ALGEBRAIC COMBINATORICS

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Volume 6, issue 2 (2023), p. 457-469.
https://doi.org/10.5802/alco.265

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Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org   www.centre-mersenne.org
e-ISSN: 2589-5486

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Abstract Let $f(z) = \sum_{n=0}^{\infty}(-1)^n z^n/n!n!$. In their 1975 paper, Carlitz, Scoville and Vaughan provided a combinatorial interpretation of the coefficients in the power series $1/f(z) = \sum_{n=0}^{\infty}\omega_n z^n/n!n!$. They proved that ω_n counts the number of pairs of permutations of the nth symmetric group $\mathcal{S}_n$ with no common ascent. This paper gives a combinatorial interpretation of a natural q-analogue of ω_n by studying the top homology of the Segre product of the subspace lattice $B_n(q)$ with itself. We also derive an equation that is analogous to a well-known symmetric function identity: $\sum_{i=0}^{n}(-1)^i e_i h_{n-i} = 0$, which then generalizes our q-analogue to a symmetric group representation result.

1. Introduction

Consider the power series $f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^n}{n!n!}$ and define the numbers $\omega_0, \omega_1, \omega_2, \dots$ by $\frac{1}{f(z)} = \sum_{n=0}^{\infty}\omega_n \frac{z^n}{n!n!}$. It follows quickly from the definition that for $n \geqslant 1$,

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}^2 \omega_k = 0. \tag{1}$$

Given $\sigma \in \mathcal{S}_n$, a permutation of $[n] \coloneqq \{1, 2, \dots, n\}$, we call $i \in [n-1]$ an *ascent* of σ if $\sigma(i) < \sigma(i+1)$. Carlitz, Scoville and Vaughan proved the following result:

Theorem 1.1. (Carlitz, Scoville, and Vaughan [5]) *The number ω_k in equation* (1) *is the number of pairs of permutations of $\mathcal{S}_k$ with no common ascent.*

Two permutations have no common ascent if they do not rise at the same position when written in one-line notation. For example, in one-line notation $(12, 21)$, $(21, 12)$, $(21, 21)$ are all the pairs of permutations of $\{1, 2\}$ with no common ascent, so we have $\omega_2 = 3$. Since the Bessel function $J_0(z)$ is essentially $f(z^2)$, Carlitz, Scoville and Vaughan's result provided a combinatorial interpretation of the coefficient ω_k in the reciprocal Bessel function.

In this paper, we will develop a q-analogue of Theorem 1.1. To that purpose, recall that $[n]_q \coloneqq q^{n-1} + q^{n-2} + \cdots + 1$ is the q-analogue of the natural number n

Manuscript received 18th December 2018, revised 7th September 2020 and 31st August 2022, accepted 31st August 2022.

Keywords. algebraic combinatorics, poset homology, shellability, symmetric functions, symmetric group representation.

Acknowledgements. The author would like to thank John Shareshian for his insightful comments and valuable advice.

and that $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$ is the q-analogue of the binomial coefficient $\binom{n}{k}$, where $[n]_q! := \prod_{i=1}^{n} [i]_q$. For a permutation $\sigma \in \mathcal{S}_n$, the *inversion statistic* is defined by

$$\mathrm{inv}(\sigma) :=\mid \{(i,j) : 1 \leqslant i < j \leqslant n \text{ and } \sigma(i) > \sigma(j)\} \mid .$$

Theorem 1.2. *Let $\mathcal{D}_n$ denote the set $\{(\sigma,\tau) \in \mathcal{S}_n \times \mathcal{S}_n \mid \sigma, \tau$ have no common ascent$\}$, and let $W_n(q) = \sum_{(\sigma,\tau)\in\mathcal{D}_n} q^{\mathrm{inv}(\sigma)+\mathrm{inv}(\tau)}$. Then for $n \geqslant 1$,*

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q^2 (-1)^i W_i(q) = 0. \tag{2}$$

Put $F(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{[n]_q![n]_q!}$. The function $F\left((\frac{z}{2(1-q)})^2 \right)$ is the q-Bessel function $J_0^{(1)}(z;q)$. The q-Bessel functions were first introduced by F. H. Jackson in 1905 and can be found in later literature (see Gasper and Rahman [6]). It follows from equation (2) that $\frac{1}{F(z)} = \sum_{n=0}^{\infty} W_n(q) \frac{z^n}{[n]_q![n]_q!}$, giving the coefficients of the reciprocal q-Bessel function a combinatorial meaning.

In Section 2, we will prove Theorem 1.2 by studying the top homology of the Segre product of the subspace lattice $B_n(q)$ with itself. From a poset homology perspective, the coefficient $W_n(q)$ is a signless Euler characteristic and counts the number of decreasing maximal chains of this Segre product poset. All definitions will be reviewed in this section.

