ALGEBRAIC COMBINATORICS

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Volume 6, issue 1 (2023), p. 165-195.
https://doi.org/10.5802/alco.245

Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

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Abstract The multiple flag variety $\mathfrak{X} = \mathrm{Gr}(\mathbb{C}^{p+q}, r) \times (\mathrm{Fl}(\mathbb{C}^p) \times \mathrm{Fl}(\mathbb{C}^q))$ can be considered as a double flag variety associated to the symmetric pair $(G, K) = (\mathrm{GL}_{p+q}(\mathbb{C}), \mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C}))$ of type AIII. We consider the diagonal action of K on $\mathfrak{X}$. There is a finite number of orbits for this action, and our first result is a description of these orbits: parametrization (by a certain set of graphs), dimensions, closure relations and cover relations.

In [5], we defined two generalized Steinberg maps from the K-orbits of $\mathfrak{X}$ to the nilpotent K-orbits in $\mathfrak{k}$ and those in the Cartan complement of $\mathfrak{k}$, respectively. The main result in the present paper is a complete, explicit description of these two Steinberg maps by means of a combinatorial algorithm which extends the classical Robinson–Schensted correspondence.

1. Introduction

1.1. A multiple flag variety and its orbital decomposition. In this paper, we consider the multiple flag variety

$$\mathfrak{X} = \mathrm{Gr}(V, r) \times \mathrm{Fl}(V^+) \times \mathrm{Fl}(V^-), \tag{1}$$

where

- $V = \mathbb{C}^{p+q}$ is equipped with a polar decomposition $V = V^+ \oplus V^-$ with $V^+ = \mathbb{C}^p \times \{0\}^q$ and $V^- = \{0\}^p \times \mathbb{C}^q$;
- $\mathrm{Gr}(V, r)$ denotes the Grassmann variety of r-dimensional subspaces of V;
- $\mathrm{Fl}(V^+)$ and $\mathrm{Fl}(V^-)$ denote the varieties of complete flags of V^+ and V^-, respectively.

Each factor of the variety $\mathfrak{X}$ has a natural action of

$$K \coloneqq \mathrm{GL}(V^+) \times \mathrm{GL}(V^-) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a \in \mathrm{GL}_p(\mathbb{C}),\ d \in \mathrm{GL}_q(\mathbb{C}) \right\} \subset \mathrm{GL}(V),$$

and $\mathfrak{X}$ is endowed with the resulting diagonal action of K.

The multiple flag variety $\mathfrak{X}$ can be written in the form

$$\mathfrak{X} = G/P \times K/B_K,$$

where $G = \mathrm{GL}(V)$, $P \subset G$ is a maximal parabolic subgroup, $B_K \subset K$ is a Borel subgroup. In this way, $\mathfrak{X}$ is a *double flag variety associated to the symmetric pair*

Manuscript received 14th October 2021, revised 19th December 2021, accepted 1st May 2022.

Keywords. double flag variety, conormal bundle, exotic moment map, nilpotent orbits, partial permutations, Robinson–Schensted correspondence, Steinberg variety.

Acknowledgements. The second author is supported by JSPS KAKENHI Grant No. #21K03184.

(G, K) in the sense of [10] and [8]. In particular, it is known from [10] that the above variety $\mathfrak{X}$ has a finite number of K-orbits.

In [3, 5], we have initiated an analogue of Steinberg theory for double flag varieties associated to symmetric pairs such as $\mathfrak{X}$. Specifically, we have defined two Steinberg maps, from the set of K-orbits of $\mathfrak{X}$ to the sets of nilpotent K-orbits of $\mathfrak{k} := \mathrm{Lie}(K)$ and of its Cartan complement $\mathfrak{s}$, respectively.

For general symmetric pairs, the calculation of the generalized Steinberg maps appears to be quite difficult. In [5], we have considered the variety $\mathfrak{X}$ of (1) in the special case where $p = q = r$ and we have computed the Steinberg maps on a special subset of K-orbits parametrized by partial permutations. The present paper deals with the variety $\mathfrak{X}$ of (1) and its K-orbits in full generality.

Here we summarize the main results achieved in this paper:

- We describe completely the decomposition of $\mathfrak{X}$ into K-orbits: we give a parametrization of the orbits (in terms of certain graphs), we provide a dimension formula, and describe the closure relations and the cover relations; see Section 2.2.
- Our main result (Theorem 2.5) is the calculation of the two Steinberg maps mentioned above. This calculation is concrete, and it is done by means of a sophisticated combinatorial algorithm that generalizes the Robinson–Schensted correspondence; see Sections 2.3–2.4.

In the following subsection, we explain the construction of the generalized Steinberg maps and we give more insight on our main result.

1.2. Conormal variety and Steinberg maps. We consider the Lie algebras

$$\mathfrak{g} = \mathfrak{gl}_{p+q}(\mathbb{C}) \supset \mathfrak{k} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a \in \mathfrak{gl}_p(\mathbb{C}),\ d \in \mathfrak{gl}_q(\mathbb{C}) \right\}$$

and a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \qquad \text{where} \qquad \mathfrak{s} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : b \in \mathrm{M}_{p,q}(\mathbb{C}),\ c \in \mathrm{M}_{q,p}(\mathbb{C}) \right\}.$$

We write $x = x_{\mathfrak{k}} + x_{\mathfrak{s}}$ with $(x_{\mathfrak{k}}, x_{\mathfrak{s}}) \in \mathfrak{k} \times \mathfrak{s}$ for the decomposition of an element $x \in \mathfrak{g}$ along the Cartan decomposition. Moreover, we identify the Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ with their duals $\mathfrak{g}^*$ and $\mathfrak{k}^*$ through the trace form.

Any partial flag $\mathcal{F} = (F_0 = 0 \subset F_1 \subset \ldots \subset F_k = V')$ of a vector space V' gives rise to a parabolic subalgebra $\mathfrak{p}(\mathcal{F})$ of $\mathfrak{gl}(V')$ and the corresponding nilradical $\mathfrak{nil}(\mathcal{F})$ defined by

$$\begin{aligned} \mathfrak{p}(\mathcal{F}) &= \mathrm{Stab}_{\mathfrak{gl}(V')}(\mathcal{F}) = \{x \in \mathfrak{gl}(V') : x(F_i) \subset F_i \quad \forall i = 1, \ldots, k\}, \\ \mathfrak{nil}(\mathcal{F}) &= \{x \in \mathfrak{gl}(V') : x(F_i) \subset F_{i-1} \quad \forall i = 1, \ldots, k\}. \end{aligned}$$

For a subspace $W \subset V'$, we denote by $\mathfrak{p}(W)$ and $\mathfrak{nil}(W)$ the (maximal) parabolic subalgebra and the nilradical associated to the partial flag $(0 \subset W \subset V')$.

As explained in [5, §3], the cotangent bundle $T^*\mathfrak{X}$ inherits a Hamiltonian action of K, which gives rise to a moment map $\mu_{\mathfrak{X}} : T^*\mathfrak{X} \to \mathfrak{k}^* = \mathfrak{k}$. The nullfiber $\mathcal{Y} = \mu_{\mathfrak{X}}^{-1}(0)$ is called a *conormal variety.* It can be described explicitly as

$$\mathcal{Y} = \{(W, \mathcal{F}^+, \mathcal{F}^-, x) \in \mathfrak{X} \times \mathfrak{gl}(V) : x \in \mathfrak{nil}(W),\ x_{\mathfrak{k}} \in \mathfrak{nil}(\mathcal{F}^+) \times \mathfrak{nil}(\mathcal{F}^-)\}.$$

Every K-orbit $\mathbb{O} \subset \mathfrak{X}$ yields a conormal bundle $T^*_{\mathbb{O}}\mathfrak{X}$ that can be realized as a (locally-closed) subvariety of $\mathcal{Y}$ given by

$$T^*_{\mathbb{O}}\mathfrak{X} = \{(W, \mathcal{F}^+, \mathcal{F}^-, x) \in \mathcal{Y} : (W, \mathcal{F}^+, \mathcal{F}^-) \in \mathbb{O}\}.$$

The variety $\mathcal{Y}$ is equidimensional of dimension $\dim \mathfrak{X}$, and its irreducible components are precisely the closures of the various conormal bundles $T^*_{\mathbb{O}}\mathfrak{X}$, since the set of orbits

$\mathfrak{X}/K$ is finite. One can find a more comprehensive introduction to the theory of conormal varieties in [5, §3] (see also [1]).

The conormal variety $\mathcal{Y}$ is equipped with two K-equivariant projections to $\mathfrak{k}$ and $\mathfrak{s}$, namely

$$\phi_{\mathfrak{k}} : \mathcal{Y} \to \mathfrak{k},\ (W, \mathcal{F}^+, \mathcal{F}^-, x) \mapsto x_{\mathfrak{k}} \quad \text{and} \quad \phi_{\mathfrak{s}} : \mathcal{Y} \to \mathfrak{s},\ (W, \mathcal{F}^+, \mathcal{F}^-, x) \mapsto x_{\mathfrak{s}}.$$

It immediately follows from the description of the conormal variety that the image of $\phi_{\mathfrak{k}}$ is contained in the cone of nilpotent elements $\mathcal{N}_{\mathfrak{k}} \subset \mathfrak{k}$. It is shown in [3, Proposition 4.2] that (for the variety $\mathfrak{X}$ considered in this paper) the image of $\phi_{\mathfrak{s}}$ is also contained in the nilpotent cone $\mathcal{N}_{\mathfrak{s}} \subset \mathfrak{s}$. It is known that both nilpotent cones $\mathcal{N}_{\mathfrak{k}}$ and $\mathcal{N}_{\mathfrak{s}}$ consist of finitely many adjoint K-orbits

Therefore, we can define two maps

$$\Phi_{\mathfrak{k}} : \mathfrak{X}/K \to \mathcal{N}_{\mathfrak{k}}/K \quad \text{and} \quad \Phi_{\mathfrak{s}} : \mathfrak{X}/K \to \mathcal{N}_{\mathfrak{s}}/K$$

in the following way: for every orbit $\mathbb{O} \in \mathfrak{X}/K$, define $\Phi_{\mathfrak{k}}(\mathbb{O}) \in \mathcal{N}_{\mathfrak{k}}/K$, resp. $\Phi_{\mathfrak{s}}(\mathbb{O}) \in \mathcal{N}_{\mathfrak{s}}/K$, as the unique nilpotent K-orbit which is open and dense in the image of the conormal bundle $T^*_{\mathbb{O}}\mathfrak{X}$ by the projection map $\phi_{\mathfrak{k}}$, resp. $\phi_{\mathfrak{s}}$. According to the terminology introduced in [5], we will refer to $\Phi_{\mathfrak{k}}$ as the *symmetrized Steinberg map* and to $\Phi_{\mathfrak{s}}$ as the *exotic Steinberg map*.

In Section 2.3, we describe the maps $\Phi_{\mathfrak{k}}$ and $\Phi_{\mathfrak{s}}$. In [5], in the special case where $p = q = r$, the images $\Phi_{\mathfrak{k}}(\mathbb{O})$ and $\Phi_{\mathfrak{s}}(\mathbb{O})$ are determined when an orbit $\mathbb{O}$ is contained in a "big cell" of $\mathfrak{X}/K$. Moreover, in Section 2.4, we describe the fibers of $\Phi_{\mathfrak{k}}$ by means of a combinatorial procedure that extends the classical Robinson–Schensted correspondence; this also generalizes [5, Theorem 7.8]. In this way, the results in the present paper are new and complement those in [5], as now we have a full description of the two Steinberg maps and a better understanding of them at the same time. Note that the results given in Section 2.3 regarding $\Phi_{\mathfrak{k}}$ and for $p = q = r$ were already announced in [4, §2] without proofs.

2. Main results

2.1. Combinatorial notation on pairs of partial permutations. By $\mathfrak{T}_{p,r}$ we denote the set of $p \times r$ matrices whose coefficients are 0 or 1, with at most one 1 in each row and each column. (If $p = r$, we recover the set of partial permutation matrices considered in [5].) By $\mathfrak{T} = \mathfrak{T}_{(p,q),r}$ we denote the set of $(p+q) \times r$ matrices of rank r (we have $r \leqslant p+q$) of the form

$$\omega = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix},$$

where $\tau_1 \in \mathfrak{T}_{p,r}$ and $\tau_2 \in \mathfrak{T}_{q,r}$. Note that the symmetric group $\mathfrak{S}_r$ acts on $\mathfrak{T}$ by right multiplication, and we denote the quotient set by $\overline{\mathfrak{T}} = \mathfrak{T}/\mathfrak{S}_r$.

In Section 2.2*, we will show that the elements of* $\overline{\mathfrak{T}}$ *parameterize the* K*-orbits of* $\mathfrak{X}$*.*

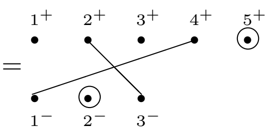
a) Graphic representation of a pair of partial permutations:

We represent any element $\omega \in \overline{\mathfrak{T}}$ by a graph $\mathcal{G}(\omega)$ obtained as follows:

- The set of vertices consists of p "positive" vertices $1^+, \ldots, p^+$ and q "negative" vertices $1^-, \ldots, q^-$, displayed along two horizontal lines.
- Put an edge between i^+ and j^- for every column of ω that contains exactly two 1's, in positions i (within the block τ_1) and $p+j$ (within the block τ_2).
- Put a mark at the vertex i^+, respectively j^-, for every column of ω that contains exactly one 1, in position i (within τ_1), respectively $p+j$ (within τ_2).

For instance, for $(p,q)=(5,3)$ and $r=4$,

(2) $$\omega = \begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ \hline 0&1&0&0\\ 0&0&0&1\\ 1&0&0&0 \end{pmatrix} \qquad \Rightarrow \qquad \mathcal{G}(\omega) =$$



.

We will say that a vertex is *free* whenever it is not a marked point nor an end point of an edge (like 1^+ and 3^+ in the above example).

In general, the assignment $\omega \mapsto \mathcal{G}(\omega)$ establishes a bijection between $\overline{\mathfrak{T}}$ and the set of graphs with vertices $\{1^+,\ldots,p^+\}\cup\{1^-,\ldots,q^-\}$, exactly r edges or marked vertices, where every vertex is incident with at most one edge, and such that there is no edge which is incident with a marked vertex or joins two vertices of the same sign.

b) Numerical invariants for the graph:

The following data associated to an element $\omega\in\overline{\mathfrak{T}}$ and its graphic representation $\mathcal{G}(\omega)$ will play a role in the statement of our main results.

- Set the *degree* of a vertex of $\mathcal{G}(\omega)$ as 0, 1, or 2, depending on whether this vertex is free, incident with an edge, or marked.

  We define $a^+(\omega)$, respectively $a^-(\omega)$, as the number of pairs of positive vertices (i^+,j^+) with $i<j$ and $\deg(i^+)<\deg(j^+)$, respectively pairs of negative vertices (i^-,j^-) with $i<j$ and $\deg(i^-)<\deg(j^-)$.

  Let $b(\omega)$ be the number of edges of $\mathcal{G}(\omega)$.

  Finally, let $c(\omega)$ be the number of *crossings*, i.e. pairs of edges (i^+,j^-), (k^+,ℓ^-) such that $i<k$ and $j>\ell$.
- For all $(i,j)\in\{0,1,\ldots,p\}\times\{0,1,\ldots,q\}$, let $r_{i,j}(\omega)$ be the number of edges and marks contained in the subgraph of $\mathcal{G}(\omega)$ formed by the vertices k^+ $(1\leqslant k\leqslant i)$, ℓ^- $(1\leqslant \ell\leqslant j)$ and the edges/marks contained within this set of vertices. Let $R(\omega)=(r_{i,j}(\omega))_{0\leqslant i\leqslant p,\,0\leqslant j\leqslant q}$ be the $(p+1)\times(q+1)$ matrix containing these numbers.

  In particular, $r_{i,0}(\omega)$ (resp. $r_{0,j}(\omega)$) is the number of marked vertices among $\{1^+,\ldots,i^+\}$ (resp. $\{1^-,\ldots,j^-\}$).

In Section 2.2, the numbers $a^\pm(\omega)$, $b(\omega)$, $c(\omega)$ appear in the dimension formula for the K-orbits of $\mathfrak{X}$, while the matrices $R(\omega)$ are used to describe the inclusion relations between orbit closures.

- We decompose $\{1,\ldots,p\}=I\sqcup L\sqcup L'$ in the following way: I, resp. L, resp. L', denotes the set of elements $i\in\{1,\ldots,p\}$ such that i^+ is a vertex of $\mathcal{G}(\omega)$ of degree 1, resp. 2, resp. 0.

  We decompose $\{1,\ldots,q\}=J\sqcup M\sqcup M'$ in the same way: J, resp. M, resp. M', consists of the elements j such that j^- has degree 1, resp. 2, resp. 0.

  Let $\sigma: J\to I$ be the bijection defined by letting $\sigma(j)=i$ if (i^+,j^-) is an edge in $\mathcal{G}(\omega)$.

Note that ω is characterized by the subsets I, L, L', J, M, M' and the bijection $\sigma: J\to I$. *In Section 2.3, these data are used to compute the symmetrized and exotic Steinberg maps, by means of a combinatorial algorithm.*

Note also that we have $b(\omega)=\#I=\#J$, and $c(\omega)$ is the number of inversions of σ.

Example 2.1. Let ω be as in (2). Then,

$$a^+(\omega)=7,\quad a^-(\omega)=1,\quad b(\omega)=2,\quad c(\omega)=1,\quad R(\omega)=\begin{pmatrix}0&0&1&1\\0&0&1&1\\0&0&1&2\\0&0&1&2\\0&1&2&3\\1&2&3&4\end{pmatrix}$$

$$I=\{2,4\},\ L=\{5\},\ L'=\{1,3\},$$

$$J=\{1,3\},\ M=\{2\},\ M'=\varnothing,\quad \sigma=\begin{pmatrix}1&3\\4&2\end{pmatrix}\in \mathrm{Bij}(J,I).$$

Note that the matrix $R(\omega)$ can also be viewed as a plane partition.

2.2. Orbit decomposition of the multiple flag variety $\mathfrak{X}$. Recall that we consider the space $V=\mathbb{C}^{p+q}$ endowed with the polar decomposition

$$V=V^+\oplus V^- \qquad \text{where} \qquad V^+=\mathbb{C}^p\times\{0\}^q \quad \text{and} \quad V^-=\{0\}^p\times\mathbb{C}^q.$$

Let

$$\mathcal{F}_0^+=(\mathbb{C}^i\times\{0\}^{p-i}\times\{0\}^q)_{i=0}^p \quad \text{and} \quad \mathcal{F}_0^-=(\{0\}^p\times\mathbb{C}^j\times\{0\}^{q-j})_{j=0}^q$$

be the standard complete flags of V^+ and V^-.

