ALGEBRAIC COMBINATORICS

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Volume 5, issue 5 (2022), p. 1165-1203.
https://doi.org/10.5802/alco.243

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www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

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Plethysm and the algebra of uniform block permutations

Rosa Orellana, Franco Saliola, Anne Schilling & Mike Zabrocki

Abstract We study the representation theory of the uniform block permutation algebra in the context of the representation theory of factorizable inverse monoids. The uniform block permutation algebra is a subalgebra of the partition algebra and is also known as the party algebra. We compute its characters and provide a Frobenius characteristic map to symmetric functions. This reveals connections of the characters of the uniform block permutation algebra and plethysms of Schur functions.

1. Introduction

The partition algebra arose in the early 1990s in the work of Martin [16, 17, 18, 15] and Jones [9] in the context of the Potts model in statistical mechanics. It is a generalization of the Temperley–Lieb algebra and can be formulated in terms of an important question in invariant theory: If a group G acts on an n-dimensional vector space V, how does $V^{\otimes k}$ decompose into irreducible representations of G? This question can be studied using the centralizer algebra $\mathrm{End}_G(V^{\otimes k})$. The partition algebra is isomorphic to this centralizer algebra when the group G is the symmetric group $\mathfrak{S}_n$ [9, 17], that is, $\mathrm{End}_{\mathfrak{S}_n}(V^{\otimes k}) \cong P_k(n)$.

Inspired by this work, Tanabe [28] considered the case when G is a unitary reflection group $G(r,p,n)$, where $G(r,1,n)$ is a group of $n \times n$ monomial matrices whose nonzero entries are r-th roots of unity and $G(r,p,n)$ is a subgroup of index p in $G(r,1,n)$. Kosuda [10, 12] studied the party algebra U_k, which corresponds to the subcase $p = 1$, $n > k$ and $r > k$. The party algebra is a subalgebra of the partition algebra $P_k(n)$. Elements in the party algebra can be viewed as bijections between blocks of the same size of two set partitions of $\{1, 2, \ldots, k\}$ of the same type. To quote from Kosuda [10]:

> Suppose that there exist two parties each of which consists of n members. The parties hold meetings splitting into several small groups. Every group consists of the members of each party of the same number. The set of such decompositions into small groups makes an algebra called the party algebra under a certain product.(1)

Manuscript received 7th January 2022, revised 12th April 2022, accepted 19th April 2022.

Keywords. partition algebra, plethysm, representation theory of semigroups, symmetric functions.

Acknowledgements. RO was partially supported by NSF grant DMS–1300512. AS was partially supported by NSF grant DMS–1760329 and DMS–2053350. FS and MZ are supported by NSERC/CNSRG.

(1)Clearly in arriving at the name 'party algebra' Kosuda was not imagining a party of introverted mathematicians for which we would likely see each vertex in the diagram isolated.

ISSN: 2589-5486

Since the block sizes of the two set partitions are required to match, this algebra is also known as the uniform block permutation algebra [4], which is the terminology we will use in this paper.

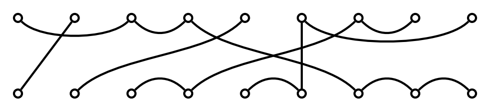


Figure 1. An example diagram representing an element of the uniform block permutation monoid U_9. The connected components of the graph visually represent the blocks of the set partitions. Each connected component contains the same number of elements in the top row as in the bottom row.

In this paper, we study the representation theory of the uniform block permutation algebra U_k. It is an interesting, nontrivial example of a factorizable inverse monoid. We use the general theory of finite inverse monoids to develop the representation theory of U_k. This relies on theorems due to Clifford [2], Munn [20] and Ponizovskiĭ [22] and the explicit constructions given in [23, 13]. The exposition and notation we follow here is found in [5, 27]. In particular, by characterizing the idempotents, the maximal subgroups, the J -classes and the L -classes of U_k, the Schützenberger representations can be employed to construct the irreducible representations of U_k. The representations that we obtain very nicely extend Young's construction of the irreducible representations of symmetric groups: instead of a symmetric group action on standard tableaux we obtain a monoid action on sequences of set-valued tableaux. The action can be described on a basis indexed by combinatorial objects using familiar and well-used relations on tableaux, rather than operations on paths in a Bratteli diagram as appears in the construction by Kosuda [12].

We also compute the irreducible characters of the uniform block permutation algebra U_k and relate them to symmetric functions by defining a Frobenius characteristic that maps a class function of U_k to an element of the k-fold tensor product of the symmetric functions. More precisely, each irreducible representation of U_k is indexed by a k-tuple of partitions $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ such that $\sum_{i=1}^{k} i|\lambda^{(i)}| = k$, and the associated symmetric function of the character of the restriction of this representation to $\mathfrak{S}_k$ is

$$s_{\lambda^{(1)}}[s_1] s_{\lambda^{(2)}}[s_2] \ldots s_{\lambda^{(k)}}[s_k],$$

where s_λ is the Schur function indexed by a partition λ and $s_\lambda[s_h]$ is the plethysm of s_λ with the Schur function s_h indexed by a single row. In this sense, the representation theory of U_k gives a novel representation theoretic approach to plethysm.

Furthermore, having the image of the characters under the Frobenius map reduces the computation of the characters to a computation on symmetric functions. In a 2005 talk, Naruse presented (without proof) several results on the characters of the Tanabe algebra, and hence as a special case the uniform block permutation algebra U_k [21]. This included character tables for U_k for small values of k. Using the symmetric function connection that we establish, these tables can be verified. We are not aware of any other proofs of Naruse's results in the literature.

In a subsequent paper, we will consider the restriction from the general linear group GL_n to the symmetric group $\mathfrak{S}_n$ which involves the same restriction coefficients as the restriction of the partition algebra $P_k(n)$ to the symmetric group $\mathfrak{S}_k$. The uniform block permutation algebra can be viewed as an intermediate step in this restriction, see [7, Section 4.1]. The restriction from the partition algebra $P_k(n)$ to the uniform block permutation algebra U_k involves the Littlewood–Richardson rule, whereas the restriction from U_k to $\mathfrak{S}_k$ involves the plethysm operation.

To conclude, let us compare our approach in this paper with existing constructions in the literature. Irreducible matrix representations of U_k were previously constructed by Kosuda [12] by defining a tower of algebras $\mathsf{U}_k \subset \mathsf{U}_{k+1}$. In Kosuda's approach [12], the rows of the matrices are indexed by paths in the Bratteli diagram of the tower of monoid algebras and the action is defined over the field $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \ldots, \sqrt{k})$. In contrast, we use the theory of finite monoids to construct the irreducible representations of U_k in terms of tuples of set-valued tableaux and the action is expressed in the basis with coefficients that are integers. The bijection between the tuples of set-valued tableaux that we use here and the path model used by Kosuda [12] is similar to the bijection described in [3] relating the path model for diagram algebras and set-valued tableaux. Set-valued tableaux also appear in the construction of the representations of the partition algebra and sub-diagram algebras by Halverson and Jacobson [6].

This paper is organized as follows. In Section 2, we introduce uniform block permutations and describe the monoid structure on them. In particular, we provide a presentation of the monoid of uniform block permutations U_k and show that it is an inverse monoid. In Section 3, we compute the maximal subgroups, $\mathcal{J}$- and $\mathcal{L}$-classes of U_k. Using Schützenberger representations, this makes it possible to construct the irreducible representations of U_k. The characters of U_k are computed in Section 4. Finally, in Section 5 the connection of the characters with symmetric functions is established.

2. The monoid of uniform block permutations

After some preliminary notation in Sections 2.1 and 2.2 on partitions and set partitions, we define uniform block permutations U_k in Section 2.3 and give its monoid structure in Section 2.4. We show in Section 2.5 that every element of U_k is a product of an idempotent and a permutation. We recall a presentation of U_k in Section 2.6 and we conclude in Section 2.7 with a proof that U_k is an inverse monoid.

2.1. Partitions. A *partition* of a positive integer k is a nonincreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers such that $\lambda_1 + \cdots + \lambda_\ell = k$. We write $|\lambda|$ for $\lambda_1 + \cdots + \lambda_\ell$ and call λ_i the *parts* of λ. The *length* of the partition λ is $\ell(\lambda) = \ell$. We write $\lambda \vdash k$ to mean that λ is a partition of k. We declare that the empty sequence () is the unique partition of 0, and we denote this by ?. We will often use *exponential notation* for partitions in which b consecutive occurrences of the part i is denoted by i^b: for example, $(4, 4, 4, 2, 1, 1, 1, 1)$ can be denoted $(1^4 2^1 4^3)$.

We use Young diagrams to represent partitions. If $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition of k, then the *Young diagram* of λ is the left-justified array of k cells (or boxes) with λ_i cells in the i-th row. We use French notation, so that the largest row is at the bottom. This may be upside down from what is sometimes used in representation theory.

For every nonnegative integer k, we define

$$I_k = \left\{ \left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}\right) : \lambda^{(i)} \text{ are partitions such that } \sum_{i=1}^{k} i|\lambda^{(i)}| = k \right\}. \tag{1}$$

We denote elements in I_k as $\vec{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(k)})$. We will see that the elements of I_k index the irreducible representations of the uniform block permutation algebra U_k. In examples the elements of I_k will be expressed by dropping the trailing empty partitions in the list of partitions so that, for instance, the element $(?, (2), ?, ?)$ of I_4 will be displayed without loss of information as $(?, (2))$.

2.2. Set partitions. A *set partition* π of a set X is a collection of nonempty subsets $\{\pi_1, \ldots, \pi_\ell\}$ of X such that $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{\ell} \pi_i = X$. We use $\pi \vdash X$ to denote that π is a set partition of X. The subsets π_i are called the *blocks* of π.

If $\pi = \{\pi_1, \ldots, \pi_\ell\}$ is a set partition of $[k] = \{1, 2, \ldots, k\}$, then we order the blocks in π using the *graded last letter order*: if A and B are two blocks, then $A \leq B$ in the graded last letter order if either $|A| < |B|$ or if $|A| = |B|$, then $\mathsf{max}(A) \leq \mathsf{max}(B)$. For example, the blocks of $\pi = \{\{4\}, \{1, 6\}, \{3, 8\}, \{2, 5, 7\}\}$ are listed in graded last letter order.

To simplify notation, we often write the set partitions by separating the blocks by vertical lines. For example, the set partition $\pi = \{\{4\}, \{1, 6\}, \{3, 8\}, \{2, 5, 7\}\}$ will also be denoted by $\pi = 4|16|38|257$.

The *type* of a set partition π, denoted $\mathsf{type}(\pi)$, is the (integer) partition formed by the sizes of the blocks of π. For example,

$$\mathsf{type}(4|16|38|257) = (3, 2, 2, 1) = (1, 2^2, 3).$$

The number of set partitions of type $\lambda = (1^{a_1}, 2^{a_2}, \ldots, k^{a_k})$ is

$$\mathsf{sp}_k(\lambda) = \frac{k!}{a_1! \cdots a_k! (1!)^{a_1} (2!)^{a_2} \cdots (k!)^{a_k}}. \tag{2}$$

Set partitions are ordered by *refinement*: if π and γ are two set partitions, then we say that π is *finer* than γ and that γ is *coarser* than π, if every block of π is a subset of some block of γ. In that case, we write $\pi \leq \gamma$.

Given a set partition $\pi = \{\pi_1, \ldots, \pi_\ell\} \vdash [k]$ and permutation $\sigma \in \mathfrak{S}_k$, we define $\sigma(\pi) = \{\sigma(\pi_1), \ldots, \sigma(\pi_\ell)\}$, where $\sigma(A) = \{\sigma(a) : a \in A\}$ for every subset $A \subseteq [k]$. Since $|A| = |\sigma(A)|$ for all $A \subseteq [k]$, it follows that π and $\sigma(\pi)$ have the same type.

2.3. Uniform block permutations. We define the set of uniform block permutations U_k and give three equivalent ways to view its elements: as set partitions of $[k] \cup [\bar{k}]$; as size-preserving bijections between the blocks of two set partitions; as two-row diagrams. We will use these interpretations interchangeably throughout the paper.

2.3.1. *Set partitions of* $[k] \cup [\bar{k}]$. For nonzero $k \in \mathsf{N}$, define

$$[k] = \{1, \ldots, k\} \qquad \text{and} \qquad [\bar{k}] = \{\bar{1}, \ldots, \bar{k}\}.$$

For each $a \in [k]$, we define $\bar{\bar{a}} = a$ so that $a \mapsto \bar{a}$ as an involution on $[k] \cup [\bar{k}]$.

Let $d = \{d_1, d_2, \ldots, d_\ell\}$ be a set partition of $[k] \cup [\bar{k}]$. We say that d is *uniform* if $|d_i \cap [k]| = |d_i \cap [\bar{k}]|$ for all $1 \leq i \leq \ell$. Let U_k be the set of uniform set partitions of $[k] \cup [\bar{k}]$:

$$\mathsf{U}_k = \left\{ d \vdash [k] \cup [\bar{k}] : d \text{ uniform} \right\}.$$

Let $\mathsf{top}(d)$ be the set partition of $[k]$ consisting of the blocks $d_i \cap [k]$ for $1 \leq i \leq \ell$ and $\mathsf{bot}(d)$ the set partition of $[k]$ containing the blocks $\overline{d_i} \cap [k]$ for $1 \leq i \leq \ell$, where $\overline{d_i} = \{\bar{a} : a \in d_i\}$. For example,

$$\begin{aligned} d &= \{\{2, \bar{4}\}, \{5, \bar{7}\}, \{1, 3, \bar{1}, \bar{2}\}, \{4, 6, \bar{3}, \bar{6}\}, \{7, 8, 9, \bar{5}, \bar{8}, \bar{9}\}\}, \\ \mathsf{top}(d) &= \{\{2\}, \{5\}, \{1, 3\}, \{4, 6\}, \{7, 8, 9\}\}, \\ \mathsf{bot}(d) &= \{\{4\}, \{7\}, \{1, 2\}, \{3, 6\}, \{5, 8, 9\}\}. \end{aligned}$$

When writing set partitions, we list the blocks in graded last letter order.

2.3.2. *Size-preserving bijections between set partitions of* $[k]$. It is useful to think of d as the size-preserving bijection $d\colon \mathsf{top}(d) \to \mathsf{bot}(d)$ that maps $d_i \cap [k]$ to $\overline{d_i} \cap [k]$. Continuing the previous example, the bijection associated with d, expressed in two-line notation, is

$$\begin{pmatrix} \{2\} & \{5\} & \{1,3\} & \{4,6\} & \{7,8,9\} \\ \{4\} & \{7\} & \{1,2\} & \{3,6\} & \{5,8,9\} \end{pmatrix}.$$

For this reason, the elements of U_k are called *uniform block permutations*. With this interpretation, it follows that the number of elements in U_k is

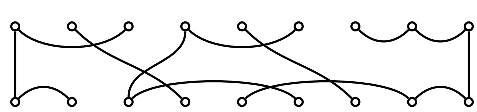
$$|\mathsf{U}_k| = \sum_{\lambda=(1^{a_1},\ldots,k^{a_k}) \quad k} \mathsf{sp}_k(\lambda)^2 a_1! \cdots a_k!.$$

Starting with $k = 0$, the sequence of $|\mathsf{U}_k|$ begins

$$1, 1, 3, 16, 131, 1496, 22482, 426833, \ldots$$

and is listed as sequence A023998 in the Online Encyclopedia of Integer Sequences [8].

2.3.3. *Diagrams.* A graph on the vertex set $[k] \quad [\bar{k}]$ *represents* a set partition d $[k] \quad [\bar{k}]$ if (the vertices of) the connected components of the graph are the blocks of d. We draw these graphs by arranging the vertices in two rows: $1, 2, \ldots, k$ appear from left to right in the top row; and $\bar{1}, \bar{2}, \ldots, \bar{k}$ from left to right in the bottom row. In this way, the graph



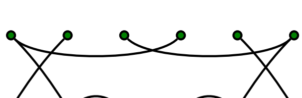
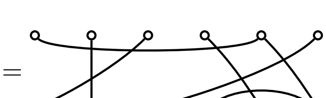
represents the set partition $\{\{1,3,\bar{1},\bar{2}\},\{2,\bar{4}\},\{4,6,\bar{3},\bar{6}\},\{5,\bar{7}\},\{7,8,9,\bar{5},\bar{8},\bar{9}\}\}$. We call this the *(two-row) diagram* of the set partition. Notice that it is possible that more than one graph represents a given set partition; therefore, a diagram represents a class of labeled graphs that have the same connected components.

2.4. Monoid structure. We next define a monoid structure on the set of all set partitions of $[k] \quad [\bar{k}]$. It follows from this definition that the product of two uniform block permutations is again a uniform block permutation, from which we obtain a monoid structure on U_k.

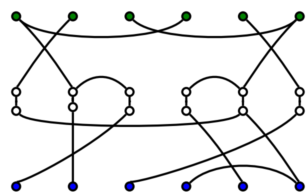
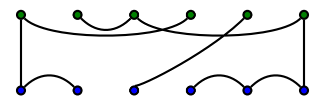
Let $d, d \quad \mathsf{U}_k$ (or more generally, any pair of set partitions $[k] \quad [\bar{k}]$), which we view as diagrams. The product dd is defined as follows:

- stack d on top of d so that the bottom vertices of d line up with the top vertices of d ;
- compute the connected components of the resulting three-row diagram;
- eliminate the vertices of the middle row from the connected components.

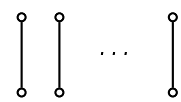
Example 2.1. We illustrate the product of the following two set partitions:

$d =$  and $d =$  .

The product dd is the set partition whose blocks correspond to the connected components of the diagram obtained by stacking the diagrams of d and d :

dd =  = .

This multiplication of diagrams is associative and the product of two uniform block permutations is a uniform block permutations, and hence makes U_k into a finite monoid whose identity element is

$$\{\{1,\bar{1}\},\{2,\bar{2}\},\dots,\{k,\bar{k}\}\} \quad = $$ .

Since connected vertices in the top row of d remain connected in dd , it follows that the set partition $\mathsf{top}(dd\ )$ is coarser than or equal to $\mathsf{top}(d)$. Similarly, the set partition $\mathsf{bot}(dd\ )$ is coarser than or equal to $\mathsf{bot}(d\ )$. Furthermore, any block of dd contains at least one block of $\mathsf{top}(d)$ and at least one block of $\mathsf{bot}(d\ )$. If $n(d)$ is the *number of blocks* in a diagram d, then for all $d, d \quad \mathsf{U}_k$,

$$n(dd\ )\ 6\ \min\{n(d), n(d\ )\}.$$

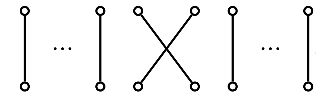
Remark 2.2 (Diagram multiplication and composition of bijections). As explained in Section 2.3.2, it is often useful to think of diagrams d as bijections $d : \mathsf{top}(d) \to \mathsf{bot}(d)$ that preserve block-size, and so we highlight some important nuances of this approach.

If $d : \mathsf{top}(d) \to \mathsf{bot}(d)$ and $d\ : \mathsf{top}(d\ ) \to \mathsf{bot}(d\ )$ satisfy $\mathsf{top}(d\ ) = \mathsf{bot}(d)$, then the composition of d and d is defined, and the resulting bijection is precisely the one associated with the product dd . In particular, in this case, dd maps a block B to the block $d\ (d(B))$.

The inverse of a bijection $d : \mathsf{top}(d) \to \mathsf{bot}(d)$ is obtained by reflecting the diagram of d across a horizontal line, which we denote by $\widetilde{d}$ (cf. Section 2.7). Note that $d\widetilde{d}$ is the identity mapping on $\mathsf{top}(d)$ and $\widetilde{d}d$ is the identity mapping on $\mathsf{bot}(d)$, which are not necessarily equal to the identity element of U_k. However, they are idempotents of U_k (cf. Lemma 2.3).

2.5. Permutation-idempotent and idempotent-permutation decompositions. We prove that every uniform block permutation can be factored as a product of a permutation and an idempotent, and also as a product of an idempotent and a permutation. We begin by embedding the symmetric group $\mathfrak{S}_k$ in U_k, then we characterize the idempotents in U_k, and finally we prove the existence of the factorizations.

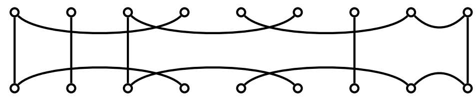
2.5.1. *Permutations.* Let $\mathfrak{S}_k$ denote the symmetric group consisting of the permutations of the set $[k]$. We identify each permutation $\sigma \quad \mathfrak{S}_k$ with the uniform block permutation $\{\{1,\overline{\sigma(1)}\},\dots,\{k,\overline{\sigma(k)}\}\}$, which we also denote by σ. Note that the diagram representing σ is the diagram with an edge connecting i and $\overline{\sigma(i)}$. (Observe that under this identification, the product of two permutations $\sigma_1\sigma_2$ maps $i \quad [k]$ to $\sigma_2(\sigma_1(i))$ instead of $\sigma_1(\sigma_2(i))$; cf. Remark 2.2.) For instance, if s_i is the permutation that swaps i and $i+1$ and fixes all other elements of $[k]$, then

$$s_i \quad = \quad \{\{1,\bar{1}\},\dots,\{i,\overline{i+1}\},\{i+1,\bar{i}\},\dots,\{k,\bar{k}\}\} \quad = $$ .

2.5.2. *Idempotents.* For every set partition π of $[k]$ we define the following element of U_k:

$$e_\pi = \{A \quad \bar{A} : A \quad \pi\} \quad \mathsf{U}_k,$$

where $\bar{A} = \{\bar{i} : i \quad A\}$. For example,

$$e_{2|7|14|36|589} =$$  .

It is not hard to see that e_π is an idempotent, and the next result proves that all the idempotents in U_k are of this form.

Lemma 2.3. *The set $E(\mathsf{U}_k) = \{e_\pi : \pi \quad [k]\}$ is a complete set of idempotents in U_k.*

Furthermore, if Π_k is the lattice of set partitions of $[k]$ viewed as a monoid with the join operation $\quad$, then the map $\pi \to e_\pi$ is monoid isomorphism from Π_k to $E(\mathsf{U}_k)$. Thus,

$$e_\pi e_\gamma = e_{\pi \quad \gamma}.$$

Proof. Suppose $d \quad \mathsf{U}_k$ is an idempotent. We will prove that $d = e_{\mathsf{top}(d)}$, where

$$e_{\mathsf{top}(d)} = \left\{ \left(d_i \cap [k]\right) \quad \overline{\left(d_i \cap [k]\right)} : d_i \text{ is a block of } d \right\}.$$

We need to prove $d_i = (d_i \cap [k]) \quad \overline{(d_i \cap [k])}$ or equivalently, $\overline{d_i \cap [k]} = d_i \cap [\overline{k}]$.