In Section 3, we define the product Frobenius characteristic map to serve as a useful tool in studying representations of the product group $\mathcal{S}_n \times \mathcal{S}_n$. We then further generalize our q-analogue to a symmetric group representation result in Section 4 (see Theorem 4.1) using the Whitney homology technique. This generalization is an analogue of the well-known symmetric function identity: $\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0$.

Finally, in Section 5 we point out that an alternative proof of Theorem 1.1 can be obtained by specializing our proof of Theorem 1.2 at $q = 1$.

2. The q-analogue of a result of Carlitz, Scoville, and Vaughan

We recall the definition of $B_n(q)$, which is a q-analogue of the subset lattice B_n. Let q be a prime power and $\mathbb{F}_q$ the finite field of q elements. Consider the n-dimensional linear vector space $\mathbb{F}_q^n$ and its subspaces. Then $B_n(q)$ is the lattice of those subspaces ordered by inclusion. The poset $B_n(q)$ is a geometric lattice, so every element is a join of atoms ([10, Example 3.10.2]). The poset $B_n(q)$ is graded with a rank function $\rho(W) :=$ the dimension of the subspace W, where a poset is said to be graded if it is pure and bounded.

An *edge labeling* of a bounded poset P is a map $\lambda : \mathcal{E}(P) \to \Lambda$, where $\mathcal{E}(P)$ is the set of covering relations $x \lessdot y$ of P and Λ is some poset. If P is a poset with an edge labeling λ, then a maximal chain $c = (\hat{0} \lessdot x_1 \lessdot \cdots \lessdot x_t \lessdot \hat{1})$ of P is *increasing* if $\lambda(\hat{0}, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_t, \hat{1})$. We call the chain c *decreasing* if there is no $i \in \{1, 2, \ldots, t\}$ such that $\lambda(x_{i-1}, x_i) < \lambda(x_i, x_{i+1})$ in Λ. For a chain c, we associate a word

$$\lambda(c) = \lambda(\hat{0}, x_1)\lambda(x_1, x_2) \cdots \lambda(x_t, \hat{1}).$$

If $\lambda(c_1)$ lexicographically precedes $\lambda(c_2)$, we say that c_1 lexicographically precedes c_2 and we denote this by $c_1 <_L c_2$.

Definition 2.1 (Björner and Wachs [2, Definition 2.1]). *An edge labeling is called an* EL-labeling (edge lexicographical labeling) *if for every interval $[x, y]$ in P,*

1. *there is a unique increasing maximal chain c in $[x, y]$, and*
2. *$c <_L c'$ for all other maximal chains c' in $[x, y]$.*

A bounded poset that admits an EL-labeling is said to be *EL-shellable*. We only need to consider pure shellability in this paper since both $B_n(q)$ and the Segre product of $B_n(q)$ with itself (see Definition 2.4) are pure and bounded. It is well known that $B_n(q)$ is EL-shellable (see [14, Exercise 3.4.7]) and a general edge-labeling for semimodular lattices is given in [10]. Here we define a specific EL-labeling of $B_n(q)$, which will be used to prove our results. Let A be the set of all atoms of $B_n(q)$. For a subspace of $\mathbb{F}_q^n$, $X \in B_n(q)$, we define $A(X) \coloneqq \{V \in A \mid V \leqslant X\}$. The following two steps define an edge-labeling on the graded poset $B_n(q)$.

1. For a 1-dimensional subspace V of $\mathbb{F}_q^n$ (an atom of $B_n(q)$), let v be a basis element of V. We define a map $f\colon A \to [n]$, $f(V) =$ the index of the right-most non-zero coordinate of v. For example, in $B_3(3)$, if $V_1 = \operatorname{span}\{\langle 1,0,1\rangle\}$ and $V_2 = \operatorname{span}\{\langle 2,1,0\rangle\}$, $f(V_1) = 3$ and $f(V_2) = 2$.

2. In the case of $B_3(3)$, if $X = \operatorname{span}\{\langle 1,0,1\rangle, \langle 2,1,0\rangle\}$, then $A(X)$ also contains $\operatorname{span}\{\langle 0,1,1\rangle\}$ and $\operatorname{span}\{\langle 2,2,1\rangle\}$. But any vector whose right-most non-zero coordinate is the first coordinate will not be in X. So $f(A(X)) = \{2,3\}$ and $\mid f(A(X)) \mid = 2$. For a k-dimensional subspace X of $\mathbb{F}_q^n$, Gaussian elimination implies the existence of a basis of X whose elements have distinct right-most non-zero coordinates and this in turn implies that $f(A(X))$ has $\dim(X)$ elements. Let Y be an element of $B_n(q)$ that covers X, then $\dim(Y) = \dim(X) + 1$. The set $f(A(Y))\backslash f(A(X))$ is a subset of $[n]$ and has exactly one element. This element will be the label of the edge (X, Y).