Every $(p+q)\times r$ matrix ω determines a subspace $[\omega]\coloneqq\operatorname{Im}\omega\subset V$, which remains the same up to permutation of the columns of ω. In particular, every $\omega\in\overline{\mathfrak{T}}$ determines a point $[\omega]$ in $\mathrm{Gr}(V,r)$, and thus a point $([\omega],\mathcal{F}_0^+,\mathcal{F}_0^-)$ in $\mathfrak{X}=\mathrm{Gr}(V,r)\times\mathrm{Fl}(V^+)\times\mathrm{Fl}(V^-)$.

Theorem 2.2. (1) *Every K-orbit in $\mathfrak{X}$ is of the form $\mathbb{O}_\omega\coloneqq K\cdot([\omega],\mathcal{F}_0^+,\mathcal{F}_0^-)$ for a unique element $\omega\in\overline{\mathfrak{T}}=\mathfrak{T}_{(p,q),r}/\mathfrak{S}_r$.*

(2) $\dim\mathbb{O}_\omega=\frac{p(p-1)}{2}+\frac{q(q-1)}{2}+a^+(\omega)+a^-(\omega)+\frac{b(\omega)(b(\omega)+1)}{2}+c(\omega)$.

(3) *$\mathbb{O}_\omega$ is the set of triples $(W,\mathcal{F}^+=(F_i^+)_{i=0}^p,\mathcal{F}^-=(F_j^-)_{j=0}^q)\in\mathfrak{X}$ satisfying the condition*

$$\dim W\cap(F_i^++F_j^-)=r_{i,j}(\omega) \quad \textit{for all } (i,j)\in\{0,\dots,p\}\times\{0,\dots,q\}.$$

(4) *$\overline{\mathbb{O}_\omega}\subset\overline{\mathbb{O}_{\omega'}}$ if and only if $r_{i,j}(\omega)\geqslant r_{i,j}(\omega')$ for all $(i,j)\in\{0,\dots,p\}\times\{0,\dots,q\}$.*

As a complement of this result, we determine the cover relations in the poset $(\{\overline{\mathbb{O}_\omega}\},\subset)$. We say that $\mathbb{O}_{\omega'}$ covers $\mathbb{O}_\omega$ if $\overline{\mathbb{O}_{\omega'}}$ strictly contains $\overline{\mathbb{O}_\omega}$ and is minimal (among the orbit closures) for this property. Equivalently, this means that $\overline{\mathbb{O}_\omega}$ is an irreducible component of the boundary $\partial\mathbb{O}_{\omega'}=\overline{\mathbb{O}_{\omega'}}\setminus\mathbb{O}_{\omega'}$.

Theorem 2.3. *The following conditions are equivalent:*

(1) *$\mathbb{O}_{\omega'}$ covers $\mathbb{O}_\omega$;*

(2) *$\dim\mathbb{O}_{\omega'}=\dim\mathbb{O}_\omega+1$ and (the graph of) ω is obtained from (the graph of) ω' by modifying the pattern of at most four vertices a^+,b^+,c^-,d^- ($a<b$, $c<d$), according to one of the cases indicated in Figure 1.*

As a consequence, the boundary of every non-closed orbit is equidimensional of codimension one.

It follows from Theorem 2.3 that the boundary $\partial\mathbb{O}_\omega\coloneqq\overline{\mathbb{O}_\omega}\setminus\mathbb{O}_\omega$ of every non-closed orbit is equidimensional of codimension one in $\overline{\mathbb{O}_\omega}$. This boundary is in general not irreducible as it already appears in Example 2.4 (a).

Example 2.4. (a) In Figure 2, we represent the elements $\omega\in\overline{\mathfrak{T}}$ (under the form of their graphic incarnations $\mathcal{G}(\omega)$) in the case where $p=q=r=2$. We indicate the dimensions of the corresponding K-orbits $\mathbb{O}_\omega$. An edge joining two parameters indicates a cover relation.

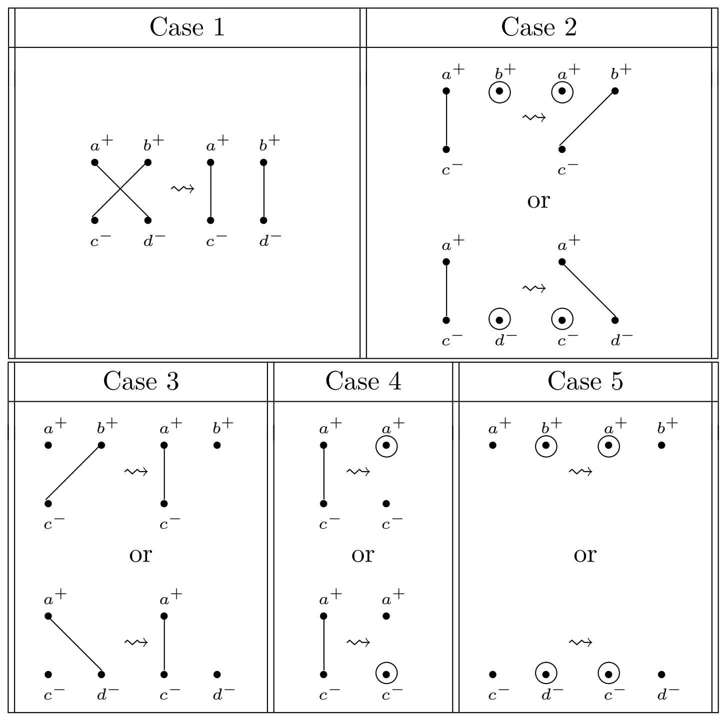


Figure 1. Elementary moves yielding cover relations in the poset $(\{\overline{\mathbb{O}_\omega}\}, \subset)$.

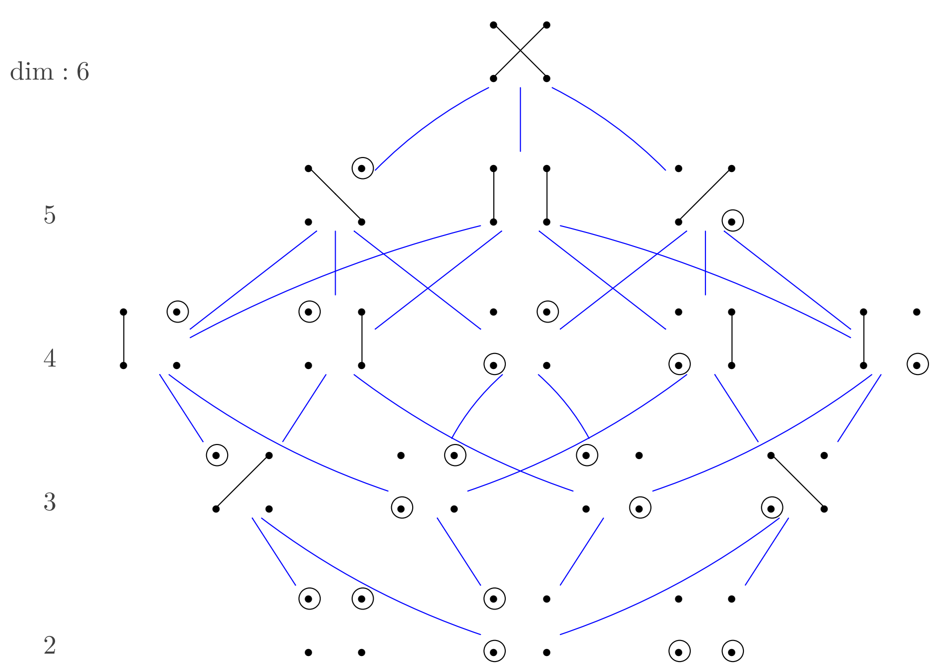
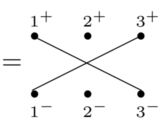
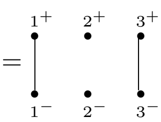


Figure 2. The parameters of the K-orbits of $\mathfrak{X}$ and the cover relations for $p = q = r = 2$.

(b) In Figure 1, the vertices a^+, b^+ or c^-, d^- involved in an elementary move that yields a cover relation are not necessarily consecutive. For example, $\mathbb{O}_{\omega'}$ covers $\mathbb{O}_\omega$ if

$\mathcal{G}(\omega') =$ 

$\mathcal{G}(\omega) =$ 

This corresponds to Case 1 of Figure 1 with $a = c = 1$, $b = d = 3$. Note however that in Case 5 of Figure 1, the vertices a^+, b^+ (resp. c^-, d^-) must be consecutive for having a cover relation.

The proofs of Theorems 2.2 and 2.3 are given in Section 3.

One ingredient for showing the parametrization of the orbits in Theorem 2.2(1) is that the orbits of a pair of Borel subgroups $B_p^+ \times B_r^+ \subset \mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_r(\mathbb{C})$ on the space of $p \times r$ matrices are parametrized by partial permutations (Lemma 3.4). This classification of orbits is also shown in [6], where dimension formulas, closure relations, and properties of closures of orbits are described.

2.3. Description of symmetrized and exotic Steinberg maps. We turn our attention to the maps $\Phi_\mathfrak{k} : \mathfrak{X}/K \to \mathcal{N}_\mathfrak{k}/K$ and $\Phi_\mathfrak{s} : \mathfrak{X}/K \to \mathcal{N}_\mathfrak{s}/K$ defined in Section 1.2. The three orbit sets arising here can be parametrized combinatorially.

- $\mathfrak{X}/K = \{\mathbb{O}_\omega : \omega \in \overline{\mathfrak{T}}\}$ (see Section 2.2).

For the other two orbit sets, the parametrization is well known (see, e.g. [2]):

- $\mathcal{N}_\mathfrak{k}$ is the nilpotent cone of the Lie algebra
$$\mathfrak{k} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a \in \mathfrak{gl}_p(\mathbb{C}),\ d \in \mathfrak{gl}_q(\mathbb{C}) \right\} \cong \mathfrak{gl}_p(\mathbb{C}) \times \mathfrak{gl}_q(\mathbb{C}),$$
and its adjoint K-orbits $\mathfrak{O}_{(\lambda,\mu)}$ are parametrized by pairs of partitions $\lambda \vdash p$ and $\mu \vdash q$ (viewed as Young diagrams) through Jordan normal form. Specifically, the number of boxes in the first k columns of λ (resp. μ) indicates the dimension of $\ker a^k$ (resp. $\ker d^k$).
- $\mathcal{N}_\mathfrak{s}$ is the nilpotent cone of
$$\mathfrak{s} = \left\{ x = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : b \in \mathrm{M}_{p,q}(\mathbb{C}),\ c \in \mathrm{M}_{q,p}(\mathbb{C}) \right\},$$
and its adjoint K-orbits $\mathfrak{O}_\Lambda$ are parametrized by signed Young diagrams Λ of signature (p, q). Specifically, the number of $+$'s (resp. $-$'s) in the first k columns of Λ indicates the dimension of $V^+ \cap \ker x^k$ (resp. $V^- \cap \ker x^k$) for $x \in \mathfrak{O}_\Lambda$.

We give a combinatorial algorithm which describes $\Phi_\mathfrak{k}$ and $\Phi_\mathfrak{s}$ completely. If $w : S \to R$ is a bijection between two sets of integers, let $(\mathrm{RS}_1(w), \mathrm{RS}_2(w))$ denote the pair of Young tableaux associated to w via the Robinson–Schensted correspondence, so that the set of entries of $\mathrm{RS}_1(w)$ (resp. $\mathrm{RS}_2(w)$) is R (resp. S) (see, e.g. [7]).

Let $\omega \in \overline{\mathfrak{T}}$, and let $I, L, L', J, M, M', \sigma$ be the corresponding data in the sense of Section 2.1. Thus we have partitions $I \sqcup L \sqcup L' = \{1, \ldots, p\}$, $J \sqcup M \sqcup M' = \{1, \ldots, q\}$, and $\sigma : J \to I$ is a bijection. We write $I = \{i_1 < \ldots < i_k\}$, $J = \{j_1 < \ldots < j_k\}$, $L = \{\ell_1 < \ldots < \ell_s\}$, $L' = \{\ell'_1 < \ldots < \ell'_{s'}\}$, $M = \{m_1 < \ldots < m_t\}$, $M' = \{m'_1 < \ldots < m'_{t'}\}$, and we consider the following permutations

$$w_{\mathfrak{k},+} = \begin{pmatrix} 1 & \cdots & s & s+1 & \cdots & s+k & s+k+1 & \cdots & p \\ \ell_s & \cdots & \ell_1 & \sigma(j_1) & \cdots & \sigma(j_k) & \ell'_{s'} & \cdots & \ell'_1 \end{pmatrix} \in \mathfrak{S}_p, \tag{3}$$

$$w_{\mathfrak{k},-} = \begin{pmatrix} 1 & \cdots & t & t+1 & \cdots & t+k & t+k+1 & \cdots & q \\ m_t & \cdots & m_1 & \sigma^{-1}(i_1) & \cdots & \sigma^{-1}(i_k) & m'_{t'} & \cdots & m'_1 \end{pmatrix} \in \mathfrak{S}_q, \tag{4}$$

and the bijections

$$w_{\mathfrak{s},+} = \begin{pmatrix} m_1 \cdots m_t & j_1 & \cdots & j_k & q+1 \cdots q+s' \\ -1 \cdots -t & \sigma(j_1) & \cdots & \sigma(j_k) & \ell'_{s'} \quad \cdots \quad \ell'_1 \end{pmatrix}, \tag{5}$$

which maps $J \cup M \cup \{q+1,\ldots,q+s'\}$ to $\{-t,\ldots,-1\} \cup I \cup L'$, and

$$w_{\mathfrak{s},-} = \begin{pmatrix} \ell_1 \cdots \ell_s & i_1 & \cdots & i_k & p+1 \cdots p+t' \\ -1 \cdots -s & \sigma^{-1}(i_1) & \cdots & \sigma^{-1}(i_k) & m'_{t'} \quad \cdots \quad m'_1 \end{pmatrix}, \tag{6}$$

which maps $I \cup L \cup \{p+1,\ldots,p+t'\}$ to $\{-s,\ldots,-1\} \cup J \cup M'$.

In the next theorem, $\#\lambda_{\leqslant c}$ denotes the number of boxes in the first c columns of a Young diagram λ, and $\#\Lambda_{\leqslant c}(+)$ (resp. $\#\Lambda_{\leqslant c}(-)$) denotes the number of $+$'s (resp. $-$'s) in the first c columns of a signed Young diagram Λ.

Theorem 2.5. *Let $\omega \in \overline{\mathfrak{T}}$, and consider the above notation.*

1. *The image of $\mathbb{O}_\omega$ by the symmetrized Steinberg map is $\Phi_{\mathfrak{k}}(\mathbb{O}_\omega) = \mathfrak{O}_{\lambda,\mu}$ where (λ,μ) is the pair of Young diagrams given by*
$$(\lambda,\mu) = \big(\mathrm{shape}(\mathrm{RS}_1(w_{\mathfrak{k},+})), \mathrm{shape}(\mathrm{RS}_1(w_{\mathfrak{k},-}))\big).$$
2. *The image of $\mathbb{O}_\omega$ by the exotic Steinberg map is $\Phi_{\mathfrak{s}}(\mathbb{O}_\omega) = \mathfrak{O}_\Lambda$ where Λ is the signed Young diagram determined as follows:*
   (a) *For every $c \geqslant 1$ even,*
   $$\#\Lambda_{\leqslant c}(+) = \#\lambda_{\leqslant c} \quad \textit{and} \quad \#\Lambda_{\leqslant c}(-) = \#\mu_{\leqslant c},$$
   where (λ,μ) is the pair of Young diagrams given in part (1).
   (b) *For every $c \geqslant 1$ odd,*
   $$\#\Lambda_{\leqslant c}(+) = s - t + \#\lambda'_{\leqslant c} \quad \textit{and} \quad \#\Lambda_{\leqslant c}(-) = t - s + \#\mu'_{\leqslant c},$$
   where (λ',μ') is the pair of Young diagrams given by
   $$(\lambda',\mu') = \big(\mathrm{shape}(\mathrm{RS}_1(w_{\mathfrak{s},+})), \mathrm{shape}(\mathrm{RS}_1(w_{\mathfrak{s},-}))\big).$$

We prove this theorem in Section 4.

Example 2.6. (a) For ω as in Example 2.1, we have $s = t = 1$,

$$w_{\mathfrak{k},+} = \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \\ 5\ 4\ 2\ 3\ 1 \end{pmatrix}, \qquad w_{\mathfrak{k},-} = \begin{pmatrix} 1\ 2\ 3 \\ 2\ 3\ 1 \end{pmatrix},$$
$$w_{\mathfrak{s},+} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & -1 & 2 & 3 & 1 \end{pmatrix}, \quad \text{and} \quad w_{\mathfrak{s},-} = \begin{pmatrix} 2 & 4 & 5 \\ 3 & 1 & -1 \end{pmatrix},$$

hence we get

$$\lambda = \ydiagram{2,1,1,1}, \quad \mu = \ydiagram{2,1}, \quad \lambda' = \ydiagram{3,1}, \quad \mu' = \ydiagram{1,1}, \quad \text{and} \quad \Lambda = \begin{ytableau} - & + \\ - & + \\ + \\ + \\ + \\ - \end{ytableau}.$$

(b) In Figure 3, we calculate the pair of Young diagrams (λ,μ) and the signed Young diagram Λ such that $\Phi_{\mathfrak{k}}(\mathbb{O}_\omega) = \mathfrak{O}_{\lambda,\mu}$ and $\Phi_{\mathfrak{s}}(\mathbb{O}_\omega) = \mathfrak{O}_\Lambda$ for all $\omega \in \overline{\mathfrak{T}}$, for $p = q = r = 2$ (the same case as in Example 2.4).

