \text{if } w(s-1) \quad w(\overline{i}) \end{cases}$ for $s \in [2, i]$,

(6) $\mathrm{b}_i(w) = n - i + 1$,

then $(wr_y) - (w) = i - 2n + 1$ by Lemma 4.11, which is our assertion. We prove (1)–(6) as follows.

(1)–(2) follow by the same method as in Step 4 of the proof of Lemma 4.30.

(3): Since $wr_y(1) = 1$, we have $\mathrm{A}_s(wr_y) = \mathrm{B}_s(wr_y) = ?$ and $\mathrm{a}_s(wr_y) = \mathrm{b}_s(wr_y) = 0$.

(4): Let $s \in [2, i]$. We first claim that $i + 1 \notin \mathrm{B}_s(wr_y)$. Indeed, we see from (q-B2) that $[k-1] \quad \{wr_y(u) \mid u \in [i]\}$ and $k \notin \{wr_y(u) \mid u \in [i]\}$. By Lemma 4.11, we have

$$wr_y(i+1) = \min([n] \smallsetminus \{wr_y(u) \mid u \in [i]\}) = k = w(\overline{i}).$$

Since $wr_y(s) = w(s-1) \quad n \quad \overline{k} = w(i) = \sigma(wr_y(i+1))$, we have $i + 1 \notin \mathrm{B}_s(wr_y)$ as claimed. It follows that

$$\mathrm{B}_s(wr_y) = \{t \in [s+1, i] \mid \underbrace{wr_y(s)}_{=w(s-1)} \quad \underbrace{\sigma(wr_y(t))}_{=\sigma(w(t-1))}\} \quad \{t \in [i+2, n] \mid \underbrace{wr_y(s)}_{=w(s-1)} \quad \underbrace{wr_y(t)}_{=\sigma(w(t))}\}.$$

We next claim that $i + 1 \notin \mathrm{B}_{s-1}(w)$. We see from Lemma 4.11 that $w(i+1) = 1$. Hence $\sigma(w(i+1)) = \overline{1} \quad w(s-1)$, which implies $i + 1 \notin \mathrm{B}_{s-1}(w)$ as claimed. Note that $i \in \mathrm{B}_{s-1}(w)$ if and only if $w(s-1) \quad w(\overline{i})$. It follows that the map

$$[s+1, n] \to [s, n], \quad t \mapsto \begin{cases} t-1 & \text{if } t \in [s+1, i], \\ t & \text{if } t \in [i+1, n], \end{cases}$$

induces a bijection from $\mathrm{B}_s(wr_y)$ to $\mathrm{B}_{s-1}(w)$ (resp. from $\mathrm{B}_s(wr_y)$ to $\mathrm{B}_{s-1}(w) \smallsetminus \{i\}$) if $w(s-1) \quad w(\overline{i})$ (resp. if $w(s-1) \quad w(\overline{i})$). This proves (4).

(5): Let $s \in [2,i]$. Since $\lfloor wr_\gamma \rfloor(i+1) = w(\overline{i})$, we have $i+1 \in \mathrm{A}_s(\lfloor wr_\gamma \rfloor)$ if and only if $w(s-1) \succ w(\overline{i})$. It follows from Lemma 4.11 and $\lfloor wr_\gamma \rfloor(t) = w(t)$ for $t \in [i+2,n]$ that $\mathrm{A}_s(\lfloor wr_\gamma \rfloor) = \mathrm{A}_{s-1}(w) \setminus \{i+1\}$ (resp. $\mathrm{A}_s(\lfloor wr_\gamma \rfloor) = \mathrm{A}_{s-1}(w)$) if $w(s-1) \succ w(\overline{i})$ (resp. if $w(s-1) \prec w(\overline{i})$). This prove (5).

(6): By Lemma 4.11 and (87), we have $w(i+1) = 1$. Hence $w(i) \succeq \overline{1} = \sigma(w(i+1))$ and $i+1 \notin \mathrm{B}_s(w)$. Also, we see from (87) that $w(i) \succ \sigma(w(t))$ for $t \in [i+2,n]$, and consequently $\mathrm{B}_s(w) = [i+2,n]$. This proves (6).

Lemma 4.32. *$i \in [2,n-1]$, $c_i(\gamma^\vee) = 2$, and* $\mathrm{Q}(i,w,\gamma)$ *imply* (q-B3).

Proof. Assume that $i \in [2,n-1]$ and $c_i(\gamma^\vee) = 2$, and that $\mathrm{Q}(i,w,\gamma)$ is true. We see from Lemmas 4.5 and 4.24 that $\mathrm{Q}(i,w,\gamma)$ is equivalent to $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = 2i-4n+1$. By the same argument as in Step 2 of the proof of Proposition 4.18 in §4.4, we have $\lfloor wr_\gamma \rfloor(1) = w(\overline{i})$, $\lfloor wr_\gamma \rfloor(2) = w(\overline{i-1})$, $\lfloor wr_\gamma \rfloor(u) = w(u-2)$ for $u \in [3,i]$, $\gamma^\vee = \varepsilon_{i-1} + \varepsilon_i$, and $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = 2i - 4n + 1 + 2(w(\overline{i}) - 1) + 2(w(\overline{i-1}) - 2)$. Hence $w(\overline{i}) = 1$ and $w(\overline{i-1}) = 2$. This implies (q-B3).

Lemma 4.33. (q-B3) *implies* $\mathrm{Q}(i,w,\gamma)$.

Proof. Assume that (q-B3) is true; note that $c_i(\gamma^\vee) = 2$. We see from Lemmas 4.5 and 4.24 that $\mathrm{Q}(i,w,\gamma)$ is equivalent to $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = 2i-4n+1$. By the same argument as in Step 2 of the proof of Proposition 4.18 in §4.4, we have $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = 2i-4n+1+2(w(\overline{i})-1)+2(w(\overline{i-1})-2)$. Since $w(\overline{i}) = 1$ and $w(\overline{i-1}) = 2$ by (q-B3), we conclude that $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = 2i - 4n + 1$.

Lemma 4.34. $\mathrm{Q}(n,w,\gamma)$ *is equivalent to* (q-B4).

Proof. By Lemmas 4.4 and 4.23, we may assume that $\gamma^\vee = \varepsilon_{n-1} + \varepsilon_n \in Q^{\vee, I \setminus \{n\}}$ and $\lfloor wr_\gamma \rfloor = wr_\gamma (z_\gamma^{I \setminus \{n\}})^{-1}$. We see from Lemmas 4.5 and 4.24 that $\mathrm{Q}(n,w,\gamma)$ is equivalent to $\ell(\lfloor wr_\gamma \rfloor) - \ell(w) = 1 - 2n$.

We first show that $\mathrm{Q}(n,w,\gamma)$ implies (q-B4). Set $J = I \setminus \{n\}$; note that J is of type A_{n-1}. We see that $n-2 \in J_{\mathrm{af}}$ satisfies the condition for $\gamma^\vee \in Q^\vee$ in Lemma 4.1; note that $J \setminus \{n-2\} = [n-3] \sqcup \{n-1\}$ is of type $A_{n-3} \times A_1$. Hence $z_\gamma^J = w_0^J w_0^{J \setminus \{n-2\}} = w_0^J w_0^{[n-3]} w_0^{\{n-1\}}$ is given by $u \mapsto u+2$ for $u \in [n-2]$, $n-1 \mapsto 1$, and $n \mapsto 2$. Then $\lfloor wr_\gamma \rfloor$ is given by $1 \mapsto w(\overline{n})$, $2 \mapsto w(\overline{n-1})$, and $u \mapsto w(u-2)$ for $u \in [3,n]$. It follows from Lemma 4.11 that

$$(88) \qquad \underbrace{w(\overline{n}) \prec w(\overline{n-1})}_{n} \prec w(1) \prec w(2) \prec \cdots \prec w(n-2) \prec \underbrace{w(n-1) \prec w(n)}_{\overline{n}}.$$

Hence $w(\overline{n}) < w(\overline{n-1}) < w(u)$ for $u \in [n-2]$, which implies $\lfloor wr_\gamma \rfloor(1) = w(\overline{n}) = 1$ and $\lfloor wr_\gamma \rfloor(2) = w(\overline{n-1}) = 2$. This proves (q-B4).

We next show that (q-B4) implies $\mathrm{Q}(n,w,\gamma)$. We see that (q-B4) yields (88). If we prove that

(1) $\mathrm{a}_s(\lfloor wr_\gamma \rfloor) = \mathrm{a}_s(w) = 0$ for $s \in [n]$, $\mathrm{b}_1(\lfloor wr_\gamma \rfloor) = \mathrm{b}_2(\lfloor wr_\gamma \rfloor) = \mathrm{b}_n(w) = 0$, $\mathrm{b}_{n-1}(w) = 1$,
(2) $\mathrm{e}_1(\lfloor wr_\gamma \rfloor) = \mathrm{e}_2(\lfloor wr_\gamma \rfloor) = 0$, $\mathrm{e}_{n-1}(w) = \mathrm{e}_n(w) = 1$, $\mathrm{e}_s(\lfloor wr_\gamma \rfloor) = \mathrm{e}_{s-2}(w)$ for $s \in [3,n]$,
(3) $\mathrm{b}_s(\lfloor wr_\gamma \rfloor) = \mathrm{b}_{s-2}(w) - 2$ for $s \in [3,n]$,

then $(\ wr_{\gamma}\ ) - \ (w) = 1 - 2n$, which implies $\mathrm{Q}(n, w, y)$. We prove (1)–(3) as follows. (1)–(2) follows from (88). We deduce from $n - 1, n \quad \mathrm{B}_{s-2}(w)$ that the map

$$\mathrm{B}_s(\ wr_{\gamma}\ ) = \{t \quad [s+1, n] \mid \underbrace{wr_{\gamma}\ (s)}_{=w(s-2)} \quad \underbrace{\sigma(\ wr_{\gamma}\ (t))}_{=\sigma(w(t-2))}\}$$

$$\to \mathrm{B}_{s-2}(w) \ \mathrm{r}\ \{n-1, n\}, \ t \to t-2$$

is bijective, which implies $\mathrm{b}_s(\ wr_{\gamma}\ ) = \mathrm{b}_{s-2}(w) - 2$ for $s \quad [3, n]$. This proves (3).

Proof of Proposition 4.21. (1) and (3)–(4) follow by the same method as in the proof of Proposition 4.8.

We prove (2). The assertion for $i = 1$ follows immediately from Proposition 4.27. Also, we can prove the assertion for $i = n$ by a similar argument to the proof of Proposition 4.13. Assume that $i \quad [2, n-1]$. Let $x, y \quad (W^{I \ \mathrm{r}\ \{i\}})_{\mathrm{af}}$, $Y_i^{B_n}(x) = (\mathrm{T}, c)$, and $Y_i^{B_n}(y) = (\mathrm{T}\ , c\ )$. It follows immediately from Propositions 4.25–4.27 that $x \quad y$ implies c 6 c and $\mathrm{T}(u) \quad \mathrm{T}\ (u + c\ - c)$ for $u \quad [i - c\ + c]$. Hence we may assume that $d := c\ - c > 0$ and $\mathrm{T}(u) \quad \mathrm{T}\ (u + d)$ for $u \quad [i - d]$. The proof is by induction on d. The assertion for $d = 0$ follows immediately from Propositions 4.25 and 4.27. Assume that $d > 1$. In what follows, we write $e = \#\{u \quad [i] \mid \mathrm{T}(u) \quad \overline{n}\}$ and $\{a_1 < a_2 < \cdots < a_{n-i+e}\} = [n] \ \mathrm{r}\ \{\mathrm{T}(u) \mid u \quad [i]\}$. We have $\overline{n} \quad \mathrm{T}(i - u + 1) \quad \overline{a_u}$ for $u \quad [e]$. Note that $\mathrm{T}(i - d) \quad n$ is equivalent to e 6 d. We have divided the proof into seven steps.

Step 1. We prove that if $d > 2$ is even, then $x \quad y$. Let $s \quad [i+1]$ be such that $\mathrm{T}(s) \quad s + 2$ and $\mathrm{T}(u) \quad u + 1$ for $u \quad [s-1]$. Define $\mathrm{T}_1, \mathrm{T}_2 \quad \mathrm{CST}_{B_n}(\ \ _i)$ by

$$\mathrm{T}_1(u) = \begin{cases} u + 2 & \text{if } u \quad [\min\{s-1, i-2\}], \\ \mathrm{T}(u) & \text{if } u \quad [s, i-2], \\ \overline{2} & \text{if } u = i - 1, \\ \overline{1} & \text{if } u = i, \end{cases}$$

$$\mathrm{T}_2(u) = \begin{cases} 1 & \text{if } u = 1, \\ 2 & \text{if } u = 2, \\ \mathrm{T}_1(u-2) & \text{if } u \quad [3, i]. \end{cases}$$

Let $x_1, x_2 \quad (W^{I \ \mathrm{r}\ \{i\}})_{\mathrm{af}}$ be such that $Y_i^{B_n}(x_1) = (\mathrm{T}_1, c)$ and $Y_i^{B_n}(x_2) = (\mathrm{T}_2, c + 2)$. We have $\mathrm{T}(u) \quad \mathrm{T}_1(u)$ for $u \quad [i]$. Hence $x \quad x_1$ by the assertion for $d = 0$. We have $x_1 \quad x_2$ by Proposition 4.27 ($\frac{\infty}{2}$-B7). It remains to prove that $x_2 \quad y$. By induction hypothesis, it suffices to show that $\mathrm{T}_2(u) \quad \mathrm{T}\ (u + d - 2)$ for $u \quad [i - d + 2]$; note that $c\ - (c + 2) = d - 2 > 0$ is even. We have $\mathrm{T}_2(1) = 1 \quad \mathrm{T}\ (1 + d - 2)$ and $\mathrm{T}_2(2) = 2 \quad \mathrm{T}\ (2 + d - 2)$. Let $u \quad [i - d + 2] \cap [3, i]$. If $u - 2 \quad [\min\{s-1, i-2\}]$, then $\mathrm{T}_2(u) = u \quad \mathrm{T}\ (u) \quad \mathrm{T}\ (u + d - 2)$. If $u - 2 \quad [s, i-2]$, then $\mathrm{T}_2(u) = \mathrm{T}(u-2) \quad \mathrm{T}\ (u + d - 2)$.

Step 2. We prove that if $x \quad y$, d is odd, and $d \quad [i-1]$, then $\mathrm{T}(i-d) \quad n$, or equivalently, e 6 d. Since d is odd, we see from Proposition 4.27 that there exists an edge $x_1 \xrightarrow{\beta} x_2$, $x_1, x_2 \quad (W^{I \ \mathrm{r}\ \{i\}})_{\mathrm{af}}$, $\beta \quad \Delta^+_{\mathrm{af}}$, in $\mathrm{SiB}^{I \ \mathrm{r}\ \{i\}}$ of type ($\frac{\infty}{2}$-B6) such that $x \quad x_1$ and $x_2 \quad y$; we may assume that there is no edges of type ($\frac{\infty}{2}$-B6) in a directed path $\mathbf{p}$ from x to x_1 in $\mathrm{SiB}^{I \ \mathrm{r}\ \{i\}}$. Write $Y_i^{B_n}(x_1) = (\mathrm{T}_1, c_1)$ and $Y_i^{B_n}(x_2) = (\mathrm{T}_2, c_2)$; note that $c_1 - c$ is even. Set $e_1 = \#\{u \quad [i] \mid \mathrm{T}_1(u) \quad \overline{n}\}$; we have $e_1 = 1$ by ($\frac{\infty}{2}$-B6). It follows from Proposition 4.27 that there exist $(c_1 - c)/2$ edges of type ($\frac{\infty}{2}$-B7) in the directed path $\mathbf{p}$, and hence $1 = e_1 > e - (c_1 - c)$. Since $c\ - c_2 > 0$ and $c_2 - c_1 = 1$, we have $d = c\ - c = (c\ - c_2) + (c_2 - c_1) + (c_1 - c) > 1 + (c_1 - c) > e$.

Step 3. We prove that if $x \quad y$, d is odd, $d \quad [i]$, and $\mathrm{T}(i-d+1) = \overline{a_d}$, then $1 \quad a_d \quad \mathrm{T}\ (a_d)$; we see from Step 2 and $\mathrm{T}(i-d+1) = \overline{a_d}$ that $e = d$ and $a_d \quad [d, i]$. We proceed by induction on d.

Assume that $d = 1$. We first prove that $1 \quad a_1$. It follows from $d = 1$, $x \quad y$, and Proposition 4.27 that there exists an edge $x_1 \xrightarrow{\beta} x_2$, $x_1, x_2 \quad (W^{I \setminus \{i\}})_{\mathrm{af}}$, $\beta \quad \Delta^+_{\mathrm{af}}$, in $\mathrm{SiB}^{I \setminus \{i\}}$ of type ($\frac{\infty}{2}$-B6) such that $x \quad x_1$ and $x_2 \quad y$. If we write $Y_i^{B_n}(x_1) = (\mathrm{T}\ , c\ )$, then $c\ = c$, $\mathrm{T}(u) \quad \mathrm{T}\ (u)$ for $u \quad [i]$, and $\mathrm{T}\ (i) \quad \overline{1}$ (see ($\frac{\infty}{2}$-B6)). Hence $\overline{a_1} = \mathrm{T}(i) \quad \mathrm{T}\ (i) \quad \overline{1}$, which gives $1 \quad a_1$.

Assume that $d = 1$. We next prove that $a_1 \quad \mathrm{T}\ (a_1)$. Suppose, contrary to our claim, that $\mathrm{T}\ (a_1) \quad a_1$. By a similar argument above, we see from Proposition 4.27 (see ($\frac{\infty}{2}$-B6)) that there exist $l \quad [2, a_1]$ and $\mathrm{T}_1, \mathrm{T}_2 \quad \mathrm{CST}_{B_n}(\quad_i)$ such that $\mathrm{T}(u) \quad \mathrm{T}_1(u)$ for $u \quad [i]$, $\mathrm{T}_1(u) = u + 1$ for $u \quad [l-2]$, $\mathrm{T}_1(i) = \overline{l}$, $\mathrm{T}_2(1) = 1$, $\mathrm{T}_2(u) = \mathrm{T}_1(u-1)$ for $u \quad [2, i]$, and $\mathrm{T}_2(u) \quad \mathrm{T}\ (u)$ for $u \quad [i]$; it follows from Lemma 4.11 that $\mathrm{T}_1(u) \quad u + 2$ for $u \quad [l-1, i-1]$. Since $a_1 - 1 \quad [l-1, i-1]$, we have $\mathrm{T}_2(a_1) = \mathrm{T}_1(a_1 - 1) \quad a_1 + 1 \quad a_1 \quad \mathrm{T}\ (a_1)$, contrary to $\mathrm{T}_2(a_1) \quad \mathrm{T}\ (a_1)$.

Assume that $d > 3$. Since $a_d \quad [d, i]$, we have $1 \quad a_d$. It remains to prove that $a_d \quad \mathrm{T}\ (a_d)$. Define $\mathrm{T}_3, \mathrm{T}_4 \quad \mathrm{CST}_{B_n}(\quad_i)$ by

$$\mathrm{T}_3(u) = \begin{cases} u+2 & \text{if } u \quad [a_2 - 2], \\ \mathrm{T}(u) & \text{if } u \quad [a_2 - 1, i-2], \\ \overline{2} & \text{if } u = i-1, \\ \overline{1} & \text{if } u = i, \end{cases}$$

$$\mathrm{T}_4(u) = \begin{cases} 1 & \text{if } u = 1, \\ 2 & \text{if } u = 2, \\ \mathrm{T}_3(u-2) & \text{if } u \quad [3, i]; \end{cases}$$

note that if $u \quad [i-2]$ and $\mathrm{T}(u) \quad a_2$, then $u \quad [a_2 - 1, i-2]$ and $\mathrm{T}_3(u) = \mathrm{T}(u)$. Let $x_4 \quad (W^{I \setminus \{i\}})_{\mathrm{af}}$ be such that $Y_i^{B_n}(x_4) = (\mathrm{T}_4, c+2)$. We see that $c\ - (c+2) = d-2$ is odd, $d - 2 \quad [i-2]$, $\mathrm{T}_4(i-(d-2)) \quad n$ (by Step 2), and $\mathrm{T}_4(i-(d-2)+1) = \mathrm{T}_3(i-d+1) = \mathrm{T}(i-d+1) = \overline{a_d}$; note that $i - d + 1 \quad [i-2]$ and $[n] \setminus \{\mathrm{T}_4(u) \mid u \quad [i-d+2]\} = \{a_3 < \cdots < a_{n-i+d}\}$. Therefore, if we prove that $x_4 \quad y$, then $a_d \quad \mathrm{T}\ (a_d)$ by induction hypothesis.

Let us prove that $x_4 \quad y$. It su ces to show that $\mathrm{T} \quad \mathrm{CST}_{B_n}(\quad_i)$ and (T, c) $(\mathrm{T}\ , c+2)$ imply $(\mathrm{T}_4, c+2) \quad (\mathrm{T}\ , c+2)$. Indeed, we see from Proposition 4.27 that there exists $x \quad (W^{I \setminus \{i\}})_{\mathrm{af}}$ such that $x \quad x \quad y$ and $Y_i^{B_n}(x\ ) = (\mathrm{T}\ , c+2)$ for some $\mathrm{T} \quad \mathrm{CST}_{B_n}(\quad_i)$. Since $(c+2) - c = 2$ is even, $x \quad x$ is equivalent to $(\mathrm{T}, c) \quad (\mathrm{T}\ , c+2)$ by Step 1. Also, $x_4 \quad x$ is equivalent to $(\mathrm{T}_4, c+2) \quad (\mathrm{T}\ , c+2)$. Therefore, $(\mathrm{T}_4, c+2) \quad (\mathrm{T}\ , c+2)$ (and $x \quad y$) yield $x_4 \quad y$.

Assume that $\mathrm{T} \quad \mathrm{CST}_{B_n}(\quad_i)$ and $(\mathrm{T}, c) \quad (\mathrm{T}\ , c+2)$. We have $\mathrm{T}_4(u) = u \quad \mathrm{T}\ (u)$ for $u \quad [a_2]$, and $\mathrm{T}_4(u) = \mathrm{T}(u-2) \quad \mathrm{T}\ (u)$ for $u \quad [a_2 + 1, i]$. Hence $(\mathrm{T}_4, c+2)$ $(\mathrm{T}\ , c+2)$.

Step 4. We prove that $d = 1$ and (ii) in Definition 4.20 (2) for (T, c) and $(\mathrm{T}\ , c\ )$ imply $x \quad y$.

We first assume that ($i = n-1$ and $\mathrm{T}(i) \quad n$) or ($\overline{n} \quad \mathrm{T}(i) \quad \overline{a_1}$). Then $n-i+e > 2$ and $\mathrm{T}(i) \quad \overline{a_2}$. Define $\mathrm{T}_1, \mathrm{T}_2 \quad \mathrm{CST}_{B_n}(\quad _i)$ by

$$\mathrm{T}_1(u) = \begin{cases} u+1 & \text{if } u \quad [a_2-2], \\ \mathrm{T}(u) & \text{if } u \quad [a_2-1, i-1], \\ \overline{a_2} & \text{if } u = i, \end{cases}$$

$$\mathrm{T}_2(u) = \begin{cases} 1 & \text{if } u = 1, \\ \mathrm{T}_1(u-1) & \text{if } u \quad [2, i]. \end{cases}$$

Let $x_1, x_2 \quad (W^{I \setminus \{i\}})_{\mathrm{af}}$ be such that $Y_i^{B_n}(x_1) = (\mathrm{T}_1, c)$ and $Y_i^{B_n}(x_2) = (\mathrm{T}_2, c\ ) = (\mathrm{T}_2, c+1)$. We have $\mathrm{T}(u) = u \quad u+1 = \mathrm{T}_1(u)$ for $u \quad [a_1-1]$, $\mathrm{T}(u) = \mathrm{T}_1(u)$ for $u \quad [a_1, i-1]$, and $\mathrm{T}(i) \quad \overline{a_2} = \mathrm{T}_1(i)$. Hence $x \quad x_1$ by the assertion for $d = 0$. We have $x_1 \quad x_2$ by Proposition 4.27 ($\frac{\infty}{2}$-B6). We have $\mathrm{T}_2(u) = u \quad \mathrm{T}\ (u)$ for $u \quad [a_2-1]$, and $\mathrm{T}_2(u) = \mathrm{T}(u-1) \quad \mathrm{T}\ (u)$ for $u \quad [a_2, i]$. Hence $x_2 \quad y$ by the assertion for $d = 0$. Consequently, we have $x \quad y$.

We next assume that $i = n-1$ and $\mathrm{T}(n-1) \quad n$; note that $n-i+e = 1$. Define $\mathrm{T}_1, \mathrm{T}_2 \quad \mathrm{CST}_{B_n}(\quad _{n-1})$ to be such that $\mathrm{T}_1(u) = \mathrm{T}(u)$ for $u \quad [n-2]$, $\mathrm{T}_1(n-1) = \overline{a_1}$, $\mathrm{T}_2(1) = 1$, and $\mathrm{T}_2(u) = \mathrm{T}_1(u-1)$ for $u \quad [2, n-1]$. Let $x_1, x_2 \quad (W^{I \setminus \{i\}})_{\mathrm{af}}$ be such that $Y_i^{B_n}(x_1) = (\mathrm{T}_1, c)$ and $Y_i^{B_n}(x_2) = (\mathrm{T}_2, c\ ) = (\mathrm{T}_2, c+1)$. By a similar argument above, we can see that $x \quad x_1 \quad x_2 \quad y$.

We next assume that $\mathrm{T}(i) = \overline{a_1}$ and $1 \quad a_1 \quad \mathrm{T}\ (a_1)$. Then $e = d$, $n-i+e > 2$, $a_2 \quad [3, i+1]$, and $u+1 \quad \mathrm{T}\ (u)$ for $u \quad [a_1, i]$ (see (ii) in Definition 4.20 (2)). Define $\mathrm{T}_3, \mathrm{T}_4 \quad \mathrm{CST}_{B_n}(\quad _i)$ by

$$\mathrm{T}_3(u) = \begin{cases} u+1 & \text{if } u \quad [a_1-2], \\ u+2 & \text{if } u \quad [a_1-1, a_2-2], \\ \mathrm{T}(u) & \text{if } u \quad [a_2-1, i], \end{cases}$$

$$\mathrm{T}_4(u) = \begin{cases} 1 & \text{if } u = 1, \\ \mathrm{T}_3(u-1) & \text{if } u \quad [2, i]. \end{cases}$$

Let $x_3, x_4 \quad (W^{I \setminus \{i\}})_{\mathrm{af}}$ be such that $Y_i^{B_n}(x_3) = (\mathrm{T}_3, c)$ and $Y_i^{B_n}(x_4) = (\mathrm{T}_4, c\ ) = (\mathrm{T}_4, c+1)$. We have $\mathrm{T}(u) = u \quad u+1 = \mathrm{T}_3(u)$ for $u \quad [a_1-2]$, $\mathrm{T}(a_1-1) = a_1-1 \quad a_1+1 = \mathrm{T}_3(a_1-1)$, $\mathrm{T}(u) = u+1 \quad u+2 = \mathrm{T}_3(u)$ for $u \quad [a_1, a_2-2]$, and $\mathrm{T}(u) = \mathrm{T}_3(u)$ for $u \quad [a_2-1, i]$. Hence $x \quad x_3$ by the assertion for $d = 0$. We have $x_3 \quad x_4$ by Proposition 4.27 ($\frac{\infty}{2}$-B6). We have $\mathrm{T}_4(u) = u \quad \mathrm{T}\ (u)$ for $u \quad [a_1-1]$, $\mathrm{T}_4(u) = \mathrm{T}_3(u-1) = u+1 \quad \mathrm{T}\ (u)$ for $u \quad [a_1, a_2-1]$, and $\mathrm{T}_4(u) = \mathrm{T}(u-1) \quad \mathrm{T}\ (u)$ for $u \quad [a_2, i]$. Hence $x_4 \quad y$ by the assertion for $d = 0$. Consequently, we have $x \quad y$.

Step 5. We prove that if d is odd, $d = e \quad [3, i]$, $\mathrm{T}(i-d+1) = \overline{a_d}$, and $1 \quad a_d \quad \mathrm{T}\ (a_d)$, then $x \quad y$. Let $\mathrm{T}_3, \mathrm{T}_4 \quad \mathrm{CST}_{B_n}(\quad _i)$ be as in Step 3, and let $x_3, x_4 \quad (W^{I \setminus \{i\}})_{\mathrm{af}}$ be such that $Y_i^{B_n}(x_3) = (\mathrm{T}_3, c)$ and $Y_i^{B_n}(x_4) = (\mathrm{T}_4, c+2)$. We have $\mathrm{T}(u) \quad \mathrm{T}_3(u)$ for $u \quad [i]$, and hence $x \quad x_3$ by the assertion for $d = 0$. We have $x_3 \quad x_4$ by Proposition 4.27 ($\frac{\infty}{2}$-B7). Hence $x \quad x_4$. By the same method as in Step 3, we see that $c\ -(c+2) = d-2$ is odd, $d-2 \quad [i-2]$, $\mathrm{T}_4(i-(d-2)) \quad n$, $\mathrm{T}_4(i-(d-2)+1) = \overline{a_d}$, and $\{a_3 < \cdots < a_{n-i+d}\} = [n] \setminus \{\mathrm{T}_4(u) \mid u \quad [i]\}$. Also, we have $\mathrm{T}_4(u) \quad \mathrm{T}\ (u+d-2)$ for $u \quad [i-d+2]$, because $\mathrm{T}_4(u) = u \quad \mathrm{T}\ (u) \quad \mathrm{T}\ (u+d-2)$ for $u \quad [\min\{a_2, i-d+2\}]$, and $\mathrm{T}_4(u) = \mathrm{T}(u-2) \quad \mathrm{T}\ (u+d-2)$ for $u \quad [a_2+1, i-d+2]$. Consequently, $(\mathrm{T}_4, c+2) \quad (\mathrm{T}\ , c\ )$ (see (ii) in Definition 4.20 (2)). By induction hypothesis (and Step 4), we have $x_4 \quad y$. Hence $x \quad y$.

Step 6. We prove that if d is odd, $d \quad [3, i]$, and $\mathrm{T}(i-d+1) \quad n$, then $x \quad y$. We proceed by induction on e; note that $e \quad [0, d-1]$.

Assume that $e = 0$. By the same method as in Step 4, we can find $x_1, x_2 \quad (W^{I \setminus \{i\}})_{\mathrm{af}}$ and $\mathrm{T}_1, \mathrm{T}_2 \quad \mathrm{CST}_{B_n}(\quad _i)$ such that $Y_i^{B_n}(x_1) = (\mathrm{T}_1, c)$, $Y_i^{B_n}(x_2) = (\mathrm{T}_2, c+1)$, $x \quad x_1 \quad x_2$, and $\mathrm{T}_2(u) \quad \mathrm{T}(u+d-1)$ for $u \quad [u-d+1]$. Since $c - (c+1) = d-1$ is even, we have $x_2 \quad y$ by Step 1. Hence $x \quad y$.

Assume that $e \quad [d-1]$. Let $\mathrm{T}_4 \quad \mathrm{CST}_{B_n}(\quad _i)$ and $x_4 \quad (W^{I \setminus \{i\}})_{\mathrm{af}}$ be as in Step 3; since $n - i + e > 2$, T_4 is well-defined. By the same method as in Step 5, we see that $x \quad x_4$, $c - (c+2) = d-2$ is odd, and $\mathrm{T}_4(u) \quad \mathrm{T}(u+d-2)$ for $u \quad [i-d+2]$. It remains to prove that $x_4 \quad y$. We have $\mathrm{T}_4(i-d+3) \quad \max\{i-d+3, \mathrm{T}(i-d+1)\} \quad n$ and $\#\{u \quad [i] \mid \mathrm{T}_4(u) \quad \overline{n}\} = \max\{0, e-2\} < e$. If $d > 5$, then $c - (c+2) \quad [3, i]$, and hence $x_4 \quad y$ by induction hypothesis. If $d = 3$, then $e \quad [2]$, $c - (c+2) = 1$, $\mathrm{T}_4(i) \quad n$, and hence $x_4 \quad y$ by Step 4. Thus $x \quad y$.

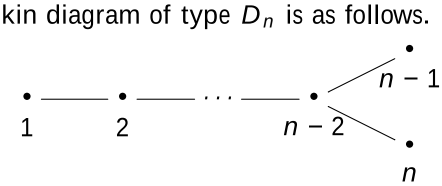
Step 7. We prove that if d is odd and $d > i$, then $x \quad y$. Note that $d > 3$ and $n - i + e > 1$.

We first assume that $n - i + e = 1$. Then $i = n-1$ and $e = 0$. In a way similar to the case of $e = 0$ in Step 6, we can see that $x \quad y$.

We next assume that $n - i + e > 2$. We proceed by induction on d. Let $\mathrm{T}_4 \quad \mathrm{CST}_{B_n}(\quad _i)$ and $x_4 \quad (W^{I \setminus \{i\}})_{\mathrm{af}}$ be as in Step 3; since $n - i + e > 2$, T_4 is well-defined. By a similar argument to Step 5, we have $x \quad x_4$ and $\mathrm{T}_4(u) \quad \mathrm{T}(u+d-2)$ for $u \quad [i-d+2]$. Note that $c - (c+2) = d-2$ is odd. If $d = 3 = i+1$, then $d-2 = 1$, $\mathrm{T}_4(i) = 2 \quad n$, and hence $x_4 \quad y$ by Step 4. If $d > 5$ and $d \quad \{i+1, i+2\}$, then $d-2 \quad [3, i]$, $\mathrm{T}_4(i-d+3) \quad \mathrm{T}_4(2) = 2 \quad n$, and hence $x_4 \quad y$ by Step 6. If $d > i+2$, then $d-2 > i$, and hence $x_4 \quad y$ by induction hypothesis. Thus $x \quad y$.

The proof of Proposition 4.21 is complete.

4.6. Type $D_n^{(1)}$. Fix an integer $n > 4$. Set $I = [n]$. We assume that the labeling of the vertices of the Dynkin diagram of type D_n is as follows.



Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be an orthonormal basis of an n-dimensional Euclidean space $\mathbb{R}^n$. Let $\Delta = \{\pm(\varepsilon_s \pm \varepsilon_t) \mid s, t \quad [n],\ s < t\}$ be a root system of type D_n, and let $\Pi = \{\alpha_s = \varepsilon_s - \varepsilon_{s+1} \mid s \quad [n-1]\} \quad \{\alpha_n = \varepsilon_{n-1} + \varepsilon_n\}$ be a simple root system of Δ.

Let W be the Weyl group of Δ. Note that W acts faithfully on $\{\pm\varepsilon_s \mid s \quad [n]\} \quad \mathbb{R}^n$. Define a partially ordered set $\mathcal{D}_n$ by

(89) $$\mathcal{D}_n = \quad 1 \quad 2 \quad \cdots \quad n-1 \quad \begin{matrix} n \\ \overline{n} \end{matrix} \quad \overline{n-1} \quad \cdots \quad \overline{2} \quad \overline{1} \quad .$$

Let $\sigma : \mathcal{D}_n \to \mathcal{D}_n$ be the bijection defined by $s \leftrightarrow \overline{s}$ for $s \quad [n]$. If we identify $\mathcal{D}_n$ with $\{\pm\varepsilon_s \mid s \quad [n]\}$ by $s = \varepsilon_s$ and $\overline{s} = -\varepsilon_s$ for $s \quad [n]$, then W can be described as follows:

(90) $$W = \{w \quad \mathfrak{S}(\mathcal{D}_n) \mid w(\sigma(s)) = \sigma(w(s)) \text{ for } s \quad [n], \text{ and } \#\{s \quad [n] \mid w(s) \quad \overline{n}\} \text{ is even}\}.$$

Note that $r_{\pm(\varepsilon_s - \varepsilon_t)} = (s\ t)(\overline{s}\ \overline{t})$ and $r_{\pm(\varepsilon_s + \varepsilon_t)} = (s\ \overline{t})(\overline{s}\ t)$ for $s, t \quad [n]$, $s < t$.