Proposition 2.2. *The edge labeling described above is an EL-labeling on the subspace lattice $B_n(q)$.*

Proof. Edges in the same chain cannot take duplicate labels since $\mathbb{F}_q^n$ is n-dimensional and any maximal chain must take all labels in $\{1, 2, \dots, n\}$. Let $[X, Y]$ be a closed interval in $B_n(q)$. All maximal chains of $[X, Y]$ will take labels from the set $f(A(Y))\backslash f(A(X))$. Let $a_1 < a_2 < \cdots < a_l$ be all the elements of $f(A(Y))\backslash f(A(X))$ arranged in increasing order. Given $a_i \in f(A(Y))\backslash f(A(X))$, there exists an atom $V_i \in A(Y)$ with $f(V_i) = a_i$. Note that V_i is a 1-dimensional subspace of $\mathbb{F}_q^n$ and the join of V_i and X is in $[X, Y]$. We build a chain according to the increasing order of a_i's, each time adjoining one 1-dimensional subspace. Then the chain $c = (X \lessdot X \vee V_1 \lessdot \cdots \lessdot X \vee V_1 \vee V_2 \vee \cdots \vee V_l = Y)$ is an increasing maximal chain of $[X, Y]$.

For the uniqueness of the increasing maximal chain, it suffices to show the uniqueness of the selection of $X \vee V_1$ since X and Y are arbitrary. Take $V_1' \in A(Y)$ with $f(V_1') = a_1$. We can find a basis vector v_1 of V_1 and a basis vector v_1' of V_1' so that the a_1th coordinate of both vectors are 1. Then $v_1 - v_1' \in Y$ and $f(\operatorname{span}\{v_1 - v_1'\}) < a_1$. Since $a_1 < a_2 < \cdots < a_l$ are all the elements of $f(A(Y))\backslash f(A(X))$, $v_1 - v_1'$ must be a vector in X. Then $X \vee V_1 = X \vee V_1'$. Therefore $X \vee V_1$ is unique. At each step of building the increasing maximal chain, there is a unique subspace that gives the connecting edge the smallest label.

Labels of all other maximal chains of $[X, Y]$ are permutations of elements in $f(A(Y))\backslash f(A(X))$, which are all lexicographically larger than the label of the unique increasing chain $c = (X \lessdot X \vee V_1 \lessdot \cdots \lessdot X \vee V_1 \vee V_2 \vee \cdots \vee V_l = Y)$. Condition (2) of Definition 2.1 is also satisfied. ☐

Under this EL-labeling, to each maximal chain of the subspace lattice $B_n(q)$, one can assign a permutation σ of $\mathcal{S}_n$. See Section 1 for the definition of the inversion statistic $\operatorname{inv}(\sigma)$.

Lemma 2.3. *The number of maximal chains of $B_n(q)$ assigned label $\sigma \in \mathcal{S}_n$ is $q^{\operatorname{inv}(\sigma)}$.*

Proof. For each 1-dimensional subspace of $\mathbb{F}_q^n$, we can pick a basis vector that has 1 on its right-most non-zero coordinate. Given $\sigma \in \mathcal{S}_n$, for each $i \in [n-1]$, let $\mathrm{inv}_{\sigma(i)}$ denote the number of j such that $1 \leqslant i < j \leqslant n$ and $\sigma(j) < \sigma(i)$. The number of ways to choose an atom W_1 such that the edge $(0, W_1)$ takes label $\sigma(1)$ is clearly $q^{\sigma(1)-1} = q^{\mathrm{inv}_{\sigma(1)}}$.

Let $k \in [n]$, assume the chain $0 \lessdot W_1 \lessdot \ldots \lessdot W_{k-1}$ has label $\sigma(1)\sigma(2)\ldots\sigma(k-1)$. We need to choose a $W_k = W_{k-1} \vee V_k$ so that the edge (W_{k-1}, W_k) takes the label $\sigma(k)$ and V_k is an atom. Pick a basis vector for V_k, call it v_k, that has 1 on the $\sigma(k)$th coordinate and all 0's after the $\sigma(k)$th coordinate. For all j such that $1 \leqslant k < j \leqslant n$ and $\sigma(j) < \sigma(k)$, W_{k-1} contains no vector whose right-most non-zero coordinate is the $\sigma(j)$th. Thus, any variation of the values on those $\sigma(j)$th coordinates of v_k will result in a different W_k. Then there are $q^{\mathrm{inv}_{\sigma(k)}}$ ways to choose a W_k. Therefore, the number of maximal chains assigned label σ is $\prod_{i=1}^{i=n} q^{\mathrm{inv}_{\sigma(i)}} = q^{\sum_{i=1}^{i=n} \mathrm{inv}_{\sigma(i)}} = q^{\mathrm{inv}(\sigma)}$. □

Let us review a simplified definition of the Segre product poset. A general definition can be found in [4].