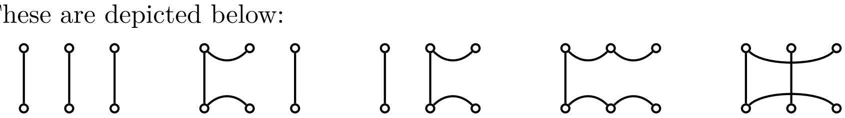
Let us first show that it suffices to prove $\mathsf{bot}(d) = \mathsf{top}(d)$. Let $\delta : \mathsf{top}(d) \to \mathsf{bot}(d)$ be the bijection associated with d that maps $d_i \cap [k]$ to $\overline{d_i} \cap [k]$. Then δ is a permutation of $\mathsf{top}(d)$ and $\delta \circ \delta$ is the bijection associated with dd. Since $dd = d$, it follows that $\delta \circ \delta = \delta$, and thus δ is the identity mapping. Hence, $\overline{d_i \cap [k]} = \overline{\delta(d_i \cap [k])} = \overline{\overline{d_i} \cap [k]} = d_i \cap [\overline{k}]$.

We now prove $\mathsf{bot}(d) = \mathsf{top}(d)$; more specifically, if $d_i \cap [k] \quad \mathsf{top}(d)$, then $d_i \cap [k]$ $\mathsf{bot}(d)$. Consider the three-row diagram constructed by stacking d on top of a second copy of d, which we denote by d . We will refer to the three vertices occurring in a column by v_a, v_a and v_a, with v_a in the top row, v_a in the middle row, and v_a in the bottom row.

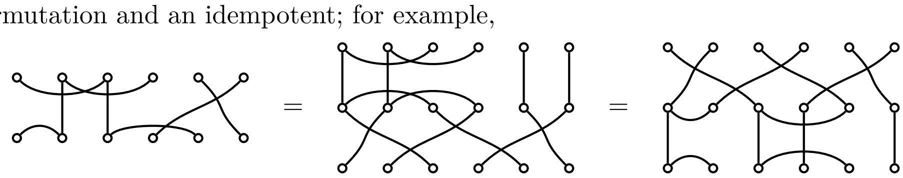
View d_i as a connected component of d and let d_i be the corresponding component of d . Write $\mathsf{top}(d_i) = \{v_1, \dots, v_\ell\}$ and $\mathsf{top}(d_i) = \{v_1, \dots, v_\ell\}$. To prove $d_i \cap [k]$ $\mathsf{bot}(d)$, it suffices to prove $v_1, \dots, v_\ell$ belong to the same connected component of d .

For each v_a, pick u_a such that u_a and v_a are in the same connected component of d . Then u_a and v_a are in the same connected component of d. Since $v_1, \dots, v_\ell \quad d_i$, and d_i is a connected component, it follows that $u_1, \dots, u_\ell$ are in the same connected component of dd . Since d is idempotent, the connected components of dd and d coincide (up to relabelling). Thus, $u_1, \dots, u_\ell$ are in the same connected component of d. Consequently, $u_1, \dots, u_\ell$ are in the same connected component of d , and thus so are $v_1, \dots, v_\ell$.

Example 2.4. There are 5 idempotents of U_3 corresponding to the 5 set partitions of [3]. These are depicted below:



2.5.3. *Permutation-idempotent and idempotent-permutation decompositions.* We now prove that every uniform block permutation can be factored as the product of a permutation and an idempotent; for example,



It turns out that the idempotents in the above decomposition are determined by d: they are $e_{\mathsf{top}(d)}$ and $e_{\mathsf{bot}(d)}$, respectively. However, the permutation is *not* unique.

Proposition 2.5. *For every $d \in \mathsf{U}_k$ and every $\sigma \in \mathfrak{S}_k$ satisfying $\sigma(B \cap [k]) = \overline{B} \cap [k]$ for all blocks B of d, we have*

$$d = e_{\mathsf{top}(d)}\, \sigma = \sigma e_{\,\mathsf{bot}(d)}.$$

Consequently,

$$\mathsf{U}_k = E(\mathsf{U}_k)\, \mathfrak{S}_k = \mathfrak{S}_k\, E(\mathsf{U}_k).$$

Remark 2.6. A monoid M is said to be *factorizable* if $M = GE$, where G is a subgroup of M and E is a set of idempotents in M. Thus, Proposition 2.5 states that U_k is factorizable.

Proof of Proposition 2.5. Recall from Section 2.3.2 that every uniform block permutation $d \in \mathsf{U}_k$ is associated with the size-preserving bijection defined by

$$\begin{aligned} \mathsf{top}(d) &\longrightarrow \mathsf{bot}(d) \\ B \cap [k] &\longrightarrow \overline{B} \cap [k] \end{aligned}$$

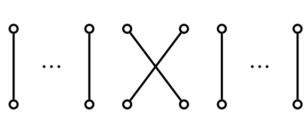
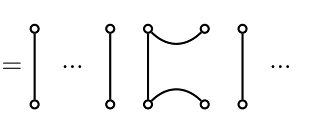
where B ranges over all blocks of d. If σ is any permutation in $\mathfrak{S}_k$ satisfying $\sigma(B \cap [k]) = \overline{B} \cap [k]$ for all blocks B, then $\sigma^{-1} \circ d$ maps each block $B \cap [k]$ to itself, and so it is the bijection associated with the idempotent $e_{\mathsf{top}(d)}$. Thus, in U_k we have $d\sigma^{-1} = e_{\mathsf{top}(d)}$.

2.5.4. *Properties of idempotents.* The following lemma collects some useful properties of the idempotents in U_k that we use throughout the paper. They can be proved directly from the definition of the product of two diagrams.

Lemma 2.7. *Let $\pi \vdash [k]$, $\tau \in \mathfrak{S}_k$ and $d \in \mathsf{U}_k$.*

(a) $\widetilde{e}_\pi = e_\pi$.
(b) $\mathsf{top}(\tau e_\pi) = \tau^{-1}(\pi)$ *and* $\mathsf{bot}(e_\pi \tau) = \tau(\pi)$.
(c) $\tau e_\pi \tau^{-1} = e_{\tau^{-1}(\pi)}$*; consequently,* $e_\pi \tau = \tau e_{\tau(\pi)}$ *and* $\tau e_\pi = e_{\tau^{-1}(\pi)}\tau$.
(d) $e_{\mathsf{top}(d)} d = d$ *and* $d e_{\mathsf{bot}(d)} = d$.
(e) $\mathsf{bot}(d e_\pi)$ *and* $\mathsf{top}(e_\pi d)$ *are coarser than or equal to* π.

2.6. Presentation of U_k. We recall here a known presentation of U_k; see [4, 10, 11]. For $1 \leqslant i < k$, set $s_i = \{\{1,\overline{1}\},\dots,\{i,\overline{i+1}\},\{i+1,\overline{i}\},\dots,\{k,\overline{k}\}\}$, which corresponds to the permutation in $\mathfrak{S}_k$ that swaps i and $i+1$, and $b_i = \{\{1,\overline{1}\},\dots,\{i,i+1,\overline{i},\overline{i+1}\},\dots,\{k,\overline{k}\}\}$. As diagrams

$s_i =$  $b_i =$ 

Then s_i, b_i for $1 \leqslant i \leqslant k-1$ generate U_k subject to the following relations:

(1) $s_i^2 = 1, \quad 1 \leqslant i \leqslant k-1$
(2) $b_i^2 = b_i, \quad 1 \leqslant i \leqslant k-1$
(3) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad 1 \leqslant i \leqslant k-2$
(4) $s_i b_{i+1} s_i = s_{i+1} b_i s_{i+1}, \quad 1 \leqslant i \leqslant k-2$
(5) $s_i s_j = s_j s_i, \quad |i-j| > 1$
(6) $b_i s_j = s_j b_i, \quad |i-j| > 1$
(7) $b_i s_i = s_i b_i = b_i, \quad 1 \leqslant i \leqslant k-1$
(8) $b_i b_j = b_j b_i, \quad 1 \leqslant i \leqslant k-1.$

2.7. U_k IS AN INVERSE MONOID. We prove that U_k is an inverse monoid, which will allow us to make use of known results for this class of monoids (see [27, Chapter 3]).

A monoid M is called an *inverse monoid* if for every $x \in M$, there exists a unique element $x^* \in M$, called the *generalized inverse* of x, satisfying $xx^*x = x$ and $x^*xx^* = x^*$.

Given a set partition d of $[k] \cup [\bar{k}]$, let $\widetilde{d}$ denote the set partition whose diagram is obtained by reflecting the diagram of d across a horizontal line. Note that if d is uniform, then $\widetilde{d}$ is also uniform. If d is a permutation in $\mathfrak{S}_k$, then $\widetilde{d}$ is precisely the inverse of the permutation. Furthermore, it can be verified directly using diagrams that

$$d\widetilde{d}d = d, \qquad \widetilde{d}d\widetilde{d} = \widetilde{d}, \qquad \widetilde{dd'} = \widetilde{d'}\,\widetilde{d}, \qquad \tilde{\tilde{d}} = d.$$

Using the fact that the idempotents of U_k commute (Lemma 2.3), one can prove that $\widetilde{d}$ is the unique element satisfying $d\widetilde{d}d = d$ and $\widetilde{d}d\widetilde{d} = \widetilde{d}$; see the proof of [27, Theorem 3.2].

PROPOSITION 2.8.

(1) *U_k is an inverse monoid, where the generalized inverse of $d \in \mathsf{U}_k$ is $\widetilde{d}$.*
(2) *$E(\mathsf{U}_k)$ is a commutative inverse monoid that is generated by $(i+1, j)b_i(i+1, j)$ for $1 \leqslant i < j \leqslant k$, where $(i+1, j)$ is the transposition in $\mathfrak{S}_k$ that swaps $i+1$ and j.*

3. IRREDUCIBLE REPRESENTATIONS OF U_k

We will develop the representation theory of U_k using results from the representation theory of finite monoids as presented in the excellent book by Steinberg [27]. We begin with a very brief overview to guide our development.

Let M be a finite monoid. Given an idempotent $e \in M$, there is a unique largest subgroup of M that contains e, which is called the maximal subgroup of M at the idempotent e and denoted by G_e. The irreducible (complex) representations of M (i.e. the simple $\mathbb{C}M$-modules) are determined by the irreducible representations of the maximal subgroups G_e. We describe the maximal subgroups of U_k in Section 3.1. Two maximal subgroups G_e and G_f of M are isomorphic if the idempotents e and f are *$\mathscr{J}$-equivalent*. This equivalence relation is defined in Section 3.2, where we determine the $\mathscr{J}$-classes of U_k. In Section 3.3 we describe the irreducible representations of the maximal subgroups of U_k. The construction makes use of an auxiliary representation called the Schützenberger representation that we describe in Section 3.4. In Section 3.5 we construct all the irreducible representations of U_k. Finally, in Section 3.6 we give a tableau model for the irreducible U_k-representations.

3.1. MAXIMAL SUBGROUPS OF U_k. As explained above, the representation theory of U_k can be expressed in terms of the representation theory of its maximal subgroups, and the subsequent results will describe their structure. The next result identifies the maximal subgroups of U_k. Recall that every idempotent of U_k is of the form e_π, where π is a set partition of $[k]$ (cf. Lemma 2.3).

LEMMA 3.1. *Let $e_\pi \in \mathsf{U}_k$ be the idempotent corresponding to a set partition $\pi \vdash [k]$. The maximal subgroup of U_k at the idempotent e_π is*

$$G_{e_\pi} = \{d \in \mathsf{U}_k : d\widetilde{d} = \widetilde{d}d = e_\pi\} = \{d \in \mathsf{U}_k : \mathsf{top}(d) = \mathsf{bot}(d) = \pi\}.$$

Proof. Let G_{e_π} be the maximal subgroup of U_k associated with e_π. Since U_k is an inverse monoid, G_{e_π} consists of all elements d of U_k such that $\widetilde{d}d = d\widetilde{d} = e_\pi$ [27, Corollary 3.6].

Suppose $\mathsf{top}(d) = \mathsf{bot}(d) = \pi$. By Proposition 2.5, there exists $\sigma \quad \mathfrak{S}_k$ satisfying $d = \sigma e_\pi = e_\pi \sigma$ and $\sigma(\pi) = \pi$. Thus,

$$\widetilde{d}d = (e_\pi \sigma^{-1})(\sigma e_\pi) = e_\pi \quad \text{and} \quad d\widetilde{d} = (\sigma e_\pi)(e_\pi \sigma^{-1}) = e_{\sigma^{-1}(\pi)} = e_\pi.$$

Conversely, suppose that $\widetilde{d}d = d\widetilde{d} = e_\pi$. Write $d = \sigma e_{\mathsf{bot}(d)}$ with $\sigma \quad \mathfrak{S}_k$. Then

$$e_\pi = \widetilde{d}d = \left(e_{\mathsf{bot}(d)}\sigma^{-1}\right)\left(\sigma e_{\mathsf{bot}(d)}\right) = e_{\mathsf{bot}(d)},$$
$$e_\pi = d\widetilde{d} = \left(\sigma e_{\mathsf{bot}(d)}\right)\left(e_{\mathsf{bot}(d)}\sigma^{-1}\right) = e_{\sigma^{-1}(\mathsf{bot}(d))},$$

which imply that $\pi = \mathsf{bot}(d)$; and $\pi = \sigma^{-1}(\mathsf{bot}(d)) = \mathsf{top}(d)$.

Next, we prove that each maximal subgroup is a direct product of symmetric groups. For any set B, let $\mathfrak{S}_B$ denote the permutation group of the elements in B.

Proposition 3.2. *Let $\pi = \{\pi_1, \ldots, \pi_\ell\} \quad [k]$.*

(1) *$G_{e_\pi} = \mathfrak{S}_{\pi^{(1)}} \times \mathfrak{S}_{\pi^{(2)}} \times \cdots \times \mathfrak{S}_{\pi^{(k)}}$, where $\pi^{(i)}$ is the set of blocks of π of size i.*
(2) *Let $B_i = \{j : |\pi_j| = i\}$. If $d \quad G_{e_\pi}$, then there exists $\tau \quad \mathfrak{S}_{B_1} \times \cdots \times \mathfrak{S}_{B_k}$ such that*

$$d = \left\{\pi_i \quad \overline{\pi_{\tau(i)}} : 1 \ 6 \ i \ 6 \ \ell\right\}.$$

Proof. If $d \quad G_{e_\pi}$ then $\mathsf{top}(d) = \mathsf{bot}(d) = \pi$. Then d is a bijection from the blocks of π to the blocks of π such that blocks of the same size map to blocks of the same size. This means that if we consider only the blocks of size i in d, there is a $\tau^{(i)} \quad \mathfrak{S}_{B_i}$ that describes the bijection for these blocks. Since this holds for any size i, the permutation $\tau := \tau^{(1)} \times \tau^{(2)} \times \cdots \times \tau^{(k)} \quad \mathfrak{S}_{B_1} \times \mathfrak{S}_{B_2} \times \cdots \times \mathfrak{S}_{B_k}$ describes the bijection for all the blocks of all sizes. Since $\pi_i \to \pi_{\tau(i)}$, the corresponding set partition has blocks $\pi_i \quad \overline{\pi}_{\tau(i)}$.

Corollary 3.3. *For $\pi \quad [k]$ with $\mathsf{type}(\pi) = (1^{a_1} 2^{a_2} \ldots k^{a_k})$, we have*

$$G_{e_\pi} \quad \mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_k}.$$

It follows from Corollary 3.3 that G_{e_π} and G_{e_γ} are isomorphic if $\mathsf{type}(\pi) = \mathsf{type}(\gamma)$. Explicitly, $\sigma^{-1} G_{e_\pi} \sigma = G_{e_\gamma}$ for any permutation $\sigma \quad \mathfrak{S}_k$ satisfying $\sigma^{-1} e_\pi \sigma = e_\gamma$.

Corollary 3.4. *If π and γ are two set partitions of $[k]$ satisfying $\mathsf{type}(\pi) = \mathsf{type}(\gamma)$, then G_{e_π} is isomorphic to G_{e_γ}. In particular, there exists a $\sigma \quad \mathfrak{S}_k$ such that $\sigma(\pi) = \gamma$ and*

$$\sigma^{-1} G_{e_\pi} \sigma = G_{e_\gamma}.$$

3.2. J-classes. Let x and y be elements of a monoid M. We say that x and y are *J-equivalent* if $MxM = MyM$. This is an equivalence relation; hence, it partitions the monoid M into classes which are called the *J-classes* of M. We denote by J_x the J-class containing x. In the next proposition we give a characterization for the J-classes of U_k and show that they are indexed by partitions of k.

Proposition 3.5. *Let k be a nonnegative integer.*

(a) *Every J-class of U_k contains an idempotent.*
(b) *Two elements $d, d \quad \mathsf{U}_k$ are in the same J-class if and only if $\mathsf{type}(\mathsf{top}(d)) = \mathsf{type}(\mathsf{top}(d\ ))$.*
(c) *The J-classes are in bijection with partitions λ of k. In particular, if $d \quad \mathsf{U}_k$ and $\mathsf{type}(\mathsf{top}(d)) = \lambda$, then*

$$J_\lambda := J_d = \{d\ : \mathsf{type}(\mathsf{top}(d\ )) = \lambda\}.$$

(d) *If $\pi \quad [k]$ and $\mathsf{type}(\pi) = \lambda$, then*

$$J_\lambda = \{\sigma e_\pi \tau : \sigma, \tau \quad \mathfrak{S}_k\}.$$

Proof. (a) Every $d \in \mathsf{U}_k$ can be written using the permutation-idempotent representation as $d = \sigma e_\pi$ for some $\sigma \in \mathfrak{S}_k$, where $\pi = \mathsf{bot}(d)$. Then $\mathsf{U}_k d \mathsf{U}_k = \mathsf{U}_k \sigma e_\pi \mathsf{U}_k = \mathsf{U}_k e_\pi \mathsf{U}_k$, where the last equality follows because σ is invertible in U_k. Thus, J_d contains e_π.

(b) Using the permutation-idempotent representation of d and d' we know that $d = \sigma e_\pi$ and $d' = \sigma' e_\gamma$ for some $\pi, \gamma \vdash [k]$ and $\sigma, \sigma' \in \mathfrak{S}_k$. Furthermore, by Lemma 2.7, $\mathsf{type}(\mathsf{top}(d)) = \mathsf{type}(\pi)$ and $\mathsf{type}(\mathsf{top}(d')) = \mathsf{type}(\gamma)$. Therefore, it suffices to prove that e_π and e_γ are in the same $\mathcal{J}$-class if and only if $\mathsf{type}(\pi) = \mathsf{type}(\gamma)$.

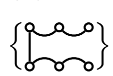
If $\mathsf{type}(\pi) = \mathsf{type}(\gamma)$, then there exists a $\tau \in \mathfrak{S}_k$ such that $\gamma = \tau(\pi)$. Also, $\tau^{-1} e_\pi \tau = e_{\tau(\pi)}$. Hence, $\mathsf{U}_k e_\gamma \mathsf{U}_k = \mathsf{U}_k \tau^{-1} e_\pi \tau \mathsf{U}_k = \mathsf{U}_k e_\pi \mathsf{U}_k$, since $\mathsf{U}_k \tau^{-1} = \mathsf{U}_k = \tau \mathsf{U}_k$ due to the fact that τ is invertible in U_k. This implies that any two idempotents e_π and e_γ such that $\mathsf{type}(\pi) = \mathsf{type}(\gamma)$ are in the same $\mathcal{J}$-class.

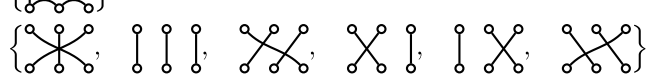
Observe that $\mathsf{U}_k e_\pi \mathsf{U}_k$ contains elements d such that $\mathsf{top}(d)$ and $\mathsf{bot}(d)$ are equal to or coarser than $\tau(\pi)$ for some $\tau \in \mathfrak{S}_k$. Assume that $\mathsf{type}(\gamma) \neq \mathsf{type}(\pi)$ with $\gamma, \pi \vdash [k]$ and $\mathsf{type}(\gamma) = (1^{b_1} 2^{b_2} \dots k^{b_k})$ and $\mathsf{type}(\pi) = (1^{a_1} 2^{a_2} \dots k^{a_k})$. Then for some i, $a_i \neq b_i$. Without loss of generality assume that $a_1 = b_1, \dots, a_{i-1} = b_{i-1}$ and $a_i < b_i$ for some i. This means that γ has more blocks of size i than π. Therefore, it is not possible for γ to be coarser than $\tau(\pi)$ for any $\tau \in \mathfrak{S}_k$. Hence, $e_\gamma \notin \mathsf{U}_k e_\pi \mathsf{U}_k$.

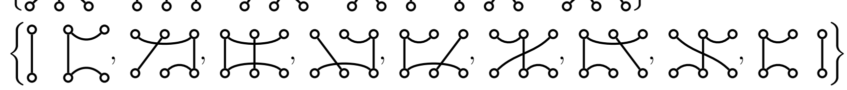
(c) This is a direct consequence of (b).

(d) Multiplying e_π on the left by a permutation results in a diagram whose top is a permutation of π and hence has the same type as π. Similarly, multiplying e_π on the right by a permutation results in a diagram with bottom that has the same type as π. Hence $\{\sigma e_\pi \tau : \sigma, \tau \in \mathfrak{S}_k\} \subseteq J_\lambda$. Conversely, suppose $d \in J_\lambda$. We may write $d = e_{\mathsf{top}(d)} \sigma$ for some $\sigma \in \mathfrak{S}_k$. Since $\mathsf{type}(\mathsf{top}(d)) = \mathsf{type}(\pi)$, there exists a $\tau \in \mathfrak{S}_k$ such that $\tau(\pi) = \mathsf{top}(d)$. Hence $d = e_{\tau(\pi)} \sigma = \tau^{-1} e_\pi \tau \sigma$, proving that $J_\lambda \subseteq \{\sigma e_\pi \tau : \sigma, \tau \in \mathfrak{S}_k\}$.

Example 3.6. There are three $\mathcal{J}$-classes for U_3:

$J_{(3)} =$ 

$J_{(1,1,1)} =$ 

$J_{(2,1)} =$ 

When constructing irreducible representations of U_k, we need only one maximal subgroup for each $\mathcal{J}$-class. It is useful to make this choice standard. Recall that the $\mathcal{J}$-classes are indexed by partitions of k. Hence, if $\lambda = (1^{a_1} 2^{a_2} \dots k^{a_k})$, we define the *representative set partition* associated to λ as

$$\pi_\lambda = \{\{1\}, \{2\}, \dots, \{a_1\}, \{a_1 + 1, a_1 + 2\}, \dots, \{a_1 + 2a_2 - 1, a_1 + 2a_2\}, \dots\}. \tag{3}$$

This is the set partition that uses $1, \dots, a_1$ for blocks of size one, $a_1 + 1, \dots, a_1 + 2a_2$ for blocks of size two, et cetera. For example if $k = 11$ and $\lambda = (1^4 2^2 3^1)$, then

$$\pi_\lambda = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5, 6\}, \{7, 8\}, \{9, 10, 11\}\}.$$

In this case, we write $G_\lambda \coloneqq G_{e_{\pi_\lambda}}$ and call it the *representative maximal subgroup* associated to λ. Note that $\{G_\lambda : \lambda \vdash k\}$ is a set of maximal subgroups of U_k with each subgroup associated with a distinct $\mathcal{J}$-class of U_k.