For $w \quad \mathfrak{S}(\mathcal{D}_n)$ and $s \quad [n]$, set

(91) $$\mathrm{A}_s(w) = \{t \quad [s+1, n] \mid w(s) \quad w(t) \text{ in } \mathcal{D}_n\}, \quad \mathrm{a}_s(w) = \#\mathrm{A}_s(w),$$

(92) $$\mathrm{B}_s(w) = \{t \quad [s+1, n] \mid w(s) \quad \sigma(w(t)) \text{ in } \mathcal{D}_n\}, \quad \mathrm{b}_s(w) = \#\mathrm{B}_s(w);$$

note that $a_n(w) = b_n(w) = 0$. The length function $\ell : W \to \mathbb{Z}_{\geq 0}$ is given by

$$(93) \qquad \ell(w) = \sum_{s=1}^{n} (a_s(w) + b_s(w))$$

for $w \in W$. The longest element of W is given by $u \mapsto \overline{u}$, $u \in [n]$, if n is even, and $u \mapsto \overline{u}$, $u \in [n-1]$, if n is odd.

Let $|\cdot| : D_n \to [n]$ be the map defined by $|s| = s$ and $|\overline{s}| = s$ for $s \in [n]$. We identify a totally ordered i-element subset $T = \{T(1) \prec T(2) \prec \cdots \prec T(i)\} \subset D_n$ with the column-strict tableau of the form (63). For $i \in I \setminus \{n-1\}$ and $w \in W$, let $T_w^{(i)} \subset D_n$ be the totally ordered i-element subset such that

$$(94) \qquad T_w^{(i)} = \left\{ T_w^{(i)}(1) \prec T_w^{(i)}(2) \prec \cdots \prec T_w^{(i)}(i) \right\} = \{w(1), w(2), \ldots, w(i)\}.$$

For $w \in W$, let $T_w^{(n-1)} \subset D_n$ be the totally ordered n-element subset such that

$$(95) \qquad \begin{aligned} T_w^{(n-1)} &= \left\{ T_w^{(n-1)}(1) \prec T_w^{(n-1)}(2) \prec \cdots \prec T_w^{(n-1)}(n) \right\} \\ &= \{w(1), w(2), \ldots, w(n-1), w(\overline{n})\}. \end{aligned}$$

For $i \in [n-2]$, let $\mathrm{CST}_{D_n}(\varpi_i)$ be the family of totally ordered i-element subsets T of D_n such that $|T(u)|$, $u \in [i]$, are all distinct. Let $\mathrm{CST}_{D_n}(\varpi_{n-1})$ (resp. $\mathrm{CST}_{D_n}(\varpi_n)$) be the family of totally ordered n-element subsets T of D_n such that $|T(u)|$, $u \in [n]$, are all distinct, and $\#\{u \in [n] \mid T(u) \succeq \overline{n}\}$ is odd (resp. even). The proof of the next lemma is standard (cf. [4, §8.1]).

Lemma 4.35. *Let $i \in I$. We have*

$$W^{I \setminus \{i\}} = \begin{cases} \{w \in W \mid w(1) \prec \cdots \prec w(i), \text{ and} \\ \qquad w(i+1) \prec \cdots \prec w(n-1) \prec w(n) \prec w(\overline{n-1})\} & \text{if } i \in [n-2], \\ \{w \in W \mid w(1) \prec \cdots \prec w(n-1) \prec w(\overline{n})\} & \text{if } i = n-1, \\ \{w \in W \mid w(1) \prec \cdots \prec w(n-1) \prec w(n)\} & \text{if } i = n. \end{cases}$$

If $w \in W^{I \setminus \{i\}}$, then $\ell(w) = \sum_{s=1}^{i}(a_s(w) + b_s(w))$ and $A_s(w) \subset [i+1, n]$ for $s \in [i]$. The map $W^{I \setminus \{i\}} \to \mathrm{CST}_{D_n}(\varpi_i)$, $w \mapsto T_w^{(i)}$, is bijective.

We see from Lemmas 4.1–4.2 and 4.35 that the map

$$(96) \qquad Y_i^{D_n} : W_{\mathrm{af}} \to \mathrm{CST}_{D_n}(\varpi_i) \times \mathbb{Z}, \; wt_\xi \mapsto \left(T_w^{(i)}, c_i(\xi) \right),$$

induces a bijection from the subset $(W^{I \setminus \{i\}})_{\mathrm{af}} \subset W_{\mathrm{af}}$ to $\mathrm{CST}_{D_n}(\varpi_i) \times \mathbb{Z}$.

Definition 4.36. *Let $i \in I$, $(T, c), (T', c') \in \mathrm{CST}_{D_n}(\varpi_i) \times \mathbb{Z}$, and $d := c' - c$. Define a partial order $\preceq$ on $\mathrm{CST}_{D_n}(\varpi_i) \times \mathbb{Z}$ as follows.*

(1) *Assume that $i \in [n-2]$. Set $(T, c) \preceq (T', c')$ if*

$$(97) \qquad (d > 0), \quad (T(u) \preceq T'(u+d) \text{ in } D_n \text{ for } u \in [i-d])$$

and one of the following conditions holds:
- (i) *d is even.*
- (ii) *d is odd and $T(i) \preceq n$.*
- (iii) *d is odd and $T(i) \succeq \overline{n}$. Let*

$$(98) \qquad \begin{aligned} a &= \min\left([n] \setminus \{ |T(u)| \mid u \in [i],\ T(u) \succeq \overline{n}\}\right), \\ b &= \min\left([n] \setminus \{ |T'(u)| \mid u \in [i]\}\right); \end{aligned}$$

note that $a \leqslant b$. *If* $d \quad [i]$, *then* $a \quad T(d)$. *If* $a < b$ *and* $k \quad [i]$ *satisfies* $T(k) \quad b \quad T(k+1)$, *then* $(T(u) \quad T(u+d-1)$ *for* $u \quad [2, \min\{k, i-d+1\}])$ *and* $(b \quad T(k+d)$ *if* $k \quad [i-d])$.

(2) *Assume that* $i \quad \{n-1, n\}$. *Set* $(T, c) \quad (T, c)$ *if*

$$(d > 0) \text{ and } (T(u) \quad T(u+2d) \text{ in } D_n \text{ for } u \quad [n-2d]). \tag{99}$$

Proposition 4.37. *Let* $i \quad I$.

1. $Y_i^{D_n} \circ \Pi^{I \setminus \{i\}} = Y_i^{D_n}$.
2. *For* $x, y \quad W_{\mathrm{af}}$, *we have* $\Pi^{I \setminus \{i\}}(x) \quad \Pi^{I \setminus \{i\}}(y)$ *in* $(W^{I \setminus \{i\}})_{\mathrm{af}}$ *if and only if* $Y_i^{D_n}(x) \quad Y_i^{D_n}(y)$ *in* $\mathrm{CST}_{D_n}(\ _i) \times \mathbb{Z}$.
3. *Let* $i \quad [n-2]$ *and* $(T, c), (T, c) \quad \mathrm{CST}_{D_n}(\ _i) \times \mathbb{Z}$. *If* $c - c > i$, *then* $(T, c) \quad (T, c)$.
4. *Let* $i \quad \{n-1, n\}$ *and* $(T, c), (T, c) \quad \mathrm{CST}_{D_n}(\ _i) \times \mathbb{Z}$. *If* $2(c - c) > n$, *then* $(T, c) \quad (T, c)$.

By combining Propositions 3.2 and 4.37 (2), we obtain the following tableau criterion for the semi-infinite Bruhat order on W_{af} of type $D_n^{(1)}$.

Theorem 4.38. *Let* $J \quad I$. *For* $x, y \quad (W^J)_{\mathrm{af}}$, *we have* $x \quad y$ *in* $(W^J)_{\mathrm{af}}$ *if and only if* $Y_i^{D_n}(x) \quad Y_i^{D_n}(y)$ *in* $\mathrm{CST}_{D_n}(\ _i) \times \mathbb{Z}$ *for all* $i \quad I \setminus J$.

The remainder of this subsection is devoted to the proof of Proposition 4.37.

The proofs of Lemmas 4.39–4.40 below are straightforward.

Lemma 4.39. *Let* $i \quad I$ *and* $\gamma \quad \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$. *We have* $\gamma \quad Q^{\ , I \setminus \{i\}}$ *if and only if one of the following conditions holds:*

1. $i = 1$ *and* $\gamma = \varepsilon_1 - \varepsilon_2 = \alpha_1$.
2. $i \quad [2, n-2]$ *and* $\gamma = \varepsilon_i - \varepsilon_{i+1} = \alpha_i$.
3. $i \quad [2, n-2]$ *and* $\gamma = \varepsilon_{i-1} + \varepsilon_i = \alpha_{i-1} + 2\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$.
4. $i = n-1$ *and* $\gamma = \varepsilon_{n-1} - \varepsilon_n = \alpha_{n-1}$.
5. $i = n$ *and* $\gamma = \varepsilon_{n-1} + \varepsilon_n = \alpha_n$.

Lemma 4.40. *Let* $i \quad I$. *We have*

$$2\langle \alpha_i, \rho - \rho_{I \setminus \{i\}} \rangle = \begin{cases} 2n - i - 1 & \text{if } i \quad [2, n-2], \\ 2n - 2 & \text{if } i \quad \{1, n-1, n\}. \end{cases} \tag{100}$$

Proposition 4.41 (cf. [4, §8.2]). *Let* $i \quad I$, $w \quad W^{I \setminus \{i\}}$, *and* $\gamma \quad \Delta^+$. *There exists a Bruhat edge* $w \to \ wr_\gamma = wr_\gamma$ *in* $\mathrm{QB}^{I \setminus \{i\}}$ *if and only if* $\gamma \quad \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$ *and one of the following statements holds.*

(b-D1) $i \quad [n-2]$, $c_i(\gamma) = 1$, *and there exists* $s \quad [i]$ *such that* $wr_\gamma(u) = w(u)$ *for* $u \quad [i] \setminus \{s\}$, $1 \quad w(s) \quad n$, *and* $wr_\gamma(s) = \min([w(s)+1, n] \setminus \{\ w(u) \mid u \quad [i],\ w(u) \quad \overline{n}\})$; *in this case, we have* $\gamma = \varepsilon_s - \varepsilon_t = \alpha_s + \alpha_{s+1} + \cdots + \alpha_t$ *for some* $t \quad [i+1, n]$.

(b-D2) $i \quad [n-2]$, $c_i(\gamma) = 1$, *and there exists* $s \quad [i]$ *such that* $wr_\gamma(u) = w(u)$ *for* $u \quad [i] \setminus \{s\}$, $\overline{n} \quad w(s) \quad \overline{1}$, *and* $wr_\gamma(s) = \sigma(\max([\ w(s) - 1] \setminus \{w(u) \mid u \quad [i],\ w(u) \quad n\}))$; *in this case, we have* $\gamma = \varepsilon_s - \varepsilon_t = \alpha_s + \alpha_{s+1} + \cdots + \alpha_t$ *for some* $t \quad [i+1, n]$.

(b-D3) $i \quad [n-2]$, $c_i(\gamma) = 1$, *and there exists* $s \quad [i]$ *such that* $wr_\gamma(u) = w(u)$ *for* $u \quad [i] \setminus \{s\}$, *and* $(w(s), wr_\gamma(s)) \quad \{(n-1, \overline{n}), (n, \overline{n-1})\}$; *in this case, we have* $\gamma = \varepsilon_s + \varepsilon_t = \alpha_s + \cdots + \alpha_{t-1} + 2\alpha_t \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ *for some* $t \quad [i+1, n-1]$, *or* $\gamma = \varepsilon_s + \varepsilon_n = \alpha_s + \cdots + \alpha_{n-2} + \alpha_n$.

(b-D4) *$i \in [2, n-2]$, $c_i(\gamma^{\vee}) = 2$, and there exist $s, t \in [i]$ such that $s < t$, $wr_{\gamma}(u) = w(u)$ for $u \in [i] \setminus \{s, t\}$ and either ($wr_{\gamma}(s) = w(s)+1 = \sigma(w(t)) = \sigma(wr_{\gamma}(t))+1 \leq n$) or ($w(s) = \sigma(wr_{\gamma}(t)) = n-1$ and $w(t) = \sigma(wr_{\gamma}(s)) = n$) holds; in this case, we have $\gamma^{\vee} = \varepsilon_s + \varepsilon_t = \alpha_s^{\vee} + \cdots + \alpha_{t-1}^{\vee} + 2\alpha_t^{\vee} + \cdots + 2\alpha_{n-2}^{\vee} + \alpha_{n-1}^{\vee} + \alpha_n^{\vee}$.*

(b-D5) *$i \in \{n-1, n\}$, $c_i(\gamma^{\vee}) = 1$, and there exist $s, t \in [i]$ such that $s < t$, $wr_{\gamma}(u) = w(u)$ for $u \in [i] \setminus \{s, t\}$ and $wr_{\gamma}(s) = w(s)+1 = \sigma(w(t)) = \sigma(wr_{\gamma}(t))+1 \leq n$; in this case, we have $\gamma^{\vee} = \varepsilon_s + \varepsilon_t = \alpha_s^{\vee} + \cdots + \alpha_{t-1}^{\vee} + 2\alpha_t^{\vee} + \cdots + 2\alpha_{n-2}^{\vee} + \alpha_{n-1}^{\vee} + \alpha_n^{\vee}$.*

(b-D6) *$i = n-1$, $c_{n-1}(\gamma^{\vee}) = 1$, and there exists $s \in [n-2]$ such that $wr_{\gamma}(u) = w(u)$ for $u \in [n] \setminus \{s, s+1\}$ and $\sigma(wr_{\gamma}(s)) = \sigma(wr_{\gamma}(s+1)) + 1 = w(s+1) = w(s) + 1 \leq n$; in this case, we have $\gamma^{\vee} = \varepsilon_s + \varepsilon_{s+1} = \alpha_s^{\vee} + 2\alpha_{s+1}^{\vee} + \cdots + 2\alpha_{n-2}^{\vee} + \alpha_{n-1}^{\vee} + \alpha_n^{\vee}$ if $s \in [n-3]$, and $\gamma^{\vee} = \varepsilon_{n-2} + \varepsilon_{n-1} = \alpha_{n-2}^{\vee} + \alpha_{n-1}^{\vee} + \alpha_n^{\vee}$ if $s = n-2$.*

(b-D7) *$i = n$, $c_n(\gamma^{\vee}) = 1$, and there exists $s \in [n-1]$ such that $wr_{\gamma}(u) = w(u)$ for $u \in [n] \setminus \{s, s+1\}$ and $\sigma(wr_{\gamma}(s)) = \sigma(wr_{\gamma}(s+1))+1 = w(s+1) = w(s)+1 \leq n$; in this case, we have $\gamma^{\vee} = \varepsilon_s + \varepsilon_{s+1} = \alpha_s^{\vee} + 2\alpha_{s+1}^{\vee} + \cdots + 2\alpha_{n-2}^{\vee} + \alpha_{n-1}^{\vee} + \alpha_n^{\vee}$ if $s \in [n-3]$, $\gamma^{\vee} = \varepsilon_{n-2} + \varepsilon_{n-1} = \alpha_{n-2}^{\vee} + \alpha_{n-1}^{\vee} + \alpha_n^{\vee}$ if $s = n-2$, and $\gamma^{\vee} = \varepsilon_{n-1} + \varepsilon_n = \alpha_n^{\vee}$ if $s = n-1$.*

Moreover, for $i \in I \setminus \{n-1\}$ and $w, v \in W^{I \setminus \{i\}}$, we have $w \preceq v$ if and only if $w(u) \preceq v(u)$ in D_n for $u \in [i]$. For $w, v \in W^{I \setminus \{n-1\}}$, we have $w \preceq v$ if and only if $w(\overline{n}) \preceq v(\overline{n})$ in D_n and $w(u) \preceq v(u)$ in D_n for $u \in [n-1]$.

For $i \in I$, $w \in W^{I \setminus \{i\}}$ and $\gamma \in \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$, let $\mathrm{Q}(i, w, \gamma)$ denote the following statement.

$\mathrm{Q}(i, w, \gamma)$**:** There exists a quantum edge $w \xrightarrow{\gamma} \lfloor wr_{\gamma} \rfloor$ in $\mathrm{QB}^{I \setminus \{i\}}$.

Proposition 4.42. *Let $i \in I$, $w \in W^{I \setminus \{i\}}$, and $\gamma \in \Delta^+ \setminus \Delta^+_{I \setminus \{i\}}$. Then $\mathrm{Q}(i, w, \gamma)$ is true if and only if one of the following statements holds.*

(q-D1) *$i \in [n-2]$ and $c_i(\gamma^{\vee}) = 1$. If we write $\{a_1 < a_2 < \cdots < a_{n-i+1}\} = [n] \setminus \{|w(u)| \mid u \in [i-1]\}$, then $a_1 = 1$, $\{w(\overline{i}), \lfloor wr_{\gamma} \rfloor(1)\} = \{1, a_2\}$ and $\lfloor wr_{\gamma} \rfloor(u) = w(u-1)$ for $u \in [2, i]$; in this case, we have $\gamma^{\vee} = \varepsilon_i - \varepsilon_{i+1} = \alpha_i^{\vee}$.*

(q-D2) *$i \in [2, n-2]$, $c_i(\gamma^{\vee}) = 2$, $\lfloor wr_{\gamma} \rfloor(1) = w(\overline{i}) = 1$, $\lfloor wr_{\gamma} \rfloor(2) = w(\overline{i-1}) = 2$, and $\lfloor wr_{\gamma} \rfloor(u) = w(u-2)$ for $u \in [3, i]$; in this case, we have $\gamma^{\vee} = \varepsilon_{i-1} + \varepsilon_i = \alpha_{i-1}^{\vee} + 2\alpha_i^{\vee} + 2\alpha_{i+1}^{\vee} + \cdots + 2\alpha_{n-2}^{\vee} + \alpha_{n-1}^{\vee} + \alpha_n^{\vee}$.*

(q-D3) *$i = n-1$, $c_{n-1}(\gamma^{\vee}) = 1$, $\lfloor wr_{\gamma} \rfloor(1) = w(n) = 1$, $\lfloor wr_{\gamma} \rfloor(2) = \sigma(w(n-1)) = 2$, $\lfloor wr_{\gamma} \rfloor(u) = w(u-2)$ for $u \in [3, n-1]$, and $\lfloor wr_{\gamma} \rfloor(n) = \sigma(w(n-2))$; in this case, we have $\gamma^{\vee} = \varepsilon_{n-1} - \varepsilon_n = \alpha_{n-1}^{\vee}$.*

(q-D4) *$i = n$, $c_n(\gamma^{\vee}) = 1$, $\lfloor wr_{\gamma} \rfloor(1) = \sigma(w(n)) = 1$, $\lfloor wr_{\gamma} \rfloor(2) = \sigma(w(n-1)) = 2$, and $\lfloor wr_{\gamma} \rfloor(u) = w(u-2)$ for $u \in [3, n]$; in this case, we have $\gamma^{\vee} = \varepsilon_{n-1} + \varepsilon_n = \alpha_n^{\vee}$.*

Before starting the proof of Proposition 4.42, we mention a consequence of Lemma 4.3 and Propositions 4.41–4.42.

Proposition 4.43. *Let $i \in I$, $x, y \in (W^{I \setminus \{i\}})_{\mathrm{af}}$, $Y_i^{D_n}(x) = (\mathrm{T}, c)$, and $Y_i^{D_n}(y) = (\mathrm{T}', c')$. There exists an edge $x \xrightarrow{\beta} y$ in $\mathrm{SiB}^{I \setminus \{i\}}$ for some $\beta \in \Delta^+_{\mathrm{af}}$ if and only if one of the following conditions holds:*

($\frac{\infty}{2}$-D1) *$i \in [n-2]$, $c' = c$, and there exists $s \in [i]$ such that $\mathrm{T}'(u) = \mathrm{T}(u)$ for $u \in [i] \setminus \{s\}$, $1 \preceq \mathrm{T}(s) \prec n$, and $\mathrm{T}'(s) = \min([\mathrm{T}(s) + 1, n] \setminus \{|\mathrm{T}(u)| \mid u \in [i], \mathrm{T}(u) \neq \overline{n}\})$.*

($\frac{\infty}{2}$-D2) *$i \in [n-2]$, $c' = c$, and there exists $s \in [i]$ such that $\mathrm{T}'(u) = \mathrm{T}(u)$ for $u \in [i] \setminus \{s\}$, $\overline{n} \preceq \mathrm{T}(s) \prec \overline{1}$, and $\mathrm{T}'(s) = \sigma(\max([|\mathrm{T}(s)| - 1] \setminus \{|\mathrm{T}(u)| \mid u \in [i], \mathrm{T}(u) \neq n\}))$.*

($\frac{\infty}{2}$-D3) $i \in [n-2]$, $c' = c$, *and there exists* $s \in [i]$ *such that* $T'(u) = T(u)$ *for* $u \in [i] \setminus \{s\}$, *and* $(T(s), T'(s)) \in \{(n-1, \overline{n}), (n, \overline{n-1})\}$.

($\frac{\infty}{2}$-D4) $i \in [2, n-2]$, $c' = c$, *and there exist* $s, t \in [i]$ *such that* $s < t$, $T'(u) = T(u)$ *for* $u \in [i] \setminus \{s, t\}$ *and* $T'(s) = T(s) + 1 = \sigma(T(t)) = \sigma(T'(t)) + 1 \preceq n$.

($\frac{\infty}{2}$-D5) $i \in \{n-1, n\}$, $c' = c$, *and there exist* $s, t \in [i]$ *such that* $s < t$, $T'(u) = T(u)$ *for* $u \in [i] \setminus \{s, t\}$ *and* $T'(s) = T(s) + 1 = \sigma(T(t)) = \sigma(T'(t)) + 1 \preceq n$.

($\frac{\infty}{2}$-D6) $i = n-1$, $c' = c$, *and there exists* $s \in [n-2]$ *such that* $T'(u) = T(u)$ *for* $u \in [n] \setminus \{s, s+1\}$ *and* $\sigma(T'(s)) = \sigma(T'(s+1)) + 1 = T(s+1) = T(s) + 1 \preceq n$.

($\frac{\infty}{2}$-D7) $i = n$, $c' = c$, *and there exists* $s \in [n-1]$ *such that* $T'(u) = T(u)$ *for* $u \in [n] \setminus \{s, s+1\}$ *and* $\sigma(T'(s)) = \sigma(T'(s+1)) + 1 = T(s+1) = T(s) + 1 \preceq n$.

($\frac{\infty}{2}$-D8) $i \in [n-2]$ *and* $c' = c + 1$. *If we write* $\{a_1 < a_2 < \cdots < a_{n-i+1}\} = [n] \setminus \{|T(u)| \mid u \in [i-1]\}$, *then* $a_1 = 1$, $\{\sigma(T(i)), T'(1)\} = \{1, a_2\}$ *and* $T'(u) = T(u-1)$ *for* $u \in [2, i]$.

($\frac{\infty}{2}$-D9) $i \in [2, n-2]$, $c' = c + 2$, $T'(1) = \sigma(T(i)) = 1$, $T'(2) = \sigma(T(i-1)) = 2$, *and* $T'(u) = T(u-2)$ *for* $u \in [3, i]$.

($\frac{\infty}{2}$-D10) $i \in \{n-1, n\}$, $c' = c + 1$, $T'(1) = \sigma(T(n)) = 1$, $T'(2) = \sigma(T(n-1)) = 2$, *and* $T'(u) = T(u-2)$ *for* $u \in [3, n]$.

EXAMPLE 4.44. (1) Let $i = 1$. ($\frac{\infty}{2}$-D8) is equivalent to $c' = c + 1$ and $(T, T') \in \left\{ \left(\boxed{\overline{1}}, \boxed{2}\right), \left(\boxed{\overline{2}}, \boxed{1}\right) \right\}$.

(2) Let $n = 4$ and $i = 2$. ($\frac{\infty}{2}$-D8) is equivalent to the condition that $c' = c + 1$ and (T, T') equals one of the following:

$$\left(\begin{array}{|c|}\hline 2\\\hline \overline{3}\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right), \left(\begin{array}{|c|}\hline 3\\\hline \overline{2}\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\right), \left(\begin{array}{|c|}\hline 4\\\hline \overline{2}\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right), \left(\begin{array}{|c|}\hline \overline{4}\\\hline \overline{2}\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline \overline{4}\\\hline\end{array}\right), \left(\begin{array}{|c|}\hline \overline{3}\\\hline \overline{2}\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline \overline{3}\\\hline\end{array}\right),$$

$$\left(\begin{array}{|c|}\hline 3\\\hline \overline{1}\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right), \left(\begin{array}{|c|}\hline 4\\\hline \overline{1}\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right), \left(\begin{array}{|c|}\hline \overline{4}\\\hline \overline{1}\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline \overline{4}\\\hline\end{array}\right), \left(\begin{array}{|c|}\hline \overline{3}\\\hline \overline{1}\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline \overline{3}\\\hline\end{array}\right), \left(\begin{array}{|c|}\hline \overline{2}\\\hline \overline{1}\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline \overline{2}\\\hline\end{array}\right).$$

($\frac{\infty}{2}$-D9) is equivalent to $c' = c + 2$, $T = \begin{array}{|c|}\hline \overline{2}\\\hline \overline{1}\\\hline\end{array}$ and $T' = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$.

We have divided the proof of Proposition 4.42 into a sequence of lemmas.

LEMMA 4.45. $Q(1, w, y)$ *implies* (q-D1).

Proof. Assume that $Q(1, w, y)$ is true. By Lemmas 4.4 and 4.39, we have $y \in \varepsilon_1 - \varepsilon_2 \in Q^{\vee, I \setminus \{1\}}$, $r_y = (1\ 2)(\overline{1}\ \overline{2})$, and $\lfloor wr_y \rfloor = wr_y(z_y^{I \setminus \{1\}})^{-1}$. Set $J = I \setminus \{1\}$; note that J is of type D_{n-1}. We see that $2 \in J_{\mathrm{af}}$ satisfies the condition for $y \in Q$ in Lemma 4.1; note that $J \setminus \{2\}$ is of type D_{n-2}. Hence $z_y^{I \setminus \{1\}} = w_0^J w_0^{J \setminus \{2\}}$ is given by $2 \to \overline{2}$, $n \to \overline{n}$, and $u \to u$ for $u \in [n] \setminus \{2, n\}$. Then $\lfloor wr_y \rfloor$ is given by $1 \to w(2)$, $2 \to w(\overline{1})$, $u \to w(u)$ for $u \in [3, n-1]$, and $n \to w(\overline{n})$. It follows from (90) and Lemma 4.35 that

$$\max\{w(2), w(\overline{1})\} \prec w(3) \prec w(4) \prec \cdots \prec w(n-1) \prec \underbrace{w(\overline{n})}_{=n} \prec w(\overline{n-1}). \tag{101}$$

Hence $\{\lfloor wr_y \rfloor(1) = w(2), w(\overline{1})\} = \{a_1 = 1, a_2 = 2\}$. This implies (q-D1).

LEMMA 4.46. $i = 1$ *and* (q-D1) *imply* $Q(1, w, y)$.

Proof. Assume that (q-D1) and $i = 1$; we have $a_1 = 1$, $a_2 = 2$, and $\{w(\overline{i}), \lfloor wr_y \rfloor(1)\} = \{1, 2\}$. We see from Lemmas 4.5 and 4.39–4.40 that $Q(1, w, y)$ is equivalent to $\ell(\lfloor wr_y \rfloor) - \ell(w) = 3 - 2n$. We check at once that (q-D1), $i = 1$, and Lemma 4.35 yield (101). If $w(1) = \overline{1}$, then $\ell(w) = 2n - 2$, $\ell(\lfloor wr_y \rfloor) = 1$, and hence

$\ell(\lfloor wr_y \rfloor) - \ell(w) = 3 - 2n$. If $w(1) = \overline{2}$, then $\ell(w) = 2n - 3$, $\ell(\lfloor wr_y \rfloor) = 0$, and hence $\ell(\lfloor wr_y \rfloor) - \ell(w) = 3 - 2n$.

Lemma 4.47. *$n > 5$, $i \in [2, n-3]$, $c_i(y) = 1$, and* $\mathrm{Q}(i, w, y)$ *imply* (q-D1).

Proof. Assume that $n > 5$, $i \in [2, n-3]$, and $c_i(y) = 1$, and that $\mathrm{Q}(i, w, y)$ is true. By Lemmas 4.4 and 4.39, we have $y = \varepsilon_i - \varepsilon_{i+1} \in Q^{\vee, I \smallsetminus \{i\}}$, $r_y = (i\ i+1)(\overline{i}\ \overline{i+1})$, and $\lfloor wr_y \rfloor = wr_y(z_y^{I \smallsetminus \{i\}})^{-1}$. Let $I \smallsetminus \{i\} = I_1 \sqcup I_2$, where $I_1 = [i-1]$ is of type A_{i-1} and $I_2 = [i+1, n]$ is of type D_{n-i}. We see that $(i-1, i+1) \in (I_1)_{\mathrm{af}} \times (I_2)_{\mathrm{af}}$ satisfies the condition for $y \in Q$ in Lemma 4.1; note that $I_1 \smallsetminus \{i-1\} = [i-2]$ is of type A_{i-2} and $I_2 \smallsetminus \{i+1\} = [i+2, n]$ is of type D_{n-i-1}. Hence $z_y^{I \smallsetminus \{i\}} = w_0^{I_1} w_0^{I_1 \smallsetminus \{i-1\}} w_0^{I_2} w_0^{I_2 \smallsetminus \{i+1\}} = (1\ 2\ \cdots\ i)(\overline{1}\ \overline{2}\ \cdots\ \overline{i})(i+1\ \overline{i+1})(n\ \overline{n})$. Then $\lfloor wr_y \rfloor$ is given by $1 \to w(i+1)$, $u \to w(u-1)$ for $u \in [2, i]$, $i+1 \to w(\overline{i})$, $u \to w(u)$ for $u \in [i+2, n-1]$, and $n \to w(\overline{n})$. It follows from Lemma 4.35 that

$$
\begin{aligned}
&w(i+1) \prec w(1) \prec w(2) \prec \cdots \prec w(i-1) \prec w(i), \\
(102) \qquad &\max\{w(\overline{i}), w(i+1)\} \prec w(i+2) \prec \cdots \prec \underbrace{w(n-1) \prec w(\overline{n}) \prec w(\overline{n-1})}_{n}.
\end{aligned}
$$

Let $\{a_1 < a_2 < \cdots < a_{n-i+1}\} = [n] \smallsetminus \{|w(u)| \mid u \in [i-1]\}$. By (102), we have $w(i+1) = \min\{w(u) \mid u \in [n]\}$, which implies $a_1 = 1$. Since $[n] \smallsetminus \{|w(u)| \mid u \in [i-1]\} = \{w(\overline{i}), w(i+1), w(i+2), \ldots, w(n-1), |w(n)|\}$, we have $\{w(\overline{i}), \lfloor wr_y \rfloor(1) = w(i+1)\} = \{a_1 = 1, a_2\}$.

Lemma 4.48. *$n > 5$, $i \in [2, n-3]$, and* (q-D1) *imply* $\mathrm{Q}(i, w, y)$.

Proof. Assume that $n > 5$, $i \in [2, n-3]$, and (q-D1) hold. We see from Lemmas 4.5 and 4.39–4.40 that $\mathrm{Q}(i, w, y)$ is equivalent to $\ell(\lfloor wr_y \rfloor) - \ell(w) = 2 - 2n + i$. We check at once that (q-D1) yields (102). We deduce from (102) that

(1) $\mathrm{a}_1(\lfloor wr_y \rfloor) = \begin{cases} 0 & \text{if } w(i+1) \prec w(\overline{i}), \\ 1 & \text{if } w(i+1) \succ w(\overline{i}), \end{cases}$

(2) $\mathrm{b}_1(\lfloor wr_y \rfloor) = \begin{cases} 0 & \text{if } w(i+1) \prec w(\overline{i}), \\ w(i+1) - 2 & \text{if } w(i+1) \succ w(\overline{i}), \end{cases}$

(3) for $s \in [2, i]$ and $t \in [s+1, i]$, $t \in \mathrm{A}_s(\lfloor wr_y \rfloor)$ and $t-1 \in \mathrm{A}_{s-1}(w)$,

(4) for $s \in [2, i]$ and $t \in [s+1, i]$, $t \in \mathrm{B}_s(\lfloor wr_y \rfloor)$ if and only if $t-1 \in \mathrm{B}_{s-1}(w)$,

(5) for $s \in [2, i]$, $i+1 \in \mathrm{A}_s(\lfloor wr_y \rfloor)$ if and only if $i \in \mathrm{B}_{s-1}(w)$,

(6) for $s \in [2, i]$, $i+1 \in \mathrm{B}_s(\lfloor wr_y \rfloor)$ if and only if $i \in \mathrm{A}_{s-1}(w)$,

(7) for $s \in [2, i]$ and $t \in [i+2, n-1]$, $t \in \mathrm{A}_s(\lfloor wr_y \rfloor)$ if and only if $t \in \mathrm{A}_{s-1}(w)$,

(8) for $s \in [2, i]$ and $t \in [i+2, n-1]$, $t \in \mathrm{B}_s(\lfloor wr_y \rfloor)$ if and only if $t \in \mathrm{B}_{s-1}(w)$,

(9) for $s \in [2, i]$, $n \in \mathrm{A}_s(\lfloor wr_y \rfloor)$ if and only if $n \in \mathrm{B}_{s-1}(w)$,

(10) for $s \in [2, i]$, $n \in \mathrm{B}_s(\lfloor wr_y \rfloor)$ if and only if $n \in \mathrm{A}_{s-1}(w)$,

(11) $i+1 \in \mathrm{A}_{s-1}(w)$ for $s \in [2, i]$,

(12) $i+1 \in \mathrm{B}_{s-1}(w)$ if and only if $s \in [i - w(i+1) + 3, i]$,

(13) $\mathrm{a}_i(w) = n - i$, $\mathrm{b}_i(w) = \begin{cases} n - i - 1 & \text{if } w(i+1) \prec w(\overline{i}), \\ n - i & \text{if } w(i+1) \succ w(\overline{i}). \end{cases}$

Hence $\ell(\lfloor wr_y \rfloor) - \ell(w) = 2 - 2n + i$, which is our assertion.

Lemma 4.49. *$n > 5$, $c_{n-2}(y) = 1$, and* $\mathrm{Q}(n-2, w, y)$ *imply* (q-D1).