Definition 2.4. *Let P be a graded poset with a rank function ρ. Then the* Segre product poset *of P with itself, denoted by $P \circ P$, is defined to be the induced subposet of the product poset $P \times P$ consisting of the pairs $(x, y) \in P \times P$ such that $\rho(x) = \rho(y)$.*

Now consider the Segre product of $B_n(q)$ with itself. Using the EL-labeling of $B_n(q)$ described right after Definition 2.1, the Segre product poset $B_n(q) \circ B_n(q)$ admits the following edge labeling. Given two elements $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ in $B_n(q) \circ B_n(q)$ satisfying the covering relation $X \lessdot Y$, we must have $X_1 \lessdot Y_1$ and $X_2 \lessdot Y_2$ in $B_n(q)$. In the EL-labeling of $B_n(q)$, suppose the edge connecting X_1 and Y_1 admits a label i and the edge connecting X_2 and Y_2 admits a label j, then the edge connecting X and Y in $B_n(q) \circ B_n(q)$ is labeled by (i, j).

Corollary 2.5. (of Proposition 2.2) *The edge-labeling of $B_n(q) \circ B_n(q)$ defined above is an EL-labeling.*

Proof. Let $[X, Y]$ be any closed interval in $B_n(q) \circ B_n(q)$. The elements $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$, where $X_1 \leqslant Y_1$ and $X_2 \leqslant Y_2$ in $B_n(q)$. So $[X_1, Y_1]$ and $[X_2, Y_2]$ are closed intervals in $B_n(q)$. Since the labeling for $B_n(q)$ is an EL-labeling, there is a unique increasing maximal chain c_1 in $[X_1, Y_1]$ that lexicographically precedes all other chains in the same interval. There is also a unique increasing maximal chain c_2 in $[X_2, Y_2]$. Then the chain in $[X, Y]$ formed by pairing elements of c_1 and c_2 of the same rank must be the unique increasing maximal chain in $[X, Y]$. Any other chain would have non-increasing labels in $[X_1, Y_1]$ or $[X_2, Y_2]$, hence is non-increasing in $[X, Y]$. This unique increasing maximal chain of $[X, Y]$ must also satisfy part (2) of Definition 2.1 because c_1 and c_2 both satisfy this condition. □

The following theorem of Björner and Wachs connects the permutations in $\mathcal{S}_n$ with the maximal chains of the Segre product poset $B_n(q) \circ B_n(q)$. Let $\hat{P}$ be the bounded extension of P. That is, $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ and $\hat{0}$ and $\hat{1}$ are attached even if P already has a bottom or a top element.

Theorem 2.6. (Björner and Wachs [3, Theorem 4.1], see also Wachs [14, Theorem 3.2.4]). *Suppose P is a poset for which $\hat{P}$ admits an EL-labeling. Then the order complex of P has the homotopy type of a wedge of spheres, where the number of i-spheres is the number of decreasing maximal $(i+2)$-chains of $\hat{P}$. The decreasing maximal $(i+2)$-chains, with $\hat{0}$ and $\hat{1}$ removed, form a basis for the cohomology $\widetilde{H}^i(P; \mathbb{Z})$.*

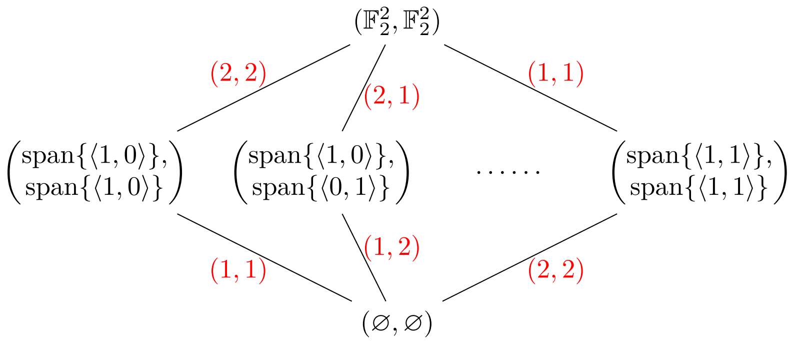

FIGURE 1. An EL-labeling of $B_2(2) \circ B_2(2)$

Since P has the homotopy type of a wedge of spheres, $\widetilde{H}^i(P;\mathbb{Z}) \cong \widetilde{H}_i(P;\mathbb{Z})$ (Wachs [14, Theorem 1.5.1]). We will use $\widetilde{H}_i(P;\mathbb{Z})$, the reduced homology of the order complex $\Delta(P)$, instead of the cohomology group in this paper.

EXAMPLE 2.7. Figure 1 is an EL-labeling of the Segre product poset $B_2(2)\circ B_2(2)$. The left-most chain is increasing with label $(12, 12)$. The decreasing (i.e. non-increasing) chains have labels $(12, 21)$, $(21, 12)$, or $(21, 21)$. We use $P_n(q)$ to denote the proper part of the Segre product poset, i.e. $P_n(q) \coloneqq B_n(q) \circ B_n(q) \smallsetminus \{\hat{0}, \hat{1}\}$. Then the decreasing chains of $B_2(2) \circ B_2(2)$ with the top and bottom elements removed form a basis of $\widetilde{H}^0(P_2(2);\mathbb{Z})$.

