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Modified Macdonald polynomials and the multispecies zero-range process: I

Arvind Ayyer, Olya Mandelshtam & James B Martin

Abstract In this paper we prove a new combinatorial formula for the modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$, motivated by connections to the theory of interacting particle systems from statistical mechanics. The formula involves a new statistic called *queue inversions* on fillings of tableaux. This statistic is closely related to the *multiline queues* which were recently used to give a formula for the Macdonald polynomials $P_\lambda(X;q,t)$. In the case $q=1$ and $X=(1,1,\ldots,1)$, that formula had also been shown to compute stationary probabilities for a particle system known as the *multispecies ASEP* on a ring, and it is natural to ask whether a similar connection exists between the modified Macdonald polynomials and a suitable statistical mechanics model. In a sequel to this work, we demonstrate such a connection, showing that the stationary probabilities of the *multispecies totally asymmetric zero-range process* (mTAZRP) on a ring can be computed using tableaux formulas with the queue inversion statistic. This connection extends to arbitrary $X=(x_1,\ldots,x_n)$; the x_i play the role of site-dependent jump rates for the mTAZRP.

1. Introduction

The theory of symmetric functions has a long history, having its origins in invariant theory, Galois theory, group theory and, of course, combinatorics. As Stanley [29, Notes in Chapter 7] remarks, the first published work on symmetric functions was a derivation of the well-known Newton–Girard identity by A. Girard [14] in 1629. The reason for the name is that it was independently rediscovered by Newton around 1666.

On the other hand, the theory of interacting particle systems is relatively modern. It was an influential paper of Spitzer [28] in 1970 that initiated the subject and set out the important questions in the field. In particular, the simple exclusion process and the zero-range process were first defined there.

The focus of this paper is a particular family of symmetric functions, the *modified Macdonald polynomials*, motivated by a new link to an interactive particle system known as the *multispecies totally asymmetric zero-range process*.

Over the last couple of decades, the theory of special functions and symmetric functions have found unexpected connections to diverse interacting particle systems. The asymmetric simple exclusion process (ASEP) has played a central role in this

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connection. Building on work of Uchiyama, Sasamoto, and Wadati [32, 27], Corteel and Williams [8] found that the partition function of the single species ASEP with open boundaries is related to moments of Askey–Wilson polynomials, and discovered combinatorial formulas for those quantities. Generalizing this work, Corteel and Williams [9] and also Cantini [2] showed that the partition function of the two-species ASEP with open boundaries is related to certain moments of Koornwinder polynomials, and soon after, the second author along with Corteel and Williams [6] found the associated combinatorial formulas. More generally, Cantini, Garbali, de Gier and Wheeler [4] found that the partition function of a multispecies variant of the ASEP with open boundaries is a specialization of a Koornwinder polynomial, but finding formulas for these quantities is still an outstanding open question.

The multispecies ASEP on a ring is another version of the ASEP that has been found to have deep connections to orthogonal polynomials. Cantini, de Gier and Wheeler [3] showed that the stationary probabilities of the ASEP on a ring are related to the nonsymmetric Macdonald polynomials, and that the partition function of the ASEP on a circle is equal to the symmetric Macdonald polynomial $P_\lambda(X;q,t)$ evaluated at $x_1 = \cdots = x_n = q = 1$. In the totally asymmetric case (at $t=0$), Ferrari and the third author [10] introduced a recursive formulation which expresses the stationary probabilities as sums over combinatorial objects known as *multiline queues*. Recently, the third author generalized these multiline queues to compute probabilities for the full ASEP [26]. Building upon that result, the second author together with Corteel and Williams further generalized these multiline queues to incorporate the parameters $x_1, \ldots, x_n, q$ to obtain formulas for $P_\lambda(X;q,t)$ and the related nonsymmetric Macdonald polynomials [7]. These formulas interpolate between probabilities of the ASEP on a ring and the Macdonald polynomials.

There are several natural bases for symmetric polynomials. The *monomial symmetric polynomials*, *elementary symmetric polynomials*, *complete homogeneous symmetric polynomials* and *power sum symmetric polynomials* are classical. Probably the single most important family is the family of *Schur polynomials*, for which there are several generalizations. The symmetric Macdonald polynomials are a remarkable two-parameter generalization of Schur polynomials that were introduced by Macdonald in 1987 [23, 24]. They are indexed by partitions λ and are denoted $P_\lambda(X;q,t)$, where $X = (x_1, x_2, \ldots)$ is the alphabet and q and t are parameters; coefficients of $P_\lambda(X;q,t)$ are in $\mathsf{Q}(q,t)$. They can be characterized as the unique basis for the ring of symmetric functions in q and t satisfying certain orthogonality and triangularity conditions.

The modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$ defined by Garsia and Haiman [12] are a special form of the P_λ's with coefficients in $\mathsf{Z}[q,t]$ obtained through a formal operation called *plethysm* from a scalar multiple of $P_\lambda(X;q,t)$. Understanding the combinatorics of these polynomials has been a fundamental area of interest in algebraic combinatorics. The most commonly used combinatorial description of $\widetilde{H}_\lambda(X;q,t)$ is a tableau formula due to Haglund, Haiman, and Loehr [15]:

$$\widetilde{H}_\lambda(X;q,t) = \sum_{\sigma:\mathrm{dg}(\lambda)\to\mathsf{Z}} t^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} x^\sigma \tag{1}$$

where the sum is over all fillings of a diagram $\mathrm{dg}(\lambda)$ whose shape is the partition λ. The *inversion* statistic $\mathrm{inv}(\sigma)$ and the *major index* statistic $\mathrm{maj}(\sigma)$ are explained in Section 2.1. Note that the form we give in (1) uses a different convention than that of [15] (in which the inv and maj statistics are exchanged and the shape of the diagram is transposed). This is justified because of the well-known identity $\widetilde{H}_\lambda(X;q,t) = \widetilde{H}_{\lambda'}(X;t,q)$ [17, Prop. 2.6].

Recently, a different combinatorial model for these polynomials has been given by Garbali and Wheeler [11]. Independently, the second author together with Corteel, Haglund, Mason, and Williams found formulas for $\widetilde{H}_\lambda(X;q,t)$ that were based on the combinatorial interpretation of plethysm applied to multiline queues [5]. One such formula was a compact version of the original tableaux formula of Haglund, Haiman, and Loehr. The authors of [5] also conjectured a formula related to (1) in which the inversion statistic is replaced by a variant form which we call the *queue inversion* statistic. Our main result is a proof of this conjectural formula:

Theorem. *The modified Macdonald polynomial may be written as*

$$\widetilde{H}_\lambda(X;q,t) = \sum_{\sigma:\mathrm{dg}(\lambda)\to\mathsf{Z}} t^{\mathrm{quinv}(\sigma)} q^{\mathrm{maj}(\sigma)} x^\sigma, \tag{2}$$

where the definition of the queue inversion statistic $\mathrm{quinv}(\sigma)$ *is again given in Section* 2.1.

Both the statistic $\mathrm{inv}(\sigma)$ appearing in (1) and the statistic $\mathrm{quinv}(\sigma)$ appearing in (2) are defined in terms of the content of particular triples of cells in the filling σ. Even though these two results look tantalizingly similar, we do not know of a bijection translating between the two statistics even in very simple cases (for instance tableaux with just two rows).

A particular motivation for our new representation is a link between the modified Macdonald polynomials at $q = 1$ and the multispecies totally asymmetric zero-range process (TAZRP) on the ring, which we establish in a companion paper [1]. The stationary probabilities of a TAZRP on a ring with n sites, with site-specific parameters $x_1, x_2, \dots, x_n$ and a global parameter t, and with particle-types determined by a partition λ, can be written (suitably normalized) as polynomials in $x_1, \dots, x_n$ and t whose sum is $\widetilde{H}_\lambda(X;1,t)$. Moreover, under the probability distribution on fillings σ of $\mathrm{dg}(\lambda)$ proportional to the weights $t^{\mathrm{quinv}(\sigma)}x^\sigma$ appearing on the right-hand side of (2), the distribution of the bottom row of the tableau projects to the stationary distribution of the TAZRP. See Section 2.2 for further details of this connection.

The plan of the article is as follows. In Section 2, we introduce necessary definitions and notation and formally state the main result mentioned above. We also give details of the link to the multispecies totally asymmetric zero-range process. Section 3 gives some further background which will be required for the proofs. In Section 4, we prove the symmetry of the function defined by the right-hand side of (2). The modified Macdonald polynomials are characterized by the three properties stated in Section 3.1. The third property will turn out to be immediate from our definition. In Section 5, we prove the first of these properties. In Section 6, we prove the second property in a certain nondegenerate case. Settling the second property in the degenerate case proves to be the main hurdle. This is taken care of in Section 7 using a bijection on certain tableaux that is described in Section 8. The latter is in turn obtained via a generating function identity on words respecting a coinversion-type statistic. This identity might be of independent interest and is proved in Section 9. In Section 10, we conclude the proof of our main result and mention various related questions.

2. Main result and related work

2.1. Definitions and main result. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition. The *diagram of type* λ, which we denote $\mathrm{dg}(\lambda)$, consists of the cells $\{(r,i), 1 \leqslant i \leqslant k, 1 \leqslant r \leqslant \lambda_i\}$, which we depict using k bottom-justified columns, where the i'th column from left to right has λ_i boxes. See Figure 1 for an illustration of $\mathrm{dg}((3,2,1,1))$. The cell (r,i) corresponds to the cell in the i'th column in the r'th row of $\mathrm{dg}(\lambda)$, where rows are labeled from bottom to top. We warn the reader that this is the opposite convention of labelling points in the plane using cartesian coordinates, but is somewhat standard in the literature. We will use this convention throughout the article.

A *filling* of type (λ, n) is a function $\sigma : \mathrm{dg}(\lambda) \to [n]$ defined on the cells of $\mathrm{dg}(\lambda)$ (where $[n] = \{1, 2, \ldots, n\}$). We also refer to a diagram together with a filling of it as a *tableau*. Let $\mathrm{Tab}(\lambda, n)$ be the set of fillings of type (λ, n), and $\mathrm{Tab}(\lambda)$ the set of functions $\sigma : \mathrm{dg}(\lambda) \to \mathsf{Z}^+$.

$$\begin{array}{|c|c|c|c|}\hline 1 \\ \cline{1-2} 3 & 2 \\ \hline 7 & 6 & 5 & 4 \\ \hline \end{array}$$

Figure 1. The diagram $\mathrm{dg}((3, 2, 1, 1))$ with the cells labeled according to the reading order in Definition 2.1. Our convention is that the cell labelled 5 has coordinates $(1, 3)$.

For σ in $\mathrm{Tab}(\lambda, n)$ or in $\mathrm{Tab}(\lambda)$, and a cell $x = (r, i)$ of $\mathrm{dg}(\lambda)$, we call $\sigma(x)$ the *content* of the cell x in the filling σ. For convenience, we also define $\sigma((\lambda_i + 1, i)) = 0$, where $(\lambda_i + 1, i)$ is the nonexistent cell above column i. When $r > 1$, define $\mathrm{South}(x)$ to be the cell $(r - 1, i)$ directly below cell x in the same column.

Definition 2.1. *Define the* reading order *on the cells of a tableau to be along the rows from right to left, taking the rows from top to bottom.*

See Figure 1 for an illustration of the reading order. We now define various statistics on fillings of tableaux.

Definition 2.2. *Let $\sigma \in \mathrm{Tab}(\lambda)$. Define the* leg *of a cell (r, i) to be the number of cells in column i above row r, i.e. $\mathrm{leg}((r, i)) = \lambda_i - r$. Define the* major index *of the filling σ to be:*

$$\mathrm{maj}(\sigma) = \sum_{x : \sigma(x) > \sigma(\mathrm{South}(x))} (\mathrm{leg}(x) + 1)$$

Definition 2.3. *Given a diagram $\mathrm{dg}(\lambda)$, a* triple *consists of either*

- *three cells $(r + 1, i)$, (r, i) and (r, j) with $i < j$; or*
- *two cells (r, i) and (r, j) with $i < j$ and $\lambda_i = r$ (in which case the triple is called a* degenerate triple*).*

Let us write $a = \sigma((r + 1, i))$, $b = \sigma((r, i))$, and $c = \sigma((r, j))$, for the contents of the cells of the triple, so that we can depict a triple along with its contents as

$$\begin{array}{c} \boxed{a} \\ \boxed{b} \end{array} \cdots \boxed{c}\,, \qquad \textit{or} \qquad \begin{array}{c} 0 \\ \boxed{b} \end{array} \cdots \boxed{c}$$

We say that a triple is a queue inversion triple*, or a* quinv triple *for short, if its entries are oriented counterclockwise when the entries are read in increasing order, with ties being broken with respect to reading order. If the triple is degenerate with content b, c, it is a* quinv *triple if and only if $b < c$. (this is equivalent to thinking of a degenerate triple as a regular triple with $a = 0$).*

Accordingly, define $\mathcal{Q}$ to be the set of contents such that

$$\mathcal{Q} = \big\{(a, b, c) \in [n]^3 : a < b < c,\ \text{or } b < c < a,\ \text{or } c < a < b,\ \text{or } a = b \neq c\big\}. \tag{3}$$

Then the triple $((r + 1, i), (r, i), (r, j))$ where $i < j$ forms a quinv triple in σ if and only if $(\sigma((r + 1, i)), \sigma((r, i)), \sigma((r, j))) \in \mathcal{Q}$.

Remark 2.4. An easy way to check if a triple is a quinv triple is: if the two entries in the same column are the same but not equal to the third one, or if all three entries are different and increasing in the counterclockwise direction, then the triple is a quinv triple. In all other cases, the triple is not a quinv triple. For example, the following are quinv triples:

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdots\boxed{3},\quad \begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array}\cdots\boxed{2},\quad \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\cdots\boxed{1},\quad \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}\cdots\boxed{2},\quad \begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array}\cdots\boxed{1},$$

whereas the following are not:

$$\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\cdots\boxed{2},\quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdots\boxed{1},\quad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\cdots\boxed{1},\quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\cdots\boxed{2},\quad \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}\cdots\boxed{1}.$$

3	2					
1	3	1	3	3		
1	1	2	1	2	3	3

Figure 2. A tableau of type $\lambda = (3,3,2,2,2,1,1)$ and $n = 3$. The weight of this filling is $x_1^5 x_2^3 x_3^6 q^5 t^{12}$.

Definition 2.5. *The* weight *of a filling* σ *is* $x^\sigma t^{\mathrm{quinv}(\sigma)} q^{\mathrm{maj}(\sigma)}$*, where:*

- $x^\sigma = \prod_{u\in \mathrm{dg}(\lambda)} \sigma(u)$ *is the content of* σ*,*
- $\mathrm{quinv}(\sigma)$ *is the total number of* quinv *triples in* σ*,*
- $\mathrm{maj}(\sigma)$ *is the major index given in Definition* 2.2.

See Figure 2 for an example of a tableau and its weight.

The main result we present in this article is the following one, which was conjectured by the second author, Corteel, Haglund, Mason, and Williams.

Theorem 2.6. *Let* λ *be a partition. The modified Macdonald polynomial can be written as*

$$\widetilde{H}_\lambda(X;q,t) = \sum_{\sigma\in \mathrm{Tab}(\lambda)} x^\sigma t^{\mathrm{quinv}(\sigma)} q^{\mathrm{maj}(\sigma)}.$$

Proof sketch. We first show that our formula is symmetric in the variables X in Theorem 4.1. The modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$ are symmetric functions that are uniquely characterized by the axioms (5), (6), and (7). Our strategy is to show that our formula satisfies these axioms. Axiom (7) is immediate from our definition. Axioms (5) and (6) are written equivalently as (8) and (9), respectively. We employ the canonical tool of superization to deal with the negative sign in the plethysm. Thus we introduce "super fillings" denoted by $\widetilde{\mathrm{Tab}}(\lambda)$ in Definition 3.9. The axioms (8) and (9) then become equivalent to (12) and (13), respectively. Theorem 5.3 proves (12) and Theorem 6.3 combined with Proposition 7.2 proves (13). See Section 10 for the formal proof.

Remark 2.7. Note that when λ has all parts distinct, one avoids the technical parts of our proof, with Corollary 6.9 proving (13), and the proof is greatly simplified. In this case, all our arguments directly follow the proof strategy of [15, Theorem 2.2].

Remark 2.8. The tableaux which we consider are closely related to the *multiline queues* mentioned in the introduction, which are used in [10], [26], [7] and elsewhere.

We briefly explain the correspondence, and the term "queue inversion". (The details are not required for the rest of the paper.)

Consider a partition $(\lambda_1, \dots, \lambda_k)$ and a filling $\sigma \in \mathrm{Tab}(\lambda, n)$. We consider a system composed of a sequence of queues in series, labelled $\lambda_1, \lambda_1 - 1, \dots, 1$, and with a collection of customers labelled $1, 2, \dots, k$.

Customer i enters the system in queue λ_i, and proceeds sequentially to queues $\lambda_i - 1, \dots, 1$. The entry $\sigma((r,i))$ represents the "time" that customer i is served at queue r and enters queue $r-1$.

Similarly, the entry $\sigma((r+1,i))$ gives the time that customer i is served at queue $r+1$ and enters queue r. Hence the time interval during which customer i is present at queue r is the (cyclic) interval $[\sigma((r,i)), \sigma((r+1,i))]$. Note that under this interpretation, $\sigma((r+1,i))$ is "earlier" than $\sigma((r,i))$; in this sense "time" runs from right to left, and wraps cyclically around.

Customers have a priority order, with customer i having higher priority than customer j for $i < j$. Take any $i < j$. If the service time $\sigma((r,j))$ of customer j at queue r is strictly within the interval $[\sigma((r,i)), \sigma((r+1,i))]$ during which customer i is present at that queue, then this contradicts the priority order. Such an event corresponds to the triple of cells $(r+1,i)$, (r,i), and (r,j) forming a quinv triple.

Putting $t = 0$ corresponds to a case where the priority order is strictly enforced (and gives positive weight only to tableaux containing no quinv triples).

Finally for completeness, we compare the queue inversion statistic defined above to the inversion statistic used in Haglund, Haiman and Loehr's formula (1), which we refer to as the HHL inversion statistic. Whereas the queue inversion statistic concerns triples of cells of the form $(r+1,i), (r,i), (r,j)$, with $i < j$ (with a degenerate triple in the case $r = \lambda_i$), the HHL inversion statistic concerns triples of the form $(r,i), (r-1,i), (r,j)$ with $i < j$ (with a degenerate triple in the case $r = 1$), which we can depict as

$$\begin{array}{ccc} \boxed{a} & & \\ \boxed{b} & \cdots & \boxed{c} \end{array} \quad \text{and} \quad \begin{array}{ccc} \boxed{a} & \cdots & \boxed{c} \\ \boxed{b} & & \end{array}$$

respectively. A triple of the latter kind is called an *inversion triple* if $(\sigma((r,i)), \sigma((r-1,i)), \sigma((r,j))) \in \mathcal{Q}$ (with $\sigma(0,i)$ taken to be ∞ for all i). Then $\mathrm{inv}(\sigma)$ is the number of inversion triples in the filling σ.

2.2. Multispecies totally asymmetric zero-range process. We consider an interacting particle system which we call the *multispecies totally asymmetric zero-range process* (or TAZRP), which appears in [31], and is a specialization of a much wider class of systems known as multispecies zero-range processes; see for example the review in [19].

Fix a partition $\lambda = (\lambda_1, \dots \lambda_k)$ and a positive integer n. Then $\mathrm{TAZRP}(\lambda, n)$ has n sites (labelled $1, 2, \dots, n$), and k particles with types $\lambda_1, \dots, \lambda_k$. Each site may be empty or may contain one or more particles. Particles of the same type are indistinguishable.

The system evolves as a continuous-time Markov chain. Any jump of the system consists of a single particle jumping from site j to site $j-1$, for some $j \in \{1, \dots, n\}$ (sites are considered cyclically mod n, so that a particle jumping out of site 1 enters site n). The rates are governed by a global parameter t and site-dependent parameters $x_1, \dots, x_n$.

For each $j \in \{1, \dots, n\}$ and each $a > 1$, a bell of level a rings at site j at rate $x_j^{-1} t^{a-1}$. When such a bell rings: if site j contains at least a particles, then the a'th highest-numbered of them jumps to site $j-1$. If j contains fewer than a particles, nothing changes.

In [1] we show that the stationary probabilities of the model are rational functions of t and $x_1, \ldots, x_n$ with non-negative integer coefficients, with common denominator $\widetilde{H}_\lambda(x_1, \ldots, x_n; 1, t)$. In fact, there is a function f from the set $\mathrm{Tab}(\lambda, n)$ to the set of TAZRP configurations, such that the stationary probability $P(\eta)$ of a configuration η is given by

$$P(\eta) = \frac{1}{\widetilde{H}_\lambda(x_1, \ldots, x_n; 1, t)} \sum_{\substack{\sigma \in \mathrm{Tab}(\lambda, n): \\ f(\sigma) = \eta}} x^\sigma t^{\mathrm{quinv}(\sigma)}. \tag{4}$$

In this sense we can say that the functions $\widetilde{H}_\lambda$ act as "partition functions" for the TAZRP. The value $f(\sigma)$ depends on the tableau σ only through its bottom row.

The proof of property (4) in [1] is done by constructing a Markov chain on the space $\mathrm{Tab}(\lambda, n)$ in which the stationary probability of a tableau σ is proportional to $x^\sigma t^{\mathrm{quinv}(\sigma)}$, and which projects via the function f to $\mathrm{TAZRP}(\lambda, n)$.

3. Background and preliminaries

The proofs of several of our results will rely on the techniques used in [15], which we present in Sections 3.1– 3.3. The final subsection 3.4 is needed for the proofs in Section 9.