3.3. Irreducible representations of the maximal subgroups. We now describe the irreducible representations of the maximal subgroups G_{e_π}. From Proposition 3.2, we know that $G_{e_\pi} = \mathfrak{S}_{\pi^{(1)}} \times \mathfrak{S}_{\pi^{(2)}} \times \cdots \times \mathfrak{S}_{\pi^{(k)}}$, where $\pi^{(i)}$ is the set of blocks of π of size i. Hence, each irreducible representation of G_{e_π} is isomorphic to a tensor product of the form $V_1 \otimes V_2 \otimes \cdots \otimes V_k$, where V_i is an irreducible representation of $\mathfrak{S}_{\pi^{(i)}}$ for all $1 \leqslant i \leqslant k$.

Recall that the irreducible representations of the symmetric group $\mathfrak{S}_{\pi^{(i)}}$ are indexed by partitions of $|\pi^{(i)}|$ and admit the following combinatorial description. As a vector space, the irreducible representation indexed by the partition $\lambda^{(i)}$ is spanned by the set of standard tableaux with entries the blocks in $\pi^{(i)}$. The action of $\mathfrak{S}_{\pi^{(i)}}$ is given by permuting the entries of the tableaux; however, the result may not be a standard tableaux, in which case one uses the Garnir relations to express the result as a linear combination of standard tableaux. For details, see [24, Chapter 2].

For a sequence of partitions $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ with $\lambda^{(i)} \vdash |\pi^{(i)}|$, let

$$V_{G_{e_\pi}}^{\vec{\lambda}} = V_{\mathfrak{S}_{\pi^{(1)}}}^{\lambda^{(1)}} \otimes \cdots \otimes V_{\mathfrak{S}_{\pi^{(k)}}}^{\lambda^{(k)}},$$

where $V_{\mathfrak{S}_{\pi^{(i)}}}^{\lambda^{(i)}}$ is the irreducible representation of $\mathfrak{S}_{\pi^{(i)}}$ indexed by $\lambda^{(i)}$. By the above discussion, this is an irreducible representation of G_{e_π} and all the irreducible representations of G_{e_π} are of this form. In particular, the irreducible representations of G_{e_π} are indexed by k-tuples of partitions $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ such that $\lambda^{(i)} \vdash |\pi^{(i)}|$ and $\sum_{i=1}^{k} i|\pi^{(i)}| = k$. This implies that $\vec{\lambda} \in I_k$ with I_k as defined in Equation (1).

The combinatorial descriptions of each of the irreducible representations in the tensor product above combine to give a combinatorial description of the irreducible representations of G_{e_π}. In order to state it, we need the following definitions.

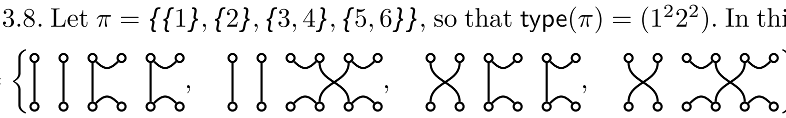
Definition 3.7. *Let $\pi \vdash [k]$. A π-tableau $\mathbf{T}$ of shape $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ is a k-tuple of tableaux $(T^{(1)}, T^{(2)}, \ldots, T^{(k)})$, where $T^{(i)}$ is a standard tableau of shape $\lambda^{(i)} \vdash |\lambda^{(i)}|$ filled with the blocks in π of size i.*

As a vector space, the irreducible G_{e_π}-representation $V_{G_{e_\pi}}^{\vec{\lambda}}$ is spanned by the set of π-tableaux of shape $\vec{\lambda}$. By Proposition 3.2, every $\tau \in G_{e_\pi}$ can be expressed uniquely as $\tau = \tau^{(1)}\tau^{(2)} \cdots \tau^{(k)}$ with $\tau^{(i)} \in \mathfrak{S}_{\pi^{(i)}}$. Then the action of τ on a π-tableau is given by

$$\tau \cdot \mathbf{T} = (\tau^{(1)} \cdot T^{(1)}, \ldots, \tau^{(k)} \cdot T^{(k)}),$$

where $\tau^{(i)} \cdot T^{(i)}$ is obtained by permuting the entries of the tableau $T^{(i)}$. The result may not be a π-tableau since it may not be standard, in which case we use the Garnir straightening relations on each component to express the result as a linear combination of standard π-tableaux (for details see [24, Section 2.6] or [1]).

Example 3.8. Let $\pi = \{\{1\}, \{2\}, \{3, 4\}, \{5, 6\}\}$, so that $\mathsf{type}(\pi) = (1^2 2^2)$. In this case,

$G_{e_\pi} =$ 

There are four irreducible representations of G_{e_π}, which are all one dimensional:

$$V_{G_{e_\pi}}^{((1,1),(1,1))} \text{ with basis } \left\{ \left(\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}, \begin{array}{|c|}\hline 56 \\ \hline 34 \\ \hline \end{array} \right) \right\},$$

$$V_{G_{e_\pi}}^{((2),(1,1))} \text{ with basis } \left\{ \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|}\hline 56 \\ \hline 34 \\ \hline \end{array} \right) \right\},$$

$$V_{G_{e_\pi}}^{((1,1),(2))} \text{ with basis } \left\{ \left(\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 34 & 56 \\ \hline \end{array} \right) \right\},$$

$$V_{G_{e_\pi}}^{((2),(2))} \text{ with basis } \left\{ \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 34 & 56 \\ \hline \end{array} \right) \right\}.$$

3.4. L-classes and Schützenberger representations. For each idempotent $e \in \mathsf{U}_k$, we define a (U_k, G_e)-bimodule $\mathsf{C}L_e$ that is known in the semigroup theory literature as the *left Schützenberger representation associated with e*. The Schützenberger representations will be used in Section 3.5 to construct irreducible U_k-representations from irreducible G_e-representations. As a vector space, $\mathsf{C}L_e$ is spanned by the elements of the L-class of e, so we begin by studying the L-classes of U_k.

Let x and y be elements of a monoid M. We say that x and y are *L-equivalent* if $Mx = My$. This is an equivalence relation; hence, it partitions M into classes which are called the *L-classes* of M. The L-class of an element x is denoted by L_x.

Proposition 3.9. *Let k be a nonnegative integer.*

(a) *Two elements $d_1, d_2 \in \mathsf{U}_k$ are in the same L-class if and only if $\mathsf{bot}(d_1) = \mathsf{bot}(d_2)$.*
(b) *Every L-class in U_k contains a unique idempotent.*
(c) *The L-classes of U_k are in bijection with the set partitions π of $[k]$. More precisely,*

$$L_\pi := L_{e_\pi} = \{d \in \mathsf{U}_k : \mathsf{bot}(d) = \pi\}.$$

(d) *For every $\lambda \vdash k$, the J-class J_λ is a disjoint union of L-classes. More precisely,*

$$J_\lambda = \biguplus_{\pi : \mathsf{type}(\pi) = \lambda} L_\pi.$$

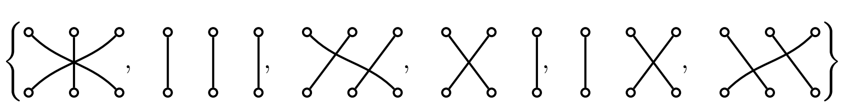
Proof. (a) By Proposition 2.5, every element $d \in \mathsf{U}_k$ can be written as $d = \sigma e_{\mathsf{bot}(d)}$ for some permutation $\sigma \in \mathfrak{S}_k$. Thus, $\mathsf{U}_k d = \mathsf{U}_k \sigma e_{\mathsf{bot}(d)} = \mathsf{U}_k e_{\mathsf{bot}(d)}$. The last equality follows since σ is invertible in U_k. Therefore, if $m \in \mathsf{U}_k e_{\mathsf{bot}(d)}$, then $\mathsf{bot}(m)$ is coarser than or equal to $\mathsf{bot}(d)$. Hence, d_1, d_2 are in the same L-class if and only if $\mathsf{bot}(d_1) = \mathsf{bot}(d_2)$.

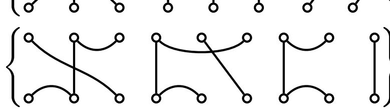
(b) From part (a) we have that an L-class L contains elements that have the same set partition π as the bottom row. Since there is a unique idempotent, namely e_π, that has bottom row π, the result follows.

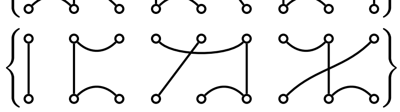
(c) Since every L-class contains a unique idempotent, the L-classes are in bijection with the set partitions of $[k]$.

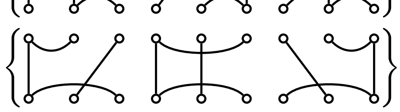
(d) By Proposition 3.5(d), $J_\lambda = \{\sigma e_\pi \tau : \sigma, \tau \in \mathfrak{S}_k \text{ and } \mathsf{type}(\pi) = \lambda\}$, while $L_\pi = \{\sigma e_\pi : \sigma \in \mathfrak{S}_k\}$. Thus, the result follows.

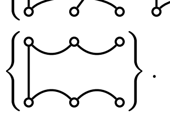
Example 3.10. There are five L-classes for U_3:

$L_{1|2|3} =$ 

$L_{12|3} =$ 

$L_{1|23} =$ 

$L_{13|2} =$ 

$L_{123} =$ 

For any $\pi \quad [k]$, let $\mathsf{C}L_\pi$ be the vector space with basis the elements of the L -class L_π. It has a *left U_k-action* defined by

$$m \quad \ell = \begin{cases} m\ell, & \text{if } m\ell \quad L_\pi, \\ 0, & \text{else,} \end{cases} \tag{4}$$

for all $m \quad \mathsf{U}_k$ and $\ell \quad L_\pi$, which is then extended C-linearly to all of $\mathsf{C}L_\pi$. The nonzero products $m \quad \ell$ can be characterized as follows.

Lemma 3.11. *Let $m \quad \mathsf{U}_k$ and $\ell \quad L_\pi$. Then $m \quad \ell = 0$ if and only if $\mathsf{bot}(m)$ is finer than $\mathsf{top}(\ell)$.*

The *right G_{e_π}-action* on $\mathsf{C}L_\pi$ is extended C-linearly from the action of G_{e_π} on L_π given by right multiplication. Although it is true for any finite monoid that the maximal subgroup of an idempotent e acts by right multiplication on the L -class of e [27, Proposition 1.10], below we provide a proof that is specific to U_k and we identify the orbits of this action.

Proposition 3.12. *Let π be a set partition of $[k]$.*

(a) *G_{e_π} acts by right multiplication on L_π and this right action is free. In other words,*
   - *if $\ell \quad L_\pi$ and $g \quad G_{e_\pi}$, then $\ell g \quad L_\pi$; and*
   - *if $\ell \quad L_\pi$ and $\ell g = \ell h$ for some $g, h \quad G_{e_\pi}$, then $g = h$.*

(b) *$d_1, d_2 \quad L_\pi$ are in the same G_{e_π}-orbit if and only if $\mathsf{top}(d_1) = \mathsf{top}(d_2)$.*

(c) *For every set partition $\gamma \quad [k]$ such that $\mathsf{type}(\gamma) = \mathsf{type}(\pi)$,*

$$L_\pi^\gamma = \{d \quad \mathsf{U}_k : \mathsf{top}(d) = \gamma \text{ and } \mathsf{bot}(d) = \pi\}$$

is an orbit for the right G_{e_π}-action on L_π, and all the orbits are of this form. Thus, the G_{e_π}-orbits in L_π are in bijection with the set partitions γ of $\mathsf{type}(\pi)$.

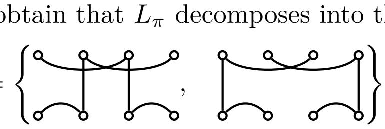
Proof. (a) Let $\ell \quad L_\pi$ and $g, h \quad G_{e_\pi}$, and think of them as bijections as in Section 2.3.2. Since $\mathsf{bot}(\ell) = \pi = \mathsf{top}(g)$, we have that ℓg is the composition of ℓ and g (see Remark 2.2). Hence, $\mathsf{bot}(\ell g) = \mathsf{bot}(g) = \pi$ and so $\ell g \quad L_\pi$. Similarly, ℓh is the composition of ℓ and h. Thus, if $\ell g = \ell h$, then $g = h$ since ℓ is a bijection.

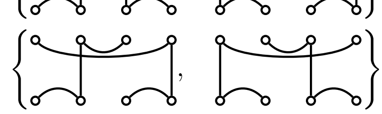
(b) If $d_2 = d_1 g$ for some $g \quad G_{e_\pi}$, then $\mathsf{top}(d_1) = \mathsf{top}(d_1 g) = \mathsf{top}(d_2)$ since multiplying on the right by an element in G_{e_π} has no effect on the top row of the diagram of d_1.

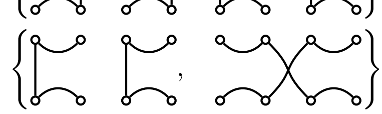
Conversely, if $d_1, d_2 \quad L_\pi$ and $\mathsf{top}(d_1) = \mathsf{top}(d_2)$, then both d_1 and d_2 are bijections from $\mathsf{top}(d_1)$ to π. Since $\widetilde{d}_1 d_2$ is a size-preserving bijection from π to itself, it is equal to an element $g \quad G_{e_\pi}$. By composing $\widetilde{d}_1 d_2 = g$ on the right with d_1, we conclude that $d_2 = d_1 g$.

(c) This follows directly from (b).

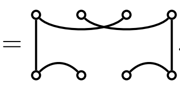
Example 3.13. Let $\pi = 12|34$. The L -class L_π contains 6 elements all with the same bottom row. The group G_{e_π} contains two elements, the identity and the permutation of the two blocks. Hence we obtain that L_π decomposes into three G_{e_π}-orbits:

$L_{12|34}^{13|24} =$  ,

$L_{12|34}^{23|14} =$  ,

$L_{12|34}^{12|34} =$  .

We will now choose *orbit representatives* of the right G_{e_π}-action of L_π. Let L^γ_π be an orbit, where $\gamma \subseteq [k]$ and $\mathsf{type}(\gamma) = \mathsf{type}(\pi)$. Think of the elements of L^γ_π as bijections $\ell\colon \gamma \to \pi$. Assume $\pi = \{\pi_1 < \cdots < \pi_r\}$ and $\gamma = \{\gamma_1 < \cdots < \gamma_r\}$ are ordered using the graded last letter order. Let $\ell^\gamma_\pi\colon \gamma \to \pi$ be the bijection that sends γ_i to π_i for all i. If $\gamma = \pi$, then this is the identity bijection and we have $\ell^\pi_\pi = e_\pi$.

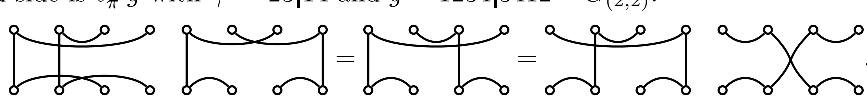
Example 3.14. The orbit representative for $L^{13|24}_{12|34}$ is $\ell^{13|24}_{12|34} =$ .

The next result describes the relationship between the actions of U_k and G_{e_π} on $\mathbf{C}L_\pi$.

Proposition 3.15. *Let $m \in \mathsf{U}_k$ and $d \in L^\gamma_\pi$. If $m \cdot d = 0$, then there exists a unique $g \in G_{e_\pi}$ such that $md = \ell^{\gamma'}_\pi g$, where $\gamma' = \sigma^{-1}(\gamma)$ for any $\sigma \in \mathfrak{S}_k$ such that $m = \sigma e_{\mathsf{bot}(m)}$.*

Proof. Let $m \in \mathsf{U}_k$ and write $m = \sigma e_{\mathsf{bot}(m)}$ for some $\sigma \in \mathfrak{S}_k$. Let $d \in L^\gamma_\pi$ be such that $m \cdot d = 0$. Then $md \in L_\pi$, and so $\mathsf{bot}(m)$ is finer than $\mathsf{top}(d) = \gamma$. Since $\mathsf{bot}(m)$ is finer than γ, we have $e_{\mathsf{bot}(m)} e_\gamma = e_\gamma$ by Lemma 2.3. By Lemma 2.7 we have $e_\gamma d = d$, which means that $md = \sigma e_{\mathsf{bot}(m)} d = \sigma d$. Since $\mathsf{top}(\sigma d) = \sigma^{-1}(\gamma)$, we have $\sigma d \in L^{\gamma'}_\pi$, where $\gamma' = \sigma^{-1}(\gamma)$. By Proposition 3.12, the right G_{e_π}-action on L_π is free and $L^{\gamma'}_\pi$ is an orbit for this action. This means that there is a unique $g \in G_{e_\pi}$ such that $md = \ell^{\gamma'}_\pi g$.

Example 3.16. Let $\pi = 12|34$ and $\gamma = 13|24$. The following diagram equation is an example of Proposition 3.15, where the left hand side product is md and the right hand side is $\ell^{\gamma'}_\pi g$ with $\gamma' = 23|14$ and $g = 12\overline{34}|34\overline{12} \in G_{(2,2)}$.



3.5. Irreducible representations of U_k. In this section, we explain how each irreducible representation of U_k is obtained by inflating an irreducible representation of one of its maximal subgroups. In Section 3.6, we will describe a tableau model for these representations.

We begin by identifying a natural indexing set of the isomorphism classes of irreducible representations of U_k.

Proposition 3.17. *The isomorphism classes of the irreducible representations of U_k are indexed by I_k as defined in* (1).

Proof. For any finite monoid M, let $\mathrm{Irr}_{\mathbf{C}}(M)$ be the set of isomorphism classes of irreducible representations of M over $\mathbf{C}$. By [27, Corollary 5.6], there is a bijection between $\mathrm{Irr}_{\mathbf{C}}(M)$ and $\bigcup_e \mathrm{Irr}_{\mathbf{C}}(G_e)$, where the idempotents e are chosen one from each $\mathcal{J}$-class of M.

Recall from Section 3.2 that $\{e_{\pi_\lambda} : \lambda \vdash k\}$ is a set of representative idempotents for the $\mathcal{J}$-classes of U_k, and that the associated maximal subgroup G_λ is isomorphic to $\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_k}$ if $\lambda = (1^{a_1} 2^{a_2} \ldots k^{a_k})$ (Corollary 3.3). Hence, the isomorphism classes of irreducible representations of G_λ are indexed by sequences of partitions $(\lambda^{(1)}, \ldots, \lambda^{(k)})$ such that $\lambda^{(i)} \vdash a_i$ and $\sum_{i=1}^k i a_i = k$ (cf. Section 3.3).

For $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ with $|\lambda^{(i)}| = a_i$, we define $\overrightarrow{\mathsf{type}}(\vec{\lambda}) = (1^{a_1} 2^{a_2} \ldots k^{a_k})$. Let $\vec{\lambda} \in I_k$ and write $\lambda = \overrightarrow{\mathsf{type}}(\vec{\lambda})$. Let $V^{\vec{\lambda}}_{G_\lambda}$ be the irreducible representation of G_λ indexed by $\vec{\lambda}$. By [27, Theorem 5.5],

$$W^{\vec{\lambda}}_{\mathsf{U}_k} = \mathsf{Ind}^{\mathsf{U}_k}_{G_\lambda}\left(V^{\vec{\lambda}}_{G_\lambda}\right) \Big/ \mathsf{rad}\left(\mathsf{Ind}^{\mathsf{U}_k}_{G_\lambda}\left(V^{\vec{\lambda}}_{G_\lambda}\right)\right)$$

is an irreducible representation of U_k. Since U_k is a finite inverse monoid, the monoid algebra $\mathsf{C}\mathsf{U}_k$ is semisimple [27, Corollary 9.4], from which it follows that $\mathsf{rad}(\mathsf{Ind}_{G_\lambda}^{\mathsf{U}_k}(V_{G_\lambda}^{\vec{\lambda}})) = 0$. Thus,

$$W_{\mathsf{U}_k}^{\vec{\lambda}} = \mathsf{Ind}_{G_\lambda}^{\mathsf{U}_k}\big(V_{G_\lambda}^{\vec{\lambda}}\big) = \mathsf{C}L_\lambda \quad {}_{\mathsf{C}G_\lambda} V_{G_\lambda}^{\vec{\lambda}},$$

where $\mathsf{C}L_\lambda$ is the left Schützenberger representation associated with the idempotent e_{π_λ} (cf. Section 3.4). Since $\mathsf{C}L_\lambda$ is a $(\mathsf{U}_k, G_\lambda)$-bimodule, the tensor product $\mathsf{C}L_\lambda \quad {}_{\mathsf{C}G_\lambda} V_{G_\lambda}^{\vec{\lambda}}$ is a left U_k-module, where for all $d \quad \mathsf{U}_k$, $\ell \quad L_\lambda$ and $v \quad V_{G_\lambda}^{\vec{\lambda}}$:

$$d \cdot (\ell \quad v) = (d \quad \ell) \quad v. \tag{5}$$

Notice that the tensor product is over $\mathsf{C}G_\lambda$, which is the case throughout this section.

We now describe a basis of $W_{\mathsf{U}_k}^{\vec{\lambda}}$. In Section 3.3, we found that a basis of $V_{G_\lambda}^{\vec{\lambda}}$ is given by the π-tableaux of shape $\vec{\lambda}$. To obtain a basis of $W_{\mathsf{U}_k}^{\vec{\lambda}}$, it suffices to tensor this basis with the orbit representatives of the right G_λ-action on L_λ, as we prove next.