PROPOSITION 2.8. *Let $W_n(q) = \sum_{(\sigma,\tau)\in\mathcal{D}_n} q^{(\mathrm{inv}(\sigma)+\mathrm{inv}(\tau))}$, where $\mathcal{D}_n$ denotes the set of pairs of permutations $(\sigma,\tau) \in \mathcal{S}_n \times \mathcal{S}_n$ with no common ascent. Then $W_n(q)$ equals the total number of decreasing maximal chains of $P_n(q) \coloneqq B_n(q)\circ B_n(q)\smallsetminus\{\hat{0},\hat{1}\}$ with respect to the labeling described above.*

Proof. An edge label $(i, j) \in [n] \times [n] \leqslant (k, l)$ if and only if $i \leqslant k$ and $j \leqslant l$. By the definition of our labeling for $B_n(q) \circ B_n(q)$, there cannot be repeat edge labels along any one chain. So a chain label is decreasing as long as the two components of any two consecutive edge labels do not increase at the same time. Each maximal chain labeling of $B_n(q) \circ B_n(q)$ corresponds to a pair of permutations of $\mathcal{S}_n$. Then labels of decreasing maximal chains are all pairs of permutations with no common ascent. Given a pair of permutations (σ, τ), the number of maximal chains assigned label (σ, τ) is $q^{\mathrm{inv}(\sigma)} \cdot q^{\mathrm{inv}(\tau)} = q^{(\mathrm{inv}(\sigma)+\mathrm{inv}(\tau))}$ by Lemma 2.3. Then the total number of decreasing maximal chains of $P_n(q)$ is

$$W_n(q) = \sum_{(\sigma,\tau)\in\mathcal{D}_n} q^{(\mathrm{inv}(\sigma)+\mathrm{inv}(\tau))}.$$

□

REMARK 2.9. The Segre product poset $B_n(q) \circ B_n(q)$ is the q-analogue of the Segre product poset $B_n \circ B_n$, agreeing with the formal definition of a q-analogue in R. Simion's paper [8]. She showed that the q-analogue of an EL-shellable poset is also EL-shellable. The EL-labeling of $B_n(q) \circ B_n(q)$ we use in this paper provides intuition and a combinatorial interpretation for $W_n(q)$.

Björner and Welker proved that if two pure posets are Cohen-Macaulay, then their Segre product is also Cohen-Macaulay (see [4, Theorem 1]). This result in particular proves that the poset $B_n(q)\circ B_n(q)$ is Cohen-Macaulay because $B_n(q)$ is. Corollary 2.5

says that $B_n(q) \circ B_n(q)$ is shellable, which is a stronger property than the Cohen-Macaulayness. Later in Section 4 we will use the fact that the Segre product poset $B_n \circ B_n$ is Cohen-Macaulay.

Proof of Theorem 1.2. The poset $P_n(q) = B_n(q) \circ B_n(q) \smallsetminus \{\hat{0}, \hat{1}\}$ is pure. By Theorem 2.6, $P_n(q)$ has the homotopy type of a wedge of $(n-2)$-spheres, and its decreasing maximal $(n-2)$-chains form a basis of the reduced $(n-2)$-nd cohomology. Since $P_n(q)$ is graded and EL-shellable, all reduced homology groups other than the top one vanish (Björner [1]). In Proposition 2.8, we defined $W_n(q)$ to be the total number of decreasing maximal chains of $P_n(q)$. Then using the Euler-Poincaré formula [14, Theorem 1.2.8] and Philip Hall's theorem (Stanley [10, Proposition 3.8.6]) we get

$$\mu_{\widehat{P_n}(q)}(\hat{0}, \hat{1}) = (-1)^n W_n(q) = \widetilde{\chi}(\Delta(P_n(q))), \tag{3}$$

where $\widehat{P_n}(q) = B_n(q) \circ B_n(q)$ denotes $P_n(q)$ with $\hat{0}$ and $\hat{1}$ adjoined.

On the other hand, by the definition of the Möbius function,

$$\mu(\hat{0}, \hat{1}) = -\sum_{\hat{0} \leqslant x < \hat{1}} \mu(\hat{0}, x).$$

Each x in $P_n(q)$ is the product of two k-dimensional subspaces X_1, X_2 of $\mathbb{F}_q^n$, for some k with $0 \leqslant k < n$. The intervals $[\hat{0}, X_1]$ and $[\hat{0}, X_2]$ are isomorphic to the poset $B_k(q)$, hence $\mu(\hat{0}, x)$ is just $\mu_{\widehat{P_k}(q)}(\hat{0}, \hat{1})$, where $P_k(q) = B_k(q) \circ B_k(q) \smallsetminus \{\hat{0}, \hat{1}\}$. The number of k-dimensional subspaces of $\mathbb{F}_q^n$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q$ (Stanley [10, Proposition 1.7.2]). So the number of distinct $x = (X_1, X_2)$ where X_1 and X_2 are k-dimensional subspaces is $\begin{bmatrix} n \\ k \end{bmatrix}_q^2$. Therefore we have

$$\mu_{\widehat{P_n}(q)}(\hat{0}, \hat{1}) = -\sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix}_q^2 \mu_{\widehat{P_i}(q)}(\hat{0}, \hat{1}) = -\sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix}_q^2 (-1)^i W_i(q).$$