3.1. The axioms uniquely characterizing $\widetilde{H}_\lambda$.

Let $\Lambda \equiv \Lambda(q, t)$ be the algebra of symmetric functions whose coefficients are rational functions in q and t. Bases of Λ are indexed by partitions $\mu = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k \geqslant 0)$. Recall that μ' denotes the conjugate partition and $\leqslant$ represents the dominance order on partitions: $\mu \leqslant \lambda$ if and only if $\mu_1 + \cdots + \mu_j \leqslant \lambda_1 + \cdots + \lambda_j$ for each $j \geqslant 1$. Recall that $\{p_\mu\}$ is the power sum basis for Λ and $\langle \cdot, \cdot \rangle$ is the Hall scalar product. Let ω be the standard involution on Λ. For symmetric functions in two alphabets X and Y, let ω_X and ω_Y be the standard involutions acting separately in the X and Y variables, respectively. For a formal power series A in indeterminates $a_1, a_2, \cdots$, in our case with coefficients in $\mathsf{Q}(q, t)$, $p_k[A]$ is the formal substitution of a_i^k for each indeterminate a_i. Then for an arbitrary $f \in \Lambda$, the *plethysm* $f[A]$ is defined by expressing f in the power sum basis and substituting $p_k[A]$ for each p_k in the expansion. By convention, we define the *plethystic alphabets* $X = x_1 + x_2 + \cdots$ and $Y = y_1 + y_2 + \cdots$, so that $f[X] = f(X)$, $f[tX] = t^d f[X]$ and $f[-X] = (-1)^d \omega(f(X))$ if f is homogeneous of degree d, $f[X + Y] = f(X, Y)$, where $f(X, Y)$ represents the concatenation of the alphabets X and Y, and $f[X(1-q)]$ is the image of f under the algebra homomorphism mapping $p_k(X)$ to $(1 - q^k)p_k(X)$. See [17, §2] for a complete description.

The modified Macdonald polynomials are the basis of Λ with coefficients in $\mathsf{Q}(q, t)$, characterized by the following triangularity and normalization axioms (derived from Macdonald's original triangularity and orthogonality axioms [17]), and symmetric functions satisfying the axioms are unique, if they exist: see [17, Proposition 2.6] and [18, Section 6.1].

$$\widetilde{H}_\lambda[X(1-t); q, t] = \sum_{\mu \geqslant \lambda'} a_{\mu\lambda}(q, t) s_\mu(X), \tag{5}$$

$$\widetilde{H}_\lambda[X(1-q); q, t] = \sum_{\mu \geqslant \lambda} b_{\mu\lambda}(q, t) s_\mu(X), \tag{6}$$

$$\left\langle \widetilde{H}_\lambda(X; q, t), s_{(n)}(X) \right\rangle = 1 \tag{7}$$

for some coefficients $a_{\mu\lambda}(q, t), b_{\mu\lambda}(q, t) \in \mathsf{Q}(q, t)$.

We first use the following facts to rewrite the axioms in a more convenient form.

(1) s_λ and m_λ are lower triangular with respect to each other: $s_\lambda \in \mathbb{Z}\{m_\mu : \mu \leqslant \lambda\}$ and $m_\lambda \in \mathbb{Z}\{s_\mu : \mu \leqslant \lambda\}$.
(2) We can write $f[X(q-1)] = \tilde{f}(qX, -X)$, where $\tilde{f}(X,Y) = \omega_Y f[X+Y]$.
(3) For a symmetric function $f(X)$ that is homogeneous of degree d and a plethystic alphabet Y, $f[-Y] = (-1)^d(\omega_Y f)[Y]$.
(4) The partial ordering on partitions is reversed by transposing: $\lambda \leqslant \mu \longleftrightarrow \mu' \leqslant \lambda'$, where $\leqslant$ is the dominance order.

Thus we rewrite the two triangularity axioms, which, along with (7) uniquely characterize $\widetilde{H}_\lambda$, in terms of the monomial basis:

$$\widetilde{H}_\lambda[X(t-1); q,t] = \sum_{\mu \leqslant \lambda} d_{\mu\lambda}(q,t) m_\mu(X) \tag{8}$$

$$\widetilde{H}_\lambda[X(q-1); q,t] = \sum_{\mu \leqslant \lambda'} c_{\mu\lambda}(q,t) m_\mu(X), \tag{9}$$

for some coefficients $c_{\mu\lambda}(q,t), d_{\mu\lambda}(q,t) \in \mathbb{Q}(q,t)$.

3.2. LLT polynomials. LLT polynomials are a well-known family of symmetric polynomials discovered by Lascoux, Leclerc, and Thibon [20]. We provide the combinatorial definition of LLT polynomials, which was introduced in [16].

Recall that the Young diagram of a partition λ is a left-justified array of cells such that the i'th row contains λ_i cells. We will number our rows from bottom to top, the so-called *French convention.*

We will identify a partition with its Young diagram. Let λ and μ be partitions with $\mu_j \leqslant \lambda_j$ for all j, i.e. $\mu \subseteq \lambda$ as Young diagrams. Then the *skew diagram* or *skew shape*, denoted $\nu = \lambda \smallsetminus \mu$, is the subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$ consisting of the cells in λ/μ. We imagine diagonals running through the cells of ν from southwest to northeast, and we define the *diagonal* of a cell $u = (i,j)$ (in row i and column j) to be the integer $d(u) = i - j + 1$.

A *semistandard Young tableau* (SSYT) of skew shape ν is a filling of the diagram ν with positive integers, denoted by the function $\rho : \nu \to \mathbb{Z}_+$, which is weakly increasing on each row of ν (from left to right) and strictly increasing on each column (from bottom to top). We denote the set of fillings of ν by $\mathrm{SSYT}(\nu)$. For a filling $\rho \in \mathrm{SSYT}(\nu)$, $\rho(u)$ denotes the entry in cell u of ν.

Let $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$ be a tuple of skew diagrams, and let

$$\mathrm{SSYT}(\boldsymbol{\nu}) = \mathrm{SSYT}(\nu^{(1)}) \times \cdots \times \mathrm{SSYT}(\nu^{(k)}).$$

We note that although the representation $\nu = \lambda/\mu$ is not unique for a given diagram ν, we need consider only the *relative positions* of the $\nu^{(i)}$'s with respect to each other. Thus for each i, we fix $\nu^{(i)}$ to be such that the site u_i in row 1, column 1 (whether or not there is a cell at that location) is positioned at the origin so that $d(u_i) = 1$.

Definition 3.1. *The set of* inversions *on* $\boldsymbol{\rho} = (\rho^{(1)}, \dots, \rho^{(k)}) \in \mathrm{SSYT}(\boldsymbol{\nu})$ *is defined as follows. Let u, v be cells in $\nu^{(i)}$ and $\nu^{(j)}$ respectively. The cells u and v form an inversion if $\rho^{(i)}(u) > \rho^{(j)}(v)$ and either*

i. *$i < j$ and $d(u) = d(v)$, or*
ii. *$i > j$ and $d(u) = d(v) + 1$.*

We denote the number of inversions of $\boldsymbol{\rho}$ by $\mathrm{inv}(\boldsymbol{\rho})$, and the monomial in x corresponding to the content of $\boldsymbol{\rho}$ by $x^{\boldsymbol{\rho}} = \prod_i \prod_{u \in \nu^{(i)}} x_{\rho^{(i)}(u)}$.

Example 3.2. Figure 3 shows a semistandard filling $\boldsymbol{\rho} = (\rho^{(1)}, \rho^{(2)}, \rho^{(3)})$ of the tuple of skew diagrams $\boldsymbol{\nu}$ where $\nu^{(1)} = (1,1)/\varnothing$, $\nu^{(2)} = (1,1)/\varnothing$, and $\nu^{(3)} = (2,2,2)/(2,1)$. There are five inversions in $\boldsymbol{\rho}$: the 3 in diagonal 3 forms an inversion with the 2 in diagonal 2, the 3 and the 2 in diagonal 2 form an inversion, the 2 in diagonal 2 forms an inversion with the bottommost 1 in diagonal 1, and the 4 in diagonal 2 forms an inversion with both 1's in diagonal 1. Thus $\mathrm{inv}(\boldsymbol{\rho}) = 5$ and $x^{\boldsymbol{\rho}} = x_1^2 x_2^2 x_3^2 x_4$.

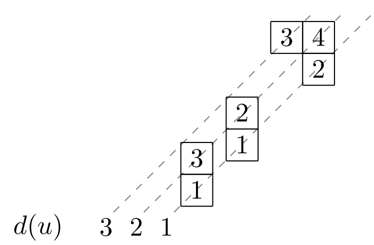


Figure 3. For $\boldsymbol{\nu} = ((1,1),(1,1),(2,2,2)/(2,1))$, we show a filling $\boldsymbol{\rho} \in \mathrm{SSYT}(\boldsymbol{\nu})$, with $\mathrm{inv}(\boldsymbol{\rho}) = 5$ and $x^{\boldsymbol{\rho}} = x_1^2 x_2^2 x_3^2 x_4$.

Definition 3.3. *Let $\boldsymbol{\nu}$ be a tuple of skew diagrams. The LLT polynomial indexed by $\boldsymbol{\nu}$ is*

$$G_{\boldsymbol{\nu}}(X;t) = \sum_{\boldsymbol{\rho} \in \mathrm{SSYT}(\boldsymbol{\nu})} t^{\mathrm{inv}(\boldsymbol{\rho})} x^{\boldsymbol{\rho}}.$$

Theorem 3.4 ([20, 16]). *The polynomial $G_{\boldsymbol{\nu}}(X;t)$ is symmetric in the variables x.*

Remark 3.5. Our Theorem 4.1 relies on Theorem 3.4. The original proof for Theorem 3.4 in [20] uses a construction in the representation theory of affine Hecke algebras, and a purely combinatorial proof of the symmetry is given in [15, Section 10: Appendix].

Definition 3.6. *Define a* ribbon *to be a connected skew diagram with no 2×2 squares. For a ribbon ν with $k = |\nu|$, we label its boxes from northwest to southeast by $1,\dots,k$, and define its* descent set*, denoted $\mathrm{Des}(\nu)$, to be the set of labels in $\{1,\dots,k-1\}$ corresponding to boxes that have a box below in the same column.*

Let w be a word with entries in Z_+. Denote the set of locations of descents in w by $\mathrm{Des}(w) = \{i : w_i > w_{i+1}\}$. Let ν be the ribbon with the same descent set as w, i.e. $\mathrm{Des}(\nu) = \mathrm{Des}(w)$. We define a ribbon corresponding to w to be a filling of the cells of ν with the entry w_i in cell i for $i = 1,\dots,k$. We call $\mathrm{ribbon}(w)$ the ribbon corresponding to w.

It is immediate from the definition that $\mathrm{ribbon}(w)$ is a SSYT.

Example 3.7. Consider the word $w = (4,3,3,4,5,3,2,1,2,2)$, which has descent set $\mathrm{Des}(w) = \{1,5,6,7\}$. The ribbon $\nu = (6,4,4,4,1)/(3,3,3)$ is the unique ribbon with the same descent set: $\mathrm{Des}(\nu) = \mathrm{Des}(w)$. Below we show ν with its cells labeled from northwest to southeast, and the corresponding SSYT $\mathrm{ribbon}(w)$ of shape ν with its boxes filled by the entries of w.

$$\nu = \begin{array}{|c|c|c|c|c|c|} \hline 1 & & & & & \\ \hline 2 & 3 & 4 & 5 & & \\ \hline & & & 6 & & \\ \hline & & & 7 & & \\ \hline & & & 8 & 9 & 10 \\ \hline \end{array} \qquad \mathrm{ribbon}(w) = \begin{array}{|c|c|c|c|c|c|} \hline 4 & & & & & \\ \hline 3 & 3 & 4 & 5 & & \\ \hline & & & 3 & & \\ \hline & & & 2 & & \\ \hline & & & 1 & 2 & 2 \\ \hline \end{array}$$

3.3. Super-alphabets and quasisymmetric function expansion. With maj, quinv and $\mathrm{Tab}(\lambda)$ as defined in Section 2, we define:

$$C_\lambda(X;q,t) = \sum_{\sigma\in\mathrm{Tab}(\lambda)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)} x^\sigma. \tag{10}$$

After showing $C_\lambda(X;q,t)$ is symmetric in the variables x_i in Section 4, we will show it satisfies (8) and (9). We do this by modifying the proof of the HHL formula in [15, Section 4] for the setting of $\mathrm{Tab}(\lambda)$, where we will consider the *superization* of C_λ. In this subsection, we adapt the well-known properties of superization to this new setting.

Let $n \geqslant 0$ and fix a subset $D \subseteq \{1,\dots,n-1\}$. Define *Gessel's quasisymmetric function* $Q_{n,D}(X)$ in the variables $X = x_1, x_2, \dots$ by

$$Q_{n,D}(X) = \sum_{\substack{a_1\leqslant a_2\leqslant\cdots\leqslant a_n\\ a_i=a_{i+1}\implies i\notin D}} x_{a_1}x_{a_2}\cdots x_{a_n},$$

where the indices are $a_i \in \mathsf{Z}_+$. Define the "super-alphabet"

$$\mathcal{A} = \mathsf{Z}_+ \cup \mathsf{Z}_- = \{\bar{1}, 1, \bar{2}, 2, \dots\}$$

consisting of positive and "negative" letters $i, \bar{i}$ of our original alphabet. One can consider any total ordering on $\mathcal{A}\cup\{0\}$; in our proofs we will use the two total orderings

$$\begin{aligned}(\mathcal{A}\cup\{0\},<_1) &= \{0<1<\bar{1}<2<\bar{2}<\cdots\},\\ (\mathcal{A}\cup\{0\},<_2) &= \{0<1<2<3<\cdots<\bar{3}<\bar{2}<\bar{1}\}.\end{aligned}$$

For any fixed total ordering $<$ on $\mathcal{A}$, we define the more general "super" quasiymmetric function $\widetilde{Q}_{n,D}(X,Y)$ in the variables $X = x_1, x_2, \dots$ and $Y = y_1, y_2, \dots$ by

$$\widetilde{Q}_{n,D}(X,Y) = \sum_{\substack{a_1\leqslant a_2\leqslant\cdots\leqslant a_n\\ a_i=a_{i+1}\in\mathsf{Z}_+\implies i\notin D\\ a_i=a_{i+1}\in\mathsf{Z}_-\implies i\in D}} z_{a_1}z_{a_2}\cdots z_{a_n}, \tag{11}$$

where the indices are $a_i \in \mathcal{A}$, and $z_i = x_i$ when $i \in \mathsf{Z}_+$ and $z_i = y_i$ when $i \in \mathsf{Z}_-$.

For $a, b \in \mathcal{A}\cup\{0\}$ and any given total ordering $<$, we will use the notation $I(a,b)$:

$$I(a,b) = \begin{cases}1, & a>b \text{ or } a=b\in\mathsf{Z}_-,\\ 0, & a<b \text{ or } a=b\in\mathsf{Z}_+.\end{cases}$$

To avoid confusion, we will use the terminology I_1 (*resp.* I_2) whenever we use the ordering $<_1$ (*resp.* $<_2$). For example,

$$\begin{aligned}&I_1(\bar{3},\bar{3}) = I_2(\bar{3},\bar{3}) = 1, && I_1(3,\bar{2}) = 1, && I_2(3,\bar{2}) = 0,\\ &I_1(2,\bar{2}) = I_2(2,\bar{2}) = 0, && I_1(\bar{2},\bar{3}) = 0, && I_2(\bar{2},\bar{3}) = 1.\end{aligned}$$

Definition 3.8. *The* superization *of a symmetric function $f(X)$ is $\widetilde{f}(X,Y) = \omega_Y f[X+Y]$.*

Definition 3.9. *Given a super alphabet $\mathcal{A} = \mathsf{Z}_+\cup\mathsf{Z}_-$ and a fixed total ordering $<$, a* super filling *of a diagram $\mathrm{dg}(\lambda)$ is a function $\sigma : \mathrm{dg}(\lambda)\to\mathcal{A}$, with the following extensions of the definitions of the maj, quinv, and $\mathcal{Q}$ from Definition 2.2, Definition 2.3, and (3), respectively. Denote the set of super fillings of $\mathrm{dg}(\lambda)$ by $\widetilde{\mathrm{Tab}}(\lambda)$.*

- maj*: If $y = \mathrm{South}(x)$ in $\mathrm{dg}(\lambda)$, then $x\in\mathrm{Des}(\sigma)$ if $I(\sigma(x),\sigma(y)) = 1$. The* maj *statistic is defined as before.*

- quinv*: If three cells x, y, z with entries $\sigma(x) = a, \sigma(y) = b, \sigma(z) = c$ form a triple in the configuration*

$$\begin{array}{l} {}^{x}\,\begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \\ {}^{y} \end{array} \cdots \; {}^{z}\,\boxed{c}$$

where z is the cell to the right of y in the same row, and $y = \mathrm{South}(x)$ if x exists, then the triple is a quinv *triple if and only if exactly* one *of the following is true:*

$$\{I(a,b) = 1,\ I(c,b) = 0,\ I(a,c) = 0\},$$

and in this case we say $(a,b,c) \in \mathcal{Q}$. It is not a quinv triple *if and only if exactly* two *of the conditions above are true.*(1) *As before, we write* $\mathrm{quinv}(\sigma)$ *as the number of* quinv *triples in σ.*

We write $|\sigma|$ to denote the regular filling with the positive alphabet, such that $|\sigma|(u) = |\sigma(u)|$ for each $u \in \mathrm{dg}(\lambda)$.

It is immediate that when $\sigma = |\sigma|$, the above definitions reduce to those of the statistics of a regular filling as given in Section 2.1. Moreover, note that the definition of quinv given above still holds for a degenerate triple, in which the cell x does not exist: as per our convention, in that case $a = 0$, so $I(a,b) = I(a,c) = 0$, and hence the triple is a quinv triple if and only if $I(c,b) = 1$.

Example 3.10. We give some examples of quinv triples in super fillings when there are repeated entries, noting that $I(a,a) = 0$ and $I(\bar{a},\bar{a}) = 1$ for any fixed ordering $<$. The following are quinv triples:

$$\begin{array}{|c|}\hline \bar{1} \\ \hline \bar{1} \\ \hline \end{array} \cdots \boxed{\bar{1}}, \quad \begin{array}{|c|}\hline 2 \\ \hline \bar{1} \\ \hline \end{array} \cdots \boxed{\bar{1}}, \quad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline \end{array} \cdots \boxed{\bar{2}}, \quad \begin{array}{|c|}\hline \bar{2} \\ \hline 1 \\ \hline \end{array} \cdots \boxed{\bar{2}},$$

whereas the following are not quinv triples:

$$\begin{array}{|c|}\hline \bar{1} \\ \hline \bar{1} \\ \hline \end{array} \cdots \boxed{2}, \quad \begin{array}{|c|}\hline \bar{1} \\ \hline \bar{1} \\ \hline \end{array} \cdots \boxed{\bar{2}}.$$

Moreover, $\begin{array}{|c|}\hline \bar{2} \\ \hline \bar{2} \\ \hline \end{array}$ is a descent, while (as before) $\begin{array}{|c|}\hline 2 \\ \hline 2 \\ \hline \end{array}$ is not. Since $I(a,a) = 0$ and $I(\bar{a},\bar{a}) = 1$ for any ordering $<$, this example is independent of the ordering.

To obtain a tableaux formula for the generating function of the superization $\widetilde{C}(X,Y;q,t)$, we present a standard construction following an analogous argument in [15] that makes use of standard fillings to give a quasisymmetric expansion of our formulas. The following proposition states a well-known property of the superization of a symmetric function that holds for any total ordering on the super-alphabet $\mathcal{A}$.

Proposition 3.11 ([16, Corollary 2.4.3]). *Let $f(x)$ be a symmetric function homogeneous of degree n, written as a sum over quasisymmetric functions as*

$$f(z) = \sum_{D \subseteq \{1,\dots,n-1\}} c_D Q_{n,D}(z).$$

The superization of f is given by

$$\widetilde{f}(x,y) = \sum_{D \subseteq \{1,\dots,n-1\}} c_D \widetilde{Q}_{n,D}(x,y).$$

(1)The reader may check that it is impossible for all or none of the conditions to be true.

We call a filling π of $\mathrm{dg}(\lambda)$ a *standard filling* if each element in $\{1,\dots,n\}$ occurs exactly once in π, where $n=|\lambda|$. A standard filling is represented by the bijection $\pi : \mathrm{dg}(\lambda)\to\{1,\dots,n\}$. The *standardization* of a super filling σ is the unique standard filling π such that $\sigma\circ\pi^{-1}$ is weakly increasing, and for each $x\in\mathcal{A}$, the restriction of π to $\sigma^{-1}(\{x\})$ is increasing with respect to the reading order if $x\in\mathsf{Z}_+$ and decreasing if $x\in\mathsf{Z}_-$. It is straightforward to check that if π is the standardization of σ, then for each $u,v\in\mathrm{dg}(\lambda)$, $I(\sigma(u),\sigma(v))=I(\pi(u),\pi(v))$, and so the statistics maj and quinv are preserved under standardization: $\mathrm{maj}(\sigma)=\mathrm{maj}(\pi)$ and $\mathrm{quinv}(\sigma)=\mathrm{quinv}(\pi)$. Note that both the standardization and the function I depend on the choice of ordering; see Example 3.12.

Example 3.12. The standardization of σ with respect to the ordering $<_1$ is π_1, and the standardization with respect to the ordering $<_2$ is π_2.

$$\sigma=\begin{array}{|c|c|c|c|c|}\cline{1-1} 2 \\ \cline{1-3} \bar{1} & \bar{2} & 1 \\ \hline 3 & 2 & \bar{1} & 1 & 2 \\ \hline \end{array}\quad,\quad \pi_1=\begin{array}{|c|c|c|c|c|}\cline{1-1} 5 \\ \cline{1-3} 4 & 8 & 1 \\ \hline 9 & 7 & 3 & 2 & 6 \\ \hline \end{array}\quad,\quad \pi_2=\begin{array}{|c|c|c|c|c|}\cline{1-1} 3 \\ \cline{1-3} 9 & 7 & 1 \\ \hline 6 & 5 & 8 & 2 & 4 \\ \hline \end{array}$$

Now consider the reading word $\mathrm{rw}(\pi)$ of a standard filling π of $\mathrm{dg}(\lambda)$, which is defined to be the sequence of entries obtained from the filling in reading order: this is a permutation of $\{1,\dots,n\}$ where $n=|\lambda|$. We call $D(\pi)\subseteq\{1,\dots,n-1\}$ the *index* of π, defined by

$$D(\pi)=\big\{i\in\{1,\dots,n-1\}\ :\ i+1 \text{ precedes } i \text{ in } \mathrm{rw}(\pi)\big\}.$$

Then π is the standardization of σ if and only if the weakly increasing function $a=\sigma\circ\pi^{-1}:\{1,\dots,n\}\to\mathcal{A}$ satisfies:

- if $a(i)=a(i+1)\in\mathsf{Z}_+$, then $i\notin D(\pi)$, and
- if $a(i)=a(i+1)\in\mathsf{Z}_-$, then $i\in D(\pi)$.