Remark 2.7. Note that the pair of bijections $(w_{\mathfrak{s},+}, w_{\mathfrak{s},-})$ of (5)–(6) determines the original element $\omega \in \overline{\mathfrak{T}}$, as the data (I,J,L,L',M,M',σ) can be recovered from this

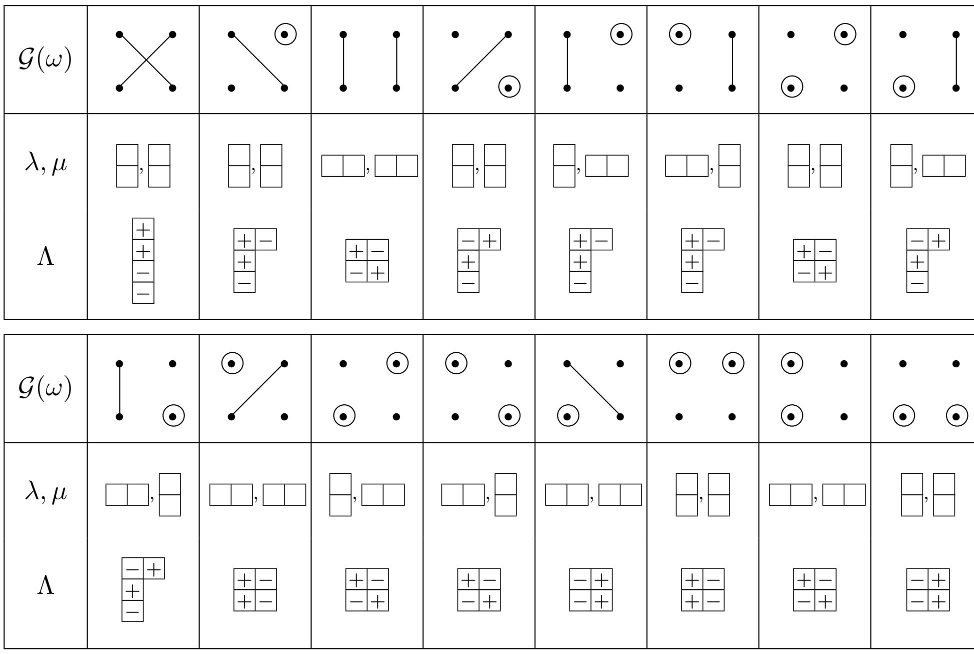
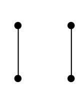
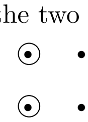


FIGURE 3. Calculation of $\Phi_{\mathfrak{k}}(\mathbb{O}_\omega) = \mathfrak{O}_{\lambda,\mu}$ and $\Phi_{\mathfrak{s}}(\mathbb{O}_\omega) = \mathfrak{O}_\Lambda$ for $p = q = r = 2$.

pair. On the contrary, the pair $(w_{\mathfrak{k},+}, w_{\mathfrak{k},-})$ of (3)–(4) does not determine ω. For instance, for the elements ω corresponding to the two graphs

 and 

we get the same pair of permutations $(w_{\mathfrak{k},+}, w_{\mathfrak{k},-}) = (\mathrm{id}_{\{1,2\}}, \mathrm{id}_{\{1,2\}})$.

The tableaux $\mathrm{RS}_1(w_{\mathfrak{k},+})$ and $\mathrm{RS}_1(w_{\mathfrak{k},-})$ involved in Theorem 2.5 can also be obtained as the result of the following combinatorial algorithms. We need more notation:

- If T, S are Young tableaux with disjoint sets of entries, we denote by $T * S$ the rectification by jeu de taquin of the skew tableau obtained by displaying S on the top right corner of T. For example,

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 6 & \\ \cline{1-1} \end{array} * \begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 7 & & \\ \cline{1-1} \end{array} = \mathrm{Rect}\left(\begin{array}{ccccc} & & 2 & 4 & 5 \\ & & 7 & & \\ 1 & 3 & & & \\ 6 & & & & \end{array}\right) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 7 & & \\ \cline{1-2} 6 & & & \\ \cline{1-1} \end{array}.$$

  If U is a third tableau whose entries do not appear in T nor S, the properties of jeu de taquin imply that $(T * S) * U = T * (S * U)$ (see [7]), hence the notation $T * S * U$ is unambiguous.
- Let $[L], [L'], [M], [M']$ denote the vertical Young tableaux whose entries are the elements in L, L', M, M', respectively.

We then have:

$$\mathrm{RS}_1(w_{\mathfrak{k},+}) = [L] * \mathrm{RS}_1(\sigma) * [L'] \tag{7}$$

and

$$\mathrm{RS}_1(w_{\mathfrak{k},-}) = [M] * \mathrm{RS}_1(\sigma^{-1}) * [M'] = [M] * \mathrm{RS}_2(\sigma) * [M']. \tag{8}$$

Remark 2.8. Assume that $p = q = r = n$ and $L = M' = \varnothing$, thus $s' = t = n - k$. This special case is the one considered in [5, §9–10] (except that the set L in the notation of [5, §9–10] corresponds to the set L' in the notation of the present paper). In this case:

(1) The tableaux $\mathrm{RS}_1(w_{\mathfrak{k},+})$ and $\mathrm{RS}_1(w_{\mathfrak{k},-})$ coincide with the tableaux $\mathrm{RS}_1(\sigma) * [L']$ and $[M] * \mathrm{RS}_2(\sigma)$ involved in [5, Theorems 7.4, 9.1, and 10.4 (1)].
(2) The skew tableau obtained from $\mathrm{RS}_1(w_{\mathfrak{s},+})$ by deleting the boxes with negative entries coincides with the skew tableau $[M] * \mathrm{RS}_2(\sigma) \triangle \mathrm{RS}_1(\sigma) * [L']$ involved in [5, Theorem 10.4 (2)]. This follows from [5, Lemma 10.9].
(3) We have just $w_{\mathfrak{s},-} = \sigma^{-1}$, hence $\mathrm{RS}_1(w_{\mathfrak{s},-}) = \mathrm{RS}_2(\sigma)$, which is the tableau involved in [5, Theorem 10.4 (3)].

Thus, Theorem 2.5 recovers the results stated in [5, Theorems 9.1 and 10.4].

2.4. An extension of the Robinson–Schensted correspondence. As pointed out in Remark 2.7, the pair of permutations $(w_{\mathfrak{k},+}, w_{\mathfrak{k},-})$ of (3)–(4), involved in the calculation of the symmetrized Steinberg map image $\Phi_{\mathfrak{k}}(\omega)$, does not fully determine the element $\omega \in \overline{\mathfrak{T}}$. A fortiori the map $\Phi_{\mathfrak{k}}$ itself is far from being injective.

In fact, we can determine the fibers of $\Phi_{\mathfrak{k}}$ in terms of a combinatorial correspondence which extends the Robinson–Schensted correspondence. The following theorem also generalizes [5, Theorem 7.6]. We use the previous notation. In addition, we write $\lambda' \mathrel{\subset\!\!\!\cdot\,} \lambda$ whenever λ' is a Young subdiagram of λ such that the skew diagram $\lambda \setminus \lambda'$ is *column strip* (i.e. it contains at most one box in each row). Hereafter, $\mathcal{P}(n)$ denotes the set of partitions $\lambda \vdash n$, also seen as Young diagrams of size $|\lambda| = n$.

Theorem 2.9. *There is an explicit bijection*

$$\mathrm{gRS} : \overline{\mathfrak{T}} \xrightarrow{\sim} \mathcal{T} := \bigsqcup_{(\lambda,\mu) \in \mathcal{P}(p) \times \mathcal{P}(q)} \mathcal{T}_{\lambda,\mu}$$

where $\mathcal{T}_{\lambda,\mu}$ is the set of 5-tuples $(T_1, T_2; \lambda', \mu'; \nu)$ satisfying

($\star$) *T_1 and T_2 are standard Young tableaux of shapes λ and μ, respectively;*
($\star\star$) *$\nu \mathrel{\subset\!\!\!\cdot\,} \lambda' \mathrel{\subset\!\!\!\cdot\,} \lambda$, $\nu \mathrel{\subset\!\!\!\cdot\,} \mu' \mathrel{\subset\!\!\!\cdot\,} \mu$, and $|\lambda'| + |\mu'| = |\nu| + r$.*

Specifically, to the element $\omega \in \overline{\mathfrak{T}}$, we associate the 5-tuple

$$\begin{aligned} \mathrm{gRS}(\omega) = (T_1, T_2; \lambda', \mu'; \nu) := \big(& [L] * \mathrm{RS}_1(\sigma) * [L'], [M] * \mathrm{RS}_2(\sigma) * [M']; \\ & \mathrm{shape}([L] * \mathrm{RS}_1(\sigma)), \mathrm{shape}([M] * \mathrm{RS}_2(\sigma)); \\ & \mathrm{shape}(\mathrm{RS}_1(\sigma))\big). \end{aligned}$$

By combining Theorems 2.5 (a) and 2.9, we get a commutative diagram

$$\begin{tikzcd} \mathfrak{X}/K \simeq \overline{\mathfrak{T}} \arrow[r, "\sim", "\mathrm{gRS}"'] \arrow[dr, "\Phi_{\mathfrak{k}}"'] & \bigsqcup_{(\lambda,\mu) \in \mathcal{P}(p) \times \mathcal{P}(q)} \mathcal{T}_{\lambda,\mu} \arrow[d] & \ni & (T_1, T_2; \lambda', \mu'; \nu) \arrow[d, mapsto] \\ & \mathcal{P}(p) \times \mathcal{P}(q) & \ni & (\lambda, \mu) \end{tikzcd}$$

from which we have that gRS restricts to a bijection $\Phi_{\mathfrak{k}}^{-1}(\lambda, \mu) \xrightarrow{\sim} \mathcal{T}_{\lambda,\mu}$.

Proof. First, we note that the considered map is well defined: the fact that $\lambda \setminus \lambda'$, $\lambda' \setminus \nu$, $\mu \setminus \mu'$, and $\mu' \setminus \nu$ are column strips follows from [7, Proposition in §1.1], and we have

$$|\lambda'| + |\mu'| = (s + k) + (t + k) = k + (k + s + t) = |\nu| + r$$

where, as before, $s = \#L$, $t = \#M$, $k = \#I = \#J$.

Next, we show that the map is bijective. Let $(T_1, T_2; \lambda', \mu'; \nu) \in \mathcal{T}_{\lambda,\mu}$. Applying twice [7, Proposition in §1.1], we find that there is a unique 6-tuple (S_1, S_2, L, L', M, M'), where S_1, S_2 are Young tableaux of shape ν and L, L', M, M' are sets of integers, such that $T_1 = [L] * S_1 * [L']$, $T_2 = [M] * S_2 * [M']$, $\lambda' = \mathrm{shape}([L] * S_1)$, and $\mu' = \mathrm{shape}([M] * S_2)$ (understanding that the contents of L, S_1, L' are disjoint, as well as those of M, S_2, M'). Let I (resp. J) be the set of entries of S_1 (resp. S_2). By the Robinson–Schensted correspondence, we get $(S_1, S_2) = (\mathrm{RS}_1(\sigma), \mathrm{RS}_2(\sigma))$ for a unique bijection $\sigma : J \to I$. Then, the data $I, L, L', J, M, M', \sigma$ determine a unique element $\omega \in \overline{\mathfrak{T}}$ (see Section 2.1) such that $\mathrm{gRS}(\omega) = (T_1, T_2; \lambda', \mu'; \nu)$. □

Example 2.10. (a) The 5-tuple corresponding to the element ω of Example 2.1 is

$$\mathrm{gRS}(\omega) = (T_1, T_2; \lambda', \mu'; \nu) = \left(\begin{ytableau} 1 & 3 \\ 2 \\ 4 \\ 5 \end{ytableau}, \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}; \begin{ytableau} \ \\ \ \\ \ \end{ytableau}, \begin{ytableau} \ & \ \end{ytableau}; \begin{ytableau} \ \\ \ \end{ytableau} \right) = \left(\begin{ytableau} *(red) 1 & 3 \\ *(red) 2 \\ *(yellow) 4 \\ 5 \end{ytableau}, \begin{ytableau} *(red) 1 & *(yellow) 3 \\ *(red) 2 \end{ytableau} \right)$$

(we encode the 5-tuple as the pair of tableaux (T_1, T_2) where the boxes of λ', μ', ν are colored).

(b) In Figure 4, we give the bijection of Theorem 2.9 in the case where $p = 3$, $q = r = 2$. In this case, the set $\overline{\mathfrak{T}}$ has 34 elements.

Remark 2.11. We point out that Singh [12] has recently developed a Robinson–Schensted correspondence for partial permutations. Specifically, he has established a bijection between the set of partial permutations of size $p \times q$ and a set of triples (Λ, P, Q) consisting of a signed Young diagram of size $p+q$ and two standard Young tableaux of sizes p and q. Note that, if $r = q$, the set of partial permutations of size $p \times q$ can be realized in a natural way as a subset of our set $\overline{\mathfrak{T}}$. One can ask whether there is a relation between the correspondence in Theorem 2.9 and the bijection given in [12, Theorem A].

We refer to [11, 15, 16] for other generalizations of the Robinson–Schensted correspondence that arise in geometry.

We derive from Theorem 2.9 an interpretation of the cardinals of the fibers $\Phi_{\mathfrak{k}}^{-1}(\mathfrak{O}_{\lambda,\mu})$ based on representation theory. If $\lambda \in \mathcal{P}(n)$, let $\rho_\lambda^{(n)}$ denote the corresponding irreducible representation of $\mathfrak{S}_n$. In this way, the (isomorphism classes of) irreducible representations $\rho_\lambda^{(p)} \boxtimes \rho_\mu^{(q)}$ of $\mathfrak{S}_p \times \mathfrak{S}_q$ are parametrized by the pairs of partitions $(\lambda, \mu) \in \mathcal{P}(p) \times \mathcal{P}(q)$. Here $\boxtimes$ stands for the outer tensor product.

For every triple of nonnegative integers (k, s, t) such that

$$s' := p - k - s \geqslant 0, \quad t' := q - k - t \geqslant 0, \quad k + s + t = r, \tag{9}$$

we consider the subgroup of $\mathfrak{S}_p \times \mathfrak{S}_q$ given by

$$\begin{aligned} H_{k,s,t} &= \{(a_1, a_2, a_3; b_1, b_2, b_3) \in (\mathfrak{S}_k \times \mathfrak{S}_s \times \mathfrak{S}_{s'}) \times (\mathfrak{S}_k \times \mathfrak{S}_t \times \mathfrak{S}_{t'}) : a_1 = b_1\} \\ &\cong \Delta\mathfrak{S}_k \times \mathfrak{S}_s \times \mathfrak{S}_{s'} \times \mathfrak{S}_t \times \mathfrak{S}_{t'}, \end{aligned}$$

where $\Delta\mathfrak{S}_k$ stands for the diagonal embedding of $\mathfrak{S}_k$ in $\mathfrak{S}_k \times \mathfrak{S}_k$. Let ε denote the signature representation of $H_{k,s,t}$ (the restriction of $\rho_{(1^p)}^{(p)} \boxtimes \rho_{(1^q)}^{(q)}$). The induced representation $\mathrm{Ind}_{H_{k,s,t}}^{\mathfrak{S}_p \times \mathfrak{S}_q} \varepsilon$ decomposes as a sum of irreducible representations

$$\mathrm{Ind}_{H_{k,s,t}}^{\mathfrak{S}_p \times \mathfrak{S}_q} \varepsilon = \bigoplus_{(\lambda,\mu) \in \mathcal{P}(p) \times \mathcal{P}(q)} (\rho_\lambda^{(p)} \boxtimes \rho_\mu^{(q)})^{m_{k,s,t}(\lambda,\mu)},$$

where $m_{k,s,t}(\lambda, \mu)$ denotes the multiplicity of $\rho_\lambda^{(p)} \boxtimes \rho_\mu^{(q)}$. The next corollary is a generalization of [5, Corollary 7.10].

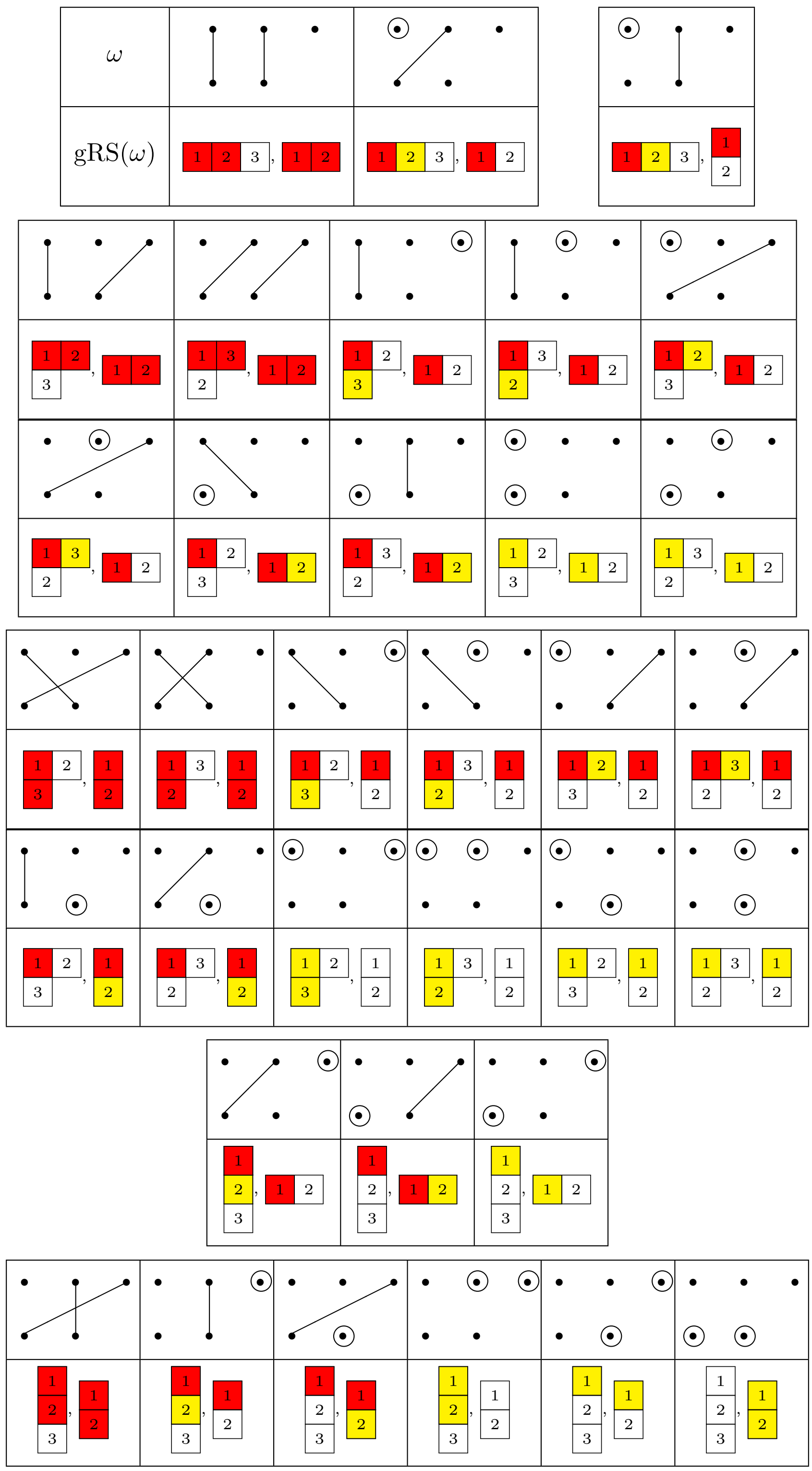


Figure 4. The correspondence $\omega \mapsto (T_1, T_2; \lambda', \mu'; \nu)$ in the case $(p, q, r) = (3, 2, 2)$.