Proof. Assume that $n > 5$ and $c_{n-2}(y) = 1$, and that $\mathrm{Q}(n-2, w, y)$ is true. By Lemmas 4.5 and 4.39–4.40, we have $\ell(\lfloor wr_y \rfloor) - \ell(w) = -n$. By Lemmas 4.4 and 4.39, we have $y = \varepsilon_{n-2} - \varepsilon_{n-1} \in Q^{\vee, I \smallsetminus \{n-2\}}$, $r_y = (n-2\ n-1)(\overline{n-2}\ \overline{n-1})$, and

$wr_y = wr_y(z_y^{I\setminus\{n-2\}})^{-1}$. Let $I \setminus \{n-2\} = I_1 \quad I_2 \quad I_2$, where $I_1 = [n-3]$ is of type A_{n-3}, $I_2 = \{n-1\}$ is of type A_1, and $I_2 = \{n\}$ is of type A_1. We see that $(n-3, n-1, n) \quad (I_1)_{\mathrm{af}} \times (I_2)_{\mathrm{af}} \times (I_2)_{\mathrm{af}}$ satisfies the condition for $y \quad Q$ in Lemma 4.1; note that $I_1 \setminus \{n-3\}$ is of type A_{n-4}. Hence $z_y^{I\setminus\{n-2\}} = w_0^{I_1} w_0^{I_1 \setminus \{n-3\}} w_0^{I_2} w_0^{I_2} = (1\ 2\ \cdots\ n-2)(\overline{1}\ \overline{2}\ \cdots\ \overline{n-2})(n-1\ \overline{n-1})(n\ \overline{n})$. Then wr_y is given by $1 \to w(n-1)$, $u \to w(u-1)$ for $u \quad [2, n-2]$, $n-1 \to w(\overline{n-2})$, and $n \to w(\overline{n})$. It follows from Lemma 4.35 that

$$(103) \qquad \begin{array}{l} w(n-1) \quad w(1) \quad w(2) \quad \cdots \quad w(n-3) \quad w(n-2), \\ \max\{w(\overline{n-2}), w(n-1)\} \quad w(\overline{n}) \quad w(\overline{n-1}). \end{array}$$

Let $\{a_1 < a_2 < a_3\} = [n] \setminus \{ w(u) \ | u \quad [n-3]\} = \{ w(\overline{n-2}) , w(n-1), w(\overline{n}) \}$. We see from (103) that $a_1 = 1$ and $w(n-1) < w(\overline{n})$. What is left is to show that $w(\overline{n-2}) \quad n$ and $w(\overline{n-2}) < w(\overline{n})$. We have the following cases:

(i) $w(\overline{n-2}) \quad w(n-1)$,
(ii) $w(\overline{n-2}) \quad w(n-1)$ and $w(n-2) \quad w(\overline{n})$,
(iii) $w(\overline{n-2}) \quad w(n-1)$ and $w(n-2) \quad w(\overline{n})$;

we will prove that (i) or (ii) holds and these imply (q-D1). It follows from (103) that

(1) $\mathrm{a}_1(wr_y) = \begin{cases} 1 & \text{if (i)}, \\ 0 & \text{if (ii) or (iii)}, \end{cases}$

(2) $\mathrm{b}_1(wr_y) = \begin{cases} w(n-1) - 2 & \text{if (i)}, \\ 0 & \text{if (ii) or (iii)}, \end{cases}$

(3) for $s \quad [2, n-2]$, $n-1 \quad \mathrm{A}_s(wr_y)$ if and only if $n-2 \quad \mathrm{B}_{s-1}(w)$,
(4) for $s \quad [2, n-2]$, $n \quad \mathrm{A}_s(wr_y)$ if and only if $n \quad \mathrm{B}_{s-1}(w)$,
(5) for $s \quad [2, n-2]$, $n-1 \,/\, \mathrm{B}_s(wr_y)$ and $n-1 \quad \mathrm{A}_{s-1}(w)$,
(6) for $s \quad [2, n-2]$, $n \quad \mathrm{B}_s(wr_y)$ if and only if $n \quad \mathrm{A}_{s-1}(w)$,
(7) for $s \quad [2, n-2]$ and $t \quad [s+1, n-2]$, $t \quad \mathrm{B}_s(wr_y)$ if and only if $t-1 \quad \mathrm{B}_{s-1}(w)$,
(8) for $s \quad [2, n-2]$, $n-1 \quad \mathrm{B}_{s-1}(w)$ if and only if $s \quad [n-(w(n-1)-1), n-2]$,

(9) $\mathrm{a}_{n-2}(w) = 2$ and $\mathrm{b}_{n-2}(w) = \begin{cases} 2 & \text{if (i)}, \\ 1 & \text{if (ii)}, \\ 0 & \text{if (iii)}. \end{cases}$

If (i) holds, then $w(n-2) = \overline{1}$. Hence $w(\overline{n-2}) \quad n$ and $w(\overline{n-2}) < w(\overline{n})$.

If (ii) holds, then $wr_y (1) = w(n-1) = 1$ and $w(\overline{n-2}) \quad \min\{w(n), w(\overline{n})\} \quad n$. Hence $w(\overline{n-2}) < w(\overline{n})$.

If (iii) holds, then $(wr_y) - (w) = 1 - n = -n$, a contradiction.

Lemma 4.50. *$n > 5$, $i = n-2$, and* (q-D1) *imply* $\mathrm{Q}(n-2, w, y)$.

Proof. Assume that $n > 5$, $i = n-2$, and (q-D1) hold. By Lemmas 4.5 and 4.39–4.40, $\mathrm{Q}(n-2, w, y)$ is equivalent to $(wr_y) - (w) = -n$. We check at once that (q-D1) and $i = n-2$ imply (103) and (i) or (ii) in the proof of Lemma 4.49. Then (1)–(9) in the proof of Lemma 4.49 yield $(wr_y) - (w) = -n$.

Lemma 4.51. *$n = 4$, $c_2(y) = 1$, and* $\mathrm{Q}(2, w, y)$ *imply* (q-D1).

Proof. Assume that $n = 4$ and $c_2(y) = 1$, and that $\mathrm{Q}(2, w, y)$ is true. By Lemmas 4.4 and 4.39, we have $y = \varepsilon_2 - \varepsilon_3 \quad Q^{,I\setminus\{2\}}$, $r_y = (2\ 3)(\overline{2}\ \overline{3})$, and $wr_y = wr_y(z_y^{I\setminus\{2\}})^{-1}$. Let $I \setminus \{2\} = I_1 \quad I_2 \quad I_2$, where $I_1 = \{1\}$, $I_2 = \{3\}$, and $I_2 = \{4\}$ are of type A_1. We see that $(1, 3, 4) \quad (I_1)_{\mathrm{af}} \times (I_2)_{\mathrm{af}} \times (I_2)_{\mathrm{af}}$ satisfies the condition for $y \quad Q$ in Lemma 4.1. Hence $z_y^{I\setminus\{2\}} = r_1 r_3 r_4 = (1\ 2)(\overline{1}\ \overline{2})(3\ \overline{3})(4\ \overline{4})$. Then wr_y is given by $1 \to w(3)$, $2 \to w(1)$, $3 \to w(\overline{2})$, and $4 \to w(\overline{4})$. It follows from Lemma 4.35 that

(104) $\quad w(3) \quad w(1) \quad w(2), \quad w(3) \quad w(4) \quad w(\overline{3}), \quad w(\overline{2}) \quad w(4) \quad w(2);$

all $w \quad W^{I \setminus \{2\}}$ satisfying (104) are listed in Table 1 below. It is easy to check that each w in the table satisfies (q-D1) for $n = 4$ and $i = 2$.

Table 1.

$w(1)$	$w(2)$	$w(3)$	$w(4)$	$wr_{\gamma}\ (1)$	$wr_{\gamma}\ (2)$	$wr_{\gamma}\ (3)$	$wr_{\gamma}\ (4)$
2	$\overline{3}$	1	$\overline{4}$	1	2	3	4
3	$\overline{2}$	1	$\overline{4}$	1	3	2	4
3	$\overline{1}$	2	$\overline{4}$	2	3	1	4
4	$\overline{2}$	1	$\overline{3}$	1	4	2	3
4	$\overline{1}$	2	$\overline{3}$	2	4	1	3
$\overline{4}$	$\overline{2}$	1	3	1	$\overline{4}$	2	$\overline{3}$
$\overline{4}$	$\overline{1}$	2	3	2	$\overline{4}$	1	$\overline{3}$
$\overline{3}$	$\overline{2}$	1	4	1	$\overline{3}$	2	$\overline{4}$
$\overline{3}$	$\overline{1}$	2	4	2	$\overline{3}$	1	$\overline{4}$
$\overline{2}$	$\overline{1}$	3	4	3	$\overline{2}$	1	$\overline{4}$

Lemma 4.52. *$n = 4$, $i = 2$, and* (q-D1) *imply* $\mathrm{Q}(2, w, \gamma)$.

Proof. Assume that $n = 4$, $i = 2$, and (q-D1) hold. By Lemmas 4.5 and 4.39–4.40, $\mathrm{Q}(2, w, \gamma)$ is equivalent to $(\ wr_{\gamma}\ ) - \ (w) = -4$. We see that (q-D1) implies $wr_{\gamma}\ (1) \quad wr_{\gamma}\ (2) = w(1) \quad w(2)$, $[4] \setminus \{\ w(1)\ \} = \{1 < a_2 < a_3\}$, and $\{w(\overline{2}),\ wr_{\gamma}\ (1)\} = \{1, a_2\}$. For this to happen, w must be one of the ten elements listed in Table 1. In either case, it is easy to check that $(\ wr_{\gamma}\ ) - \ (w) = -4$.

Lemma 4.53. *$n > 5$, $c_2(\gamma\ ) = 2$, and* $\mathrm{Q}(2, w, \gamma)$ *imply* (q-D2).

Proof. Assume that $n > 5$ and $c_2(\gamma\ ) = 2$, and that $\mathrm{Q}(2, w, \gamma)$ is true. By Lemmas 4.5 and 4.39–4.40, we have $(\ wr_{\gamma}\ ) - \ (w) = 7 - 4n$. By Lemmas 4.4 and 4.39, we have $\gamma \ = \varepsilon_1 + \varepsilon_2 \quad Q^{\ ,I \setminus \{2\}}$, $r_{\gamma} = (1\ \overline{2})(\overline{1}\ 2)$, and $\ wr_{\gamma} \ = wr_{\gamma}(z_{\gamma}^{I \setminus \{2\}})^{-1}$. Let $I \setminus \{2\} = I_1 \quad I_2$, where $I_1 = \{1\}$ is of type A_1 and $I_2 = [3, n]$ is of type D_{n-2}. We see that $(0, 0) \quad (I_1)_{\mathrm{af}} \times (I_2)_{\mathrm{af}}$ satisfies the condition for $\gamma \quad Q$ in Lemma 4.1. Hence $\ wr_{\gamma} \ = wr_{\gamma}$ acts by $1 \to w(\overline{2})$, $2 \to w(\overline{1})$, and $u \to w(u)$ for $u \quad [3, n]$. The proof will be divided into three steps.

Step 1. We show that $w(1) \quad w(2) \quad n$ leads to a contradiction. Suppose that $w(1) \quad w(2) \quad n$. Then $w(n) \quad n$, by (90). It follows from Lemma 4.35 that $\mathrm{a}_1(\ wr_{\gamma}\ ) = \mathrm{a}_2(\ wr_{\gamma}\ ) = n - 2$, $\mathrm{b}_1(\ wr_{\gamma}\ ) = n - w(2) + 1$, $\mathrm{b}_2(\ wr_{\gamma}\ ) = n - w(1) - 1$, $\mathrm{a}_1(w) = w(1) - 1$, $\mathrm{a}_2(w) = w(2) - 2$, and $\mathrm{b}_1(w) = \mathrm{b}_2(w) = 0$. Hence $(\ wr_{\gamma}\ ) - \ (w) = 4n - 1 - 2w(1) - 2w(2) > 7 - 4n$, a contradiction.

Step 2. We show that $w(1) \quad n$ and $\overline{n} \quad w(2)$ lead to a contradiction. Suppose that $w(1) \quad n$ and $\overline{n} \quad w(2)$. Then $w(\overline{n}) \quad n$, by (90). We have the following cases:

(i) $w(1) \quad w(\overline{2}) \quad w(\overline{n}) \quad n$,
(ii) $w(1) \quad w(\overline{n}) \quad w(\overline{2}) \quad n$,
(iii) $w(\overline{2}) \quad w(1) \quad w(\overline{n}) \quad n$,
(iv) $w(\overline{2}) \quad w(\overline{n}) \quad w(1) \quad n$,
(v) $w(\overline{n}) \quad w(1) \quad w(\overline{2}) \quad n$,
(vi) $w(\overline{n}) \quad w(\overline{2}) \quad w(1) \quad n$.

It follows from Lemma 4.35 that

$$(1)\quad \mathrm{a}_1(\ wr_y\ ) = \begin{cases} w(\overline{2}) - 1 & \text{if (iii) or (iv)},\\ w(\overline{2}) - 2 & \text{if (i) or (vi)},\\ w(\overline{2}) - 3 & \text{if (ii) or (v)}, \end{cases}$$

$$(2)\quad \mathrm{a}_2(\ wr_y\ ) = \begin{cases} n - 2 & \text{if (i), (ii), or (iii)},\\ n - 3 & \text{if (iv), (v), or (vi)}, \end{cases}$$

$$(3)\quad \mathrm{b}_1(\ wr_y\ ) = \begin{cases} 0 & \text{if (iii) or (iv)},\\ 1 & \text{if (i) or (vi)},\\ 2 & \text{if (ii) or (v)}, \end{cases}$$

$$(4)\quad \mathrm{b}_2(\ wr_y\ ) = \begin{cases} n - w(1) - 1 & \text{if (i) or (ii)},\\ n - w(1) & \text{if (iii) or (v)},\\ n - w(1) + 1 & \text{if (iv) or (vi)}, \end{cases}$$

$$(5)\quad \mathrm{a}_1(w) = \begin{cases} w(1) - 3 & \text{if (iv) or (vi)},\\ w(1) - 2 & \text{if (iii) or (v)},\\ w(1) - 1 & \text{if (i) or (ii)}, \end{cases}$$

$$(6)\quad \mathrm{a}_2(w) = \begin{cases} n - 3 & \text{if (ii), (v) or (vi)},\\ n - 2 & \text{if (i), (iii) or (iv)}, \end{cases}$$

$$(7)\quad \mathrm{b}_1(w) = \begin{cases} 0 & \text{if (i) or (ii)},\\ 1 & \text{if (iii) or (v)},\\ 2 & \text{if (iv) or (vi)}, \end{cases}$$

$$(8)\quad \mathrm{b}_2(w) = \begin{cases} n - w(\overline{2}) - 1 & \text{if (iii) or (iv)},\\ n - w(\overline{2}) & \text{if (i) or (vi)},\\ n - w(\overline{2}) + 1 & \text{if (ii) or (v)}. \end{cases}$$

Hence

$$(105)\qquad (\ wr_y\ ) - \ (w) = \begin{cases} 2w(\overline{2}) - 2w(1) - 1 & \text{if (i), (ii), or (v)},\\ 2w(\overline{2}) - 2w(1) + 1 & \text{if (iii), (iv) or (vi)}. \end{cases}$$

If (i), (ii), or (v) hold, then $2w(\overline{2}) - 2w(1) - 1 > 0$, contrary to $(\ wr_y\ ) - \ (w) = 7 - 4n < 0$. If (iii), (iv), or (vi) hold, then $1 \le w(\overline{2}) \le w(1) \le n$ and $2w(\overline{2}) - 2w(1) + 1 > 3 - 2n$, contrary to $(\ wr_y\ ) - \ (w) = 7 - 4n$ and $n > 5$.

Step 3. By Steps 1–2, we have $\overline{n} \quad w(1) \quad w(2)$; note that $w(n) \quad n$, by (90). It remains to prove that $w(\overline{1}) = 2$ and $w(\overline{2}) = 1$. It follows from Lemma 4.35 that $\mathrm{a}_1(\ wr_y\ ) = w(\overline{2}) - 1$, $\mathrm{a}_2(\ wr_y\ ) = w(\overline{1}) - 2$, $\mathrm{b}_1(\ wr_y\ ) = \mathrm{b}_2(\ wr_y\ ) = 0$, $\mathrm{a}_1(w) = \mathrm{a}_2(w) = n - 2$, $\mathrm{b}_1(w) = (n - 1) - (w(\overline{1}) - 2)$, and $\mathrm{b}_2(w) = (n - 2) - (w(\overline{2}) - 1)$. Hence $(\ wr_y\ ) - \ (w) = 7 - 4n + 2(w(\overline{2}) - 1) + 2(w(\overline{1}) - 2)$; note that $w(\overline{2}) - 1 > 0$ and $w(\overline{1}) - 2 > 0$. Since $(\ wr_y\ ) - \ (w) = 7 - 4n$, we conclude that $w(\overline{1}) = 2$ and $w(\overline{2}) = 1$.

The proof of Lemma 4.53 is complete.

Lemma 4.54. *$n > 5$, $i = 2$, and* (q-D2) *imply* $\mathrm{Q}(2, w, y)$.

Proof. Assume that $n > 5$, $i = 2$, and (q-D2) hold. By Lemmas 4.5 and 4.39–4.40, $\mathrm{Q}(2, w, y)$ is equivalent to $(\ wr_y\ ) - \ (w) = 7 - 4n$. It follows from Lemma 4.35 that $w = r_y = (1\ \overline{2})(\overline{1}\ 2)$ and $\ wr_y\ = e$. We check at once that $(\ wr_y\ ) - \ (w) = 7 - 4n$.

Lemma 4.55. *$n > 5$, $i \in [3, n-3]$, $c_i(\gamma) = 2$, and $\mathrm{Q}(i, w, \gamma)$ imply* (q-D2).

Proof. Assume that $n > 5$, $i \in [3, n-3]$, and $c_i(\gamma) = 2$, and that $\mathrm{Q}(i, w, \gamma)$ is true. By Lemmas 4.5 and 4.39–4.40, we have $\ell(wr_\gamma) - \ell(w) = 3 - 4n + 2i$. By Lemmas 4.4 and 4.39, we have $\gamma = \varepsilon_{i-1} + \varepsilon_i \in Q^{\vee, I \setminus \{i\}}$, $r_\gamma = (i-1\ \bar{i})(\overline{i-1}\ i)$, and $wr_\gamma = wr_\gamma(z_\gamma^{I \setminus \{i\}})^{-1}$. Let $I \setminus \{i\} = I_1 \sqcup I_2$, where $I_1 = [i-1]$ is of type A_{i-1} and $I_2 = [i+1, n]$ is of type D_{n-i}. We see that $(i-2, 0) \in (I_1)_{\mathrm{af}} \times (I_2)_{\mathrm{af}}$ satisfies the condition for $\gamma \in Q$ in Lemma 4.1; note that $I_1 \setminus \{i-2\}$ is of type $A_{i-3} \times A_1$. Hence $z_\gamma^{I \setminus \{i\}} = w_0^{I_1} w_0^{I_1 \setminus \{i-2\}}$ is given by $1 \to i-1$, $2 \to i$, $u \to u-2$ for $u \in [3, i]$, and $u \to u$ for $u \in [i+1, n]$. Then wr_γ is given by $1 \to w(\bar{i})$, $2 \to w(\overline{i-1})$, $u \to w(u-2)$ for $u \in [3, i]$, and $u \to w(u)$ for $u \in [i+1, n]$. It follows from Lemma 4.35 that

$$(106)\qquad \begin{aligned} &w(\bar{i}) < \underbrace{w(\overline{i-1})}_{n} < w(1) < w(2) < \cdots < \underbrace{w(i-1)}_{\bar{n}} < w(i),\\ &w(i+1) < w(i+2) < \cdots < \underbrace{w(n-1)}_{n} < w(n) < w(\overline{n-1}). \end{aligned}$$

We have the following cases:

(i) $w(n) < w(i-1) < w(i)$,
(ii) $w(i-1) < w(n) < w(i)$,
(iii) $w(i-1) < w(i) < w(n)$;

we will prove that (i) holds. It follows from (106) that

(1) $\mathrm{a}_1(wr_\gamma) = w(\bar{i}) - 1$, $\mathrm{a}_2(wr_\gamma) = w(\overline{i-1}) - 2$,

(2) $\mathrm{b}_1(wr_\gamma) = \begin{cases} 0 & \text{if (i) or (ii)},\\ 1 & \text{if (iii)}, \end{cases}$ $\mathrm{b}_2(wr_\gamma) = \begin{cases} 0 & \text{if (i)},\\ 1 & \text{if (ii) or (iii)}, \end{cases}$

(3) for $s \in [3, i]$, $\mathrm{a}_s(wr_\gamma) = \mathrm{a}_{s-2}(w)$ and $\mathrm{b}_s(wr_\gamma) = \mathrm{b}_{s-2}(w) - 2$,

(4) $\mathrm{a}_{i-1}(w) = \begin{cases} n-i & \text{if (i)},\\ n-i-1 & \text{if (ii) or (iii)}, \end{cases}$ $\mathrm{a}_i(w) = \begin{cases} n-i & \text{if (i) or (ii)},\\ n-i-1 & \text{if (iii)}, \end{cases}$

(5) $\mathrm{b}_{i-1}(w) = n - i + 1 - (w(\overline{i-1}) - 2)$ and $\mathrm{b}_i(w) = n - i - (w(\bar{i}) - 1)$.

Hence

$$(107)\qquad \ell(wr_\gamma) - \ell(w) = \begin{cases} 3 - 4n + 2i + 2(w(\bar{i}) - 1) + 2(w(\overline{i-1}) - 2) & \text{if (i)},\\ 5 - 4n + 2i + 2(w(\bar{i}) - 1) + 2(w(\overline{i-1}) - 2) & \text{if (ii)},\\ 7 - 4n + 2i + 2(w(\bar{i}) - 1) + 2(w(\overline{i-1}) - 2) & \text{if (iii)}; \end{cases}$$

note that $w(\bar{i}) - 1 \geq 0$ and $w(\overline{i-1}) - 2 \geq 0$. Since $\ell(wr_\gamma) - \ell(w) = 3 - 4n + 2i$, we have (i), $w(\bar{i}) = 1$, and $w(\overline{i-1}) = 2$. This implies (q-D2).

Lemma 4.56. *$n > 5$, $i \in [3, n-3]$, and* (q-D2) *imply* $\mathrm{Q}(i, w, \gamma)$.

Proof. Assume that $n > 5$, $i \in [3, n-3]$, and (q-D2) hold. By Lemmas 4.5 and 4.39–4.40, $\mathrm{Q}(i, w, \gamma)$ is equivalent to $\ell(wr_\gamma) - \ell(w) = 3 - 4n + 2i$. We check at once that (q-D2) yields $w(n) < w(i-1) < w(i)$ and (106). As in the proof of Lemma 4.55, we have $\ell(wr_\gamma) - \ell(w) = 3 - 4n + 2i + 2(w(\bar{1}) - 1) + 2(w(\overline{i-1}) - 2)$. Since $w(\bar{1}) = 1$ and $w(\overline{i-1}) = 2$, we conclude that $\ell(wr_\gamma) - \ell(w) = 3 - 4n + 2i$.

Lemma 4.57. *$n > 5$, $c_{n-2}(\gamma) = 2$, and $\mathrm{Q}(n-2, w, \gamma)$ imply* (q-D2).

Proof. Assume that $n > 5$ and $c_{n-2}(\gamma) = 2$, and that $\mathrm{Q}(n-2, w, \gamma)$ is true. By Lemmas 4.5 and 4.39–4.40, we have $\ell(wr_\gamma) - \ell(w) = -1 - 2n$. By Lemmas 4.4 and 4.39, we have $\gamma = \varepsilon_{n-3} + \varepsilon_{n-2} \in Q^{\vee, I \setminus \{n-2\}}$, $r_\gamma = (n-3\ \overline{n-2})(\overline{n-3}\ n-2)$, and $wr_\gamma = wr_\gamma(z_\gamma^{I \setminus \{n-2\}})^{-1}$. Let $I \setminus \{n-2\} = I_1 \sqcup I_2 \sqcup I_2'$, where $I_1 = [n-3]$ is of type A_{n-3}, $I_2 = \{n-1\}$ is of type A_1, and $I_2 = \{n\}$ is of type A_1. We see that $(n-4, 0, 0)$ $(I_1)_{\mathrm{af}} \times (I_2)_{\mathrm{af}} \times (I_2)_{\mathrm{af}}$ satisfies the condition for y Q in Lemma 4.1; note that I_1 r $\{n-4\}$ is of type $A_{n-5} \times A_1$. Hence $z_y^{I \text{ r } \{n-2\}} = w_0^{I_1} w_0^{I_1 \text{ r } \{n-4\}}$ is given by $u \to u+2$ for u $[n-4]$, $n-3 \to 1$, $n-2 \to 2$, $n-1 \to n-1$, and $n \to n$. Then wr_y is given by $1 \to w(\overline{n-2})$, $2 \to w(\overline{n-3})$, $u \to w(u-2)$ for u $[3, n-2]$, $n-1 \to w(n-1)$, and $n \to w(n)$. It follows from Lemma 4.35 that

$$\begin{aligned} (108) \qquad & w(\overline{n-2}) \quad w(\overline{n-3}) \quad w(1) \quad w(2) \quad \cdots \quad w(n-3) \quad w(n-2), \\ & w(n-1) \quad w(n) \quad w(\overline{n-1}). \end{aligned}$$

Analysis similar to that in the proof of Lemma 4.55 shows that $w(\overline{n-2}) = 1$ and $w(\overline{n-3}) = 2$. This implies (q-D2).

LEMMA 4.58. *$n > 5$, $i = n-2$, and* (q-D2) *imply* $\mathrm{Q}(n-2, w, y)$.

Proof. Assume that $n > 5$, $i = n-2$, and (q-D2) hold. By Lemmas 4.5 and 4.39–4.40, $\mathrm{Q}(n-2, w, y)$ is equivalent to $(wr_y) - (w) = -1-2n$. We see that $i = n-2$ and (q-D2) imply (108). Hence $\mathrm{a}_1(wr_y) = \mathrm{a}_2(wr_y) = \mathrm{b}_1(wr_y) = \mathrm{b}_2(wr_y) = 0$, $\mathrm{A}_s(wr_y) = \mathrm{A}_{s-2}(w)$ for s $[3, n-2]$, $\mathrm{A}_{n-3}(w) = \mathrm{A}_{n-2}(w) = \mathrm{B}_{n-2}(w) = \{n-1, n\}$, and $\mathrm{B}_{n-3}(w) = \{n-2, n-1, n\}$. Also, for s $[3, n-2]$, we see that $n-3, n-2$ $\mathrm{B}_{s-2}(w)$ and the map below is bijective.

$$(109) \qquad \mathrm{B}_s(wr_y) \to \mathrm{B}_{s-2}(w) \text{ r } \{n-3, n-2\}, \quad \begin{matrix} t \to t-2 & \text{if } t \quad [s+1, n-2], \\ t \to t & \text{if } t \quad \{n-1, n\}. \end{matrix}$$

This implies $\mathrm{b}_s(wr_y) = \mathrm{b}_{s-2}(w) - 2$ for s $[3, n-2]$. Combining these yields $(wr_y) - (w) = -1-2n$.

LEMMA 4.59. *$n = 4$, $c_2(y) = 2$, and $\mathrm{Q}(2, w, y)$ are equivalent to* (q-D2).

Proof. The assertion follows by the same method as in the proof of Lemmas 4.53–4.54; in this case, we have $w = (1\ \overline{2})(\overline{1}\ 2)$ and $wr_y = e$.

LEMMA 4.60. $\mathrm{Q}(n-1, w, y)$ *implies* (q-D3).

Proof. Assume that $\mathrm{Q}(n-1, w, y)$ is true. By Lemmas 4.4 and 4.39, we have $y = \varepsilon_{n-1} - \varepsilon_n$ $Q^{\vee, I \text{ r } \{n-1\}}$, $r_y = (n-1\ n)(\overline{n-1}\ \overline{n})$, and $wr_y = wr_y(z_y^{I \text{ r } \{n-1\}})^{-1}$. Set J r $\{n-1\}$; note that J is of type A_{n-1}. We see that $n-2$ J_{af} satisfies the condition for y Q in Lemma 4.1; note that J r $\{n-2\}$ is of type $A_{n-3} \times A_1$. Hence $z_y^{I \text{ r } \{n-1\}}$ is given by $u \to u+2$ for u $[n-3]$, $n-2 \to \overline{n}$, $n-1 \to 1$, and $n \to \overline{2}$. Then wr_y is given by $1 \to w(n)$, $2 \to w(\overline{n-1})$, $u \to w(u-2)$ for u $[3, n-1]$, and $n \to w(\overline{n-2})$. It follows from Lemma 4.35 that

$$(110) \qquad w(n) \quad w(\overline{n-1}) \quad w(1) \quad w(2) \quad \cdots \quad w(n-2) \quad w(n-1) \quad w(\overline{n}),$$

which implies $w(n) = 1$ and $w(n-1) = \overline{2}$. This completes the proof.

LEMMA 4.61. (q-D3) *implies* $\mathrm{Q}(n-1, w, y)$.

Proof. Assume that (q-D3) is true. By Lemmas 4.5 and 4.39–4.40, $\mathrm{Q}(n-1, w, y)$ is equivalent to $(wr_y) - (w) = 3-2n$. We check at once that (q-D3) implies (110). It follows that

(1) $\mathrm{a}_1(wr_y) = \mathrm{a}_2(wr_y) = \mathrm{b}_1(wr_y) = \mathrm{b}_2(wr_y) = 0$,

(2) for s $[3, n-1]$, $\mathrm{A}_s(wr_y = \begin{cases} ? & \text{if } w(s-2) \quad w(\overline{n-2}), \\ \{n\} & \text{if } w(s-2) \quad w(\overline{n-2}), \end{cases}$ $\mathrm{A}_{s-2}(w) = \{n\}$,

(3) for s $[3, n-1]$, $b_s(wr_y) - b_{s-2}(w) = \begin{cases} -1 & \text{if } w(s-2) \quad w(\overline{n-2}), \\ -2 & \text{if } w(s-2) \quad w(\overline{n-2}), \end{cases}$

(4) $A_{n-2}(w) = A_{n-1}(w) = \{n\}$, $B_{n-2}(w) = \{n-1\}$, $B_{n-1}(w) = ?$.

We give the proof only for (3). Let s $[3, n-1]$. We have $n-1$ $B_{s-2}(w)$ and n / $B_{s-2}(w)$. We see that $n-2$ $B_{s-2}(w)$ if and only if $w(s-2)$ $w(\overline{n-2})$. Then the map

$$B_s(wr_y) \to B_{s-2}(w) \text{ r } \{n-2, n-1\}, \; t \to t-2, \tag{111}$$

is bijective. Combining these yields (3). By (1)–(4) above, we have $(wr_y) - (w) = 3 - 2n$.

Lemma 4.62. $Q(n, w, y)$ *implies* (q-D4).

Proof. Assume that $Q(n, w, y)$ is true. By Lemmas 4.4 and 4.39, we have $y = \varepsilon_{n-1} + \varepsilon_n$ $Q^{,I \text{ r } \{n\}}$, $r_y = (n-1\ \overline{n})(\overline{n-1}\ n)$, and $wr_y = wr_y(z_y^{I \text{ r } \{n\}})^{-1}$. Set $J = I$ r $\{n\} = [n-1]$; note that J is of type A_{n-1}. We see that $n-2$ J_{af} satisfies the condition for y Q in Lemma 4.1; note that J r $\{n-2\}$ is of type $A_{n-3} \times A_1$. Hence $z_y^{I \text{ r } \{n\}}$ is given by $u \to u+2$ for u $[n-2]$, $n-1 \to 1$, and $n \to 2$. Then wr_y is given by $1 \to w(\overline{n})$, $2 \to w(\overline{n-1})$, and $u \to w(u-2)$ for u $[3, n]$. It follows from Lemma 4.35 that

$$w(\overline{n}) \quad w(\overline{n-1}) \quad w(1) \quad w(2) \quad \cdots \quad w(n-2) \quad w(n-1) \quad w(n), \tag{112}$$

which implies $w(n) = \overline{1}$ and $w(n-1) = \overline{2}$. This completes the proof.

Lemma 4.63. (q-D4) *implies* $Q(n, w, y)$.

Proof. Assume that (q-D4) is true. By Lemmas 4.5 and 4.39–4.40, $Q(n, w, y)$ is equivalent to $(wr_y) - (w) = 3 - 2n$. We check at once that (q-D4) implies (112). Hence $A_s(wr_y) = A_s(w) = ?$ for s $[n]$, $B_1(wr_y) = B_2(wr_y) = B_n(w) = ?$, $B_{n-1}(w) = \{n\}$, and the map $B_s(wr_y) \to B_{s-2}(w)$ r $\{n-1, n\}$, $t \to t-2$, is bijective for s $[3, n]$. Since $n-1, n$ $B_{s-2}(w)$ for s $[3, n]$, we have $b_s(wr_y) = b_{s-2}(w) - 2$ for s $[3, n]$. It follows that $(wr_y) - (w) = 3 - 2n$.

Proof of Proposition 4.37. (1) and (3)–(4) follow by the same method as in the proof of Proposition 4.8.

We prove (2). We can prove the assertion for i $\{1, n-1, n\}$ by a similar argument to the proof of Proposition 4.13. Assume that i $[2, n-2]$. Let x, y $(W^{I \text{ r } \{i\}})_{\text{af}}$, $Y_i^{D_n}(x) = (T, c)$, and $Y_i^{D_n}(y) = (T, c)$. It follows immediately from Propositions 4.41–4.43 that x y implies c 6 c and $T(u)$ $T(u + c - c)$ for u $[i - c + c]$. Hence we may assume that $d := c - c > 0$ and $T(u)$ $T(u + d)$ for u $[i - d]$. If d is even, then the assertion follows by the same method as in Step 1 of the proof of Proposition 4.21. In what follows, assume that $d > 1$ is odd. Write $a = \min([n] \text{ r } \{ T(u) \mid u \ [i], T(u) \ \overline{n}\})$ and $b = \min([n] \text{ r } \{ T(u) \mid u \ [i]\})$ (see (98) in Definition 4.36 (1)). Let $\{a_1 < a_2 < \cdots < a_{n-i+1}\} = [n]$ r $\{ T(u) \mid u$ $[i-1]\}$. We divide the proof into four steps.

Step 1. We show that $T(i)$ n implies x y. Define T_1, T_2 $\mathrm{CST}_{D_n}(\ _i)$ by

$$T_1(u) = \begin{cases} u+1 & \text{if } u \quad [a_2 - 2], \\ T(u) & \text{if } u \quad [a_2 - 1, i-1], \\ \overline{a_2} & \text{if } u = i, \end{cases} \tag{113}$$

$$T_2(u) = \begin{cases} u & \text{if } u \quad [a_2 - 1], \\ T(u-1) & \text{if } u \quad [a_2, i]. \end{cases} \tag{114}$$

Let $x_1, x_2 \quad (W^{I \text{ r } \{i\}})_{\mathrm{af}}$ be such that $Y_i^{D_n}(x_1) = (\mathrm{T}_1, c)$ and $Y_i^{D_n}(x_2) = (\mathrm{T}_2, c+1)$. Since $\mathrm{T}(u) \quad \mathrm{T}_1(u)$ for $u \quad [i]$, we have $x \quad x_1$ by the assertion for $d = 0$. We have $x_1 \quad x_2$ by Proposition 4.43 ($\frac{\infty}{2}$-D8). We have $x_2 \quad y$ because $c \ - (c+1) = d - 1 > 0$ is even and $\mathrm{T}_2(u) \quad \mathrm{T}\ (u + d - 1)$ for $u \quad [i - d + 1]$. Combining these we conclude that $x \quad y$.

Step 2. We show that ($\mathrm{T}(i) \quad \overline{n}$, $a = b$, and $d > i$) or ($\mathrm{T}(i) \quad \overline{n}$, $a = b$, $d \quad [i]$, and $a \quad \mathrm{T}\ (d)$) imply $x \quad y$. Assume that $\mathrm{T}(i) \quad \overline{n}$ and $a = b$; note that $a_1 = 1$ holds. We claim that $a \quad \mathrm{T}(1)$. Indeed, if $\mathrm{T}(1) \quad \overline{n}$, then $a \quad \mathrm{T}(1)$. If $\mathrm{T}(1) \quad n$, then $\mathrm{T}(1) \quad [n] \text{ r } \{ \ \mathrm{T}(u) \ \mid u \quad [i], \ \mathrm{T}(u) \quad \overline{n}\}$ and hence $a \quad \mathrm{T}(1)$ by minimality of a. Since $a = b = \ \mathrm{T}(u)$ for $u \quad [i]$, we conclude that $a \quad \mathrm{T}(1)$. Thus $\mathrm{T}_2 \quad \mathrm{CST}_{D_n}(\ _i)$ below is well-defined.

$$\mathrm{T}_1(u) = \begin{cases} \mathrm{T}(u) & \text{if } u \quad [i-1], \\ \max\{\overline{a_2}, \mathrm{T}(i)\} & \text{if } u = i, \end{cases} \tag{115}$$

$$\mathrm{T}_2(u) = \begin{cases} a & \text{if } u = 1, \\ \mathrm{T}(u-1) & \text{if } u \quad [2, i]. \end{cases} \tag{116}$$

Let $x_1, x_2 \quad (W^{I \text{ r } \{i\}})_{\mathrm{af}}$ be such that $Y_i^{D_n}(x_1) = (\mathrm{T}_1, c)$ and $Y_i^{D_n}(x_2) = (\mathrm{T}_2, c+1)$. Write $\{b_1 < b_2 < \cdots < b_{n-i+1}\} = [n] \text{ r } \{ \ \mathrm{T}_1(u) \ \mid u \quad [i-1]\}$; note that $a_\nu = b_\nu$ for $\nu \quad [n-i+1]$. Since $\mathrm{T}(u) \quad \mathrm{T}_1(u)$ for $u \quad [i]$, we have $x \quad x_1$ by the assertion for $d = 0$. We claim that $x_1 \quad x_2$. Indeed, if $\ \mathrm{T}(i) \ = a_1 = 1$, then $\mathrm{T}_1(i) = \overline{1}$ and $\mathrm{T}_2(1) = a = a_2$, which imply $b_1 = \ \mathrm{T}_1(i) \ = 1$ and $b_2 = \mathrm{T}_2(1)$. If $a_2 \ 6 \ \ \mathrm{T}(i) \$, then $\mathrm{T}_1(i) = \overline{a_2}$ and $a = 1$, which imply $b_1 = \mathrm{T}_2(1) = 1$ and $b_2 = \ \mathrm{T}_1(i) \$. Proposition 4.43 ($\frac{\infty}{2}$-D8) now yields $x_1 \quad x_2$ as claimed. We claim that $x_2 \quad y$. Indeed, $c \ - (c+1) = d - 1 > 0$ is even. If $d > i$, then the condition (97) is trivial, and hence $x_2 \quad y$. If $d \quad [i]$ and $a \quad \mathrm{T}\ (d)$, then $\mathrm{T}_2(1) = a \quad \mathrm{T}\ (d) = \mathrm{T}\ (1 + (d-1))$, $\mathrm{T}_2(u) = \mathrm{T}(u-1) \quad \mathrm{T}\ (u + (d-1))$ for $u \quad [2, i - d + 1]$, and hence $x_2 \quad y$ as claimed. Combining these we conclude that $x \quad y$.