Example 3.13. In Example 3.12, $D(\pi_1)=\{3,4,7\}$ and $D(\pi_2)=\{2,6,8\}$. We check that $\sigma\circ\pi_1^{-1}=(1,1,\bar{1},\bar{1},2,2,2,\bar{2},3)$ implies $1,5,6\notin D(\pi_1)$ and $3\in D(\pi_1)$ which is indeed true, and similarly $\sigma\circ\pi_2^{-1}=(1,1,2,2,2,3,\bar{2},\bar{1},\bar{1})$ implies $1,3,4\notin D(\pi_2)$ and $8\in D(\pi_2)$, also true.

Thus we obtain, by Theorem 4.1 and Proposition 3.11, the following proposition, corresponding to [15, Proposition 4.3] with our statistic quinv replacing the "inv" statistic in the latter.

Proposition 3.14 ([15, Proposition 4.3]). *Let λ be a partition of n. The polynomial $C_\lambda(x;q,t)$ has the following quasisymmetric expansion as a sum over standard fillings π of $\mathrm{dg}(\lambda)$:*

$$C_\lambda(x;q,t)=\sum_{\pi} q^{\mathrm{maj}(\pi)}t^{\mathrm{quinv}(\pi)}Q_{n,D(\pi)}(x).$$

The superization of $C_\lambda(x;q,t)$ has the expansion

$$\widetilde{C}_\lambda(x,y;q,t)=\sum_{\pi} q^{\mathrm{maj}(\pi)}t^{\mathrm{quinv}(\pi)}\widetilde{Q}_{n,D(\pi)}(x,y).$$

This has the following formula in terms of super fillings:

$$\widetilde{C}_\lambda(x,y;q,t)=\sum_{\sigma\in\mathrm{dg}(\lambda)\to\mathcal{A}} q^{\mathrm{maj}(\sigma)}t^{\mathrm{quinv}(\sigma)}z^{\sigma},$$

where $z_i=x_i$ if $i\in\mathsf{Z}_+$ and $z_i=y_i$ if $i\in\mathsf{Z}_-$, and statistics quinv *and* maj *on super fillings $\sigma\in\mathrm{dg}(\lambda)\to\mathcal{A}$ given as in Definition* 3.9.

Denote the set of super fillings $\{\sigma \in \mathrm{dg}(\lambda) \to \mathcal{A}\}$ by $\widetilde{\mathrm{Tab}}(\lambda)$. We use the identities $C_\lambda[X(t-1);q,t] = \widetilde{C}_\lambda(tX,-X;q,t)$ and $C_\lambda[X(q-1);q,t] = \widetilde{C}_\lambda(qX,-X;q,t)$ to obtain

$$C_\lambda[X(t-1);q,t] = \sum_{\sigma \in \widetilde{\mathrm{Tab}}(\lambda)} (-1)^{m(\sigma)} q^{\mathrm{maj}(\sigma)} t^{p(\sigma)+\mathrm{quinv}(\sigma)} x^{|\sigma|}, \tag{12}$$

$$C_\lambda[X(q-1);q,t] = \sum_{\sigma \in \widetilde{\mathrm{Tab}}(\lambda)} (-1)^{m(\sigma)} q^{p(\sigma)+\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)} x^{|\sigma|} \tag{13}$$

where $p(\sigma) = |\{u : \sigma(u) \in \mathsf{Z}_+\}|$ and $m(\sigma) = |\{u : \sigma(u) \in \mathsf{Z}_-\}|$ are the numbers of positive and negative entries in the super filling σ, respectively. Note that these formulas are valid for any total ordering chosen on $\mathcal{A}$.

3.4. Basic facts from q-series. We recall some standard facts about q-series and combinatorics of words. Recall that the *q-numbers* are given by

$$[n] \equiv [n]_q \coloneqq \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1},$$

for $n \in \mathsf{N}$. For us, q will be a fixed formal variable, and we will not specify it for notational convenience. The *q-factorial* is then

$$[n]! \coloneqq [1][2]\cdots[n],$$

and the *q-binomial coefficient* is

$$\begin{bmatrix} n \\ k \end{bmatrix} \coloneqq \frac{[n]!}{[k]![n-k]!}.$$

Although it is not obvious that this is a polynomial, this can be seen from the initial conditions $\begin{bmatrix} n \\ n \end{bmatrix} = \begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ and the generalized Pascal triangle recurrence

$$\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix} + q^{m+1} \begin{bmatrix} n \\ m+1 \end{bmatrix}. \tag{14}$$

We also note that $\begin{bmatrix} n \\ m \end{bmatrix} = 0$ if $n > 0$ and either $m < 0$ or $m > n$, $\begin{bmatrix} 0 \\ m \end{bmatrix} = \delta_{m,0}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix} = 1$. The *$q$-multinomial coefficient* is defined similarly. Suppose $\alpha = (\alpha_1, \dots, \alpha_n)$ is a tuple of nonnegative integers. Recall that $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then

$$\begin{bmatrix} |\alpha| \\ \alpha_1, \dots, \alpha_n \end{bmatrix} \coloneqq \frac{[|\alpha|]!}{[\alpha_1]! \cdots [\alpha_n]!}.$$

For $\alpha = (\alpha_1, \dots, \alpha_n)$ a tuple of positive integers, let W_α be the set of words in the alphabet $[n]$ with α_i occurrences of the letter i, $1 \leqslant i \leqslant n$. For a word $w \in W_\alpha$, define $\mathrm{coinv}(w) = |\{(i,j) \ : \ i < j,\ w_i < w_j\}|$ to be the number of coinversions of w. The following result is classical.

Proposition 3.15 ([30, Proposition 1.7.1]). *The* coinv *generating function of W_α is*

$$\sum_{w \in W_\alpha} q^{\mathrm{coinv}(w)} = \begin{bmatrix} |\alpha| \\ \alpha_1, \dots, \alpha_n \end{bmatrix}.$$

Strictly speaking, the result above is usually stated for the inv generating function for the number of inversions, but there is an easy bijection showing that the same result holds for the coinversion generating function as well. We now list a few of the standard q-series identities that we will need in our proofs. The first is the well-known q-binomial theorem.

Proposition 3.16 ([30, Equation (1.87)]). *For n a nonnegative integer,*

$$\sum_{i=0}^{n} x^i q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix} = \prod_{j=0}^{n-1} (1 + xq^j).$$

The celebrated q-Chu–Vandermonde identity will prove very useful for us. We write it in the form more tractable for our purposes.

Theorem 3.17 ([13, Equations (1.5.2) and (1.5.3)]). *For m, n, k nonnegative integers,*

$$\sum_{i=0}^{k} \begin{bmatrix} m \\ k-i \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix} q^{i(m-k+i)} = \begin{bmatrix} m+n \\ k \end{bmatrix}.$$

This is valid even when $k > \max(m, n)$. In particular, if $k > m + n$, then both sides are zero.

The q-Chu–Vandermonde identity is also valid when m, n are negative integers. In that case, a formulation useful for us will be the following.

Corollary 3.18. *For m, n, k nonnegative integers such that $m > n > k$,*

$$\sum_{i=k}^{m-n+k} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} m-i \\ n-k \end{bmatrix} q^{i(n-k+1)} = \begin{bmatrix} m+1 \\ n+1 \end{bmatrix} q^{k(n-k+1)}.$$

The last is a telescoping sum, which can be derived from the fundamental recurrence (14) for the q-binomial coefficients.

Proposition 3.19. *For m, n, k nonnegative integers such that $m + 1 > n > k$,*

$$\sum_{i=k}^{n} \begin{bmatrix} m-i \\ n-i \end{bmatrix} q^{i(m-n+1)} = \begin{bmatrix} m-k+1 \\ n-k \end{bmatrix} q^{k(m-n+1)}.$$

Note that both sides are equal to 1 when $m = n - 1$, since by convention, $\begin{bmatrix} -1 \\ 0 \end{bmatrix} = 1$.

4. Proof that $C_\lambda(X, q, t)$ is symmetric

This section is devoted to proving the following theorem.

Theorem 4.1. *The polynomial $C_\lambda(X; q, t)$ is symmetric in the variables x_i.*

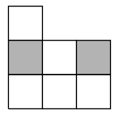
We will prove Theorem 4.1 by expanding C_λ in terms of the LLT polynomials $G_{\boldsymbol{\nu}}(X; t)$ defined in Section 3.2.

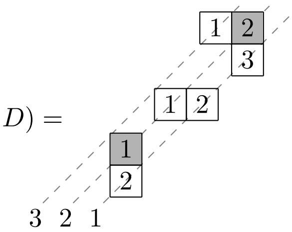
Let λ be a partition with k parts, let $\widehat{\mathrm{dg}}(\lambda) = \{(r, j) \in \mathrm{dg}(\lambda) : r > 1\}$ be the cells in $\mathrm{dg}(\lambda)$ not contained in the bottom row, and let $D \subseteq \widehat{\mathrm{dg}}(\lambda)$ be any subset. We define $\widehat{\boldsymbol{\nu}}(\lambda, D) = (\nu^{(k)}, \ldots, \nu^{(2)}, \nu^{(1)})$ to be a tuple of k ribbons (see Definition 3.6) such that

$$\mathrm{Des}(\nu^{(j)}) = \{\lambda_j - r + 1 \ : \ (r, j) \in D\},$$

where the ribbons are arranged from southwest to northeast, and such that the southeast-most cell of each ribbon is aligned on diagonal 1. In other words, the j'th ribbon has length λ_j and its descent set $\mathrm{Des}(\nu^{(j)})$ corresponds to the restriction of D to column j of $\mathrm{dg}(\lambda)$ when read from top to bottom. See Example 4.2.

Example 4.2. Let $\lambda = (3, 2, 2)$ with the subset of descents chosen to be $D = \{(2, 1), (2, 3)\}$, indicated by the shaded boxes in the figure below. Then $\mathrm{Des}(\nu^{(1)}) = \{2\}$, $\mathrm{Des}(\nu^{(2)}) = ?$, $\mathrm{Des}(\nu^{(3)}) = \{1\}$, and $\widehat{\boldsymbol{\nu}}(\lambda, D) = \{\nu^{(3)}, \nu^{(2)}, \nu^{(1)}\}$ is the tuple of three ribbons shown below from left to right, with the cells of each ribbon labeled in reading order and descents marked by shaded boxes.

$\mathrm{dg}(\lambda) =$ 

$\widehat{\boldsymbol{\nu}}(\lambda, D) =$ 

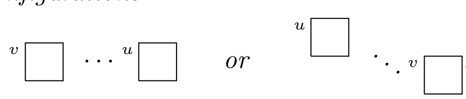
We now refine (10) by splitting it into fillings with a given descent set.

Definition 4.3. *Let λ be a partition and $\sigma \in \mathrm{Tab}(\lambda)$. A pair of cells $u = (r, i)$ and $v = (r', j)$ is said to be* attacking *if*

i. $r = r'$ *and* $i > j$, *or*

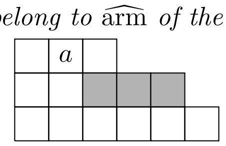
ii. $r = r' + 1$ *and* $i < j$,

i.e. in the following configurations:



If $\sigma(u) > \sigma(v)$ for a pair of attacking cells u, v, they form an attacking inversion*. If $\sigma(u) \neq \sigma(v)$ for every pair of attacking cells $u, v \in \mathrm{dg}(\lambda)$, we call σ a* non-attacking *filling.*

Denote the number of attacking inversions in $\sigma \in \mathrm{Tab}(\lambda)$ by $\widehat{\mathrm{inv}}(\sigma)$. For a cell $u = (r, i) \in \mathrm{dg}(\lambda)$, we define $\widehat{\mathrm{arm}}(u)$ to be the number of cells $(r-1, j) \in \mathrm{dg}(\lambda)$ such that $j > i$, i.e. the shaded cells belong to $\widehat{\mathrm{arm}}$ of the cell labeled a below.



Remark 4.4. The $\widehat{\cdot}$ symbol will help differentiate $\widehat{\mathrm{inv}}$ and $\widehat{\mathrm{arm}}$ from the traditional definitions of the notions of inversions and arms that appear in the proofs of the corresponding HHL formulas in [15, Section 3]. Note that we have called $\widehat{\mathrm{inv}}$ "attacking inversions" because they correspond to inversions occurring in "attacking cells" with respect to the reading order, to parallel the terminology used in [15] when referring to their version of attacking cells and inversions.

We will be grouping tableaux by their descent sets, indexed by subsets $D \subseteq \widehat{\mathrm{dg}}(\lambda)$. For each subset $D \subseteq \widehat{\mathrm{dg}}(\lambda)$, define

$$F_{\lambda,D}(X;t) = \sum_{\substack{\sigma \in \mathrm{Tab}(\lambda) \\ \mathrm{Des}(\sigma) = D}} t^{\widehat{\mathrm{inv}}(\sigma)} x^{\sigma}. \tag{15}$$

Now, we rewrite C_λ in terms of the $F_{\lambda,D}$'s.

Definition 4.5. *Let $U \subseteq \mathrm{dg}(\lambda)$ be a subset of cells. Define*

$$\mathrm{maj}(U) = \sum_{u \in U} (\mathrm{leg}(u) + 1)$$

and

$$\widehat{\mathrm{arm}}(U) = \sum_{u \in U} \widehat{\mathrm{arm}}(u).$$

Lemma 4.6.

$$C_\lambda(X;q,t) = \sum_{D \subseteq \widehat{\mathrm{dg}}(\lambda)} q^{\mathrm{maj}(D)} t^{-\widehat{\mathrm{arm}}(D)} F_{\lambda,D}(X;t).$$

Proof. We write (10) as a sum over descent sets:

$$C_\lambda(X;q,t) = \sum_{D \subseteq \widehat{\mathrm{dg}}(\lambda)} q^{\mathrm{maj}(D)} \sum_{\substack{\sigma \in \mathrm{Tab}(\lambda)\\ \mathrm{Des}(\sigma)=D}} t^{\mathrm{quinv}(\sigma)} x^\sigma.$$

Now suppose $\sigma \in \mathrm{Tab}(\lambda)$ with $D \coloneqq \mathrm{Des}(\sigma)$. We will show that

$$\mathrm{quinv}(\sigma) = \widehat{\mathrm{inv}}(\sigma) - \sum_{u \in D} \widehat{\mathrm{arm}}(u). \tag{16}$$

Consider a triple $(x,y,z) \in \mathrm{dg}(\lambda)$ with respective contents $a \coloneqq \sigma(x)$, $b \coloneqq \sigma(y)$, $c \coloneqq \sigma(z)$ in the configuration

$$\begin{array}{cccc} {\scriptstyle x} & \boxed{a} & & \\ {\scriptstyle y} & \boxed{b} & \cdots & {\scriptstyle z}\ \boxed{c} \end{array}.$$

(The triple may be degenerate, in which case $a = 0$.)

By comparing to (3), $(a,b,c) \in \mathcal{Q}$ if and only if exactly one of the following is true:

$$\left\{\begin{matrix} x \in D, \\ (y,z) \text{ do not form an attacking inversion,} \\ (z,x) \text{ do not form an attacking inversion} \end{matrix}\right\}. \tag{17}$$

First suppose $x \notin D$. If $(a,b,c) \in \mathcal{Q}$, then exactly one of (y,z) or (z,x) contributes to $\widehat{\mathrm{inv}}(\sigma)$ according to (17). If $(a,b,c) \notin \mathcal{Q}$, then neither (y,z) or (z,x) contributes to $\widehat{\mathrm{inv}}(\sigma)$. The contribution of the triple (x,y,z) to the left hand side and the right hand side of (16) match in each case.

Now suppose $x \in D$. If $(a,b,c) \in \mathcal{Q}$, then both (y,z) and (z,x) contribute to $\widehat{\mathrm{inv}}(\sigma)$, but since $z \in \widehat{\mathrm{arm}}(x)$, the total contribution to the right hand side in (16) is 1, which matches the contribution to the left hand side. If $(a,b,c) \notin \mathcal{Q}$, then exactly one of (y,z) or (z,x) contributes to $\widehat{\mathrm{inv}}(\sigma)$, but since $z \in \widehat{\mathrm{arm}}(x)$, the total contribution to the right hand side in (16) is zero, again matching the contribution to the left hand side.

Every pair of cells u,v where $u \in D$ and $v \in \widehat{\mathrm{arm}}(u)$, and every attacking pair that contributes to $\widehat{\mathrm{inv}}(\sigma)$, corresponds to some triple in σ. Thus every term in the left hand side is accounted for, and (16) follows.

We will now describe a weight-preserving map $\widehat{\mathrm{LLT}}$ from $\mathrm{Tab}(\lambda)$ with a fixed descent set D to fillings $\mathrm{SSYT}(\widehat{\boldsymbol{\nu}}(\lambda, D))$.

Definition 4.7. *For each descent set $D \subseteq \{(r,j) \in \widehat{\mathrm{dg}}(\lambda)\}$, define the map*

$$\widehat{\mathrm{LLT}} : \{\sigma \in \mathrm{Tab}(\lambda) : \mathrm{Des}(\sigma) = D\} \to \mathrm{SSYT}(\widehat{\boldsymbol{\nu}}(\lambda, D))$$

as follows. Let $\sigma \in \mathrm{Tab}(\lambda)$, and let $c_1, \dots, c_k$ be its columns from left to right, where each column contains the entries in σ from top to bottom: $c_i = (\sigma(\lambda_i, i), \dots, \sigma(2,i), \sigma(1,i))$. Set $\rho^{(i)} = \mathrm{ribbon}(c_i)$. Define $\boldsymbol{\rho} = (\rho^{(k)}, \rho^{(k-1)}, \dots, \rho^{(1)})$ to be the tuple of ribbons corresponding to the columns in reverse order, such that the last boxes of each ribbon are on the same diagonal, and set $\widehat{\mathrm{LLT}}(\sigma) = \boldsymbol{\rho}$. See Figure 4 for an example.

Remark 4.8. The careful reader may observe that the map $\widehat{\mathrm{LLT}}$ is identical to the corresponding map described in the proof of [15, Proposition 3.4] if the columns of the filling of $\mathrm{dg}(\lambda)$ are taken in reverse order to be mapped to ribbons.

From the definitions of LLT inversions in Definition 3.1 and attacking inversions in fillings of diagrams in Definition 4.3, the following result is immediate. See Figure 4 for an example.

$$\sigma = \begin{array}{|c|ccc}\hline 3 & & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \\ \hline 2 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} & \\ \hline \end{array}$$

$\widehat{\mathrm{LLT}}(\sigma) =$

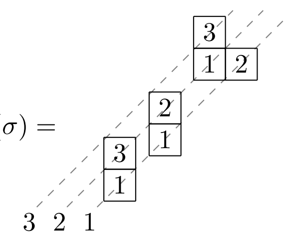


Figure 4. The filling $\widehat{\mathrm{LLT}}(\sigma)$ corresponding to $\sigma \in \mathrm{Tab}(\lambda)$. The numbering of the diagonals is shown on the bottom. One can check that $\mathrm{inv}(\widehat{\mathrm{LLT}}(\sigma)) = \widehat{\mathrm{inv}}(\sigma) = 5$. Moreover, the pairs of inversions in $\widehat{\mathrm{LLT}}(\sigma)$ correspond precisely to the attacking inversions in σ.

Lemma 4.9. *Let $\sigma \in \mathrm{Tab}(\lambda)$ and $\boldsymbol{\rho} = \widehat{\mathrm{LLT}}(\sigma)$. Then*

$$\widehat{\mathrm{inv}}(\sigma) = \mathrm{inv}(\boldsymbol{\rho}).$$

Thus we have defined a weight preserving bijection from $\mathrm{Tab}(\lambda)$ with descent set $D \subseteq \{(r,j) \in \widehat{\mathrm{dg}}(\lambda)\}$ to $\mathrm{SSYT}(\widehat{\boldsymbol{\nu}}(\lambda, D))$. (The reverse map follows easily from reversing Definition 4.7.) From this we obtain our final key lemma.

Lemma 4.10. *Let λ be a partition, and let $D \subseteq \{(r,j) \in \widehat{\mathrm{dg}}(\lambda)\}$. Then*

$$F_{\lambda,D}(X;t) = G_{\widehat{\boldsymbol{\nu}}(\lambda,D)}(X;t).$$

Theorem 4.1 follows directly from Theorem 3.4 and Lemma 4.10.

5. Proof that $C_\lambda(X;q,t)$ satisfies (8)

In this section we will use the ordering $<_1$ on $\mathcal{A}$ and construct a sign-reversing, weight-preserving involution Ψ on super fillings $\widetilde{\mathrm{Tab}}(\lambda)$, which will cancel out all terms involving x^μ if $\mu \not\geq \lambda$.

We begin by defining the following map.

Definition 5.1. *Let $u \in \mathrm{dg}(\lambda)$. For $\sigma \in \widetilde{\mathrm{Tab}}(\lambda)$, define the map Φ_u by*

$$\Phi_u(\sigma(w)) = \begin{cases} \sigma(w), & w \neq u, \\ \overline{\sigma(w)}, & w = u. \end{cases}$$

In other words, Φ_u is the map that negates the content of the cell u.

We first recall Definition 4.3 for attacking cells in our setting. A pair of cells $u = (r,i)$ and $v = (r',j)$ is attacking if $r = r'$, or $r = r'+1$ and $i < j$, i.e. they are either in the same row, or the one to the right is one row below. If u,v is an attacking pair of cells, u and v are said to attack each other.