Corollary 2.12. *For every pair of partitions $(\lambda,\mu)\in\mathcal{P}(p)\times\mathcal{P}(q)$, we have*

$$\#\Phi_{\mathfrak{k}}^{-1}(\mathfrak{O}_{\lambda,\mu})=\sum_{(k,s,t)} m_{k,s,t}(\lambda,\mu)\dim\rho_\lambda^{(p)}\boxtimes\rho_\mu^{(q)},$$

where the sum is over triples (k,s,t) satisfying (9).

Proof. Note that $\varepsilon=\mathbf{1}\boxtimes\varepsilon'$, where $\mathbf{1}$ is the trivial representation of $\Delta\mathfrak{S}_k$ and ε' is the signature representation of $\mathfrak{S}_s\times\mathfrak{S}_{s'}\times\mathfrak{S}_t\times\mathfrak{S}_{t'}$. Let

$$\tilde{H}_{k,s,t}=(\mathfrak{S}_k\times\mathfrak{S}_s\times\mathfrak{S}_{s'})\times(\mathfrak{S}_k\times\mathfrak{S}_t\times\mathfrak{S}_{t'}).$$

The intermediate induced representation $\mathrm{Ind}_{H_{k,s,t}}^{\tilde{H}_{k,s,t}}\varepsilon$ can be written as

$$\begin{aligned}\mathrm{Ind}_{H_{k,s,t}}^{\tilde{H}_{k,s,t}}\varepsilon&=(\mathrm{Ind}_{\Delta\mathfrak{S}_k}^{\mathfrak{S}_k\times\mathfrak{S}_k}\mathbf{1})\boxtimes\varepsilon'=\mathbb{C}\mathfrak{S}_k\boxtimes\rho_{(1^s)}^{(s)}\boxtimes\rho_{(1^{s'})}^{(s')}\boxtimes\rho_{(1^t)}^{(t)}\boxtimes\rho_{(1^{t'})}^{(t')}\\&=\bigoplus_{\nu\in\mathcal{P}(k)}(\rho_\nu^{(k)}\boxtimes\rho_{(1^s)}^{(s)}\boxtimes\rho_{(1^{s'})}^{(s')})\boxtimes(\rho_\nu^{(k)}\boxtimes\rho_{(1^t)}^{(t)}\boxtimes\rho_{(1^{t'})}^{(t')})^*,\end{aligned}$$

where $\mathbb{C}\mathfrak{S}_k=\bigoplus_{\nu\in\mathcal{P}(k)}\rho_\nu^{(k)}\boxtimes(\rho_\nu^{(k)})^*$ is the regular representation of $\mathfrak{S}_k\times\mathfrak{S}_k$. Applying twice the Pieri rule (see [7, §2.2 and §7.3]), we have

$$\mathrm{Ind}_{\mathfrak{S}_k\times\mathfrak{S}_s\times\mathfrak{S}_{s'}}^{\mathfrak{S}_p}(\rho_\nu^{(k)}\boxtimes\rho_{(1^s)}^{(s)}\boxtimes\rho_{(1^{s'})}^{(s')})=\bigoplus_{\substack{\lambda'\in\mathcal{P}(k+s)\\ \text{with }\nu\subset\lambda'}}\ \bigoplus_{\substack{\lambda\in\mathcal{P}(p)\\ \text{with }\lambda'\subset\lambda}}\rho_\lambda^{(p)}.$$

Since $\mathrm{Ind}_{H_{k,s,t}}^{\mathfrak{S}_p\times\mathfrak{S}_q}\varepsilon=\mathrm{Ind}_{\tilde{H}_{k,s,t}}^{\mathfrak{S}_p\times\mathfrak{S}_q}\mathrm{Ind}_{H_{k,s,t}}^{\tilde{H}_{k,s,t}}\varepsilon$, we deduce that

$$m_{k,s,t}(\lambda,\mu)=\#\{(\nu,\lambda',\mu')\in\mathcal{P}(k)\times\mathcal{P}(k+s)\times\mathcal{P}(k+t):\nu\subset\lambda'\subset\lambda,\ \nu\subset\mu'\subset\mu\}$$

for all triples (k,s,t). Note also that $\dim\rho_\lambda^{(p)}\boxtimes\rho_\mu^{(q)}$ is the number of pairs of standard Young tableaux (T_1,T_2) of shape λ and μ, respectively. The claimed equality now follows from Theorem 2.9. ☐

Corollary 2.13. *The total number of K-orbits in $\mathfrak{X}$ is given by*

$$\#\mathfrak{X}/K=\sum_{(k,s,t)}\dim\mathrm{Ind}_{H_{k,s,t}}^{\mathfrak{S}_p\times\mathfrak{S}_q}\varepsilon=\sum_{(k,s,t)}\binom{p}{k,s,s'}\binom{q}{k,t,t'}k!,$$

where the sums are over triples (k,s,t) satisfying (9).

3. On the decomposition of $\mathfrak{X}$ into K-orbits

The purpose of this section is to prove the results stated in Theorems 2.2 and 2.3, regarding the decomposition of $\mathfrak{X}=\mathrm{Gr}(V,r)\times\mathrm{Fl}(V^+)\times\mathrm{Fl}(V^-)$ into orbits of $K=\mathrm{GL}(V^+)\times\mathrm{GL}(V^-)$.

By $B_k^+\subset\mathrm{GL}_k(\mathbb{C})$ we denote the subgroup of invertible upper triangular matrices in $\mathrm{GL}_k(\mathbb{C})$. Then

$$B_K=\left\{\begin{pmatrix}a&0\\0&d\end{pmatrix}:a\in B_p^+,\ d\in B_q^+\right\}=B_p^+\times B_q^+\subset\mathrm{GL}(V^+)\times\mathrm{GL}(V^-)$$

is a Borel subgroup of K. Recall that $\mathcal{F}_0^+$ and $\mathcal{F}_0^-$ denote the standard flags of V^+ and V^-. Thus, B_K is the stabilizer of the pair $(\mathcal{F}_0^+,\mathcal{F}_0^-)$ for the action of K on the product of flag varieties $\mathrm{Fl}(V^+)\times\mathrm{Fl}(V^-)$. In Section 3.1, we observe that there is a one-to-one correspondence between the orbit sets $\mathfrak{X}/K$ and $\mathrm{Gr}(V,r)/B_K$, which preserves the inclusion relations between closures of orbits. This fact is a useful ingredient in the rest of the section.

In Section 3.2, we show the parametrization of orbits and the dimension formula stated in Theorem 2.2(1)–(3). In Section 3.3, we describe the closure relations of orbits by proving Theorems 2.2(4) and 2.3. In Section 3.4, we make further remarks and mention relations with the existing literature.

3.1. A preliminary lemma. We will use the following lemma.

Lemma 3.1. (1) *The mapping* $\mathrm{Gr}(V,r) \to \mathfrak{X}$, $W \mapsto (W, \mathcal{F}_0^+, \mathcal{F}_0^-)$ *induces a one-to-one correspondence between the orbit sets*

$$\Xi : \mathrm{Gr}(V,r)/B_K \to \mathfrak{X}/K,\ \mathcal{O} = B_K \cdot W \mapsto \Xi(\mathcal{O}) = K \cdot (W, \mathcal{F}_0^+, \mathcal{F}_0^-).$$

(2) *If* $\mathbb{O} = \Xi(\mathcal{O}) \subset \mathfrak{X}$ *is the* K*-orbit corresponding to* $\mathcal{O} \subset \mathrm{Gr}(V,r)$*, then* $\mathbb{O} \cong K \times^{B_K} \mathcal{O}$*. In particular,* $\dim \mathbb{O} = \dim \mathcal{O} + \dim K/B_K$.

(3) *The correspondence preserves the closure relations. Namely, if* $\mathcal{O}_1, \mathcal{O}_2$ *are* B_K*-orbits of* $\mathrm{Gr}(V,r)$ *and if* $\mathbb{O}_1 = \Xi(\mathcal{O}_1)$ *and* $\mathbb{O}_2 = \Xi(\mathcal{O}_2)$ *are the corresponding* K*-orbits of* $\mathfrak{X}$*, then*

$$\overline{\mathcal{O}_1} \subset \overline{\mathcal{O}_2} \iff \overline{\mathbb{O}_1} \subset \overline{\mathbb{O}_2}.$$

Lemma 3.1 is a consequence of the following lemma (which applies to a general connected reductive group K). As already used in Lemma 3.1, $K \times^Q X$ stands for the quotient of $K \times X$ by the action of Q given by $q \cdot (k,x) = (kq^{-1}, qx)$, and $[k,x]$ denotes the class of (k,x) in this quotient.

Lemma 3.2. *Let* K *be a connected reductive group and let* $Q \subset K$ *be a parabolic subgroup. Let* X *be an algebraic variety endowed with an action of* K*. Consider the diagonal action of* K *on* $\mathbb{X} \coloneqq K/Q \times X$*. Note that there is an isomorphism* $\chi : \mathbb{X} \to K \times^Q X$ *given by* $\chi(kQ, x) = [k, k^{-1}x]$*. Also we consider the closed immersion* $\iota : X \to \mathbb{X}$, $x \mapsto (Q, x)$.

(1) *There is a one-to-one correspondence (in fact an isomorphism of partially ordered sets)*

$$\Xi : \{Q\text{-stable subsets } M \subset X\} \to \{K\text{-stable subsets } N \subset \mathbb{X}\}$$

given by $\Xi(M) = K \cdot \iota(M)$*. The inverse bijection is given by* $N \mapsto \iota^{-1}(N)$*. Moreover,* Ξ *restricts to a one-to-one correspondence between the orbit sets* X/Q *and* $\mathbb{X}/K$.

(2) *Every* Q*-stable subset* $M \subset X$ *yields a subset* $K \times^Q M \subset K \times^Q X$*, and we have* $\chi(\Xi(M)) = K \times^Q M$.

(3) *Let* $N = \Xi(M)$ *for some* Q*-stable subset* M*. Then,* M *is closed in* X *if and only if* N *is closed in* $\mathbb{X}$*. More generally, we have* $\overline{N} = \Xi(\overline{M})$.

Though this lemma is well known, we give a proof for the sake of completeness.

Proof. (1) The map Ξ is certainly well defined. For a Q-stable subset $M \subset X$, the inclusion $M \subset \iota^{-1}(K \cdot \iota(M)) = \iota^{-1}(\Xi(M))$ is clear, while for $x \in \iota^{-1}(\Xi(M))$ there are $k \in K$ and $y \in M$ such that $\iota(x) = k \cdot \iota(y)$, which means that $(Q,x) = (kQ, ky)$, whence $k \in Q$ and $x = ky \in Q \cdot M = M$. We have shown that $M = \iota^{-1}(\Xi(M))$.

For a K-stable subset $N \subset \mathbb{X}$, given $x \in \iota^{-1}(N)$ and $q \in Q$ we have

$$\iota(qx) = (Q, qx) = q \cdot (Q, x) \in N,$$

hence $qx \in \iota^{-1}(N)$; this shows that $\iota^{-1}(N)$ is Q-stable.

Since N is K-stable, the inclusion $\Xi(\iota^{-1}(N)) = K \cdot \iota(\iota^{-1}(N)) \subset N$ is clear, while for $(kQ, x) \in N$ we have $\iota(k^{-1}x) = (Q, k^{-1}x) = k^{-1} \cdot (kQ, x) \in N$, hence $(kQ, x) \in K \cdot \iota(\iota^{-1}(N)) = \Xi(\iota^{-1}(N))$. This shows that $N = \Xi(\iota^{-1}(N))$.

We have thus shown that Ξ is a bijection, with inverse bijection given by $N \mapsto \iota^{-1}(N)$. Note also that the implications

$$(M \subset M' \Rightarrow \Xi(M) \subset \Xi(M')) \quad \text{and} \quad (N \subset N' \Rightarrow \iota^{-1}(N) \subset \iota^{-1}(N'))$$

are clear, which show that Ξ is in fact an isomorphism of posets.

Finally, if $M = Q\cdot x$ is a Q-orbit, then $\Xi(M) = K\cdot(Q\cdot\iota(x)) = K\cdot\iota(x)$ is a K-orbit. If $N = K\cdot(kQ, x)$ is a K-orbit, then $N = K\cdot(Q, k^{-1}x) = \Xi(Q\cdot(k^{-1}x))$ is the image of a Q-orbit. The proof of part (1) is complete.

(2) First we note that, if $M \subset X$ is Q-stable, then the quotient $K\times^Q M = (K\times M)/Q$ coincides with the subset $\{[k,x]\in K\times^Q X : x\in M\}\subset K\times^Q X$. This subset can also be written as

$$K\times^Q M = \{\chi(kQ, kx) : k\in K,\ x\in M\} = \chi(K\cdot\iota(M)) = \chi(\Xi(M)).$$

(3) Let $N = \Xi(M)$. If N is closed, then $M = \iota^{-1}(N)$ is closed. Conversely, assume that M is closed. Then, $\iota(M)\subset\mathbb{X}$ is closed and Q-stable, and this implies that the set $\{(k,\xi)\in K\times\mathbb{X} : k^{-1}\cdot\xi\in\iota(M)\}$ is closed as well as its image in $K/Q\times\mathbb{X}$. Note that

$$N = K\cdot\iota(M) = \mathrm{pr}_2(\{(kQ,\xi)\in K/Q\times\mathbb{X} : k^{-1}\cdot\xi\in\iota(M)\}).$$

Since K/Q is complete, we conclude that N is closed.

More generally, using that $\Xi(\overline{M})$ and $\Xi^{-1}(\overline{N})$ are closed, and the fact that Ξ is an isomorphism of posets, we get

$$\overline{N} = \overline{\Xi(M)} \subset \Xi(\overline{M}) = \Xi(\overline{\Xi^{-1}(N)}) \subset \Xi(\Xi^{-1}(\overline{N})) = \overline{N},$$

hence $\overline{N} = \Xi(\overline{M})$, as claimed. □

Remark 3.3. In Lemma 3.2, the assumption that Q is parabolic is used only in part (3). Also note that the isomorphism χ is K-equivariant when K acts on $\mathbb{X} = K/Q\times X$ diagonally and on $K\times^Q X$ by left multiplication.

3.2. Parametrization and dimension formula – proof of Theorem 2.2(1)–(3).

As before, we denote by $B_p^+\subset \mathrm{GL}_p(\mathbb{C})$ (resp. $B_r^+\subset\mathrm{GL}_r(\mathbb{C})$) the Borel subgroup of upper triangular matrices. We need two lemmas. The first one is an analogue of [5, Proposition 6.3]. It is also shown in [6, p. 390], but we give a proof for the sake of completeness.

Lemma 3.4. *Every $p\times r$ matrix can be written in the form $b_1\tau b_2$ for some $b_1\in B_p^+$, $b_2\in B_r^+$, and a unique $\tau\in\mathfrak{T}_{p,r}$.*

Proof. Through Gauss elimination, any $p\times r$ matrix a can be transformed into a matrix $\tau\in\mathfrak{T}_{p,r}$ by a series of operations consisting of multiplying a row (resp. a column) by a nonzero scalar or adding to a row (resp. to a column) another row (resp. column) situated below it (resp. on its left). These operations correspond to multiplying on the left (resp. on the right) by an element of B_p^+ (resp. B_r^+). Hence the double coset $B_p^+ a B_r^+$ contains an element $\tau\in\mathfrak{T}_{p,r}$, which means that $a\in B_p^+\tau B_r^+$.

For every pair $(i,j)\in\{1,\ldots,p\}\times\{1,\ldots,r\}$, the mapping $a\mapsto\beta_{i,j}(a) \coloneqq \operatorname{rank}(a_{k,\ell})_{\substack{i\leqslant k\leqslant p\\ 1\leqslant\ell\leqslant j}}$ is constant on the set $B_p^+\tau B_r^+$, and we have

$$\beta_{i,j}(a) = \beta_{i,j}(\tau) = \#\{1\text{'s within the submatrix } (\tau_{k,\ell})_{\substack{i\leqslant k\leqslant p\\ 1\leqslant\ell\leqslant j}}\}.$$

This implies that two different elements $\tau,\tau'\in\mathfrak{T}_{p,r}$ cannot belong to the same double coset $B_p^+ a B_r^+$, whence the uniqueness. □

The second lemma is analogous to [5, Lemma 8.2].

Lemma 3.5. *For every $\tau\in\mathfrak{T}_{p,r}$, there is a permutation $w\in\mathfrak{S}_r$ such that $\tau w B_r^+\subset B_p^+\tau w$.*

Proof. By $(e_1^\ell, \dots, e_\ell^\ell)$ we denote the standard basis of $\mathbb{C}^\ell$. By $e_{i,j}^\ell$ we denote the elementary $\ell \times \ell$ matrix whose (i,j) coefficient is 1 and the other coefficients are 0. By $\mathrm{diag}(t_1, \dots, t_\ell)$, we denote the diagonal matrix with coefficients $t_1, \dots, t_\ell$ along the diagonal.

Let $k = \operatorname{rank} \tau$. Let $1 \leqslant i_1 < \dots < i_k \leqslant p$ be such that $\operatorname{Im} \tau = \langle e_{i_1}^p, \dots, e_{i_k}^p \rangle$. We choose $w \in \mathfrak{S}_r$ such that $\tau w(e_j^r) = 0$ for $1 \leqslant j \leqslant r-k$ and $\tau w(e_{r-k+j}^r) = e_{i_j}^p$ for $1 \leqslant j \leqslant k$.

For every diagonal matrix $t = \mathrm{diag}(t_1, \dots, t_r) \in B_r^+$, we have

$$\tau w t = t' \tau w \in B_p^+ \tau w$$

for any diagonal matrix $t' = \mathrm{diag}(t'_1, \dots, t'_p) \in B_p^+$ such that $t'_{i_j} = t_{r-k+j}$ for all $j \in \{1, \dots, k\}$. For every transvection $u = 1_r + x e_{j,\ell}^r \in B_r^+$ where $1 \leqslant j < \ell \leqslant r$, we have

$$\tau w u = \begin{cases} \tau w & \text{if } j \leqslant r-k, \\ (1_p + x e^p_{i_{j-(r-k)}, i_{\ell-(r-k)}}) \tau w & \text{if } j > r-k, \end{cases}$$

hence $\tau w u \in B_p^+ \tau w$ in each case. Since B_r^+ is generated by the diagonal matrices and the transvections, we conclude that $\tau w B_r^+ \subset B_p^+ \tau w$. $\square$

Now we are ready to prove parts (1) and (3) of Theorem 2.2.