Step 3. Assume that $\mathrm{T}(i) \quad \overline{n}$ and $a < b$; note that $\mathrm{T}(1) = a$ holds. Let $k \quad [i]$ be such that $\mathrm{T}(k) \quad b \quad \mathrm{T}(k+1)$. In this case, we show that the condition (iii) in Definition 4.36 (1) implies $x \quad y$. Define $\mathrm{T}_1, \mathrm{T}_2 \quad \mathrm{CST}_{D_n}(\ _i)$ by

$$\mathrm{T}_1(u) = \begin{cases} \mathrm{T}(u+1) & \text{if } u \quad [k-1], \\ b & \text{if } u = k, \\ \mathrm{T}(u) & \text{if } u \quad [k+1, i-1], \\ \max\{\overline{a_2}, \mathrm{T}(i)\} & \text{if } u = i, \end{cases} \tag{117}$$

$$\mathrm{T}_2(u) = \begin{cases} a & \text{if } u = 1, \\ \mathrm{T}_1(u-1) & \text{if } u \quad [2, i]. \end{cases} \tag{118}$$

If $k = 1$, then $\mathrm{T}_2(2) = \mathrm{T}_1(1) = b \quad a$. If $k \quad [2, i]$, then $\mathrm{T}_2(2) = \mathrm{T}_1(1) = \mathrm{T}(2) \quad \mathrm{T}(1) = a$. Therefore T_2 is well-defined. Let $x_1, x_2 \quad (W^{I \text{ r } \{i\}})_{\mathrm{af}}$ be such that $Y_i^{D_n}(x_1) = (\mathrm{T}_1, c)$ and $Y_i^{D_n}(x_2) = (\mathrm{T}_2, c+1)$. Write $\{b_1 < b_2 < \cdots < b_{n-i+1}\} = [n] \text{ r } \{ \ \mathrm{T}_1(u) \ \mid u \quad [i-1]\}$. Since $\mathrm{T}(u) \quad \mathrm{T}_1(u)$ for $u \quad [i]$, we have $x \quad x_1$ by the assertion for $d = 0$. We claim that $x_1 \quad x_2$. Indeed, if $\ \mathrm{T}(i) \ = a_1 = 1$, then $b_1 = \ \mathrm{T}_1(i) \ = 1$ and $b_2 = \mathrm{T}_2(1)$. If $\ \mathrm{T}(i) \ = a_1 > 1$, then $\mathrm{T}_1(i) = \mathrm{T}(i)$, $a = 1$, and $b = a_2$, which imply $b_1 = \mathrm{T}_2(1) = 1$ and $b_2 = \ \mathrm{T}_1(i) \$. If $\ \mathrm{T}(i) \ > a_2$, then $\mathrm{T}_1(i) = \overline{a_2}$, $a = 1$, and $b = a_1$, which imply $b_1 = \mathrm{T}_2(1) = 1$ and $b_2 = \ \mathrm{T}_1(i) \$. Proposition 4.43 ($\frac{\infty}{2}$-D8) now yields $x_1 \quad x_2$ as claimed. We claim that $x_2 \quad y$. Indeed, $c \ - (c+1) = d - 1 > 0$ is even. If $d > i$, then the condition (97) is trivial, and hence $x_2 \quad y$. If $d \quad [i]$, $a \quad \mathrm{T}\ (d)$, ($\mathrm{T}(u) \quad \mathrm{T}\ (u + d - 1)$ for $u \quad [2, \min\{k, i - d + 1\}]$) and ($b \quad \mathrm{T}\ (k + d)$ if $k \quad [i - d]$), then $T_2(1) = a \quad T(d) = T(1 + (d-1))$, $T_2(u) = T_1(u-1) = T(u) \quad T(u + (d-1))$ for $u \quad [2, \min\{k, i-d+1\}]$, $T_2(k+1) = b \quad T(k+d) = T(k+1+(d-1))$ if $k \quad [i-d]$, $T_2(u) = T(u-1) \quad T(u-1+d) = T(u+(d-1))$ for $u \quad [k+2, i-d+1]$, and hence $x_2 \quad y$ as claimed. Combining these we conclude that $x \quad y$.

Step 4. We show that $x \quad y$ and $T(i) \quad \overline{n}$ imply $(T, c) \quad (T, c)$. By Proposition 4.37 (3), we may assume that $d \quad [i]$. Let $T_1, T_2 \quad CST_{D_n}(\quad_i)$ and $x_1, x_2 \quad (W^{I\, r\, \{i\}})_{af}$ be as in Steps 2–3. By the arguments in Steps 2–3, the edge $x_1 \to x_2$ is of type $(\frac{\infty}{2}$-D8). Note that $(T, c) \quad (T, c)$ is equivalent to $T_2(u) \quad T(u+d-1)$ for $u \quad [i-d+1]$ (see (iii) in Definition 4.36 (1)). Since d is odd, we see from Proposition 4.43 that there exists an edge $x_3 \xrightarrow{\beta} x_4$, $x_3, x_4 \quad (W^{I\, r\, \{i\}})_{af}$, $\beta \quad \Delta^+_{af}$, in $SiB^{I\, r\, \{i\}}$ of type $(\frac{\infty}{2}$-D8) such that $x \quad x_3$ and $x_4 \quad y$; we may assume that there is no edges of type $(\frac{\infty}{2}$-D8) in a directed path from x to x_3 in $SiB^{I\, r\, \{i\}}$. Write $Y_i^{D_n}(x_3) = (T_3, c)$ and $Y_i^{D_n}(x_4) = (T_4, c+1)$. We see from Proposition 4.43 that $c - c > 0$ and $c - (c+1) > 0$ are both even. Set $d = c - c$ and $d = c - (c+1)$; note that $d + d + 1 = d$ and $d, d \quad [0, d-1]$. Since $x_4 \quad y$, we have $T_4(u) \quad T(u+d)$ for $u \quad [i-d]$. Hence $(T_2(u) \quad T(u+d-1)$ for $u \quad [i-d+1])$ follows from $(T_2(u) \quad T_4(u+d)$ for $u \quad [i-d])$.

We first claim that $T_2(1) \quad T_4(1)$ if $d = 0$. Assume that $d = 0$. We first assume that $T(i) \quad \overline{1}$. Then $T_1(i) \quad \overline{1}$ and $T_2(1) = 1$ by $(\frac{\infty}{2}$-D8). Hence $T_2(1) \quad T_4(1)$. We next assume that $T(i) = \overline{1}$. Then $T_3(i) = \overline{1}$, because $x \quad x_3$ and $d = 0$. If we write $\{c_1 < c_2 < \cdots < c_{n-i+1}\} = [n]\ r\ \{\ T_3(u) \quad | u \quad [i-1]\}$, then $c_1 = 1$ and $T_4(1) = c_2$ by $(\frac{\infty}{2}$-D8). Also, we have $T_2(1) = a = a_2$ by $(\frac{\infty}{2}$-D8); note that if $a < b$, then $a_2 = b_2$, where b_2 is as in Step 3, because $T(1) = a$ and we assume that $T(i) = \overline{1}$. By the definition of a, $\{T(u) \mid u \quad [i-a_2+1, i]\} = \{\overline{1}, \overline{2}, \ldots, \overline{a_2-1}\}$. Hence $\{T_3(u) \mid u \quad [i-a_2+1, i]\} = \{\overline{1}, \overline{2}, \ldots, \overline{a_2-1}\}$, because $x \quad x_3$ and $d = 0$. This implies a_2 6 c_2. Therefore $T_2(1) \quad T_4(1)$.

We next claim that $T_2(1) \quad T_4(1+d)$ if $d = 0$. Assume that $d \quad [d-1]$ or $d = 0$. Set

$$
(119) \qquad \begin{aligned} \{p_1 < p_2 < \cdots < p_\mu\} &= [b-1]\ r\ \{\ T(u) \quad | u \quad [i-d],\ T(u) \quad \overline{n}\}, \\ \{q_1 < q_2 < \cdots < q_\nu\} &= [b-1]\ r\ \{\ T_3(u) \quad | u \quad [i],\ T_3(u) \quad \overline{n}\}. \end{aligned}
$$

If $[b-1]\ r\ \{\ T_3(u) \quad | u \quad [i],\ T_3(u) \quad \overline{n}\} = ?$, then $b \quad T_3(u)$ for $u \quad [i]$. In particular, we have $T_2(1) = a \quad b \quad T_3(d) = T_4(1+d)$ if $d = 0$, which is our claim. Therefore we may assume that $\nu > 1$. We see from $(\frac{\infty}{2}$-D8) that $q_1 \quad T_4(1) \quad T_4(2) = T_3(1)$, which implies $q_u \quad T_3(u)$ for $u \quad [\nu]$. In particular, we have $b \quad T_3(u)$ for $u \quad [\nu, i]$. If $d \quad [\nu, i-1]$, then $T_2(1) = a \quad b \quad T_3(d) = T_4(1+d)$, which is our claim. Therefore we may assume that $\nu > 2$ and $d \quad [0, \nu-1]$. Then $(q_u \quad T_3(u)$ for $u \quad [\nu])$ gives $(q_{u+1} \quad T_3(u)$ for $u \quad [\nu-1])$. By minimality of b, we have $[b-1] \quad \{\ T(u) \quad | u \quad [i]\}$. Hence there exists $l \quad [0, d]$ such that

$$
(120) \qquad \{p_1 < p_2 < \cdots < p_\mu\} = \{T(u) \mid u \quad [\mu - l]\} \quad \{\ T(u) \quad | u \quad [i-l+1, i]\},
$$

which implies $T(u) \quad p_{u+l}$ for $u \quad [\mu - l]$. Since l 6 d, we have $T(u) \quad p_{u+d}$ for $u \quad [\mu - d]$. Since $x \quad x_3$, we have $T(u) > T_3(u+d)$ if $u \quad [i-d]$ and $T(u) \quad \overline{n}$. Hence $\mu > \nu$ and p_u 6 q_u for $u \quad [\nu]$. Combining these gives

$$
(121) \quad T(u) \quad p_{u+d} \quad q_{u+d} \quad T_3(u+d-1) = T_4(u+d) \text{ for } u \quad [\nu - d] \text{ if } d = 0.
$$

In particular, $T_2(1) = a \quad T(1) \quad T_4(1+d)$ if $d = 0$, which is our claim. Similarly, we have

$$
(122) \qquad T(u+1) \quad T_3(u) \text{ for } u \quad [\nu - 1] \text{ if } d = 0.
$$

We next claim that $\mathrm{T}_1(u) \preceq \mathrm{T}_3(u+d)$ for $u \in [i-d-1]$. If $u \in [i-d-1]$ satisfies $\mathrm{T}_1(u) = \mathrm{T}(u)$, then $\mathrm{T}_1(u) \preceq \mathrm{T}_3(u+d)$ since $x \preceq x_3$. It remains to prove that $\mathrm{T}_1(u) \preceq \mathrm{T}_3(u+d)$ for $u \in [k] \cap [i-d-1]$ under the assumption that $a < b$, where $k \in [i]$ is as in Step 3. Recall that $b \preceq \mathrm{T}_3(u)$ for $u \in [\max\{1, \nu\}, i]$. If $\nu \in [0,1]$, then $\mathrm{T}_1(u) \preceq b \preceq \mathrm{T}_3(u+d)$ for $u \in [k] \cap [i-d-1]$. Therefore we may assume that $\nu > 2$. We first assume that $d \in [d-1]$. If $u \in [k-1] \cap [\nu-d]$, then $\mathrm{T}_1(u) = \mathrm{T}(u+1) \preceq \mathrm{T}_3(u+d)$ by (121). If $k \in [\nu-d-1]$, then $\mathrm{T}_1(k) = b \preceq \mathrm{T}(k+1) \preceq \mathrm{T}_3(k+d)$ by (121). If $k = \nu - d$, then $\mathrm{T}_1(k) = b \preceq \mathrm{T}_3(\nu) = \mathrm{T}_3(k+d)$. If $u \in [\nu-d+1, k]$, then $\mathrm{T}_1(u) \preceq b \preceq \mathrm{T}_3(u+d)$. We next assume that $d = 0$. If $u \in [k-1] \cap [\nu-1]$, then $\mathrm{T}_1(u) = \mathrm{T}(u+1) \preceq \mathrm{T}_3(u)$ by (122). If $k \in [\nu-1]$, then $\mathrm{T}_1(k) = b \preceq \mathrm{T}(k+1) \preceq \mathrm{T}_3(k)$ by (122). If $u \in [\nu, k]$, then $\mathrm{T}_1(u) \preceq b \preceq \mathrm{T}_3(u)$.

We finally claim that $\mathrm{T}_2(u) \preceq \mathrm{T}_4(u+d)$ for $u \in [2, i-d]$. Let $u \in [2, i-d]$; note that $u-1 \in [i-d-1]$. By the above, we have $\mathrm{T}_2(u) = \mathrm{T}_1(u-1) \preceq \mathrm{T}_3(u-1+d) = \mathrm{T}_4(u+d)$.

The proof of Proposition 4.37 is complete.

5. Tableau model for crystal bases of level-zero representations

In this section, we apply the results in §4 to crystal bases of level-zero representations of $\mathbf{U}$ of type $B_n^{(1)}$, $C_n^{(1)}$, and $D_n^{(1)}$. We introduce quantum Kashiwara–Nakashima columns (see Definitions 5.15 and 5.20) and semi-infinite Kashiwara–Nakashima tableaux (see Definitions 5.12, 5.17, and 5.22). We will see that these tableaux give combinatorial models for crystal bases of level-zero fundamental representations and level-zero extremal weight modules. When $\mathbf{U}$ is of type $B_n^{(1)}$ or $D_n^{(1)}$, we give an explicit description of the crystal isomorphisms among three different realizations of the crystal basis of a level-zero fundamental representation by quantum Lakshmibai–Seshadri paths (see §5.1), quantum Kashiwara–Nakashima columns, and (ordinary) Kashiwara–Nakashima columns.

5.1. Quantum Lakshmibai–Seshadri paths.

In this subsection, we give a brief exposition of quantum Lakshmibai–Seshadri paths (see [25] for details). Assume that $\mathbf{U}$ is of untwisted affine type.

Let $\lambda \in P^+$. Recall the notation J_λ in (29). For a rational number $0 < a \leqslant 1$, define $\mathrm{QB}(\lambda; a)$ to be the subgraph of QB^{J_λ} with the same vertex set but having only the edges of the form

$$w \xrightarrow{\gamma} v \text{ with } a\langle \gamma^\vee, \lambda\rangle \in \mathbb{Z}; \tag{123}$$

note that $\mathrm{QB}(\lambda; 1) = \mathrm{QB}^{J_\lambda}$. A quantum Lakshmibai–Seshadri path of shape λ is, by definition, a pair $(\boldsymbol{w}; \boldsymbol{a})$ of a sequence $\boldsymbol{w} : w_1, w_2, \dots, w_l$ of elements in W^{J_λ} and an increasing sequence $\boldsymbol{a} : 0 = a_0 < a_1 < \cdots < a_l = 1$ of rational numbers such that there exists a directed path from w_{u+1} to w_u in $\mathrm{QB}(\lambda; a_u)$ for $u \in [l-1]$. Let $\mathrm{QLS}(\lambda)$ denote the set of quantum Lakshmibai–Seshadri paths of shape λ. We call an element $(\boldsymbol{w}; \boldsymbol{a}) \in \mathrm{QLS}(\lambda)$ a Lakshmibai–Seshadri path of shape λ if there exists a directed path from w_{u+1} to w_u in $\mathrm{QB}(\lambda; a_u)$ not having quantum edges for $u \in [l-1]$. Let $\mathrm{LS}(\lambda)$ denote the set of Lakshmibai–Seshadri paths of shape λ.

In the same manner as in §2.5 we can define maps $\mathrm{wt} : \mathrm{QLS}(\lambda) \to P$, $e_i, f_i : \mathrm{QLS}(\lambda) \to \mathrm{QLS}(\lambda) \sqcup \{\mathbf{0}\}$, and $\varepsilon_i, \phi_i : \mathrm{QLS}(\lambda) \to \mathbb{Z}_{\geqslant 0}$ for $i \in I_{\mathrm{af}}$.

Theorem 5.1 ([12, 15, 27, 25]). *Let* $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$.

(1) *The set* $\mathrm{QLS}(\lambda)$ *equipped with the maps* $\mathrm{wt}, e_i, f_i, \varepsilon_i, \phi_i$, $i \in I_{\mathrm{af}}$, *is a* $\mathbf{U}'$-*crystal. The* $\mathbf{U}'$-*crystal* $\mathrm{QLS}(\lambda)$ *is isomorphic to the crystal basis of the tensor product* $\bigotimes_{i \in I} W(\varpi_i)^{\otimes m_i}$ *of level-zero fundamental representations* (*see* §2.4).

(2) *The set* $\mathrm{LS}(\lambda)$ *equipped with the maps* $\mathrm{wt}, e_i, f_i, \varepsilon_i, \phi_i$, $i \in I$, *is a* $\mathfrak{g}$-*crystal. The* $\mathfrak{g}$-*crystal* $\mathrm{LS}(\lambda)$ *is isomorphic to the crystal basis of the integrable highest weight module of highest weight* λ *over the quantized universal enveloping algebra associated with* $\mathfrak{g}$.

5.2. Quantum Bruhat graphs and Maya diagrams. Throughout this subsection, we assume that $\mathbf{U}$ is of type $B_n^{(1)}$, $C_n^{(1)}$, or $D_n^{(1)}$. The aim of this subsection is to give descriptions of $\mathrm{QLS}(\varpi_i)$ and $\mathrm{LS}(\varpi_i)$ in terms of Maya diagrams.

Let $i \in I$. Note that $J_{\varpi_i} = I \setminus \{i\}$. We see from $c_i(\gamma^\vee) = \langle \gamma^\vee, \varpi_i \rangle \in \{0, 1, 2\}$ for $\gamma \in \Delta^+$ that the graph $\mathrm{QB}(\varpi_i; a)$ has at least one edge only if $a \in \{1/2, 1\}$. Hence each element of $\mathrm{QLS}(\varpi_i)$ is of the form $(w; 0, 1)$ or $(v, w; 0, 1/2, 1)$ for some $w, v \in W^{I \setminus \{i\}}$. For simplicity of notation, we write $(w, w) = (w; 0, 1)$ and $(v, w) = (v, w; 0, 1/2, 1)$. The next lemma follows immediately from the results in §4 (see Propositions 4.17, 4.25–4.26, and 4.41–4.42).

Lemma 5.2. (1) *If* $\mathbf{U}$ *is of type* $C_n^{(1)}$, *then* $\mathrm{QLS}(\varpi_i) = \mathrm{LS}(\varpi_i)$.

(2) *If* ($\mathbf{U}$ *is of type* $B_n^{(1)}$ *and* $i = n$), ($\mathbf{U}$ *is of type* $C_n^{(1)}$ *and* $i = 1$), *or* ($\mathbf{U}$ *is of type* $D_n^{(1)}$ *and* $i \in \{1, n-1, n\}$), *then* $w = v$ *for all* $(w, v) \in \mathrm{QLS}(\varpi_i)$, $\mathrm{QLS}(\varpi_i) = \mathrm{LS}(\varpi_i)$, *and* $W^{I \setminus \{i\}} \to \mathrm{QLS}(\varpi_i)$, $w \mapsto (w; 0, 1)$, *is bijective.*

We say that $i \in I$ is minuscule if $\langle \gamma^\vee, \varpi_i \rangle \in \{0, 1\}$ for all $\gamma \in \Delta^+$; we see that $i \in I$ is minuscule if and only if it satisfies the assumption of Lemma 5.2 (2).

Unless otherwise stated we assume that $i \in I$ is not minuscule. We call a subset $J \subset [n]$ a segment if there exist $j, k \in [n]$ such that $j \leq k$ and $J = [j, k]$. For segments $J = [j, k]$ and $J' = [j', k']$, we write $J < J'$ if $k + 1 < j'$. Let S_i be the family of all sequences $(J_1 < \cdots < J_\mu)$, $\mu \geq 1$, of segments such that $\sum_{s=1}^{\mu} \# J_s = i$. It is easy to check that for $w \in W^{I \setminus \{i\}}$ there exists a unique $J(w) = (J_1 < \cdots < J_\mu) \in S_i$ such that $\{ |w(u)| \mid u \in [i]\} = J_1 \sqcup \cdots \sqcup J_\mu$. For $J = (J_1 < \cdots < J_\mu) \in S_i$, set $W^{I \setminus \{i\}}[J] = \{w \in W^{I \setminus \{i\}} \mid J(w) = J\}$. It follows that

$$W^{I \setminus \{i\}} = \bigsqcup_{J \in S_i} W^{I \setminus \{i\}}[J]. \tag{124}$$

Let 2^{J_ν} denote the power set of J_ν for $\nu \in [\mu]$. We call an element of $\prod_{\nu=1}^{\mu} 2^{J_\nu}$ a Maya diagram. We see that the next map is bijective.

$$\begin{aligned} M : W^{I \setminus \{i\}}[J] &\to \prod_{\nu=1}^{\mu} 2^{J_\nu}, \\ w &\mapsto M(w) = (J_\nu \cap \{ -w(u) \mid u \in [i],\ w(u) \in [\overline{n}]\})_{\nu=1}^{\mu}. \end{aligned} \tag{125}$$

The next lemma follows immediately from the results in §4 (see (b-C3), (b-B3), (b-B5), (q-B3), (b-D4) and (q-D2) in Propositions 4.17, 4.25–4.26, and 4.41–4.42).

Lemma 5.3. *Assume that* $i \in I$ *is not minuscule. Let* $w, v \in W^{I \setminus \{i\}}$. *If* $(v, w) \in \mathrm{QLS}(\varpi_i)$, *then* $J(w) = J(v)$.

Let $J \in S_i$. Define $\mathrm{QB}(\varpi_i; 1/2)[J]$ to be the induced subgraph of $\mathrm{QB}(\varpi_i; 1/2)$ with the vertex set $W^{I \setminus \{i\}}[J]$. It follows from Lemma 5.3 that

$$\mathrm{QB}(\varpi_i; 1/2) = \bigsqcup_{J \in S_i} \mathrm{QB}(\varpi_i; 1/2)[J]. \tag{126}$$

Let $J \subset [n]$ be a segment. We concern the following conditions for Maya diagrams $M, N \in 2^J$.

(M1) There exists $j \in [n-1]$ such that $j \in N$, $j+1 \in M$, and $M \setminus \{j+1\} = N \setminus \{j\}$.
(M2) $n \in N$ and $M = N \setminus \{n\}$.
(M3) $n-1, n \in N$ and $M = N \setminus \{n-1, n\}$.
(M4) $1, 2 \in M$ and $N = M \setminus \{1, 2\}$.

Definition 5.4. *Assume that $i \in I$ is not minuscule. Let $J \subset [n]$ be a segment such that $\#J \leqslant i$. For W of type B_n, C_n, or D_n, define $\mathrm{M}(\varpi_i; 1/2)[J]$ to be the directed graph whose vertex set is 2^J and edges are given as follows. Let $M, N \in 2^J$.*

1. *Assume that W is of type B_n. There exists an edge $M \to N$ if (M1), (M2), or (M4) hold.*
2. *Assume that W is of type C_n. There exists an edge $M \to N$ if (M1) holds.*
3. *Assume that W is of type D_n. There exists an edge $M \to N$ if (M1), (M3), or (M4) hold.*

Write $M \trianglelefteq N$ if there exists a directed path from M to N in $\mathrm{M}(\varpi_i; 1/2)[J]$. Write $M \trianglelefteq' N$ if there exists a directed path from M to N in $\mathrm{M}(\varpi_i; 1/2)[J]$ not having edges of type (M4). We see that $\trianglelefteq$ and $\trianglelefteq'$ define partial orders on 2^J; note that if (W is of type C_n) or ($\{1, 2\} \not\subset J$), then $\trianglelefteq$ is identical to $\trianglelefteq'$.

The next lemma is an easy consequence of the definition of partial orders $\trianglelefteq$ and $\trianglelefteq'$ on 2^J.

Lemma 5.5. *Assume that $i \in I$ is not minuscule. Let $J \subset [n]$ be a segment such that $\#J \leqslant i$. Let $M, N \in 2^J$. Write $M = \{m_1 < m_2 < \cdots < m_r\}$ and $N = \{n_1 < n_2 < \cdots < n_s\}$.*

1. *Assume that (W is of type C_n), (W is of type B_n, $n \notin J$, and $\{1, 2\} \not\subset J$), or (W is of type D_n and $\{1, 2\} \not\subset J$). We have $M \trianglelefteq N$ if and only if $r = s$ and $m_\nu > n_\nu$ for $\nu \in [r]$.*
2. *Assume that W is of type B_n and $n \in J$. We have $M \trianglelefteq N$ if and only if $r \leqslant s$ and $m_\nu > n_\nu$ for $\nu \in [r]$.*
3. *Assume that W is of type D_n and $n-1, n \in J$. We have $M \trianglelefteq N$ if and only if $s - r \in 2\mathbb{Z}_{>0}$ and $m_\nu > n_\nu$ for $\nu \in [r]$.*
4. *Assume that (W is of type B_n, $n \notin J$, and $\{1, 2\} \subset J$) or (W is of type D_n, $n-1, n \notin J$, and $\{1, 2\} \subset J$). We have $M \trianglelefteq N$ if and only if $r - s \in 2\mathbb{Z}_{>0}$ and $m_{r-\nu} > n_{s-\nu}$ for $\nu \in [0, s-1]$. We have $M \trianglelefteq' N$ if and only if $r = s$ and $m_\nu > n_\nu$ for $\nu \in [r]$.*

Definition 5.6. *Assume that $i \in I$ is not minuscule. Let $J = (J_1 < \cdots < J_\mu) \in S_i$. Define $\mathrm{M}(\varpi_i; 1/2)[J]$ to be the directed graph whose vertex set is $\bigsqcup_{\nu=1}^{\mu} 2^{J_\nu}$ and edges are given as follows: for $(M_\nu), (N_\nu) \in \bigsqcup_{\nu=1}^{\mu} 2^{J_\nu}$, set $(M_\nu) \to (N_\nu)$ if there exists $s \in [\mu]$ such that $M_s \to N_s$ in $\mathrm{M}(\varpi_i; 1/2)[J_s]$ and $M_t = N_t$ for $t \in [\mu] \setminus \{s\}$. Set $(M_\nu) \trianglelefteq (N_\nu)$ (resp. $(M_\nu) \trianglelefteq' (N_\nu)$) if $M_\nu \trianglelefteq N_\nu$ (resp. $M_\nu \trianglelefteq' N_\nu$) for all $\nu \in [\mu]$.*

The next lemma is an immediate consequence of (b-C3), (b-B3), (b-B5), (q-B3), (b-D4), and (q-D2) in Propositions 4.17, 4.25–4.26, and 4.41–4.42.

Lemma 5.7. *Assume that $i \in I$ is not minuscule. Let $J = (J_1 < \cdots < J_\mu) \in S_i$. The map (125) induces an isomorphism*

$$\mathrm{QB}(\varpi_i; 1/2)[J] \xrightarrow{\cong} \mathrm{M}(\varpi_i; 1/2)[J] \tag{127}$$

of directed graphs.

We see that an edge in $\mathrm{M}(\varpi_i; 1/2)[J]$ of type (M1)–(M3) (resp. (M4)) corresponds to a Bruhat edge (resp. a quantum edge) in $\mathrm{QB}^{I \setminus \{i\}}$. By Lemma 5.7, we can define a partial order $\trianglelefteq$ (resp. $\trianglelefteq'$) on $W^{I \setminus \{i\}}$ as follows. Let $w, v \in W^{I \setminus \{i\}}$. If $J(w) = J(v)$, then w and v are incomparable to each other. Assume that $J(w) = J(v)$. Recall the map M in (125). Set $w \trianglelefteq v$ (resp. $w \trianglelefteq' v$) if $M(w) \trianglelefteq M(v)$ (resp. $M(w) \trianglelefteq' M(v)$). We have thus proved that

(128) $$\mathrm{QLS}(\varpi_i) = \{(v,w) \mid w, v \in W^{I \setminus \{i\}},\ w \trianglelefteq v\},$$

(129) $$\mathrm{LS}(\varpi_i) = \{(v,w) \mid w, v \in W^{I \setminus \{i\}},\ w \trianglelefteq' v\}.$$

From now on, we freely identify w, $\mathrm{T}^{(i)}_w$, and $M(w)$ with each other for $w \in W^{I \setminus \{i\}}$.

For $(v,w) \in \mathrm{QLS}(\varpi_i)$, let $d_i(v,w) \in \mathbb{Z}_{>0}$ be the number of edges of type (M4) in a directed path from w to v in $\mathrm{QB}(\varpi_i; 1/2)$; for convenience we define $d_i(v,w) = 0$ for $(v,w) \in \mathrm{QLS}(\varpi_i)$ if $i \in I$ is minuscule. We see from the next lemma that $d_i(v,w)$ is independent of the choice of a directed path from w to v in $\mathrm{QB}(\varpi_i; 1/2)$.

Lemma 5.8. *Assume that $i \in I$ is not minuscule. Let $(v,w) \in \mathrm{QLS}(\varpi_i)$; we see from Lemma 5.3 that $J(w) = J(v)$. Write $J(w) = J(v) = (J_1 < \cdots < J_\mu)$, $r = \#\{u \in [i] \mid w(u) \in J_1,\ w(u) \in \overline{n}\}$, and $s = \#\{u \in [i] \mid v(u) \in J_1,\ v(u) \in \overline{n}\}$; note that if $M(w) = (M_\nu)$ and $M(v) = (N_\nu)$, then $r = \# M_1$ and $s = \# N_1$. If (W is of type C_n) or ($\{1,2\} \not\subset J_1$), then $d_i(v,w) = 0$. If (W is of type B_n or D_n) and ($\{1,2\} \subset J_1$), then $d_i(v,w) = (r-s)/2$.*

Proof. The assertion follows from Lemma 5.5 (4).

Let $\mathrm{QLS}(\varpi_i)_{\mathrm{af}} = \mathrm{QLS}(\varpi_i) \times \mathbb{Z}$ be the affinization of the $\mathbf{U}$-crystal $\mathrm{QLS}(\varpi_i)$ (see §2.2); we have $\mathrm{wt}((v,w),c) = (1/2)(v\varpi_i + w\varpi_i) - c\delta \in P_{\mathrm{af}}$ for $(v,w) \in \mathrm{QLS}(\varpi_i)$ and $c \in \mathbb{Z}$. The next lemma holds for all $i \in I$.

Lemma 5.9. *Let $i \in I$ and $m \in \mathbb{Z}_{>0}$.*

(1) *We have an isomorphism*

(130) $$\begin{aligned} \mathrm{QLS}(\varpi_i)_{\mathrm{af}} &\to \mathbb{B}^{\frac{\infty}{2}}(\varpi_i), \\ ((v,w),c) &\mapsto \left(vT^{I\setminus\{i\}}_{(c+d_i(v,w))\alpha_i^\vee}, wT^{I\setminus\{i\}}_{(c-d_i(v,w))\alpha_i^\vee}; 0, \frac{1}{2}, 1 \right) \end{aligned}$$

of $\mathbf{U}$-crystals, where $w, v \in W^{I\setminus\{i\}}$ and $c \in \mathbb{Z}$; we understand that $(x, x; 0, 1/2, 1) = (x; 0, 1)$ for $x \in (W^{I\setminus\{i\}})_{\mathrm{af}}$.

(2) *The crystal basis $B(m\varpi_i)$ is isomorphic to the subcrystal of $\mathrm{QLS}(\varpi_i)_{\mathrm{af}}^{\otimes m}$ consisting of the elements $\bigotimes_{\nu=1}^{m}((v_\nu, w_\nu), c_\nu)$ such that*

(131) $$w_\nu T^{I\setminus\{i\}}_{(c_\nu - d_i(v_\nu,w_\nu))\alpha_i^\vee} \succeq v_{\nu+1} T^{I\setminus\{i\}}_{(c_{\nu+1}+d_i(v_{\nu+1},w_{\nu+1}))\alpha_i^\vee} \text{ in } (W^{I\setminus\{i\}})_{\mathrm{af}}$$

for $\nu \in [m-1]$.

Proof. (1): If $i \in I$ is minuscule, then the proof is straightforward (see also Remark 5.10 below).

Assume that $i \in I$ is not minuscule. By Lemmas 4.3 and 5.5–5.8, we check at once that the map (130) is well-defined, and is an isomorphism of $\mathbf{U}$-crystals.

(2): The assertion follows from (1) and [10, Theorem 3.4].

Remark 5.10. Let $i \in I$. By Theorem 5.1 (1) and [16, Theorem 5.17 (vii)–(viii)], we have a unique isomorphism $\mathrm{QLS}(\varpi_i)_{\mathrm{af}} \cong B(\varpi_i)$ of $\mathbf{U}$-crystals. By Theorem 2.9 (2), we have a unique isomorphism $B(\varpi_i) \cong \mathbb{B}^{\frac{\infty}{2}}(\varpi_i)$ of $\mathbf{U}$-crystals. The map (130) equals the composition of these isomorphisms.

5.3. Type $C_n^{(1)}$. Throughout this subsection, we assume that $\mathfrak{g}$ and W are of type C_n. Recall that $I = [n]$, $\Delta = \{\pm(\varepsilon_s \pm \varepsilon_t) \mid s, t \in [n],\ s < t\} \cup \{\pm 2\varepsilon_s \mid s \in [n]\}$, and $\Pi = \{\alpha_s = \varepsilon_s - \varepsilon_{s+1} \mid s \in [n-1]\} \cup \{\alpha_n = 2\varepsilon_n\}$. The highest root is $\theta = 2\varepsilon_1 = 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n$; we have $\theta^\vee = \varepsilon_1$. We identify ϖ_i with $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i$ for $i \in I$.

Let $i \in [n]$. A map $\mathsf{C} : [i] \to \mathcal{C}_n$ is, by definition, a Kashiwara–Nakashima C_n-column (KN C_n-column for short) of shape ϖ_i if

(KN-C1) $\mathsf{C}(1) \prec \mathsf{C}(2) \prec \cdots \prec \mathsf{C}(i)$,
(KN-C2) if $t = \mathsf{C}(p) = \sigma(\mathsf{C}(q)) \in [n]$ for some $p, q \in [i]$, then $q - p > i - t$.

Let $\mathrm{KN}_{C_n}(\varpi_i)$ be the set of KN C_n-columns of shape ϖ_i. We sometimes identify $\mathsf{C} \in \mathrm{KN}_{C_n}(\varpi_i)$ with its image $\{\mathsf{C}(u) \mid u \in [i]\} \subset \mathcal{C}_n$.

Let $\mathsf{C} : [i] \to \mathcal{C}_n$ be a map satisfying (KN-C1). Let $I_\mathsf{C} = \{z_1 \prec z_2 \prec \cdots \prec z_k\}$ be the set of $z \in \mathcal{C}_n$ such that $z \preceq n$ and $\{z, \overline{z}\} \subset \mathsf{C}$. We say that C can be split if there exists a subset $J_\mathsf{C} = \{y_1 > y_2 > \cdots > y_k\} \subset [n]$ such that

(i) $y_1 = \max\{y \in \mathcal{C}_n \mid y \prec z_1,\ y \notin \mathsf{C},\ \overline{y} \notin \mathsf{C}\}$,
(ii) $y_v = \max\{y \in \mathcal{C}_n \mid y \prec \min\{y_{v-1}, z_v\},\ y \notin \mathsf{C},\ \overline{y} \notin \mathsf{C}\}$ for $v \in [2, k]$.