Definition 5.2. *Let $\sigma \in \mathrm{Ta\hat{b}}(\lambda)$. We define $\Psi(\sigma)$ as follows.*

- *If there is no pair of attacking cells u,v in σ such that $|\sigma(u)| = |\sigma(v)|$, then set $\Psi(\sigma) = \sigma$. In this case, σ is called a non-attacking super filling.*
- *Otherwise let a be the smallest integer such that $|\sigma(x)| = |\sigma(y)| = a$ for some pair x,y of attacking cells in σ. Let v be the last cell in reading order among all such attacking pairs, and let u be the last cell in reading order that attacks v and such that $|\sigma(u)| = a$. Then $\Psi(\sigma) = \Phi_u(\sigma)$.*

We have defined Ψ such that if σ is not a fixed point, the map flips the sign of the entry at a designated cell u that depends only on $|\sigma|$: thus $\Psi(\Psi(\sigma)) = \sigma$. The following theorem is our main result in this section.

Theorem 5.3.

$$C_\lambda[X(t-1); q, t] = \sum_{\substack{\sigma \in \widetilde{\mathrm{Tab}}(\lambda)\\ \sigma : \Psi(\sigma)=\sigma}} (-1)^{m(\sigma)} q^{\mathrm{maj}(\sigma)} t^{p(\sigma)+\mathrm{quinv}(\sigma)} x^{|\sigma|}$$
$$= \sum_{\mu \leqslant \lambda} c_{\mu\lambda}(q,t) m_\mu.$$

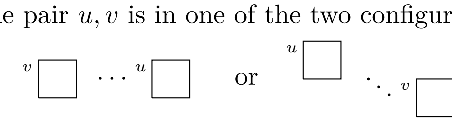
The proof of the theorem above will follow from the next two lemmas.

Lemma 5.4. *For any $\sigma \in \widetilde{\mathrm{Tab}}(\lambda)$, we have* $\mathrm{maj}(\Psi(\sigma)) = \mathrm{maj}(\sigma)$.

Lemma 5.5. *If σ is not a fixed point of Ψ so that $\Psi(\sigma) = \Phi_u(\sigma)$ for some cell u, then $\Psi = \Phi_u$ changes the number of* quinv *triples by exactly one:*

$$\mathrm{quinv}(\Phi_u(\sigma)) = \mathrm{quinv}(\sigma) + \begin{cases} 1, & \text{if } \sigma(u) \in \mathsf{Z}_+, \\ -1, & \text{if } \sigma(u) \in \mathsf{Z}_-. \end{cases}$$

Since u precedes v in reading order in the last attacking pair with entries equal to a in absolute value, the pair u, v is in one of the two configurations:



We first make some simple but key observations.

Proposition 5.6. *Suppose $\sigma \in \widetilde{\mathrm{Tab}}(\lambda)$ is not a fixed point of $\Psi(\sigma)$, and let u be such that $\Psi(\sigma) = \Phi_u(\sigma)$. Let u and v be an attacking pair for σ as given in Definition* 5.2*, and $|\sigma(u)| = |\sigma(v)| = a$.*

1. *If there exists a cell y below u, then $|\sigma(y)| \neq a$.*
2. *For any b we have $I_1(b, a) = I_1(b, \bar{a})$.*
3. *For any b such that $|b| \neq a$, we have $I_1(a, b) = I_1(\bar{a}, b)$.*

Proof. For (1), note that y forms an attacking pair together with v, and since the pair y, v comes after the pair u, v in the reading order, $|\sigma(y)| \neq a$. (2) and (3) are easily verified by going through the cases.

Proof of Lemma 5.4. We will show that Ψ preserves the descent set of σ, which implies maj is also preserved:

$$\mathrm{Des}(\Psi(\sigma)) = \mathrm{Des}(\sigma).$$

If σ is a fixed point of Ψ, there is nothing to prove, so suppose $\Psi(\sigma) = \Phi_u(\sigma)$ for some cell u. Since Φ_u is an involution, let us assume without loss of generality that $\sigma(u) = a \in \mathsf{Z}_+$. As u is the only cell that changed in $\Phi_u(\sigma)$, the only cells that might have changed whether or not they are descents are u and the cell directly above it. Let us consider the cells x and y directly above and below u, if they exist. Set $a = \sigma(u), b = \sigma(y), c = \sigma(x)$. Then we obtain the transition:

$$\begin{array}{c|c|} {\scriptstyle x} & c \\ \hline {\scriptstyle u} & a \\ \hline {\scriptstyle y} & b \\ \hline \end{array} \quad \xrightarrow{\Phi_u} \quad \begin{array}{c|c|} {\scriptstyle x} & c \\ \hline {\scriptstyle u} & \bar{a} \\ \hline {\scriptstyle y} & b \\ \hline \end{array} \tag{18}$$

Since $|b| \neq a$ by Proposition 5.6(1), we have $u \in \mathrm{Des}(\sigma)$ if and only if $u \in \mathrm{Des}(\Phi_u(\sigma))$. By Proposition 5.6(3), x (if it exists) is a descent for σ if and only if it is a descent for $\Phi_u(\sigma)$. Thus $x \in \mathrm{Des}(\sigma)$ if and only if $x \in \mathrm{Des}(\Phi_u(\sigma))$, thus concluding the proof.

Proof of Lemma 5.5. Let us assume without loss of generality that $\sigma(u) = a \in \mathsf{Z}_+$. The only triples whose contribution to $\operatorname{quinv}(\sigma)$ is different from $\operatorname{quinv}(\Phi_u(\sigma))$ are ones that include u. Moreover, since $I_1(a,b) = I_1(\bar a, b)$ for any b such that $|b| \neq a$ by Proposition 5.6(3), the only triples we need to consider are ones that include u plus another entry with absolute value equal to a. We examine the following three possible types of triples containing u, for some entries b, c in cells y, x respectively.

Case 1:
$$\begin{array}{l} {}^{u}\boxed{a} \\ {}^{x}\boxed{c}\ \cdots{}^{y}\boxed{b} \end{array} \quad \xrightarrow{\Phi_u} \quad \begin{array}{l} {}^{u}\boxed{\bar a} \\ {}^{x}\boxed{c}\ \cdots{}^{y}\boxed{b} \end{array}$$

Since $|c| \neq a$ by Proposition 5.6(1), suppose $|b| = a$. For both $b = a$ and $b = \bar a$, we have $I_1(a,b) = 0$, $I_1(\bar a, b) = 1$, and also $I_1(b,c) = I_1(a,c) = I_1(\bar a, c)$ by Proposition 5.6(3). Then, exactly two of the conditions
$$\{I_1(a,c) = 1,\ I_1(b,c) = 0,\ I_1(a,b) = 0\}$$
are true in σ and similarly, exactly one of the conditions
$$\{I_1(\bar a,c) = 1,\ I_1(b,c) = 0,\ I_1(\bar a,b) = 0\}$$
is true in $\Phi_u(\sigma)$; hence the triple (u,x,y) is a quinv triple in $\Phi_u(\sigma)$, but not in σ.

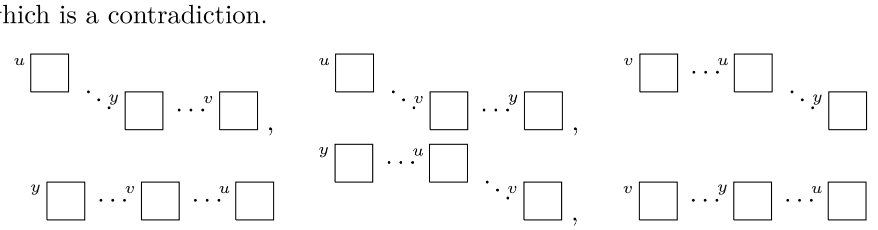
Case 2:
$$\begin{array}{l} {}^{y}\boxed{b} \\ {}^{u}\boxed{a}\ \cdots{}^{x}\boxed{c} \end{array} \quad \xrightarrow{\Phi_u} \quad \begin{array}{l} {}^{y}\boxed{b} \\ {}^{u}\boxed{\bar a}\ \cdots{}^{x}\boxed{c} \end{array}$$

For any x, y (including when y does not exist) we have $I_1(c,a) = I_1(c,\bar a)$ and $I_1(b,a) = I_1(b,\bar a)$ by Proposition 5.6(2), and so the triple (y,u,x) is trivially a quinv triple in σ if and only if it is one in $\Phi_u(\sigma)$.

Case 3:
$$\begin{array}{l} {}^{x}\boxed{c} \\ {}^{y}\boxed{b}\ \cdots{}^{u}\boxed{a} \end{array} \quad \xrightarrow{\Phi_u} \quad \begin{array}{l} {}^{x}\boxed{c} \\ {}^{y}\boxed{b}\ \cdots{}^{u}\boxed{\bar a} \end{array}$$

Again, $I_1(c,a) = I_1(c,\bar a)$ for all c by Proposition 5.6(2), and $I_1(a,b) = 1 - I_1(\bar a, b)$ if and only if $|b| = a$. As in Case 1, if $|b| = a$, the triple (x,y,u) is a quinv triple in $\Phi_u(\sigma)$, but not in σ.

Fix y to be the cell with entry $b \coloneqq \sigma(y)$ in the figures above. Next we show that if Case 1 or Case 3 occurs for a triple with $|\sigma(y)| = a$, then necessarily $y = v$, i.e. u and y is precisely the last attacking pair in reading order whose entries have absolute value a. It is easy to check this by considering all the following possible configurations of u, v, y below (the first row showing Case 1 and the second row showing Case 3), that if $v \neq y$, the last pair of attacking cells with absolute value a must then include y, which is a contradiction.



Therefore, every triple in σ contributes to $\operatorname{quinv}(\sigma)$ if and only if it also contributes to $\operatorname{quinv}(\Phi_u(\sigma))$, with the exception of exactly one triple, namely the unique triple that contains both u and v. Since we have assumed $\sigma(u) = a \in \mathsf{Z}_+$, this triple does not contribute to $\operatorname{quinv}(\sigma)$, but does contribute to $\operatorname{quinv}(\Phi_u(\sigma))$, so we obtain $\operatorname{quinv}(\Phi_u(\sigma)) = \operatorname{quinv}(\sigma) + 1$, giving us the desired expression.

Proof of Theorem 5.3. By Lemmas 5.4 and 5.5, if σ is not a fixed point of Ψ, then $\Psi(\sigma) = \Phi_u(\sigma)$ for some cell u, so that $p(\Phi_u(\sigma)) = p(\sigma) + \begin{cases} -1, & \text{if } \sigma(u) \in \mathsf{Z}_+, \\ 1, & \text{if } \sigma(u) \in \mathsf{Z}_-. \end{cases}$ Thus

$$q^{\mathrm{maj}(\Psi(\sigma))} t^{p(\Psi(\sigma))+\mathrm{quinv}(\Psi(\sigma))} = q^{\mathrm{maj}(\sigma)} t^{p(\sigma)+\mathrm{quinv}(\sigma)},$$

and so the contribution to the right hand side of (12) of $\Psi(\sigma)$ differs by a negative sign from the contribution of σ, and so these terms cancel each other out. Hence we obtain the first equality.

For the second equality, we use the fact that the fixed points of Ψ are precisely the non-attacking super fillings of $\mathrm{dg}(\lambda)$, which implies in particular that in each row, an entry appears at most once in absolute value. Suppose the monomial x^α appears as a term in the sum. Since the polynomial is symmetric, let us assume $\alpha = (\alpha_1, \alpha_2, \ldots)$ such that $\alpha_1 \geqslant \alpha_2 \geqslant \cdots$. For each j, $\alpha_1 + \cdots + \alpha_j$ is the number of entries in σ with absolute value at most j. Recall that the rows of $\mathrm{dg}(\lambda)$ are of lengths $(\lambda'_1, \lambda'_2, \ldots)$ and columns are of lengths $(\lambda_1, \lambda_2, \ldots)$. Consequently, counting by rows, the number of entries with value at most j counted by $\alpha_1 + \cdots + \alpha_j$ cannot exceed $\sum_i \min(\lambda'_i, j) = \lambda_1 + \cdots + \lambda_j$, which is the condition that $\alpha \leqslant \lambda$.

6. Proof that $C_\lambda(X; q, t)$ satisfies (9) in the nondegenerate case

In this section, we will use the ordering $<_2$ on $\mathcal{A}$ and describe sign-reversing, weight-preserving maps on super fillings $\widetilde{\mathrm{Tab}}$, which will cancel out all terms in (13) involving x^μ if $\mu \not\leqslant \lambda'$. We will follow the strategy of the proof of the corresponding result for HHL tableaux in [15, Section 5.2]. However, our proof will deviate for a particular subset of fillings, which we describe below as Φ-*degenerate* fillings, and will treat separately in Section 7.

Definition 6.1. *For $\sigma \in \widetilde{\mathrm{Tab}}(\lambda)$, let $a \in \mathsf{Z}_+$ be the smallest positive integer such that there exists a cell $(r, j) \in \mathrm{dg}(\lambda)$ with $|\sigma((r, j))| = a$ such that $r > a$, if such an a exists. We call such an a the* distinguished label *of σ. If a exists, we call the first cell in* Tab *reading order whose absolute value is a the* distinguished cell *of σ, denoted by $u(\sigma)$, and we define*

$$\Upsilon := \Phi_{u(\sigma)}.$$

Moreover, σ falls into one of the following three categories:

(a) *if no such a exists, we say σ is* Φ-trivial*.*
(b) *if the distinguished cell $u(\sigma)$ does not belong to any degenerate triples (regardless of whether they are* quinv *triples), we say σ is* Φ-nondegenerate*.*
(c) *if the cell $u(\sigma)$ is part of any degenerate triples (regardless of whether they are* quinv *triples), we say σ is* Φ-degenerate*. If r is the row containing $u(\sigma)$, we call the set of cells in row r that form degenerate triples the* degenerate segment*.*

See Figure 5 for examples of all three.

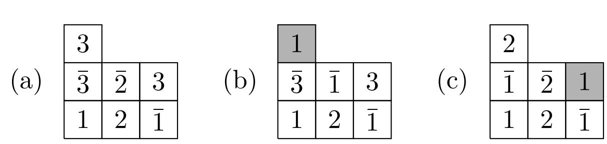


Figure 5. (a) Φ-trivial, (b) Φ-nondegenerate, and (c) Φ-degenerate. The grey box marks the distinguished cell if such exists. The degenerate segment in (c) is composed of the two cells at the tops of columns 2 and 3 in row 2.

Importantly, Φ-trivial fillings satisfy the following restriction on their content.

Lemma 6.2. *If σ is a Φ-trivial filling of* $\mathrm{dg}(\lambda)$ *with partition content μ, then $\mu \leqslant \lambda'$.*

Proof. Let $(1^{\mu_1}, 2^{\mu_2}, \cdots)$ be the content of $|\sigma|$ where $\mu_1 \geqslant \mu_2 \geqslant \cdots$. If σ is Φ-nondegenerate, then for each $j \geqslant 1$, all j's and $\bar{j}$'s must be contained in rows 1 through j, and so $\mu_1 + \cdots + \mu_j \leqslant \lambda'_1 + \cdots + \lambda'_j$ (the length of row j of $\mathrm{dg}(\lambda)$ is λ'_j). This is precisely the condition that $\mu \leqslant \lambda'$.

The main result in this section will be the following theorem.

Theorem 6.3.

$$C_\lambda[X(q-1);q,t] = \sum_{\substack{\sigma \in \widetilde{\mathrm{Tab}}(\lambda) \\ \sigma \text{ is } \Phi\text{-trivial}}} (-1)^{m(\sigma)} q^{p(\sigma)+\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)} x^{|\sigma|} \tag{19}$$

$$= \sum_{\mu \leqslant \lambda'} c_{\mu\lambda}(q,t) m_\mu \tag{20}$$

for some coefficients $c_{\mu\lambda}(q,t) \in \mathsf{Z}(q,t)$.

The outline of our proof is as follows. First we show that Υ is an involution on Φ-nondegenerate fillings which will cancel the terms coming from those fillings in (13). This mirrors the corresponding involution from [15, Section 5.2]. Next, in Section 7 we will prove the existence of a bijection on Φ-degenerate fillings, which will cancel the terms coming from those fillings in (13). By Theorem 7.2, only terms arising from Φ-trivial fillings remain, giving (19), which is equivalent to (20) by Lemma 6.2.

Recall the map Φ_u from Definition 5.1. As we will see in the following lemma, under total ordering $<_2$, for certain u's, the action of Φ_u on σ has the property that it increases maj by exactly one, and all triples that are not contained in the degenerate segment contribute to $\mathrm{quinv}(\sigma)$ if and only if they contribute to $\mathrm{quinv}(\Phi_u(\sigma))$.

When σ is not Φ-trivial, the following two lemmas present two important properties of $\Phi_u(\sigma)$ that hold for certain choices of the cell u.

Lemma 6.4. *Suppose σ is not Φ-trivial with distinguished label a, let r be the row containing the distinguished cell, and let u be any cell in row r with $|\sigma(u)| = a$. Then,*

$$\mathrm{maj}(\Phi_u(\sigma)) = \mathrm{maj}(\sigma) + \begin{cases} 1, & \textit{if } \sigma(u) = a, \\ -1, & \textit{if } \sigma(u) = \bar{a}. \end{cases} \tag{21}$$

Lemma 6.5. *Suppose σ is not Φ-trivial with distinguished label a, and let r be the row containing the distinguished cell. Then*

(1) *if σ is Φ-nondegenerate and u is the distinguished cell, or*
(2) *if σ is Φ-degenerate and u is any cell in the degenerate segment of row r with $|\sigma(u)| = a$,*

the following holds: every nondegenerate triple containing u in $\Phi_u(\sigma)$ is a quinv *triple if and only if it is also a* quinv *triple in σ.*

Lemmas 6.4 and 6.5 will be proved later in this section.

Example 6.6. Let σ be the filling in Figure 5(b). The distinguished label is $a = 1$ and $u = (3,1)$ is the distinguished cell (i.e. the 1 in the topmost row), then $\Phi_u(\sigma)$ is the filling

$$\begin{array}{|c|c|c|} \cline{1-1} \bar{1} & \multicolumn{2}{c}{} \\ \hline \bar{3} & \bar{1} & 3 \\ \hline 1 & 2 & \bar{1} \\ \hline \end{array}$$

Moreover, $\mathrm{maj}(\sigma) = 3$, $\mathrm{maj}(\Phi_u(\sigma)) = 4$, and $\mathrm{quinv}(\sigma) = \mathrm{quinv}(\Phi_u(\sigma)) = 3$.

Remark 6.7. The knowledgeable reader may observe that a similar involution $\Phi_{u'}$ was defined in [15, Section 5.2], where u' was chosen to be the topmost, *leftmost* cell of $\mathrm{dg}(\lambda)$ whose absolute value matches the distinguished label of σ.

We make a key observation.

Lemma 6.8. *Let a be the distinguished label and r the row containing the distinguished cell in a filling σ. Let x be any cell in either row $r-1$ or r, and let y be any cell in row $r+1$ or higher. Then $|\sigma(x)| > a$ and $|\sigma(y)| > a$.*

Proof. Since a is by definition the smallest positive integer such that there exists a cell in row $j > a$ with absolute value a, a row greater than or equal to $r-1$ cannot contain any cells with absolute value strictly less than a. Moreover, since u is chosen such that r is maximal, rows $r+1$ and higher cannot contain any cells with absolute value less than or equal to a.

Proof of Lemma 6.4. The proof is identical to that of [15, Lemma 5.2]. Let u be any cell in row r with $|\sigma(u)| = a$. Without loss of generality, assume $\sigma(u) = a$. The only possible change in the descent set of σ by the action of Φ_u can occur at the cell u or the cell directly above u if such exists. Consider the cells x and y with contents $c \coloneqq \sigma(x)$ and $b \coloneqq \sigma(y)$ directly above and below u, respectively, if such exist. Then we obtain a transition exactly analogous to (18).

By Lemma 6.8, $|b| > a$ and $|c| > a$. Thus $I_2(c, a) = I_2(\bar{a}, b) = 1$ and $I_2(c, \bar{a}) = I_2(a, b) = 0$, so necessarily $x \in \mathrm{Des}(\sigma)$ and $u \notin \mathrm{Des}(\sigma)$, while $x \notin \mathrm{Des}(\Phi_u(\sigma))$ and $u \in \mathrm{Des}(\Phi_u(\sigma))$, implying that $\mathrm{maj}(\Phi_u(\sigma)) = \mathrm{maj}(\sigma) + 1$. A similar argument proves the statement in the case where the cell x may not exist.

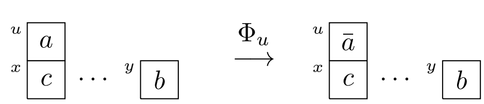
Proof of Lemma 6.5. We adapt the proof of [15, Lemma 5.2] to the setting of $\widetilde{\mathrm{Tab}}$. Let u be a cell in row r with $|\sigma(u)| = a$, such that either

(1) σ is Φ-nondegenerate and u is the distinguished cell, or
(2) σ is Φ-degenerate and u is in the degenerate segment of row r.

Without loss of generality, assume $\sigma(u) = a$.

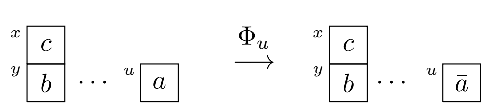
Clearly the only triples that may be affected by Φ_u are those containing the cell u. Let us inspect the three possible *nondegenerate* configurations of such triples. Let x, y be the other two cells in the triple, with contents $b \coloneqq \sigma(y)$ and $c \coloneqq \sigma(x)$; since the triples we examine are nondegenerate, both x and y must exist. We invoke Lemma 6.8.

Case 1:



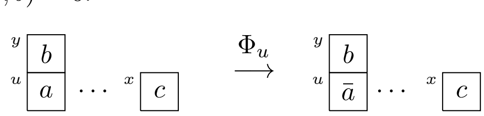
Both $|b| > a$ and $|c| > a$. Then $I_2(a, c) = I_2(a, b) = 0$ and $I_2(\bar{a}, c) = I_2(\bar{a}, b) = 1$. Thus the triple (u, x, y) is a quinv triple both in σ and in $\Phi(\sigma)$ if and only if $I_2(b, c) = 1$.