Proof of Theorem 2.2(1) and (3). Every $W \in \mathrm{Gr}(V, r)$ is the image of a matrix

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \text{with} \quad a_1 \in \mathrm{M}_{p,r}(\mathbb{C}),\ a_2 \in \mathrm{M}_{q,r}(\mathbb{C}),\ \operatorname{rank} a = r.$$

By Lemma 3.4, there are $\tau_1 \in \mathfrak{T}_{p,r}$, $b_1 \in B_p^+$, $b_2 \in B_r^+$ such that $a_1 = b_1 \tau_1 b_2$. Moreover, by Lemma 3.5, there is $w \in \mathfrak{S}_r$ such that $\tau_1 w B_r^+ \subset B_p^+ \tau_1 w$. We have

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 \tau_1 w \\ a'_2 \end{pmatrix} w^{-1} b_2$$

for some $a'_2 \in \mathrm{M}_{q,r}(\mathbb{C})$. Applying again Lemma 3.4, there are $\tau_2 \in \mathfrak{T}_{q,r}$, $b_3 \in B_r^+$, and $b_4 \in B_q^+$ such that $a'_2 = b_4 \tau_2 b_3$. Moreover, there is $b'_1 \in B_p^+$ such that $\tau_1 w b_3^{-1} = b'_1 \tau_1 w$. This yields

$$a = \begin{pmatrix} b_1 b'_1 \tau_1 w \\ b_4 \tau_2 \end{pmatrix} b_3 w^{-1} b_2 = \begin{pmatrix} b_1 b'_1 & 0 \\ 0 & b_4 \end{pmatrix} \begin{pmatrix} \tau_1 w \\ \tau_2 \end{pmatrix} b_3 w^{-1} b_2$$

hence

$$W = \operatorname{Im} a \in B_K \cdot [\omega] \quad \text{where} \quad \omega = \begin{pmatrix} \tau_1 w \\ \tau_2 \end{pmatrix} \in \mathfrak{T}_{(p,q),r}.$$

This implies that $\mathrm{Gr}(V, r) = \bigcup_{\omega \in \overline{\mathfrak{T}}} B_K \cdot [\omega]$, hence $\mathfrak{X} = \bigcup_{\omega \in \overline{\mathfrak{T}}} \mathbb{O}_\omega$ in view of Lemma 3.1.

The mappings

$$\xi = (W, (F_i^+)_{i=0}^p, (F_j^-)_{j=0}^q) \mapsto d_{i,j}(\xi) \coloneqq \dim W \cap (F_i^+ + F_j^-),$$

for $(i,j) \in \{0, \dots, p\} \times \{0, \dots, q\}$, are constant on every K-orbit of $\mathfrak{X}$. Let $(e_1^+, \dots, e_p^+)$ (resp. $(e_1^-, \dots, e_q^-)$) be the standard basis of $V^+ = \mathbb{C}^p \times \{0\}^q$ (resp. $V^- = \{0\}^p \times \mathbb{C}^q$), so that the standard flags $\mathcal{F}_0^\pm$ are given by $\mathcal{F}_0^+ = (\langle e_1^+, \dots, e_i^+ \rangle)_{i=0}^p$ and $\mathcal{F}_0^- = (\langle e_1^-, \dots, e_j^- \rangle)_{j=0}^q$. For every $\omega \in \overline{\mathfrak{T}}$, the definition of the graph $\mathcal{G}(\omega)$ implies that the subspace

$$[\omega] \cap (\langle e_1^+, \dots, e_i^+ \rangle + \langle e_1^-, \dots, e_j^- \rangle)$$

is spanned by the vectors e_k^+ $(1 \leqslant k \leqslant i)$ such that $\mathcal{G}(\omega)$ has a mark at k^+, the vectors e_ℓ^- $(1 \leqslant \ell \leqslant j)$ such that $\mathcal{G}(\omega)$ has a mark at ℓ^-, and the linear combinations $e_k^+ + e_\ell^-$

($1 \leqslant k \leqslant i$ and $1 \leqslant \ell \leqslant j$) such that $\mathcal{G}(\omega)$ has an edge joining the vertices k^+ and ℓ^-. This implies that

$$d_{i,j}(([\omega], \mathcal{F}_0^+, \mathcal{F}_0^-)) = \dim[\omega] \cap (\langle e_1^+, \dots, e_i^+ \rangle + \langle e_1^-, \dots, e_j^- \rangle) = r_{i,j}(\omega).$$

We deduce that

$$\mathbb{O}_\omega \subset \{\xi = (W, \mathcal{F}^+, \mathcal{F}^-) \in \mathfrak{X} : d_{i,j}(\xi) = r_{i,j}(\omega) \ \text{ for all } i,j\} \quad \text{for all } \omega \in \overline{\mathfrak{T}}. \tag{10}$$

If ω, ω' are two different elements of the set $\overline{\mathfrak{T}}$, then their graphs $\mathcal{G}(\omega)$, $\mathcal{G}(\omega')$ must be different, hence the matrices $R(\omega) = (r_{i,j}(\omega))$ and $R(\omega') = (r_{i,j}(\omega'))$ are different. From (10), it follows that the orbits $\mathbb{O}_\omega$ and $\mathbb{O}_{\omega'}$ are disjoint. Therefore, $\mathfrak{X}$ is the disjoint union of the orbits $\mathbb{O}_\omega$ for $\omega \in \overline{\mathfrak{T}}$. This also implies that the inclusion in (10) must be an equality for all $\omega \in \overline{\mathfrak{T}}$. This establishes Theorem 2.2(1) and (3). □

Proof of Theorem 2.2(2). Lemma 3.1 implies

$$\dim \mathbb{O}_\omega = \dim B_K \cdot [\omega] + \dim K/B_K = \dim B_K \cdot [\omega] + \binom{p}{2} + \binom{q}{2}. \tag{11}$$

Let $\mathfrak{b}_K = \mathrm{Lie}(B_K) = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x \in \mathfrak{b}_p^+,\ y \in \mathfrak{b}_q^+ \right\}$, where $\mathfrak{b}_p^+ = \mathrm{Lie}(B_p^+) \subset \mathrm{M}_p(\mathbb{C})$ and $\mathfrak{b}_q^+ = \mathrm{Lie}(B_q^+) \subset \mathrm{M}_q(\mathbb{C})$ are the subspaces of upper triangular matrices. We have

$$\begin{aligned} \dim B_K \cdot [\omega] &= \dim B_K - \dim\{b \in B_K : b([\omega]) = [\omega]\} \\ &= \dim \mathfrak{b}_K - \dim\{z \in \mathfrak{b}_K : z([\omega]) \subset [\omega]\}. \end{aligned} \tag{12}$$

As before, we denote by $(e_1^+, \dots, e_p^+)$, resp. $(e_1^-, \dots, e_q^-)$, the standard basis of $V^+ = \mathbb{C}^p \times \{0\}^q$, resp. $V^- = \{0\}^p \times \mathbb{C}^q$. The linear space $[\omega]$ is spanned by the vectors e_a^+ with $a \in \{1, \dots, p\}$ such that the graph $\mathcal{G}(\omega)$ has a mark at a^+, the vectors e_c^- with $c \in \{1, \dots, q\}$ such that there is a mark at c^-, and the linear combinations $e_a^+ + e_c^-$ for all pairs $(a, c) \in \{1, \dots, p\} \times \{1, \dots, q\}$ such that $\mathcal{G}(\omega)$ has an edge joining a^+ and c^-. This implies that a matrix $z = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in \mathfrak{b}_K$ satisfies $z([\omega]) \subset [\omega]$ if and only if the upper triangular matrices $x = (x_{i,j})_{1 \leqslant i,j \leqslant p}$ and $y = (y_{i,j})_{1 \leqslant i,j \leqslant q}$ satisfy the following equations:

(1) For every $a \in \{1, \dots, p\}$ such that $\mathcal{G}(\omega)$ has a mark at a^+, we must have $x_{i,a} = 0$ for all $i < a$ such that there is no mark at i^+.
(2) For every $c \in \{1, \dots, q\}$ such that $\mathcal{G}(\omega)$ has a mark at c^-, we must have $y_{j,c} = 0$ for all $j < c$ such that there is no mark at j^-.
(3) For every pair $(a, c) \in \{1, \dots, p\} \times \{1, \dots, q\}$ such that $\mathcal{G}(\omega)$ has an edge (a^+, c^-), we must have $x_{i,a} = 0$ for all $i < a$ such that i^+ is a free vertex (i.e. not marked nor incident with an edge) in $\mathcal{G}(\omega)$, and we must have $y_{j,c} = 0$ for all $j < c$ such that j^- is a free vertex in $\mathcal{G}(\omega)$.
(4) For (a, c) as in (3), we must also have $x_{i,a} = y_{j,c}$ for all pair $(i, j) \in \{1, \dots, a\} \times \{1, \dots, c\}$ such that (i^+, j^-) is an edge in $\mathcal{G}(\omega)$, i.e. for all edge which is situated on the left of (a^+, c^-) or coincides with (a^+, c^-) itself. Finally, we get one more equation $x_{i,a} = 0$, or resp. $y_{j,c} = 0$, for every edge (i^+, j^-) which has a crossing with (a^+, c^-), i.e. such that $i < a$ and $c < j$, resp. $i > a$ and $j < c$.

We have listed linearly independent equations which characterize the subspace $\{z \in \mathfrak{b}_K : z([\omega]) \subset [\omega]\} \subset \mathfrak{b}_K$. With the notation of Theorem 2.2(2), the above items (1)–(3) yield $a^+(\omega) + a^-(\omega)$ equations, while the item (4) yields $\frac{b(\omega)(b(\omega)+1)}{2} + c(\omega)$ equations. This implies that

$$\dim\{z \in \mathfrak{b}_K : z([\omega]) \subset [\omega]\} = \dim \mathfrak{b}_K - \Big(a^+(\omega) + a^-(\omega) + \frac{b(\omega)(b(\omega)+1)}{2} + c(\omega)\Big).$$

Combining this equality with (11) and (12), we get the dimension formula stated in Theorem 2.2(2). ☐

3.3. Closure relations – proof of Theorems 2.2(4) and 2.3. For $\omega, \omega' \in \overline{\mathfrak{T}}$, we write $\omega \preceq \omega'$ if we have $r_{i,j}(\omega) \geqslant r_{i,j}(\omega')$ for all $(i,j) \in \{0,\ldots,p\} \times \{0,\ldots,q\}$. This clearly endows $\overline{\mathfrak{T}}$ with a partial order and, for showing Theorem 2.2(4), we have to show that this order characterizes the inclusion relations between orbit closures in $\mathfrak{X}$. We need four lemmas.

Lemma 3.6. *Assume that ω is obtained from ω' by one of the elementary moves described in Figure 1. Then, the following relations hold:*

$$\omega \prec \omega' \qquad \text{and} \qquad \mathbb{O}_\omega \subset \overline{\mathbb{O}_{\omega'}}.$$

Proof. We consider the cases described in Figure 1.

- In Case 1, we have $r_{i,j}(\omega) = r_{i,j}(\omega') + 1$ if $a \leqslant i < b$ and $c \leqslant j < d$, and we have $r_{i,j}(\omega) = r_{i,j}(\omega')$ otherwise. Hence $r_{i,j}(\omega) \geqslant r_{i,j}(\omega')$ for all i, j, and this implies that $\omega \prec \omega'$.
- In Case 2, upper subcase (resp. lower subcase), we have $r_{i,j}(\omega) = r_{i,j}(\omega') + 1$ if $a \leqslant i < b$ and $j < c$ (resp. $i < a$ and $c \leqslant j < d$), and $r_{i,j}(\omega) = r_{i,j}(\omega')$ otherwise. Hence, again, we get $\omega \prec \omega'$.
- In Case 3, upper subcase (resp. lower subcase), we have $r_{i,j}(\omega) = r_{i,j}(\omega') + 1$ if $a \leqslant i < b$ and $c \leqslant j$ (resp. $a \leqslant i$ and $c \leqslant j < d$) and $r_{i,j}(\omega) = r_{i,j}(\omega')$ otherwise. Whence $\omega \prec \omega'$.
- In Case 4, upper subcase (resp. lower subcase), we have $r_{i,j}(\omega) = r_{i,j}(\omega') + 1$ if $a \leqslant i$ and $j < c$ (resp. $i < a$ and $c \leqslant j$) and $r_{i,j}(\omega) = r_{i,j}(\omega')$ otherwise. Once again this yields $\omega \prec \omega'$.
- In Case 5, upper subcase (resp. lower subcase), we have $r_{i,j}(\omega) = r_{i,j}(\omega') + 1$ if $a \leqslant i < b$ (resp. $c \leqslant j < d$) and $r_{i,j}(\omega) = r_{i,j}(\omega')$ otherwise, and once again we deduce that $\omega \prec \omega'$ in this case.

In each case, we have shown that $\omega \prec \omega'$.

As before, we denote by $e^k_{i,j}$ the $k \times k$ elementary matrix with 1 at position (i,j) and 0 elsewhere. Then, let $u^k_{i,j}(t) = 1_k + te^k_{i,j}$ and $\delta^k_i(t) = 1_k + (t-1)e^k_{i,i}$. For $t \in \mathbb{C}^*$, we consider the matrix h_t given by

$$h_t = \begin{pmatrix} A_t & 0 \\ 0 & D_t \end{pmatrix},$$

where A_t and D_t are blocks of respective sizes $p \times p$ and $q \times q$ given by

$$A_t = \begin{cases} u^p_{a,b}(-t)\delta^p_a(t) & \text{in Cases } 1,\ 2^+, \\ u^p_{a,b}(t) & \text{in Cases } 3^+,\ 5^+, \\ \delta^p_a(t) & \text{in Cases } 3^-,\ 4^+, \\ 1_p & \text{in Cases } 2^-, 4^-, 5^-, \end{cases} \qquad D_t = \begin{cases} u^q_{c,d}(t)\delta^q_c(-t) & \text{in Cases } 1,\ 2^-, \\ u^q_{c,d}(t) & \text{in Cases } 3^-,\ 5^-, \\ \delta^q_c(t) & \text{in Cases } 3^+,\ 4^-, \\ 1_q & \text{in Cases } 2^+, 4^+, 5^+. \end{cases}$$

Here the notation N^+ (resp. N^-) refers to the upper (resp. lower) subcase of Case N in Figure 1. In each case, we obtain a subset $\{h_t\}_{t\in\mathbb{C}^*} \subset K$ such that

$$([\omega], \mathcal{F}^+_0, \mathcal{F}^-_0) = \lim_{t\to\infty} h_t \cdot ([\omega'], \mathcal{F}^+_0, \mathcal{F}^-_0),$$

and this shows that the inclusion $\mathbb{O}_\omega \subset \overline{\mathbb{O}_{\omega'}}$ holds. ☐

Lemma 3.7. *For every $\omega, \omega' \in \overline{\mathfrak{T}}$, the following implication holds:*

$$\mathbb{O}_\omega \subset \overline{\mathbb{O}_{\omega'}} \quad \Longrightarrow \quad \omega \preceq \omega'.$$

Proof. Assume that $\mathbb{O}_\omega \subset \overline{\mathbb{O}_{\omega'}}$. For each pair $(i,j) \in \{0,\dots,p\} \times \{0,\dots,q\}$, the mapping

$$\mathfrak{X} \to \mathbb{Z}_{\geqslant 0}, \quad (W,(F_k^+)_{k=0}^p,(F_\ell^-)_{\ell=0}^q) \mapsto \dim W \cap (F_i^+ + F_j^-)$$

is upper semicontinuous. Thus, in view of Theorem 2.2(3), we have

$$\mathbb{O}_\omega \subset \overline{\mathbb{O}_{\omega'}} \subset \{(W,(F_k^+)_{k=0}^p,(F_\ell^-)_{\ell=0}^q) \in \mathfrak{X} : \dim W \cap (F_i^+ + F_j^-) \geqslant r_{i,j}(\omega')\},$$

whereas

$$\mathbb{O}_\omega \subset \{(W,(F_k^+)_{k=0}^p,(F_\ell^-)_{\ell=0}^q) \in \mathfrak{X} : \dim W \cap (F_i^+ + F_j^-) = r_{i,j}(\omega)\}.$$

This yields $r_{i,j}(\omega) \geqslant r_{i,j}(\omega')$ for all pair (i,j), hence $\omega \preceq \omega'$. $\square$

Lemma 3.8. *For every $\omega,\omega'' \in \overline{\mathfrak{T}}$ such that $\omega \prec \omega''$, there is $\omega' \in \overline{\mathfrak{T}}$ with $\omega \preceq \omega' \preceq \omega''$ such that one of the pairs (ω,ω'), (ω',ω'') fits in one of the cases described in Figure* 1.

Proof. We reason by induction on $p+q \geqslant 0$, with immediate initialization if $p+q=0$. Assume that $\omega,\omega'' \in \overline{\mathfrak{T}}$ are such that $\omega \prec \omega''$. In particular we have $\omega \neq \omega''$, which forces $r \geqslant 1$, i.e. the graphs $\mathcal{G}(\omega)$ and $\mathcal{G}(\omega'')$ have at least one edge or marked vertex.

In the case where $\mathcal{G}(\omega)$ and $\mathcal{G}(\omega'')$ have one common edge or mark – call it x, by removing this edge or mark together with the corresponding vertices (or vertex), and after renumbering of the vertices, we obtain subgraphs $\mathcal{G}(\breve{\omega}) = \mathcal{G}(\omega) \setminus x$ and $\mathcal{G}(\breve{\omega}'') = \mathcal{G}(\omega'') \setminus x$ associated to smaller sized matrices $\breve{\omega}$ and $\breve{\omega}''$, and we still have $\breve{\omega} \prec \breve{\omega}''$ due to the definition of the relation $\preceq$. The induction hypothesis yields $\breve{\omega}'$ with $\breve{\omega} \preceq \breve{\omega}' \preceq \breve{\omega}''$, whose associated graph $\mathcal{G}(\breve{\omega}')$ yields $\mathcal{G}(\breve{\omega})$ or is yielded by $\mathcal{G}(\breve{\omega}'')$ through one of the elementary moves described in Figure 1. There is an element $\omega' \in \overline{\mathfrak{T}}$ such that $\mathcal{G}(\breve{\omega}') = \mathcal{G}(\omega') \setminus x$, and this element satisfies the requirements of the lemma. In conclusion,

(13) we may assume $\mathcal{G}(\omega)$ and $\mathcal{G}(\omega'')$ have no common edge nor marked vertex.