Proposition 3.18. *Let $\vec{\lambda} \quad I_k$, $\lambda = \overrightarrow{\mathsf{type}}(\vec{\lambda})$ and $\pi = \pi_\lambda$. Let $\{\ell_\pi^\gamma : \gamma \quad [k], \mathsf{type}(\gamma) = \lambda\}$ be the orbit representatives of the right G_λ-action on L_λ as defined in Section 3.4, and let $\mathsf{B}_{\vec{\lambda}}(G_\lambda)$ be a basis for the irreducible G_λ-representation $V_{G_\lambda}^{\vec{\lambda}}$ indexed by $\vec{\lambda}$. Then a basis for the irreducible U_k-representation $W_{\mathsf{U}_k}^{\vec{\lambda}}$ is*

$$\mathsf{B}_{\vec{\lambda}}(\mathsf{U}_k) := \left\{\ell_\pi^\gamma \quad \mathbf{T} : \gamma \quad [k], \mathsf{type}(\gamma) = \lambda \text{ and } \mathbf{T} \quad \mathsf{B}_{\vec{\lambda}}(G_\lambda)\right\}.$$

Proof. Since $\mathsf{B}_{\vec{\lambda}}(G_\lambda)$ is a basis of $V_{G_\lambda}^{\vec{\lambda}}$, it follows that $W_{\mathsf{U}_k}^{\vec{\lambda}}$ is spanned by $\ell \quad \mathbf{T}$ with $\ell \quad L_\lambda$ and $\mathbf{T} \quad \mathsf{B}_{\vec{\lambda}}(G_\lambda)$. By Proposition 3.15, if $d \quad \ell = 0$, then there is a unique $g \quad G_\lambda$ and $\gamma \quad [k]$ satisfying $\mathsf{type}(\gamma) = \lambda$ and $d \quad \ell = \ell_\pi^\gamma g$. Thus,

$$d \cdot \big(\ell \quad \mathbf{T}\big) = \ell_\pi^\gamma g \quad \mathbf{T} = \ell_\pi^\gamma \quad g\mathbf{T},$$

which proves that $W_{\mathsf{U}_k}^{\vec{\lambda}}$ is spanned by elements of the form $\ell_\pi^\gamma \quad \mathbf{T}$.

Furthermore, since $\{\ell_\pi^\gamma : \gamma \quad [k], \mathsf{type}(\gamma) = \lambda\}$ is a basis of $\mathsf{C}L_\pi$ as a right G_λ-module and $\mathsf{B}_{\vec{\lambda}}(G_\lambda)$ is a C-basis for the irreducible G_λ-representation $V_{G_\lambda}^{\vec{\lambda}}$, then $\mathsf{B}_{\vec{\lambda}}(\mathsf{U}_k)$ is linearly independent as a vector space over C.

As a consequence of identifying that the basis is indexed by a pair consisting of a set partition $\gamma \quad [k]$ such that $\mathsf{type}(\gamma) = \overrightarrow{\mathsf{type}}(\vec{\lambda})$ and a π-tableau of shape $\vec{\lambda}$, we have the following formula for the dimension of the irreducible representation of U_k.

Corollary 3.19. *Let $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}) \quad I_k$ and $\lambda = \overrightarrow{\mathsf{type}}(\vec{\lambda})$, then*

$$\dim W_{\mathsf{U}_k}^{\vec{\lambda}} = \mathsf{sp}_k(\lambda) f^{\lambda^{(1)}} f^{\lambda^{(2)}} \cdots f^{\lambda^{(k)}},$$

where $\mathsf{sp}_k(\lambda)$ is equal to the number of set partitions of type λ (see Equation (2)) and f^λ is equal to the number of standard tableaux of shape λ.

Example 3.20. There are five irreducible U_3-representations. We give their bases below:

$$W_{\mathsf{U}_3}^{((3))} = \mathrm{span}\left\{\ell_{1|2|3}^{1|2|3} \quad \left(\boxed{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}}\right)\right\},$$

$$W_{\mathsf{U}_3}^{((2,1))} = \mathrm{span}\left\{\ell_{1|2|3}^{1|2|3} \quad \left(\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\right), \ell_{1|2|3}^{1|2|3} \quad \left(\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\right)\right\},$$

$$W_{\mathsf{U}_3}^{((1,1,1))} = \mathrm{span}\left\{\ell_{1|2|3}^{1|2|3} \quad \left(\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}\right)\right\},$$

$$W_{U_3}^{((1),(1))} = \operatorname{span}\left\{\ell^{1|23}_{1|23} \quad \left(\boxed{1}, \boxed{23}\right), \ell^{2|13}_{1|23} \quad \left(\boxed{1}, \boxed{23}\right), \ell^{3|12}_{1|23} \quad \left(\boxed{1}, \boxed{23}\right)\right\},$$
$$W_{U_3}^{(?,?,(1))} = \operatorname{span}\left\{\ell^{123}_{123} \quad \left(?, ?, \boxed{123}\right)\right\}.$$

Example 3.21. In this example, we illustrate the action of an element in U_k on a basis element. To demonstrate with an example that is sufficiently large, take $k = 17$ and represent the labels 10 through 17 by the letters a through h.

Let $\vec{\lambda} = ((2,1),(2,2),(1,1)) \quad I_{17}$ so that $\lambda = \overrightarrow{\mathsf{type}}(\vec{\lambda}) = (1^3\,2^4\,3^2)$. Choose our basis element to be $\ell^{\gamma}_{\pi} \quad \mathbf{T} \quad W^{\vec{\lambda}}_{U_{17}}$, where

$$\ell^{\gamma}_{\pi} = 2\,\bar{1} \,|\, 7\,\bar{2} \,|\, g\,\bar{3} \,|\, 1\,3\,\bar{4}\,\bar{5} \,|\, 5\,b\,\bar{6}\,\bar{7} \,|\, 6\,d\,\bar{8}\,\bar{9} \,|\, 9\,e\,\bar{a}\,\bar{b} \,|\, 4\,a\,c\,\bar{c}\,\bar{d}\,\bar{e} \,|\, 8\,f\,h\,\bar{f}\,\bar{g}\,\bar{h}$$

and

$$\mathbf{T} = \left(\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 67 & ab \\ \hline 45 & 89 \\ \hline \end{array}, \quad \begin{array}{|c|}\hline fgh \\ \hline cde \\ \hline \end{array} \right).$$

Now any element $d \quad U_{17}$ such that the number of blocks in $d\ell^{\gamma}_{\pi}$ is smaller than the number of blocks in ℓ^{γ}_{π} will act as 0.

As an example then, let us consider the action of an element d such that the number of blocks in $d\ell^{\gamma}_{\pi}$ is equal to the number of blocks of ℓ^{γ}_{π}. That is, $\mathsf{bot}(d)$ must be finer than γ. Let

$$d = 2\,\bar{8} \,|\, 8\,\bar{2} \,|\, 9\,\bar{g} \,|\, a\,\bar{d} \,|\, b\,\bar{7} \,|\, c\,\bar{6} \,|\, e\,\bar{a} \,|\, f\,\bar{3} \,|\, h\,\bar{1} \,|\, 1\,4\,\bar{5}\,\bar{b} \,|\, 6\,7\,\bar{9}\,\bar{e} \,|\, 3\,d\,\bar{4}\,\bar{c} \,|\, 5\,g\,\bar{f}\,\bar{h}.$$

Then the action of d on $\ell^{\gamma}_{\pi} \quad \mathbf{T}$ is $\ell^{\gamma}_{\pi} \quad g \cdot \mathbf{T}$, where

$$\gamma \ = 8 \,|\, 9 \,|\, b \,|\, 1\,4 \,|\, 6\,7 \,|\, a\,c \,|\, f\,h \,|\, 3\,d\,e \,|\, 2\,5\,g$$

and

$$g = 1\,\bar{1} \,|\, 2\,\bar{3} \,|\, 3\,\bar{2} \,|\, 4\,5\,\bar{6}\,\bar{7} \,|\, 6\,7\,\bar{a}\,\bar{b} \,|\, 8\,9\,\bar{8}\,\bar{9} \,|\, a\,b\,\bar{4}\,\bar{5} \,|\, c\,d\,e\,\bar{c}\,\bar{d}\,\bar{e} \,|\, f\,g\,h\,\bar{f}\,\bar{g}\,\bar{h} \quad G_{e_\pi},$$

since $d\ell^{\gamma}_{\pi} = \ell^{\gamma}_{\pi}\, g$. We must then apply some straightening relations on $g \cdot \mathbf{T}$ to express it as a linear combination of the basis elements.

3.6. Tableau model for the irreducible U_k-representations.

We prove that the basis of $W^{\vec{\lambda}}_{U_k}$ in Proposition 3.18 is in bijection with certain sequences of set-valued tableaux and we describe the action of U_k directly in terms of these sequences.

Definition 3.22. *A* uniform tableau *of shape $\vec{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)}) \quad I_k$ is a sequence of tableaux $\mathbf{S} = (S^{(1)}, \dots, S^{(k)})$ such that:*

(1) *$S^{(i)}$ is a tableau of shape $\lambda^{(i)}$ filled with subsets of $[k]$ of size i;*
(2) *$S^{(i)}$ is standard, i.e. increasing along rows and columns in the last letter order; and*
(3) *the subsets appearing in $\mathbf{S}$ form a set partition of $[k]$.*

We define $T_{\vec{\lambda}}$ to be the set of uniform tableaux of shape $\vec{\lambda}$.

Example 3.23. Here are the elements in $T_{((1),(1,1))}$:

$$\left(\boxed{1}, \begin{array}{|c|}\hline 45 \\ \hline 23 \\ \hline\end{array}\right), \left(\boxed{1}, \begin{array}{|c|}\hline 35 \\ \hline 24 \\ \hline\end{array}\right), \left(\boxed{1}, \begin{array}{|c|}\hline 25 \\ \hline 34 \\ \hline\end{array}\right), \left(\boxed{2}, \begin{array}{|c|}\hline 45 \\ \hline 13 \\ \hline\end{array}\right), \left(\boxed{2}, \begin{array}{|c|}\hline 35 \\ \hline 14 \\ \hline\end{array}\right),$$
$$\left(\boxed{2}, \begin{array}{|c|}\hline 15 \\ \hline 34 \\ \hline\end{array}\right), \left(\boxed{3}, \begin{array}{|c|}\hline 45 \\ \hline 12 \\ \hline\end{array}\right), \left(\boxed{3}, \begin{array}{|c|}\hline 25 \\ \hline 14 \\ \hline\end{array}\right), \left(\boxed{3}, \begin{array}{|c|}\hline 15 \\ \hline 24 \\ \hline\end{array}\right), \left(\boxed{4}, \begin{array}{|c|}\hline 35 \\ \hline 12 \\ \hline\end{array}\right),$$
$$\left(\boxed{4}, \begin{array}{|c|}\hline 25 \\ \hline 13 \\ \hline\end{array}\right), \left(\boxed{4}, \begin{array}{|c|}\hline 15 \\ \hline 23 \\ \hline\end{array}\right), \left(\boxed{5}, \begin{array}{|c|}\hline 34 \\ \hline 12 \\ \hline\end{array}\right), \left(\boxed{5}, \begin{array}{|c|}\hline 24 \\ \hline 13 \\ \hline\end{array}\right), \left(\boxed{5}, \begin{array}{|c|}\hline 14 \\ \hline 23 \\ \hline\end{array}\right).$$

We now define an action of U_k on the vector space $\mathbb{C}\mathcal{T}_{\vec{\lambda}}$ consisting of formal linear combinations of the uniform tableaux in $\mathcal{T}_{\vec{\lambda}}$ with complex coefficients and then show that it is isomorphic to the irreducible representation $W^{\vec{\lambda}}_{\mathsf{U}_k}$.

Recall that U_k is generated by s_i and b_i, where $1 \leqslant i \leqslant k-1$, as described in Section 2.6. For $\mathbf{S} \in \mathcal{T}_{\vec{\lambda}}$, let

$$
b_i\mathbf{S} = \begin{cases} \mathbf{S}, & \text{if } i \text{ and } i+1 \text{ are in the same cell in } \mathbf{S}, \\ 0, & \text{otherwise,} \end{cases} \tag{6}
$$

and let $s_i\mathbf{S}$ be obtained from $\mathbf{S}$ by interchanging i and $i+1$. It is possible that $s_i\mathbf{S}$ is not standard, in which case we apply the Garnir straightening relations (we illustrate this in Example 3.26 and refer the reader to [24, Section 2.6] or [1] for details) to obtain a linear combination of elements in $\mathcal{T}_{\vec{\lambda}}$. It is straightforward to verify that the relations in Section 2.6 hold so that $\mathbb{C}\mathcal{T}_{\vec{\lambda}}$ is a representation of U_k.

For the next result, we remind the reader that the blocks of set partitions are ordered using the graded last letter order and that the elements $\ell^\gamma_\pi \otimes \mathbf{T}$, where $\mathsf{type}(\gamma) = \lambda$ and $\mathbf{T} \in \mathcal{B}_{\vec{\lambda}}(G_\lambda)$, form a basis of $W^{\vec{\lambda}}_{\mathsf{U}_k}$ (Proposition 3.18).

Theorem 3.24. *Let $\vec{\lambda} \in \mathcal{I}_k$ and write $\lambda = \overrightarrow{\mathsf{type}}(\vec{\lambda})$ and $\pi = \pi_\lambda$. For $\ell^\gamma_\pi \otimes \mathbf{T} \in \mathcal{B}_{\vec{\lambda}}(\mathsf{U}_k)$, let $\rho(\ell^\gamma_\pi \otimes \mathbf{T})$ be the sequence of tableaux obtained from $\mathbf{T}$ by replacing the block $\pi_i \in \pi$ with the block $\gamma_i \in \gamma$. Then ρ extends linearly to an isomorphism of representations $\rho\colon W^{\vec{\lambda}}_{\mathsf{U}_k} \to \mathbb{C}\mathcal{T}_{\vec{\lambda}}$.*

Proof. Let $\mathbf{S} = \rho(\ell^\gamma_\pi \otimes \mathbf{T})$. If we view ℓ^γ_π as a bijection from γ to π, then $\mathbf{S}$ is obtained by applying the inverse bijection to the entries in $\mathbf{T}$. In particular, ρ is invertible.

Notice that $\mathbf{S} \in \mathcal{T}_{\vec{\lambda}}$: first of all, $\mathbf{S}$ has the same shape $\vec{\lambda}$ as $\mathbf{T}$; its entries are the blocks of γ; each $S^{(i)}$ is filled with blocks of the same size i since ℓ^γ_π preserves the sizes of the blocks; finally, since the blocks are ordered using graded last letter order, each $S^{(i)}$ is standard since we have replaced the entries of $\mathbf{T}$ with blocks in the same order as those of π.

Now we show that the action of U_k commutes with ρ. It suffices to show this for the generators s_i and b_i. By definition,

$$
b_i \cdot (\ell^\gamma_\pi \otimes \mathbf{T}) = (b_i \odot \ell^\gamma_\pi) \otimes \mathbf{T} = \begin{cases} b_i\ell^\gamma_\pi \otimes \mathbf{T}, & \text{if } b_i\ell^\gamma_\pi \in L_\pi, \\ 0, & \text{otherwise.} \end{cases}
$$

Note that $b_i\ell^\gamma_\pi \in L_\pi$ if and only if $b_i\ell^\gamma_\pi = \ell^\gamma_\pi$, or equivalently, if and only if i and $i+1$ are in the same block of γ. Thus,

$$
b_i \cdot (\ell^\gamma_\pi \otimes \mathbf{T}) = \begin{cases} \ell^\gamma_\pi \otimes \mathbf{T}, & \text{if } i \text{ and } i+1 \text{ are in the same block of } \gamma, \\ 0, & \text{otherwise.} \end{cases}
$$

Comparing with Equation (6), it follows that $\rho(b_i \cdot (\ell^\gamma_\pi \otimes \mathbf{T})) = b_i \cdot \rho(\ell^\gamma_\pi \otimes \mathbf{T})$.

Next, we consider the action of s_i. Tracing through the definitions, we have

$$
s_i \cdot (\ell^\gamma_\pi \otimes \mathbf{T}) = (s_i \odot \ell^\gamma_\pi) \otimes \mathbf{T} = s_i\ell^\gamma_\pi \otimes \mathbf{T},
$$

where the last equality follows from the observation that $s_i\ell^\gamma_\pi \in L_\pi$ because $\mathsf{bot}(s_i\ell^\gamma_\pi) = \pi$.

To describe $s_i\ell^\gamma_\pi$ explicitly, write $\gamma = \{\gamma_1, \ldots, \gamma_\ell\}$ and $\pi = \{\pi_1, \ldots, \pi_\ell\}$ with the blocks order using graded last letter order, and recall that ℓ^γ_π is the bijection that maps γ_h to π_h. If exchanging i and $i+1$ in γ does not change the order of the blocks (i.e. $s_i(\gamma_1) < \cdots < s_i(\gamma_\ell)$ in graded last letter order), then $s_i\ell^\gamma_\pi = \ell^{s_i(\gamma)}_\pi$ so that $s_i \cdot (\ell^\gamma_\pi \otimes \mathbf{T}) = \ell^{s_i(\gamma)}_\pi \otimes \mathbf{T}$. Its image under ρ is obtained from $\mathbf{T}$ by replacing each block π_j appearing in $\mathbf{T}$ with $s_i(\gamma_j)$, which is precisely the definition of the action of s_i. Thus, $\rho(s_i \cdot (\ell^{\gamma}_{\pi} \quad \mathbf{T})) = s_i \cdot \rho(\ell^{\gamma}_{\pi} \quad \mathbf{T})$.

Otherwise, there exist blocks γ_j and γ_{j+1} with $\max(\gamma_j) = i$, $\max(\gamma_{j+1}) = i+1$, $|\gamma_j| = |\gamma_{j+1}|$ and

$$s_i(\gamma) = \big\{ s_i(\gamma_1), s_i(\gamma_2), \dots, s_i(\gamma_{j-1}), s_i(\gamma_{j+1}), s_i(\gamma_j), s_i(\gamma_{j+2}), \dots, s_i(\gamma_\ell) \big\}, \tag{7}$$

where the blocks are listed in graded last letter order. Then $s_i \ell^{\gamma}_{\pi} = \ell^{s_i(\gamma)}_{\pi} g$, where

$$g = \begin{pmatrix} \pi_1 & \cdots & \pi_j & \pi_{j+1} & \cdots & \pi_\ell \\ \pi_1 & \cdots & \pi_{j+1} & \pi_j & \cdots & \pi_\ell \end{pmatrix}$$

is the permutation in G_λ that exchanges π_j and π_{j+1}. Therefore, the image of $s_i \cdot (\ell^{\gamma}_{\pi} \quad \mathbf{T}) = \ell^{s_i(\gamma)}_{\pi} \quad g \cdot \mathbf{T}$ under ρ is obtained from $\mathbf{T}$ by exchanging π_j and π_{j+1} and then each π_h is replaced with the *block in position* h of $s_i(\gamma)$ (as listed in Equation (7)). Thus, $\rho(s_i \cdot (\ell^{\gamma}_{\pi} \quad \mathbf{T}))$ is again obtained from $\rho(\ell^{\gamma}_{\pi} \quad \mathbf{T})$ by interchanging i and $i+1$.

Example 3.25. Under the bijection ρ described in Theorem 3.24 the basis elements of $W^{((1),(1))}_{U_3}$ correspond to the tableaux in $\mathcal{T}_{\vec{\lambda}}$ as follows:

$$\begin{aligned}
\ell^{1|23}_{1|23} \quad \left(\boxed{1}, \boxed{23}\right) &\to \left(\boxed{1}, \boxed{23}\right), \\
\ell^{2|13}_{1|23} \quad \left(\boxed{1}, \boxed{23}\right) &\to \left(\boxed{2}, \boxed{13}\right), \\
\ell^{3|12}_{1|23} \quad \left(\boxed{1}, \boxed{23}\right) &\to \left(\boxed{3}, \boxed{12}\right).
\end{aligned}$$

Example 3.26. Let $\vec{\lambda} = ((2,1),(2,2),(1,1)) \quad I_{17}$ so that $\lambda = \overrightarrow{\mathsf{type}}(\vec{\lambda}) = (1^3\,2^4\,3^2)$. As in Example 3.21, we represent 10 through 17 by the letters a through h. Consider

$$\mathbf{S} = \left(\begin{array}{|c|c} \hline g & \\ \hline 2 & \multicolumn{1}{c|}{7} \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 5b & 9e \\ \hline 13 & 6d \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 8fh \\ \hline 4ac \\ \hline \end{array} \right)$$

which is the image under ρ of the basis element in Example 3.21. Consider the action of

$$d = 2\,\overline{8} \mid 8\,\overline{2} \mid 9\,\overline{g} \mid a\,\overline{d} \mid b\,\overline{7} \mid c\,\overline{6} \mid e\,\overline{a} \mid f\,\overline{3} \mid h\,\overline{1} \mid 1\,4\,\overline{5}\,\overline{b} \mid 6\,7\,\overline{9}\,\overline{e} \mid 3\,d\,\overline{4}\,\overline{c} \mid 5\,g\,\overline{f}\,\overline{h}$$

on the uniform tableau $\mathbf{S}$. Since $\mathsf{bot}(d)$ is finer than the set partition of the entries of $\mathbf{S}$, the result is non-zero and is equal to

$$\left(\begin{array}{|c|c} \hline 9 & \\ \hline 8 & \multicolumn{1}{c|}{b} \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 14 & 67 \\ \hline fh & ac \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 25g \\ \hline 3de \\ \hline \end{array} \right).$$

This is not a basis element because the middle tableau is not standard with respect to the graded last letter order. We then apply some straightening relations to express it as a linear combination of the basis elements. The interested reader may then compute that the action of d on $\mathbf{S}$ is equal to the following linear combination:

$$\left(\begin{array}{|c|c} \hline 9 & \\ \hline 8 & \multicolumn{1}{c|}{b} \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline ac & fh \\ \hline 14 & 67 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 25g \\ \hline 3de \\ \hline \end{array} \right) - \left(\begin{array}{|c|c} \hline 9 & \\ \hline 8 & \multicolumn{1}{c|}{b} \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 67 & fh \\ \hline 14 & ac \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 25g \\ \hline 3de \\ \hline \end{array} \right).$$

4. The characters of U_k

The last two sections of this paper are devoted to a careful analysis of the characters of U_k. This development will allow us to give an expression of the character values in terms of symmetric functions in Section 5 and make explicit the connection between plethysm and the restriction of U_k-modules to the symmetric group $\mathfrak{S}_k \quad U_k$.

In this section, we describe the characters for the irreducible U_k-representations that were presented in the previous section. In general, the characters of finite monoids were studied by McAlister [19]. Here we use the notation described in [27, Chapter 7].

4.1. Generalized conjugacy classes. Let M be a finite monoid. For every $m \in M$, the subsemigroup of M generated by m contains a unique idempotent that we denote m^ω (see [27, Corollary 1.2]). One can think of ω as representing the smallest positive integer such that m^ω is an idempotent. Two elements m and n in M are *conjugate* if there exist $x, x' \in M$ such that $xx'x = x$, $x'xx' = x'$, $x'x = m^\omega$, $xx' = n^\omega$ and $xm^{\omega+1}x' = n^{\omega+1}$. This is an equivalence relation whose equivalence classes are called the *generalized conjugacy classes* of M. Notice that m and $m^{\omega+1}$ are conjugate for all $m \in M$ [27, Chapter 7].

By [27, Proposition 7.4], there is a bijection between the generalized conjugacy classes of M and the union of the sets of conjugacy classes of the maximal subgroups $G_{e_1}, \ldots, G_{e_s}$, where $e_1, \ldots, e_s$ are idempotents chosen one from each $\mathcal{J}$-class of M that contains an idempotent. The bijection is obtained by intersecting a generalized conjugacy class of M with the conjugacy classes of G_{e_i}: exactly one of these intersections is nonempty. In particular, to select a set of representatives of the generalized conjugacy classes of M, it suffices to take one element from each of the conjugacy classes of the maximal subgroups $G_{e_1}, \ldots, G_{e_s}$.