Case 2:



Since $|c| > a$, we have $I_2(c, a) = 1$ and $I_2(c, \bar{a}) = 0$. Since $|b| > a$, we have $I_2(a, b) = 0$ and $I_2(\bar{a}, b) = 1$. Thus the triple (x, y, u) is a quinv triple both in σ and in $\Phi(\sigma)$ if and only if $I_2(c, b) = 0$.

Case 3:



This case can only occur if σ is Φ-nondegenerate, since otherwise u is necessarily in the degenerate segment, contradicting the existence of the cell y. Thus we assume u is the distinguished cell, which means it is the first cell in reading order with content equal to a or $\bar{a}$. Consequently $|c| > a$ because x precedes u in the reading order, and so $I_2(c, a) = 1$ and $I_2(c, \bar{a}) = 0$. Since $|b| > a$, we have $I_2(b, a) = 1$ and $I_2(b, \bar{a}) = 0$. Thus, again, the triple (y, u, x) is a quinv triple both in σ and in $\Phi(\sigma)$ if and only if $I_2(b, c) = 1$.

In each of these cases, all nondegenerate triples containing u contribute to $\operatorname{quinv}(\sigma)$ if and only if they also contribute to $\operatorname{quinv}(\Phi_u(\sigma))$, as desired.

Corollary 6.9. *Let $\sigma \in \widetilde{\operatorname{Tab}}(\lambda)$ be Φ-nondegenerate.*

(i) Υ *is an involution on* $\widetilde{\operatorname{Tab}}(\lambda)$*, and*
(ii) $x^{|\Upsilon(\sigma)|}q^{p(\Upsilon(\sigma))+\operatorname{maj}(\Upsilon(\sigma))}t^{\operatorname{quinv}(\Upsilon(\sigma))} = x^{|\sigma|}q^{p(\sigma)+\operatorname{maj}(\sigma)}t^{\operatorname{quinv}(\sigma)}$.

Proof. The fact that Υ is an involution is immediate, since $|\Upsilon(\sigma)| = |\sigma|$, and so both fillings have the same distinguished label. Hence $u(\Upsilon(\sigma)) = u(\sigma)$, meaning that the same distinguished cell is chosen for $\Upsilon(\sigma)$ as for σ.

For part (ii), $|\Upsilon(\sigma)| = |\sigma|$ by definition. Without loss of generality, assume $\sigma(u) \in \mathsf{Z}_+$. Then by Lemma 6.4 and Lemma 6.5, $p(\Upsilon(\sigma)) = p(\sigma) - 1$, $\operatorname{maj}(\Upsilon(\sigma)) = \operatorname{maj}(\sigma) + 1$, and $\operatorname{quinv}(\sigma) = \operatorname{quinv}(\Upsilon(\sigma))$ since σ is Φ-nondegenerate, so the equality follows.

Unfortunately, when $\sigma \in \widetilde{\operatorname{Tab}}(\lambda)$ is Φ-degenerate, it may be the case that no choice of the cell u for which $|\sigma(u)| = a$ will be such that Φ_u preserves quinv and changes maj by 1. For example, going back to the filling σ in Figure 5(c), which has maj $= 3$ and quinv $= 3$, we list below the outputs of $\Phi_u(\sigma)$ where we set u to be each of the four cells whose content is 1 or $\bar{1}$, with the highlighted cell indicating the choice of u.

2			2			2			2		
1	$\bar{2}$	1	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	1	$\bar{1}$	$\bar{2}$	1
1	2	$\bar{1}$	1	2	$\bar{1}$	$\bar{1}$	2	$\bar{1}$	1	2	1

maj=2, quinv=2   maj=4, quinv=4   maj=3, quinv=2   maj=3, quinv=2

Thus we need to do something more subtle. In the following section, we will describe with some technical manipulations how to define a bijection on Φ-degenerate fillings that preserves quinv and changes maj by 1.

7. A bijection on Φ-degenerate tableaux

In this section, we *demonstrate the existence* of a bijection Θ on Φ-degenerate super fillings with the following properties. Let $\sigma \in \widetilde{\operatorname{Tab}}$.

i. $x^{|\Theta(\sigma)|} = x^{|\sigma|}$ and $p(\Theta(\sigma)) = p(\sigma) \pm 1$,
ii. $\operatorname{maj}(\Theta(\sigma)) = \operatorname{maj}(\sigma) \mp 1$, and
iii. $\operatorname{quinv}(\Theta(\sigma)) = \operatorname{quinv}(\sigma)$.

Note that we do not succeed in constructing such an bijection, and it remains an open question to find an explicit bijection.

Definition 7.1. *For a Φ-degenerate filling σ, define the* degenerate word *to be the content of the degenerate segment containing the distinguished cell, read from left to right. For example, in Figure* 5*(c), the degenerate word is* $(\bar{2}, 1)$*.*

The main result of this section is the following.

Theorem 7.2. *For a partition λ, the sum over the Φ-degenerate subset of fillings in $\widetilde{\mathrm{Tab}}(\lambda)$ in the right hand side of* (13)*, is zero.*

$$\sum_{\substack{\sigma\in\widetilde{\mathrm{Tab}}(\lambda)\\ \sigma \text{ is } \Phi\text{-degenerate}}} (-1)^{m(\sigma)} q^{p(\sigma)+\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)} x^{|\sigma|} = 0.$$

The outline of our proof is as follows. Let a be the distinguished label of a Φ-degenerate filling $\sigma \in \widetilde{\mathrm{Tab}}$. We will show in Proposition 8.6 that there exists a quinv-preserving map $\Theta : \widetilde{\mathrm{Tab}} \to \widetilde{\mathrm{Tab}}$ that preserves the content of the filling in absolute value (in particular Θ preserves the content in all the letters that are not a or $\bar{a}$), such that $\Theta(\sigma)$ has exactly one more or one less $\bar{a}$ as σ. Next in Definition 7.3 we define entry-swapping operators $\{\tau_j\}$ on $\widetilde{\mathrm{Tab}}$ that swap the topmost entries in columns $j, j+1$ (i.e. they swap entries in the degenerate word) while preserving the maj and quinv in the rest of the filling (Lemmas 7.5 and 7.6). Suppose the distinguished cell $u(\sigma)$ lies in the degenerate segment with degenerate word w. There is then a unique product of operators $\tau_w = \tau_{i_1}\tau_{i_2}\dots\tau_{i_s}$ that can be applied to $\Upsilon(\sigma) = \Phi_{u(\sigma)}(\sigma)$ such that

- the degenerate word of $\tau_w(\Upsilon(\sigma))$ matches $\phi(w)$, and
- $\widehat{\sigma} = \tau_w(\Upsilon(\sigma))$ satisfies

$$q^{p(\widehat{\sigma})+\mathrm{maj}(\widehat{\sigma})} t^{\mathrm{quinv}(\widehat{\sigma})} = q^{p(\sigma)+\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)}.$$

7.1. The operators τ_j. We first define the entry-swapping operator τ_j that swaps the topmost entries (if they are distinct) in columns $j, j+1$ of equal height while preserving certain statistics. This operator is modified for use in our setting from its original definition in [22].

Definition 7.3 (Definition of the operator τ_j). *Suppose two columns j and $j+1$ have equal height, i.e. $\lambda_j = \lambda_{j+1} = k$. We define an operator τ_j which exchanges contents of certain cells between columns j and $j+1$.*

Write $\sigma(r,j) = a_r$ and $\sigma(r,j+1) = b_r$ for $r = 1,\dots,k$.

Define $r_{\max}$ to be the largest $r \in \{2,\dots,k\}$ with the following property: either (a_r, a_{r-1}, b_{r-1}) and (b_r, a_{r-1}, b_{r-1}) are both in $\mathcal{Q}$, or both are not in $\mathcal{Q}$. (*That is, we look for the largest value of r such that exchanging the entries a_r and b_r makes no difference to whether $((r,j),(r-1,j),(r-1,j+1))$ is a quinv triple.*) *If there is no such r, let $r_{\max} = 1$.*

Now let the operator τ_j swap the entries between columns j and $j+1$ in rows i with $r_{\max} \leqslant i \leqslant k$; i.e. for those values of i,

$$\tau_j(\sigma)(i,j) = b_i, \qquad \tau_j(\sigma)(i,j+1) = a_i,$$

while all other entries are the same in σ and in $\tau_j(\sigma)$, as in the picture below, where we denote $\ell \coloneqq r_{\max}$.

$$\begin{array}{l} \\ \text{row } k \\ \vdots \\ \text{row } \ell \\ \text{row } \ell-1 \\ \vdots \\ \\ \end{array}
\begin{array}{c} \sigma \\ \begin{array}{|c|c|} \hline a_k & b_k \\ \hline \vdots & \vdots \\ \hline a_\ell & b_\ell \\ \hline a_{\ell-1} & b_{\ell-1} \\ \hline \vdots & \vdots \\ \hline \end{array} \\ \scriptstyle j \quad j+1 \end{array}
\xrightarrow{\tau_j}
\begin{array}{c} \tau_j(\sigma) \\ \begin{array}{|c|c|} \hline b_k & a_k \\ \hline \vdots & \vdots \\ \hline b_\ell & a_\ell \\ \hline a_{\ell-1} & b_{\ell-1} \\ \hline \vdots & \vdots \\ \hline \end{array} \\ \scriptstyle j \quad j+1 \end{array}$$

Put another way: we swap the pair in row k, and iteratively, if we have swapped the pair in row i and this made a difference to whether the triple $(i,j),(i-1,j),(i-1,j+1)$ is a quinv triple, then we swap in row $i-1$ also.

Example 7.4. Suppose σ has columns $j, j+1$ as shown below. Then $k=5$ and $r_{\max}=3$ since both $(2,3,4)\in\mathcal{Q}$ and $(3,3,4)\in\mathcal{Q}$. Thus applying the operator τ_j gives the following. The cells whose content was swapped are shown in grey.

$$\begin{array}{c} \sigma \\ \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 2 & 3 \\ \hline 2 & 3 \\ \hline 3 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \\ j \quad j{+}1 \end{array} \xrightarrow{\ \tau_j\ } \begin{array}{c} \tau_j(\sigma) \\ \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 3 & 2 \\ \hline 3 & 2 \\ \hline 3 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \\ j \quad j{+}1 \end{array}$$

It is straightforward to check that τ_j is an involution: let σ be a filling with $\sigma'=\tau_j(\sigma)$ such that the entries in rows $\ell,\cdots,k$ are swapped between columns j and $j+1$, meaning that the $r_{\max}$ for σ is ℓ. One verifies that for any $c\in\mathcal{A}$, if $a_k\neq b_k$, then $(c,a_k,b_k)\in\mathcal{Q}$ if and only if $(c,b_k,a_k)\notin\mathcal{Q}$. On the other hand, if $a_k=b_k\in\mathsf{Z}_+$, then $(c,a_k,b_k)\notin\mathcal{Q}$, and if $a_k=b_k\in\mathsf{Z}_-$, then $(c,a_k,b_k)\in\mathcal{Q}$, and so, necessarily, $r_{\max}>k+1$. Therefore, if a swap at row $k+1$ changes the contribution to quinv of the triple of cells (x_{k+1},x_k,y_k), then $a_k\neq b_k$, and hence this change will be reversed by a swap at row k. Thus it is immediate that σ' has the same value for $r_{\max}$, and so τ_j acting on σ' reverses the swaps of the cells in rows $\ell,\dots,k$, recovering σ.

Proving the following Lemmas 7.5 and 7.6 regarding the properties of the action of τ_j on super fillings $\widetilde{\mathrm{Tab}}$ is also straightforward, but technical.

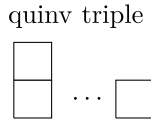
Lemma 7.5. *Let $\sigma\in\widetilde{\mathrm{Tab}}$ and let j be such that the columns $j,j+1$ are of equal height k. Then*

$$\operatorname{maj}(\tau_j(\sigma))=\operatorname{maj}(\sigma). \tag{22}$$

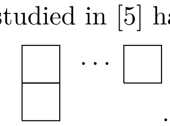
Lemma 7.6. *Let $\sigma\in\widetilde{\mathrm{Tab}}$ and let j be such that the columns $j,j+1$ are of equal height k. Let $x=(k,j)$ and $y=(k,j+1)$ be the cells at the tops of columns j and $j+1$. Then*

$$\operatorname{quinv}(\tau_j(\sigma))=\operatorname{quinv}(\sigma)+\begin{cases}1, & \sigma(x)>\sigma(y),\\ -1, & \sigma(x)<\sigma(y),\\ 0, & \sigma(x)=\sigma(y).\end{cases} \tag{23}$$

Remark 7.7. We point out that our definition of τ_j is very similar to that of a similar operator in [5, Definition 3.7], with two important differences. First, τ_j necessarily swaps the topmost entries in the columns if they are different and acts trivially if they are identical, instead of swapping the first pair of non-identical entries (from the bottom) as in [5]. Second, in our case swaps propagate downwards from row k, whereas in the corresponding operator defined in [5], the swaps propagate upwards. This has to do with the fact that a quinv triple has configuration



whereas the corresponding triple studied in [5] has configuration



.

Proof of Lemma 7.5. The proof is almost identical to that of [5, Lemma 3.10], but we present it here as a warm-up for Lemma 7.6, in the more nuanced setting of super fillings.

If $k = 1$, there is no contribution to maj from columns $j, j+1$, so assume $k > 1$. Consider the entries in columns $j, j+1$ in the two rows $\ell, \ell-1$, such that $1 < \ell \leqslant k$, as shown below.

$$\begin{array}{r|c|c|} \multicolumn{1}{r}{} & \multicolumn{1}{c}{\scriptstyle j} & \multicolumn{1}{c}{\scriptstyle j+1} \\ \cline{2-3} \text{row } \ell & a & b \\ \cline{2-3} \text{row } \ell-1 & c & d \\ \cline{2-3} \end{array}$$

If a, b are not swapped by τ_j, there is nothing to check. If both a, b and c, d are swapped by τ_j, the contribution to maj from the cells $(\ell, j), (\ell, j+1)$ remains the same. Thus the only case we need to check is when $\ell = r_{\max}$ so that the swapping procedure terminates at row ℓ and a, b are swapped, but c, d are not. Thus assume τ_j sends $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ to $\begin{array}{|c|c|}\hline b & a \\ \hline c & d \\ \hline\end{array}$.

For the remainder of this section, set $I = I_2$. Since $\ell = r_{\max}$, we have that the triples (a, c, d) and (b, c, d) are are either both in $\mathcal{Q}$, or neither is in $\mathcal{Q}$. Following the condition in Definition 3.9 of a triple to be in $\mathcal{Q}$, let us consider the sets

$$ACD = \begin{Bmatrix} I(a,c) = 1 \\ I(a,d) = 0 \\ I(d,c) = 0 \end{Bmatrix} \quad \text{and} \quad BCD = \begin{Bmatrix} I(b,c) = 1 \\ I(b,d) = 0 \\ I(d,c) = 0 \end{Bmatrix},$$

where we know that either both ACD and BCD contain exactly one condition that is true, or they both contain exactly two that are true. Recall that a pair of cells u, v in σ with $v = \mathrm{South}(u)$ forms a descent if $I(\sigma(u), \sigma(v)) = 1$. We consider two cases based on whether or not $I(d, c) = 0$.

Case 1: Suppose $I(d, c) = 0$. If the two triples are in $\mathcal{Q}$, both of the remaining conditions in ACD and BCD must be false, and so we must have $I(a, c) = I(b, c) = 0$ and $I(a, d) = I(b, d) = 1$. This implies the cells $(\ell, j), (\ell, j+1)$ make the same contribution to maj in σ and $\tau_j(\sigma)$. On the other hand, if the two triples are not in $\mathcal{Q}$, exactly one of the two remaining conditions in both ACD and BCD must be true, and so we must have $I(a, c) = I(a, d)$ and $I(b, c) = I(b, d)$. This also results in the same contribution to maj from the cells $(\ell, j), (\ell, j+1)$ in σ and $\tau_j(\sigma)$.

Case 2: Suppose $I(d, c) = 1$. This case is similar. If the two triples are in $\mathcal{Q}$, we must have $I(a, c) = I(a, d)$ and $I(b, c) = I(b, d)$ since exactly one of the two remaining conditions must be true. If the two triples are not in $\mathcal{Q}$, both of the two remaining conditions must be true, and so we must have $I(a, c) = I(b, c) = 1$ and $I(a, d) = I(b, d) = 0$. Both cases result in the same contribution to maj from the cells $(\ell, j), (\ell, j+1)$.

We see finally that even though the locations of the descents may change after applying τ_j, the total contribution to maj in columns $j, j+1$ remains constant.

Proof of Lemma 7.6. The proof follows by a similar argument as that of [5, Lemma 3.11], though the details are a bit nuanced when we work with super fillings.

Consider the columns $j, j+1$ of σ shown below with $x = (k, j)$, $y = (k, j+1)$, and $a_k = \sigma(x)$, $b_k = \sigma(y)$. Assume $a_k \neq b_k$, since otherwise τ_j is trivial. Let $\ell := r_{\max}$, so τ_j swaps all pairs of entries in rows $j, j+1$ from row k down to row ℓ. If $\ell = 1$, then the columns $j, j+1$ simply switched places, which trivially proves the claim. Thus assume $\ell > 1$, and let c, d be the cells in row $\ell - 1$ in columns $j, j+1$ respectively.

$$
\begin{array}{lccccccc}
 & & \sigma & & & & \tau_j(\sigma) & \\
\text{row } k & \begin{array}{|c|c|}\hline a_k & b_k \\ \hline \vdots & \vdots \\ \hline a_\ell & b_\ell \\ \hline c & d \\ \hline \end{array} & & \begin{array}{|c|}\hline f_1 \\ \hline \vdots \\ \hline f_\ell \\ \hline f \\ \hline \end{array} & \xrightarrow{\tau_j} & \begin{array}{|c|c|}\hline b_k & a_k \\ \hline \vdots & \vdots \\ \hline b_\ell & a_\ell \\ \hline c & d \\ \hline \end{array} & & \begin{array}{|c|}\hline f_1 \\ \hline \vdots \\ \hline f_\ell \\ \hline f \\ \hline \end{array} \\
\text{row } \ell & & & & & & & \\
\text{row } \ell - 1 & & & & & & & \\
 & {\scriptstyle j \quad j+1} & & {\scriptstyle s} & & {\scriptstyle j \quad j+1} & & {\scriptstyle s}
\end{array}
$$

Since only columns $j, j+1$ are affected by τ_j, we only need to examine the contribution in $\mathrm{quinv}(\sigma)$ and $\mathrm{quinv}(\tau_j(\sigma))$ of triples with at least one entry in one of those columns. By construction, with the exception of the degenerate triple formed by the entries (a_k, b_k), all triples in σ in columns $j, j+1$ make the same contribution in $\mathrm{quinv}(\sigma)$ as the corresponding triples in the same locations of $\tau_j(\sigma)$ do in $\mathrm{quinv}(\tau_j(\sigma))$. Moreover, triples containing cells in columns to the left of j will not be affected either, nor will any triples with all cells below row ℓ. It remains for us to examine the triples containing cells in rows greater than or equal to $\ell - 1$ in one of the columns $j, j+1$, and a cell in a column s for some $s > j+1$. Both σ and $\tau_j(\sigma)$ have the triples $((i+1, j), (i, j), (i, s))$ and $((i+1, j+1), (i, j+1), (i, s))$ with the same contents (a_{i+1}, a_i, f_i) and (b_{i+1}, b_i, f_i) for $\ell \leqslant i \leqslant k$ (per our convention we set $\sigma(a_{k+1}) = \sigma(b_{k+1}) = 0$). Therefore the only triples whose content was altered are the triples $((\ell, j), (\ell-1, j), (\ell-1, s))$ and $((\ell, j+1), (\ell-1, j+1), (\ell-1, s))$ with respective contents (a_ℓ, c, f) and (b_ℓ, d, f) in σ, which ended up with respective content (b_ℓ, c, f) and (a_ℓ, d, f) in $\tau_j(\sigma)$. We will now check that the total number of quinv triples is preserved in each of these pairs. To simplify notation, we will drop the subscripts and write $a = a_\ell$ and $b = b_\ell$.

We begin with a key observation. Since $\ell = r_{\max}$, we have that either both (a, c, d) and (b, c, d) are in $\mathcal{Q}$, or neither is in $\mathcal{Q}$. Equivalently, this means that in the sets

$$
ACD = \begin{Bmatrix} I(a,c) = 1 \\ I(a,d) = 0 \\ I(d,c) = 0 \end{Bmatrix} \quad \text{and} \quad BCD = \begin{Bmatrix} I(b,c) = 1 \\ I(b,d) = 0 \\ I(d,c) = 0 \end{Bmatrix},
$$

either exactly one condition is true in both, or exactly two are true in both. By studying the cases in the proof of Lemma 7.5, we get that either both $I(a, c) = I(a, d)$ and $I(b, c) = I(b, d)$, or both $I(a, c) \neq I(a, d)$ and $I(b, c) \neq I(b, d)$.

Let us now consider the triples (a, c, f), (b, d, f), and (b, c, f), (a, d, f), respectively corresponding to the sets

$$
ACF = \begin{Bmatrix} I(a,c) = 1 \\ I(a,f) = 0 \\ I(f,c) = 0 \end{Bmatrix}, \ BDF = \begin{Bmatrix} I(b,d) = 1 \\ I(b,f) = 0 \\ I(f,d) = 0 \end{Bmatrix} \ \text{v/s} \ BCF = \begin{Bmatrix} I(b,c) = 1 \\ I(b,f) = 0 \\ I(f,c) = 0 \end{Bmatrix}, \ ADF = \begin{Bmatrix} I(a,d) = 1 \\ I(a,f) = 0 \\ I(f,d) = 0 \end{Bmatrix}.
$$

We split our argument into two cases. To eliminate extra notation, let us identify the triples with their corresponding sets.

In the first case, let $I(a, c) = I(a, d)$ and $I(b, c) = I(b, d)$. If $I(f, c) = I(f, d)$, then ACF (*resp.* BDF) has the same number of true conditions as ADF (*resp.* BCF), which means the pairs make the same contribution to $\mathrm{quinv}(\sigma)$ as to $\mathrm{quinv}(\tau_j(\sigma))$. Thus let us assume $I(f, c) \neq I(f, d)$. In the table below, we examine the cases recalling (from Definition 3.9) that there is no contradiction in either ACF or ADF arising from the latter having either all conditions true or all conditions false. Our notation is as follows: when we write, say, Set $\Rightarrow X$, this means that X must be true in order to avoid a contradiction in Set.