Notation: It is convenient to encode the set of edges and marks of the graph $\mathcal{G}(\omega)$ in the following way:

$$\begin{aligned} E(\omega) \coloneqq \{&(a,c) \in \{1,\dots,p\} \times \{1,\dots,q\} : (a^+,c^-) \text{ is an edge in } \mathcal{G}(\omega)\} \\ &\cup\{(a,0) : a \in \{1,\dots,p\},\ \mathcal{G}(\omega) \text{ has a mark at } a^+\} \\ &\cup\{(0,c) : c \in \{1,\dots,q\},\ \mathcal{G}(\omega) \text{ has a mark at } c^-\}. \end{aligned}$$

Then we note that

$$r_{i,j}(\omega) = \#E(\omega) \cap (\{0,\dots,i\} \times \{0,\dots,j\}) \quad \text{for all } i,j. \tag{14}$$

We define the set $E(\omega'')$ relative to ω'' in the same way. Both sets $E(\omega)$ and $E(\omega'')$ have r elements, in particular they are nonempty.

We choose an element $(a_0,c_0) \in E(\omega'')$ with the minimal possible value of c_0. If $c_0 \neq 0$, then c_0^- is not a free vertex in $\mathcal{G}(\omega'')$: it is marked if $a_0 = 0$ or incident with an edge (a_0^+,c_0^-) if $a_0 \neq 0$. Moreover, the minimality of c_0 guarantees then that every vertex c^- with $c < c_0$ is free in $\mathcal{G}(\omega'')$, and $\mathcal{G}(\omega'')$ contains no mark at a^+ for all $a \in \{1,\dots,p\}$ (because $(a,0)$ cannot belong to $E(\omega'')$, due to the minimality of c_0).

If $c_0 = 0$, then a_0^+ is a marked vertex in $\mathcal{G}(\omega'')$. There may be more than one marked vertex of this type in $\mathcal{G}(\omega'')$, and we choose a_0 minimal for this property. Thus, in any situation, we have $r_{a_0,c_0}(\omega'') = 1$.

Case 1 ($c_0 \neq 0$ and $r_{a_0,c_0-1}(\omega) \geqslant 1$)
The condition means that $\mathcal{G}(\omega)$ has an edge or mark within the vertices $\{i^+ : 1 \leqslant i \leqslant a_0\} \cup \{j^- : 1 \leqslant j < c_0\}$. In other words, we can find a pair $(a_1,c_1) \in$

$E(\omega)$ with $0 \leqslant a_1 \leqslant a_0$ and $0 \leqslant c_1 < c_0$. Note that $(a_0, c_1) \neq (0,0)$ since $(a_1, c_1) \neq (0,0)$. There is an element $\omega' \in \overline{\mathfrak{T}}$ such that

$$E(\omega') = (E(\omega'') \smallsetminus \{(a_0, c_0)\}) \cup \{(a_0, c_1)\}.$$

This incorporates several situations, and in each one the graph $\mathcal{G}(\omega')$ is deduced from $\mathcal{G}(\omega'')$ through one of the elementary moves depicted in Figure 1:

- If $a_0 \neq 0$ and $c_1 \neq 0$ (resp. $c_1 = 0$), then $\mathcal{G}(\omega')$ is obtained from $\mathcal{G}(\omega'')$ by replacing the edge (a_0^+, c_0^-) by an edge (a_0^+, c_1^-) (resp. by a mark at a_0^+), whereas c_0^- becomes a free vertex. This corresponds to Case 3 – lower subcase (resp. Case 4 – upper subcase) in Figure 1.
- If $a_0 = 0$, then $\mathcal{G}(\omega'')$ has a mark at c_0^-, and $\mathcal{G}(\omega')$ is obtained by replacing this mark by a mark at c_1^-, whereas c_0^- becomes a free vertex. This corresponds to Case 5 – lower subcase in Figure 1.

In each situation, we get $\omega' \prec \omega''$ in view of Lemma 3.6. For $(i,j) \in \{0, \ldots, p\} \times \{0, \ldots, q\}$, we have $r_{i,j}(\omega') = r_{i,j}(\omega'')$ (hence $r_{i,j}(\omega') \leqslant r_{i,j}(\omega)$) unless $i \geqslant a_0$ and $c_1 \leqslant j < c_0$. If $i \geqslant a_0$ and $c_1 \leqslant j < c_0$, we have

$$r_{i,j}(\omega') = r_{i,j}(\omega'') + 1 = 1 \leqslant r_{i,j}(\omega)$$

(the second equality is due to the minimality of c_0, while the inequality follows from (14) and the fact that $(a_1, c_1) \in E(\omega)$). We conclude that the inequality $r_{i,j}(\omega') \leqslant r_{i,j}(\omega)$ holds for all pair (i,j), hence $\omega \preceq \omega'$, and the element ω' satisfies all the requirements of the lemma.

Case 2 ($c_0 = 0$ or $r_{a_0, c_0 - 1}(\omega) = 0$)
This condition implies that the set $E(\omega)$ contains no pair of the form (a, c) with $0 \leqslant a \leqslant a_0$ and $0 \leqslant c < c_0$ (see (14)). Note also that $(a_0, c_0) \notin E(\omega)$ (due to (13)).

Since $r_{a_0, c_0}(\omega) \geqslant r_{a_0, c_0}(\omega'') = 1$, there is a pair $(a_0', c_0) \in E(\omega)$ with $0 \leqslant a_0' < a_0$. In particular this forces $a_0 \neq 0$.

The fact that $a_0 \neq 0$ implies that a_0^+ is a vertex in $\mathcal{G}(\omega)$. Either a_0^+ is incident with an edge/marked in $\mathcal{G}(\omega)$, in which case $E(\omega)$ contains an element of the form (a_0, d_0) with $c_0 < d_0 \leqslant q$, or a_0^+ is a free vertex in $\mathcal{G}(\omega)$, in which case we set $d_0 = q + 1$.

We choose $a_1 \in \{a_0', \ldots, a_0 - 1\}$ maximal such that $(a_1, c_1) \in E(\omega)$ for some c_1 with $c_0 \leqslant c_1 < d_0$. There is an element $\omega' \in \overline{\mathfrak{T}}$ such that

$$E(\omega') = \begin{cases} (E(\omega) \smallsetminus \{(a_1, c_1), (a_0, d_0)\}) \cup \{(a_1, d_0), (a_0, c_1)\} & \text{if } d_0 \leqslant q, \\ (E(\omega) \smallsetminus \{(a_1, c_1)\}) \cup \{(a_0, c_1)\} & \text{if } d_0 = q + 1. \end{cases}$$

In each situation, the graph $\mathcal{G}(\omega')$ yields $\mathcal{G}(\omega)$ by one of the moves of Figure 1:

- In the case where $d_0 \leqslant q$, the graph $\mathcal{G}(\omega)$ has an edge (a_0^+, d_0^-). If $a_1, c_1 \neq 0$, then (a_1^+, c_1^-) is also an edge in $\mathcal{G}(\omega)$, and the relation between $\mathcal{G}(\omega')$ and $\mathcal{G}(\omega)$ is as depicted in Case 1 of Figure 1. If $c_1 = 0$ (resp. $a_1 = 0$), then $\mathcal{G}(\omega)$ has a mark at a_1^+ (resp. c_1^-) and the relation with $\mathcal{G}(\omega')$ is as in Case 2 - upper subcase (resp. lower subcase) of Figure 1.
- In the case where $d_0 = q + 1$, the vertex a_0^+ is a free vertex in $\mathcal{G}(\omega)$. The relation between $\mathcal{G}(\omega')$ and $\mathcal{G}(\omega)$ is as described in Case 3 - upper subcase, Case 5 - upper subcase, or Case 4 - lower subcase of Figure 1, depending on whether $a_1, c_1 \neq 0$, $c_1 = 0$ (and $a_1 \neq 0$), or $a_1 = 0$ (and $c_1 \neq 0$).

In particular we have $\omega \prec \omega'$ (by Lemma 3.6).

For all $(i,j) \in \{0,\ldots,p\} \times \{0,\ldots,q\}$, we have $r_{i,j}(\omega') = r_{i,j}(\omega)$ unless $a_1 \leqslant i < a_0$ and $c_1 \leqslant j < d_0$, in which case we have $r_{i,j}(\omega') = r_{i,j}(\omega) - 1$. In the latter situation, we nevertheless have

$$r_{i,j}(\omega) = r_{a_0,j}(\omega) \geqslant r_{a_0,j}(\omega'') = 1 + r_{a_0-1,j}(\omega'') \geqslant 1 + r_{i,j}(\omega'')$$

(where the first equality is due to the maximality of a_1). Thus, the inequality $r_{i,j}(\omega') \geqslant r_{i,j}(\omega'')$ holds for all pair (i,j), and therefore we have $\omega' \preceq \omega''$. The element ω' satisfies the required conditions. This completes the proof of the lemma. □

Lemma 3.9. *If* $\mathbb{O}_{\omega'}$ *covers* $\mathbb{O}_\omega$*, then* $\dim \mathbb{O}_{\omega'} = \dim \mathbb{O}_\omega + 1$.

Proof. Note that $\mathbb{O}_{\omega'}$ covers $\mathbb{O}_\omega$ if and only if $\overline{\mathbb{O}_\omega}$ is an irreducible component of $\overline{\mathbb{O}_{\omega'}} \setminus \mathbb{O}_{\omega'}$. The conclusion of the lemma is implied by the following general fact, taking also Lemma 3.1 into account.

Fact: Given a connected solvable algebraic group acting on an algebraic variety, the boundary $\partial O = \overline{O} \setminus O$ of each (non closed) orbit is equidimensional of codimension 1 in $\overline{O}$.

A proof of this fact can be found in [14, Lemmas 2.12–2.13]. □

Now we are in position to proceed with the proof of Theorems 2.2(4) and 2.3.

Proof of Theorem 2.2(4). The "only if" part is shown in Lemma 3.7. For the inverse implication, let $\omega, \omega' \in \overline{\mathfrak{T}}$ be such that $\omega \preceq \omega'$. Repeated applications of Lemma 3.8 yield a sequence of elements

$$\omega = \omega_0 \prec \omega_1 \prec \cdots \prec \omega_\ell = \omega'$$

such that (ω_{k-1}, ω_k) fits in one of the cases of Figure 1 for all k. Then, Lemma 3.6 shows that the following sequence of inclusions holds:

$$\overline{\mathbb{O}_{\omega_0}} \subset \overline{\mathbb{O}_{\omega_1}} \subset \cdots \subset \overline{\mathbb{O}_{\omega_\ell}}.$$

In particular, we get the desired inclusion $\overline{\mathbb{O}_\omega} \subset \overline{\mathbb{O}_{\omega'}}$. □

Proof of Theorem 2.3. Assume that condition (1) of Theorem 2.3 holds. First, the equality $\dim \mathbb{O}_{\omega'} = \dim \mathbb{O}_\omega + 1$ follows from Lemma 3.9. Next, we have in particular $\omega \prec \omega'$ in view of Theorem 2.2(4). Lemma 3.8 yields an element $\omega_0 \in \overline{\mathfrak{T}}$ with $\omega \preceq \omega_0 \preceq \omega'$ and such that (ω, ω_0) or (ω_0, ω') fits in one of the cases of Figure 1. By Theorem 2.2(4) again, we get $\overline{\mathbb{O}_\omega} \subset \overline{\mathbb{O}_{\omega_0}} \subset \overline{\mathbb{O}_{\omega'}}$, and therefore $\omega = \omega_0$ or $\omega_0 = \omega'$, due to the assumption that $\mathbb{O}_{\omega'}$ covers $\mathbb{O}_\omega$. In both cases this implies that the pair (ω, ω') fits in one of the cases of Figure 1, that is, the graph $\mathcal{G}(\omega)$ is obtained from $\mathcal{G}(\omega')$ by one of the moves listed in Figure 1. This yields condition (2) of Theorem 2.3.

Conversely, assume (2). By Lemma 3.6, the inclusion $\overline{\mathbb{O}_\omega} \subset \overline{\mathbb{O}_{\omega'}}$ holds. This inclusion, combined with the fact that $\dim \mathbb{O}_{\omega'} = \dim \mathbb{O}_\omega + 1$, implies that $\mathbb{O}_{\omega'}$ covers $\mathbb{O}_\omega$. □

3.4. Further remarks. (a) In view of Lemma 3.1, the results shown in Section 3 establish the properties of the K-orbits on $\mathfrak{X}$ as well as of the B_K-orbits on $\mathrm{Gr}(V,r)$. Specifically, we obtain the decomposition

$$\mathrm{Gr}(V,r) = \bigsqcup_{\omega \in \overline{\mathfrak{T}}} B_K \cdot [\omega].$$

Note that $\mathrm{Gr}(V,r)$ is a fortiori a union of finitely many orbits for the action of K. The description of these K-orbits is well known, and it can be related to the decomposition into B_K-orbits in the following way.

Given $\omega \in \overline{\mathfrak{T}}$, we have introduced a matrix $R(\omega) = (r_{i,j}(\omega))_{\substack{0\leqslant i\leqslant p\\ 0\leqslant j\leqslant q}}$ which determines the orbit $B_K\cdot[\omega]$ (Theorem 2.2(3)) and its closure relations with other orbits (Theorem 2.2(4)). In particular, the pair of integers $(r_{p,0}(\omega), r_{0,q}(\omega))$ can be expressed as

$$(r_{p,0}(\omega), r_{0,q}(\omega)) = (\dim[\omega] \cap V^+, \dim[\omega] \cap V^-),$$

and we have actually

$$K \cdot [\omega] = \{W \in \mathrm{Gr}(V, r) : (\dim W \cap V^+, \dim W \cap V^-) = (r_{p,0}(\omega), r_{0,q}(\omega))\}.$$

Thus

$$K \cdot [\omega] = K \cdot [\omega'] \iff (r_{p,0}(\omega), r_{0,q}(\omega)) = (r_{p,0}(\omega'), r_{0,q}(\omega')).$$

Moreover,

$$K \cdot [\omega] \subset \overline{K \cdot [\omega']} \iff \big(r_{p,0}(\omega) \geqslant r_{p,0}(\omega') \quad \text{and} \quad r_{0,q}(\omega) \geqslant r_{0,q}(\omega')\big).$$

Note that the number $s := r_{p,0}(\omega)$ (resp. $t := r_{0,q}(\omega)$) is the number of marks among the positive (resp. negative) vertices of the graph $\mathcal{G}(\omega)$. In view of Theorem 2.2(2), the B_K-orbit $B_K\cdot[\omega]$ is dense in its K-saturation $K\cdot[\omega]$ if and only if the degree of vertices is nondecreasing from left to right along the row of positive (resp. negative) vertices of $\mathcal{G}(\omega)$ (i.e. marked vertices are located on the right and free vertices are located on the left), and each pair of edges has a crossing. Thus there are $\binom{k}{2}$ crossings, where $k := r - (s+t)$ is the number of edges, and we have

$$\dim K \cdot [\omega] = \dim B_K \cdot [\omega] = (s+k)(p-s) + (t+k)(q-t) - k^2.$$

For example, if $p = q = r = 2$, the variety $\mathrm{Gr}(V, r)$ is the union of six K-orbits. In Figure 5 we indicate the graphs $\mathcal{G}(\omega)$ corresponding to the B_K-orbits $B_K \cdot [\omega]$ which are dense in their K-saturation $K \cdot [\omega]$, the dimensions of the K-orbits, and the cover relations; this diagram is deduced from Figure 2 given in Example 2.4.

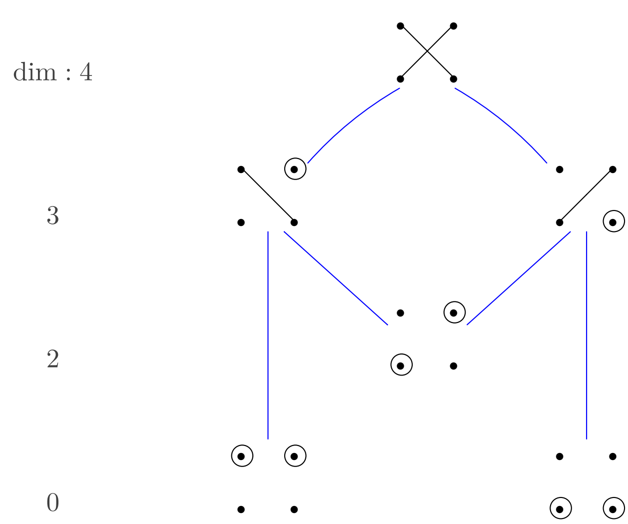


Figure 5. The parameters of the K-orbits of $\mathrm{Gr}(V, r)$ and the cover relations for $p = q = r = 2$.

(b) In [9], Matsuki and Oshima classify the orbit set $K\backslash G/B$, where $B \subset G = \mathrm{GL}_{p+q}(\mathbb{C})$ is a Borel subgroup. It appears that our parametrization of $\mathfrak{X}/K \cong B_K\backslash G/P$ resembles to theirs; here $P \subset G$ denotes the maximal parabolic subgroup obtained as the stabilizer of an r-dimensional subspace, and which satisfies $B \subset P$. In particular, by gluing orbits, Matsuki and Oshima's classification yields a parametrization of $K\backslash G/P$ which coincides with the one described in part (a) above. We can speculate on a deeper relation between the two orbit sets $K\backslash G/B$ and $B_K\backslash G/P$. We also refer to [15], where the image of the moment map for the action of K on G/B is considered.