Consequently,

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q^2 (-1)^i W_i(q) = 0.$$

By Proposition 2.8, $W_i(q) = \sum_{(\sigma,\tau) \in \mathcal{D}_i} q^{(\mathrm{inv}(\sigma) + \mathrm{inv}(\tau))}$ is the number of decreasing maximal chains of $P_i(q)$, where, as above, $\mathcal{D}_i$ denotes the set of pairs of permutations $(\sigma, \tau) \in \mathcal{S}_i \times \mathcal{S}_i$ with no common ascent. □

Corollary 2.10. *The Euler characteristic of the Segre product of the subspace lattice $B_n(q) \circ B_n(q)$ is $(-1)^n W_n(q)$.*

Proof. See equation (3) in the proof of Theorem 1.2. □

3. The product Frobenius characteristic map

The Frobenius characteristic map is often used to study representations of the symmetric group. Here we will define a product Frobenius characteristic map to help understand representations of $\mathcal{S}_n \times \mathcal{S}_n$. Therefore, let us consider two sets of variables $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$. Following Sagan's notations [7], R^n denotes the space of class functions on $\mathcal{S}_n$ and $R = \oplus_n R^n$. We will use $R^{m,n}$ to denote the space of class functions on $\mathcal{S}_m \times \mathcal{S}_n$ and let $R_{2d} = \oplus_{m,n} R^{m,n}$. Let Λ^n be the space of homogeneous degree n symmetric functions. Then $\Lambda(x) = \oplus_n \Lambda^n(x)$ and $\Lambda(y) = \oplus_n \Lambda^n(y)$ denote the rings of symmetric functions in variables $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ respectively. Given $\mu \vdash n$ with $\mu = (1^{m_1} 2^{m_2} \ldots)$, we write $z_\mu = \prod_{i=1}^{i=n} i^{m_i} m_i!$.

Let us recall the definition of the usual characteristic map.

Definition 3.1. *The* (Frobenius) characteristic map $ch^n : R^n \to \Lambda^n$ *is defined by*

$$ch^n(\chi) = \sum_{\mu \vdash n} z_\mu^{-1} \chi_\mu p_\mu,$$

where χ_μ is the value of χ on the class μ and p_μ is the power sum symmetric function. Define $ch \coloneqq \oplus_n ch^n$.

Now we define a product characteristic map.

Definition 3.2. *Let χ be a class function on $\mathcal{S}_m \times \mathcal{S}_n$. The* product Frobenius characteristic map $ch : R_{2d} \to \Lambda(x) \otimes \Lambda(y)$ *is defined as:*

$$ch(\chi) = \sum_{(\mu,\lambda)\vdash(m,n)} z_\mu^{-1} z_\lambda^{-1} \chi_{(\mu,\lambda)} p_\mu(x) p_\lambda(y), \tag{4}$$

where $\chi_{(\mu,\lambda)}$ is the value of χ on the class (μ, λ) and p_μ, p_λ are power sum symmetric functions. The class (μ, λ) is indexed by a partition $\mu \vdash m$ and a partition $\lambda \vdash n$ that tell us the cycle types of elements of $\mathcal{S}_m$ and $\mathcal{S}_n$ respectively.

Proposition 3.3. *For a character $f \otimes g$ of $\mathcal{S}_m \times \mathcal{S}_n$, where f is a character of $\mathcal{S}_m$ and g a character of $\mathcal{S}_n$, the product Frobenius characteristic* $\mathrm{ch}(f \otimes g)$ *equals* $\mathrm{ch}(f)(x)\mathrm{ch}(g)(y)$.

Proof. Equation (4) gives us

$$\mathrm{ch}(f \otimes g) = \sum_{(\mu,\lambda)\vdash(m,n)} z_\mu^{-1} z_\lambda^{-1} (f \otimes g)_{(\mu,\lambda)} p_\mu(x) p_\lambda(y).$$

For a conjugacy class $(\mu, \lambda) \vdash (m, n)$, let $\sigma \in \mathcal{S}_m$ have cycle type μ and $\tau \in \mathcal{S}_n$ have cycle type λ. The character value

$$(f \otimes g)_{(\mu,\lambda)} = (f \otimes g)(\sigma, \tau) = f(\sigma)g(\tau) = f_\mu g_\lambda,$$

where the second equality is by [7, Theorem 1.11.2]. Then

$$\begin{aligned}
\mathrm{ch}(f \otimes g) &= \sum_{(\mu,\lambda)\vdash(m,n)} z_\mu^{-1} z_\lambda^{-1} f_\mu g_\lambda p_\mu(x) p_\lambda(y) \\
&= \sum_{\mu\vdash m} z_\mu^{-1} f_\mu p_\mu(x) \sum_{\lambda\vdash n} z_\lambda^{-1} g_\lambda p_\lambda(y) \\
&= \mathrm{ch}(f)(x)\mathrm{ch}(g)(y).
\end{aligned}$$

□

Because the product Frobenius characteristic map is an extension of the usual (Frobenius) characteristic map, we keep the notation ch for product Frobenius characteristic map even though ch was previously defined to be $\oplus_n \mathrm{ch}^n$ in various literature (Sagan [7], Stanley [9]). The meaning of ch will be clear in the given context.