	$I(f,c)=0$ and $I(f,d)=1$	$I(f,c)=1$ and $I(f,d)=0$
$I(a,c)=I(a,d)=1$	$ACF \Rightarrow I(a,f)=1$	$ADF \Rightarrow I(a,f)=1$
$I(a,c)=I(a,d)=0$	$ADF \Rightarrow I(a,f)=0$	$ACF \Rightarrow I(a,f)=0$

This table implies that necessarily, $I(a,c)=I(a,d)=I(a,f)$. An analogous argument yields that necessarily, $I(b,d)=I(b,f)=I(b,c)$. From here it is straightforward to check that for any choice of $I(f,c)$ and $I(f,d)$, ACF and BDF make the same contribution to $\mathrm{quinv}(\sigma)$, as BCF and ADF to $\mathrm{quinv}(\tau_j(\sigma))$, and so the total contribution of the pair ACF, BDF to $\mathrm{quinv}(\sigma)$ matches the contribution of the pair BCF, ADF to $\mathrm{quinv}(\tau_j(\sigma))$.

In the second case, let $I(a,c) \neq I(a,d)$ and $I(b,c) \neq I(b,d)$. Because maj is preserved, we must also have that $I(a,c) \neq I(b,d)$, implying that $I(a,c)=I(b,c)$ and $I(b,d)=I(a,d)$. We must also consider all choices for $I(f,c)$ and $I(f,d)$. We complete the argument by carefully considering all possible cases in the table below. We use some additional shorthand notation: when we write, say, $\{\cancel{ACF}, ADF\}$, this means ACF is not a quinv triple, but ADF is one. When we write, say, $ACF = ADF$ *dep. on* $I(a,f)$, this means that the value of $I(a,f)$ determines both ACF and ADF (and they are equal in either case).

	$I(f,c)=0$, $I(f,d)=0$	$I(f,c)=0$, $I(f,d)=1$	$I(f,c)=1$, $I(f,d)=0$	$I(f,c)=1$, $I(f,d)=1$
$I(a,c)=1$, $I(b,c)=1$, $I(b,d)=0$, $I(a,d)=0$	$ACF \Rightarrow I(a,f)=1$ $\Rightarrow \{\cancel{ACF},\ ADF\}$ $BCF \Rightarrow I(b,f)=1$ $\Rightarrow \{BDF,\ \cancel{BCF}\}$	$ACF \Rightarrow$ $I(a,f)=1$, contradicts ADF	$ACF = ADF$ dep. on $I(a,f)$ $BCF = BDF$ dep. on $I(b,f)$	$ADF \Rightarrow I(a,f)=0$ $\Rightarrow \{\cancel{ACF}, ADF\}$ $BDF \Rightarrow I(b,f)=0$ $\Rightarrow \{BDF,\ \cancel{BCF}\}$
$I(a,c)=0$, $I(b,c)=0$, $I(b,d)=1$, $I(a,d)=1$	$ADF \Rightarrow I(a,f)=1$ $\Rightarrow \{ACF,\ \cancel{ADF}\}$ $BDF \Rightarrow I(b,f)=1$ $\Rightarrow \{\cancel{BDF},\ BCF\}$	$ACF = ADF$ dep. on $I(a,f)$ $BCF = BDF$ dep. on $I(b,f)$	$ACF \Rightarrow$ $I(a,f)=0$, contradicts ADF	$ACF \Rightarrow I(a,f)=0$ $\Rightarrow \{ACF, \cancel{ADF}\}$ $BCF \Rightarrow I(b,f)=0$ $\Rightarrow \{\cancel{BDF},\ BCF\}$

Again, in every possible case, the contribution of the pair ACF, BDF to $\mathrm{quinv}(\sigma)$ matches the contribution of the pair BCF, ADF to $\mathrm{quinv}(\tau_j(\sigma))$. Thus we conclude that the only triple whose contribution to $\mathrm{quinv}(\sigma)$ is different from its contribution to $\mathrm{quinv}(\tau_j(\sigma))$ is the degenerate triple (x,y) at the top of the columns $j, j+1$. If $\sigma(x)=\sigma(y)$, $\tau_j(\sigma)=\sigma$. If $\sigma(x) \neq \sigma(y)$ and if (x,y) was a degenerate quinv triple before the swap, swapping them will cause $\tau_i(\sigma)$ to lose one quinv triple, and gain one otherwise.

Remark 7.8. We will be working with permutations, and corresponding reduced expressions of those permutations to products of the operators τ_j. Unfortunately, the operators do not satisfy braid relations, i.e. in general $\tau_j\tau_{j+1}\tau_j(\sigma) \neq \tau_{j+1}\tau_j\tau_{j+1}(\sigma)$. Thus we will need to choose a canonical reduced expression for a given permutation to make our construction well-defined. One way to do this uses the *positive distinguished subexpression*, or PDS, of a reduced expression, defined in [25]. We denote by $\mathrm{PDS}(\pi)$ the PDS corresponding to a permutation π. We will refer the reader to [5, Section 3] for a detailed treatment of this notion; in our setting it suffices to assert that $\mathrm{PDS}(\pi)$ is a unique reduced expression for any permutation π.

8. From a bijection on words to a bijection on super fillings

In this section, we prove Proposition 8.6, which states the existence of a bijection on the set of Φ-degenerate fillings $\widetilde{\mathrm{Tab}}$, using the super-alphabet $\mathcal{A}$ and total ordering $<_2$ on that alphabet. Proposition 8.6 is needed in the proof of Theorem 7.2.

We begin with some definitions. Let $n \geqslant 3$ be a positive integer and let $\beta = (\beta_2, \dots, \beta_{n-1})$ be a tuple of positive integers that will represent the fixed content in the letters $\{2, \dots, n-1\}$ of the words we consider. Denote $|\beta| = \sum_i \beta_i$ and let $N > |\beta|$ be a positive integer. For convenience, let $L = N - |\beta|$. We will consider words of length N in the alphabet $[n] = \{1, \dots, n\}$ as follows. For $0 \leqslant k \leqslant L$, define

$$W_k \equiv W_k^{(\beta,L)} = \left\{ w \in [n]^N \;\middle|\; \begin{matrix} c_n(w) = k, c_1(w) = L-k, \\ c_i(w) = \beta_i \text{ for } 2 \leqslant i \leqslant n-1 \end{matrix} \right\}, \tag{24}$$

where $c_i(w)$ counts the number of times the letter i appears in w. We say such words are of type β.

Example 8.1. For example, with $n = 3, N = 4, L = 3$ and $\beta = (1)$, we have

$$\begin{aligned} &W_0 = \{1112, 1121, 1211, 2111\},\\ &W_1 = \{1132, 1123, 1213, 2113, 1312, 1321, 1231, 2131, 3112, 3121, 3211, 2311\},\\ &W_2 = \{1332, 1323, 1233, 2133, 3132, 3123, 3213, 2313, 3312, 3321, 3231, 2331\},\\ &W_3 = \{3332, 3323, 3233, 2333\}. \end{aligned} \tag{25}$$

Recall that a *coinversion* of a word w is a pair (i, j), $i < j$, such that $w_i < w_j$, and $\mathrm{coinv}(w)$ is the number of such pairs. In our applications, we define the map ρ_α that maps a word v with content α in the super-alphabet $\mathcal{A} = \{1, 2, \dots, \bar{2}, \bar{1}\}$ under total ordering $<_2$ to a word w in the letters $\{1, 2, \dots, n\}$, such that the order of the letters is preserved under the mapping, and such that the smallest letter $a = \min|\alpha|$ is mapped to 1 and $\bar{a}$ is mapped to n, as follows. Set $\ell = \max|\alpha| - \min|\alpha| + 2$, and let $n = 2(\ell - 1)$. The map ρ_α will send the unbarred letters in v to the letters $1, \dots, \ell - 1$ and the barred letters to $\ell, \dots, n$ by mapping $j \in v$ as follows:

$$\rho_\alpha(j) = \begin{cases} j - a + 1 & \text{if } j \in \mathsf{Z}_+, \\ n - |j| + a & \text{if } j \in \mathsf{Z}_-. \end{cases}$$

Now, for a fixed $1 \leqslant \ell \leqslant n$, let $w \in W_k$ (i.e. $c_n(w) = k$). Define

$$\mathrm{quinv}^{(\ell)}(w) = \mathrm{coinv}(w) + \binom{c_\ell(w)}{2} + \cdots + \binom{c_{n-1}(w)}{2} + \binom{k}{2}. \tag{26}$$

Thus if the preimage $v = \rho_\alpha^{-1}(w)$ in $\mathcal{A}$ corresponds to the degenerate word of some $\sigma \in \widetilde{\mathrm{Tab}}$, $\mathrm{quinv}^{(\ell)}(w)$ is equal to the contribution from the degenerate word v to $\mathrm{quinv}(\sigma)$ (under the ordering $<_2$). We may slightly abuse notation by writing $\mathrm{quinv}^{(\ell)}(v) \coloneqq \mathrm{quinv}^{(\ell)}(\rho_\alpha(v))$ to describe this contribution to $\mathrm{quinv}(\sigma)$.

Example 8.2. Let $v = (\bar{5}, \bar{2}, 3, 3, \bar{3}, \bar{2}, 3, 5, \bar{3})$ be a word in the alphabet $\mathcal{S} = \{3, 5, \bar{5}, \bar{3}, \bar{2}\}$ with content $\alpha = \{\bar{2}, \bar{2}, 3, 3, 3, \bar{3}, \bar{3}, 5, \bar{5}\}$. Then $a = 2$, $\ell = 5 - 2 + 2 = 5$, $n = 8$, and $w = \rho_\alpha(v) = (5, 8, 2, 2, 7, 8, 2, 4, 7)$ with $k = 2, c_5(w) = 1, c_6(w) = 2$, and $c_7(w) = 0$. Indeed, $\mathrm{quinv}(v) = 18$ according to $<_2$, $\mathrm{coinv}(w) = 16$, and $\mathrm{quinv}^{(\ell)}(w) = 16 + \binom{1}{2} + \binom{2}{2} + \binom{0}{2} + \binom{2}{2} = 18$.

The proof of Proposition 8.6 relies on the existence of a $\mathrm{quinv}^{(\ell)}$-preserving bijection ϕ on $W \equiv \cup_{k=0}^{L} W_k$ that satisfies the desired properties. This result is stated below, but first we need some additional notation. For $0 \leqslant k \leqslant L$, partition W_k from (24) into two subsets as follows. For $w \in W_k$, let $p_n(w)$ be the position of the leftmost n in w from the left and $p_1(w)$ be the position of the rightmost 1 from the right, ignoring all n's. If $c_j(w) = 0$, set $p_j(w) = \infty$ for $j = 1, n$. Then define

$$\begin{aligned} W_k^{\leqslant} &= \{w \in W_k \mid p_n(w) \leqslant p_1(w)\} \\ W_k^{>} &= \{w \in W_k \mid p_n(w) > p_1(w)\} \end{aligned}$$

Note that by our convention, $W_0^{>} = W_0$ and $W_L^{\leqslant} = W_L$, with $W_0^{\leqslant} = W_L^{>} = \varnothing$.

Example 8.3. Let $n = 3, N = 4, L = 3$, and $\beta = (1)$. Then

$$
\begin{aligned}
W_0^{>} &= \{11\mathbf{1}2, 112\mathbf{1}, 121\mathbf{1}, 211\mathbf{1}\},\\
W_1^{\leqslant} &= \{1\mathbf{31}2, \mathbf{3}1\mathbf{1}2, \mathbf{3}12\mathbf{1}, \mathbf{3}21\mathbf{1}\},\\
W_1^{>} &= \{1\mathbf{13}2, 1\mathbf{1}2\mathbf{3}, 12\mathbf{13}, 21\mathbf{13}, 1\mathbf{3}2\mathbf{1}, 12\mathbf{31}, 21\mathbf{31}, 2\mathbf{3}1\mathbf{1}\},\\
W_2^{\leqslant} &= \{\mathbf{13}23, \mathbf{13}32, \mathbf{31}23, \mathbf{31}32, \mathbf{3}2\mathbf{1}3, \mathbf{3}23\mathbf{1}, \mathbf{3}3\mathbf{1}2, \mathbf{3}32\mathbf{1}\},\\
W_2^{>} &= \{\mathbf{1}2\mathbf{3}3, 2\mathbf{13}3, 2\mathbf{31}3, 2\mathbf{3}3\mathbf{1}\},\\
W_3^{\leqslant} &= \{\mathbf{3}332, \mathbf{3}323, \mathbf{3}233, 2\mathbf{3}33\}.
\end{aligned}
$$

The leftmost n and the rightmost 1 are marked in bold. For instance, the pairs $(p_n(w), p_1(w))$ for the words in $W_2^{>}$ are $\{(3,2),(3,2),(2,1),(2,1)\}$, and for the words in $W_2^{\leqslant}$, they are $\{(2,2),(2,2),(1,2),(1,2),(1,1),(1,1),(1,2),(1,1)\}$.

Let $W^{\leqslant} = \bigcup_k W_k^{\leqslant}$ and $W^{>} = \bigcup_k W_k^{>}$.

Theorem 8.4. *There exists a bijection $\phi : W^{>} \to W^{\leqslant}$ satisfying the following conditions:*

(1) *ϕ maps $W_k^{>}$ to $W_{k+1}^{\leqslant}$ bijectively for $0 \leqslant k \leqslant L-1$.*
(2) *$\mathrm{quinv}^{(\ell)}(w) = \mathrm{quinv}^{(\ell)}(\phi(w))$ for all $w \in W^{>}$, and*
(3) *The subword of w in the letters $2, \dots, n-1$ is preserved by ϕ.*

Section 9 is devoted to the proof of Theorem 8.4, as it is quite technical.

Now, fix the content α, which determines n, ℓ, L. For $0 \leqslant k \leqslant L$, define the sets

$$
V_k = \{\rho_\alpha^{-1}(w) : w \in W_k\}, \quad V_k^{\leqslant} = \{\rho_\alpha^{-1}(w) : w \in W_k^{\leqslant}\}, \quad V_k^{>} = \{\rho_\alpha^{-1}(w) : w \in W_k^{>}\},
$$

$$
V^{\leqslant} = \bigcup_k V_k^{\leqslant}, \quad V^{>} = \bigcup_k V_k^{>}.
$$

Define also the bijection $\phi_\alpha : V^{>} \to V^{\leqslant}$ by $\phi_\alpha(v) := \rho_\alpha^{-1} \circ \phi \circ \rho_\alpha(v)$. It is immediate that ϕ_α satisfies the conditions (1)–(2) from Theorem 8.4 when all W's replaced by V's. It also satisfies the analog of condition (3): the subword of v in the letters between $a+1$ and $\overline{a+1}$ is preserved (where $a = \min|\alpha|$).

Let σ be a Φ-degenerate filling with degenerate word v that contains the distinguished label. We split the set of Φ-degenerate fillings into two disjoint sets that are denoted by $\mathcal{V}^{>}$ and $\mathcal{V}^{\leqslant}$:

- $\sigma \in \mathcal{V}^{>}$ if $v \in V^{>}$, and
- $\sigma \in \mathcal{V}^{\leqslant}$ if $v \in V^{\leqslant}$.

Example 8.5. Consider the set of words

$$
\{\bar{1}\bar{1}2, \bar{1}2\bar{1}, 2\bar{1}\bar{1}, \bar{1}12, \bar{1}21, 2\bar{1}1, 1\bar{1}2, 12\bar{1}, 21\bar{1}, 112, 121, 211\}.
$$

Here $V^{>} = \{211, 121, 2\bar{1}1, 112, 21\bar{1}, 12\bar{1}\}$ with $\mathrm{quinv}^{(\ell)}$-generating functions $1+q+q^2$ and $q+q^2+q^3$ for $V_0^{>}, V_1^{\leqslant}$ and for $V_1^{>}, V_2^{\leqslant}$, respectively. The bijection ϕ_α from Theorem 8.4 is unique on the corresponding sets $W^{>}$ and $W^{\leqslant}$, thus the elements of $V^{>}$ are mapped to the elements of $V^{\leqslant}$ as follows:

$$
\begin{aligned}
&\phi_\alpha(211) = \bar{1}21,\ \mathrm{quinv}^{(\ell)} = 0, \quad &&\phi_\alpha(121) = \bar{1}12,\ \mathrm{quinv}^{(\ell)} = 1,\\
&\phi_\alpha(2\bar{1}1) = \bar{1}\bar{1}2,\ \mathrm{quinv}^{(\ell)} = 1, \quad &&\phi_\alpha(112) = 1\bar{1}2,\ \mathrm{quinv}^{(\ell)} = 2,\\
&\phi_\alpha(21\bar{1}) = \bar{1}2\bar{1},\ \mathrm{quinv}^{(\ell)} = 2, \quad &&\phi_\alpha(12\bar{1}) = 2\bar{1}\bar{1},\ \mathrm{quinv}^{(\ell)} = 3.
\end{aligned}
$$

Consider the fillings σ_1, σ_2 shown below. In both, the degenerate word contains the distinguished label:

$$\sigma_1 = \begin{array}{|c|c|c|c|c}\cline{1-1} 2 & \multicolumn{4}{c}{} \\ \cline{1-4} 1 & 2 & 1 & \bar{1} & \\ \cline{1-4} 3 & 1 & \bar{2} & \bar{1} & \\ \hline 2 & 1 & 2 & 1 & \multicolumn{1}{c|}{\bar{2}} \\ \hline \end{array}\;, \qquad \sigma_2 = \begin{array}{|c|c|c|c|c}\cline{1-1} 2 & \multicolumn{4}{c}{} \\ \cline{1-4} 1 & 1 & \bar{1} & 2 & \\ \cline{1-4} 3 & 1 & \bar{2} & \bar{1} & \\ \hline 2 & 1 & 2 & 1 & \multicolumn{1}{c|}{\bar{2}} \\ \hline \end{array}$$

The filling σ_1 belongs to $\mathcal{V}^{>}$ since $21\bar{1} \in V^{>}$, and the filling σ_2 belongs to $\mathcal{V}^{6}$ since $1\bar{1}2 \in V^{6}$.

We are now ready to state the main result of this section.

Proposition 8.6. *There exists a map Θ on the subset $\mathcal{V}^{>}$ of Φ-degenerate fillings σ such that*

(1) $\Theta \;:\; \mathcal{V}^{>} \longrightarrow \mathcal{V}^{6}$ *is a bijection,*
(2) $\mathrm{maj}(\Theta(\sigma)) = \mathrm{maj}(\sigma) + 1$*,*
(3) $\mathrm{quinv}(\Theta(\sigma)) = \mathrm{quinv}(\sigma)$*,*
(4) $x^{|\Theta(\sigma)|} = x^{|\sigma|}$ *and* $p(\Theta(\sigma)) = p(\sigma) - 1$*.*

Proof. Let σ be a Φ-degenerate filling of $\mathrm{dg}(\lambda)$ with distinguished label a contained in degenerate word v. We will define the map Θ acting on $\sigma \in \mathcal{V}^{>}$ with the following steps.

(i) Set $u(\sigma)$ to be the first cell in reading order with content a in σ (the distinguished cell).
(ii) Let $\widehat{v}$ be the degenerate word of $\Upsilon(\sigma) = \Phi_{u(\sigma)}(\sigma)$. ($\widehat{v}$ is equal to v, except that the rightmost a is flipped to become $\bar{a}$.)
(iii) Let α be the content of v. Let π be a canonical permutation acting on positions to get from $\widehat{v}$ to $\phi_\alpha(v)$, i.e. $\widehat{v}\cdot\pi = \phi_\alpha(v)$. (For instance, π could be canonically defined as follows: begin by fixing the leftmost entry of $\phi_\alpha(v)$ that doesn't match $\widehat{v}$ by a series of adjacent transpositions. Continue in this way until no mismatched entries remain.) Let $\mathrm{PDS}(\pi) = s_{i_1}s_{i_2}\dots s_{i_m}$ be the unique reduced word which is the PDS corresponding to π (see Remark 7.8).
(iv) Let d be the number of columns in $\mathrm{dg}(\lambda)$ strictly to the left of the degenerate segment. Set $\Theta = \Upsilon \circ \tau_{i_1+d} \circ \tau_{i_2+d} \circ \cdots \circ \tau_{i_m+d}$ acting on the right, so that
$$\Theta(\sigma) = \tau_{i_m+d} \circ \cdots \circ \tau_{i_2+d} \circ \tau_{i_1+d} \circ \Upsilon(\sigma).$$

We prove that Θ satisfies the properties (1)–(4) in order.

Θ *satisfies* (1)*:* The degenerate word of $\Theta(\sigma)$ is by construction equal to $\phi_\alpha(v)$. Thus $v \in V^{>}$ implies that $\Theta(\sigma) \in \mathcal{V}^{6}$. Next we note that the cell $u(\sigma)$ that is flipped in $\sigma \in \mathcal{V}^{>}$ by Υ is determined uniquely, as is the product $\tau_{i_m+d} \circ \cdots \circ \tau_{i_2+d} \circ \tau_{i_1+d}$ of operators applied to $\Upsilon(\sigma)$. The operators τ_j are involutions, so this construction is well-defined. Thus we obtain the reverse map Θ^{-1} from $\mathcal{V}^{6}$ to $\mathcal{V}^{>}$ by simply reversing the operators in the map Θ.

Let $\sigma \in \mathcal{V}^{6}$ be a filling of $\mathrm{dg}(\lambda)$, and again let a be the distinguished label and $v \in V^{6}$ the degenerate word of σ. Call $\widehat{u}(\sigma)$ the cell in the diagram $\mathrm{dg}(\lambda)$ that contains the rightmost entry a in $\phi_\alpha^{-1}(v) \in V^{>}$. Call v^* the word obtained by flipping the rightmost a in $\phi(v)$ to become $\bar{a}$. Then define the permutation π such that $v^*\pi = v$, and let $\mathrm{PDS}(\pi) = s_{i_1}s_{i_2}\dots s_{i_m}$. The reader may now check that the map from $\sigma \in W^{6}$ to $\Theta^{-1}(\sigma)$ given by
$$\Theta^{-1}(\sigma) = \Phi_{\widehat{u(\sigma)}} \circ \tau_{i_1+d} \circ \tau_{i_2+d} \circ \cdots \circ \tau_{i_m+d}(\sigma),$$
is the inverse of the map from $\sigma \in \mathcal{V}^{>}$ to $\mathcal{V}^{6}$, since $\widehat{u}(\sigma)$ is by design precisely equal to $u(\Theta^{-1}(\sigma))$ as defined in (i). Hence the construction of Θ is an injection.