We now apply the above to U_k. Since U_k is an inverse monoid, every $x \in \mathsf{U}_k$ has a (unique) generalized inverse $\widetilde{x}$ satisfying $x\widetilde{x}x = x$ and $\widetilde{x}x\widetilde{x} = \widetilde{x}$ (cf. Section 2.7). Therefore, two elements c and d are conjugate in U_k if and only if there exists $x \in \mathsf{U}_k$ such that $\widetilde{x}x = c^\omega$, $x\widetilde{x} = d^\omega$ and $xc^{\omega+1}\widetilde{x} = d^{\omega+1}$.

We next define a notion of "cycle type" for the elements of U_k, which will allow us to determine whether two elements are conjugate in U_k. First, let d be an element of a maximal subgroup G_{e_π}. Then d is a permutation of the blocks of π that maps blocks of size i to blocks of size i for every $1 \leqslant i \leqslant k$. Letting $d^{(i)}$ denote the restriction of d to the blocks of size i of π, we define the *cycle type* of d to be $\mathsf{cycletype}(d) = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)})$, where $\mu^{(i)}$ is the cycle type of the permutation $d^{(i)}$. For an arbitrary element $x \in \mathsf{U}_k$, we define its *cycle type* to be the cycle type of $x^{\omega+1} \in G_{x^\omega}$. In other words, $\mathsf{cycletype}(x) = \mathsf{cycletype}(x^{\omega+1})$.

Proposition 4.1. *Two elements $c, d \in \mathsf{U}_k$ are conjugate if and only if*

$$\mathsf{cycletype}(c) = \mathsf{cycletype}(d).$$

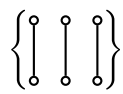
Proof. Suppose $\mathsf{cycletype}(c) = \mathsf{cycletype}(d) = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)})$. Hence by definition $\mathsf{cycletype}(c^{\omega+1}) = \mathsf{cycletype}(d^{\omega+1})$. Then $c^\omega = e_\pi$ and $d^\omega = e_\gamma$ for some set partitions π and γ. Moreover, by the definition of cycle type, π and γ must have type $(1^{a_1}2^{a_2}\ldots k^{a_k})$, where $a_i = |\mu^{(i)}|$ for all $1 \leqslant i \leqslant k$. By Corollary 3.4, there exists a permutation $\sigma \in \mathfrak{S}_k$ such that $\widetilde{\sigma}G_{e_\pi}\sigma = G_{e_\gamma}$; note that $\widetilde{\sigma} = \sigma^{-1}$ for permutations. Thus, $\widetilde{\sigma}c^{\omega+1}\sigma$ and $d^{\omega+1}$ both belong to G_{e_γ} and they both have the same cycle type. Hence, they are conjugate in G_{e_γ}, which implies they are conjugate in U_k: explicitly, there exists $y \in G_{e_\gamma}$ such that $\widetilde{y}(\widetilde{\sigma}c^{\omega+1}\sigma)y = d^{\omega+1}$, and so the element $x = \widetilde{\sigma y}$ satisfies $x\widetilde{x} = \widetilde{y}y = e_\gamma = d^\omega$, $\widetilde{x}x = \sigma e_\gamma \widetilde{\sigma} = e_\pi = c^\omega$ and $xc^{\omega+1}\widetilde{x} = d^{\omega+1}$.

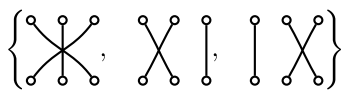
Conversely, suppose c and d are conjugate in U_k. Then there exists $x \in \mathsf{U}_k$ such that $\widetilde{x}x = c^\omega$, $x\widetilde{x} = d^\omega$, and $xc^{\omega+1}\widetilde{x} = d^{\omega+1}$. Let $\pi = \mathsf{bot}(x)$ and $\gamma = \mathsf{top}(x)$ so that $c^\omega = e_\pi$ and $d^\omega = e_\gamma$. Then $\mathsf{type}(\pi) = \mathsf{type}(\gamma)$ because x is a bijection from π to γ that preserves block sizes. By Corollary 3.4, there exists a permutation $\sigma \in \mathfrak{S}_k$ such that $\widetilde{\sigma}G_{e_\pi}\sigma = G_{e_\gamma}$. Thus, $\widetilde{\sigma}c^{\omega+1}\sigma$ and $d^{\omega+1}$ both belong to G_{e_γ} and they are conjugate in U_k. It follows that they are conjugate in G_{e_γ} and so they have the same cycle type since G_{e_γ} is a group of permutations. Hence, $\mathsf{cycletype}(c^{\omega+1}) = \mathsf{cycletype}(\widetilde{\sigma}c^{\omega+1}\sigma) = \mathsf{cycletype}(d^{\omega+1})$.

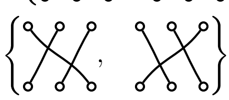
The above gives a straightforward algorithm for computing the cycle type of any $d \quad \mathsf{U}_k$. To find the cycle type of d, we compute $d^{\omega+1}$ and then find the cycle type of $d^{\omega+1} \quad G_{d^\omega}$. For $\vec{\mu} \quad I_k$, define

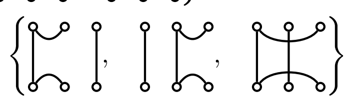
$$C_{\vec{\mu}} = \{x \quad \mathsf{U}_k : \mathsf{cycletype}(x) = \vec{\mu}\} .$$

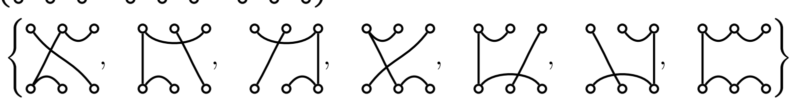
Example 4.2. The generalized conjugacy classes in U_3 are listed below:

$C_{((1^3))} =$ 

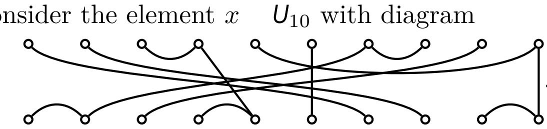
$C_{((2,1))} =$ 

$C_{((3))} =$ 

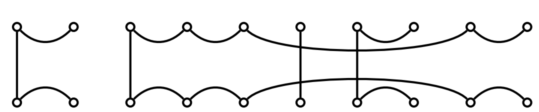
$C_{((1),(1))} =$ 

$C_{(?,?,(1))} =$ 

Example 4.3. Consider the element $x \quad \mathsf{U}_{10}$ with diagram



We can then check that 4 is the smallest integer such that x^4 is idempotent and $x^4 = e_\pi$, where $\pi = \{\{6\},\{1,2\},\{7,8\},\{3,4,5,9,10\}\}$. The element x^4 has the following diagram:



We deduce that $\overrightarrow{\mathsf{type}}(\mathsf{cycletype}(x)) = \mathsf{type}(\pi) = (5,2,2,1)$. Since $x^5 = xe_\pi = e_\pi x$ is

$$\left\{\{6,\overline{6}\},\{1,2,\overline{7},\overline{8}\},\{7,8,\overline{1},\overline{2}\},\{2,3,4,9,10,\overline{2},\overline{3},\overline{4},\overline{9},\overline{10}\}\right\} ,$$

we conclude that $\mathsf{cycletype}(x) = \mathsf{cycletype}(x^5) = ((1),(2),?,?,(1))$ because x acts on e_π by permuting the two sets of size 2 (in a cycle) and fixing the sets of size 1 and 5.

4.2. Representative conjugacy class element. In this section, we describe for each $\vec{\mu} \quad I_k$ a representative conjugacy class element, denoted $d_{\vec{\mu}}$, contained in the generalized conjugacy class $C_{\vec{\mu}}$ consisting of all elements of cycle type $\vec{\mu}$. As shown in the previous section, we can choose $d_{\vec{\mu}}$ to be an element of a representative maximal subgroup of U_k.

For any set $A = \{x_1,\dots,x_{|A|}\}$ and $\mu = (\mu_1,\dots,\mu_\ell) \quad |A|$, we define the *representative element of cycle type* μ in $\mathfrak{S}_A$ to be

$$d_\mu^A = (x_1x_2\cdots x_{\mu_1})(x_{\mu_1+1}x_{\mu_1+2}\cdots x_{\mu_1+\mu_2})\cdots(x_{|A|-\mu_\ell+1}x_{|A|-\mu_\ell+2}\cdots x_{|A|}),$$

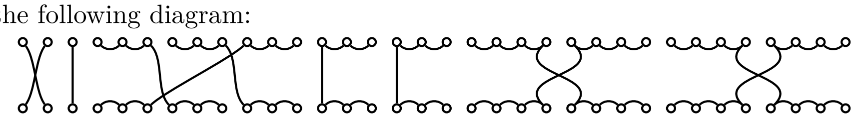
where d_μ^A is expressed in cycle notation. If A is clear from the context, we write $d_\mu = d_\mu^A$.

Let $\vec{\mu} = (\mu^{(1)}, \ldots, \mu^{(k)}) \in I_k$ and $\mu = \overrightarrow{\mathsf{type}}(\vec{\mu})$. Recall that $G_\mu = \mathfrak{S}_{\pi^{(1)}} \times \cdots \times \mathfrak{S}_{\pi^{(k)}}$, where $\pi^{(i)}$ is the set of blocks of size i in π_μ. Since $d^{\pi^{(i)}}_{\mu^{(i)}} \in \mathfrak{S}_{\pi^{(i)}}$ for all $1 \leq i \leq k$, we define

$$d_{\vec{\mu}} = d^{\pi^{(1)}}_{\mu^{(1)}} d^{\pi^{(2)}}_{\mu^{(2)}} \cdots d^{\pi^{(k)}}_{\mu^{(k)}} \in G_\mu.$$

Proposition 4.4. *Let $\vec{\mu} \in I_k$. Then* $\mathsf{cycletype}(d_{\vec{\mu}}) = \vec{\mu}$.

Example 4.5. Let $k = 34$ and $\vec{\mu} = ((2,1), ?, (3,1,1), (2,2))$. Then $d_{\vec{\mu}}$ is represented by the following diagram:



.

4.3. Characters of U_k. We give a formula for the characters of the irreducible representations of U_k in terms of the characters of the irreducible representations of the representative maximal subgroups G_λ.

For $\vec{\lambda} \in I_k$, let $W^{\vec{\lambda}}_{\mathsf{U}_k}$ denote the irreducible representation of U_k defined in Section 3.5. Its *character* is the function $\chi^{\vec{\lambda}}_{\mathsf{U}_k} : \mathsf{U}_k \to \mathsf{C}$ defined by

$$\chi^{\vec{\lambda}}_{\mathsf{U}_k}(d) = \mathrm{trace}\left(\rho_{\mathsf{B}}(d)\right),$$

where B is any basis of $W^{\vec{\lambda}}_{\mathsf{U}_k}$ and $\rho_{\mathsf{B}}(d)$ denotes the matrix representing the action of d with respect to B. Since trace is unchanged under change of basis, this definition does not depend on the chosen basis.

As in the case of group characters, monoid characters are constant on generalized conjugacy classes [27, Proposition 7.9]. Therefore, we need only determine the value of $\chi^{\vec{\lambda}}_{\mathsf{U}_k}$ on the representative conjugacy class elements $d_{\vec{\mu}}$ defined in Section 4.2. We begin by expressing this in terms of the characters $\chi^{\vec{\lambda}}_{G_\lambda}$ of the irreducible representations $V^{\vec{\lambda}}_{G_\lambda}$. Recall the refinement order on set partitions defined in Section 2.2.

Proposition 4.6. *Let $\vec{\lambda} \in I_k$ and $\lambda = \overrightarrow{\mathsf{type}}(\vec{\lambda})$. If $d_{\vec{\mu}} \in \mathsf{U}_k$ is an element of the generalized conjugacy class $C_{\vec{\mu}} \subseteq \mathsf{U}_k$ indexed by $\vec{\mu} \in I_k$, then*

$$\chi^{\vec{\lambda}}_{\mathsf{U}_k}(d_{\vec{\mu}}) = \sum_{d \in C(d_{\vec{\mu}};\lambda)} \chi^{\vec{\lambda}}_{G_\lambda}(\sigma_d),$$

where

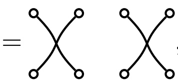
- $C(d_{\vec{\mu}};\lambda) = \{d : d \geq d_{\vec{\mu}},\ \mathsf{top}(d) = \mathsf{bot}(d),\ \mathsf{type}(\mathsf{top}(d)) = \lambda\}$*, and*
- σ_d *is the unique element in G_λ satisfying $d\ell^{\mathsf{top}(d)}_{\pi_\lambda} = \ell^{\mathsf{top}(d)}_{\pi_\lambda}\sigma_d$.*

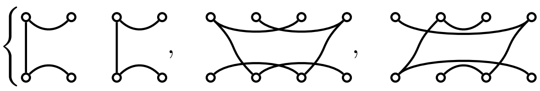
Proof. We compute the trace of $d_{\vec{\mu}}$ acting on the basis $\{\ell^\gamma_{\pi_\lambda} \otimes \mathbf{T} : \mathsf{type}(\gamma) = \lambda, \mathbf{T} \in \mathsf{B}_{\vec{\lambda}}(G_\lambda)\}$ of Proposition 3.18. Recall that the action of $d_{\vec{\mu}}$ on $\ell^\gamma_{\pi_\lambda} \otimes \mathbf{T}$ is given by

$$d_{\vec{\mu}} \cdot (\ell^\gamma_{\pi_\lambda} \otimes \mathbf{T}) = (d_{\vec{\mu}} \ell^\gamma_{\pi_\lambda}) \otimes \mathbf{T} = \begin{cases} d_{\vec{\mu}}\ell^\gamma_{\pi_\lambda} \otimes \mathbf{T}, & \text{if } \mathsf{bot}(d_{\vec{\mu}}\ell^\gamma_{\pi_\lambda}) = \pi_\lambda, \\ 0, & \text{otherwise.} \end{cases}$$

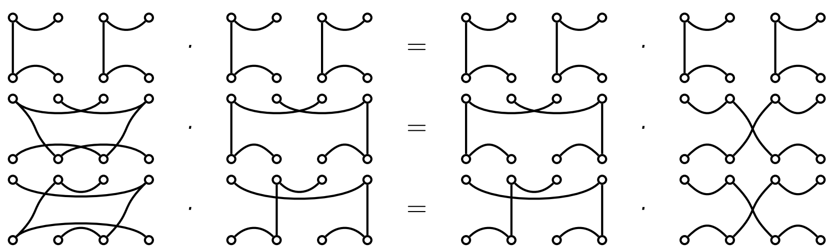
Writing $\mu = \overrightarrow{\mathsf{type}}(\vec{\mu})$ and noting that $d_{\vec{\mu}}$ is a permutation of the blocks of π_μ (since $d_{\vec{\mu}} \in G_\mu$), it follows that $\mathsf{bot}(d_{\vec{\mu}}\ell^\gamma_{\pi_\lambda}) = \pi_\lambda$ if and only if $\pi_\mu = \mathsf{bot}(d_{\vec{\mu}})$ is finer than γ. Thus, if π_μ is not finer than γ, then there is no contribution to the trace.

Suppose that π_μ is finer than γ. Then we can merge blocks of $d_{\vec{\mu}}$ to obtain a diagram d such that $\mathsf{bot}(d) = \gamma$. Notice that the diagram d satisfies $d_{\vec{\mu}}\ell^\gamma_{\pi_\lambda} = d\ell^\gamma_{\pi_\lambda}$. By Proposition 3.15, there is a unique $\sigma_d \in G_\lambda$ satisfying $d\ell^\gamma_{\pi_\lambda} = \ell^{\mathsf{top}(d)}_{\pi_\lambda}\sigma_d$. Therefore, $d_{\vec{\mu}} \cdot (\ell^\gamma_{\pi_\lambda} \otimes \mathbf{T}) = \ell^{\mathsf{top}(d)}_{\pi_\lambda} \otimes \sigma_d\mathbf{T}$, which contributes to the computation of the trace only when $\mathsf{top}(d) = \gamma$. In this case, $d_{\vec{\mu}}$ maps $\ell^\gamma_{\pi_\lambda} \otimes \mathbf{T}$ to $\ell^\gamma_{\pi_\lambda} \otimes \sigma_d\mathbf{T}$ and so the trace is equal to the trace of the mapping $\mathbf{T} \to \sigma_d \mathbf{T}$, which is precisely $\chi^{\vec{\lambda}}_{G_\lambda}(\sigma_d)$ because $\mathsf{B}_{\vec{\lambda}}(G_\lambda)$ is a basis of $V^{\vec{\lambda}}_{G_\lambda}$.

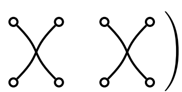
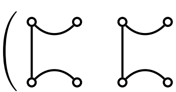
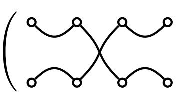
Example 4.7. Let $\vec{\lambda} = (?, (1,1))$ and $\vec{\mu} = ((2,2))$. Then $\lambda = (2,2)$, $d_{\vec{\mu}} =$ , and

$$C(d_{\vec{\mu}};\lambda) = \Big\{$$  $$\Big\}.$$

For each $d \quad C(d_{\vec{\mu}};\lambda)$, there is a unique $\sigma_d \quad G_\lambda$ satisfying $d\ell^{\mathsf{top}(d)}_{\pi_\lambda} = \ell^{\mathsf{top}(d)}_{\pi_\lambda}\sigma_d$, which we illustrate below.



Therefore,

$$\chi^{\vec{\lambda}}_{\mathsf{U}_4}\Bigg($$  $$\Bigg) = \chi^{\vec{\lambda}}_{G_\lambda}\Bigg($$  $$\Bigg) + 2\chi^{\vec{\lambda}}_{G_\lambda}\Bigg($$  $$\Bigg) = -1.$$

4.4. Reformulation of the characters of U_k. As observed in Example 4.7, there can be elements in $C(d_{\vec{\mu}};\lambda)$ that have the same cycle type in G_λ. For $\vec{\nu}$ with $\overrightarrow{\mathsf{type}}(\vec{\nu}) = \lambda$, let

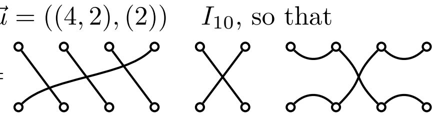
$$C(d_{\vec{\mu}};\vec{\nu}) = \{d \quad C(d_{\vec{\mu}};\lambda) \,:\, \mathsf{cycletype}(d) = \vec{\nu}\}$$

and define $b^{\vec{\nu}}_{\vec{\mu}} = |C(d_{\vec{\mu}};\vec{\nu})|$. Then rewriting the formula in Proposition 4.6, we have

$$\chi^{\vec{\lambda}}_{\mathsf{U}_k}(d_{\vec{\mu}}) = \sum_{\substack{\vec{\nu} \quad I_k \\ \overrightarrow{\mathsf{type}}(\vec{\nu}) = \lambda}} b^{\vec{\nu}}_{\vec{\mu}}\, \chi^{\vec{\lambda}}_{G_\lambda}(d_{\vec{\nu}}). \tag{8}$$

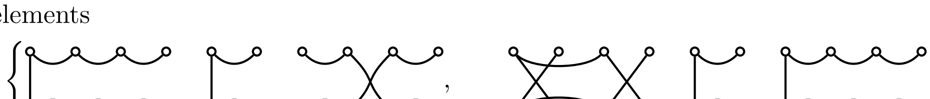
Our next goal is to find an explicit formula for the multiplicities $b^{\vec{\nu}}_{\vec{\mu}}$.

Example 4.8. Consider $\vec{\mu} = ((4,2),(2)) \quad I_{10}$, so that

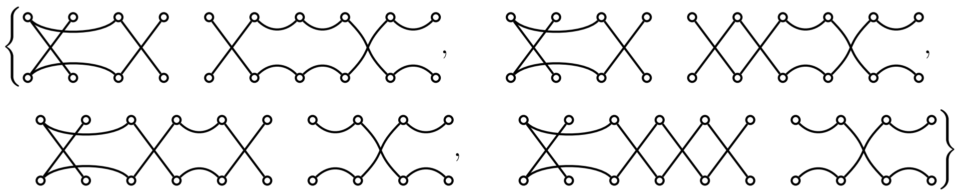
$$d_{\vec{\mu}} =$$  .

We compute $C(d_{\vec{\mu}};\vec{\nu})$ for three different choices of $\vec{\nu}$. These examples will then be used later to demonstrate how the algebraic formulas capture the enumeration of these sets.

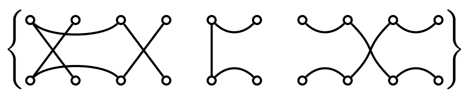
First consider $\vec{\nu} = (?, (2,1), ?, (1)) \quad I_{10}$. Starting with the above diagram for $d_{\vec{\mu}}$, we see that there are two ways of creating a block of size 4, either by adding edges to the first cycle or the last cycle. Therefore, $C(d_{\vec{\mu}};\vec{\nu})$ consists of the following two elements



Next consider $\vec{\nu} = (?, (2), (2)) \quad I_{10}$. To make a cycle of length 2 with three vertices, the cycle of length two in the middle of the diagram can either connect to the first or the last cycle and it can connect in two ways. Therefore, there are 4 elements in the set $C(d_{\vec{\mu}};\vec{\nu})$



Finally, consider $\vec{\nu} = (\emptyset, (2,2,1))$. There is precisely one way to add edges to $d_{\vec{\mu}}$ to obtain an element of cycle type $\vec{\nu}$ and so $C(d_{\vec{\mu}};\vec{\nu})$ is the following singleton set



Let $A_1, \ldots, A_r$ be the blocks of a set partition of $[k]$ so that $|A_i| = |A_j|$ for all $1 \leqslant i, j \leqslant r$. Suppose that d consists of the blocks $\{A_i, \overline{A_{i+1}}\}$ for $1 \leqslant i \leqslant r-1$ and $\{A_r, \overline{A_1}\}$. If $|A_i| = q$ and $A_i = \{(i-1)q+1, (i-1)q+2, \ldots, iq\}$ for all i, then we refer to this particular cycle as the *canonical r-cycle*. Notice that d is an r-cycle in the sense that if we ignore bars and present the permutation as a two line array, then d has the form

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ A_2 & A_3 & \cdots & A_1 \end{pmatrix}.$$

In general, a permutation of the blocks $A_1, A_2, \ldots, A_r$ is called an *r-cycle* if there exists a $\sigma \in \mathfrak{S}_r$ such that $A_i \to A_{\sigma(i)}$ and σ is an r-cycle in $\mathfrak{S}_r$.