Define $r\mathsf{C}, l\mathsf{C} \in \mathrm{CST}_{C_n}(\varpi_i)$ to be such that $r\mathsf{C} = (\mathsf{C} \setminus \{\overline{z} \mid z \in I_\mathsf{C}\}) \cup \{\overline{y} \mid y \in J_\mathsf{C}\}$ and $l\mathsf{C} = (\mathsf{C} \setminus I_\mathsf{C}) \cup J_\mathsf{C}$.

Define a $\mathfrak{g}$-crystal structure on $\mathrm{KN}_{C_n}(\varpi_i)$ as follows (cf. [18, §4]). Let $\mathsf{C} \in \mathrm{KN}_{C_n}(\varpi_i)$. If we set $\varepsilon_{\overline{s}} = -\varepsilon_s$ for $s \in [n]$, then the weight of C is

$$\mathrm{wt}(\mathsf{C}) = \sum_{u \in [i]} \varepsilon_{\mathsf{C}(u)}. \tag{132}$$

Let $j \in I$. Note that only the letters $j, j+1, \overline{j+1}, \overline{j}$ may be changed in C when we apply e_j or f_j. Moreover, the actions of e_j and f_j are uniquely determined from $\mathsf{C} \cap \{j, j+1, \overline{j+1}, \overline{j}\}$. Hence, by omitting the letters not being in $\{j, j+1, \overline{j+1}, \overline{j}\}$, we can illustrate the actions of f_j for $j \in [n-1]$ by

$$\boxed{1} \xrightarrow{f_1} \boxed{2} \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} \boxed{n-1} \xrightarrow{f_{n-1}} \boxed{n}, \tag{133}$$

$$\boxed{\overline{1}} \xleftarrow{f_1} \boxed{\overline{2}} \xleftarrow{f_2} \cdots \xleftarrow{f_{n-2}} \boxed{\overline{n-1}} \xleftarrow{f_{n-1}} \boxed{\overline{n}}, \tag{134}$$

$$\begin{array}{|c|}\hline j \\ \hline \overline{j+1} \\ \hline \end{array} \xrightarrow{f_j} \begin{array}{|c|}\hline j+1 \\ \hline \overline{j+1} \\ \hline \end{array} \xrightarrow{f_j} \begin{array}{|c|}\hline j+1 \\ \hline \overline{j} \\ \hline \end{array}, \tag{135}$$

$$\begin{array}{|c|}\hline j \\ \hline j+1 \\ \hline \overline{j+1} \\ \hline \end{array} \xrightarrow{f_j} \begin{array}{|c|}\hline j \\ \hline j+1 \\ \hline \overline{j} \\ \hline \end{array}, \qquad \begin{array}{|c|}\hline j \\ \hline \overline{j+1} \\ \hline \overline{j} \\ \hline \end{array} \xrightarrow{f_j} \begin{array}{|c|}\hline j+1 \\ \hline \overline{j+1} \\ \hline \overline{j} \\ \hline \end{array}; \tag{136}$$

set $f_j\mathsf{C} = \mathbf{0}$ otherwise. Similarly, the action of f_n is illustrated by

$$\boxed{n} \xrightarrow{f_n} \boxed{\overline{n}}; \tag{137}$$

set $f_n \mathsf{C} = \mathbf{0}$ otherwise. The maps e_j, $j \in I$, are defined to be such that the condition (C6) in §2.2 holds. For $\mathsf{C} \in \mathrm{KN}_{C_n}(\varpi_i)$ and $j \in I$, set $\varepsilon_j(\mathsf{C}) = \max\{k \in \mathbb{Z}_{>0} \mid e_j^k \mathsf{C} = \mathbf{0}\}$ and $\phi_j(\mathsf{C}) = \max\{k \in \mathbb{Z}_{>0} \mid f_j^k \mathsf{C} = \mathbf{0}\}$.

The next lemma is a reformulation of [34, Theorem A.1] in terms of Maya diagrams.

Lemma 5.11. *Assume that $\mathfrak{g}$ and W are of type C_n. For a map $\mathsf{C} : [i] \to \mathcal{C}_n$ satisfying* (KN-C1)*, we have $\mathsf{C} \in \mathrm{KN}_{C_n}(\varpi_i)$ if and only if C can be split. The map*

$$\mathrm{KN}_{C_n}(\varpi_i) \to \mathrm{LS}(\varpi_i), \; \mathsf{C} \mapsto (r\mathsf{C}, l\mathsf{C}), \tag{138}$$

is an isomorphism of $\mathfrak{g}$-crystals. The inverse of (138) *is given as follows. Let $(v, w) \in \mathrm{LS}(\varpi_i)$, $J(w) = J(v) = (J_1 < \cdots < J_\mu) \in S_i$ and $M(w) = (M_\nu)$, $M(v) = (N_\nu) \in \prod_{\nu=1}^{\mu} 2^{J_\nu}$. The inverse image of (v, w) is*

$$\begin{aligned} \mathsf{C} &= \{v(u) \mid u \in [i],\ v(u) \in n\} \sqcup \{w(u) \mid u \in [i],\ w(u) \in \overline{n}\} \\ &= \bigsqcup_{\nu=1}^{\mu} ((J_\nu \setminus N_\nu) \sqcup \{\overline{z} \mid z \in M_\nu\}); \end{aligned} \tag{139}$$

we have $I_{\mathsf{C}} = \bigsqcup_{\nu=1}^{\mu}(M_\nu \setminus N_\nu)$ and $J_{\mathsf{C}} = \bigsqcup_{\nu=1}^{\mu}(N_\nu \setminus M_\nu)$.

By Lemma 5.2 (1), we have $\mathrm{QLS}(\varpi_i) = \mathrm{LS}(\varpi_i) = \mathrm{KN}_{C_n}(\varpi_i)$. Hence $\mathrm{KN}_{C_n}(\varpi_i)$ inherits a $\mathbf{U}'$-crystal structure from $\mathrm{QLS}(\varpi_i)$. We see that only the letters $1, \overline{1}$ may be changed in C when we apply e_0 or f_0, and the actions of e_0 and f_0 are uniquely determined from $\mathsf{C} \cap \{1, \overline{1}\}$. The action of f_0 is illustrated by

$$\boxed{\overline{1}} \xrightarrow{f_0} \boxed{1}\,; \tag{140}$$

set $f_0 \mathsf{C} = \mathbf{0}$ otherwise. The map e_0 is defined to be such that the condition (C6) in §2.2 holds.

For a tuple $(\mathsf{C}_1, \mathsf{C}_2, \ldots, \mathsf{C}_m)$ of columns, let $\mathsf{C}_1\mathsf{C}_2\cdots\mathsf{C}_m$ denote the tableau whose ν-th column is C_ν. Recall the partial order $\preceq$ on $\mathrm{CST}_{C_n}(\varpi_i) \times \mathbb{Z}$ (see Definition 4.12).

Definition 5.12. *Let $i \in I$ and $m \in \mathbb{Z}_{>0}$.*

(1) *Let $\mathsf{T} = (\mathsf{T}_1\mathsf{T}_2\cdots\mathsf{T}_m, (c_1, c_2, \ldots, c_m))$, where $\mathsf{T}_\nu \in \mathrm{KN}_{C_n}(\varpi_1) = \mathrm{CST}_{C_n}(\varpi_1)$ and $c_\nu \in \mathbb{Z}$ for $\nu \in [m]$. We call T a semi-infinite KN C_n-tableau of shape $m\varpi_1$ if*

$$(\mathsf{T}_\nu, c_\nu) \preceq (\mathsf{T}_{\nu+1}, c_{\nu+1}) \tag{141}$$

in $\mathrm{CST}_{C_n}(\varpi_1) \times \mathbb{Z}$ for $\nu \in [m-1]$.

(2) *Assume that $i \in [2, n]$. Let $\mathsf{T} = (\mathsf{C}_1\mathsf{C}_2\cdots\mathsf{C}_m, (c_1, c_2, \ldots, c_m))$, where $\mathsf{C}_\nu \in \mathrm{KN}_{C_n}(\varpi_i)$ and $c_\nu \in \mathbb{Z}$ for $\nu \in [m]$. We call T a semi-infinite KN C_n-tableau of shape $m\varpi_i$ if*

$$(l\mathsf{C}_\nu, c_\nu) \preceq (r\mathsf{C}_{\nu+1}, c_{\nu+1}) \tag{142}$$

in $\mathrm{CST}_{C_n}(\varpi_i) \times \mathbb{Z}$ for $\nu \in [m-1]$.

Let $\mathrm{Y}_{C_n}^{\frac{\infty}{2}}(m\varpi_i)$ be the set of semi-infinite KN C_n-tableaux of shape $m\varpi_i$. For $\lambda = \sum_{i \in I} m_i\varpi_i \in P^+$, set $\mathrm{Y}_{C_n}^{\frac{\infty}{2}}(\lambda) = \prod_{i \in I} \mathrm{Y}_{C_n}^{\frac{\infty}{2}}(m_i\varpi_i)$. We call an element of $\mathrm{Y}_{C_n}^{\frac{\infty}{2}}(\lambda)$ a semi-infinite KN C_n-tableau of shape λ.

Let $\mathrm{KN}_{C_n}(\varpi_i)_{\mathrm{af}}$ denote the affinization of the $\mathbf{U}'$-crystal $\mathrm{KN}_{C_n}(\varpi_i)$ (see §2.2). Combining Theorem 2.8, Proposition 4.13 (2), Lemma 5.9, and Definition 5.12 we obtain the following theorem; note that $d_i(v, w) = 0$ for all $i \in I$ and $(v, w) \in \mathrm{QLS}(\varpi_i)$, by Lemma 5.8.

Theorem 5.13. *Assume that* $\mathbf{U}$ *is of type* $C_n^{(1)}$*. Let* $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$*. For each* $i \in I$*, the image of the map*

$$\begin{aligned} \mathrm{Y}^{\frac{\infty}{2}}_{C_n}(m_i \varpi_i) &\to \mathrm{KN}_{C_n}(\varpi_i)_{\mathrm{af}}^{\otimes m_i}, \\ (\mathsf{C}_1\mathsf{C}_2\cdots\mathsf{C}_{m_i}, (c_1, c_2, \ldots, c_{m_i})) &\mapsto \bigotimes_{\nu \in [m_i]} (\mathsf{C}_\nu, c_\nu), \end{aligned} \tag{143}$$

is a $\mathbf{U}$*-subcrystal. Hence we can define a* $\mathbf{U}$*-crystal structure on* $\mathrm{Y}^{\frac{\infty}{2}}_{C_n}(m_i \varpi_i)$ *to be such that the map* (143) *is a strict embedding of* $\mathbf{U}$*-crystals. In particular,* $\mathrm{Y}^{\frac{\infty}{2}}_{C_n}(\lambda)$ *is a* $\mathbf{U}$*-subcrystal of* $\bigotimes_{i \in I} \mathrm{KN}_{C_n}(\varpi_i)_{\mathrm{af}}^{\otimes m_i}$*. Then* $\mathrm{Y}^{\frac{\infty}{2}}_{C_n}(\lambda)$ *is isomorphic, as a* $\mathbf{U}$*-crystal, to the crystal basis* $B(\lambda)$*.*

5.4. Type $B_n^{(1)}$. Throughout this subsection, we assume that $\mathfrak{g}$ and W are of type B_n. Recall that $I = [n]$, $\Delta = \{\pm(\varepsilon_s \pm \varepsilon_t) \mid s, t \in [n],\ s < t\} \cup \{\pm\varepsilon_s \mid s \in [n]\}$, and $\Pi = \{\alpha_s = \varepsilon_s - \varepsilon_{s+1} \mid s \in [n-1]\} \cup \{\alpha_n = \varepsilon_n\}$. The highest root is $\theta = \varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$; we have $\theta^\vee = \theta$. We identify ϖ_i with $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i$ if $i \in [n-1]$, and with $\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)$ if $i = n$.

Set

$$\mathcal{B}_n = \{1 \prec 2 \prec \cdots \prec n \prec 0 \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}\}. \tag{144}$$

Define $\sigma : \mathcal{B}_n \to \mathcal{B}_n$ by $\sigma(0) = 0$, $\sigma(u) = \overline{u}$, and $\sigma(\overline{u}) = u$ for $u \in [n]$.

Let $i \in [n-1]$. A map $\mathsf{C} : [i] \to \mathcal{B}_n$ is, by definition, a Kashiwara–Nakashima B_n-column (KN B_n-column for short) of shape ϖ_i if

(KN-B1) $\mathsf{C}(1) \preceq \mathsf{C}(2) \preceq \cdots \preceq \mathsf{C}(i)$,
(KN-B2) if $0 \neq \mathsf{C}(u)$ or $\mathsf{C}(u+1) \neq 0$, then $\mathsf{C}(u) \prec \mathsf{C}(u+1)$ for $u \in [i-1]$,
(KN-B3) if $t = \mathsf{C}(p) = \sigma(\mathsf{C}(q)) \in [n]$ for some $p, q \in [i]$, then $q - p > i - t$;

note that $0 \in \mathcal{B}_n$ is the unique element that may appear in C more than once. Let $\mathrm{KN}_{B_n}(\varpi_i)$ be the set of KN B_n-columns of shape ϖ_i. We sometimes identify $\mathsf{C} \in \mathrm{KN}_{B_n}(\varpi_i)$ with the multiset $\{\mathsf{C}(u) \mid u \in [i]\}$. For convenience we also denote by $\mathrm{KN}_{B_n}(\varpi_n) = \mathrm{CST}_{B_n}(\varpi_n)$.

Let $i \in [n-1]$. Let $\mathsf{C} : [i] \to \mathcal{B}_n$ be a map satisfying (KN-B1)–(KN-B2). Let $I_{\mathsf{C}} = \{z_1 \succeq z_2 \succeq \cdots \succeq z_k\}$ be the multiset of $z \in \mathsf{C}$ such that $z \preceq 0$ and $\{z, \sigma(z)\} \subset \mathsf{C}$; note that $\{\mathsf{C}(u) \mid u \in [i], \mathsf{C}(u) = 0\}$ is a multisubset of I_{C}. We say that C can be split if there exists a subset $J_{\mathsf{C}} = \{y_1 > y_2 > \cdots > y_k\} \subset [n]$ such that

(i) $y_1 = \max\{y \in \mathcal{B}_n \mid y \prec z_1,\ y \notin \mathsf{C},\ \overline{y} \notin \mathsf{C}\}$,
(ii) $y_\nu = \max\{y \in \mathcal{B}_n \mid y \prec \min\{y_{\nu-1}, z_\nu\},\ y \notin \mathsf{C},\ \overline{y} \notin \mathsf{C}\}$ for $\nu \in [2, k]$.

Define $r\mathsf{C}, l\mathsf{C} \in \mathrm{CST}_{B_n}(\varpi_i)$ to be such that $r\mathsf{C} = (\mathsf{C} \setminus \{\sigma(z) \mid z \in I_{\mathsf{C}}\}) \sqcup \{\sigma(y) \mid y \in J_{\mathsf{C}}\}$ and $l\mathsf{C} = (\mathsf{C} \setminus I_{\mathsf{C}}) \sqcup J_{\mathsf{C}}$.

Define a $\mathfrak{g}$-crystal structure on $\mathrm{KN}_{B_n}(\varpi_i)$ for $i \in [n-1]$ as follows (cf. [18, §5]); for the definition of a $\mathfrak{g}$-crystal structure on $\mathrm{KN}_{B_n}(\varpi_n)$ we refer the reader to [21, §2.3]. The maps $\mathrm{wt}, \varepsilon_j, \phi_j$ for $j \in I$ and e_j, f_j for $j \in [n-1]$ are defined in the same manner as those for $\mathrm{KN}_{C_n}(\varpi_i)$. Note that only the letters $n, 0, \overline{n}$ may be changed in C when we apply e_n or f_n. Moreover, the actions of e_n and f_n are uniquely determined from the multiset $\{C(u) \mid u \in [i],\ C(u) \in \{n, 0, \overline{n}\}\}$. The action of f_n is illustrated by

$$(145)\qquad \boxed{n} \xrightarrow{f_n} \boxed{0} \xrightarrow{f_n} \boxed{\overline{n}}\,, \qquad \begin{array}{|c|}\hline n \\ \hline 0 \\ \hline \vdots \\ \hline 0 \\ \hline \end{array} \xrightarrow{f_n} \begin{array}{|c|}\hline 0 \\ \hline \vdots \\ \hline \vdots \\ \hline 0 \\ \hline \end{array} \xrightarrow{f_n} \begin{array}{|c|}\hline 0 \\ \hline \vdots \\ \hline 0 \\ \hline \overline{n} \\ \hline \end{array}\,;$$

set $f_n C = \mathbf{0}$ otherwise. The map e_n is defined to be such that the condition (C6) in §2.2 holds.

The next lemma is a reformulation of [21, Corollary 3.1.11 and Remark 3.1.13] in terms of Maya diagrams.

Lemma 5.14. *Assume that $\mathfrak{g}$ and W are of type B_n.*

1. *The map $\mathrm{LS}(\varpi_n) \to \mathrm{KN}_{B_n}(\varpi_n) = \mathrm{CST}_{B_n}(\varpi_n)$, $(w; 0, 1) \mapsto \mathrm{T}^{(n)}_w$, is an isomorphism of $\mathfrak{g}$-crystals.*
2. *Assume that $i \in [n-1]$. For a map $C : [i] \to B_n$ satisfying (KN-B1)–(KN-B2), we have $C \in \mathrm{KN}_{B_n}(\varpi_i)$ if and only if C can be split. The map*

$$(146)\qquad \mathrm{KN}_{B_n}(\varpi_i) \to \mathrm{LS}(\varpi_i),\ C \mapsto (rC, lC),$$

is an isomorphism of $\mathfrak{g}$-crystals. The inverse of (146) is given as follows. Let $(v, w) \in \mathrm{LS}(\varpi_i)$, $J(w) = J(v) = (J_1 < \cdots < J_\mu) \in S_i$, $M(w) = (M_\nu)$, $M(v) = (N_\nu) \in \prod_{\nu=1}^{\mu} 2^{J_\nu}$, and $f = \# N_\mu - \# M_\mu \in \mathbb{Z}_{>0}$. The inverse image of (v, w) is

$$C = \{v(u) \mid u \in [i],\ v(u) \in n\} \sqcup \{w(u) \mid u \in [i],\ w(u) \in \overline{n}\} \sqcup \{\underbrace{0, 0, \ldots, 0}_{f \text{ times}}\}$$

$$(147)\qquad = \bigsqcup_{\nu=1}^{\mu} ((J_\nu \setminus N_\nu) \sqcup \{\overline{z} \mid z \in M_\nu\}) \sqcup \{\underbrace{0, 0, \ldots, 0}_{f \text{ times}}\};$$

we have $I_C = \bigsqcup_{\nu=1}^{\mu}(M_\nu \setminus N_\nu) \sqcup \{\underbrace{0, 0, \ldots, 0}_{f \text{ times}}\}$ and $J_C = \bigsqcup_{\nu=1}^{\mu}(N_\nu \setminus M_\nu)$.

Set

$$(148)\qquad \tilde{B}_n = B_n \sqcup \{\overline{0}\} = \left\{ 1 \prec 2 \prec \cdots \prec n \prec 0 \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1} \prec \overline{0} \right\}.$$

Definition 5.15. *Let $i \in [n-1]$. A map $\tilde{C} : [i] \to \tilde{B}_n$ is, by definition, a quantum Kashiwara–Nakashima B_n-column (QKN B_n-column for short) of shape ϖ_i if*

(QKN-B1) *there exists $m \in \mathbb{Z}_{\geq 0}$ such that $2m \leq i$ and*

$$(149)\qquad \begin{aligned} &\tilde{C}(1) \prec \tilde{C}(2) \prec \cdots \prec \tilde{C}(i-2m) \\ &\prec \overline{0} = \tilde{C}(i-2m+1) = \cdots = \tilde{C}(i-1) = \tilde{C}(i), \end{aligned}$$

(QKN-B2) *the map $C : [i-2m] \to B_n$, $u \mapsto \tilde{C}(u)$, is a KN B_n-column of shape ϖ_{i-2m};*

in this case, we write $\tilde{C} = C \sqcup \{\underbrace{\overline{0}, \overline{0}, \ldots, \overline{0}}_{2m \text{ times}}\}$ for brevity. Let $\mathrm{QKN}_{B_n}(\varpi_i)$ be the set of QKN B_n-columns of shape ϖ_i. We sometimes identify $\tilde{C} \in \mathrm{QKN}_{B_n}(\varpi_i)$ with the multiset $\{\tilde{C}(u) \mid u \in [i]\}$. For convenience we also denote by $\mathrm{QKN}_{B_n}(\varpi_n) = \mathrm{CST}_{B_n}(\varpi_n)$.

Let $i \in [n-1]$. Let $\tilde{C} \in \mathrm{QKN}_{B_n}(\varpi_i)$, and let $C \in \mathrm{KN}_{B_n}(\varpi_{i-2m})$ be as in (QKN-B2). Write $\{x_1 < x_2 < \cdots < x_{n-i+2m}\} = [n] \setminus \{ rC(u) \mid u \in [i-2m]\}$ and set $K_{\tilde{C}} = \{x_1 < x_2 < \cdots < x_{2m}\}$; note that x_ν, $\nu \in [2m]$, are uniquely determined by

(i) $x_1 = \min\{x \in B_n \mid x \succeq 1,\ x \notin rC,\ \overline{x} \notin rC\}$,

(ii) $x_\nu = \min\{x \in B_n \mid x \succ x_{\nu-1},\ x \notin r\mathrm{C},\ \overline{x} \notin r\mathrm{C}\}$ *for* $\nu \in [2, 2m]$.

Define $r\tilde{\mathrm{C}} = K_{\tilde{\mathrm{C}}} \sqcup r\mathrm{C}$ *and* $l\tilde{\mathrm{C}} = \{\overline{x} \mid x \in K_{\tilde{\mathrm{C}}}\} \sqcup l\mathrm{C}$*; note that* $r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}} \in \mathrm{CST}_{B_n}(\varpi_i)$ *(cf.* [6]*; see also* [26, Algorithm 4.1]*).*

Let $i \in [n-1]$. Define a $\mathbf{U}'$-crystal structure on $\mathrm{QKN}_{B_n}(\varpi_i)$ as follows. Let $\tilde{\mathrm{C}} = \mathrm{C} \sqcup \{\overline{0}, \dots, \overline{0}\} \in \mathrm{QKN}_{B_n}(\varpi_i)$ and $j \in I$. Set $\mathrm{wt}(\tilde{\mathrm{C}}) = \mathrm{wt}(\mathrm{C})$, $e_j\tilde{\mathrm{C}} = e_j\mathrm{C} \sqcup \{\overline{0}, \dots, \overline{0}\}$, and $f_j\tilde{\mathrm{C}} = f_j\mathrm{C} \sqcup \{\overline{0}, \dots, \overline{0}\}$; we understand that $e_j\tilde{\mathrm{C}} = \mathbf{0}$ (resp. $f_j\tilde{\mathrm{C}} = \mathbf{0}$) if $e_j\mathrm{C} = \mathbf{0}$ (resp. $f_j\mathrm{C} = \mathbf{0}$).

Recall that $z_k = \min I_{\mathrm{C}}$ and $\# K_{\tilde{\mathrm{C}}} = 2m$, where $m \in \mathbb{Z}_{>0}$ is as in (QKN-B1). The actions of e_0 and f_0 are uniquely determined from I_{C}, J_{C}, $K_{\tilde{\mathrm{C}}}$, and $\mathrm{C} \cap \{1, 2, \overline{2}, \overline{1}, z_k, \overline{z_k}\}$. Let $y_k = \min J_{\mathrm{C}}$. If $m > 0$, let $x_1 = \min K_{\tilde{\mathrm{C}}}$ and $x_2 = \min(K_{\tilde{\mathrm{C}}} \setminus \{x_1\})$. The action of f_0 is illustrated as follows.

(i) Assume that $m = 0$ and $y_k \notin \{1, 2\}$. Set

$$\begin{array}{|c|}\hline \overline{2} \\ \hline\end{array} \xrightarrow{f_0} \begin{array}{|c|}\hline 1 \\ \hline\end{array}, \qquad \begin{array}{|c|}\hline \overline{1} \\ \hline\end{array} \xrightarrow{f_0} \begin{array}{|c|}\hline 2 \\ \hline\end{array}, \qquad \begin{array}{|c|}\hline \overline{2} \\ \hline \overline{1} \\ \hline\end{array} \xrightarrow{f_0} \begin{array}{|c|}\hline \overline{0} \\ \hline \overline{0} \\ \hline\end{array}, \tag{150}$$

and set $f_j\tilde{\mathrm{C}} = \mathbf{0}$ otherwise.

(ii) Assume that $m > 0$ and $y_k \notin \{1, 2\}$. Set

$$\begin{gathered} \begin{array}{|c|}\hline \overline{2} \\ \hline \overline{1} \\ \hline\end{array} \xrightarrow{f_0} \begin{array}{|c|}\hline \overline{0} \\ \hline \overline{0} \\ \hline\end{array} \xrightarrow{f_0} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \\ \begin{array}{|c|}\hline \overline{2} \\ \hline \overline{0} \\ \hline \overline{0} \\ \hline\end{array} \xrightarrow{f_0} \begin{array}{|c|}\hline 1 \\ \hline x_2 \\ \hline \overline{x_2} \\ \hline\end{array}, \qquad \begin{array}{|c|}\hline \overline{1} \\ \hline \overline{0} \\ \hline \overline{0} \\ \hline\end{array} \xrightarrow{f_0} \begin{array}{|c|}\hline 2 \\ \hline x_2 \\ \hline \overline{x_2} \\ \hline\end{array}, \end{gathered} \tag{151}$$

and set $f_j\tilde{\mathrm{C}} = \mathbf{0}$ otherwise.

(iii) Assume that $y_k \in \{1, 2\}$. Set

$$\begin{array}{|c|}\hline z_k \\ \hline \overline{z_k} \\ \hline \overline{2} \\ \hline\end{array} \xrightarrow{f_0} \begin{array}{|c|}\hline 1 \\ \hline \overline{0} \\ \hline \overline{0} \\ \hline\end{array}, \qquad \begin{array}{|c|}\hline z_k \\ \hline \overline{z_k} \\ \hline \overline{1} \\ \hline\end{array} \xrightarrow{f_0} \begin{array}{|c|}\hline 2 \\ \hline \overline{0} \\ \hline \overline{0} \\ \hline\end{array}, \tag{152}$$

and set $f_j\tilde{\mathrm{C}} = \mathbf{0}$ otherwise.

The map e_0 is defined to be such that the condition (C6) in §2.2 holds. For $\tilde{\mathrm{C}} \in \mathrm{QKN}_{B_n}(\varpi_i)$ and $j \in I_{\mathrm{af}}$, set $\varepsilon_j(\tilde{\mathrm{C}}) = \max\{k \in \mathbb{Z}_{>0} \mid e_j^k\tilde{\mathrm{C}} = \mathbf{0}\}$ and $\phi_j(\tilde{\mathrm{C}}) = \max\{k \in \mathbb{Z}_{>0} \mid f_j^k\tilde{\mathrm{C}} = \mathbf{0}\}$. It is Theorem 5.16 (2) below that makes these definitions allowable.

Similarly, we can define a $\mathbf{U}'$-crystal structure on $\mathrm{QKN}_{B_n}(\varpi_n)$. The maps e_0, f_0 is given as follows. Let $\tilde{\mathrm{C}} \in \mathrm{QKN}_{B_n}(\varpi_n)$. We see that only the letters $1, 2, \overline{2}, \overline{1}$ may be changed in $\tilde{\mathrm{C}}$ when we apply e_0 or f_0, and the actions of e_0 and f_0 are uniquely determined from $\tilde{\mathrm{C}} \cap \{1, 2, \overline{2}, \overline{1}\}$. The action of f_0 is illustrated by

$$\begin{array}{|c|}\hline \overline{2} \\ \hline \overline{1} \\ \hline\end{array} \xrightarrow{f_0} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}; \tag{153}$$

set $f_0\tilde{C} = \mathbf{0}$ otherwise. The map e_0 is defined to be such that the condition (C6) in §2.2 holds.

Theorem 5.16. *Assume that* $\mathbf{U}$ *is of type* $B_n^{(1)}$.

(1) *The set* $\mathrm{QKN}_{B_n}(\varpi_n)$ *equipped with the maps* $\mathrm{wt}, e_j, f_j, \varepsilon_j, \phi_j$, $j \in I_{\mathrm{af}}$, *is a* $\mathbf{U}$*-crystal. The map* $\mathrm{QLS}(\varpi_n) = \mathrm{LS}(\varpi_n) \to \mathrm{QKN}_{B_n}(\varpi_n) = \mathrm{CST}_{B_n}(\varpi_n)$, $(w; 0, 1) \mapsto \mathrm{T}_w^{(n)}$, *is an isomorphism of* $\mathbf{U}$*-crystals.*

(2) *Assume that* $i \in [n-1]$. *The set* $\mathrm{QKN}_{B_n}(\varpi_i)$ *equipped with the maps* $\mathrm{wt}, e_j, f_j, \varepsilon_j, \phi_j$, $j \in I_{\mathrm{af}}$, *is a* $\mathbf{U}$*-crystal. The map*

$$\mathrm{QKN}_{B_n}(\varpi_i) \to \mathrm{QLS}(\varpi_i), \quad \tilde{\mathrm{C}} \mapsto (r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}), \tag{154}$$

is an isomorphism of $\mathbf{U}$*-crystals. The inverse of* (154) *is given as follows. Let* $(v, w) \in \mathrm{QLS}(\varpi_i)$, $J(w) = J(v) = (J_1 < \cdots < J_\mu) \subset S_i$ *and* $M(w) = (M_\nu)$, $M(v) = (N_\nu) \in \prod_{\nu=1}^{\mu} 2^{J_\nu}$. *Set* $f = \# N_\mu - \# M_\mu \in \mathbb{Z}_{>0}$ *if* $n \in J_\mu$, *and set* $f = 0$ *otherwise. Set* $2m = \# M_1 - \# N_1 \in 2\mathbb{Z}_{>0}$ *if* $1, 2 \in J_1$, *and set* $m = 0$ *otherwise; we see from Lemma* 5.8 *that* $m = d_i(v, w)$. *Define*

(i) $\{y^1 < y^2 < \cdots < y^l\} = N_1 \setminus M_1$,
(ii) $z^1 = \min\{z \in J_1 \mid z \succeq y^1,\ z \notin M_1 \setminus N_1\}$, *and*
(iii) $z^\nu = \min\{z \in J_1 \mid z \succ \max\{y^\nu, z^{\nu-1}\},\ z \notin M_1 \setminus N_1\}$ *for* $\nu \in [2, l]$.

The inverse image of (v, w) *is*

$$\begin{aligned}\tilde{\mathrm{C}} = {} & (J_1 \setminus (M_1 \cup N_1)) \sqcup \{z^\nu, \overline{z^\nu} \mid \nu \in [l]\} \sqcup \{\overline{z} \mid z \in M_1 \cap N_1\} \\ & \sqcup \bigsqcup_{\nu=2}^{\mu} ((J_\nu \setminus N_\nu) \sqcup \{\overline{z} \mid z \in M_\nu\}) \sqcup \{\underbrace{0, 0, \ldots, 0}_{f \text{ times}}\} \sqcup \{\underbrace{\overline{0}, \overline{0}, \ldots, \overline{0}}_{2m \text{ times}}\};\end{aligned} \tag{155}$$

if we set $\mathrm{C} = \tilde{\mathrm{C}} \setminus \{\underbrace{\overline{0}, \overline{0}, \ldots, \overline{0}}_{2m \text{ times}}\}$, *then* $\mathrm{C} \in \mathrm{KN}_{B_n}(\varpi_{i-2m})$ *and*

$$I_{\mathrm{C}} = \{z^\nu \mid \nu \in [l]\} \sqcup \bigsqcup_{\nu=2}^{\mu} (M_\nu \setminus N_\nu) \sqcup \{\underbrace{0, 0, \ldots, 0}_{f \text{ times}}\}, \quad J_{\mathrm{C}} = \bigsqcup_{\nu=1}^{\mu} (N_\nu \setminus M_\nu). \tag{156}$$

(3) *Assume that* $i \in [n-1]$. *The map*

$$\mathrm{QKN}_{B_n}(\varpi_i) \to \bigsqcup_{m=0}^{\lfloor \frac{i}{2} \rfloor} \mathrm{KN}_{B_n}(\varpi_{i-2m}), \quad \tilde{\mathrm{C}} = \mathrm{C} \sqcup \{\underbrace{\overline{0}, \overline{0}, \ldots, \overline{0}}_{2m \text{ times}}\} \mapsto \mathrm{C}, \tag{157}$$

is an isomorphism of $\mathfrak{g}$*-crystals, where* C *and* m *are as in* (QKN-B1)–(QKN-B2) *for* $\tilde{\mathrm{C}}$. *Here we understand that* $\mathrm{KN}_{B_n}(\varpi_0) = \{?\}$ *is a* $\mathfrak{g}$*-crystal isomorphic to the crystal basis of the trivial module. The inverse image of* $\mathrm{C} \in \mathrm{KN}_{B_n}(\varpi_{i-2m})$ *under the map* (157) *is* $\tilde{\mathrm{C}} = \mathrm{C} \sqcup \{\underbrace{\overline{0}, \overline{0}, \ldots, \overline{0}}_{2m \text{ times}}\}$.

The proof of (1) and (3) in Theorem 5.16 is straightforward (cf. [7, Lemma 2.7 (i)]). In §5.6, we will give the proof for Theorem 5.16 (2).

Recall the partial order $\preceq$ on $\mathrm{CST}_{B_n}(\varpi_i) \times \mathbb{Z}$ (see Definition 4.20).

Definition 5.17. *Let* $i \in I$ *and* $m \in \mathbb{Z}_{>0}$.

(1) *Let* $\mathbf{T} = (\mathrm{T}_1 \mathrm{T}_2 \cdots \mathrm{T}_m, (c_1, c_2, \ldots, c_m))$, *where* $\mathrm{T}_\nu \in \mathrm{QKN}_{B_n}(\varpi_n) = \mathrm{CST}_{B_n}(\varpi_n)$ *and* $c_\nu \in \mathbb{Z}$ *for* $\nu \in [m]$. *We call* $\mathbf{T}$ *a semi-infinite KN* B_n*-tableau of shape* $m\varpi_n$ *if*

$$(\mathrm{T}_\nu, c_\nu) \succeq (\mathrm{T}_{\nu+1}, c_{\nu+1}) \tag{158}$$

in $\mathrm{CST}_{B_n}(\varpi_i) \times \mathbb{Z}$ *for* $\nu \in [m-1]$.

(2) *Assume that $i \in [n-1]$. Let $\mathrm{T} = (\tilde{\mathrm{C}}_1\tilde{\mathrm{C}}_2 \cdots \tilde{\mathrm{C}}_m, (c_1, c_2, \dots, c_m))$, where $\tilde{\mathrm{C}}_\nu \in \mathrm{QKN}_{B_n}(\varpi_i)$ and $c_\nu \in \mathbb{Z}$ for $\nu \in [m]$. We call T a semi-infinite KN B_n-tableau of shape $m\varpi_i$ if*

$$(l\tilde{\mathrm{C}}_\nu, c_\nu - d_i(r\tilde{\mathrm{C}}_\nu, l\tilde{\mathrm{C}}_\nu)) \succeq (r\tilde{\mathrm{C}}_{\nu+1}, c_{\nu+1} + d_i(r\tilde{\mathrm{C}}_{\nu+1}, l\tilde{\mathrm{C}}_{\nu+1})) \tag{159}$$

in $\mathrm{CST}_{B_n}(\varpi_i) \times \mathbb{Z}$ for $\nu \in [m-1]$.

Let $\mathrm{Y}^{\frac{\infty}{2}}_{B_n}(m\varpi_i)$ be the set of semi-infinite KN B_n-tableaux of shape $m\varpi_i$. For $\lambda = \sum_{i \in I} m_i\varpi_i \in P^+$, set $\mathrm{Y}^{\frac{\infty}{2}}_{B_n}(\lambda) = \bigotimes_{i \in I} \mathrm{Y}^{\frac{\infty}{2}}_{B_n}(m_i\varpi_i)$. We call an element of $\mathrm{Y}^{\frac{\infty}{2}}_{B_n}(\lambda)$ a semi-infinite KN B_n-tableau of shape λ.