Finally, by Theorem 8.4 we have that for each $k > 0$, the number of elements in $\mathcal{V}^{>}$ with exactly k $\bar{a}$'s, the desired content, and a fixed number of quinv triples equals the number of elements in $\mathcal{V}^{\leqslant}$ with exactly $k+1$ $\bar{a}$'s, from which we conclude that Θ is also a surjection.

Θ *satisfies* (2)*:* By Lemma 6.4, when u is any cell in the degenerate segment of σ with $\sigma(u) = a$, the operator Φ_u acts on σ by increasing the maj by exactly one. By Lemma 7.5 the operators τ_j leave the maj unchanged.

Θ *satisfies* (3)*:* By Lemma 6.5, when u is any cell in the degenerate segment of σ with $|\sigma(u)| = a$, all triples with the exception of degenerate triples containing u contribute to $\mathrm{quinv}(\sigma)$ if and only if they also contribute to $\mathrm{quinv}(\Phi_u(\sigma))$. Observe that the degenerate triples containing u are precisely the elements in v, the degenerate word of σ. For ℓ defined as in 8 for the degenerate word v, the contribution to $\mathrm{quinv}(\sigma)$ from the degenerate triples in v is precisely equal to $\mathrm{quinv}^{(\ell)}(v)$ of (26). Therefore let us write $\mathrm{quinv}(\sigma) = \mathrm{quinv}'(\sigma) + \mathrm{quinv}^{(\ell)}(v)$, where $\mathrm{quinv}'(\sigma)$ is the contribution to $\mathrm{quinv}(\sigma)$ coming from all triples but the degenerate triples in v. We have $\mathrm{quinv}'(\Phi_u(\sigma)) = \mathrm{quinv}'(\sigma)$ by Lemma 6.5, and $\mathrm{quinv}'(\tau_j(\sigma)) = \mathrm{quinv}'(\sigma)$ by Lemma 7.6 for all τ_j acting on columns that contain the degenerate word v. By construction we have that $\Theta(\sigma)$ has degenerate word $\phi_\alpha(v)$, and since $\mathrm{quinv}^{(\ell)}(v) = \mathrm{quinv}^{(\ell)}(\phi_\alpha(v))$ by Theorem 8.4, we have thus $\mathrm{quinv}(\Theta(\sigma)) = \mathrm{quinv}(\sigma)$, which completes our argument.

Θ *satisfies* (4)*:* The content of σ is changed only by the operator Υ, which flips exactly one entry from positive to negative, thus reducing $p(\sigma)$ by 1, while leaving the absolute value of the content of σ unchanged.

Theorem 7.2 follows as an immediate corollary.

Example 8.7. Consider σ_1 from Example 8.5. The degenerate word is $v = 21\bar{1} \in V^{>}$, $\widehat{v} = 2\bar{1}\bar{1}$, and $\phi_\alpha(v) = \bar{1}2\bar{1}$ (here $\alpha = (1)$). Thus the permutation to get from $\widehat{v}$ to $\phi_\alpha(v)$ is $\pi = 213$, which has corresponding PDS $\pi = s_1$, and $u(\sigma_1)$ is the first cell with content 1 in reading order. Thus we apply τ_1 to σ_1 after flipping the 1 to obtain

$$\Theta(\sigma_1) = \tau_1 \circ \Phi_{u(\sigma_1)}(\sigma_1) = \begin{array}{|c|c|c|c|c|} \cline{1-1} 2 \\ \cline{1-4} 1 & \bar{1} & 2 & \bar{1} \\ \cline{1-4} 3 & \bar{2} & 1 & \bar{1} \\ \hline 2 & 1 & 2 & 1 & \bar{2} \\ \hline \end{array} \in \mathcal{V}^{\leqslant}.$$

For another example, consider

$$\sigma_3 = \begin{array}{|c|c|c|c|c|} \cline{1-1} 2 \\ \cline{1-4} 1 & 1 & 2 & 1 \\ \cline{1-4} 3 & \bar{2} & 1 & \bar{1} \\ \hline 2 & 1 & 2 & 1 & \bar{2} \\ \hline \end{array} \in \mathcal{V}^{>}.$$

Then $v = 121, \widehat{v} = 12\bar{1}$, and $\phi_\alpha(v) = 12\bar{1}$, so that the right-acting permutation π such that $\widehat{v}\pi = \phi(v)$ is $\pi = s_2 s_1$. The number of columns left of the degenerate segment is $\ell = 1$, and the cell $u(\sigma_3) = (3,4)$. Thus $\Theta(\sigma_2) = \tau_2 \circ \tau_3 \circ \Phi_{(3,4)}(\sigma_3)$:

$$\sigma_3 \xrightarrow{\Phi_{(3,4)}} \begin{array}{|c|c|c|c|c|} \cline{1-1} 2 \\ \cline{1-4} 1 & 1 & 2 & \bar{1} \\ \cline{1-4} 3 & \bar{2} & 1 & \bar{1} \\ \hline 2 & 1 & 2 & 1 & \bar{2} \\ \hline \end{array} \xrightarrow{\tau_3} \begin{array}{|c|c|c|c|c|} \cline{1-1} 2 \\ \cline{1-4} 1 & 1 & \bar{1} & 2 \\ \cline{1-4} 3 & \bar{2} & \bar{1} & 1 \\ \hline 2 & 1 & 2 & 1 & \bar{2} \\ \hline \end{array} \xrightarrow{\tau_2} \begin{array}{|c|c|c|c|c|} \cline{1-1} 2 \\ \cline{1-4} 1 & \bar{1} & 1 & 2 \\ \cline{1-4} 3 & \bar{2} & \bar{1} & 1 \\ \hline 2 & 1 & 2 & 1 & \bar{2} \\ \hline \end{array} = \Theta(\sigma_3) \in \mathcal{V}^{\leqslant}.$$

9. A bijection on words

The aim of this section is to prove Theorem 8.4: the existence of the bijection $\phi : W^{>} \to W^{\leqslant}$ for words in the alphabet $[n]$ with a fixed type $\beta = (\beta_2, \ldots, \beta_{n-1})$ in the letters $\{2,\dots,n-1\}$. We denote the set of such words by $W = W(\beta)$. For the proofs, we ask the reader to refer to the terminology and basic results on q-series from 3.4.

First off, for such a map ϕ to exist, the $\mathrm{quinv}^{(\ell)}$ generating function of W must be a multiple of 2. To see this, recall the definition of $\mathrm{quinv}^{(\ell)}$ from (26) and write

$$\begin{aligned}\sum_{w\in W} q^{\mathrm{quinv}^{(\ell)}(w)} &= \sum_{k=0}^{L}\sum_{w\in W_k} q^{\mathrm{coinv}(w)+\binom{\beta_\ell}{2}+\cdots+\binom{\beta_{n-1}}{2}+\binom{k}{2}}\\ &= q^{\binom{\beta_\ell}{2}+\cdots+\binom{\beta_{n-1}}{2}}\sum_{k=0}^{L} q^{\binom{k}{2}}\begin{bmatrix}N\\ L-k,\beta_2,\dots,\beta_{n-1},k\end{bmatrix}\\ &= q^{\binom{\beta_\ell}{2}+\cdots+\binom{\beta_{n-1}}{2}}\begin{bmatrix}N\\ L,\beta_2,\dots,\beta_{n-1}\end{bmatrix}\sum_{k=0}^{L} q^{\binom{k}{2}}\begin{bmatrix}L\\ k\end{bmatrix},\end{aligned}$$

where we have used Proposition 3.15 in the second line. Recall that W is the set of words of length N in the alphabet $\{1,\dots,n\}$ with content given by $(L-k,\beta_2,\dots,\beta_{n-1},k)$ for $L = N-(\beta_2+\cdots+\beta_{n-1})$ and k ranging from 0 to L. For $j=2,\dots,n-1$, the content in the letters j is fixed to be $c_j(w)=\beta_j$ for all words $w\in W$, while k corresponds to the number of letters n in words of W. Now, the sum is a direct application of the q-binomial theorem, Proposition 3.16, with $x=1$ and we find that

$$\sum_{w\in W} q^{\mathrm{quinv}^{(\ell)}(w)} = 2q^{\binom{\beta_\ell}{2}+\cdots+\binom{\beta_{n-1}}{2}}\begin{bmatrix}N\\ L,\beta_2,\dots,\beta_{n-1}\end{bmatrix}\prod_{j=1}^{L-1}(1+q^j),$$

as claimed.

The following theorem shall lead us to a proof of Theorem 8.4.

THEOREM 9.1. *For n,N,L,β, k and ℓ as above, we have the identities*

$$\sum_{w\in W_k^{>}} q^{\mathrm{quinv}^{(\ell)}(w)} = q^{\binom{\beta_\ell}{2}+\cdots+\binom{\beta_{n-1}}{2}+\binom{k+1}{2}}\begin{bmatrix}N\\ L,\beta_2,\dots,\beta_{n-1}\end{bmatrix}\begin{bmatrix}L-1\\ k\end{bmatrix}, \tag{27}$$

$$\sum_{w\in W_k^{6}} q^{\mathrm{quinv}^{(\ell)}(w)} = q^{\binom{\beta_\ell}{2}+\cdots+\binom{\beta_{n-1}}{2}+\binom{k}{2}}\begin{bmatrix}N\\ L,\beta_2,\dots,\beta_{n-1}\end{bmatrix}\begin{bmatrix}L-1\\ k-1\end{bmatrix}. \tag{28}$$

Proof of Theorem 8.4. As a consequence of Theorem 9.1, the $\mathrm{quinv}^{(\ell)}$ generating functions of $W_k^{>}$ and W_{k+1}^{6} are equal, thus proving (2).

To show (3), let $w=(w_1,\dots,w_N)\in W_k$ and let w' be obtained from w by replacing all occurrences of the letters from 2 through $n-1$ by 2, and all the occurrences of n by 3. Let w'' be the subword of w consisting of the letters from 2 through $n-1$ in w. All we need to do is to show that the $\mathrm{quinv}^{(\ell)}$ generating functions for every fixed w'' are the same. First of all, it is clear that $\mathrm{coinv}(w)=\mathrm{coinv}(w')+\mathrm{coinv}(w'')$. Moreover, the question of whether w belongs to W_k^{6} or $W_k^{>}$ is the same as that of w'. Thus we may refine the sums appearing in (27) and (28) by fixing the subword w'' and summing (26) over all words w' of length N with $N-L$ 2's. This is implicitly done in the proof of Theorem 9.1. Comparing (27) for k and (28) for $k+1$, we see that the common q-multinomial coefficient corresponds to the sum of $\mathrm{coinv}(w'')$ over all possible subwords w''. The other factors match for $W_k^{>}$ and W_{k+1}^{6}, completing the proof of (3).

EXAMPLE 9.2. As an illustration of Theorem 9.1, consider Example 8.3 and let $L=3$. We compute the $\mathrm{quinv}^{(\ell)}$ generating functions with $\ell=2$ for the sets $W_k^{>}$ and W_k^{6}

for $0 \leqslant k \leqslant 3$, denoted by $g_k^{>}(q)$ and $g_k^{\leqslant}(q)$, respectively:

$$\begin{aligned}
g_0^{>}(q) &= g_1^{\leqslant}(q) = 1+q+q^2+q^3,\\
g_1^{>}(q) &= g_2^{\leqslant}(q) = q+2q^2+2q^3+2q^4+q^5,\\
g_2^{>}(q) &= g_3^{\leqslant}(q) = q^3+q^4+q^5+q^6.
\end{aligned}$$

The bijection ϕ claimed in Theorem 8.4 is unique for the sets $W_0^{>} \xrightarrow{\phi} W_1^{\leqslant}$ (namely, the words $1112, 1121, 1211, 2111$ are mapped to $1312, 3112, 3121, 3211$, respectively) and $W_2^{>} \xrightarrow{\phi} W_3^{\leqslant}$ (the words $1233, 2133, 2313, 2331$ are mapped to $2333, 3233, 3323, 3332$, respectively). However the bijection ϕ is not unique for the sets $W_1^{>} \xrightarrow{\phi} W_2^{\leqslant}$.

Although Theorem 9.1 looks as if it should be known, we have not seen this in the literature before. The result follows from a more general result, which we now state. For $w \in W_k$, recall the definition of $p_1(w)$ and $p_n(w)$ given before Example 8.3.

Theorem 9.3. *Let N, L, β, k, and ℓ be as above. Then*

$$\sum_{\substack{w\in W_k^{>}\\ p_1(w)=i}} q^{\mathrm{quinv}^{(\ell)}(w)} = q^{\binom{\beta_\ell}{2}+\cdots+\binom{\beta_{n-1}}{2}+\binom{k+1}{2}+(i-1)L}\begin{bmatrix} N-L\\ \beta_2,\dots,\beta_{n-1}\end{bmatrix} \times \begin{bmatrix} N-i\\ k, N-L-i+1, L-k-1\end{bmatrix}, \tag{29}$$

$$\sum_{\substack{w\in W_k^{\leqslant}\\ p_n(w)=j}} q^{\mathrm{quinv}^{(\ell)}(w)} = q^{\binom{\beta_\ell}{2}+\cdots+\binom{\beta_{n-1}}{2}+\binom{k}{2}+(j-1)L}\begin{bmatrix} N-L\\ \beta_2,\dots,\beta_{n-1}\end{bmatrix} \times \begin{bmatrix} N-j\\ k-1, N-L-j+1, L-k\end{bmatrix}. \tag{30}$$

These two identities imply the corresponding ones in Theorem 9.1.

Proof of Theorem 9.1. We use the telescoping sum identity in Proposition 3.19 to sum (29) and (30) over i and j, to obtain the identities (27) and (28), respectively.

We prove Theorem 9.3 using the lemmas stated after, which are in turn proved in the following two subsections.

Proof of Theorem 9.3. It will suffice to prove Theorem 9.3 for the case of $n=3$. To see this, let $w=(w_1,\dots,w_N)\in W_k$ and w' be obtained from w by replacing all occurrences of the letters from 2 through $n-1$ by 2, and all the occurrences of n by 3. Let w'' be the subword of w consisting of the letters from 2 through $n-1$ in w. Then, it is clear that $\mathrm{coinv}(w)=\mathrm{coinv}(w')+\mathrm{coinv}(w'')$. Moreover, the question of whether w belongs to $W_k^{\leqslant}$ or $W_k^{>}$ is the same as that of w'. We can then rewrite (29) as

$$\sum_{\substack{w\in W_k^{>}\\ p_1(w)=i}} q^{\mathrm{quinv}^{(\ell)}(w)} = q^{\binom{\beta_\ell}{2}+\cdots+\binom{\beta_{n-1}}{2}+\binom{k}{2}}\begin{bmatrix} N-L\\ \beta_2,\dots,\beta_{n-1}\end{bmatrix}\sum_{w'\in V_k^{>}} q^{\mathrm{coinv}(w')},$$

where $V_k^{>} = \left(W_k^{(N-L),L}\right)^{>}$.

We now restrict to the case $n = 3$ and $\beta = (\beta_2) = (N - L)$. We first consider (29). By summing the two cases in Lemma 9.4, we see that, for $1 \leqslant i \leqslant N + 1 - L$,

$$\sum_{\substack{w \in W_k^{>} \\ p_1(w) = i}} q^{\mathrm{coinv}(w)} = q^{iL+k-L} \begin{bmatrix} N-k-i \\ L-k-1 \end{bmatrix} \begin{bmatrix} N-i \\ k \end{bmatrix} = q^{iL+k-L} \begin{bmatrix} L-1 \\ k \end{bmatrix} \begin{bmatrix} N-i \\ L-1 \end{bmatrix}.$$

Now, the i-sum can be performed using Proposition 3.19. We then simplify to obtain the result. For (30), the argument is similar. Summing the two cases in Lemma 9.5 shows that, for $1 \leqslant j \leqslant N + 1 - L$,

$$\begin{aligned} \sum_{\substack{w \in W_k^{\leqslant} \\ p_3(w) = j}} q^{\mathrm{coinv}(w)} &= q^{jL-L} \begin{bmatrix} N-j-k+1 \\ L-k \end{bmatrix} \begin{bmatrix} N-j \\ k-1 \end{bmatrix} \\ &= q^{jL-L} \begin{bmatrix} L-1 \\ k-1 \end{bmatrix} \begin{bmatrix} N-j \\ L-1 \end{bmatrix}. \end{aligned}$$

Again, the j-sum can be performed using Proposition 3.19. This completes the proof.

Let $p'_1(w)$ be the actual position of the rightmost 1 from the right in w. For both lemmas, there are two subcases to consider, either $p_3(w) < N + 1 - p'_1(w)$ or not. That is to say, the leftmost 3 is to the left of the rightmost 1 or not. The first lemma is for the subset $W_k^{>}$.

Lemma 9.4. *Let $n = 3$, and N, L and k be as above. Then,*

(1) *The generating function*

$$\sum_{\substack{w \in W_k^{>} \\ p_1(w) = i \\ p_3(w) < N+1-p'_1(w)}} q^{\mathrm{coinv}(w)} = q^{(i-1)L} \begin{bmatrix} N-k-i \\ L-k-1 \end{bmatrix} \left(\begin{bmatrix} N-i \\ k \end{bmatrix} q^k - \begin{bmatrix} k+i-1 \\ k \end{bmatrix} q^{k(N-k-2i+2)} \right), \tag{31}$$

if $1 \leqslant i \leqslant \lfloor (N-k)/2 \rfloor$, and zero otherwise.

(2) *For $1 \leqslant i \leqslant \lfloor (N-k)/2 \rfloor$,*

$$\sum_{\substack{w \in W_k^{>} \\ p_1(w) = i \\ p_3(w) > N+1-p'_1(w)}} q^{\mathrm{coinv}(w)} = q^{(i-1)L+k(N-k-2i+2)} \begin{bmatrix} N-k-i \\ L-k-1 \end{bmatrix} \begin{bmatrix} k+i-1 \\ k \end{bmatrix}, \tag{32}$$

and for $\lfloor (N-k)/2 \rfloor + 1 \leqslant i \leqslant N + 1 - L$,

$$\sum_{\substack{w \in W_k^{>} \\ p_1(w) = i \\ p_3(w) > N+1-p'_1(w)}} q^{\mathrm{coinv}(w)} = q^{(i-1)L+k} \begin{bmatrix} N-k-i \\ L-k-1 \end{bmatrix} \begin{bmatrix} N-i \\ k \end{bmatrix}. \tag{33}$$

The second lemma is for the subset $W_k^{\leqslant}$.

Lemma 9.5. *Let $n = 3$, and N, L and k be as above. Then,*

(1) *The generating function*

$$\sum_{\substack{w\in W_k^{\leqslant}\\ p_3(w)=j\\ p_3(w)<N+1-p_1'(w)}} q^{\mathrm{coinv}(w)} = \begin{bmatrix} N-j\\ k-1\end{bmatrix}\left(\begin{bmatrix} N-j-k+1\\ L-k\end{bmatrix} q^{(j-1)L} - \begin{bmatrix} j-1\\ L-k\end{bmatrix} q^{(j-1)k+(L-k)(N-j-k+1)}\right), \tag{34}$$

if $1\leqslant j\leqslant \lfloor (N-k+1)/2\rfloor$, *and zero otherwise.*

(2) *For* $L-k+1\leqslant j\leqslant \lfloor (N-k+1)/2\rfloor$,

$$\sum_{\substack{w\in W_k^{\leqslant}\\ p_3(w)=j\\ p_3(w)>N+1-p_1'(w)}} q^{\mathrm{coinv}(w)} = q^{(j-1)k+(L-k)(N-j-k+1)}\begin{bmatrix} j-1\\ L-k\end{bmatrix}\begin{bmatrix} N-j\\ k-1\end{bmatrix}, \tag{35}$$

and for $\lfloor (N-k+1)/2\rfloor+1\leqslant j\leqslant N+1-L$,

$$\sum_{\substack{w\in W_k^{\leqslant}\\ p_3(w)=j\\ p_3(w)>N+1-p_1'(w)}} q^{\mathrm{coinv}(w)} = q^{(j-1)L}\begin{bmatrix} N-j-k+1\\ L-k\end{bmatrix}\begin{bmatrix} N-j\\ k-1\end{bmatrix}. \tag{36}$$

9.1. Proof of Lemma 9.4. We will now prove the cases of Lemma 9.4 separately. We first begin with the second case, since it is conceptually simpler.

Proof of Lemma 9.4(2). Suppose $w\in W_k^{>}$ with $p_3(w)=j$, $p_1'(w)=i+k$ and $j>N+1-i-k$. Then w can be written as $w=w_1 1 2^{i+j+k-N-2} 3 w_2$, where $w_1\in\{1,2\}^{N-i-k}$ with $L-k-1$ 1's and $w_2\in\{2,3\}^{N-j}$ with $k-1$ 3's. As a result, $p_1(w)=i$ and

$$\mathrm{coinv}(w)=\mathrm{coinv}(w_1)+\mathrm{coinv}(w_2)+(L-k)(i+k-1)+k(j+k-L-1).$$

Notice that the smallest (resp. largest) possible value of i is 1 (resp. $N-L+1$), namely when $j=N+2-i-k$ and w_2 contains no 2's (resp. w_1 contains no 2's). Let the left hand sides of (32) and (33) be denoted $f(i)$. By definition of $W_k^{>}$, we have $j>i$. Therefore,

$$f(i)=\sum_{j=(i+1)\vee(N-i-k+2)}^{N-k+1}\sum_{w_1,w_2} q^{\mathrm{coinv}(w_1)+\mathrm{coinv}(w_2)+(L-k)(i+k-1)+k(j+k-L-1)},$$

where $a\vee b$ denotes the maximum of a and b, and the sums over words w_1 and w_2 are performed using Proposition 3.15 to give

$$f(i)=\sum_{j=(i+1)\vee(N-i-k+2)}^{N-k+1}\begin{bmatrix} N-i-k\\ L-k-1\end{bmatrix}\begin{bmatrix} N-j\\ k-1\end{bmatrix} q^{(L-k)(i+k-1)+k(j+k-L-1)}.$$

Now, the j-sum can be performed using Proposition 3.19. However, the result depends on the lower limit. If $1\leqslant i\leqslant \lfloor (N-k)/2\rfloor$, then the lower limit is $N-i-k+2$, and otherwise, it is $i-k+1$. These two different cases prove (32) and (33).