Recall that the *induction product* $f \circ g$ is the induction of $f \otimes g$ from $\mathcal{S}_m \times \mathcal{S}_n$ to $\mathcal{S}_{m+n}$. A fundamental property of the usual characteristic map is the following:

Proposition 3.4. (Stanley [9, Proposition 7.18.2]) *The Frobenius characteristic map* $\mathrm{ch} : R \to \Lambda$ *is a bijective ring homomorphism, i.e.,* ch *is one-to-one and onto, and satisfies*

$$\mathrm{ch}(f \circ g) = \mathrm{ch}(f)\mathrm{ch}(g).$$

Remark 3.5. The product Frobenius characteristic on the tensor product of characters, $\mathrm{ch}(f \otimes g) = \mathrm{ch}(f)(x)\mathrm{ch}(g)(y)$, is a symmetric function in $\Lambda^m(x) \otimes \Lambda^n(y)$, while the usual Frobenius characteristic on the induction product of characters, $\mathrm{ch}(f \circ g) = \mathrm{ch}(f)(x)\mathrm{ch}(g)(x)$, is a symmetric function in Λ^{m+n}.

We would like the product Frobenius characteristic map to be a homomorphism as well. Given an $\mathcal{S}_k \times \mathcal{S}_l$-module V with character ψ and an $\mathcal{S}_m \times \mathcal{S}_n$-module W with character ϕ, $\psi \otimes \phi$ is the character of $V \otimes W$, which is a representation of $(\mathcal{S}_k \times \mathcal{S}_l) \times (\mathcal{S}_m \times \mathcal{S}_n)$. We want to produce a character of $\mathcal{S}_{k+m} \times \mathcal{S}_{l+n}$.

Definition 3.6. *For ψ and ϕ as given above, we define the* induction product $\psi \circ \phi$ *to be $\psi \otimes \phi \uparrow_{(\mathcal{S}_k \times \mathcal{S}_l)\times(\mathcal{S}_m \times \mathcal{S}_n)}^{\mathcal{S}_{k+m}\times\mathcal{S}_{l+n}}$. The induction product on characters extends to all class functions on R_{2d} by (bi)linearity.*

Proposition 3.7. *Let ψ be a class function on $\mathcal{S}_k \times \mathcal{S}_l$, and ϕ a class function on $\mathcal{S}_m \times \mathcal{S}_n$. The product Frobenius characteristic map* ch $: R_{2d} \to \Lambda(x) \otimes \Lambda(y)$ *is a bijective ring homomorphism, i.e.,* ch *is one-to-one and onto, and satisfies*

$$\mathrm{ch}(\psi \circ \phi) = \mathrm{ch}(\psi)\mathrm{ch}(\phi).$$

Before proving this proposition, we need to establish the following lemma:

Lemma 3.8. *Given two groups A and B, and their subgroups $F < A$ and $G < B$, if f is the character of a representation of F and g is the character of a representation of G, then*

$$f \otimes g \uparrow_{F\times G}^{A\times B} = f \uparrow_F^A \otimes g \uparrow_G^B .$$

Proof. Suppose $F < A$ has coset representatives $\{s_1, s_2, ..., s_q\}$, and $G < B$ has coset representatives $\{t_1, t_2, ..., t_r\}$. Then $\{(s_i, t_j) : i \in [q], j \in [r]\}$ is a set of coset representatives for $F \times G < A \times B$. For $(\sigma, \tau) \in A \times B$,

$$\begin{aligned} f \otimes g \uparrow_{F\times G}^{A\times B} ((\sigma,\tau)) &= \sum_{i,j} f \otimes g\big((s_i^{-1}, t_j^{-1})(\sigma,\tau)(s_i,t_j)\big) \\ &= \sum_i f(s_i^{-1}\sigma s_i) \sum_j g(t_j^{-1}\tau t_j) \\ &= f \uparrow_F^A (\sigma) g \uparrow_G^B (\tau) \\ &= f \uparrow_F^A \otimes g \uparrow_G^B ((\sigma,\tau)). \end{aligned}$$