Let $\mathrm{QKN}_{B_n}(\varpi_i)_{\mathrm{af}}$ denote the affinization of the $\mathbf{U}'$-crystal $\mathrm{QKN}_{B_n}(\varpi_i)$ (see §2.2). Combining Theorem 2.8, Proposition 4.21 (2), Lemma 5.9, and Definition 5.17 we obtain the following theorem.

Theorem 5.18. *Assume that $\mathbf{U}$ is of type $B_n^{(1)}$. Let $\lambda = \sum_{i \in I} m_i\varpi_i \in P^+$. For each $i \in I$, the image of the map*

$$\begin{aligned} &\mathrm{Y}^{\frac{\infty}{2}}_{B_n}(m_i\varpi_i) \to \mathrm{QKN}_{B_n}(\varpi_i)_{\mathrm{af}}^{\otimes m_i}, \\ &(\mathrm{C}_1\mathrm{C}_2 \cdots \mathrm{C}_{m_i}, (c_1, c_2, \dots, c_{m_i})) \mapsto \bigotimes_{\nu \in [m_i]} (\mathrm{C}_\nu, c_\nu), \end{aligned} \tag{160}$$

is a $\mathbf{U}$-subcrystal. Hence we can define a $\mathbf{U}$-crystal structure on $\mathrm{Y}^{\frac{\infty}{2}}_{B_n}(m_i\varpi_i)$ to be such that the map (160) is a strict embedding of $\mathbf{U}$-crystals. In particular, $\mathrm{Y}^{\frac{\infty}{2}}_{B_n}(\lambda)$ is a $\mathbf{U}$-subcrystal of $\bigotimes_{i \in I} \mathrm{QKN}_{B_n}(\varpi_i)_{\mathrm{af}}^{\otimes m_i}$. Then $\mathrm{Y}^{\frac{\infty}{2}}_{B_n}(\lambda)$ is isomorphic, as a $\mathbf{U}$-crystal, to the crystal basis $\mathcal{B}(\lambda)$.

5.5. Type $D_n^{(1)}$. Throughout this subsection, we assume that $\mathfrak{g}$ and W are of type D_n. Recall that $I = [n]$, $\Delta = \{\pm(\varepsilon_s \pm \varepsilon_t) \mid s, t \in [n],\ s < t\}$, and $\Pi = \{\alpha_s = \varepsilon_s - \varepsilon_{s+1} \mid s \in [n-1]\} \cup \{\alpha_n = \varepsilon_{n-1} + \varepsilon_n\}$. The highest root is $\varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$. We identify ϖ_i with $\varepsilon_1 + \varepsilon_2 + \cdots \varepsilon_i$ if $i \in [n-2]$, with $\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{n-1} - \varepsilon_n)$ if $i = n-1$, and with $\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{n-1} + \varepsilon_n)$ if $i = n$.

Let $i \in [2, n-2]$. A map $\mathrm{C} : [i] \to \mathcal{D}_n$ is, by definition, a Kashiwara–Nakashima D_n-column (KN D_n-column for short) of shape ϖ_i if

(KN-D1) $\mathrm{C}(1) \not\succeq \mathrm{C}(2) \not\succeq \cdots \not\succeq \mathrm{C}(i)$,
(KN-D2) if $t = \mathrm{C}(p) = \sigma(\mathrm{C}(q)) \in [n]$ for some $p, q \in [i]$, then $|q - p| > i - t$;

note that $n, \overline{n} \in \mathcal{D}_n$ may appear in C more than once. Let $\mathrm{KN}_{D_n}(\varpi_i)$ be the set of KN D_n-columns of shape ϖ_i.

Let $i \in [2, n-2]$. Let $\mathrm{C} : [i] \to \mathcal{D}_n$ be a map satisfying (KN-D1). Define $\mathrm{C}' : [i] \to \mathcal{B}_n$ to be such that $\mathrm{C}'(u) = \mathrm{C}'(u+1) = 0$ if $\mathrm{C}(u) = \overline{n}$ and $\mathrm{C}(u+1) = n$ for $u \in [i]$, and $\mathrm{C}'(u) = \mathrm{C}(u)$ otherwise. We say that C can be split if C' can be split. If $\mathrm{C} \in \mathrm{KN}_{D_n}(\varpi_i)$ can be split, then we write $r\mathrm{C} = r\mathrm{C}'$ and $l\mathrm{C} = l\mathrm{C}'$; note that $r\mathrm{C}, l\mathrm{C} \in \mathrm{CST}_{D_n}(\varpi_i)$. For convenience we also denote by $\mathrm{KN}_{D_n}(\varpi_i) = \mathrm{CST}_{D_n}(\varpi_i)$ for $i \in \{1, n-1, n\}$.

Define a $\mathfrak{g}$-crystal structure on $\mathrm{KN}_{D_n}(\varpi_i)$ for $i \in [n-2]$ as follows (cf. [18, §6]); for the $\mathfrak{g}$-crystal structure on $\mathrm{KN}_{D_n}(\varpi_i)$ for $i \in \{n-1, n\}$ we refer the reader to [21, §2.3]. The maps $\mathrm{wt}, \varepsilon_j, \phi_j$ for $j \in I$ and e_j, f_j for $j \in [n-2]$ are defined in the same manner as those for $\mathrm{KN}_{C_n}(\varpi_i)$. Note that only the letters $n-1, n, \overline{n}, \overline{n-1}$ may be changed in C when we apply e_{n-1}, e_n, f_{n-1}, or f_n. To define the action of f_{n-1}, let $\{\mathrm{C}(u) \mid u \in [i],\ \mathrm{C}(u) \in \{n-1, n, \overline{n}, \overline{n-1}\}\} = \{\mathrm{C}(u_1) \otimes \mathrm{C}(u_2) \otimes \cdots \otimes \mathrm{C}(u_p)\}$, where $1 \leqslant u_1 < u_2 < \cdots < u_p \leqslant i$, and continue deleting a successive pair $a \otimes b$ such that $(a, b) \in \{(\overline{n}, n), (n-1, n), (\overline{n}, \overline{n-1}), (n-1, \overline{n-1})\}$ from $\mathrm{C}(u_1) \otimes \mathrm{C}(u_2) \otimes \cdots \otimes \mathrm{C}(u_p)$

until no such pair exists. Let $\mathrm{C}(v_1) \otimes \mathrm{C}(v_2) \otimes \cdots \otimes \mathrm{C}(v_q)$ be the resulting sequence. It follows that

(161)
$$\begin{array}{|c|}\hline \mathrm{C}(v_1) \\ \hline \vdots \\ \hline \mathrm{C}(v_q) \\ \hline \end{array} \in \left\{ ?, \boxed{n-1}, \boxed{n}, \boxed{\overline{n}}, \boxed{\overline{n-1}}, \begin{array}{|c|}\hline n \\ \hline \overline{n} \\ \hline \end{array}, \begin{array}{|c|}\hline n-1 \\ \hline \overline{n} \\ \hline \end{array}, \begin{array}{|c|}\hline n \\ \hline \overline{n-1} \\ \hline \end{array} \right\}.$$

Then the action of f_{n-1} is uniquely determined from (161), and only one of the entries in (161) may be changed in C when we apply f_{n-1}. This is illustrated by

(162)
$$\boxed{n-1} \xrightarrow{f_{n-1}} \boxed{n}, \qquad \boxed{\overline{n}} \xrightarrow{f_{n-1}} \boxed{\overline{n-1}},$$
$$\begin{array}{|c|}\hline n-1 \\ \hline \overline{n} \\ \hline \end{array} \xrightarrow{f_{n-1}} \begin{array}{|c|}\hline n \\ \hline \overline{n} \\ \hline \end{array} \xrightarrow{f_{n-1}} \begin{array}{|c|}\hline n \\ \hline \overline{n-1} \\ \hline \end{array};$$

set $f_{n-1}\mathrm{C} = \mathbf{0}$ otherwise. We next define the action of f_n. Similarly, continue deleting a successive pair $a \otimes b$ such that $(a, b) \in \{(n, \overline{n}), (n-1, \overline{n}), (n, \overline{n-1}), (n-1, \overline{n-1})\}$ from $\mathrm{C}(u_1) \otimes \mathrm{C}(u_2) \otimes \cdots \otimes \mathrm{C}(u_p)$. Let $\mathrm{C}(v_1) \otimes \mathrm{C}(v_2) \otimes \cdots \otimes \mathrm{C}(v_r)$ be the resulting sequence. It follows that

(163)
$$\begin{array}{|c|}\hline \mathrm{C}(v_1) \\ \hline \vdots \\ \hline \mathrm{C}(v_r) \\ \hline \end{array} \in \left\{ ?, \boxed{n-1}, \boxed{n}, \boxed{\overline{n}}, \boxed{\overline{n-1}}, \begin{array}{|c|}\hline \overline{n} \\ \hline n \\ \hline \end{array}, \begin{array}{|c|}\hline n-1 \\ \hline n \\ \hline \end{array}, \begin{array}{|c|}\hline \overline{n} \\ \hline \overline{n-1} \\ \hline \end{array} \right\}.$$

Then the action of f_n is uniquely determined from (163), and only one of the entries in (163) may be changed in C when we apply f_n. This is illustrated by

(164)
$$\boxed{n-1} \xrightarrow{f_n} \boxed{\overline{n}}, \qquad \boxed{n} \xrightarrow{f_n} \boxed{\overline{n-1}},$$
$$\begin{array}{|c|}\hline n-1 \\ \hline n \\ \hline \end{array} \xrightarrow{f_n} \begin{array}{|c|}\hline \overline{n} \\ \hline n \\ \hline \end{array} \xrightarrow{f_n} \begin{array}{|c|}\hline \overline{n} \\ \hline \overline{n-1} \\ \hline \end{array};$$

set $f_n\mathrm{C} = \mathbf{0}$ otherwise.

The next lemma is a reformulation of [21, Corollary 3.1.11 and Remark 3.1.13] in terms of Maya diagrams.

Lemma 5.19. *Assume that $\mathfrak{g}$ and W are of type D_n.*

(1) *Assume that* $i \in \{1, n-1, n\}$. *The map* $\mathrm{LS}(\varpi_i) \to \mathrm{KN}_{D_n}(\varpi_i) = \mathrm{CST}_{D_n}(\varpi_i)$, $(w; 0, 1) \mapsto \mathrm{T}_w^{(i)}$, *is an isomorphism of* $\mathfrak{g}$*-crystals.*

(2) *Assume that* $i \in [2, n-2]$. *For a map* $\mathrm{C} : [i] \to \mathcal{D}_n$ *satisfying* (KN-D1), *we have* $\mathrm{C} \in \mathrm{KN}_{D_n}(\varpi_i)$ *if and only if* C *can be split. Hence the map* $\mathrm{KN}_{D_n}(\varpi_i) \to \mathrm{KN}_{B_n}(\varpi_i)$, $\mathrm{C} \mapsto \mathrm{C}$, *is injective. The map*

$$\mathrm{KN}_{D_n}(\varpi_i) \to \mathrm{LS}(\varpi_i),\ \mathrm{C} \mapsto (r\mathrm{C}, l\mathrm{C}), \tag{165}$$

is an isomorphism of $\mathfrak{g}$*-crystals. The inverse of* (165) *is given as follows. Let* $(v, w) \in \mathrm{LS}(\varpi_i)$, $J(w) = J(v) = (J_1 < \cdots < J_\mu) \in S_i$, $M(w) = (M_\nu)$, $M(v) = (N_\nu) \in \prod_{\nu=1}^{\mu} 2^{J_\nu}$, *and* $f = \# N_\mu - \# M_\mu \in 2\mathbb{Z}_{\geq 0}$. *The inverse image of* (v, w) *is the KN* D_n*-column* C *such that*

$$\begin{aligned} \mathrm{C} &= \{v(u) \mid u \in [i],\ v(u) \preceq n\} \sqcup \{w(u) \mid u \in [i],\ w(u) \succeq \overline{n}\} \sqcup \{\underbrace{0, 0, \ldots, 0}_{f \text{ times}}\} \\ &= \bigsqcup_{\nu=1}^{\mu} ((J_\nu \setminus N_\nu) \sqcup \{\overline{z} \mid z \in M_\nu\}) \sqcup \{\underbrace{0, 0, \ldots, 0}_{f \text{ times}}\}; \end{aligned} \tag{166}$$

we have $I_{\mathrm{C}} = \bigsqcup_{\nu=1}^{\mu}(M_\nu \setminus N_\nu) \sqcup \{\underbrace{0, 0, \ldots, 0}_{f \text{ times}}\}$ *and* $J_{\mathrm{C}} = \bigsqcup_{\nu=1}^{\mu}(N_\nu \setminus M_\nu)$.

Set

$$\tilde{\mathcal{D}}_n = \mathcal{D}_n \sqcup \{\overline{0}\} = \left\{ 1 \prec 2 \prec \cdots \prec n-1 \prec \begin{matrix} n \\ \overline{n} \end{matrix} \prec \overline{n-1} \prec \cdots \prec \overline{2} \prec \overline{1} \prec \overline{0} \right\}. \tag{167}$$

Definition 5.20. *Let* $i \in [2, n-2]$. *A map* $\tilde{\mathrm{C}} : [i] \to \tilde{\mathcal{D}}_n$ *is, by definition, a quantum Kashiwara–Nakashima* D_n*-column (QKN* D_n*-column for short) of shape* ϖ_i *if*

(QKN-D1) *there exists* $m \in \mathbb{Z}_{\geq 0}$ *such that* $2m \leq i$ *and*

$$\begin{aligned} &\tilde{\mathrm{C}}(1) \preceq \tilde{\mathrm{C}}(2) \preceq \cdots \preceq \tilde{\mathrm{C}}(i-2m) \\ &\prec \overline{0} = \tilde{\mathrm{C}}(i-2m+1) = \cdots = \tilde{\mathrm{C}}(i-1) = \tilde{\mathrm{C}}(i), \end{aligned} \tag{168}$$

(QKN-D2) *the map* $\mathrm{C} : [i-2m] \to \mathcal{D}_n$, $u \mapsto \tilde{\mathrm{C}}(u)$, *is a KN* D_n*-column of shape* ϖ_{i-2m};

in this case, we write $\tilde{\mathrm{C}} = \mathrm{C} \sqcup \{\underbrace{\overline{0}, \overline{0}, \ldots, \overline{0}}_{2m \text{ times}}\}$ *for brevity. Let* $\mathrm{QKN}_{D_n}(\varpi_i)$ *be the set of QKN* D_n*-columns of shape* ϖ_i. *For convenience we also denote by* $\mathrm{QKN}_{D_n}(\varpi_i) = \mathrm{CST}_{D_n}(\varpi_i)$ *for* $i \in \{1, n-1, n\}$.

For a map $\tilde{\mathrm{C}} : [i] \to \tilde{\mathcal{D}}_n$, *define* $\tilde{\mathrm{C}}^{\natural} : [i] \to \tilde{\mathcal{B}}_n$ *to be such that* $\tilde{\mathrm{C}}^{\natural}(u) = \tilde{\mathrm{C}}^{\natural}(u+1) = 0$ *if* $\tilde{\mathrm{C}}(u) = \overline{n}$ *and* $\tilde{\mathrm{C}}(u+1) = n$ *for* $u \in [i]$, *and* $\tilde{\mathrm{C}}^{\natural}(u) = \tilde{\mathrm{C}}(u)$ *otherwise. We see that if* $\tilde{\mathrm{C}} = \mathrm{C} \sqcup \{\underbrace{\overline{0}, \overline{0}, \ldots, \overline{0}}_{2m \text{ times}}\} \in \mathrm{QKN}_{D_n}(\varpi_i)$, *then* $\tilde{\mathrm{C}}^{\natural} \in \mathrm{QKN}_{B_n}(\varpi_i)$ *and, in consequence,* $\tilde{\mathrm{C}}^{\natural} = \mathrm{C}^{\natural} \sqcup \{\underbrace{\overline{0}, \overline{0}, \ldots, \overline{0}}_{2m \text{ times}}\}$. *For* $\tilde{\mathrm{C}} \in \mathrm{QKN}_{D_n}(\varpi_i)$, *define* $r\tilde{\mathrm{C}} = r\tilde{\mathrm{C}}^{\natural}$ *and* $l\tilde{\mathrm{C}} = l\tilde{\mathrm{C}}^{\natural}$; *note that* $r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}} \in \mathrm{CST}_{D_n}(\varpi_i)$.

Define a $\mathbf{U}'$-crystal structure on $\mathrm{QKN}_{D_n}(\varpi_i)$ in the same manner as that on $\mathrm{QKN}_{B_n}(\varpi_i)$.

Theorem 5.21. *Assume that* $\mathbf{U}$ *is of type* $D_n^{(1)}$.

(1) *Assume that* $i \in \{1, n-1, n\}$. *The set* $\mathrm{QKN}_{D_n}(\varpi_i)$ *equipped with the maps* $\mathrm{wt}, e_j, f_j, \varepsilon_j, \phi_j$, $j \in I_{\mathrm{af}}$, *is a* $\mathbf{U}'$*-crystal. The map* $\mathrm{QLS}(\varpi_i) = \mathrm{LS}(\varpi_i) \to \mathrm{QKN}_{D_n}(\varpi_i) = \mathrm{CST}_{D_n}(\varpi_i)$, $(w; 0, 1) \mapsto \mathrm{T}_w^{(i)}$, *is an isomorphism of* $\mathbf{U}'$*-crystals.*

(2) *Assume that* $i \in [2, n-2]$. *The set* $\mathrm{QKN}_{D_n}(\varpi_i)$ *equipped with the maps* $\mathrm{wt}, e_j, f_j, \varepsilon_j, \phi_j$, $j \in I_{\mathrm{af}}$, *is a* $\mathbf{U}'$*-crystal. The map*

$$\mathrm{QKN}_{D_n}(\varpi_i) \to \mathrm{QLS}(\varpi_i),\ \tilde{\mathsf{C}} \mapsto (r\tilde{\mathsf{C}}, l\tilde{\mathsf{C}}), \tag{169}$$

is an isomorphism of $\mathbf{U}'$*-crystals. The inverse of* (169) *is given as follows. Let* $(v, w) \in \mathrm{QLS}(\varpi_i)$, $J(w) = J(v) = (J_1 < \cdots < J_\mu) \in S_i$ *and* $M(w) = (M_\nu)$, $M(v) = (N_\nu) \in \prod_{\nu=1}^{\mu} 2^{J_\nu}$. *Set* $f = \# N_\mu - \# M_\mu \in 2\mathbb{Z}_{>0}$ *if* $n-1, n \in J_\mu$, *and set* $f = 0$ *otherwise. Set* $2m = \# M_1 - \# N_1 \in 2\mathbb{Z}_{>0}$ *if* $1, 2 \in J_1$, *and set* $m = 0$ *otherwise; we see from Lemma* 5.8 *that* $m = d_i(v, w)$. *Define*

(i) $\{y^1 < y^2 < \cdots < y^l\} = N_1 \setminus M_1$,
(ii) $z^1 = \min\{z \in J_1 \mid z \ge y^1,\ z \notin M_1 \setminus N_1\}$, *and*
(iii) $z^\nu = \min\{z \in J_1 \mid z \ge \max\{y^\nu, z^{\nu-1}\},\ z \notin M_1 \setminus N_1\}$ *for* $\nu \in [2, l]$.

Let C *be the KN* D_n*-column of shape* ϖ_{i-2m} *such that*

$$\begin{aligned}\mathsf{C} &= (J_1 \setminus (M_1 \cup N_1)) \sqcup \{z^\nu, \overline{z^\nu} \mid \nu \in [l]\} \sqcup \{\overline{z} \mid z \in M_1 \cap N_1\} \\ &\sqcup \bigsqcup_{\nu=2}^{\mu} ((J_\nu \setminus N_\nu) \sqcup \{\overline{z} \mid z \in M_\nu\}) \sqcup \{\underbrace{0, 0, \ldots, 0}_{f \text{ times}}\};\end{aligned} \tag{170}$$

we have $I_{\mathsf{C}} = \{z^\nu \mid \nu \in [l]\} \sqcup \bigsqcup_{\nu=2}^{\mu}(M_\nu \setminus N_\nu) \sqcup \{\underbrace{0, 0, \ldots, 0}_{f \text{ times}}\}$ *and* $J_{\mathsf{C}} = \bigsqcup_{\nu=1}^{\mu}(N_\nu \setminus M_\nu)$. *The inverse image of* (v, w) *is* $\tilde{\mathsf{C}} = \mathsf{C} \sqcup \{\underbrace{\overline{0}, \overline{0}, \ldots, \overline{0}}_{2m \text{ times}}\}$.

(3) *Assume that* $i \in [2, n-2]$. *The map*

$$\mathrm{QKN}_{D_n}(\varpi_i) \to \bigsqcup_{m=0}^{\lfloor i/2 \rfloor} \mathrm{KN}_{D_n}(\varpi_{i-2m}),\ \tilde{\mathsf{C}} = \mathsf{C} \sqcup \{\underbrace{\overline{0}, \overline{0}, \ldots, \overline{0}}_{2m \text{ times}}\} \mapsto \mathsf{C}, \tag{171}$$

is an isomorphism of $\mathfrak{g}$*-crystals, where* C *and* m *are as in* (QKN-D1)–(QKN-D2) *for* $\tilde{\mathsf{C}}$. *Here we understand that* $\mathrm{KN}_{D_n}(\varpi_0) = \{?\}$ *is a* $\mathfrak{g}$*-crystal isomorphic to the crystal basis of the trivial module. The inverse image of* $\mathsf{C} \in \mathrm{KN}_{D_n}(\varpi_{i-2m})$ *under the map* (171) *is* $\tilde{\mathsf{C}} = \mathsf{C} \sqcup \{\underbrace{\overline{0}, \overline{0}, \ldots, \overline{0}}_{2m \text{ times}}\}$.

The proof of (1) and (3) in Theorem 5.21 is straightforward (cf. [7, Lemma 2.7 (ii)]). Theorem 5.21 (2) may be proved in much the same way as Theorem 5.16 (2) (see §5.6); the details are left to the reader.

Recall the partial order $\succeq$ on $\mathrm{CST}_{D_n}(\varpi_i) \times \mathbb{Z}$ (see Definition 4.36).

DEFINITION 5.22. *Let* $i \in I$ *and* $m \in \mathbb{Z}_{>0}$.

(1) *Assume that* $i \in \{1, n-1, n\}$. *Let* $\mathsf{T} = (\mathsf{T}_1\mathsf{T}_2\cdots\mathsf{T}_m, (c_1, c_2, \ldots, c_m))$, *where* $\mathsf{T}_\nu \in \mathrm{CST}_{D_n}(\varpi_i)$ *and* $c_\nu \in \mathbb{Z}$ *for* $\nu \in [m]$. *We call* T *a semi-infinite KN* D_n*-tableau of shape* $m\varpi_i$ *if*

$$(\mathsf{T}_\nu, c_\nu) \succeq (\mathsf{T}_{\nu+1}, c_{\nu+1}) \tag{172}$$

in $\mathrm{CST}_{D_n}(\varpi_i) \times \mathbb{Z}$ *for* $\nu \in [m-1]$.

(2) *Assume that* $i \in [2, n-2]$. *Let* $\mathsf{T} = (\tilde{\mathsf{C}}_1\tilde{\mathsf{C}}_2\cdots\tilde{\mathsf{C}}_m, (c_1, c_2, \ldots, c_m))$, *where* $\tilde{\mathsf{C}}_\nu \in \mathrm{QKN}_{D_n}(\varpi_i)$ *and* $c_\nu \in \mathbb{Z}$ *for* $\nu \in [m]$. *We call* T *a semi-infinite KN* D_n*-tableau of shape* $m\varpi_i$ *if*

$$(l\tilde{\mathsf{C}}_\nu, c_\nu - d_i(r\tilde{\mathsf{C}}_\nu, l\tilde{\mathsf{C}}_\nu)) \succeq (r\tilde{\mathsf{C}}_{\nu+1}, c_{\nu+1} + d_i(r\tilde{\mathsf{C}}_{\nu+1}, l\tilde{\mathsf{C}}_{\nu+1})) \tag{173}$$

in $\mathrm{CST}_{D_n}(\varpi_i) \times \mathbb{Z}$ *for* $\nu \in [m-1]$.

Let $Y^{\frac{\infty}{2}}_{D_n}(m\ \ _i)$ *be the set of semi-infinite KN* D_n*-tableaux of shape* $m\ \ _i$*. For* $\lambda = \sum_{i\ \ I} m_i\ \ _i\ \ P^+$*, set* $Y^{\frac{\infty}{2}}_{D_n}(\lambda) = \prod_{i\ \ I} Y^{\frac{\infty}{2}}_{D_n}(m_i\ \ _i)$*. We call an element of* $Y^{\frac{\infty}{2}}_{D_n}(\lambda)$ *a semi-infinite KN* D_n*-tableau of shape* λ*.*

Let $\mathrm{QKN}_{D_n}(\ \ _i)_{\mathrm{af}}$ denote the affinization of the $\mathbf{U}$-crystal $\mathrm{QKN}_{D_n}(\ \ _i)$ (see §2.2). Combining Theorem 2.8, Proposition 4.37 (2), Lemma 5.9, and Definition 5.22 we obtain the following theorem.

Theorem 5.23. *Let* $\lambda = \sum_{i\ \ I} m_i\ \ _i\ \ P^+$*. For each* $i\ \ I$*, the image of the map*

$$(174)\qquad \begin{aligned} &Y^{\frac{\infty}{2}}_{D_n}(m_i\ \ _i) \to \mathrm{QKN}_{D_n}(\ \ _i)_{\mathrm{af}}^{m_i},\\ &(\mathrm{C}_1\mathrm{C}_2\cdots\mathrm{C}_{m_i},(c_1,c_2,\dots,c_{m_i})) \to \bigotimes_{\nu\ \ [m_i]}(\mathrm{C}_\nu,c_\nu),\end{aligned}$$

is a $\mathbf{U}$*-subcrystal. Hence we can define a* $\mathbf{U}$*-crystal structure on* $Y^{\frac{\infty}{2}}_{D_n}(m_i\ \ _i)$ *to be such that the map* (174) *is a strict embedding of* $\mathbf{U}$*-crystals. In particular,* $Y^{\frac{\infty}{2}}_{D_n}(\lambda)$ *is a* $\mathbf{U}$*-subcrystal of* $\bigotimes_{i\ \ I}\mathrm{QKN}_{D_n}(\ \ _i)_{\mathrm{af}}^{m_i}$*. Then* $Y^{\frac{\infty}{2}}_{D_n}(\lambda)$ *is isomorphic, as a* $\mathbf{U}$*-crystal, to the crystal basis* $B(\lambda)$*.*

5.6. Proof of Theorem 5.16 (2). Throughout this subsection, we assume that $\mathfrak{g}$ and W are of type B_n, and that $i\ \ [n-1]$. This subsection is devoted to the proof of Theorem 5.16 (2). We have divided the proof into a sequence of lemmas.

Recall the notation and terminology in §5.2 and §5.4. A segment is a subset of $[n]$ of the form $[j,k]$ with $j,k\ \ [n]$ and $j \leqslant k$. For segments $J = [j,k]$ and $J' = [j',k']$, write $J < J'$ if $k+1 < j'$. $\mathcal{S}_i$ is the family of all sequences $(J_1 < \cdots < J_\mu)$, $\mu \geqslant 1$, of segments such that $\sum_{\nu=1}^{\mu} \# J_\nu = i$. For $\mathbf{J} = (J_1 < \cdots < J_\mu)\ \ \mathcal{S}_i$, write $\bigsqcup\mathbf{J} = J_1 \sqcup \cdots \sqcup J_\mu$. For $\mathbf{J},\mathbf{J}'\ \ \mathcal{S}_i$, we have $\mathbf{J} = \mathbf{J}'$ if and only if $\bigsqcup\mathbf{J} = \bigsqcup\mathbf{J}'$. For $w\ \ W^{I \setminus \{i\}}$, let $\mathbf{J}(w)\ \ \mathcal{S}_i$ be such that $\bigsqcup\mathbf{J}(w) = \{\ w(u)\ \mid u\ \ [i]\}$. For $w\ \ W^{I\setminus\{i\}}$, with $\mathbf{J}(w) = (J_1 < \cdots < J_\mu)\ \ \mathcal{S}_i$, write $M(w) = (J_\nu \cap \{\ w(u)\ \mid u\ \ [i],\ w(u)\ \ \overline{n}\})_{\nu=1}^{\mu}\ \ \prod_{\nu=1}^{\mu} 2^{J_\nu}$ (see (125)).

Let $\tilde{\mathrm{C}} = \mathrm{C}\ \ \{\underbrace{\overline{0},\dots,\overline{0}}_{2m\text{ times}}\}\ \ \mathrm{QKN}_{B_n}(\ \ _i)$, with $\mathrm{C}\ \ \mathrm{KN}_{B_n}(\ \ _{i-2m})$ (see Definition 5.15). $I_\mathrm{C} = \{z_1\ \ z_2\ \ \cdots\ \ z_k\}$ is the multiset of $z\ \ \mathrm{C}$ such that $z\ \ 0$ and $\{z,\sigma(z)\}\ \ \mathrm{C}$. $J_\mathrm{C} = \{y_1 > y_2 > \cdots > y_k\}$ is a subset of $[n]$ such that $\# I_\mathrm{C} = \# J_\mathrm{C}$ and

(i) $y_1 = \max\{y\ \ B_n \mid y\ \ z_1,\ y \notin \mathrm{C},\ \overline{y} \notin \mathrm{C}\}$,

(ii) $y_\nu = \max\{y\ \ B_n \mid y\ \ \min\{y_{\nu-1},z_\nu\},\ y \notin \mathrm{C},\ \overline{y}\notin \mathrm{C}\}$ for $\nu\ \ [2,k]$.

We have

$$(175)\qquad \begin{aligned} r\mathrm{C} &= (\mathrm{C} \setminus \{\sigma(z) \mid z\ \ I_\mathrm{C}\})\ \ \{\sigma(y) \mid y\ \ J_\mathrm{C}\}\ \ \mathrm{CST}_{B_n}(\ \ _{i-2m}),\\ l\mathrm{C} &= (\mathrm{C} \setminus I_\mathrm{C})\ \ J_\mathrm{C}\ \ \mathrm{CST}_{B_n}(\ \ _{i-2m}).\end{aligned}$$

Then $K_{\tilde{\mathrm{C}}} = \{x_1 < x_2 < \cdots < x_{2m}\}$, where $\{x_1 < x_2 < \cdots < x_{n-i+2m}\} = [n] \setminus \{\ r\mathrm{C}(u)\ \mid u\ \ [i-2m]\}$. Note that $\# K_{\tilde{\mathrm{C}}} = 2m$. We have

$$(176)\qquad \begin{aligned} r\tilde{\mathrm{C}} &= K_{\tilde{\mathrm{C}}}\ \ r\mathrm{C}\ \ \mathrm{CST}_{B_n}(\ \ _i),\\ l\tilde{\mathrm{C}} &= \{\overline{x} \mid x\ \ K_{\tilde{\mathrm{C}}}\}\ \ l\mathrm{C}\ \ \mathrm{CST}_{B_n}(\ \ _i).\end{aligned}$$

By (75) and Lemma 4.11, there exist $w,v\ \ W^{I\setminus\{i\}}$ such that $\mathrm{T}^{(i)}_w = l\tilde{\mathrm{C}}$ and $\mathrm{T}^{(i)}_v = r\tilde{\mathrm{C}}$. We have $w(u) = l\tilde{\mathrm{C}}(u)$ and $v(u) = r\tilde{\mathrm{C}}(u)$ for $u\ \ [i]$. Likewise, there exist $w',v'\ \ W^{I\setminus\{i-2m\}}$ such that $\mathrm{T}^{(i-2m)}_{w'} = l\mathrm{C}$ and $\mathrm{T}^{(i-2m)}_{v'} = r\mathrm{C}$. It follows that

$$(177)\qquad \begin{aligned} \bigsqcup\mathbf{J}(w) &= \{\ l\tilde{\mathrm{C}}(u)\ \mid u\ \ [i]\} = K_{\tilde{\mathrm{C}}}\ \ \{\ l\mathrm{C}(u)\ \mid u\ \ [i]\} = K_{\tilde{\mathrm{C}}}\ \ \bigsqcup\mathbf{J}(w'),\\ \bigsqcup\mathbf{J}(v) &= \{\ r\tilde{\mathrm{C}}(u)\ \mid u\ \ [i]\} = K_{\tilde{\mathrm{C}}}\ \ \{\ r\mathrm{C}(u)\ \mid u\ \ [i]\} = K_{\tilde{\mathrm{C}}}\ \ \bigsqcup\mathbf{J}(v').\end{aligned}$$

We know from Lemmas 5.3 and 5.14 (2) that $(v, w) \in \mathrm{LS}(\varpi_{i-2m})$ and $J(w) = J(v)$. Hence $J(w) = J(v)$, by (177).

In what follows, we freely identify w, $\mathrm{T}_w^{(i)}$, and $M(w)$ with each other for $w \in W^{I \setminus \{i\}}$. Write $J(\mathrm{T}_w^{(i)}) = J(w)$ and $M(\mathrm{T}_w^{(i)}) = M(w)$ for $w \in W^{I \setminus \{i\}}$. From what has already been proved, we have $J(l\tilde{\mathrm{C}}) = J(r\tilde{\mathrm{C}})$ for $\tilde{\mathrm{C}} \in \mathrm{QKN}_{B_n}(\varpi_i)$.

Lemma 5.24. *If $\tilde{\mathrm{C}} \in \mathrm{QKN}_{B_n}(\varpi_i)$, then $(r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) \in \mathrm{QLS}(\varpi_i)$.*

Proof. Let $\tilde{\mathrm{C}} = \mathrm{C} \sqcup \{\underbrace{\overline{0}, \ldots, \overline{0}}_{2m \text{ times}}\} \in \mathrm{QKN}_{B_n}(\varpi_i)$, with $\mathrm{C} \in \mathrm{KN}_{B_n}(\varpi_{i-2m})$. Write $J(l\tilde{\mathrm{C}}) = J(r\tilde{\mathrm{C}}) = (J_1 < \cdots < J_\mu) \in S_i$, $J(l\mathrm{C}) = J(r\mathrm{C}) = (J_1 < \cdots < J_\kappa) \in S_{i-2m}$, $M(l\tilde{\mathrm{C}}) = (M_\nu)$, $M(r\tilde{\mathrm{C}}) = (N_\nu) \in \prod_{\nu=1}^{\mu} 2^{J_\nu}$, and $M(l\mathrm{C}) = (M_\nu)$, $M(r\mathrm{C}) = (N_\nu) \in \prod_{\nu=1}^{\kappa} 2^{J_\nu}$. We know from (177) that $\bigsqcup_{\nu=1}^{\mu} J_\nu = K_{\tilde{\mathrm{C}}} \sqcup \bigsqcup_{\nu=1}^{\kappa} J_\nu$. Recall that J_1 is a segment. It follows from the definition of $K_{\tilde{\mathrm{C}}}$ that there exists $\tau \in [0, \kappa]$ such that $J_1 = K_{\tilde{\mathrm{C}}} \sqcup \bigsqcup_{\nu=1}^{\tau} J_\nu$, $M_1 = K_{\tilde{\mathrm{C}}} \sqcup \bigsqcup_{\nu=1}^{\tau} M_\nu$, $N_1 = \bigsqcup_{\nu=1}^{\tau} N_\nu$, $J_\nu = J_{\nu+\tau-1}$, $M_\nu = M_{\nu+\tau-1}$, $N_\nu = N_{\nu+\tau-1}$ for $\nu \in [2, \mu]$, and $\mu + \tau - 1 = \kappa$.

We first assume that $n \in J_\tau$ $(\subset J_1)$. Then $1 \notin J_1$ since J_1 is a segment such that $\# J_1 \le i < n$. Hence $1 \notin K_{\tilde{\mathrm{C}}} \subset J_1$. By the definition of $K_{\tilde{\mathrm{C}}}$, this implies $K_{\tilde{\mathrm{C}}} = \emptyset$ and $m = 0$. It follows that $(r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) = (r\mathrm{C}, l\mathrm{C}) \in \mathrm{LS}(\varpi_i) \subset \mathrm{QLS}(\varpi_i)$.