Proof of Lemma 9.4(1). Suppose $w\in W_k^{>}$ with $p_3(w)=j$, $p_1'(w)=i+b_3$ and $j<N+1-i-b_3$. Then w can be written as $w=w_1 3 w_2 1 w_3$, where $w_1\in\{1,2\}^{j-1}$, $w_2\in\{1,2,3\}^{N-i-b_3-j}$ and $w_3\in\{2,3\}^{i+b_3-1}$ with a total of $L-k-1$ 1's in w_1 and w_2 and a total of $k-1$ 3's in w_2 and w_3. Suppose a_1 is the number of 1's in w_1 and b_3 is the number of 3's in w_3, so $p_1(w)=i$. To get an upper bound on the number of 2's in w_3 (which is equal to $i-1$), notice that $a_1\leqslant L-k-1$, and there

are $N-L+a_1-i-j+2$ 2's in w_2, so in particular, $(N-L+a_1+1)-(i+j-1) \geqslant 0$. By definition of $W_k^{>}$, we have $i<j$ and therefore

$$2i \leqslant i+j-1 \leqslant N-L+a_1+1 \leqslant N-k,$$

from which the condition follows.

After this split, the number of coinversions of w is

$$\begin{aligned}\mathrm{coinv}(w) &= \mathrm{coinv}(w_1)+\mathrm{coinv}(w_2)+\mathrm{coinv}(w_3)+(L-k-a_1)(i+b_3-1)\\ &\quad + k(j-1)+a_1(N-j-L+1+a_1)+b_3(N-L-i-j+a_1+2).\end{aligned}$$

Let the left hand side of (31) be denoted $f(i)$. We now follow the strategy of the proof of Lemma 9.4(2). The sums over all words w_1,w_2,w_3 will give us appropriate q-multinomial coefficients by Proposition 3.15. Then we will obtain

$$\begin{aligned}f(i) = \sum_{b_3=0}^{(k-1)\wedge(N-i)} \sum_{j=i+1}^{N-i-b_3} \sum_{a_1=0}^{(j-1)\wedge(L-k-1)} &\begin{bmatrix} j-1 \\ a_1 \end{bmatrix} \begin{bmatrix} i+b_3-1 \\ b_3 \end{bmatrix} \\ &\times \begin{bmatrix} N-i-j-b_3 \\ L-k-a_1-1, N-i-j-L+a_1+2, k-1-b_3 \end{bmatrix} \\ &\quad \times q^{(L-k-a_1)(i+b_3-1)+k(j-1)+a_1(N-j-L+1+a_1)+b_3(N-L-i-j+a_1+2)},\end{aligned}$$

where $a \wedge b$ means the minimum of a and b. Let us clarify the limits of the sums first. The b_3-sum is bounded above by the minimum of the available number of 3's and the fact that w_2 is of nonnegative length. Now, j is at least $i+1$ by definition of the set $W_k^{>}$ and is at most $N-i-b_3$ again because of w_2. Then, a_1 is at most the minimum of the available number of 1's and the length of w_1.

It is common parlance to say that the (upper and lower) limits of a hypergeometric sum are *natural* if the terms vanish outside the range. We can see that the lower limits of the b_3 and a_1 sums are natural. Since both upper limits of the b_3-sum as well as the a_1-sum are natural, the answer is independent of which of them is larger. Therefore, we will only write one of them.

The a_1-sum can now be performed using Theorem 3.17 and we end up with

$$\begin{aligned}f(i) = \sum_{b_3=0}^{k-1} \sum_{j=i+1}^{N-i-b_3} q^{(L-k)(i+b_3-1)+k(j-1)+b_3(N-L-i-j+2)} &\begin{bmatrix} i+b_3-1 \\ b_3 \end{bmatrix} \\ &\times \begin{bmatrix} N-i-b_3-j \\ k-1-b_3 \end{bmatrix} \begin{bmatrix} N-i-k \\ L-k-1 \end{bmatrix}.\end{aligned}$$

We now find that the j-sum can be performed using Proposition 3.19. The result is then

$$f(i) = \sum_{b_3=0}^{k-1} q^{(i-1)L+b_3(N-k-2i+1)+k} \begin{bmatrix} i+b_3-1 \\ i-1 \end{bmatrix} \begin{bmatrix} N-2i-b_3 \\ N-k-2i \end{bmatrix} \begin{bmatrix} N-i-k \\ L-k-1 \end{bmatrix}.$$

Now, notice that if the upper limit of the b_3-sum were k, we could have applied Corollary 3.18 after replacing b_3 by b_3+i-1. We thus add and subtract the term for $b_3=k$, perform the b_3-sum and simplify to obtain the result.

9.2. Proof of Lemma 9.5. We will now prove the cases of Lemma 9.5 separately. We again begin with the second case first, since it is conceptually simpler.

Proof of Lemma 9.5(2). Suppose $w \in W_k^{\leqslant}$ with $p_3(w)=j$, $p_1'(w)=i+k$ and $j > N+1-i-k$. Then w can be written as $w = w_1 1 2^{i+j+k-N-2} 3 w_2$, where $w_1 \in \{1,2\}^{N-i-k}$

with $L-k-1$ 1's and $w_2 \in \{2,3\}^{N-j}$ with $k-1$ 3's. As a result, $p_1(w)=i$ and

$$\operatorname{coinv}(w)=\operatorname{coinv}(w_1)+\operatorname{coinv}(w_2)+(L-k)(i+k-1)+k(j+k-L-1).$$

Notice that the smallest possible value of j is $L-k+1$, namely when $i=N+2-j-k$ and w_1 contains no 2's. To see the largest possible value of j, note that $j \leqslant i$ by definition of $W_k^{\leqslant}$ and the maximum value of i is $N-L+1$ when w_1 contains no 2's. Let the left hand sides of (35) and (36) be denoted $f(j)$. Therefore,

$$f(j)=\sum_{i=j\vee(N-j-k+2)}^{N-L+1}\sum_{w_1,w_2} q^{\operatorname{coinv}(w_1)+\operatorname{coinv}(w_2)+(L-k)(i-1)+k(j-1)},$$

and the sums over words w_1 and w_2 are performed using Proposition 3.15 to give

$$f(j)=\sum_{i=j\vee(N-j-k+2)}^{N-L+1}\begin{bmatrix}N-i-k\\L-k-1\end{bmatrix}\begin{bmatrix}N-j\\k-1\end{bmatrix}q^{(L-k)(i-1)+k(j-1)}.$$

Now, the i-sum can be performed using Proposition 3.19. However, the result depends on the lower limit. If $L-k+1 \leqslant j \leqslant \lfloor (N-k+1)/2 \rfloor$, then the lower limit is $N-j-k+2$, and otherwise, it is j. These two different cases prove (35) and (36).

Proof of Lemma 9.5(1). Suppose $w \in W_k^{\leqslant}$ with $p_3(w)=j$, $p_1'(w)=i+b_3$ and $j < N+1-i-b_3$. Then w can be written as $w=w_1 3 w_2 1 w_3$, where $w_1 \in \{1,2\}^{j-1}$, $w_2 \in \{1,2,3\}^{N-i-b_3-j}$ and $w_3 \in \{2,3\}^{i+b_3-1}$ with a total of $L-k-1$ 1's in w_1 and w_2, and a total of $k-1$ 3's in w_2 and w_3. Suppose a_1 is the number of 1's in w_1 and b_3 is the number of 3's in w_3, so $p_1(w)=i$. To get an upper bound on j, which is the position of the first 3, notice that $a_1 \leqslant L-k-1$ and there are $N-L+a_1-i-j+2 \geqslant 0$ 2's in w_2. By definition of $W_k^{\leqslant}$, we have $j \leqslant i$, and therefore

$$2j \leqslant i+j \leqslant N-L+a_1+2 \leqslant N-k+1,$$

from which the condition follows.

After this split, the number of coinversions of w is

$$\begin{aligned}\operatorname{coinv}(w)=\operatorname{coinv}(w_1)+\operatorname{coinv}(w_2)+\operatorname{coinv}(w_3)+(L-k-a_1)(i+b_3-1)\\+k(j-1)+a_1(N-j-L+1+a_1)+b_3(N-L-i-j+a_1+2).\end{aligned}$$

Let the left hand side of (31) be denoted $f(j)$. We now follow the strategy of the proof of Lemma 9.5(2). The sums over all words w_1,w_2,w_3 will give us appropriate q-multinomial coefficients by Proposition 3.15. Then we will obtain

$$\begin{aligned}f(j)=\sum_{i=j}^{N-k-j+1}\sum_{b_3=0}^{(k-1)\wedge(N-i-j)}\sum_{a_1=0}^{(j-1)\wedge(L-k-1)}\begin{bmatrix}j-1\\a_1\end{bmatrix}\begin{bmatrix}i+b_3-1\\b_3\end{bmatrix}\\ \times\begin{bmatrix}N-i-j-b_3\\L-k-a_1-1,N-i-j-L+a_1+2,k-1-b_3\end{bmatrix}\\ \times q^{(L-k-a_1)(i+b_3-1)+k(j-1)+a_1(N-j-L+1+a_1)+b_3(N-L-i-j+a_1+2)}.\end{aligned}$$

Again, we need to clarify the limits. The lower limit of i is j by definition of $W_k^{\leqslant}$ and the upper limit of i comes from the fact that the number of 3's in w_2 is upper bounded by $N-i-j-b_3$. The upper limits of a_1 and b_3 are explained by the same reasoning as the similar sum in the proof of Lemma 9.4(1).

The a_1-sum can now be performed using Theorem 3.17 and we end up with

$$f(j) = \sum_{i=j}^{N-k-j+1} \sum_{b_3=0}^{k-1} q^{(L-k)(i+b_3-1)+k(j-1)+b_3(N-L-i-j+2)} \begin{bmatrix} i+b_3-1 \\ b_3 \end{bmatrix} \times \begin{bmatrix} N-i-b_3-j \\ k-1-b_3 \end{bmatrix} \begin{bmatrix} N-i-k \\ L-k-1 \end{bmatrix}.$$

We now find that the b_3-sum can be performed using Corollary 3.18 after replacing b_3 by b_3+i-1. The result is then

$$f(j) = \sum_{i=j}^{N-k-j+1} q^{(i-1)(L-k)+k(j-1)} \begin{bmatrix} N-j \\ k-1 \end{bmatrix} \begin{bmatrix} N-i-k \\ L-k-1 \end{bmatrix},$$

We now rewrite the limits of the i-sum as $\sum_{i=j}^{N-L+1} - \sum_{i=N-k-j+2}^{N-L+1}$, use Proposition 3.19 and simplify to complete the proof.

10. Conclusion and further questions

We are finally equipped to prove the main result of this paper, Theorem 2.6, stating that $\widetilde{H}_\lambda(X;q,t) = C_\lambda(X;q,t)$ where

$$C_\lambda(X;q,t) = \sum_{\sigma\in\mathrm{Tab}(\lambda)} x^\sigma t^{\mathrm{quinv}(\sigma)} q^{\mathrm{maj}(\sigma)}.$$

Proof of Theorem 2.6. By Theorem 4.1, $C_\lambda(X;q,t)$ is symmetric in the variables x_i. Axioms (5), (6), and (7) uniquely characterize and define the modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$. Axiom (7) is equivalent to the condition that the coefficient of $m_{(n)}$ in $\widetilde{H}_\lambda(X;q,t)$ is 1, since $s_{(n)} = h_{(n)}$ and $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu}$. This is immediate for $C_\lambda(X;q,t)$, as a filling σ such that $x^\sigma = x_j^n$ consists of all equal entries j and has $\mathrm{maj}(\sigma) = \mathrm{quinv}(\sigma) = 0$. Axioms (5) and (6) are written equivalently as (8) and (9), respectively. Those can in turn be rewritten in terms of $\widetilde{C}_\lambda(X;q,t)$, the superization of $C_\lambda(X;q,t)$, as (12) and (13), respectively. (12) is true by Theorem 5.3. (13) is true by Theorem 6.3. This completes the proof.

Our work leads to several natural questions, which we mention below.

Question 10.1. *From* (1) *and* (2)*, it follows that the modified Macdonald polynomial $\widetilde{H}_\lambda(X;q,t)$ can be expressed as a sum over tableaux either of queue inversion weights or of HHL weights. In these weights, the terms for the content and the major index are the same. The only difference is in the notion of a "triple", as discussed at the end of Section* 2.1. *Therefore it is natural to ask for a bijective proof of the equality of these sums.*

We conjecture something stronger: there is a bijection from $\mathrm{Tab}(\lambda,n)$ to itself that preserves the row content of the fillings and the major index, and sends the quinv statistic to the HHL inv statistic. To formally state this, we define row-equivalency classes for the fillings.

Definition 10.2. *Let σ,τ be fillings of* $\mathrm{dg}(\lambda)$*. We say σ and τ are* row-equivalent *if for every row of σ, its entries are a permutation of the entries of the corresponding row in τ. We write $\sigma\sim\tau$ when σ and τ are row-equivalent, and we denote by $[\sigma]$ the class of row-equivalent fillings that σ belongs to.*

The following conjecture is sufficient to prove Theorem 2.6 via [15, Theorem 2.2], since it would give a weight-preserving bijection from queue inversion weights to HHL weights.

Conjecture 10.3. *Given λ and n, there exists a bijection $\delta : \mathrm{Tab}(\lambda, n) \to \mathrm{Tab}(\lambda, n)$ such that for all σ,*

i. $\delta(\sigma) \sim \sigma$,
ii. $\mathrm{maj}(\delta(\sigma)) = \mathrm{maj}(\sigma)$,
iii. $\mathrm{inv}(\delta(\sigma)) = \mathrm{quinv}(\sigma)$.

Conjecture 10.3 implies, for a row-equivalency class $[T]$, that

$$\sum_{\sigma \in [T]} t^{\mathrm{quinv}(\sigma)} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma \in [T]} t^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)}.$$

Example 10.4. Consider the row-equivalency class $[T] = (\{1\}, \{2, 2, 3\}, \{1, 1, 2\})$ for $\lambda = (3, 2, 2)$. In Figure 6, we show the tableaux $F \in [T]$ with their queue inversion weights $t^{\mathrm{quinv}(F)} q^{\mathrm{maj}(F)}$ above and HHL weights $t^{\mathrm{inv}(F)} q^{\mathrm{maj}(F)}$ below. Observe that the sums of the weights are equal. Moreover, there is a bijection δ such that the queue inversion weight of F matches the HHL weight of $\delta(F)$.

	$\begin{smallmatrix}1&&\\2&2&3\\1&1&2\end{smallmatrix}$	$\begin{smallmatrix}1&&\\2&2&3\\1&2&1\end{smallmatrix}$	$\begin{smallmatrix}1&&\\2&2&3\\2&1&1\end{smallmatrix}$	$\begin{smallmatrix}1&&\\2&3&2\\1&1&2\end{smallmatrix}$	$\begin{smallmatrix}1&&\\2&3&2\\1&2&1\end{smallmatrix}$	$\begin{smallmatrix}1&&\\2&3&2\\2&1&1\end{smallmatrix}$	$\begin{smallmatrix}1&&\\3&2&2\\1&1&2\end{smallmatrix}$	$\begin{smallmatrix}1&&\\3&2&2\\1&2&1\end{smallmatrix}$	$\begin{smallmatrix}1&&\\3&2&2\\2&1&1\end{smallmatrix}$
quinv	t^2q^4	t^3q^3	t^4q^2	t^2q^3	tq^4	t^3q^2	tq^3	t^2q^3	q^4
HHL	q^4	t^2q^3	t^3q^2	tq^3	tq^4	t^4q^2	t^2q^3	t^3q^3	t^2q^4

Figure 6. For $\lambda = (3, 2, 2)$ and $n = 3$, all fillings of type (λ, n) in the row-equivalency class $[T] = (\{1\}, \{2, 2, 3\}, \{1, 1, 2\})$. Below are the corresponding queue inversion weights and HHL weights.

Question 10.5. *In Section 9, we demonstrated in Theorem 8.4 the existence of a quinv-preserving bijection between sets of words respecting certain conditions. We have not been able to find such a bijection and we think it would be interesting to find one.*

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Note added in proof: Recently, Loehr has announced a bijective proof of Theorems 9.1 and 9.3 in [21].

References

[1] Arvind Ayyer, Olya Mandelshtam, and James B. Martin, *Modified Macdonald polynomials and the multispecies zero range process: II*, (2022), `https://arxiv.org/abs/2209.09859`.

[2] Luigi Cantini, *Asymmetric simple exclusion process with open boundaries and Koornwinder polynomials*, Annales Henri Poincaré, 4, vol. 18, Springer, 2017, pp. 1121–1151.

[3] Luigi Cantini, Jan de Gier, and Michael Wheeler, *Matrix product formula for Macdonald polynomials*, J. Phys. A **48** (2015), no. 38, article no. 384001 (25 pages).

[4] Luigi Cantini, Alexandr Garbali, Jan de Gier, and Michael Wheeler, *Koornwinder polynomials and the stationary multi-species asymmetric exclusion process with open boundaries*, J. Phys. A **49** (2016), no. 44, article no. 444002 (23 pages).

[5] Sylvie Corteel, James Haglund, Olya Mandelshtam, Sarah Mason, and Lauren K. Williams, *Compact formulas for Macdonald polynomials and quasisymmetric Macdonald polynomials*, Sel. Math. **28** (2022), article no. 32 (33 pages).

[6] Sylvie Corteel, Olya Mandelshtam, and Lauren K. Williams, *Combinatorics of the two-species ASEP and Koornwinder moments*, Adv. Math. **321** (2017), 160–204.

[7] ———, *From multiline queues to Macdonald polynomials via the exclusion process*, Amer. J. Math. **144** (2019), 395–436.

[8] Sylvie Corteel and Lauren K. Williams, *Tableaux combinatorics for the asymmetric exclusion process and Askey-Wilson polynomials*, Duke Math. J. **159** (2011), no. 3, 385–415.

[9] ———, *Macdonald-Koornwinder moments and the two-species exclusion process*, Selecta Math. (N.S.) **24** (2018), no. 3, 2275–2317.

[10] Pablo A. Ferrari and James B. Martin, *Stationary distributions of multi-type totally asymmetric exclusion processes*, Ann. Probab. **35** (2007), 807–832.

[11] Alexandr Garbali and Michael Wheeler, *Modified Macdonald polynomials and integrability*, Comm. Math. Phys. **374** (2020), no. 3, 1809–1876.

[12] Adriano M. Garsia and Mark Haiman, *A graded representation model for Macdonald's polynomials.*, Proceedings of the National Academy of Sciences **90** (1993), no. 8, 3607–3610.

[13] George Gasper and Mizan Rahman, *Basic hypergeometric series*, vol. 96, Cambridge University Press, 2004.

[14] Albert Girard, *Invention nouvelle en l'algèbre*, Amsterdam, 1629.

[15] James Haglund, Mark Haiman, and Nick Loehr, *A combinatorial formula for Macdonald polynomials*, J. Amer. Math. Soc **18** (2004), 735–761.

[16] James Haglund, Mark Haiman, Nick Loehr, Jeff B. Remmel, and Alexander Ulyanov, *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. **126** (2005), no. 2, 195–232.

[17] Mark Haiman, *Macdonald polynomials and geometry*, in New perspectives in algebraic combinatorics, Math. Sci. Res. Inst. Publ., vol. 38, Cambridge University Press, 1999, pp. 207–254.

[18] ———, *Notes on Macdonald polynomials and the geometry of Hilbert schemes*, in Symmetric functions 2001: surveys of developments and perspectives, NATO Sci. Ser. II Math. Phys. Chem., vol. 74, Kluwer Acad. Publ., Dordrecht, 2002, pp. 1–64.

[19] Atsuo Kuniba, Masato Okado, and Satoshi Watanabe, *Integrable structure of multispecies zero range process*, SIGMA Symmetry Integrability Geom. Methods Appl. **13** (2017), article no. 044 (29 pages).

[20] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon, *Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties*, J. Math. Phys. **38** (1997), no. 2, 1041–1068.

[21] Nicholas A. Loehr, *Bijective proofs of some coinversion identities related to Macdonald polynomials*, 2022, `https://arxiv.org/abs/2210.11286`.

[22] Nick Loehr and Elizabeth Niese, *A bijective proof of a factorization formula for specialized Macdonald polynomials*, Ann. Comb. **16** (2012), no. 4, 815–828.

[23] Ian Macdonald, *A new class of symmetric functions*, Sém. Lothar. Combin **20** (1988).

[24] ———, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications.

[25] Bethany Marsh and Konstanze Rietsch, *Parametrizations of flag varieties*, Represent. Theory **8** (2004), 212–242 (electronic).

[26] James B. Martin, *Stationary distributions of the multi-type ASEP*, Electron. J. Probab. **25** (2020), 41 pp.

[27] Tomohiro Sasamoto, *One-dimensional partially asymmetric simple exclusion process with open boundaries: orthogonal polynomials approach*, J. Phys. A **32** (1999), no. 41, 7109–7131.

[28] Frank Spitzer, *Interaction of Markov processes*, Adv. Math. **5** (1970), no. 2, 246 – 290.

[29] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.

[30] ———, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, 2012.

[31] Yoshihiro Takeyama, *Algebraic construction of multi-species q-boson system*, Arxiv preprint `https://arxiv.org/abs/1507.02033`, 2015.

[32] Masaru Uchiyama, Tomohiro Sasamoto, and Miki Wadati, *Asymmetric simple exclusion process with open boundaries and Askey-Wilson polynomials*, J. Phys. A **37** (2004), no. 18, 4985–5002.

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