4. Calculation of symmetrized and exotic Steinberg maps

4.1. Conormal direction. As shown in Theorem 2.2, every K-orbit in $\mathfrak{X}$ takes the form

$$\mathbb{O}_\omega = K \cdot ([\omega], \mathcal{F}_0^+, \mathcal{F}_0^-)$$

for a matrix $\omega = \begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix} \in \overline{\mathfrak{T}} = \mathfrak{T}_{(p,q),r}/\mathfrak{S}_r$, where $[\omega] \in \mathrm{Gr}(V, r)$ stands for the image of ω and $(\mathcal{F}_0^+, \mathcal{F}_0^-) \in \mathrm{Fl}(V^+)\times\mathrm{Fl}(V^-)$ is the pair of standard flags. With the notation of Section 1.2 we have

$$\mathfrak{nil}([\omega]) = \{x \in \mathfrak{gl}(V) : \operatorname{Im} x \subset [\omega] \subset \operatorname{Ker} x\}, \quad \mathfrak{nil}(\mathcal{F}_0^+) = \mathfrak{n}_p^+, \quad \text{and} \quad \mathfrak{nil}(\mathcal{F}_0^-) = \mathfrak{n}_q^+$$

where $\mathfrak{n}_k^+ \subset \mathfrak{gl}_k(\mathbb{C})$ denotes the subalgebra of strictly upper-triangular matrices. Hence, the conormal bundle to the orbit $\mathbb{O}_\omega$ is obtained as

$$T^*_{\mathbb{O}_\omega}\mathfrak{X} = K \cdot \{([\omega], \mathcal{F}_0^+, \mathcal{F}_0^-, x) : x \in \mathcal{D}_\omega\}$$

where

$$\mathcal{D}_\omega := \left\{ x = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathfrak{gl}(V) : (a,d) \in \mathfrak{n}_p^+ \times \mathfrak{n}_q^+,\ \operatorname{Im} x \subset [\omega] \subset \operatorname{Ker} x \right\}. \tag{15}$$

This immediately implies:

Lemma 4.1. *Let $\omega \in \overline{\mathfrak{T}}$. Then, $\Phi_{\mathfrak{k}}(\mathbb{O}_\omega)$, respectively $\Phi_{\mathfrak{s}}(\mathbb{O}_\omega)$, is characterized as being the unique K-orbit of $\mathcal{N}_{\mathfrak{k}}$, resp. $\mathcal{N}_{\mathfrak{s}}$, which intersects the space $\{x_{\mathfrak{k}} : x \in \mathcal{D}_\omega\}$, resp. $\{x_{\mathfrak{s}} : x \in \mathcal{D}_\omega\}$, along a dense open subset.*

For the computation of the maps $\Phi_{\mathfrak{k}}$ and $\Phi_{\mathfrak{s}}$, we need a more detailed description of the conormal direction $\mathcal{D}_\omega$. We use the following notation: if a is a $k \times \ell$ matrix, then for every subsets $R \subset \{1, \ldots, k\}$ and $S \subset \{1, \ldots, \ell\}$, we denote by $(a)_{R,S}$ the submatrix of a formed by the coefficients $a_{i,j}$ with $i \in R$, $j \in S$, and we view it as a linear map from $\mathbb{C}^S$ to $\mathbb{C}^R$. Recall from Section 2.1 that ω gives rise to decompositions $\overline{p} := \{1, \ldots, p\} = I \cup L \cup L'$ and $\overline{q} := \{1, \ldots, q\} = J \cup M \cup M'$ and to a bijection $\sigma : J \to I$ which we view as a linear map from $\mathbb{C}^J$ to $\mathbb{C}^I$.

Lemma 4.2. *A matrix $x = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ (with $a \in \mathfrak{n}_p^+$, $d \in \mathfrak{n}_q^+$) belongs to $\mathcal{D}_\omega$ if and only if it satisfies the following conditions:*

$$\begin{cases} (a)_{\overline{p},L} = 0,\ (c)_{\overline{q},L} = 0,\ (b)_{\overline{p},M} = 0,\ (d)_{\overline{q},M} = 0,\\ \qquad (b)_{\overline{p},J} = -(a)_{\overline{p},I}\sigma,\ (d)_{\overline{q},J} = -(c)_{\overline{q},I}\sigma, \end{cases} \tag{16}$$

$$\begin{cases} (a)_{L',\overline{p}} = 0,\ (b)_{L',\overline{q}} = 0,\ (c)_{M',\overline{p}} = 0,\ (d)_{M',\overline{q}} = 0,\\ \qquad (a)_{I,\overline{p}} = \sigma(c)_{J,\overline{p}},\ (b)_{I,\overline{q}} = \sigma(d)_{J,\overline{q}}. \end{cases} \tag{17}$$

Proof. Let $(e_1^+, \ldots, e_p^+)$ be the standard basis of $V^+ = \mathbb{C}^p \times \{0\}^q$ and let $(e_1^-, \ldots, e_q^-)$ be the standard basis of $V^- = \{0\}^p \times \mathbb{C}^q$. Then

$$[\omega] = \langle e_i^+ : i \in L;\quad e_j^- : j \in M;\quad e_{\sigma(j)}^+ + e_j^- : j \in J\rangle. \tag{18}$$

For every matrix x such that $a \in \mathfrak{n}_p^+$ and $d \in \mathfrak{n}_q^+$, we have $x \in \mathcal{D}_\omega$ if and only if $\operatorname{Im} x \subset [\omega] \subset \operatorname{Ker} x$, and in view of (18) the second inclusion is equivalent to (16) while the first inclusion is equivalent to (17). □

Figure 6 illustrates the form of the elements in the conormal direction $\mathcal{D}_\omega$. The matrix is represented blockwise with blocks indicating the submatrices $(X)_{R,S}$ relative to the subsets $R, S \in \{I, J, L, L', M, M'\}$. Note that in the decompositions $I \cup L \cup L' = \overline{p}$ and $J \cup M \cup M' = \overline{q}$ the subsets are not consecutive, and the matrix is thus represented modulo permutation within the rows and the columns. In particular, it is required in addition that the blocks a and d be strictly upper triangular.

$$\begin{array}{c} \\ I \\ L \\ L' \\ J \\ M \\ M' \end{array}\begin{array}{c} \begin{array}{cccccc} I & L & L' & J & M & M' \end{array} \\ \left(\begin{array}{c|c|c||c|c|c} \sigma c_1 & 0 & \sigma c_2 & -\sigma c_1 \sigma & 0 & \sigma d_2 \\ \hline a_3 & 0 & a_4 & -a_3\sigma & 0 & b_4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline c_1 & 0 & c_2 & -c_1\sigma & 0 & d_2 \\ \hline c_3 & 0 & c_4 & -c_3\sigma & 0 & d_4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)\end{array}$$

FIGURE 6. Form of the elements $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ (with $a \in \mathfrak{n}_p^+$, $d \in \mathfrak{n}_q^+$) belonging to the conormal direction $\mathcal{D}_\omega$.

4.2. A REVIEW OF THE ORBIT SETS, AND AN INVOLUTION. As explained in Section 2.3, the orbits of K in the nilpotent cone

$$\mathcal{N}_\mathfrak{k} \subset \mathfrak{k} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : (a, d) \in \mathfrak{gl}_p(\mathbb{C}) \times \mathfrak{gl}_q(\mathbb{C}) \right\}$$

are parametrized by pairs of partitions $(\lambda, \mu) \in \mathcal{P}(p) \times \mathcal{P}(q)$, and we denote by $\mathfrak{O}_{\lambda,\mu}$ the orbit corresponding to the pair (λ, μ). Note that

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathfrak{O}_{\lambda,\mu} \Longleftrightarrow \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix} \in \mathfrak{O}_{\mu,\lambda}, \tag{19}$$

though here the notation $\mathfrak{O}_{\mu,\lambda}$ refers to an orbit of $K^* := \mathrm{GL}_q(\mathbb{C}) \times \mathrm{GL}_p(\mathbb{C})$ on $\mathfrak{gl}_q(\mathbb{C}) \times \mathfrak{gl}_p(\mathbb{C})$.

Recall also from Section 2.3 that the orbits of K in the nilpotent cone

$$\mathcal{N}_\mathfrak{s} \subset \mathfrak{s} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : (b, c) \in \mathrm{M}_{p,q}(\mathbb{C}) \times \mathrm{M}_{q,p}(\mathbb{C}) \right\}$$

are parametrized by signed Young diagrams of signature (p, q), and we denote by $\mathfrak{O}_\Lambda$ the orbit corresponding to Λ. We have

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in \mathfrak{O}_\Lambda \Longleftrightarrow \begin{pmatrix} 0 & c \\ b & 0 \end{pmatrix} \in \mathfrak{O}_{\Lambda^*} \tag{20}$$

where Λ^* denotes the signed Young diagram of signature (q, p) obtained from Λ by switching the $+$'s and the $-$'s, and $\mathfrak{O}_{\Lambda^*}$ is an orbit of the group K^*.

As shown in Theorem 2.2, the orbits of K in

$$\mathfrak{X} = \mathrm{Gr}(V, r) \times \mathrm{Fl}(V^+) \times \mathrm{Fl}(V^-)$$

are parametrized by the elements of $\overline{\mathfrak{T}}$, and $\mathbb{O}_\omega$ is the orbit corresponding to ω. If $\omega = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$ is an element of $\overline{\mathfrak{T}} = \mathfrak{T}_{(p,q),r}/\mathfrak{S}_r$, then $\omega^* := \begin{pmatrix} \tau_2 \\ \tau_1 \end{pmatrix}$ is an element of $\overline{\mathfrak{T}}^* := \mathfrak{T}_{(q,p),r}/\mathfrak{S}_r$ which thus yields an orbit $\mathbb{O}_{\omega^*}$ of K^* in a suitable multiple flag variety $\mathfrak{X}^*$. The graphic representation $\mathcal{G}(\omega^*)$ of ω^* is obtained from $\mathcal{G}(\omega)$ by switching the two rows of vertices, i.e. by relabeling every vertex i^+ (resp. j^-) as i^- (resp. j^+).

This implies that, if $(I, J, L, L', M, M', \sigma)$ are the data corresponding to ω in the sense of Section 2.1, then the relevant data for ω^* are

$$(I^*, J^*, L^*, L'^*, M^*, M'^*, \sigma) = (J, I, M, M', L, L', \sigma^{-1}). \tag{21}$$

Finally, by the description of the conormal direction in (15), we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{D}_\omega \iff \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in \mathcal{D}_{\omega^*}. \tag{22}$$

The observations made in (19), (20), (22) combined with Lemma 4.1 yield the following statement:

Lemma 4.3. (1) *If $\Phi_{\mathfrak{k}}(\mathbb{O}_\omega) = \mathfrak{O}_{\lambda,\mu}$, then $\Phi_{\mathfrak{k}}(\mathbb{O}_{\omega^*}) = \mathfrak{O}_{\mu,\lambda}$.*
(2) *If $\Phi_{\mathfrak{s}}(\mathbb{O}_\omega) = \mathfrak{O}_\Lambda$, then $\Phi_{\mathfrak{s}}(\mathbb{O}_{\omega^*}) = \mathfrak{O}_{\Lambda^*}$.*

There is an abuse of notation in that statement, since we use the notation $\Phi_{\mathfrak{k}}$ and $\Phi_{\mathfrak{s}}$ to designate also the symmetrized and exotic Steinberg maps relative to the symmetric pair $(G, K^*) = (\mathrm{GL}_{p+q}(\mathbb{C}), \mathrm{GL}_q(\mathbb{C}) \times \mathrm{GL}_p(\mathbb{C}))$.

4.3. Symmetrized Steinberg map $\Phi_{\mathfrak{k}}$. For a permutation $w \in \mathfrak{S}_k$ we consider the space

$$\mathfrak{n}_k^+ \cap ({}^w\mathfrak{n}_k^+) \coloneqq \mathfrak{n}_k^+ \cap (\mathrm{Ad}(w)(\mathfrak{n}_k^+)) = \{a \in \mathfrak{n}_k^+ : w^{-1}aw \in \mathfrak{n}_k^+\}.$$

The following is a well-known fact from classical Steinberg theory.

Theorem 4.4 ([13]). *The unique nilpotent orbit $\mathcal{O}_\lambda \subset \mathfrak{gl}_k(\mathbb{C})$ which intersects the space $\mathfrak{n}_k^+ \cap ({}^w\mathfrak{n}_k^+)$ along a dense open subset is the one corresponding to the Young diagram $\lambda = \mathrm{shape}(\mathrm{RS}_1(w)) = \mathrm{shape}(\mathrm{RS}_2(w))$, where $(\mathrm{RS}_1(w), \mathrm{RS}_2(w))$ denotes the pair of Young tableaux associated to w via the Robinson–Schensted correspondence.*

In our situation, we consider the permutations $w_{\mathfrak{k},+} \in \mathfrak{S}_p$ and $w_{\mathfrak{k},-} \in \mathfrak{S}_q$ of (3) and (4), associated to an element $\omega \in \overline{\mathfrak{T}}$.

Lemma 4.5. *Let $a \in \mathfrak{n}_p^+$. The following conditions are equivalent:*

(1) *There are matrices b, c, d such that $x \coloneqq \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{D}_\omega$;*
(2) *$\sigma^{-1}(a)_{I,I}\sigma$ is strictly upper triangular, $(a)_{L',\overline{p}} = 0$, and $(a)_{\overline{p},L} = 0$;*
(3) *$a \in \mathfrak{n}_p^+ \cap ({}^{w_{\mathfrak{k},+}}\mathfrak{n}_p^+)$.*

Proof. The equivalence between conditions (1) and (2) is implied by Lemma 4.2 (see Figure 6). Given $a \in \mathfrak{n}_p^+$, condition (2) is equivalent to:

$$\left(1 \leqslant i < j \leqslant p \quad \text{and} \quad \begin{cases} i \in L' \\ \text{or} \quad j \in L \\ \text{or} \quad (i, j \in I \ \text{ and } \ \sigma^{-1}(i) > \sigma^{-1}(j)) \end{cases}\right) \implies a_{i,j} = 0.$$

Condition (3) is equivalent to:

$$1 \leqslant i < j \leqslant p \quad \text{and} \quad w_{\mathfrak{k},+}^{-1}(i) > w_{\mathfrak{k},+}^{-1}(j) \implies a_{i,j} = 0.$$

By definition of $w_{\mathfrak{k},+}$ (see (3)), for $1 \leqslant i < j \leqslant p$, we have

$$w_{\mathfrak{k},+}^{-1}(i) > w_{\mathfrak{k},+}^{-1}(j) \iff \begin{cases} i \in L' \\ \text{or} \quad j \in L \\ \text{or} \quad (i, j \in I \ \text{ and } \ \sigma^{-1}(i) > \sigma^{-1}(j)). \end{cases}$$

Therefore, conditions (2) and (3) are equivalent. $\square$

Proof of Theorem 2.5(1). Let

$$\Phi_{\mathfrak{k}}(\mathbb{O}_\omega) = \mathfrak{O}_{\lambda,\mu} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : (a,d) \in \mathcal{O}_\lambda \times \mathcal{O}_\mu \subset \mathfrak{gl}_p(\mathbb{C}) \times \mathfrak{gl}_q(\mathbb{C}) \right\}.$$

By Lemmas 4.1 and 4.5, the nilpotent orbit $\mathcal{O}_\lambda \subset \mathfrak{gl}_p(\mathbb{C})$ is characterized as being the unique $\mathrm{GL}_p(\mathbb{C})$-orbit which intersects the space

$$\left\{ a \in \mathfrak{n}_p^+ : \exists b, c, d \text{ such that } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{D}_\omega \right\} = \mathfrak{n}_p^+ \cap ({}^{w_{\mathfrak{k},+}}\mathfrak{n}_p^+)$$

along a dense open subset. By Theorem 4.4, this implies that

$$\lambda = \mathrm{shape}(\mathrm{RS}_1(w_{\mathfrak{k},+})).$$

By (21) and Lemma 4.3, we also deduce that

$$\mu = \mathrm{shape}(\mathrm{RS}_1(w_{\mathfrak{k},-})).$$

The proof of Theorem 2.5(1) is complete. □

4.4. Exotic Steinberg map $\Phi_{\mathfrak{s}}$. We introduce notation which extend our notation on matrices. Given subsets R, S of integers, we let $\mathrm{M}_{R,S}(\mathbb{C})$ denote the space of linear homomorphisms $x : \mathbb{C}^S \to \mathbb{C}^R$, which can be viewed as well as matrices of coefficients $(x_{i,j})_{(i,j)\in R\times S}$. Let $\mathfrak{n}_R^+ \subset \mathrm{M}_{R,R}(\mathbb{C})$ be the subspace of endomorphisms that are strictly upper triangular as matrices.

If R, S are respectively subsets of R', S', we denote by

$$\eta_{R,S}^{R',S'} : \mathrm{M}_{R,S}(\mathbb{C}) \to \mathrm{M}_{R',S'}(\mathbb{C}), \quad x \mapsto \hat{x} = (\hat{x}_{i,j})_{(i,j)\in R'\times S'} \tag{23}$$

the linear morphism which maps a matrix x to its *extension by zero* given by $\hat{x}_{i,j} = x_{i,j}$ if $(i,j) \in R \times S$ and $\hat{x}_{i,j} = 0$ otherwise.

A bijection $w : S \to R$ yields an element of $\mathrm{M}_{R,S}(\mathbb{C})$ also denoted by w by abuse of notation. In addition, through the Robinson–Schensted algorithm, w gives rise to a pair of Young tableaux $(\mathrm{RS}_1(w), \mathrm{RS}_2(w))$ of the same shape, whose respective sets of entries are R and S.

The following is a reformulation of Theorem 4.4.

Proposition 4.6. *Given a bijection $w : S \to R$, the Jordan normal form of a general element x in the space*

$$\mathfrak{n}_R^+ \cap ({}^{w}\mathfrak{n}_S^+) := \{x \in \mathfrak{n}_R^+ : w^{-1}xw \in \mathfrak{n}_S^+\}$$

is given by the Young diagram $\mathrm{shape}(\mathrm{RS}_1(w)) = \mathrm{shape}(\mathrm{RS}_2(w))$. *In other words, for all $k \geqslant 1$,* $\dim\ker x^k$ *is the number of boxes in the first k columns of* $\mathrm{RS}_1(w)$.

Take an element $\omega \in \overline{\mathfrak{T}}$ with corresponding data $(I, J, L, L', M, M', \sigma)$. As in Section 2.3, we write $J = \{j_1 < \ldots < j_k\}$, $L' = \{\ell'_1 < \ldots < \ell'_{s'}\}$, $M = \{m_1 < \ldots < m_t\}$. Moreover, we denote

$$S = J \cup M \cup \{q+1, \ldots, q+s'\} \quad \text{and} \quad R = \{-t, \ldots, -1\} \cup I \cup L'.$$

We will consider the bijection $w := w_{\mathfrak{s},+} : S \to R$ defined in (5).

We denote $\varsigma = \eta_{I,J}^{I\cup L', J\cup M}(\sigma)$ and $\tau = \eta_{I,J}^{\overline{p},\overline{q}}(\sigma) = \eta_{I\cup L', J\cup M}^{\overline{p},\overline{q}}(\varsigma)$ (see (23)), that is,

$$\varsigma = \begin{array}{c c} & \begin{array}{cc} J & M \end{array} \\ \begin{array}{c} I \\ L' \end{array} & \left(\begin{array}{c|c} \sigma & 0 \\ \hline 0 & 0 \end{array}\right) \end{array}, \qquad \tau = \begin{array}{c c} & \begin{array}{ccc} J & M & M' \end{array} \\ \begin{array}{c} I \\ L \\ L' \end{array} & \left(\begin{array}{c|c|c} \sigma & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} \tag{24}$$

(after changing the order of rows and columns).