We next assume that $n \notin J_\tau$ (or equivalently, $n \notin J_\nu$ for $\nu \in [\tau]$), and show that $(M_\nu) \succeq (N_\nu)$ in $\prod_{\nu=1}^{\mu} 2^{J_\nu}$ (see Definition 5.6 and (128)). Since $(r\mathrm{C}, l\mathrm{C}) \in \mathrm{LS}(\varpi_{i-2m})$, we see from (129) that $M_\nu \succeq N_\nu$ in 2^{J_ν} for $\nu \in [\kappa]$. In particular, it follows from Definition 5.4, $\{1,2\} \not\subset J_\nu$, $J_\nu = J_{\nu+\tau-1}$, $M_\nu = M_{\nu+\tau-1}$, and $N_\nu = N_{\nu+\tau-1}$ for $\nu \in [2, \mu]$ that $M_\nu \succeq N_\nu$ in 2^{J_ν} for $\nu \in [2, \mu]$. Also, $n \notin J_\tau$, Lemma 5.5 (1) and (4) yield $\# M_\nu = \# N_\nu$ for $\nu \in [\tau]$. Since $M_1 = K_{\tilde{\mathrm{C}}} \sqcup \bigsqcup_{\nu=1}^{\tau} M_\nu$ and $N_1 = \bigsqcup_{\nu=1}^{\tau} N_\nu$, we have $\# M_1 - \# N_1 = \# K_{\tilde{\mathrm{C}}} = 2m \in 2\mathbb{Z}_{>0}$. Write $\bigsqcup_{\nu=1}^{\tau} M_\nu = \{m_1 < m_2 < \cdots < m_s\} \subset M_1 = \{m_1 < m_2 < \cdots < m_r\}$ and $N_1 = \{n_1 < n_2 < \cdots < n_s\}$, where $r = s + 2m$. It is clear that $m_{r-\nu} > m_{s-\nu}$ for $\nu \in [0, s-1]$. Since $M_\nu \succeq N_\nu$ in 2^{J_ν} for $\nu \in [\tau]$, we have $m_{s-\nu} > n_{s-\nu}$ for $\nu \in [0, s-1]$, by Lemma 5.5 (1) and (4). Consequently, $m_{r-\nu} > n_{s-\nu}$ for $\nu \in [0, s-1]$, and hence $M_1 \succeq N_1$ in 2^{J_1}, by Lemma 5.5 (1) and (4). We conclude that $(M_\nu) \succeq (N_\nu)$ in $\prod_{\nu=1}^{\mu} 2^{J_\nu}$, and so $(r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) \in \mathrm{QLS}(\varpi_i)$, by (128).

Lemma 5.25. *Let $(v, w) \in \mathrm{QLS}(\varpi_i)$, $J(w) = J(v) = (J_1 < \cdots < J_\mu) \in S_i$, and $M(w) = (M_\nu)$, $M(v) = (N_\nu) \in \prod_{\nu=1}^{\mu} 2^{J_\nu}$. Write $\{y^1 < y^2 < \cdots < y^l\} = N_1 \setminus M_1$. The elements $z^1 = \min\{z \in J_1 \mid z \ge y^1,\ z \in M_1 \setminus N_1\}$ and $z^\nu = \min\{z \in J_1 \mid z \ge \max\{y^\nu, z^{\nu-1}\},\ z \in M_1 \setminus N_1\}$, $\nu \in [2, l]$, are well-defined* (*see* (i)–(iii) *in Theorem* 5.16 (2)).

Proof. We see from Lemma 5.5 (1)–(2) and (4) that our assertion follows from $(M_1 \setminus N_1) \succeq (N_1 \setminus M_1)$ in 2^{J_1}. Write (temporarily) $J_1 = [p, q]$ and $M_1 \setminus N_1 = \{w^1 < w^2 < \cdots < w^k\}$.

We first assume that ($n \notin J_1$ and $\{1,2\} \not\subset J_1$) or ($n \in J_1$). Since $M_1 \succeq N_1$ by (128), we see from Lemma 5.5 (1)–(2) that $\#(M_1 \setminus N_1) - \#(N_1 \setminus M_1) = \# M_1 - \# N_1 \le 0$ and $\#(M_1 \cap [p, w]) \le \#(N_1 \cap [p, w])$ for all $w \in J_1$. On the contrary, suppose that $(M_1 \setminus N_1) \not\succeq (N_1 \setminus M_1)$ in 2^{J_1}. By Lemma 5.5 (1)–(2), there exists $r \in [k]$ such that $w^r < y^r$ and $w^\nu > y^\nu$ for $\nu \in [r-1]$. This implies $\#(M_1 \cap [p, w^r]) - \#(N_1 \cap [p, w^r]) = 1 > 0$, a contradiction.

We next assume that $\{1,2\} \subset J_1$. Since $M_1 \succeq N_1$ by (128), we see from Lemma 5.5 (4) that $\#(M_1 \setminus N_1) - \#(N_1 \setminus M_1) = \# M_1 - \# N_1 \in 2\mathbb{Z}_{>0}$ and $\#(M_1 \cap [y, q]) > \#(N_1 \cap [y, q])$ for all $y \in [p, q]$. On the contrary, suppose that $(M_1 \setminus N_1) \not\succeq (N_1 \setminus M_1)$ in 2^{J_1}. By Lemma 5.5 (4), there exists $r \in [0, l-1]$ such that $w^{k-r} < y^{l-r}$ and $w^{k-\nu} > y^{l-\nu}$

for $\nu \in [0, r-1]$. This implies $\#(M_1 \cap [y^{l-r}, q]) - \#(N_1 \cap [y^{l-r}, q]) = -1 < 0$, a contradiction.

Lemma 5.26. *Under the hypotheses of Lemma* 5.25*, set* $f = \#N_\mu - \#M_\mu \in \mathbb{Z}_{>0}$ *if* $n \in J_\mu$*, and set* $f = 0$ *otherwise. Set* $2m = \#M_1 - \#N_1 \in 2\mathbb{Z}_{>0}$ *if* $1, 2 \in J_1$*, and set* $m = 0$ *otherwise. Then*

$$\tilde{C} = (J_1 \setminus (M_1 \sqcup N_1)) \sqcup \{z^\nu, \overline{z^\nu} \mid \nu \in [l]\} \sqcup \{\overline{z} \mid z \in M_1 \cap N_1\}$$

$$\sqcup \bigsqcup_{\nu=2}^{\mu} ((J_\nu \setminus N_\nu) \sqcup \{\overline{z} \mid z \in M_\nu\}) \sqcup \{\underbrace{0, \dots, 0}_{f \text{ times}}\} \sqcup \{\underbrace{\overline{0}, \dots, \overline{0}}_{2m \text{ times}}\} \tag{178}$$

is a QKN B_n*-column of shape* ϖ_i *(see* (155)*).*

Proof. It suffices to prove that

$$C = (J_1 \setminus (M_1 \sqcup N_1)) \sqcup \{z^\nu, \overline{z^\nu} \mid \nu \in [l]\} \sqcup \{\overline{z} \mid z \in M_1 \cap N_1\}$$

$$\sqcup \bigsqcup_{\nu=2}^{\mu} ((J_\nu \setminus N_\nu) \sqcup \{\overline{z} \mid z \in M_\nu\}) \sqcup \{\underbrace{0, \dots, 0}_{f \text{ times}}\} \tag{179}$$

is a KN B_n-column of shape ϖ_{i-2m}. It is easily seen that C satisfies (KN-B1)–(KN-B2) (see §5.4). By Lemma 5.14 (2), we only need to show that C can be split. We see at once that $I_C = \{z^\nu \mid \nu \in [l]\} \sqcup \bigsqcup_{\nu=2}^{\mu}(M_\nu \setminus N_\nu) \sqcup \{\underbrace{0, \dots, 0}_{f \text{ times}}\}$. We show that J_C is given by $\bigsqcup_{\nu=1}^{\mu}(N_\nu \setminus M_\nu)$.

We see that Lemma 5.25 (and (i)–(iii) in Theorem 5.16 (2)) imply

(i) $y^l = \max\{y \in J_1 \mid y \leq z^l,\ y \in N_1 \setminus M_1\}$,
(ii) $y^\nu = \max\{y \in J_1 \mid y \leq \min\{y^{\nu+1}, z^\nu\},\ y \in N_1 \setminus M_1\}$ for $\nu \in [l-1]$.

It follows from (179) that for $y \in J_1$ we have $y \in N_1 \setminus M_1$ if and only if $y \notin C$ and $\overline{y} \notin C$. Since J_1 is a segment, $N_1 \setminus M_1 \subset J_1$, and $z^\nu \in J_1$ for $\nu \in [l]$, we can rewrite (i)–(ii) as

(i') $y^l = \max\{y \in B_n \mid y \leq z^l,\ y \notin C,\ \overline{y} \notin C\}$,
(ii') $y^\nu = \max\{y \in B_n \mid y \leq \min\{y^{\nu+1}, z^\nu\},\ y \notin C,\ \overline{y} \notin C\}$ for $\nu \in [l-1]$.

Let $\tau \in [2, \mu]$. Write $(M_\mu \setminus N_\mu) \sqcup \{\underbrace{0, \dots, 0}_{f \text{ times}}\} = \{z_1^\mu \geq z_2^\mu \geq \cdots \geq z_k^\mu\}$, $M_\tau \setminus N_\tau = \{z_1^\tau > z_2^\tau > \cdots > z_k^\tau\}$ if $\tau = \mu$, and $N_\tau \setminus M_\tau = \{y_1^\tau > y_2^\tau > \cdots > y_k^\tau\}$. Similarly to (i')–(ii'), we can deduce from $N_\tau \trianglerighteq M_\tau$, Lemmas 5.5 and 5.14 (2) that

(τ-i) $y_1^\tau = \max\{y \in B_n \mid y \leq z_1^\tau,\ y \notin C,\ \overline{y} \notin C\}$,
(τ-ii) $y_\nu^\tau = \max\{y \in B_n \mid y \leq \min\{y_{\nu-1}^\tau, z_\nu^\tau\},\ y \notin C,\ \overline{y} \notin C\}$ for $\nu \in [2, k]$.

We conclude from $\#I_C = \#\bigsqcup_{\nu=1}^{\mu}(N_\nu \setminus M_\nu)$, (i')–(ii'), and ($\tau$-i)–($\tau$-ii) for $\tau \in [2, \mu]$ that $J_C = \bigsqcup_{\nu=1}^{\mu}(N_\nu \setminus M_\nu)$. This completes the proof.

By Lemmas 5.24–5.26, we obtain the maps $\Phi_i : \mathrm{QKN}_{B_n}(\varpi_i) \to \mathrm{QLS}(\varpi_i)$, $\tilde{C} \mapsto (r\tilde{C}, l\tilde{C})$, and $\Psi_i : \mathrm{QLS}(\varpi_i) \to \mathrm{QKN}_{B_n}(\varpi_i)$, $(v, w) \mapsto \tilde{C}$, where $\tilde{C}$ is defined as (155) or (178).

Lemma 5.27. *The maps* Φ_i *and* Ψ_i *are inverses of each other.*

Proof. By Lemma 5.14 (2), the map Φ_i is injective. The proof is completed by showing that $(\Phi_i \circ \Psi_i)(v, w) = (v, w)$ for $(v, w) \in \mathrm{QLS}(\varpi_i)$.

Let $(v, w) \in \mathrm{QLS}(\varpi_i)$ and $J(w) = J(v) = (J_1 < \cdots < J_\mu) \in S_i$. If $\{1, 2\} \not\subset J_1$, then we conclude from Definition 5.4 and (129) that $(v, w) \in \mathrm{LS}(\varpi_i)$, hence that

$(\Phi_i \circ \Psi_i)(v,w) = (v,w)$ by Lemma 5.14 (2). Therefore we may and do assume that $\{1,2\} \subset J_1$.

Let $\tilde{C} = C \sqcup \{\underbrace{\overline{0},\dots,\overline{0}}_{2m \text{ times}}\} = \Psi_i(v,w) \in \mathrm{QKN}_{B_n}(\lambda_i)$, with $C \in \mathrm{KN}_{B_n}(\lambda_{i-2m})$, and let $(r\tilde{C}, l\tilde{C}) = \Phi_i(\tilde{C}) \in \mathrm{QLS}(\lambda_i)$. Write $M(w) = (M_\nu)$, $M(v) = (N_\nu) \in \prod_{\nu=1}^{\mu} 2^{J_\nu}$, $J(l\tilde{C}) = J(r\tilde{C}) = (\tilde{J}_1 < \cdots < \tilde{J}_{\tilde{\mu}}) \in S_i$, and $M(l\tilde{C}) = (\tilde{M}_\nu)$, $M(r\tilde{C}) = (\tilde{N}_\nu) \in \prod_{\nu=1}^{\tilde{\mu}} 2^{\tilde{J}_\nu}$. Note that $\tilde{C}$ and C are described by (178)–(179). Our claim is that $(r\tilde{C}, l\tilde{C}) = (v,w)$. It suffices to show that $\mu = \tilde{\mu}$, $J_\nu = \tilde{J}_\nu$, $M_\nu = \tilde{M}_\nu$, and $N_\nu = \tilde{N}_\nu$ for $\nu \in [\mu]$.

We know from (the proof of) Lemma 5.26 that $I_C = \{z^\nu \mid \nu \in [l]\} \sqcup \bigsqcup_{\nu=2}^{\mu}(M_\nu \setminus N_\nu) \sqcup \{\underbrace{0,\dots,0}_{f \text{ times}}\}$ and $J_C = \bigsqcup_{\nu=1}^{\mu}(N_\nu \setminus M_\nu)$. By (175)–(176), we have

$$r\tilde{C} = \underbrace{K_{\tilde{C}} \sqcup (J_1 \setminus (M_1 \cup N_1)) \sqcup \{z^\nu \mid \nu \in [l]\} \sqcup \{\overline{z} \mid z \in N_1\}}_{= r\tilde{C}_1} \sqcup \bigsqcup_{\nu=2}^{\mu} (\underbrace{(J_\nu \setminus N_\nu) \sqcup \{\overline{z} \mid z \in N_\nu\}}_{= r\tilde{C}_\nu}), \tag{180}$$

$$l\tilde{C} = \underbrace{\{\overline{x} \mid x \in K_{\tilde{C}}\} \sqcup (J_1 \setminus M_1) \sqcup \{\overline{z^\nu} \mid \nu \in [l]\} \sqcup \{\overline{z} \mid z \in M_1 \cap N_1\}}_{= l\tilde{C}_1} \sqcup \bigsqcup_{\nu=2}^{\mu} (\underbrace{(J_\nu \setminus M_\nu) \sqcup \{\overline{z} \mid z \in M_\nu\}}_{= l\tilde{C}_\nu}); \tag{181}$$

set $r\tilde{C}_1 = K_{\tilde{C}} \sqcup (J_1 \setminus (M_1 \cup N_1)) \sqcup \{z^\nu \mid \nu \in [l]\} \sqcup \{\overline{z} \mid z \in N_1\}$, $l\tilde{C}_1 = \{\overline{x} \mid x \in K_{\tilde{C}}\} \sqcup (J_1 \setminus M_1) \sqcup \{\overline{z^\nu} \mid \nu \in [l]\} \sqcup \{\overline{z} \mid z \in M_1 \cap N_1\}$, $r\tilde{C}_\nu = (J_\nu \setminus N_\nu) \sqcup \{\overline{z} \mid z \in N_\nu\}$, and $l\tilde{C}_\nu = (J_\nu \setminus M_\nu) \sqcup \{\overline{z} \mid z \in M_\nu\}$ for $\nu \in [2,\mu]$. If we prove that

$$\{|z| \mid z \in r\tilde{C}_\nu\} = \{|z| \mid z \in l\tilde{C}_\nu\} = J_\nu \tag{182}$$

for $\nu \in [\mu]$, then we see at once from (180)–(181) that $\mu = \tilde{\mu}$, $J_\nu = \tilde{J}_\nu$ for $\nu \in [\mu]$, $M_\nu = \tilde{M}_\nu$ for $\nu \in [2,\mu]$, and $N_\nu = \tilde{N}_\nu$ for $\nu \in [\mu]$; the proof for $M_1 = \tilde{M}_1$ will be given later. By (180)–(181), it is easy to check that (182) holds for $\nu \in [2,\mu]$.

We give the proof only for $\{|z| \mid z \in l\tilde{C}_1\} = J_1$; the proof for $\{|z| \mid z \in r\tilde{C}_1\} = J_1$ is similar. Note that

$$\{|z| \mid z \in l\tilde{C}_1\} = K_{\tilde{C}} \sqcup (J_1 \setminus M_1) \sqcup \{z^\nu \mid \nu \in [l]\} \sqcup (M_1 \cap N_1). \tag{183}$$

It is clear that $(J_1 \setminus M_1) \sqcup \{z^\nu \mid \nu \in [l]\} \sqcup (M_1 \cap N_1) \subset J_1$. Recall that J_1 is a segment. Therefore, by the definition of $K_{\tilde{C}}$, it suffices to show that $\#\{|z| \mid z \in l\tilde{C}_1\} = \#J_1$. Since $l = \#(N_1 \setminus M_1)$ and $\#K_{\tilde{C}} = \#M_1 - \#N_1$, we have $\#M_1 = \#K_{\tilde{C}} + l + \#(N_1 \cap M_1)$. From this, $M_1 \subset J_1$, and (183), we conclude that $\#\{|z| \mid z \in l\tilde{C}_1\} = \#J_1$.

It remains to prove that $M_1 = \tilde{M}_1$. We have

$$\begin{aligned} \tilde{M}_1 &= \{|z| \mid z \in l\tilde{C}_1,\ z \in \overline{n}\} \quad \text{(by definition)} \\ &= K_{\tilde{C}} \sqcup \{z^\nu \mid \nu \in [l]\} \sqcup (M_1 \cap N_1) \quad \text{(by (181))} \\ &= M_1 \quad \text{(by (182)–(183))}. \end{aligned}$$

This completes the proof.

Let $\tilde{C} \in \mathrm{QKN}_{B_n}(\lambda_i)$, $\Phi_i(\tilde{C}) = (r\tilde{C}, l\tilde{C}) \in \mathrm{QLS}(\lambda_i)$, $J(l\tilde{C}) = J(r\tilde{C}) = (J_1 < \cdots < J_\mu) \in S_i$, and $M(l\tilde{C}) = (M_\nu)$, $M(r\tilde{C}) = (N_\nu) \in \prod_{\nu=1}^{\mu} 2^{J_\nu}$. We continue to use the notation $r\tilde{C}_\nu$, $l\tilde{C}_\nu$, $\nu \in [\mu]$, in (180)–(181). We see from (128)–(129) that $(r\tilde{C}_1, l\tilde{C}_1)$

$\mathrm{QLS}(\varpi_{\# J_1})$ and $(r\tilde{\mathrm{C}}_\nu, l\tilde{\mathrm{C}}_\nu) \in \mathrm{LS}(\varpi_{\# J_\nu})$ for $\nu \in [2, \mu]$. Write $\tilde{\mathrm{C}}_1 = \Psi_{\# J_1}(r\tilde{\mathrm{C}}_1, l\tilde{\mathrm{C}}_1) \in \mathrm{QKN}_{B_n}(\varpi_{\# J_1})$ and $\tilde{\mathrm{C}}_\nu = \Psi_{\# J_\nu}(r\tilde{\mathrm{C}}_\nu, l\tilde{\mathrm{C}}_\nu) \in \mathrm{KN}_{B_n}(\varpi_{\# J_\nu})$ for $\nu \in [2, \mu]$. It follows from (the proofs of) Lemmas 5.26–5.27 that

$$\tilde{\mathrm{C}} = \Psi_i(r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) = \bigsqcup_{\nu=1}^{\mu} \Psi_{\# J_\nu}(r\tilde{\mathrm{C}}_\nu, l\tilde{\mathrm{C}}_\nu) = \bigsqcup_{\nu=1}^{\mu} \tilde{\mathrm{C}}_\nu. \tag{184}$$

Set $p_\nu = \min J_\nu$ and $J_\nu = J_\nu \cup \{p_\nu - 1\} \subset I_{\mathrm{af}}$ for $\nu \in [\mu]$.

Lemma 5.28. *With the notation above, we have*

$$f_j\tilde{\mathrm{C}} = \begin{cases} f_0\tilde{\mathrm{C}}_1 \sqcup \bigsqcup_{\nu \in [2,\mu]} \tilde{\mathrm{C}}_\nu & \text{if } j = 0 \text{ and } f_0\tilde{\mathrm{C}}_1 = \mathbf{0}, \\ f_j\tilde{\mathrm{C}}_\tau \sqcup \bigsqcup_{\nu \in [\mu] \setminus \{\tau\}} \tilde{\mathrm{C}}_\nu & \text{if } j \in J_\tau \setminus \{0\} \text{ and } f_j\tilde{\mathrm{C}}_\tau = \mathbf{0} \text{ for some } \tau \in [\mu], \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{185}$$

Proof. The assertion for $j \in I$ is immediate from the definition of f_j (see (133)–(136) and (145)). Assume that $j = 0$, and show that $f_0\tilde{\mathrm{C}}$ is determined from $\tilde{\mathrm{C}}_1$ (and J_1). We follow the notation in (150)–(152).

If $m = 0$ and $y_k \notin \{1, 2\}$, then $f_0\tilde{\mathrm{C}}$ is determined from $\tilde{\mathrm{C}} \cap \{1, 2, \overline{2}, \overline{1}\}$ (see (150)). We see from (184) that $\tilde{\mathrm{C}} \cap \{1, 2, \overline{2}, \overline{1}\} \subset \tilde{\mathrm{C}}_1$. Hence the assertion follows.

If $m > 0$ and $y_k \notin \{1, 2\}$, then $f_0\tilde{\mathrm{C}}$ is determined from $\tilde{\mathrm{C}} \cap \{1, 2, \overline{2}, \overline{1}\}$ and x_2 (see (151)). Since $\tilde{\mathrm{C}} \cap \{1, 2, \overline{2}, \overline{1}\} \subset \tilde{\mathrm{C}}_1$ and $x_2 \in K_{\tilde{\mathrm{C}}} \subset J_1$, the assertion follows.

If $y_k \in \{1, 2\}$, then $f_0\tilde{\mathrm{C}}$ is determined from $\tilde{\mathrm{C}} \cap \{1, 2, \overline{2}, \overline{1}\}$ and z_k (see (152)). Since $\tilde{\mathrm{C}} \cap \{1, 2, \overline{2}, \overline{1}\} \subset \tilde{\mathrm{C}}_1$, it follows that $y_k \in J_1$, hence that $z_k \in J_1$ by the proof of Lemma 5.26, and the assertion follows.

Let $\eta = (v, w) \in \mathrm{QLS}(\varpi_i)$ and $j \in I_{\mathrm{af}}$. Recall from (31)–(32) that

$$\overline{\eta}(t) = \begin{cases} tv\varpi_i & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ \frac{1}{2}v\varpi_i + \left(t - \frac{1}{2}\right)w\varpi_i & \text{for } \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

and $h_j^\eta(t) = \langle \alpha_j^\vee, \overline{\eta}(t)\rangle$ for $0 \leqslant t \leqslant 1$; note that the function $h_j^\eta(t)$ is uniquely determined from $\langle \alpha_j^\vee, v\varpi_i\rangle$ and $\langle \alpha_j^\vee, w\varpi_i\rangle$. We see from (35)–(36) that $f_j(v, w)$ is determined from $h_j^\eta(t)$. Consequently, $f_j(v, w) \in \{\mathbf{0}, (r_jv, w), (v, r_jw), (r_jv, r_jw)\}$ is determined from $\langle \alpha_j^\vee, v\varpi_i\rangle, \langle \alpha_j^\vee, w\varpi_i\rangle \in \{-2, -1, 0, 1, 2\}$. Note that $h_j^\eta(1)$ is an integer. We know from [25, Proposition 4.1.12] that all local minima of $h_j^\eta(t)$ are integers. Hence $(\langle \alpha_j^\vee, v\varpi_i\rangle, \langle \alpha_j^\vee, w\varpi_i\rangle)$ is neither $(-2, -1), (-2, 1), (-1, -2), (-1, 0), (-1, 1), (-1, 2), (0, -1), (0, 1), (1, -2), (1, 0), (1, 2), (2, -1)$, nor $(2, 1)$. It follows that

$$f_j(v, w) = \begin{cases} (r_jv, w) & \text{if } (\langle \alpha_j^\vee, v\varpi_i\rangle, \langle \alpha_j^\vee, w\varpi_i\rangle) \in \{(2, 0), (2, 2)\}, \\ (v, r_jw) & \text{if } (\langle \alpha_j^\vee, v\varpi_i\rangle, \langle \alpha_j^\vee, w\varpi_i\rangle) \in \{(-2, 2), (0, 2)\}, \\ (r_jv, r_jw) & \text{if } (\langle \alpha_j^\vee, v\varpi_i\rangle, \langle \alpha_j^\vee, w\varpi_i\rangle) = (1, 1), \end{cases} \tag{186}$$

and $f_j(v, w) = \mathbf{0}$ if $(\langle \alpha_j^\vee, v\varpi_i\rangle, \langle \alpha_j^\vee, w\varpi_i\rangle)$ is either

$$(-2, -2), (-2, 0), (-1, -1), (0, -2), (0, 0), (1, -1), \text{ or } (2, -2). \tag{187}$$

We now give a restatement of (186)–(187) in terms of tableaux. Let $(r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) \in \mathrm{QLS}(\varpi_i)$. Recall that W_{af} acts on $\mathcal{C}_n = \{\pm\varepsilon_s \mid s \in [n]\}$ by $W_{\mathrm{af}} \to \mathfrak{S}(\mathcal{C}_n)$, $r_0 \mapsto (1\ \overline{2})(\overline{1}\ 2)$, $r_j \mapsto (j\ j+1)(\overline{j}\ \overline{j+1})$ for $j \in [n-1]$, and $r_n \mapsto (n\ \overline{n})$.

We first assume that $j \in [n-1]$. Write $[l\tilde{\mathrm{C}}]_j = l\tilde{\mathrm{C}} \cap \{j, j+1, \overline{j+1}, \overline{j}\}$ and $[r\tilde{\mathrm{C}}]_j = r\tilde{\mathrm{C}} \cap \{j, j+1, \overline{j+1}, \overline{j}\}$. If $[r\tilde{\mathrm{C}}]_j$ (resp. $[l\tilde{\mathrm{C}}]_j$) $\in \{\{j\}, \{j, \overline{j+1}\}\}$, then define $r_j r\tilde{\mathrm{C}}$ (resp. $r_j l\tilde{\mathrm{C}}$) $\in \mathrm{CST}_{B_n}(\varpi_i)$ to be such that $(r_j r\tilde{\mathrm{C}})(u) = r_j(r\tilde{\mathrm{C}}(u))$ (resp. $(r_j l\tilde{\mathrm{C}})(u) = r_j(l\tilde{\mathrm{C}}(u))$) for $u \in [i]$. Recall that $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$. Since $\varpi_i = \sum_{s=1}^{i} \varepsilon_s$, we have $w\varpi_i = \sum_{u=1}^{i} \varepsilon_{l\mathrm{C}(u)}$ and $v\varpi_i = \sum_{u=1}^{i} \varepsilon_{r\mathrm{C}(u)}$; we understand that $\varepsilon_{\overline{s}} = -\varepsilon_s$ for $s \in [n]$. Hence we can rewrite (186)–(187) as

$$
(188)\quad f_j(r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) = \begin{cases} (r_j r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) & \text{if } [r\tilde{\mathrm{C}}]_j = \{j, \overline{j+1}\} \\ & \text{and } [l\tilde{\mathrm{C}}]_j \in \{\{j, j+1\}, \{j, \overline{j+1}\}, \{\overline{j}, \overline{j+1}\}\}, \\ (r\tilde{\mathrm{C}}, r_j l\tilde{\mathrm{C}}) & \text{if } [r\tilde{\mathrm{C}}]_j \in \{\{j, j+1\}, \{\overline{j}, j+1\}, \{\overline{j}, \overline{j+1}\}\} \\ & \text{and } [l\tilde{\mathrm{C}}]_j = \{j, \overline{j+1}\}, \\ (r_j r\tilde{\mathrm{C}}, r_j l\tilde{\mathrm{C}}) & \text{if } [r\tilde{\mathrm{C}}]_j = [l\tilde{\mathrm{C}}]_j \in \{\{j\}, \{\overline{j+1}\}\}, \end{cases}
$$

and $f_j(r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) = \mathbf{0}$ if $([r\tilde{\mathrm{C}}]_j, [l\tilde{\mathrm{C}}]_j)$ is either

$$
(189)\quad \begin{aligned} &(\{j+1, \overline{j}\}, \{j+1, \overline{j}\}), (\{j+1, \overline{j}\}, \{j, j+1\}), (\{j+1, \overline{j}\}, \{\overline{j+1}, \overline{j}\}), \\ &(\{\overline{j}\}, \{\overline{j}\}), (\{j+1\}, \{j+1\}), (\{j, j+1\}, \{j+1, \overline{j}\}), \\ &(\{\overline{j+1}, \overline{j}\}, \{j+1, \overline{j}\}), (?, ?), (\{j, j+1\}, \{j, j+1\}), \\ &(\{j, j+1\}, \{\overline{j+1}, \overline{j}\}), (\{\overline{j+1}, \overline{j}\}, \{j, j+1\}), (\{\overline{j+1}, \overline{j}\}, \{\overline{j+1}, \overline{j}\}), \\ &(\{j\}, \{\overline{j}\}), (\{\overline{j+1}\}, \{j+1\}), \text{ or } (\{j, \overline{j+1}\}, \{j+1, \overline{j}\}). \end{aligned}
$$

We next assume that $j = n$. Write $[l\tilde{\mathrm{C}}]_n = l\tilde{\mathrm{C}} \cap \{n, \overline{n}\}$ and $[r\tilde{\mathrm{C}}]_n = r\tilde{\mathrm{C}} \cap \{n, \overline{n}\}$. If $[r\tilde{\mathrm{C}}]_n$ (resp. $[l\tilde{\mathrm{C}}]_n$) $= \{n\}$, then define $r_n r\tilde{\mathrm{C}}$ (resp. $r_n l\tilde{\mathrm{C}}$) $\in \mathrm{CST}_{B_n}(\varpi_i)$ to be such that $(r_n r\tilde{\mathrm{C}})(u) = r_n(r\tilde{\mathrm{C}}(u))$ (resp. $(r_n l\tilde{\mathrm{C}})(u) = r_n(l\tilde{\mathrm{C}}(u))$) for $u \in [i]$. Note that $\alpha_n = 2\varepsilon_n$. Similarly to (188)–(189), we have

$$
(190)\quad f_n(r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) = \begin{cases} (r_n r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) & \text{if } [r\tilde{\mathrm{C}}]_n = [l\tilde{\mathrm{C}}]_n = \{n\}, \\ (r\tilde{\mathrm{C}}, r_n l\tilde{\mathrm{C}}) & \text{if } [r\tilde{\mathrm{C}}]_n = \{\overline{n}\} \text{ and } [l\tilde{\mathrm{C}}]_n = \{n\}, \\ \mathbf{0} & \text{if } ([r\tilde{\mathrm{C}}]_n, [l\tilde{\mathrm{C}}]_n) \in \{(?, ?), (\{n\}, \{\overline{n}\}), (\{\overline{n}\}, \{\overline{n}\})\}. \end{cases}
$$

We finally assume that $j = 0$. Write $[l\tilde{\mathrm{C}}]_0 = l\tilde{\mathrm{C}} \cap \{1, 2, \overline{2}, \overline{1}\}$ and $[r\tilde{\mathrm{C}}]_0 = r\tilde{\mathrm{C}} \cap \{1, 2, \overline{2}, \overline{1}\}$. If $[r\tilde{\mathrm{C}}]_0$ (resp. $[l\tilde{\mathrm{C}}]_0$) $\in \{\{\overline{2}\}, \{\overline{1}\}, \{\overline{2}, \overline{1}\}\}$, then define $r_0 r\tilde{\mathrm{C}}$ (resp. $r_0 l\tilde{\mathrm{C}}$) $\in \mathrm{CST}_{B_n}(\varpi_i)$ to be such that $(r_0 r\tilde{\mathrm{C}})(u) = r_0(r\tilde{\mathrm{C}}(u))$ (resp. $(r_0 l\tilde{\mathrm{C}})(u) = r_0(l\tilde{\mathrm{C}}(u))$) for $u \in [i]$. Note that $\theta = \varepsilon_1 + \varepsilon_2$ and $\langle \alpha_0, \lambda \rangle = -\langle \theta, \lambda \rangle = -\langle \varepsilon_1 + \varepsilon_2, \lambda \rangle$ for $\lambda \in P$. Similarly to (188)–(189), we have

$$
(191)\quad f_0(r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) = \begin{cases} (r_0 r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) & \text{if } [r\tilde{\mathrm{C}}]_0 = \{\overline{2}, \overline{1}\} \text{ and } [l\tilde{\mathrm{C}}]_0 \in \{\{1, \overline{2}\}, \{2, \overline{1}\}, \{\overline{2}, \overline{1}\}\}, \\ (r\tilde{\mathrm{C}}, r_0 l\tilde{\mathrm{C}}) & \text{if } [r\tilde{\mathrm{C}}]_0 \in \{\{1, 2\}, \{1, \overline{2}\}, \{2, \overline{1}\}\} \text{ and } [l\tilde{\mathrm{C}}]_0 = \{\overline{2}, \overline{1}\}, \\ (r_0 r\tilde{\mathrm{C}}, r_0 l\tilde{\mathrm{C}}) & \text{if } [r\tilde{\mathrm{C}}]_0 = [l\tilde{\mathrm{C}}]_0 \in \{\{\overline{1}\}, \{\overline{2}\}\}, \end{cases}
$$

and $f_0(r\tilde{\mathrm{C}}, l\tilde{\mathrm{C}}) = \mathbf{0}$ if $([r\tilde{\mathrm{C}}]_0, [l\tilde{\mathrm{C}}]_0)$ is either

$$
(192)\quad \begin{aligned} &(\{1, 2\}, \{1, 2\}), (\{1, 2\}, \{1, \overline{2}\}), (\{1, 2\}, \{2, \overline{1}\}), (\{1\}, \{1\}), (\{2\}, \{2\}), \\ &(\{1, \overline{2}\}, \{1, 2\}), (\{2, \overline{1}\}, \{1, 2\}), (?, ?), (\{2, \overline{1}\}, \{2, \overline{1}\}), (\{2, \overline{1}\}, \{1, \overline{2}\}), \\ &(\{1, \overline{2}\}, \{2, \overline{1}\}), (\{1, \overline{2}\}, \{1, \overline{2}\}), (\{\overline{1}\}, \{1\}), (\{\overline{2}\}, \{2\}), \text{ or } (\{\overline{2}, \overline{1}\}, \{1, 2\}). \end{aligned}
$$

Lemma 5.29. *Keep the notation above. If* $f_j(r\tilde{\mathrm{C}}_\tau, l\tilde{\mathrm{C}}_\tau) = \mathbf{0}$ *for some* $j \in I$ *and* $\tau \in [\mu]$, *then* $j \in J_\tau$. *If* $f_0(r\tilde{\mathrm{C}}_\tau, l\tilde{\mathrm{C}}_\tau) = \mathbf{0}$ *for some* $\tau \in [\mu]$, *then* $\tau = 1$. *In these cases, write*

$(r\tilde{C}_\tau, l\tilde{C}_\tau) = f_j(r\tilde{C}_\tau, l\tilde{C}_\tau) \in \mathrm{QLS}(\varpi_{\# J_\tau})$. *For* $j \in I_{\mathrm{af}}$, *we have*

$$f_j(r\tilde{C}, l\tilde{C}) = \Big(r\tilde{C}_\tau \sqcup \bigsqcup_{\nu \in [\mu] \setminus \{\tau\}} r\tilde{C}_\nu,\ l\tilde{C}_\tau \sqcup \bigsqcup_{\nu \in [\mu] \setminus \{\tau\}} l\tilde{C}_\nu \Big) \tag{193}$$

if $f_j(r\tilde{C}_\tau, l\tilde{C}_\tau) \neq \mathbf{0}$ *for some* $\tau \in [\mu]$, *and* $f_j(r\tilde{C}, l\tilde{C}) = \mathbf{0}$ *otherwise.*

Proof. First part follows immediately from (188)–(192). For (193), we give the proof only for the case that $j \in J_\tau \setminus \{0, n\}$, $r\tilde{C}_\tau \cap \{j, j+1, \overline{j+1}, \overline{j}\} = l\tilde{C}_\tau \cap \{j, j+1, \overline{j+1}, \overline{j}\} = \{j\}$, and $(r\tilde{C}_\tau, l\tilde{C}_\tau) = f_j(r\tilde{C}_\tau, l\tilde{C}_\tau) = (r_j r\tilde{C}_\tau, r_j l\tilde{C}_\tau) \neq \mathbf{0}$ for some $\tau \in [\mu]$ (see (188)); the proofs for the other cases are similar. It follows from $(J_1 < \cdots < J_\mu) \in S_i$ and (180)–(181) that $[r\tilde{C}]_j = [l\tilde{C}]_j = \{j\}$. By (188), $f_j(r\tilde{C}, l\tilde{C}) = (r_j r\tilde{C}, r_j l\tilde{C})$. Since $r_j r\tilde{C}_\nu = r\tilde{C}_\nu$ and $r_j l\tilde{C}_\nu = l\tilde{C}_\nu$ for $\nu \in [\mu] \setminus \{\tau\}$, we have $r_j r\tilde{C} = r_j r\tilde{C}_\tau \sqcup \bigsqcup_{\nu \in [\mu] \setminus \{\tau\}} r\tilde{C}_\nu$ and $r_j l\tilde{C} = r_j l\tilde{C}_\tau \sqcup \bigsqcup_{\nu \in [\mu] \setminus \{\tau\}} l\tilde{C}_\nu$, which is the desired conclusion.