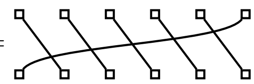
Lemma 4.9. *Let $d \in \mathsf{U}_k$ be the canonical r-cycle permuting blocks $A_1, \ldots, A_r$ with $|A_i| = \frac{k}{r}$ for all i. If we take unions of blocks of d to get a new diagram d' satisfying $\mathsf{top}(d') = \mathsf{bot}(d')$, then d' is an s-cycle where s divides r and the blocks of $\mathsf{top}(d')$ are all of size $\frac{k}{s}$.*

Proof. Suppose d' is a uniform block permutation obtained by taking the union of blocks of d and $\mathsf{top}(d') = \mathsf{bot}(d') = \{B_1, \ldots, B_s\}$, where $s \leqslant r$. Without loss of generality, suppose that B_1 is the block containing A_1 and B_1 is the union of r_1 blocks. Then $B_1 = A_1 \cup A_{i_2} \cup \cdots \cup A_{i_{r_1}}$ with $1 < i_2 < \cdots < i_{r_1}$. If two indices $1, i_2, \ldots, i_r$ are adjacent, then by the assumption that d is the canonical cycle and $\mathsf{top}(d') = \mathsf{bot}(d')$, B_1 would be the union of all A_i. In this case d' is a cycle of length $s = 1$. Otherwise, since d is the canonical r-cycle, we have that $A_2 \cup A_{i_2+1} \cup \cdots \cup A_{i_{r_1}+1}$ is the image block of B_1 in $\mathsf{bot}(d')$ which is the union of r_1 blocks, we call this block B_2. By a similar argument, there are blocks $B_3, B_4, \ldots, B_{i_2-1}$ all of size r_1 containing $A_3, A_4, \ldots, A_{i_2-1}$, respectively. Notice that B_{i_2-1} contains A_r since d maps A_{i_2-1} to A_{i_2} and A_r to A_1, so that in d' A_{i_2} and A_1 are in the same block, i.e. B_1, and this block connects to the block that contains A_{i_2-1} and A_r. Similar arguments show that A_{r-1} is contained in B_{i_2-2}, etc. So, $s = i_2 - 1$ is the total number of blocks in $\mathsf{top}(d')$. In addition, the blocks $B_1, B_2, \ldots, B_{i_2-1}$ contain blocks $A_1, A_2, \ldots, A_{(i_2-1)r_1}$ and they are all of the same size. This means that $r = (i_2-1)r_1$, hence $|B_i| = r_1|A_i| = \frac{r|A_i|}{s} = \frac{k}{s}$. By the description above we see that $d' = \begin{pmatrix} B_1 & B_2 & \ldots & B_s \\ B_2 & B_3 & \ldots & B_1 \end{pmatrix}$ is an s-cycle. ∎

There are two takeaways from the proof of Lemma 4.9. The first is that taking a union of blocks of the canonical r-cycle results in an s-cycle. The second take away is that if we label the blocks of d' as in the proof of Lemma 4.9, then $B_i = \bigcup_j A_j$ where j is congruent to i mod s. In addition, the proof implies that the union of blocks of the canonical r-cycle results in an s-cycle, where s divides r and there is only one possible way to get an s-cycle.

Example 4.10. In this example we use squares for the vertices to indicate that the vertices represent sets $A_1, A_2, \ldots, A_6$ all of the same size and containing consecutive integers. Let

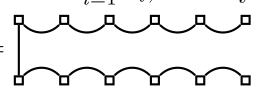
$$d = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ A_2 & A_3 & A_4 & A_5 & A_6 & A_1 \end{pmatrix} =$$



.

Then there are three possible ways to take unions of the blocks of d to get diagrams d with $\mathsf{top}(d\ ) = \mathsf{bot}(d\ )$. We can take the union of all the blocks to get one single block on top and on the bottom, i.e., $B = \ {}_{i=1}^{6} A_i$, which yields

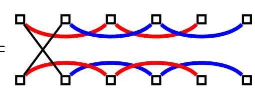
$$d\ = \begin{pmatrix} B \\ B \end{pmatrix} =$$



.

Alternatively, we can get a two cycle by taking the following unions $B_1 = A_1 \quad A_3 \quad A_5$ and $B_2 = A_2 \quad A_4 \quad A_6$, which gives

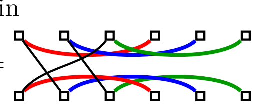
$$d\ = \begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix} =$$



.

Finally, we can get a three cycle by taking the following unions $B_1 = A_1 \quad A_4$ and $B_2 = A_2 \quad A_5$ and $B_3 = A_3 \quad A_6$ to obtain

$$d\ = \begin{pmatrix} B_1 & B_2 & B_3 \\ B_2 & B_3 & B_1 \end{pmatrix} =$$



.

Lemma 4.11. *Let r, t be positive integers. Let $d \quad \mathsf{U}_k$ consist of t r-cycles so that the r blocks in each cycle are labeled $B_1^{(i)}, \ldots, B_r^{(i)}$ for 1 6 i 6 t and the r-cycles have the form:*

$$\begin{pmatrix} B_1^{(i)} & B_2^{(i)} & \cdots & B_r^{(i)} \\ B_2^{(i)} & B_3^{(i)} & \cdots & B_1^{(i)} \end{pmatrix},$$

as obtained in the proof of Lemma 4.9. There are r^{t-1} ways to take unions of the blocks of d to get a diagram d that is an r-cycle and $\mathsf{top}(d\ ) = \mathsf{bot}(d\ )$. Moreover, the blocks of d consist of disjoint unions of the blocks of d, where there is exactly one block from each r-cycle in d and hence all blocks of d have the same size.

Proof. The proof is by induction on t. The lemma is trivially true when $t = 1$. Suppose that d consists of two r-cycles, the first permutes blocks $B_1^{(1)}, \ldots, B_r^{(1)}$ and the other permutes blocks $B_1^{(2)}, \ldots, B_r^{(2)}$. Notice that there are exactly r ways to form an r-cycle if the union consists of one block in the first cycle with one block in the second cycle. Since $B_1^{(1)} \quad B_j^{(2)}$, for any 1 6 j 6 r, can be a block in the union, by the structure of d and the requirement that $\mathsf{top}(d\ ) = \mathsf{bot}(d\ )$ we obtain that $B_2^{(1)} \quad B_{j+1}^{(2)}$ is a block and in general $B_i^{(1)} \quad B_{j+i-1}^{(2)}$ are the blocks permuted by d , where $j+i-1$ is taken mod r. This is an r-cycle because there are r blocks and we have that d maps $B_i^{(1)} \quad B_{j+i-1}^{(2)}$ to the block $B_{i+1}^{(1)} \quad B_{j+i}^{(2)}$ and, if $i = r$, then it maps to $B_1^{(1)} \quad B_j^{(2)}$.

To see that these are the only ways to obtain an r cycle, we argue by contradiction. If the unions contain more blocks from each r-cycle, d has fewer than r blocks; and if the union contains more blocks from only one cycle, then the cycles are not all of the same size. In either case we cannot get an r-cycle of blocks all of the same size.

Now if we have $t > 2$ r-cycles, we know by induction that there are r^{t-2} ways in which the first $t-1$ form an r-cycle. Now for each way of forming this union there are r ways to take the union with the last r-cycle.

In our next Proposition we want to count the number of elements in $C(d_{\vec\mu};\vec\nu)$ for any $\vec\mu,\vec\nu \vdash I_k$. Here, the parts of both $\vec\mu$ and $\vec\nu$ represent the sizes of cycles. We think of each component in $\vec\mu=(\mu^{(1)},\dots,\mu^{(k)})$ in exponential notation with $m_i(\mu^{(a)})$ the number of cycles of length i in $d_{\vec\mu}$ permuting blocks of size a. For any partition μ of k, let $\mathfrak{m}(\mu)=(m_1(\mu),m_2(\mu),\dots,m_k(\mu))$, where $m_i(\mu)$ is the multiplicity of i in μ. For $\vec\mu \vdash I_k$, define $\mathfrak{m}(\vec\mu)\coloneqq(\mathfrak{m}(\mu^{(1)}),\mathfrak{m}(\mu^{(2)}),\dots,\mathfrak{m}(\mu^{(k)}))$, where we think of $\mathfrak{m}(\mu^{(i)})$ as a sequence of length k by adding trailing zeros if necessary. Given two vectors $\vec v=(v_1,\dots,v_k)$ and $\vec u=(u_1,\dots,u_k)$ we define $\binom{\vec v}{\vec u}=\binom{v_1}{u_1}\binom{v_2}{u_2}\dots\binom{v_k}{u_k}$. Given any vector partition $\vec\tau=(\tau^{(1)},\dots,\tau^{(k)})$ we define

$$\binom{\mathfrak{m}(\vec\mu)}{\mathfrak{m}(\vec\tau)}\coloneqq\binom{\mathfrak{m}(\mu^{(1)})}{\mathfrak{m}(\tau^{(1)})}\binom{\mathfrak{m}(\mu^{(2)})}{\mathfrak{m}(\tau^{(2)})}\cdots\binom{\mathfrak{m}(\mu^{(k)})}{\mathfrak{m}(\tau^{(k)})}.$$

In a similar way, the multinomial coefficient generalizes to any vectors. If $\vec u_1,\vec u_2,\dots,\vec u_\ell$ are vectors such that $\vec u_1+\vec u_2+\cdots+\vec u_\ell=\vec v$, then

$$\binom{\vec v}{\vec u_1,\vec u_2,\dots,\vec u_\ell}=\binom{\vec v}{\vec u_1}\binom{\vec v-\vec u_1}{\vec u_2}\binom{\vec v-\vec u_1-\vec u_2}{\vec u_3}\cdots\binom{\vec u_\ell}{\vec u_\ell}.$$

For any vector partition $\vec\mu \vdash I_k$, let $\ell(\vec\mu)=\sum_{a=1}^k\ell(\mu^{(a)})$ be the sum of the length of the partitions in $\vec\mu$ and $\mathfrak{m}(\vec\mu)!\coloneqq\prod_{i=1}^k m_1(\mu^{(i)})!m_2(\mu^{(i)})!\cdots m_k(\mu^{(i)})!$ be the product of the factorial of the multiplicities of the parts of $\vec\mu$. Furthermore, for a partition μ, we define $\mathfrak{m}(\mu)!=m_1(\mu)!m_2(\mu)!\cdots m_k(\mu)!$.

Proposition 4.12. *For $\vec\mu,\vec\nu \vdash I_k$,*

(9)
$$b_{\vec\mu}^{\vec\nu}=\frac{1}{\mathfrak{m}(\vec\nu)!}\sum_{\vec\tau(\bullet,\bullet)}\binom{\mathfrak{m}(\vec\mu)}{\mathfrak{m}(\vec\tau(1,1)),\mathfrak{m}(\vec\tau(2,1)),\dots,\mathfrak{m}(\vec\tau(k,k))}\prod_{\substack{1\leq j\leq k\\ 1\leq i\leq \ell(\nu^{(j)})}}(\nu_i^{(j)})^{\ell(\vec\tau(i,j))-1},$$

where the sum is over all sequences $\vec\tau(\bullet,\bullet)$ of $\vec\tau(i,j)$'s such that for each $\nu_i^{(j)}\neq 0$, $\vec\tau(i,j) \vdash I_j$ and $\vec\mu=\biguplus_{i,j}\nu_i^{(j)}\vec\tau(i,j)$.

Proof. Recall that $b_{\vec\mu}^{\vec\nu}$ is the number of ways to take the union of the cycles in $d_{\vec\mu}$ so that the result d' satisfies $\mathsf{top}(d')=\mathsf{bot}(d')$ and $\mathsf{cycletype}(d')=\vec\nu$.

By Lemma 4.9 and Lemma 4.11 we know that every r cycle in $d_{\vec\mu}$ can contribute to exactly one cycle of length s, where s divides r. Fix $1\leq j\leq k$ and $1\leq i\leq\ell(\nu^{(j)})$, and let $C_{i,j}$ be a cycle of length $\nu_i^{(j)}$ in d', an element in $C(d_{\vec\mu};\vec\nu)$. The cycle $C_{i,j}$ is obtained as the union of cycles in $d_{\vec\mu}$. We record which cycles we use to form $C_{i,j}$ in a vector partition $\vec\tau(i,j)$ that satisfies the following conditions:

1. For every part $\vec\tau(i,j)_t^{(a)}$ of $\vec\tau(i,j)$ there exists a nonzero $\mu_{t'}^{(a)}$ such that $\mu_{t'}^{(a)}=\nu_i^{(j)}\vec\tau(i,j)_t^{(a)}$, where $1\leq a\leq k$, $1\leq t\leq\ell(\vec\tau(i,j)^{(a)})$ and $1\leq t'\leq\ell(\mu^{(a)})$.
2. The total number of cycles from $d_{\vec\mu}$ used to construct $C_{i,j}$ is $\ell(\vec\tau(i,j))$.
3. The blocks that make up $C_{i,j}$ have $\sum_{a=1}^j|\vec\tau(i,j)^{(a)}|\cdot a=j$ elements and so $\vec\tau(i,j) \vdash I_j$.

Condition (1) simply says that if a cycle $\mu_{t'}^{(a)}$ is used in the union that gives $C_{i,j}$, then we get a part in $\vec\tau(i,j)$. Condition (2) is a consequence of condition (1). Condition (3) follows because $C_{i,j}$ is a cycle that permutes blocks of size j, $\vec\tau(i,j)_t^{(a)}$ represents the number of blocks in a cycle of length $\mu_{t'}^{(a)}$ that were unioned to get a cycle of length $\nu_i^{(j)}$.

To count all possible ways to union the cycles of $d_{\vec\mu}$ to form a diagram $d' \in C(d_{\vec\mu};\vec\nu)$ we order the cycle lengths: $\nu_i^{(j)}<\nu_{i'}^{(j')}$ if $j<j'$, or if $j=j'$ and $i<i'$. If it is

possible to union the blocks of $d_{\vec{\mu}}$ to get d , then there exists a sequence of $\vec{\tau}(i,j)$ such that $\vec{\mu} = \biguplus_{i,j} \nu_i^{(j)} \vec{\tau}(i,j)$ and each $\vec{\tau}(i,j) \quad I_j$. For each such $\vec{\tau}(\bullet,\bullet)$ there are $\binom{\mathfrak{m}(\vec{\mu})}{\mathfrak{m}(\vec{\tau}(1,1)),\mathfrak{m}(\vec{\tau}(2,1)),\ldots,\mathfrak{m}(\vec{\tau}(k,k))}$ ways to choose the cycles from $d_{\vec{\mu}}$ to form the vector partitions $\vec{\tau}(i,j)$. Once the cycles are chosen from $d_{\vec{\mu}}$, we use Lemma 4.9 to get cycles of length $\nu_i^{(j)}$ and then by Lemma 4.11 there are $(\nu_i^{(j)})^{\ell(\vec{\tau}(i,j))-1}$ ways to union the cycles indexed by $\vec{\tau}(i,j)$ since by condition (2) there are a total of $\ell(\vec{\tau}(i,j))$ such cycles. If the cycles were ordered as described above, there are

$$\sum_{\vec{\tau}(\bullet,\bullet)} \binom{\mathfrak{m}(\vec{\mu})}{\mathfrak{m}(\vec{\tau}(1,1)),\mathfrak{m}(\vec{\tau}(2,1)),\ldots,\mathfrak{m}(\vec{\tau}(k,k))} \prod_{\substack{1\,6\,j\,6\,k \\ 1\,6\,i\,6\,\ell(\nu^{(j)})}} (\nu_i^{(j)})^{\ell(\vec{\tau}(i,j))-1} \tag{10}$$

ways to obtain elements $d \quad C(d_{\vec{\mu}};\vec{\nu})$.

However, we want to count the number of ways of obtaining elements of cycle type $\vec{\nu}$, where the elements of the same length are indistinguishable. In the case when $\nu_i^{(j)} = \nu_i^{(j)}$ the order in which the cycles in $d_{\vec{\mu}}$ are chosen to form the cycles of this length is interchangeable. Therefore, we need to divide by the ways to permute the cycles of a fixed length. That is we divide Equation (10) by $\mathfrak{m}(\vec{\nu})!$ to enumerate $b_{\vec{\mu}}^{\vec{\nu}}$.

Example 4.13. In Example 4.8 we gave three examples of constructing the elements of $C(d_{\vec{\mu}};\vec{\nu})$ with $\vec{\mu} = ((4,2),(2))$. Here, we show how Proposition 4.12 enumerates these sets.

First, we choose $\vec{\nu} = (?,(2,1),?,(1)) \quad I_{10}$ and we compute that there are two possible $\vec{\tau}(\bullet,\bullet) = (\vec{\tau}(1,2),\vec{\tau}(2,2),\vec{\tau}(1,4))$ satisfying

$$\big((4,2),(2)\big) = 2\cdot\vec{\tau}(1,2) \biguplus 1\cdot\vec{\tau}(2,2) \biguplus 1\cdot\vec{\tau}(1,4)\,;$$

namely,

$$\vec{\tau}(\bullet,\bullet) = \Big(\big((2),?\big),\big((2),?\big),\big(?,(2)\big)\Big) \quad\text{and}\quad \vec{\tau}(\bullet,\bullet) = \Big(\big(?,(1)\big),\big((2),?\big),\big((4),?\big)\Big).$$

Note that $\mathfrak{m}(\vec{\nu})! = 1$ and in both cases the summands in Equation (10) are equal to 1. Hence $b_{\vec{\mu}}^{\vec{\nu}} = 2$.

Now consider $\vec{\nu} = (?,(2),(2)) \quad I_{10}$. There are again two possible $\vec{\tau}(\bullet,\bullet) = (\vec{\tau}(1,2),\vec{\tau}(1,3))$ such that

$$\big((4,2),(2)\big) = 2\cdot\vec{\tau}(1,2) \biguplus 2\cdot\vec{\tau}(1,3)$$

and this time they are

$$\vec{\tau}(\bullet,\bullet) = \Big(\big((2),?\big),\big((1),(1)\big)\Big) \quad\text{and}\quad \vec{\tau}(\bullet,\bullet) = \Big(\big(?,(1)\big),\big((2,1),?\big)\Big).$$

We have again that $\mathfrak{m}(\vec{\nu})! = 1$, however this time we see that the summands for Equation (10) are both equal to 2, hence $b_{\vec{\mu}}^{\vec{\nu}} = 4$.

Finally, $\vec{\nu} = (?,(2,2,1)) \quad I_{10}$, there are two $\vec{\tau}(\bullet,\bullet) = (\vec{\tau}(1,2),\vec{\tau}(2,2),\vec{\tau}(3,2))$ satisfying

$$\big((4,2),(2)\big) = 2\cdot\vec{\tau}(1,2) \biguplus 2\cdot\vec{\tau}(2,2) \biguplus 1\cdot\vec{\tau}(3,2)\,;$$

namely,

$$\vec{\tau}(\bullet,\bullet) = \Big(\big((2),?\big),\big(?,(1)\big),\big((2),?\big)\Big) \quad\text{and}\quad \vec{\tau}(\bullet,\bullet) = \Big(\big(?,(1)\big),\big((2),?\big),\big((2),?\big)\Big).$$

The summands of Equation (10) are both 1 for these $\vec{\tau}(\bullet,\bullet)$, but we now have $\mathfrak{m}(\vec{\nu})! = 2$, hence Equation (9) says that $b_{\vec{\mu}}^{\vec{\nu}} = 1$.

For any partition $\mu = (1^{a_1} 2^{a_2} \dots r^{a_r})$ we define

$$z_\mu = 1^{a_1} a_1! 2^{a_2} a_2! \dots r^{a_r} a_r!.$$

Notice that we can rewrite this as follows:

$$z_\mu = \mathfrak{m}(\mu)! \prod_{1 \leqslant i \leqslant \ell(\mu)} \mu_i \ ,$$

which is the product of the parts of μ times the factorials of the multiplicities. Then for $\vec{\mu} \quad I_k$ we set

$$\mathbf{z}_{\vec{\mu}} = z_{\mu^{(1)}} z_{\mu^{(2)}} \cdots z_{\mu^{(k)}}.$$

Corollary 4.14. *For $\vec{\mu}, \vec{\nu} \quad I_k$,*

$$b^{\vec{\nu}}_{\vec{\mu}} = \frac{1}{\mathbf{z}_{\vec{\nu}}} \sum_{\vec{\tau}(\bullet,\bullet)} \frac{\mathbf{z}_{\vec{\mu}}}{\prod_{i,j} \mathbf{z}_{\vec{\tau}(i,j)}}, \tag{11}$$

where the sum is over all $\vec{\tau}(\bullet,\bullet)$ such that for each $1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant \ell(\nu^{(j)})$, $\vec{\tau}(i,j) \quad I_j$ and $\vec{\mu} = \biguplus_{i,j} \nu_i^{(j)} \vec{\tau}(i,j)$.

Proof. In Proposition 4.12 we showed

$$b^{\vec{\nu}}_{\vec{\mu}} = \frac{1}{\mathfrak{m}(\vec{\nu})!} \sum_{\vec{\tau}(\bullet,\bullet)} \binom{\mathfrak{m}(\vec{\mu})}{\mathfrak{m}(\vec{\tau}(1,1)), \mathfrak{m}(\vec{\tau}(2,1)), \dots, \mathfrak{m}(\vec{\tau}(k,k))} \prod_{\substack{1 \leqslant j \leqslant k \\ 1 \leqslant i \leqslant \ell(\nu^{(j)})}} (\nu_i^{(j)})^{\ell(\vec{\tau}(i,j)) - 1}.$$

We multiply the numerator and denominator of the right-hand side by $\nu_i^{(j)}$ for all (i,j) satisfying $1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant \ell(\nu^{(j)})$. In the denominator, we get $\mathbf{z}_{\vec{\nu}}$ and in the numerator all the powers of the $\nu_i^{(j)}$ increase by 1; therefore, we have

$$b^{\vec{\nu}}_{\vec{\mu}} = \frac{1}{\mathbf{z}_{\vec{\nu}}} \sum_{\vec{\tau}(\bullet,\bullet)} \binom{\mathfrak{m}(\vec{\mu})}{\mathfrak{m}(\vec{\tau}(1,1)), \mathfrak{m}(\vec{\tau}(2,1)), \dots, \mathfrak{m}(\vec{\tau}(k,k))} \prod_{\substack{1 \leqslant j \leqslant k \\ 1 \leqslant i \leqslant \ell(\nu^{(j)})}} (\nu_i^{(j)})^{\ell(\vec{\tau}(i,j))}.$$

Note that

$$(\nu_i^{(j)})^{\ell(\vec{\tau}(i,j))} = \prod_{\substack{1 \leqslant a \leqslant k \\ 1 \leqslant t \leqslant \ell(\tau(i,j)^{(a)})}} \frac{\mu_{t}^{(a)}}{\tau(i,j)_t^{(a)}},$$

where for each t, $\mu_{t}^{(a)}$ is the cycle length corresponding to $\tau(i,j)_t^{(a)}$ used in the construction of the cycle of length $\nu_i^{(j)}$, i.e., cycles that satisfy $\mu_{t}^{(a)} = \nu_i^{(j)} \tau(i,j)_t^{(a)}$ for some t and t . Once we substitute this expression for $\nu_i^{(j)}$, in the numerator we set the product of all the parts of $\vec{\mu}$ and in the denominator we get products of the parts of all $\vec{\tau}(i,j)$ for all i and j. Now expanding the multinomial coefficient and regrouping gives Equation (11).