Lemma 4.7. *Let* $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{D}_\omega$*, so that* $x_{\mathfrak{k}} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ *and* $x_{\mathfrak{s}} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$*. Then, for every* $m \geqslant 0$*, we have*

$$(x_{\mathfrak{s}})^{2m} = (-1)^m (x_{\mathfrak{k}})^{2m} = (-1)^m \begin{pmatrix} a^{2m} & 0 \\ 0 & d^{2m} \end{pmatrix} \quad \text{and} \quad (x_{\mathfrak{s}})^{2m+1} = (-1)^m \begin{pmatrix} 0 & * \\ (c\tau)^{2m} c & 0 \end{pmatrix}$$

(*where the symbol* $*$ *stands for some matrix in* $\mathrm{M}_{p,q}(\mathbb{C})$*, whose precise description is not needed*).

Proof. Every element $x \in \mathcal{D}_\omega$ is such that $x^2 = 0$. On the other hand, we can see that $(x^2)_{\mathfrak{k}} = (x_{\mathfrak{k}})^2 + (x_{\mathfrak{s}})^2$. The formula for $(x_{\mathfrak{s}})^{2m}$ ensues.

It readily follows that

$$(x_{\mathfrak{s}})^{2m+1} = (-1)^m \begin{pmatrix} 0 & a^{2m} b \\ d^{2m} c & 0 \end{pmatrix}.$$

For every $\ell \geqslant 1$, using the notation of Figure 6, we can see that $(-1)^\ell d^\ell c$ is the matrix (written blockwise)

$$\begin{array}{cc} & \begin{array}{ccc} I & L & L' \end{array} \\ \begin{array}{c} J \\ M \\ M' \end{array} & \left(\begin{array}{c|c|c} (c_1\sigma)^\ell c_1 & 0 & (c_1\sigma)^\ell c_2 \\ \hline c_3\sigma(c_1\sigma)^{\ell-1} c_1 & 0 & c_3\sigma(c_1\sigma)^{\ell-1} c_2 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array}, \tag{25}$$

and this coincides with $(c\tau)^\ell c$. Letting $\ell = 2m$, this yields the formula claimed for $(x_{\mathfrak{s}})^{2m+1}$ in the case of $m \geqslant 1$. The claimed formula is immediate if $m = 0$. $\square$

We consider the following extension-by-zero mappings:

$$\begin{tikzcd}[column sep=small] & \mathrm{M}_{J\cup M, I\cup L'}(\mathbb{C}) \arrow[dl, "\eta_1 := \eta^{\overline{q},\overline{p}}_{J\cup M, I\cup L'}"'] \arrow[dr, "\eta_2 := \eta^{S,R}_{J\cup M, I\cup L'}"] & \\ \mathrm{M}_{\overline{q},\overline{p}}(\mathbb{C}) = \mathrm{M}_{q,p}(\mathbb{C}) & & \mathrm{M}_{S,R}(\mathbb{C}). \end{tikzcd}$$

In addition, we consider the subspace $\Gamma := \{\gamma \in \mathrm{M}_{J\cup M, I\cup L'}(\mathbb{C}) : \varsigma\gamma \in \mathfrak{n}^+_{I\cup L'},\ \gamma\varsigma \in \mathfrak{n}^+_{J\cup M}\}$ (where ς is as in (24)).

Lemma 4.8. *The maps* η_1 *and* η_2 *restrict to linear isomorphisms*

$$(\eta_1)|_\Gamma : \Gamma \xrightarrow{\sim} \left\{ c \in \mathrm{M}_{q,p}(\mathbb{C}) : \exists a, b, d \text{ such that } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{D}_\omega \right\}$$

and

$$(\eta_2)|_\Gamma : \Gamma \xrightarrow{\sim} \{ z \in \mathrm{M}_{S,R}(\mathbb{C}) : wz \in \mathfrak{n}^+_R,\ zw \in \mathfrak{n}^+_S \} = \{ z \in \mathrm{M}_{S,R}(\mathbb{C}) : wz \in \mathfrak{n}^+_R \cap ({}^w\mathfrak{n}^+_S) \},$$

where $w = w_{\mathfrak{s},+}$ *is the bijection defined in* (5)*. Moreover, for every* $\gamma \in \Gamma$ *such that* $c = \eta_1(\gamma)$ *and* $z = \eta_2(\gamma)$*, we have*

$$\eta_1((\gamma\varsigma)^\ell \gamma) = (c\tau)^\ell c \quad \text{and} \quad \eta_2((\gamma\varsigma)^\ell \gamma) = (zw)^\ell z = w^{-1}(wz)^{\ell+1} \quad \text{for all } \ell \geqslant 0,$$

where ς *and* τ *are given by* (24).

Proof. The space Γ consists of the matrices

$$\gamma = \begin{array}{cc} & \begin{array}{cc} I & L' \end{array} \\ \begin{array}{c} J \\ M \end{array} & \left(\begin{array}{c|c} c_1 & c_2 \\ \hline c_3 & c_4 \end{array}\right) \end{array} \in \mathrm{M}_{J\cup M, I\cup L'}(\mathbb{C}) \tag{26}$$

(written blockwise) such that

$$\varsigma\gamma = \begin{array}{c|c} & \begin{array}{cc} I & L' \end{array} \\ \begin{array}{c} I \\ L' \end{array} & \left(\begin{array}{c|c} \sigma c_1 & \sigma c_2 \\ \hline 0 & 0 \end{array}\right) \end{array} \quad \text{and} \quad \gamma\varsigma = \begin{array}{c|c} & \begin{array}{cc} J & M \end{array} \\ \begin{array}{c} J \\ M \end{array} & \left(\begin{array}{c|c} c_1\sigma & 0 \\ \hline c_3\sigma & 0 \end{array}\right) \end{array} \tag{27}$$

are strictly upper triangular (before rearranging the rows and the columns). Moreover, for every $\ell \geqslant 1$, we have

$$(\gamma\varsigma)^\ell\gamma = \begin{array}{cc} & \begin{array}{cc} I & L' \end{array} \\ \begin{array}{c} J \\ M \end{array} & \left(\begin{array}{c|c} (c_1\sigma)^\ell c_1 & (c_1\sigma)^\ell c_2 \\ \hline c_3\sigma(c_1\sigma)^{\ell-1}c_1 & c_3\sigma(c_1\sigma)^{\ell-1}c_2 \end{array}\right) \end{array}. \tag{28}$$

One can see from Figure 6 that the space

$$\mathcal{C} := \left\{ c \in \mathrm{M}_{q,p}(\mathbb{C}) : \exists a,b,d \text{ such that } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{D}_\omega \right\}$$

consists of the matrices of the form

$$c = \begin{array}{cc} & \begin{array}{ccc} I & L & L' \end{array} \\ \begin{array}{c} J \\ M \\ M' \end{array} & \left(\begin{array}{c|c|c} c_1 & 0 & c_2 \\ \hline c_3 & 0 & c_4 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array}$$

such that the matrices

$$\begin{array}{cc} & \begin{array}{ccc} I & L & L' \end{array} \\ \begin{array}{c} I \\ L \\ L' \end{array} & \left(\begin{array}{c|c|c} \sigma c_1 & 0 & \sigma c_2 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} \in \mathrm{M}_p(\mathbb{C})$$

$$\text{and} \quad \begin{array}{cc} & \begin{array}{ccc} J & M & M' \end{array} \\ \begin{array}{c} J \\ M \\ M' \end{array} & \left(\begin{array}{c|c|c} -c_1\sigma & 0 & 0 \\ \hline -c_3\sigma & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} \in \mathrm{M}_q(\mathbb{C})$$

are strictly upper triangular (before rearranging the rows and the columns). From the above description of Γ, these conditions are equivalent to having that $c = \eta_1(\gamma)$ for some $\gamma \in \Gamma$. Hence $\mathcal{C} = \eta_1(\Gamma)$, and this implies that η_1 restricts to a linear isomorphism $(\eta_1)|_\Gamma : \Gamma \to \mathcal{C}$. Moreover, by comparing the expressions of $(c\tau)^\ell c$ and $(\gamma\varsigma)^\ell\gamma$ given in (25) and (28), respectively, one can see that the equality $\eta_1((\gamma\varsigma)^\ell\gamma) = (c\tau)^\ell c$ holds for all $\ell \geqslant 0$ whenever $\eta_1(\gamma) = c$.

We have $R = R' \cup I \cup L'$ and $S = J \cup M \cup S'$ where $R' := \{-i\}_{i=1}^t$ and $S' := \{q+j\}_{j=1}^{s'}$. Let z be an element in $\mathrm{M}_{S,R}(\mathbb{C})$, and let us write it (blockwise) as

$$z = \begin{array}{cc} & \begin{array}{ccc} R' & I & L' \end{array} \\ \begin{array}{c} J \\ M \\ S' \end{array} & \left(\begin{array}{c|c|c} z_1 & z_2 & z_3 \\ \hline z_4 & z_5 & z_6 \\ \hline z_7 & z_8 & z_9 \end{array}\right) \end{array}. \tag{29}$$

The bijection $w = w_{\mathfrak{s},+} : S \to R$ of (5)) can be written in the following matrix form

$$w = \begin{array}{c} \\ R' \\ I \\ L' \end{array}\!\!\begin{array}{c} \begin{array}{ccc} J & M & S' \end{array} \\ \left(\begin{array}{c|c|c} 0 & \sigma_0 & 0 \\ \hline \sigma & 0 & 0 \\ \hline 0 & 0 & \sigma_0' \end{array}\right) \end{array}$$

where the block $\sigma_0 \in \mathrm{M}_{R',M}(\mathbb{C})$ is yielded by the unique decreasing bijection $M \to R'$; this corresponds to a block with 1's on the antidiagonal and 0's elsewhere. The block $\sigma_0' \in \mathrm{M}_{L',S'}(\mathbb{C})$ is defined in the same way. We get

$$wz = \begin{array}{c} \\ R' \\ I \\ L' \end{array}\!\!\begin{array}{c} \begin{array}{ccc} R' & I & L' \end{array} \\ \left(\begin{array}{c|c|c} \sigma_0 z_4 & \sigma_0 z_5 & \sigma_0 z_6 \\ \hline \sigma z_1 & \sigma z_2 & \sigma z_3 \\ \hline \sigma_0' z_7 & \sigma_0' z_8 & \sigma_0' z_9 \end{array}\right) \end{array} \quad\text{and}\quad zw = \begin{array}{c} \\ J \\ M \\ S' \end{array}\!\!\begin{array}{c} \begin{array}{ccc} J & M & S' \end{array} \\ \left(\begin{array}{c|c|c} z_2\sigma & z_1\sigma_0 & z_3\sigma_0' \\ \hline z_5\sigma & z_4\sigma_0 & z_6\sigma_0' \\ \hline z_8\sigma & z_7\sigma_0 & z_9\sigma_0' \end{array}\right) \end{array}.$$

Note that we have $i < j$ whenever $(i,j) \in R' \times (I \cup L')$. We have also $i < j$ whenever $(i,j) \in (J \cup M) \times S'$. We obtain that $(wz, zw) \in \mathfrak{n}_R^+ \times \mathfrak{n}_S^+$ if and only if the following conditions are satisfied:

$$\begin{cases} \bullet \text{ the blocks } z_1, z_4, z_7, z_8, z_9 \text{ are zero,} \\ \bullet \text{ the submatrices } \begin{array}{c} \\ I \\ L' \end{array}\!\!\begin{array}{c} \begin{array}{cc} I & L' \end{array} \\ \left(\begin{array}{c|c} \sigma z_2 & \sigma z_3 \\ \hline 0 & 0 \end{array}\right) \end{array} \text{ and } \begin{array}{c} \\ J \\ M \end{array}\!\!\begin{array}{c} \begin{array}{cc} J & M \end{array} \\ \left(\begin{array}{c|c} z_2\sigma & 0 \\ \hline z_5\sigma & 0 \end{array}\right) \end{array} \text{ are} \\ \text{strictly upper triangular.} \end{cases}$$

Combining these observations with the description of the space Γ given above, we conclude that the elements in the space $\mathcal{Z} := \{z \in \mathrm{M}_{S,R}(\mathbb{C}) : wz \in \mathfrak{n}_R^+,\ zw \in \mathfrak{n}_S^+\}$ are exactly of the form $z = \eta_2(\gamma)$ for $\gamma \in \Gamma$ (see (26)–(27)) and, therefore, η_2 restricts to an isomorphism $(\eta_2)|_\Gamma : \Gamma \to \mathcal{Z}$ as asserted.

Finally, let $z = \eta_2(\gamma)$ for $\gamma \in \Gamma$ written as in (26). In the notation of (29), this means that $z_2 = c_1$, $z_3 = c_2$, $z_5 = c_3$, $z_6 = c_4$, and the other blocks of z are zero. In view of the expression for zw and $(\gamma\varsigma)^\ell\gamma$ given in (28), this yields

$$(zw)^\ell z = \begin{array}{c} \\ J \\ M \\ S' \end{array}\!\!\begin{array}{c} \begin{array}{ccc} R' & I & L' \end{array} \\ \left(\begin{array}{c|c|c} 0 & (c_1\sigma)^\ell c_1 & (c_1\sigma)^\ell c_2 \\ \hline 0 & c_3\sigma(c_1\sigma)^{\ell-1}c_1 & c_3\sigma(c_1\sigma)^{\ell-1}c_2 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} = \eta_2((\gamma\varsigma)^\ell\gamma)$$

for all $\ell \geqslant 1$. The lemma is proved. □

Proof of Theorem 2.5(2). Let $\Phi_{\mathfrak{k}}(\mathbb{O}_\omega) = \mathfrak{O}_{\lambda,\mu}$ and $\Phi_{\mathfrak{s}}(\mathbb{O}_\omega) = \mathfrak{O}_\Lambda$. Let

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a general element of $\mathcal{D}_\omega$, so that $x_{\mathfrak{k}} \in \mathfrak{O}_{\lambda,\mu}$ and $x_{\mathfrak{s}} \in \mathfrak{O}_\Lambda$ (see Lemma 4.1). It follows from Lemma 4.7 that, for every $\ell \geqslant 1$, the number $\#\Lambda_{\leqslant \ell}(+)$ of $+$'s in the first ℓ columns of the signed Young diagram Λ is equal to

$$\dim\ker a^{2m} \text{ if } \ell = 2m \text{ is even,} \quad \text{respectively} \quad \dim\ker(c\tau)^{2m}c \text{ if } \ell = 2m+1 \text{ is odd}$$

(with τ from (24)). If $\ell = 2m$ is even, Theorem 2.5 (1) shows that this number is equal to the number $\#\lambda_{\leqslant 2m}$ of boxes in the first $2m$ columns of λ. This confirms the first assertion made in Theorem 2.5(2) (a).

Now assume that $\ell = 2m+1$ is odd. By Lemma 4.8, we have $c = \eta_1(\gamma)$ with $\gamma \in \Gamma$. Moreover, we can assume that $z \coloneqq \eta_2(\gamma)$ is general in $\{z \in \mathrm{M}_{S,R}(\mathbb{C}) : wz \in \mathfrak{n}_R^+ \cap ({}^w\mathfrak{n}_S^+)\}$, with $w = w_{\mathfrak{s},+}$ as in (5). Hence, by Proposition 4.6, this element z is such that

$$\dim\ker(wz)^{2m+1} = \#\lambda'_{\leqslant 2m+1}$$

with $\lambda' = \mathrm{shape}(\mathrm{RS}_1(w))$. Note also that $\dim\ker(wz)^{2m+1} = \dim\ker(zw)^{2m}z$. In view of the previous observations, and by Lemma 4.8, we get

$$\begin{aligned}\#\Lambda_{\leqslant 2m+1}(+) &= \dim\ker(c\tau)^{2m}c \\ &= \#L + \dim\ker(\gamma\varsigma)^{2m}\gamma \\ &= \#L + \dim\ker(zw)^{2m}z - \#R' \\ &= s - t + \#\lambda'_{\leqslant 2m+1}.\end{aligned}$$

This establishes the claim in Theorem 2.5(2) (b) regarding the number of $+$'s. The formulas regarding the number of $-$'s stated in Theorem 2.5(2) (a)–(b) can now be deduced, by invoking Lemma 4.3 and taking (21) into account. The proof of Theorem 2.5(2) is complete. ☐

References

[1] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Modern Birkhäuser Classics, Birkhäuser Boston, Ltd., Boston, MA, 2010, Reprint of the 1997 edition.

[2] David H. Collingwood and William M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993.

[3] Lucas Fresse and Kyo Nishiyama, *On the exotic Grassmannian and its nilpotent variety*, Represent. Theory **20** (2016), 451–481, [Paging previously given as 1–31].

[4] ———, *Orbit embedding for double flag varieties and Steinberg maps*, in Lie groups, number theory, and vertex algebras, Contemp. Math., vol. 768, Amer. Math. Soc., [Providence], RI, [2021] ©2021, pp. 21–42.

[5] ———, *A generalization of Steinberg theory and an exotic moment map*, Int. Math. Res. Not. IMRN (2022), no. 1, 1–62.

[6] William Fulton, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. **65** (1992), no. 3, 381–420.

[7] ———, *Young tableaux. with applications to representation theory and geometry*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997.

[8] Xuhua He, Kyo Nishiyama, Hiroyuki Ochiai, and Yoshiki Oshima, *On orbits in double flag varieties for symmetric pairs*, Transform. Groups **18** (2013), no. 4, 1091–1136.

[9] Toshihiko Matsuki and Toshio Ōshima, *Embeddings of discrete series into principal series*, in The orbit method in representation theory (Copenhagen, 1988), Progr. Math., vol. 82, Birkhäuser Boston, Boston, MA, 1990, pp. 147–175.

[10] Kyo Nishiyama and Hiroyuki Ochiai, *Double flag varieties for a symmetric pair and finiteness of orbits*, J. Lie Theory **21** (2011), no. 1, 79–99.

[11] Daniele Rosso, *Classic and mirabolic Robinson-Schensted-Knuth correspondence for partial flags*, Canad. J. Math. **64** (2012), no. 5, 1090–1121.

[12] Rahul Singh, *A Robinson-Schensted correspondence for partial permutations*, 2020, `https://arxiv.org/abs/2010.13918`.

[13] Robert Steinberg, *An occurrence of the Robinson-Schensted correspondence*, J. Algebra **113** (1988), no. 2, 523–528.

[14] Dmitry A. Timashev, *A generalization of the Bruhat decomposition*, Izv. Ross. Akad. Nauk Ser. Mat. **58** (1994), no. 5, 110–123.

[15] Peter E. Trapa, *Generalized Robinson-Schensted algorithms for real groups*, Internat. Math. Res. Notices (1999), no. 15, 803–834.

[16] Roman Travkin, *Mirabolic Robinson-Schensted-Knuth correspondence*, Selecta Math. (N.S.) **14** (2009), no. 3-4, 727–758.

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