Lemma 5.30. $\mathrm{QKN}_{B_n}(\varpi_i)$ *is a* $\mathbf{U}'$*-crystal, and the map* Φ_i *is a morphism of* $\mathbf{U}'$*-crystals.*

Proof. We need to show that the set $\mathrm{QKN}_{B_n}(\varpi_i) \sqcup \{\mathbf{0}\}$ is stable under the maps e_j, f_j, $j \in I_{\mathrm{af}}$, and that the map Φ_i satisfies the conditions (CM2)–(CM3) in §2.2. We give the proof only for the equality $\Phi_i(f_j\tilde{C}) = f_j\Phi_i(\tilde{C})$ for $\tilde{C} = C \sqcup \{\underbrace{\overline{0}, \ldots, \overline{0}}_{2m \text{ times}}\} \in \mathrm{QKN}_{B_n}(\varpi_i)$ and $j \in I_{\mathrm{af}}$, where we understand that $\Phi_i(\mathbf{0}) = \mathbf{0}$ and $f_j\mathbf{0} = \mathbf{0}$; the other statements are left to the reader.

We continue to use the notation above. If $j = 0$ and $j \notin J_\tau$ for all $\tau \in [\mu]$, then $\Phi_i(f_j\tilde{C}) = \mathbf{0} = f_j\Phi_i(\tilde{C})$ by Lemmas 5.28–5.29 (see also (133)–(136) and (145)). If $j \in J_\tau$ for some $\tau \in [2, \mu]$, then $f_j\tilde{C}_\tau = \Psi_{\# J_\tau}(f_j\Phi_{\# J_\tau}(\tilde{C}_\tau))$ by $\tilde{C}_\tau \in \mathrm{KN}_{B_n}(\varpi_{\# J_\tau})$ and Lemma 5.14 (2), and hence $f_j\tilde{C} = \Psi_i(f_j\Phi_i(\tilde{C}))$ by (184) and Lemmas 5.28–5.29, which implies $\Phi_i(f_j\tilde{C}) = f_j\Phi_i(\tilde{C})$, by Lemma 5.27. It remains to prove the assertion for $j \in J_1 \sqcup \{0\}$. By (184) and Lemmas 5.28–5.29, it suffices to show that $\Phi_{\# J_1}(f_j\tilde{C}_1) = f_j\Phi_{\# J_1}(\tilde{C}_1)$. Therefore there is no loss of generality in assuming that $\mu = 1$, $J_1 = [p_1, p_1 + i - 1]$, $\tilde{C} = \tilde{C}_1$, and $j \in J_1 \sqcup \{0\}$.

We first assume that $j = 0$. If $p_1 > 2$, then $m = 0$, $\tilde{C} = C \in \mathrm{KN}_{B_n}(\varpi_i)$, and the assertion follows from Lemma 5.14 (2). Therefore we can assume that $p_1 = 1$ and $J_1 = [i]$. In particular, $j = n$ and

$$\{j, j+1\} \cap [i] \subset I_C \sqcup J_C \sqcup K_{\tilde{C}} \sqcup (\underbrace{\{\, C(u) \mid u \in [i-2m]\} \setminus I_C}_{= C \setminus I_C}); \tag{194}$$

for simplicity of notation, we write $C \setminus I_C = \{\, C(u) \mid u \in [i-2m]\} \setminus I_C$. Recall that $[\tilde{C}]_j = \tilde{C} \cap \{j, j+1, \overline{j+1}, \overline{j}\}$, $[l\tilde{C}]_j = l\tilde{C} \cap \{j, j+1, \overline{j+1}, \overline{j}\}$, and $[r\tilde{C}]_j = r\tilde{C} \cap \{j, j+1, \overline{j+1}, \overline{j}\}$. It follows from $\{\, l\tilde{C}(u) \mid u \in [i]\} = \{\, r\tilde{C}(u) \mid u \in [i]\} = J_1 = [i]$ that $\{\, u \mid u \in [l\tilde{C}]_j\} = \{\, u \mid u \in [r\tilde{C}]_j\} \in \{\{j\}, \{j, j+1\}\}$. If $f_j\tilde{C} \neq \mathbf{0}$, then we write $f_j\tilde{C} = C' \sqcup \{\overline{0}, \ldots, \overline{0}\}$ and $\Phi_i(f_j\tilde{C}) = (rf_j\tilde{C}, lf_j\tilde{C})$; recall from (175)–(176) that

$$\begin{aligned} rf_j\tilde{C} &= K_{f_j\tilde{C}} \sqcup (C' \setminus \{\sigma(z) \mid z \in I_{C'}\}) \sqcup \{\sigma(y) \mid y \in J_{C'}\}, \\ lf_j\tilde{C} &= \{\overline{x} \mid x \in K_{f_j\tilde{C}}\} \sqcup (C' \setminus I_{C'}) \sqcup J_{C'}. \end{aligned} \tag{195}$$

If $f_j(r\tilde{C}, l\tilde{C}) \neq \mathbf{0}$, then we write $(r\tilde{C}', l\tilde{C}') = f_j(r\tilde{C}, l\tilde{C})$, $[l\tilde{C}']_j = l\tilde{C}' \cap \{j, j+1, \overline{j+1}, \overline{j}\}$, and $[r\tilde{C}']_j = r\tilde{C}' \cap \{j, j+1, \overline{j+1}, \overline{j}\}$.

Since $[\tilde{C}]_j \subset \{j, j+1, \overline{j+1}, \overline{j}\}$, we have the following sixteen cases.

Case 1. Assume that $[\tilde{C}]_j = \emptyset$. By (133)–(136), $f_j\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_j = \emptyset$ and (194) that $(j = i \in K_{\tilde{C}})$, $(j, j+1 \in K_{\tilde{C}})$, or $(j \in K_{\tilde{C}}$ and $j+1 \in J_C)$, hence that

$([r\tilde{C}]_j, [l\tilde{C}]_j)$ is either $(\{j\}, \{\overline{j}\})$, $(\{j, j+1\}, \{\overline{j+1}, \overline{j}\})$, or $(\{j, \overline{j+1}\}, \{j+1, \overline{j}\})$. By (189), $f_j(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 2. Assume that $[\tilde{C}]_j = \{j\}$. By (133), $f_j\tilde{C} = (\tilde{C} \text{ r } \{j\}) \quad \{j+1\}$. It follows from $[\tilde{C}]_j = \{j\}$ and (194) that $j \quad C \text{ r } I_C$, and that $(j = i)$, $(j+1 \quad K_{\tilde{C}})$, or $(j+1 \quad J_C)$, hence that $([r\tilde{C}]_j, [l\tilde{C}]_j)$ is either $(\{j\}, \{j\})$, $(\{j, j+1\}, \{j, \overline{j+1}\})$, or $(\{j, \overline{j+1}\}, \{j, j+1\})$.

If $j = i$ and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{j\}, \{j\})$, then $([r\tilde{C}\,]_j, [l\tilde{C}\,]_j) = (\{j+1\}, \{j+1\})$, by (188). It follows from $f_j\tilde{C} = (\tilde{C} \text{ r } \{j\}) \quad \{j+1\}$ that $C = (C \text{ r } \{j\}) \quad \{j+1\}$, $I_C = I_C$, $J_C = J_C$, and $K_{f_j\tilde{C}} = K_{\tilde{C}}$; note that $J(rf_j\tilde{C}) = J(lf_j\tilde{C}) = ([i-1] < \{i+1\})$. By (195), $\Phi_i(f_j\tilde{C}) = (r\tilde{C}\,, l\tilde{C}\,)$.

If $j+1 \quad K_{\tilde{C}}$ and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{j, j+1\}, \{j, \overline{j+1}\})$, then $([r\tilde{C}\,]_j, [l\tilde{C}\,]_j) = (\{j, j+1\}, \{j+1, \overline{j}\})$, by (188). It follows from $f_j\tilde{C} = (\tilde{C} \text{ r } \{j\}) \quad \{j+1\}$ that $C = (C \text{ r } \{j\}) \quad \{j+1\}$, $I_C = I_C$, $J_C = J_C$, and $K_{f_j\tilde{C}} = (K_{\tilde{C}} \text{ r } \{j+1\}) \quad \{j\}$. By (195), $\Phi_i(f_j\tilde{C}) = (r\tilde{C}\,, l\tilde{C}\,)$.

If $j+1 \quad J_C$ and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{j, \overline{j+1}\}, \{j, j+1\})$, then $([r\tilde{C}\,]_j, [l\tilde{C}\,]_j) = (\{j+1, \overline{j}\}, \{j, j+1\})$ by (188). It follows from $f_j\tilde{C} = (\tilde{C} \text{ r } \{j\}) \quad \{j+1\}$ that $C = (C \text{ r } \{j\}) \quad \{j+1\}$, $I_C = I_C$, $J_C = (J_C \text{ r } \{j+1\}) \quad \{j\}$, and $K_{f_j\tilde{C}} = K_{\tilde{C}}$. By (195), $\Phi_i(f_j\tilde{C}) = (r\tilde{C}\,, l\tilde{C}\,)$.

Case 3. Assume that $[\tilde{C}]_j = \{j+1\}$. By (133)–(136), $f_j\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_j = \{j+1\}$ and (194) that $j+1 \quad C \text{ r } I_C$, and that $(j \quad K_{\tilde{C}})$ or $(j \quad J_C)$, hence that $([r\tilde{C}]_j, [l\tilde{C}]_j)$ is either $(\{j, j+1\}, \{j+1, \overline{j}\})$ or $(\{j+1, \overline{j}\}, \{j, j+1\})$. By (189), $f_j(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 4. Assume that $[\tilde{C}]_j = \{\overline{j+1}\}$. By (134), $f_j\tilde{C} = (\tilde{C} \text{ r } \{\overline{j+1}\}) \quad \{\overline{j}\}$. It follows from $[\tilde{C}]_j = \{\overline{j+1}\}$ and (194) that $j+1 \quad C \text{ r } I_C$, and that $(j \quad K_{\tilde{C}})$ or $(j \quad J_C)$, hence that $([r\tilde{C}]_j, [l\tilde{C}]_j)$ is either $(\{j, \overline{j+1}\}, \{\overline{j}, \overline{j+1}\})$ or $(\{\overline{j}, \overline{j+1}\}, \{j, \overline{j+1}\})$.

If $j \quad K_{\tilde{C}}$ and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{j, \overline{j+1}\}, \{\overline{j}, \overline{j+1}\})$, then $([r\tilde{C}\,]_j, [l\tilde{C}\,]_j) = (\{j+1, \overline{j}\}, \{\overline{j}, \overline{j+1}\})$, by (188). It follows from $f_j\tilde{C} = (\tilde{C} \text{ r } \{\overline{j+1}\}) \quad \{\overline{j}\}$ that $C = (C \text{ r } \{\overline{j+1}\}) \quad \{\overline{j}\}$, $I_C = I_C$, $J_C = J_C$, and $K_{f_j\tilde{C}} = (K_{\tilde{C}} \text{ r } \{j\}) \quad \{j+1\}$. By (195), $\Phi_i(f_j\tilde{C}) = (r\tilde{C}\,, l\tilde{C}\,)$.

If $j \quad J_C$ and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{\overline{j}, \overline{j+1}\}, \{j, \overline{j+1}\})$, then $([r\tilde{C}\,]_j, [l\tilde{C}\,]_j) = (\{\overline{j}, \overline{j+1}\}, \{j+1, \overline{j}\})$, by (188). It follows from $f_j\tilde{C} = (\tilde{C} \text{ r } \{\overline{j+1}\}) \quad \{\overline{j}\}$ that $C = (C \text{ r } \{\overline{j+1}\}) \quad \{\overline{j}\}$, $I_C = I_C$, $J_C = (J_C \text{ r } \{j\}) \quad \{j+1\}$, and $K_{f_j\tilde{C}} = K_{\tilde{C}}$. By (195), $\Phi_i(f_j\tilde{C}) = (r\tilde{C}\,, l\tilde{C}\,)$.

Case 5. Assume that $[\tilde{C}]_j = \{\overline{j}\}$. By (133)–(136), $f_j\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_j = \{\overline{j}\}$ and (194) that $j \quad C \text{ r } I_C$, and that $(j = i)$, $(j+1 \quad K_{\tilde{C}})$, or $(j+1 \quad J_C)$, hence that $([r\tilde{C}]_j, [l\tilde{C}]_j)$ is either $(\{\overline{j}\}, \{\overline{j}\})$, $(\{j+1, \overline{j}\}, \{\overline{j+1}, \overline{j}\})$, or $(\{\overline{j+1}, \overline{j}\}, \{j+1, \overline{j}\})$. By (189), $f_j(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 6. Assume that $[\tilde{C}]_j = \{j, j+1\}$. By (133)–(136), $f_j\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_j = \{j, j+1\}$ and (194) that $j, j+1 \quad C \text{ r } I_C$ and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{j, j+1\}, \{j, j+1\})$. By (189), $f_j(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 7. Assume that $[\tilde{C}]_j = \{j, \overline{j+1}\}$. By (135), $f_j\tilde{C} = (\tilde{C} \text{ r } \{j\}) \quad \{j+1\}$. It follows from $[\tilde{C}]_j = \{j, \overline{j+1}\}$ and (194) that $j, j+1 \quad C \text{ r } I_C$ and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{j, \overline{j+1}\}, \{j, \overline{j+1}\})$. By (188), $([r\tilde{C}\,]_j, [l\tilde{C}\,)]_j) = (\{j+1, \overline{j}\}, \{j, \overline{j+1}\})$. It follows from $f_j\tilde{C} = (\tilde{C} \text{ r } \{j\}) \quad \{j+1\}$ that $C = (C \text{ r } \{j\}) \quad \{j+1\}$, $I_C = I_C \quad \{j+1\}$, $J_C = J_C \quad \{j\}$, and $K_{f_j\tilde{C}} = K_{\tilde{C}}$. By (195), $\Phi_i(f_j\tilde{C}) = (r\tilde{C}\,, l\tilde{C}\,)$.

Case 8. Assume that $[\tilde{C}]_j = \{j, \overline{j}\}$. By (133)–(136), $f_j\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_j = \{j, \overline{j}\}$ and (194) that $j \quad I_C$, and that $(j = i)$, $(j+1 \quad K_{\tilde{C}})$, or $(j+1 \quad J_C)$, hence that $([r\tilde{C}]_j, [l\tilde{C}]_j)$ is either $(\{j\}, \{\overline{j}\})$, $(\{j, j+1\}, \{\overline{j+1}, \overline{j}\})$, or $(\{j, \overline{j+1}\}, \{j+1, \overline{j}\})$. By (189), $f_j(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 9. Assume that $[\tilde{C}]_j = \{j+1, \overline{j+1}\}$. By (135), $f_j\tilde{C} = (\tilde{C} \text{ r } \{\overline{j+1}\}) \quad \{\overline{j}\}$. It follows from $[\tilde{C}]_j = \{j+1, \overline{j+1}\}$ and (194) that $j+1 \quad I_C$, $j \quad J_C$, and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{j+1, \overline{j}\}, \{j, \overline{j+1}\})$. By (188), $([r\tilde{C}]_j, [l\tilde{C}])]_j) = (\{j+1, \overline{j}\}, \{j+1, \overline{j}\})$. It follows from $f_j\tilde{C} = (\tilde{C} \text{ r } \{\overline{j+1}\}) \quad \{\overline{j}\}$ that $C = (C \text{ r } \{\overline{j+1}\}) \quad \{\overline{j}\}$, $I_C = I_C \text{ r } \{j+1\}$, $J_C = J_C \text{ r } \{j\}$, and $K_{f_j\tilde{C}} = K_{\tilde{C}}$. By (195), $\Phi_i(f_j\tilde{C}) = (r\tilde{C}, l\tilde{C})$.

Case 10. Assume that $[\tilde{C}]_j = \{j+1, \overline{j}\}$. By (133)–(136), $f_j\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_j = \{j+1, \overline{j}\}$ and (194) that $j, j+1 \quad C \text{ r } I_C$ and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{j+1, \overline{j}\}, \{j+1, \overline{j}\})$. By (189), $f_j(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 11. Assume that $[\tilde{C}]_j = \{\overline{j+1}, \overline{j}\}$. By (133)–(136), $f_j\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_j = \{\overline{j+1}, \overline{j}\}$ and (194) that $j, j+1 \quad C \text{ r } I_C$ and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{\overline{j+1}, \overline{j}\}, \{\overline{j+1}, \overline{j}\})$. By (189), $f_j(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 12. Assume that $[\tilde{C}]_j = \{j, j+1, \overline{j+1}\}$. By (136), $f_j\tilde{C} = (\tilde{C} \text{ r } \{\overline{j+1}\}) \quad \{\overline{j}\}$. It follows from $[\tilde{C}]_j = \{j, j+1, \overline{j+1}\}$ and (194) that $j+1 \quad I_C$, $j \quad C \text{ r } I_C$, and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{j, j+1\}, \{j, \overline{j+1}\})$. By (188), $([r\tilde{C}]_j, [l\tilde{C}])]_j) = (\{j, j+1\}, \{j+1, \overline{j}\})$. It follows from $f_j\tilde{C} = (\tilde{C} \text{ r } \{\overline{j+1}\}) \quad \{\overline{j}\}$ that $C = (C \text{ r } \{\overline{j+1}\}) \quad \{\overline{j}\}$, $I_C = (I_C \text{ r } \{j+1\}) \quad \{j\}$, $J_C = J_C$, and $K_{f_j\tilde{C}} = K_{\tilde{C}}$. By (195), $\Phi_i(f_j\tilde{C}) = (r\tilde{C}, l\tilde{C})$.

Case 13. Assume that $[\tilde{C}]_j = \{j, j+1, \overline{j}\}$. By (133)–(136), $f_j\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_j = \{j, j+1, \overline{j}\}$ and (194) that $j \quad I_C$, $j+1 \quad C \text{ r } I_C$, and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{j, j+1\}, \{j+1, \overline{j}\})$. By (189), $f_j(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 14. Assume that $[\tilde{C}]_j = \{j, \overline{j+1}, \overline{j}\}$. By (136), $f_j\tilde{C} = (\tilde{C} \text{ r } \{j\}) \quad \{j+1\}$. It follows from $[\tilde{C}]_j = \{j, \overline{j+1}, \overline{j}\}$ and (194) that $j \quad I_C$, $j+1 \quad C \text{ r } I_C$, and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{j, \overline{j+1}\}, \{\overline{j}, \overline{j+1}\})$. By (188), $([r\tilde{C}]_j, [l\tilde{C}])]_j) = (\{j+1, \overline{j}\}, \{\overline{j}, \overline{j+1}\})$. It follows from $f_j\tilde{C} = (\tilde{C} \text{ r } \{j\}) \quad \{j+1\}$ that $C = (C \text{ r } \{j\}) \quad \{j+1\}$, $I_C = (I_C \text{ r } \{j\}) \quad \{j+1\}$, $J_C = J_C$, and $K_{f_j\tilde{C}} = K_{\tilde{C}}$. By (195), $\Phi_i(f_j\tilde{C}) = (r\tilde{C}, l\tilde{C})$.

Case 15. Assume that $[\tilde{C}]_j = \{j+1, \overline{j+1}, \overline{j}\}$. By (133)–(136), $f_j\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_j = \{j+1, \overline{j+1}, \overline{j}\}$ and (194) that $j+1 \quad I_C$, $j \quad C \text{ r } I_C$, and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{j+1, \overline{j}\}, \{\overline{j+1}, \overline{j}\})$. By (189), $f_j(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 16. Assume that $[\tilde{C}]_j = \{j, j+1, \overline{j+1}, \overline{j}\}$. By (133)–(136), $f_j\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_j = \{j, j+1, \overline{j+1}, \overline{j}\}$ and (194) that $j, j+1 \quad I_C$ and $([r\tilde{C}]_j, [l\tilde{C}]_j) = (\{j, j+1\}, \{\overline{j+1}, \overline{j}\})$. By (189), $f_j(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

We next assume that $j = 0$. Similarly to (194), we have

(196) $$\{1, 2\} \cap J_1 \quad I_C \quad J_C \quad K_{\tilde{C}} \quad (\underbrace{\{\, C(u) \mid u \quad [i-2m]\} \text{ r } I_C}_{= \; C \text{ r } I_C}).$$

Recall that $[\tilde{C}]_0 = \tilde{C} \cap \{1, 2, \overline{2}, \overline{1}\}$, $[l\tilde{C}]_0 = l\tilde{C} \cap \{1, 2, \overline{2}, \overline{1}\}$, and $[r\tilde{C}]_0 = r\tilde{C} \cap \{1, 2, \overline{2}, \overline{1}\}$. If $p_1 > 2$, then $f_0\tilde{C} = \mathbf{0}$ by (150)–(152), and $f_0(r\tilde{C}, l\tilde{C}) = \mathbf{0}$ by $[l\tilde{C}]_0 = [r\tilde{C}]_0 = ?$ and (192). Therefore we can assume that $J_1 = [i]$ or $J_1 = [2, i+1]$; note that if $p_1 = 2$, then $m = 0$. If $f_0\tilde{C} = \mathbf{0}$, then we continue to use the notation $f_0\tilde{C} = C \quad \{\overline{0}, \dots, \overline{0}\}$ and (195) for $j = 0$. If $f_0(r\tilde{C}, l\tilde{C}) = \mathbf{0}$, then we write $(r\tilde{C}, l\tilde{C}) = f_0(r\tilde{C}, l\tilde{C})$, $[l\tilde{C}]_0 =$

$l\tilde{C} \cap \{1,2,\overline{2},\overline{1}\}$, and $[r\tilde{C}]_0 = r\tilde{C} \cap \{1,2,\overline{2},\overline{1}\}$. Recall that $z_k = \min I_C$ and $y_k = \min J_C$. If $m > 0$, then $x_1 = \min K_{\tilde{C}}$ and $x_2 = \min(K_{\tilde{C}}$ r $\{x_1\})$.

We see from (QKN-B2) and (KN-B3) that $\{1,2\} \cap I_C = ?$ implies $[\tilde{C}]_0 = \{2,\overline{2}\}$. Hence we have the following ten cases.

Case 1. Assume that $[\tilde{C}]_0 = ?$. It follows from (196) that $(1,2 \quad K_{\tilde{C}})$, $(1 \quad K_{\tilde{C}}$ and $2 = y_k \quad J_C)$, or $(2 = p_1 = y_k \quad J_C)$, hence that $([r\tilde{C}]_0, [l\tilde{C}]_0)$ is either $(\{1,2\}, \{\overline{2},\overline{1}\})$, $(\{1,\overline{2}\}, \{2,\overline{1}\})$, or $(\{\overline{2}\}, \{2\})$.

If $1,2 \quad K_{\tilde{C}}$ and $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{1,2\}, \{\overline{2},\overline{1}\})$, then $m > 0$ and $y_k \ / \ \{1,2\}$. By (151), $f_0\tilde{C} = (\tilde{C}$ r $\{\overline{0},\overline{0}\}) \quad \{1,2\}$. By (191), $([r\tilde{C}\ ]_0, [l\tilde{C}\ ]_0) = (\{1,2\}, \{1,2\})$. It follows from $f_0\tilde{C} = (\tilde{C}$ r $\{\overline{0},\overline{0}\}) \quad \{1,2\}$ that $C \ = C \quad \{1,2\}$, $I_C \ = I_C$, $J_C \ = J_C$, and $K_{f_0\tilde{C}} = K_{\tilde{C}}$ r $\{1,2\}$. By (195), $\Phi_i(f_0\tilde{C}) = (r\tilde{C}\ , l\tilde{C}\ )$.

Assume that $2 = y_k \quad J_C$, and that $([r\tilde{C}]_0, [l\tilde{C}]_0)$ is either $(\{1,\overline{2}\}, \{2,\overline{1}\})$ or $(\{\overline{2}\}, \{2\})$. By (150)–(152), $f_0\tilde{C} = \mathbf{0}$. By (192), $f_0(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 2. Assume that $[\tilde{C}]_0 = \{1\}$. By (150)–(152), $f_0\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_0 = \{1\}$ and (196) that $1 \quad C$ r I_C, and that $(i = 1)$, $(2 \quad K_{\tilde{C}})$, or $(2 \quad J_C)$, hence that $([r\tilde{C}]_0, [l\tilde{C}]_0)$ is either $(\{1\}, \{1\})$, $(\{1,2\}, \{1,\overline{2}\})$, or $(\{1,\overline{2}\}, \{1,2\})$. By (192), $f_0(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 3. Assume that $[\tilde{C}]_0 = \{2\}$. By (150)–(152), $f_0\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_0 = \{2\}$ and (196) that $2 \quad C$ r I_C, and that $(p_1 = 2)$, $(1 \quad K_{\tilde{C}})$, or $(1 \quad J_C)$, hence that $([r\tilde{C}]_0, [l\tilde{C}]_0)$ is either $(\{2\}, \{2\})$, $(\{1,2\}, \{2,\overline{1}\})$, or $(\{2,\overline{1}\}, \{1,2\})$. By (192), $f_0(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 4. Assume that $[\tilde{C}]_0 = \{\overline{2}\}$. It follows from (196) that $2 \quad C$ r I_C, and that $(p_1 = 2)$, $(1 = x_1 \quad K_{\tilde{C}})$, or $(1 = y_k \quad J_C)$, hence that $([r\tilde{C}]_0, [l\tilde{C}]_0)$ is either $(\{\overline{2}\}, \{\overline{2}\})$, $(\{1,\overline{2}\}, \{\overline{2},\overline{1}\})$, or $(\{\overline{2},\overline{1}\}, \{1,\overline{2}\})$.

If $p_1 = 2$ and $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{\overline{2}\}, \{\overline{2}\})$, then $m = 0$ and $y_k \ / \ \{1,2\}$. By (150), $f_0\tilde{C} = (\tilde{C}$ r $\{\overline{2}\}) \quad \{1\}$. By (191), $([r\tilde{C}\ ]_0, [l\tilde{C}\ ]_0) = (\{1\}, \{1\})$. It follows from $f_0\tilde{C} = (\tilde{C}$ r $\{\overline{2}\}) \quad \{1\}$ that $C \ = (C$ r $\{\overline{2}\}) \quad \{1\}$, $I_C \ = I_C$, $J_C \ = J_C$, and $K_{f_0\tilde{C}} = K_{\tilde{C}} = ?$; note that $J(rf_0\tilde{C}) = J(lf_0\tilde{C}) = (\{1\} < [3, i+1])$. By (195), $\Phi_i(f_0\tilde{C}) = (r\tilde{C}\ , l\tilde{C}\ )$.

If $1 = x_1 \quad K_{\tilde{C}}$ and $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{1,\overline{2}\}, \{\overline{2},\overline{1}\})$, then $m > 0$ and $y_k \ / \ \{1,2\}$. By (151), $f_0\tilde{C} = (\tilde{C}$ r $\{\overline{2},\overline{0},\overline{0}\}) \quad \{1, x_2, \overline{x_2}\}$. By (191), $([r\tilde{C}\ ]_0, [l\tilde{C}\ ]_0) = (\{1,\overline{2}\}, \{1,2\})$. It follows from $f_0\tilde{C} = (\tilde{C}$ r $\{\overline{2},\overline{0},\overline{0}\}) \quad \{1, x_2, \overline{x_2}\}$ that $C \ = (C$ r $\{\overline{2}\}) \quad \{1, x_2, \overline{x_2}\}$, $I_C \ = I_C \quad \{x_2\}$, $J_C \ = J_C \quad \{2\}$, and $K_{f_0\tilde{C}} = K_{\tilde{C}}$ r $\{1, x_2\}$. By (195), $\Phi_i(f_0\tilde{C}) = (r\tilde{C}\ , l\tilde{C}\ )$.

Assume that $1 = y_k \quad J_C$ and $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{\overline{2},\overline{1}\}, \{1,\overline{2}\})$. By (152), $f_0\tilde{C} = (\tilde{C}$ r $\{z_k, \overline{z_k}, \overline{2}\}) \quad \{1,\overline{0},\overline{0}\}$. By (191), $([r\tilde{C}\ ]_0, [l\tilde{C}\ ]_0) = (\{1,2\}, \{1,\overline{2}\})$. It follows from $f_0\tilde{C} = (\tilde{C}$ r $\{z_k, \overline{z_k}, \overline{2}\}) \quad \{1,\overline{0},\overline{0}\}$ that $C \ = (C$ r $\{z_k, \overline{z_k}, \overline{2}\}) \quad \{1\}$, $I_C \ = I_C$ r $\{z_k\}$, $J_C \ = J_C$ r $\{1\}$, and $K_{f_0\tilde{C}} = K_{\tilde{C}} \quad \{2, z_k\}$. By (195), $\Phi_i(f_0\tilde{C}) = (r\tilde{C}\ , l\tilde{C}\ )$.

Case 5. Assume that $[\tilde{C}]_0 = \{\overline{1}\}$. It follows from (196) that $1 \quad C$ r I_C, and that $(i = 1)$, $(2 = x_1 \quad K_{\tilde{C}})$, or $(2 = y_k \quad J_C)$, hence that $([r\tilde{C}]_0, [l\tilde{C}]_0)$ is either $(\{\overline{1}\}, \{\overline{1}\})$, $(\{2,\overline{1}\}, \{\overline{2},\overline{1}\})$, or $(\{\overline{2},\overline{1}\}, \{2,\overline{1}\})$.

If $i = 1$ and $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{\overline{1}\}, \{\overline{1}\})$, then $m = 0$ and $y_k \ / \ \{1,2\}$. By (150), $f_0\tilde{C} = (\tilde{C}$ r $\{\overline{1}\}) \quad \{2\} = \{2\}$. By (191), $([r\tilde{C}\ ]_0, [l\tilde{C}\ ]_0) = (r\tilde{C}\ , l\tilde{C}\ ) = (\{2\}, \{2\})$. It follows from $f_0\tilde{C} = \{2\}$ that $C \ = \{2\}$ and $I_C \ = J_C \ = K_{f_0\tilde{C}} = ?$; note that $J(rf_0\tilde{C}) = J(lf_0\tilde{C}) = (\{2\})$. By (195), $\Phi_i(f_0\tilde{C}) = (r\tilde{C}\ , l\tilde{C}\ )$.

If $2 = x_1 \quad K_{\tilde{C}}$ and $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{2,\overline{1}\}, \{\overline{2},\overline{1}\})$, then $m > 0$ and $y_k \ / \ \{1,2\}$. By (151), $f_0\tilde{C} = (\tilde{C}$ r $\{\overline{1},\overline{0},\overline{0}\}) \quad \{2, x_2, \overline{x_2}\}$. By (191), $([r\tilde{C}\ ]_0, [l\tilde{C}\ ]_0) = (\{2,\overline{1}\}, \{1,2\})$. It follows from $f_0\tilde{C} = (\tilde{C} \setminus \{\overline{1},\overline{0},\overline{0}\}) \cup \{2, x_2, \overline{x_2}\}$ that $C = (C \setminus \{\overline{1}\}) \cup \{2, x_2, \overline{x_2}\}$, $I_C = I_C \cup \{x_2\}$, $J_C = J_C \cup \{1\}$, and $K_{f_0\tilde{C}} = K_{\tilde{C}} \setminus \{2, x_2\}$. By (195), $\Phi_i(f_0\tilde{C}) = (r\tilde{C}, l\tilde{C})$.

Assume that $2 = y_k \in J_C$ and $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{\overline{2},\overline{1}\}, \{2,\overline{1}\})$. By (152), $f_0\tilde{C} = (\tilde{C} \setminus \{z_k, \overline{z_k}, \overline{1}\}) \cup \{2,\overline{0},\overline{0}\}$. By (191), $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{1,2\}, \{2,\overline{1}\})$. It follows from $f_0\tilde{C} = (\tilde{C} \setminus \{z_k, \overline{z_k}, \overline{1}\}) \cup \{2,\overline{0},\overline{0}\}$ that $C = (C \setminus \{z_k, \overline{z_k}, \overline{1}\}) \cup \{2\}$, $I_C = I_C \setminus \{z_k\}$, $J_C = J_C \setminus \{2\}$, and $K_{f_0\tilde{C}} = K_{\tilde{C}} \cup \{1, z_k\}$. By (195), $\Phi_i(f_0\tilde{C}) = (r\tilde{C}, l\tilde{C})$.

Case 6. Assume that $[\tilde{C}]_0 = \{1,2\}$. By (150)–(152), $f_0\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_0 = \{1,2\}$ and (196) that $1,2 \in C \setminus I_C$ and $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{1,2\}, \{1,2\})$. By (192), $f_0(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 7. Assume that $[\tilde{C}]_0 = \{1,\overline{2}\}$. By (150)–(152), $f_0\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_0 = \{1,\overline{2}\}$ and (196) that $1,2 \in C \setminus I_C$ and $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{1,\overline{2}\}, \{1,\overline{2}\})$. By (192), $f_0(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 8. Assume that $[\tilde{C}]_0 = \{2,\overline{2}\}$. By (150)–(152), $f_0\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_0 = \{2,\overline{2}\}$ that $2 = z_k \in I_C$, $1 = y_k \in J_C$, and $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{2,\overline{1}\}, \{1,\overline{2}\})$. By (192), $f_0(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 9. Assume that $[\tilde{C}]_0 = \{2,\overline{1}\}$. By (150)–(152), $f_0\tilde{C} = \mathbf{0}$. It follows from $[\tilde{C}]_0 = \{2,\overline{1}\}$ and (196) that $1,2 \in C \setminus I_C$ and $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{2,\overline{1}\}, \{2,\overline{1}\})$. By (192), $f_0(r\tilde{C}, l\tilde{C}) = \mathbf{0}$.

Case 10. Assume that $[\tilde{C}]_0 = \{\overline{2},\overline{1}\}$. By (150)–(151), $f_0\tilde{C} = (\tilde{C} \setminus \{\overline{2},\overline{1}\}) \cup \{\overline{0},\overline{0}\}$. It follows from $[\tilde{C}]_0 = \{\overline{2},\overline{1}\}$ and (196) that $1,2 \in C \setminus I_C$ and $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{\overline{2},\overline{1}\}, \{\overline{2},\overline{1}\})$. By (191), $([r\tilde{C}]_0, [l\tilde{C}]_0) = (\{1,2\}, \{\overline{2},\overline{1}\})$. It follows from $f_0\tilde{C} = (\tilde{C} \setminus \{\overline{2},\overline{1}\}) \cup \{\overline{0},\overline{0}\}$ that $C = C \setminus \{\overline{2},\overline{1}\}$, $I_C = I_C$, $J_C = J_C$, and $K_{f_0\tilde{C}} = K_{\tilde{C}} \cup \{1,2\}$. By (195), $\Phi_i(f_0\tilde{C}) = (r\tilde{C}, l\tilde{C})$.

The proof is complete.

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