5. Characters and symmetric functions

This section presents a formula for the irreducible characters of U_k in terms of symmetric functions.

5.1. Class functions and a scalar product. A *class function* of U_k is a map $\alpha : \mathsf{U}_k \to \mathsf{C}$ that is constant on the generalized conjugacy classes of U_k. In light of Proposition 4.1, $\alpha : \mathsf{U}_k \to \mathsf{C}$ is a class function of U_k if $x, y \in \mathsf{U}_k$,

$$\mathsf{cycletype}(x) = \mathsf{cycletype}(y) \text{ implies } \alpha(x) = \alpha(y)\ .$$

Denote the set of class functions on U_k by $\mathrm{Cl}(\mathsf{U}_k)$. The class functions form a C-algebra under pointwise product (also called the Kronecker product).

Let $\vec{\mu} \in I_k$ and define $\mathfrak{I}_{\vec{\mu}} : \mathsf{U}_k \to \mathsf{C}$ to be the indicator function of the generalized conjugacy class indexed by $\vec{\mu}$. That is, for $x \in \mathsf{U}_k$,

$$\mathfrak{I}_{\vec{\mu}}(x) = \begin{cases} 1, & \text{if } \mathsf{cycletype}(x) = \vec{\mu}, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

These functions form a basis of the C-vector space of class functions of U_k.

The restriction of a class function α of U_k to its representative maximal subgroups G_λ results in a class function of G_λ. Moreover, by [27, Proposition 7.5 and Proposition 7.6], the function

$$\mathsf{Res}\colon\ \mathrm{Cl}(\mathsf{U}_k) \longrightarrow \prod_{\lambda \vdash k} \mathrm{Cl}(G_\lambda) \tag{13}$$

defined by

$$\mathsf{Res}(\alpha) = \alpha|_{\prod_{\lambda \vdash k} G_\lambda}$$

is an isomorphism of C-algebras.

As with the class functions of a finite group, there is a scalar product defined on the class functions of finite monoids. The *scalar product* of two class functions $\alpha, \beta \in \mathrm{Cl}(\mathsf{U}_k)$ is

$$\langle \alpha, \beta \rangle_{\mathrm{Cl}(\mathsf{U}_k)} = \sum_{\lambda \vdash k} \frac{1}{|G_\lambda|} \sum_{x \in G_\lambda} \alpha(x)\overline{\beta(x)}. \tag{14}$$

The indicator functions defined in Equation 12 form an orthogonal basis with respect to this scalar product. That is,

$$\left\langle \mathfrak{I}_{\vec{\lambda}}, \mathfrak{I}_{\vec{\mu}} \right\rangle_{\mathrm{Cl}(\mathsf{U}_k)} = \begin{cases} \frac{1}{\mathbf{z}_{\vec{\lambda}}}, & \text{if } \vec{\lambda} = \vec{\mu}, \\ 0, & \text{else.} \end{cases}$$

The irreducible characters of U_k also form a basis of $\mathrm{Cl}(\mathsf{U}_k)$ [27, Proposition 7.9 and 7.10]; note however that they are not orthogonal with respect to this scalar product.

5.2. Symmetric functions on multiple alphabets. Let $\mathbf{X} = X_1, X_2, \ldots$ be an infinite number of alphabets. We define the polynomial ring

$$\mathsf{Sym}_{\mathbf{X}} := \mathsf{C}[p_i[X_j] \mid i, j > 1],$$

where the degree of $p_i[X_j]$ is ij. If $\mu = (1^{a_1} 2^{a_2} \ldots r^{a_r})$ is a partition, then we define

$$p_\mu[X_j] := p_1[X_j]^{a_1} p_2[X_j]^{a_2} \cdots p_r[X_j]^{a_r};$$

for $\vec{\mu} \in I_k$, we define

$$\mathbf{p}_{\vec{\mu}}[\mathbf{X}] := p_{\mu^{(1)}}[X_1] p_{\mu^{(2)}}[X_2] \cdots p_{\mu^{(k)}}[X_k],$$

which is an element in $\mathsf{Sym}_{\mathbf{X}}$ of degree k.

For any $f[\mathbf{X}] \in \mathsf{Sym}_{\mathbf{X}}$, we say that $f[\mathbf{X}]$ is of *homogeneous degree k* if $f[\mathbf{X}]$ is in the linear span of $\{\mathbf{p}_{\vec{\mu}}[\mathbf{X}]\}_{\vec{\mu} \in I_k}$ for some non-negative integer k. When $f[\mathbf{X}]$ is of homogeneous degree k, we denote the degree by $\deg(f[\mathbf{X}]) = k$. The subspace of

$\mathsf{Sym}_{\mathbf{X}}$ of elements of homogeneous degree k is denoted by $\mathsf{Sym}_{\mathbf{X},k}$ and it has the set of elements $\{\mathbf{p}_{\vec{\mu}}[\mathbf{X}]\}_{\vec{\mu} \in I_k}$ as a linear basis. Therefore, we have that

$$\mathsf{Sym}_{\mathbf{X}} = \bigoplus_{k>0} \mathsf{Sym}_{\mathbf{X},k}$$

is a graded ring such that if $f[\mathbf{X}] \in \mathsf{Sym}_{\mathbf{X},k}$ and $g[\mathbf{X}] \in \mathsf{Sym}_{\mathbf{X},\ell}$, then $f[\mathbf{X}]g[\mathbf{X}] \in \mathsf{Sym}_{\mathbf{X},k+\ell}$.

We define a scalar product on $\mathsf{Sym}_{\mathbf{X}}$ as follows:

$$\left\langle \mathbf{p}_{\vec{\lambda}}[\mathbf{X}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \right\rangle = \begin{cases} \mathbf{z}_{\vec{\mu}}, & \text{if } \vec{\lambda} = \vec{\mu}, \\ 0, & \text{else.} \end{cases} \tag{15}$$

Note that $\left\langle \mathbf{p}_{\vec{\lambda}}[\mathbf{X}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \right\rangle = \prod_{i=1}^{k} \left\langle p_{\lambda^{(i)}}[X_i], p_{\mu^{(i)}}[X_i] \right\rangle$, where on the right hand side the scalar products are in the usual ring of symmetric functions for which the power sum functions form an orthogonal basis. In particular, for any element $f[\mathbf{X}] \in \mathsf{Sym}_{\mathbf{X}}$,

$$\left\langle f[\mathbf{X}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \right\rangle \text{ is equal to the coefficient of } \frac{\mathbf{p}_{\vec{\mu}}[\mathbf{X}]}{\mathbf{z}_{\vec{\mu}}} \text{ in } f[\mathbf{X}]. \tag{16}$$

Define also an analogue of the *Schur basis*. For $\vec{\mu} \in I_k$, set

$$\mathbf{s}_{\vec{\mu}}[\mathbf{X}] := s_{\mu^{(1)}}[X_1] s_{\mu^{(2)}}[X_2] \cdots s_{\mu^{(k)}}[X_k] \ ,$$

where $s_{\mu^{(i)}}[X_i]$ is the Schur function over the alphabet X_i. It follows that

$$\left\langle \mathbf{s}_{\vec{\lambda}}[\mathbf{X}], \mathbf{s}_{\vec{\mu}}[\mathbf{X}] \right\rangle = \begin{cases} 1, & \text{if } \vec{\lambda} = \vec{\mu}, \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Since we know for $\lambda, \mu \vdash a$, that the coefficient $\left\langle s_\lambda[X], p_\mu[X] \right\rangle = \chi^{\lambda}_{\mathfrak{S}_a}(\mu)$, it follows that

$$\left\langle \mathbf{s}_{\vec{\lambda}}[\mathbf{X}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \right\rangle = \chi^{\lambda^{(1)}}_{\mathfrak{S}_{a_1}}(\mu^{(1)}) \chi^{\lambda^{(2)}}_{\mathfrak{S}_{a_2}}(\mu^{(2)}) \cdots \chi^{\lambda^{(k)}}_{\mathfrak{S}_{a_k}}(\mu^{(k)}), \tag{18}$$

where $a_i = |\lambda^{(i)}|$ for $1 \leqslant i \leqslant k$.

Note that the value on the right hand side of Equation (18) is equal to the irreducible character indexed by $\vec{\lambda}$ evaluated at an element of cycle type $\vec{\mu}$ of the maximal subgroup $\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_k} \simeq G_\lambda$, where $\lambda = \overrightarrow{\mathsf{type}}(\vec{\lambda}) = \overrightarrow{\mathsf{type}}(\vec{\mu}) = (1^{a_1} 2^{a_2} \dots k^{a_k})$.

5.3. A Frobenius characteristic map for the maximal subgroups G_λ. By Corollary 3.3, for each partition $\lambda = (1^{a_1} 2^{a_2} \dots k^{a_k})$, the maximal subgroup G_λ is isomorphic to

$$G_\lambda \simeq \mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_k} \ .$$

The usual Frobenius map for $\mathfrak{S}_r$ sends the irreducible character of $\mathfrak{S}_r$ indexed by the partition $\mu \vdash r$ to the Schur function indexed by that partition. We denote this map by

$$\phi_{\mathfrak{S}_r}(\chi^{\mu}_{\mathfrak{S}_r}) = s_\mu \ .$$

Since the maximal subgroups are isomorphic to direct products of symmetric groups, the Frobenius map can be extended to G_λ by mapping the class functions of G_λ to the k-fold tensor product of the symmetric functions. Under this map, the image of the irreducible character of G_λ indexed by $\vec{\lambda} \in I_k$ with $\overrightarrow{\mathsf{type}}(\vec{\lambda}) = \lambda = (1^{a_1} 2^{a_2} \dots k^{a_k})$ is

$$\phi_{G_\lambda}\left(\chi^{\vec{\lambda}}_{G_\lambda}\right) = \phi_{G_\lambda}\left(\chi^{\lambda^{(1)}}_{\mathfrak{S}_{a_1}} \chi^{\lambda^{(2)}}_{\mathfrak{S}_{a_2}} \cdots \chi^{\lambda^{(k)}}_{\mathfrak{S}_{a_k}}\right) = s_{\lambda^{(1)}}[X_1] s_{\lambda^{(2)}}[X_2] \cdots s_{\lambda^{(k)}}[X_k] = \mathbf{s}_{\vec{\lambda}}[\mathbf{X}] \ .$$

Equation (18) states that for $\vec{\lambda}, \vec{\mu} \in I_k$ with $\overrightarrow{\mathsf{type}}(\vec{\lambda}) = \overrightarrow{\mathsf{type}}(\vec{\mu}) = \lambda$,

$$\chi^{\vec{\lambda}}_{G_\lambda}(d_{\vec{\mu}}) = \left\langle \mathbf{s}_{\vec{\lambda}}[\mathbf{X}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \right\rangle$$

which by Equation (16) is equal to the coefficient $\frac{\mathbf{p}_{\vec{\mu}}[\mathbf{X}]}{\mathbf{z}_{\vec{\mu}}}$ in $\mathbf{s}_{\vec{\lambda}}[\mathbf{X}]$. Furthermore, by Equation (17), the images of the irreducible characters are all (even if they are not characters of the same maximal subgroup) orthonormal in $\mathsf{Sym}_{\mathbf{X},k}$.

5.4. A Frobenius characteristic map for U_k. We now extend the characteristic map from the class functions on the maximal subgroups to the ring of class functions on U_k.

The *Frobenius characteristic map* $\phi_{\mathsf{U}_k} : \mathrm{Cl}(\mathsf{U}_k) \to \mathsf{Sym}_{\mathbf{X},k}$ is defined on the basis of indicator functions by

$$\phi_{\mathsf{U}_k}(\mathfrak{I}_{\vec{\mu}}) = \frac{\mathbf{p}_{\vec{\mu}}[\mathbf{X}]}{\mathbf{z}_{\vec{\mu}}}\ , \tag{19}$$

and extended linearly to all of $\mathrm{Cl}(\mathsf{U}_k)$. Then for any U_k-class function $\psi_{\mathsf{U}_k} : \mathsf{U}_k \to \mathsf{C}$,

$$\phi_{\mathsf{U}_k}(\psi_{\mathsf{U}_k}) = \sum_{\vec{\mu}\ \ I_k} \psi_{\mathsf{U}_k}(d_{\vec{\mu}}) \frac{\mathbf{p}_{\vec{\mu}}[\mathbf{X}]}{\mathbf{z}_{\vec{\mu}}}, \tag{20}$$

where $d_{\vec{\mu}}$ is a representative element from the generalized conjugacy class $C_{\vec{\mu}}$.

Recall from Equation (13) that we have an isomorphism $\mathrm{Cl}(\mathsf{U}_k)\quad \prod_{\lambda\ \ k} \mathrm{Cl}(G_\lambda)$ that is given by mapping a class function of U_k to the restrictions to maximal subgroups. A similar result also holds for the Frobenius characteristic map. Since $d_{\vec{\mu}}$ belongs to exactly one maximal subgroup, Equation (19) implies that

$$\phi_{\mathsf{U}_k}(f) = \sum_{\lambda\ \ k} \phi_{G_\lambda}(f|_{G_\lambda}) \tag{21}$$

for any $f\quad \mathrm{Cl}(\mathsf{U}_k)$.

Since the images of the irreducible characters of G_λ are the symmetric functions $\mathbf{s}_{\vec{\lambda}}[\mathbf{X}]$ and this basis is orthonormal in $\mathsf{Sym}_{\mathbf{X},k}$, Equation (21) implies the following important property of the Frobenius images that we have defined here.

Proposition 5.1. *Let $\vec{\lambda}\quad I_k$, $\lambda = \overrightarrow{\mathsf{type}}(\vec{\lambda})$, and let χ be a character of U_k. The multiplicity of $\chi^{\vec{\lambda}}_{G_\lambda}$ in the restriction of χ from U_k to G_λ is equal to*

$$\left\langle \phi_{\mathsf{U}_k}(\chi),\, \mathbf{s}_{\vec{\lambda}}[\mathbf{X}] \right\rangle .$$

Proof. By Equation (21),

$$\begin{aligned}
\left\langle \phi_{\mathsf{U}_k}(\chi), \mathbf{s}_{\vec{\lambda}}[\mathbf{X}] \right\rangle &= \left\langle \phi_{\mathsf{U}_k}(\chi), \phi_{G_\lambda}(\chi^{\vec{\lambda}}_{G_\lambda}) \right\rangle \\
&= \sum_{\gamma\ \ k} \left\langle \phi_{G_\gamma}(\chi|_{G_\gamma}), \phi_{G_\lambda}(\chi^{\vec{\lambda}}_{G_\lambda}) \right\rangle \\
&= \left\langle \phi_{G_\lambda}(\chi|_{G_\lambda}), \phi_{G_\lambda}(\chi^{\vec{\lambda}}_{G_\lambda}) \right\rangle
\end{aligned}$$

because the Frobenius images of the irreducible characters of G_λ are orthogonal to the Frobenius images of the irreducible characters of G_γ if $\gamma = \lambda$. Since $\phi_{G_\lambda}(\chi^{\vec{\lambda}}_{G_\lambda}) = \mathbf{s}_{\vec{\lambda}}[\mathbf{X}]$,

$$\left\langle \phi_{\mathsf{U}_k}(\chi), \mathbf{s}_{\vec{\lambda}}[\mathbf{X}] \right\rangle = \left\langle \phi_{G_\lambda}(\chi|_{G_\lambda}), \phi_{G_\lambda}(\chi^{\vec{\lambda}}_{G_\lambda}) \right\rangle = \left\langle \phi_{G_\lambda}(\chi|_{G_\lambda}), \mathbf{s}_{\vec{\lambda}}[\mathbf{X}] \right\rangle,$$

which is the coefficient of $\mathbf{s}_{\vec{\lambda}}[\mathbf{X}]$ in $\phi_{G_\lambda}(\chi|_{G_\lambda})$; or in other words, it is the multiplicity of the irreducible G_λ-representation indexed by $\vec{\lambda}\quad I_k$ in $\chi|_{G_\lambda}$.

Moreover, the Frobenius characteristic function ϕ_{U_k} has the property that

$$\left\langle \mathfrak{I}_{\vec{\mu}}, \mathfrak{I}_{\vec{\lambda}} \right\rangle_{\mathrm{Cl}(\mathsf{U}_k)} = \left\langle \phi_{\mathsf{U}_k}(\mathfrak{I}_{\vec{\mu}}), \phi_{\mathsf{U}_k}(\mathfrak{I}_{\vec{\lambda}}) \right\rangle$$

where, to be clear, on the left hand side of the equation the inner product is on the class functions from Equation (14) and on the right hand side the inner product is on the symmetric functions from Equation (15). Hence ϕ_{U_k} is an isometry with respect to the inner products on the class functions of U_k and the inner product on $\mathsf{Sym}_{\mathbf{X},k}$.

Let $\mathbf{1}_{\mathsf{U}_r}$ denote the trivial character for U_r (this is the irreducible character indexed by $(?, \dots, ?, (1)) \quad I_r$ with $r-1$ copies of $?$). This is a class function with the property that $\mathbf{1}_{\mathsf{U}_r}(a) = 1$ for all $a \quad \mathsf{U}_r$. Then let

$$E_r := E_r[X_1, X_2, \dots, X_r] \coloneqq \phi_{\mathsf{U}_r}(\mathbf{1}_{\mathsf{U}_r}) \tag{22}$$

$$= \sum_{\vec{\mu} \quad I_r} \frac{\mathbf{p}_{\vec{\mu}}[\mathbf{X}]}{\mathbf{z}_{\vec{\mu}}} \tag{23}$$

$$= \sum_{(1^{a_1} 2^{a_2} \cdots r^{a_r}) \quad r} s_{a_1}[X_1] s_{a_2}[X_2] \cdots s_{a_r}[X_r] \ . \tag{24}$$

The symmetric function E_r is the generating function for the character values for the trivial representation of U_r. It will serve as a building block in a formula for the other the irreducible characters of U_r.

The notation we have been using for symmetric functions can be extended to allow substituting an expression in place of a set of variables. This is called *plethystic notation* and more details can be found in [14], but for our purposes the following should suffice. For an element $A \quad \mathsf{Sym}_{\mathbf{X}}$, let $p_k[A]$ be the element obtained by first expressing A in the power sum basis and then replacing each $p_r[X_i]$ appearing in the expression with $p_{kr}[X_i]$. Since every symmetric function $f \quad \mathsf{Sym}$ is a polynomial in the power sum elements $p_1, p_2, p_3, \dots$, we define $f[A]$ to be the element obtained from f by replacing each p_i with $p_i[A]$. This notation is consistent with the expressions we have used thus far once we identify X_i and $p_1[X_i]$.

Remark 5.2. This notational extension is useful for providing a generating function for Equation (24). For any $r > 0$ and expressions A and B, we have

$$s_0[A] = 1, \qquad s_r[A+B] = \sum_{i=0}^{r} s_i[A] s_{r-i}[B], \qquad s_r[t^i X_i] = t^{ri} s_r[X_i].$$

The middle expression above is sometimes known in the literature as the *alphabet addition formula*. Applying these to expand the expression below in t, we have

$$s_k[1 + tX_1 + t^2 X_2 + \cdots + t^k X_k] = 1 + E_1 t + E_2 t^2 + \cdots + E_k t^k + \cdots + s_k[X_k] t^{k^2} \ .$$

The coefficient of t^r for $r > k$ in the expression on the right hand side above are symmetric functions that are not necessarily equal to E_r since they will depend on both r and k.

We will use the shorthand notation

$$\mathbf{s}_{\vec{\lambda}}[\mathbf{E}] \coloneqq s_{\lambda^{(1)}}[E_1] s_{\lambda^{(2)}}[E_2] \cdots s_{\lambda^{(k)}}[E_k]$$

and

$$\mathbf{p}_{\vec{\lambda}}[\mathbf{E}] \coloneqq p_{\lambda^{(1)}}[E_1] p_{\lambda^{(2)}}[E_2] \cdots p_{\lambda^{(k)}}[E_k] \ .$$

Then as a corollary to Equation (11), we have that the coefficients $b^{\vec{\nu}}_{\vec{\mu}}$ are given by the following symmetric function expression.

Corollary 5.3. *For* $\vec{\mu}, \vec{\nu} \quad I_k$,

$$b^{\vec{\nu}}_{\vec{\mu}} = \frac{1}{\mathbf{z}_{\vec{\nu}}} \quad \mathbf{p}_{\vec{\nu}}[\mathbf{E}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \quad . \tag{25}$$

Proof. This is found by expanding $\mathbf{p}_{\vec{\nu}}[\mathbf{E}]$ in the power sum basis and taking the coefficient of $\mathbf{p}_{\vec{\mu}}[\mathbf{X}]$ to show that it agrees with Equation (11) using Equation (15).

Using Equation (23),

$$\frac{1}{\mathbf{z}_{\vec{\nu}}}\prod_{a=1}^{k} p_{\nu^{(a)}}[E_a] = \frac{1}{\mathbf{z}_{\vec{\nu}}}\prod_{j=1}^{k}\prod_{i=1}^{\ell(\nu^{(j)})} p_{\nu_i^{(j)}}\left[\sum_{\vec{\tau}(i,j)\quad I_j}\frac{\mathbf{p}_{\vec{\tau}(i,j)}[\mathbf{X}]}{\mathbf{z}_{\vec{\tau}(i,j)}}\right] \tag{26}$$

$$= \frac{1}{\mathbf{z}_{\vec{\nu}}}\prod_{j=1}^{k}\prod_{i=1}^{\ell(\nu^{(j)})}\sum_{\vec{\tau}(i,j)\quad I_j}\frac{\mathbf{p}_{\nu_i^{(j)}\vec{\tau}(i,j)}[\mathbf{X}]}{\mathbf{z}_{\vec{\tau}(i,j)}}, \tag{27}$$

where in the terms of the sum, we note that $\nu_i^{(j)}\quad \mathsf{Z}_{>0}$ and the expression $\nu_i^{(j)}\vec{\tau}(i,j)$ is to be interpreted as $(\nu_i^{(j)}\vec{\tau}(i,j)^{(1)}, \nu_i^{(j)}\vec{\tau}(i,j)^{(2)}, \ldots, \nu_i^{(j)}\vec{\tau}(i,j)^{(k)})$ where for a positive integer a and a partition $\lambda \quad r$ we have $a\lambda = (a\lambda_1, a\lambda_2, \ldots, a\lambda_{\ell(\lambda)}) \quad ar$.

The coefficient of $\frac{\mathbf{p}_{\vec{\mu}}[\mathbf{X}]}{\mathbf{z}_{\vec{\mu}}}$ is equal to the sum of the coefficients such that for each 1 6 a 6 k,

$$\biguplus_{j=1}^{k}\biguplus_{i=1}^{\ell(\nu^{(j)})}\nu_i^{(j)}\vec{\tau}(i,j)^{(a)} = \vec{\mu}^{(a)}\ . \tag{28}$$

More specifically the coefficient of $\frac{p_{\vec{\mu}}[\mathbf{X}]}{\mathbf{z}_{\vec{\mu}}}$ in Equation (27) is equal to

$$\frac{1}{\mathbf{z}_{\vec{\nu}}}\sum_{\vec{\tau}(\bullet,\bullet)}\frac{\mathbf{z}_{\vec{\mu}}}{\prod_{j=1}^{k}\prod_{i=1}^{\ell(\nu^{(j)})}\mathbf{z}_{\vec{\tau}(i,j)}}, \tag{29}$$

where the sum is over all sequences of partitions $\vec{\tau}(i,j)\quad I_j$ for 1 6 j 6 k, 1 6 i 6 $\ell(\nu^{(j)})$ such that Equation (28) holds.

As a consequence, we have the following symmetric function expression for the character table of the uniform block permutation algebra $\mathsf{C}U_k$.

THEOREM 5.4. *Let $\chi_{\mathsf{U}_k}^{\vec{\lambda}}$ be the irreducible character of U_k indexed by $\vec{\lambda}\quad I_k$. For $\vec{\mu}\quad I_k$, let $d_{\vec{\mu}}\quad \mathsf{U}_k$ be any element such that* $\mathsf{cycletype}(d_{\vec{\mu}}) = \vec{\mu}$. *Then*

$$\chi_{\mathsf{U}_k}^{\vec{\lambda}}(d_{\vec{\mu}}) = \left\langle \mathbf{s}_{\vec{\lambda}}[\mathbf{E}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}]\right\rangle\ . \tag{30}$$

As a consequence,

$$\phi_{\mathsf{U}_k}(\chi_{\mathsf{U}_k}^{\vec{\lambda}}) = \mathbf{s}_{\vec{\lambda}}[\mathbf{E}]\ . \tag{31}$$

Proof. Since characters are class functions, they are constant on generalized conjugacy classes and so it suffices to prove the result for the conjugacy class representatives $d_{\vec{\mu}}$ defined in Section 4.2. By Equation (15), $\{\mathbf{p}_{\vec{\nu}}[\mathbf{X}]\}_{\vec{\nu}}$ is an orthogonal basis, hence for any symmetric function alphabet $\mathbf{Y} = Y_1, Y_2, Y_3, \ldots$,

$$\mathbf{s}_{\vec{\lambda}}[\mathbf{Y}] = \sum_{\vec{\nu}}\left\langle \mathbf{s}_{\vec{\lambda}}[\mathbf{Y}], \mathbf{p}_{\vec{\nu}}[\mathbf{Y}]\right\rangle\frac{\mathbf{p}_{\vec{\nu}}[\mathbf{Y}]}{\mathbf{z}_{\vec{\nu}}}\ .$$

We can expand

$$\left\langle \mathbf{s}_{\vec{\lambda}}[\mathbf{E}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}]\right\rangle = \sum_{\vec{\mu}}\left\langle \mathbf{s}_{\vec{\lambda}}[\mathbf{E}], \mathbf{p}_{\vec{\nu}}[\mathbf{E}]\right\rangle\left\langle\frac{\mathbf{p}_{\vec{\nu}}[\mathbf{E}]}{\mathbf{z}_{\vec{\nu}}}, \mathbf{p}_{\vec{\mu}}[\mathbf{X}]\right\rangle\ . \tag{32}$$

This last expression is equal to $\chi_{\mathsf{U}_k}^{\vec{\lambda}}(d_{\vec{\mu}})$ by Equations (25), (18) and (8). Equation (31) follows from (20) and the fact that for any f

$$f = \sum_{\vec{\nu}}\quad f, \mathbf{p}_{\vec{\nu}}[\mathbf{X}]\quad\frac{\mathbf{p}_{\vec{\nu}}[\mathbf{X}]}{\mathbf{z}_{\vec{\nu}}}\ .$$

Theorem 5.4 allows us to compute the character table for U_k using symmetric function computations. In order to write down the character table, we define a total order on the elements in I_k. To do this we first define the *reverse lexicographic order* on partitions: we say $\lambda \leqslant_{\mathrm{rl}} \mu$ if we have $\lambda_i > \mu_i$ at the first index i where λ and μ differ. This is a total order. For example, $(5) \leqslant_{\mathrm{rl}} (4,1) \leqslant_{\mathrm{rl}} (3,2) \leqslant_{\mathrm{rl}} (3,1,1) \leqslant_{\mathrm{rl}} (2,2,1) \leqslant_{\mathrm{rl}} (2,1,1,1) \leqslant_{\mathrm{rl}} (1,1,1,1,1)$. Now for $\vec{\lambda}, \vec{\mu} \in I_k$, we say $\vec{\lambda} < \vec{\mu}$ if $\overrightarrow{\mathsf{type}}(\vec{\lambda}) <_{\mathrm{rl}} \overrightarrow{\mathsf{type}}(\vec{\mu})$ or if $\overrightarrow{\mathsf{type}}(\vec{\lambda}) = \overrightarrow{\mathsf{type}}(\vec{\mu})$ and there exists an $1 \leqslant i \leqslant k$ such that $\lambda^{(j)} = \mu^{(j)}$ for all $1 \leqslant j < i$ and $\lambda^{(i)} <_{\mathrm{rl}} \mu^{(i)}$.

Example 5.5. The matrices are presented below with the rows and columns ordered from smallest to largest from the top row of the matrix to the bottom. The elements of I_k are presented compactly by dropping a layer of enclosing parentheses and commas.

The character table of U_2 is

$$\begin{array}{l|c|rr}
(\emptyset,1) & 1 & 1 & 1 \\ \hline
(2) & 0 & 1 & 1 \\
(11) & 0 & -1 & 1
\end{array}\,.$$

The character table of U_3 is

$$\begin{array}{l|c|c|rrr}
(\emptyset,\emptyset,1) & 1 & 1 & 1 & 1 & 1 \\ \hline
(1,1) & 0 & 1 & 0 & 1 & 3 \\ \hline
(3) & 0 & 0 & 1 & 1 & 1 \\
(21) & 0 & 0 & -1 & 0 & 2 \\
(111) & 0 & 0 & 1 & -1 & 1
\end{array}\,.$$

The character table of U_4 is

$$\begin{array}{l|c|c|rr|rr|rrrrr}
(\emptyset,\emptyset,\emptyset,1) & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
(1,\emptyset,1) & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 2 & 4 \\ \hline
(\emptyset,2) & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 3 & 1 & 3 \\
(\emptyset,11) & 0 & 0 & -1 & 1 & 1 & 1 & -1 & 0 & -1 & 1 & 3 \\ \hline
(2,1) & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & 6 \\
(11,1) & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -2 & 0 & 6 \\ \hline
(4) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
(31) & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 3 \\
(22) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 2 \\
(211) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 3 \\
(1111) & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 1
\end{array}\,.$$

5.5. Factorizations of the character table of U_k.

In [26, Section 7], Steinberg describes two factorizations of the character table for finite inverse semigroups in terms of the character tables of its maximal subgroups. We describe both of these factorizations here in Proposition 5.7 and Proposition 5.9 below.

The first factorization uses an upper uni-triangular matrix B_k with non-negative integer entries. The general description for the entries in B_k is discussed in [26, Proposition 7.1] in which Steinberg remarks that computing this matrix is in general a "daunting task." In [25, Corollary 3.7], Solomon computes this factorization for the character table of the rook monoid. For the uniform block permutation monoid U_k, our formula for the entries of B_k in terms of symmetric functions will follow from the results in Section 4.4.

The second factorization uses a different upper uni-triangular with non-negative integer entries U_k. Its entries are the multiplicities of the irreducible representations of the maximal subgroups when we restrict an irreducible representation of U_k to its maximal subgroups. Our interest in this matrix arises because of a relation with the operation of plethysm described in Corollary 5.11.

Throughout this section we assume that I_k is totally ordered as in Section 5.4. This order satisfies the condition that if $\{e_{\pi_\lambda} \mid \lambda \quad k\}$ are the idempotent representatives for the J -classes of U_k, then $\mathsf{U}_k e_{\pi_\mu} \mathsf{U}_k \quad \mathsf{U}_k e_{\pi_\nu} \mathsf{U}_k$ implies μ 6$_{\mathrm{rl}}$ ν. In particular, the largest element is $\mu = (1^k)$ since $e_{\pi_{(1^k)}}$ is the identity element of U_k, and $\nu = (k)$ is the smallest element since $e_{\pi_{(k)}}$ has one block and $e_{\pi_{(k)}}$ is the only element in $\mathsf{U}_k e_{\pi_{(k)}} \mathsf{U}_k$.

Let X_k be the character table of U_k, which we view as a matrix, and whose entries are denoted by $X_{\vec{\lambda},\vec{\mu}}$. The following result summarizes the properties of X_k proved in the previous section. They will be used to factor X_k as the product of two matrices.

Proposition 5.6. *Let $X_k = \big(X_{\vec{\lambda},\vec{\mu}}\big)_{\vec{\lambda},\vec{\mu} \quad I_k}$ be the character table of U_k viewed as a matrix. Then*

$$X_{\vec{\lambda},\vec{\mu}} = \chi^{\vec{\lambda}}_{\mathsf{U}_k}(d_{\vec{\mu}}) = \big\langle \mathbf{s}_{\vec{\lambda}}[\mathbf{E}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \big\rangle$$

and X_k is upper block diagonal with respect to the total order on I_k defined in Section 5.4.

Define $A_k = \big(A_{\vec{\lambda},\vec{\mu}}\big)_{\vec{\lambda},\vec{\mu} \quad I_k}$ to be the block diagonal matrix whose diagonal blocks are the character tables of the maximal subgroups of U_k; explicitly, $A_{\vec{\lambda},\vec{\mu}} = 0$ if $\overrightarrow{\mathsf{type}}(\vec{\lambda}) = \overrightarrow{\mathsf{type}}(\vec{\mu})$, and otherwise $A_{\vec{\lambda},\vec{\mu}} = \chi^{\vec{\lambda}}_{G_\lambda}(d_{\vec{\mu}})$, where $\lambda = \overrightarrow{\mathsf{type}}(\vec{\lambda})$. By Equation (18),

$$A_{\vec{\lambda},\vec{\mu}} = \chi^{\vec{\lambda}}_{G_\lambda}(d_{\vec{\mu}}) = \big\langle \mathbf{s}_{\vec{\lambda}}[\mathbf{X}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \big\rangle \ .$$

Define a second square matrix $B_k = \big(B_{\vec{\lambda},\vec{\mu}}\big)_{\vec{\lambda},\vec{\mu} \quad I_k}$, with the entries from Corollary 5.3,

$$B_{\vec{\mu},\vec{\nu}} = b^{\vec{\nu}}_{\vec{\mu}} = \left\langle \frac{\mathbf{p}_{\vec{\nu}}[\mathbf{E}]}{\mathbf{z}_{\vec{\nu}}}, \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \right\rangle .$$

Proposition 5.7. *The matrix A_k is block diagonal, B_k is upper uni-triangular with non-negative integer entries, and*

$$X_k = A_k \cdot B_k \ .$$

Proof. The statement that $X_k = A_k \cdot B_k$ is a restatement of Equations (8) and (32).

If $\overrightarrow{\mathsf{type}}(\vec{\nu}) = \overrightarrow{\mathsf{type}}(\vec{\mu})$, then $b^{\vec{\nu}}_{\vec{\mu}}$ is the number of ways of merging parts of $d_{\vec{\mu}}$ to obtain an element of cycle type $\vec{\nu}$. There is of course one way to do this if $\vec{\nu} = \vec{\mu}$ and zero ways if $\overrightarrow{\mathsf{type}}(\vec{\mu}) <_{\mathrm{rl}} \overrightarrow{\mathsf{type}}(\vec{\nu})$.

Example 5.8. If $k = 2$, the character table for the maximal subgroups in block diagonal form and the matrix B are

$$A_2 = \begin{array}{l|c|cc} (?,1) & 1 & 0 & 0 \\ \hline (2) & 0 & 1 & 1 \\ (11) & 0 & -1 & 1 \end{array} \qquad\qquad B_2 = \begin{array}{l|c|cc} (?,1) & 1 & 1 & 1 \\ \hline (2) & 0 & 1 & 0 \\ (11) & 0 & 0 & 1 \end{array} .$$

If $k = 3$, the character table for the maximal subgroups in block diagonal form and the matrix B are

$$A_3 = \begin{array}{l|c|c|ccc} (?,?,1) & 1 & 0 & 0 & 0 & 0 \\ \hline (1,1) & 0 & 1 & 0 & 0 & 0 \\ \hline (3) & 0 & 0 & 1 & 1 & 1 \\ (21) & 0 & 0 & -1 & 0 & 2 \\ (111) & 0 & 0 & 1 & -1 & 1 \end{array} \qquad\qquad B_3 = \begin{array}{l|c|c|ccc} (?,?,1) & 1 & 1 & 1 & 1 & 1 \\ \hline (1,1) & 0 & 1 & 0 & 1 & 3 \\ \hline (3) & 0 & 0 & 1 & 0 & 0 \\ (21) & 0 & 0 & 0 & 1 & 0 \\ (111) & 0 & 0 & 0 & 0 & 1 \end{array} .$$

If $k = 4$, the character table for the maximal subgroups in block diagonal form is

$$A_4 = \begin{array}{l|c|c|cc|cc|ccccc}
(\text{?},\text{?},\text{?},1) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
(1,\text{?},1) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
(\text{?},2) & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\text{?},11) & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
(2,1) & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
(11,1) & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline
(4) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
(31) & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 3 \\
(22) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 2 \\
(211) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 3 \\
(1111) & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 1
\end{array}$$

and the matrix B_4 of values $b_{\vec\mu}^{\vec\lambda}$ is

$$B_4 = \begin{array}{l|c|c|cc|cc|ccccc}
(\text{?},\text{?},\text{?},1) & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
(1,\text{?},1) & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 2 & 4 \\ \hline
(\text{?},2) & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
(\text{?},11) & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 3 \\ \hline
(2,1) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 0 \\
(11,1) & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 6 \\ \hline
(4) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
(31) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
(22) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
(211) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
(1111) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array} .$$

The second factorization arises from the isomorphism in Equation (13), which is induced by restricting class functions of U_k to the maximal subgroups G_λ. By [27, Theorem 6.5], the matrix corresponding to the restriction isomorphism, which we denote by U_k, is upper triangular with 1s on the diagonal. Since $\mathsf{C}\mathsf{U}_k$ is semisimple, the entries of the matrix U_k are the multiplicities of the irreducible representations of the maximal subgroups when we restrict an irreducible representation of U_k to the maximal subgroups. Thus, U_k is sometimes known as the *decomposition matrix*.

Proposition 5.9. *Define the matrix* $U_k = \left(U_{\vec\lambda,\vec\mu}\right)_{\vec\lambda,\vec\mu\ \ I_k}$ *by*

$$U_{\vec\lambda,\vec\mu} = \left\langle \mathbf{s}_{\vec\lambda}[\mathbf{E}], \mathbf{s}_{\vec\mu}[\mathbf{X}] \right\rangle \ .$$

Then U_k is upper uni-triangular with non-negative integer entries, and

$$X_k = U_k \cdot A_k \ ,$$

where X_k is the character table of U_k (see Proposition 16*), and A_k is the block diagonal matrix whose blocks are the character tables of the maximal subgroups (see Proposition* 5.7*).*

Proof. For $\vec\lambda, \vec\nu \quad I_k$ and $\nu = \overrightarrow{\mathsf{type}}(\vec\nu)$,

$$U_{\vec\lambda,\vec\nu} = \left\langle \phi_{\mathsf{U}_k}(\chi_{\mathsf{U}_k}^{\vec\lambda}), \phi_{G_\nu}(\chi_{G_\nu}^{\vec\nu}) \right\rangle = \left\langle \mathbf{s}_{\vec\lambda}[\mathbf{E}], \mathbf{s}_{\vec\nu}[\mathbf{X}] \right\rangle \ .$$

By Proposition 5.1, the entries of this matrix are multiplicities of irreducible representations in a restriction and hence they are non-negative integers; more precisely, $U_{\vec\lambda,\vec\nu}$ is the multiplicity of the irreducible G_ν-representation $V_{G_\nu}^{\vec\nu}$ in the restriction of the irreducible U_k-representation $W_{\mathsf{U}_k}^{\vec\lambda}$ to the maximal subgroup G_ν.

The factorization $X_k = U_k \cdot A_k$ is a consequence of the fact that $\{\mathbf{s}_{\vec{\nu}}[\mathbf{X}]\}_{\vec{\nu} \quad I_k}$ is an orthonormal basis of $\mathsf{Sym}_{\mathbf{X},k}$, so that

$$\left\langle \mathbf{s}_{\vec{\lambda}}[\mathbf{E}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \right\rangle = \sum_{\vec{\nu} \quad I_k} \left\langle \mathbf{s}_{\vec{\lambda}}[\mathbf{E}], \mathbf{s}_{\vec{\nu}}[\mathbf{X}] \right\rangle \quad \mathbf{s}_{\vec{\nu}}[\mathbf{X}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \quad .$$

Now if we examine the expansion of $\mathbf{s}_{\vec{\lambda}}[\mathbf{E}]$, then if $\overrightarrow{\mathsf{type}}(\vec{\nu}) <_{\mathrm{rl}} \overrightarrow{\mathsf{type}}(\vec{\lambda})$, there exists an r such that the multiplicity of r in $\overrightarrow{\mathsf{type}}(\vec{\nu})$ is $a > 0$ and in $\overrightarrow{\mathsf{type}}(\vec{\lambda})$ it is smaller than a. This implies that the degree in X_r in the symmetric function $\phi_{G_\mu}(\chi^{\vec{\mu}}_{G_\mu})$ is a but that all terms in $\mathbf{s}_{\vec{\lambda}}[\mathbf{E}] = \phi_{\mathsf{U}_k}(\chi^{\vec{\lambda}}_{\mathsf{U}_k})$ have degree in X_r smaller than a and hence $U_{\vec{\lambda},\vec{\nu}} = 0$.

Note that $\mathbf{s}_{\vec{\lambda}}[\mathbf{E}]$ is equal to $\mathbf{s}_{\vec{\lambda}}[\mathbf{X}]$ plus terms that are of not of the same degree in the same variables as $\mathbf{s}_{\vec{\lambda}}[\mathbf{X}]$. Therefore, $U_{\vec{\lambda},\vec{\lambda}} = 1$ and $U_{\vec{\lambda},\vec{\nu}} = 0$ if $\vec{\lambda} = \vec{\nu}$. We conclude that U_k is upper uni-triangular.

Example 5.10. For $k = 2, 3, 4$, the matrices corresponding to the multiplicities of an irreducible representation in the restriction from the uniform block permutation algebra to the maximal subgroups are given by

$$U_2 = \begin{array}{l|c|cc} (?,1) & 1 & 1 & 0 \\ \hline (2) & 0 & 1 & 0 \\ (11) & 0 & 0 & 1 \end{array} \; ,$$

$$U_3 = \begin{array}{l|c|c|ccc} (?,?,1) & 1 & 1 & 1 & 0 & 0 \\ \hline (1,1) & 0 & 1 & 1 & 1 & 0 \\ \hline (3) & 0 & 0 & 1 & 0 & 0 \\ (21) & 0 & 0 & 0 & 1 & 0 \\ (111) & 0 & 0 & 0 & 0 & 1 \end{array} \; ,$$

and

$$U_4 = \begin{array}{l|c|c|cc|cc|ccccc} (?,?,?,1) & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline (1,?,1) & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline (?,2) & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ (?,11) & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline (2,1) & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ (11,1) & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline (4) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ (31) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ (22) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ (211) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ (1111) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \; .$$

To dispel the impression that the entries $U_{\vec{\lambda},\vec{\mu}}$ are always 0 or 1, we note that

$$U_{(?,(1),(1)),((3),(1))} = \quad s_1[E_2]s_1[E_3], s_3[X_1]s_1[X_2] \quad = 2.$$

To indicate the importance of the decomposition matrix of U_k, we note in the following result that some of the entries of the matrix U_k correspond to the Schur expansion of certain symmetric function expressions involving plethysm. One objective of this research is to give a description of the decomposition matrix in order to provide an interpretation of these coefficients.

Corollary 5.11. *For $\mu \quad k$ and $\vec{\lambda} \quad I_k$, the multiplicity of the irreducible $\mathfrak{S}_k$-module $V^{\mu}_{\mathfrak{S}_k}$ in the restriction of the irreducible U_k-module $W^{\vec{\lambda}}_{\mathsf{U}_k}$ to $\mathfrak{S}_k$ is equal to*

$$\left\langle s_{\lambda^{(1)}}[s_1] s_{\lambda^{(2)}}[s_2] \cdots s_{\lambda^{(k)}}[s_k], \; s_\mu \right\rangle \; . \tag{33}$$

Proof. Since $G_{(1^k)} = \mathfrak{S}_k$, Proposition 5.1 states that the multiplicity of $V^{\mu}_{\mathfrak{S}_k}$ in $\mathrm{Res}^{\mathsf{U}_k}_{\mathfrak{S}_k} W^{\vec{\lambda}}_{\mathsf{U}_k}$ is

$$\left\langle \phi_{\mathsf{U}_k}(\chi^{\vec{\lambda}}_{\mathsf{U}_k}), \phi_{G_{(1^k)}}(\chi^{\mu}_{\mathfrak{S}_k}) \right\rangle = \left\langle \mathbf{s}_{\vec{\lambda}}[\mathbf{E}], s_\mu[X_1] \right\rangle . \tag{34}$$

Now $\mathbf{s}_{\vec{\lambda}}[\mathbf{E}]$ has symmetric functions involving the alphabets $X_2, X_3, \ldots, X_k$ while $s_\mu[X_1]$ does not. Thus, the value in Equation (34) is not changed if we set each of those alphabets equal to 0 in $\mathbf{s}_{\vec{\lambda}}[\mathbf{E}]$. From Equation (24), we know

$$E_r\big|_{X_2=X_3=\cdots=X_k=0} = s_r[X_1],$$

and therefore the right hand side of Equation (34) is equal to

$$\left\langle s_{\lambda^{(1)}}[s_1[X_1]] s_{\lambda^{(2)}}[s_2[X_1]] \cdots s_{\lambda^{(k)}}[s_k[X_1]],\ s_\mu[X_1] \right\rangle,$$

which is the same as Equation (33) upon dropping the reference to the variables X_1.

Acknowledgements. This project benefitted tremendously from the support by the American Institute for Mathematics (AIM) through their AIM SQuaRE program. We thank Laura Colmenarejo for discussions.

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