For the second and fourth equalities, see [7, Theorem 1.11.2]. ☐

Proof of Proposition 3.7. The bijectiveness of the product Frobenius characteristic map follows from the definition of ch and the fact that the power sums $p_\mu(x)p_\lambda(y)$ form a $\mathbb{Q}$-basis for $\Lambda(x) \otimes \Lambda(y)$. Next we will show that the map is a homomorphism. Suppose $\psi = \sum_{i,j} a_{ij} \psi_k^{(i)} \otimes \psi_l^{(j)}$ such that $\psi_k^{(i)}$'s and $\psi_l^{(j)}$'s are irreducible characters of representations of $\mathcal{S}_k$ and $\mathcal{S}_l$ respectively. Similarly, $\phi = \sum_{u,v} b_{uv} \phi_m^{(u)} \otimes \phi_n^{(v)}$. For any $\sigma_k \in \mathcal{S}_k$, $\sigma_l \in \mathcal{S}_l$, $\tau_m \in \mathcal{S}_m$, and $\tau_n \in \mathcal{S}_n$, we have

$$\begin{aligned} \psi \otimes \phi\big((\sigma_k,\sigma_l),(\tau_m,\tau_n)\big) &= \Big(\sum_{i,j} a_{ij}\psi_k^{(i)}(\sigma_k)\psi_l^{(j)}(\sigma_l)\Big)\Big(\sum_{u,v} b_{uv}\phi_m^{(u)}(\tau_m)\phi_n^{(v)}(\tau_n)\Big) \\ &= \sum_{i,j,u,v} a_{ij}b_{uv}\psi_k^{(i)}(\sigma_k)\phi_m^{(u)}(\tau_m)\psi_l^{(j)}(\sigma_l)\phi_n^{(v)}(\tau_n) \\ &= \sum_{i,j,u,v} a_{ij}b_{uv}(\psi_k^{(i)} \otimes \phi_m^{(u)}) \otimes (\psi_l^{(j)} \otimes \phi_n^{(v)})(\sigma_k,\tau_m,\sigma_l,\tau_n). \end{aligned}$$

Thus, $\psi \otimes \phi = \sum_{i,j,u,v} a_{ij} b_{uv} (\psi_k^{(i)} \otimes \phi_m^{(u)}) \otimes (\psi_l^{(j)} \otimes \phi_n^{(v)})$. So,

$$\begin{aligned}
\psi \circ \phi &= \psi \otimes \phi \uparrow_{(\mathcal{S}_k \times \mathcal{S}_l) \times (\mathcal{S}_m \times \mathcal{S}_n)}^{\mathcal{S}_{k+m} \times \mathcal{S}_{l+n}} \\
&= \sum_{i,j,u,v} a_{ij} b_{uv} (\psi_k^{(i)} \otimes \phi_m^{(u)}) \otimes (\psi_l^{(j)} \otimes \phi_n^{(v)}) \uparrow_{\mathcal{S}_k \times \mathcal{S}_m \times \mathcal{S}_l \times \mathcal{S}_n}^{\mathcal{S}_{k+m} \times \mathcal{S}_{l+n}} \\
&= \sum_{i,j,u,v} a_{ij} b_{uv} (\psi_k^{(i)} \otimes \phi_m^{(u)}) \uparrow_{\mathcal{S}_k \times \mathcal{S}_m}^{\mathcal{S}_{k+m}} \otimes (\psi_l^{(j)} \otimes \phi_n^{(v)}) \uparrow_{\mathcal{S}_l \times \mathcal{S}_n}^{\mathcal{S}_{l+n}} \\
&= \sum_{i,j,u,v} a_{ij} b_{uv} (\psi_k^{(i)} \circ \phi_m^{(u)}) \otimes (\psi_l^{(j)} \circ \phi_n^{(v)})
\end{aligned}$$

by Lemma 3.8. Now take the product Frobenius characteristic of both sides of the above equation. For clarity, we keep track of variables x and y. By Proposition 3.3 and then Proposition 3.4 we get

$$\begin{aligned}
\mathrm{ch}(\psi \circ \phi)(x,y) &= \sum_{i,j,u,v} a_{ij} b_{uv} \mathrm{ch}(\psi_k^{(i)} \circ \phi_m^{(u)})(x) \mathrm{ch}(\psi_l^{(j)} \circ \phi_n^{(v)})(y) \\
&= \sum_{i,j,u,v} a_{ij} b_{uv} \mathrm{ch}(\psi_k^{(i)})(x) \mathrm{ch}(\phi_m^{(u)})(x) \mathrm{ch}(\psi_l^{(j)})(y) \mathrm{ch}(\phi_n^{(v)})(y) \\
&= \sum_{i,j} a_{ij} \mathrm{ch}(\psi_k^{(i)})(x) \mathrm{ch}(\psi_l^{(j)})(y) \sum_{u,v} b_{uv} \mathrm{ch}(\phi_m^{(u)})(x) \mathrm{ch}(\phi_n^{(v)})(y) \\
&= \mathrm{ch}(\psi)(x,y) \mathrm{ch}(\phi)(x,y).
\end{aligned}$$

□

4. A symmetric function analogue

Using the product Frobenius characteristic map, we derive an equation that is analogous to a well-known symmetric function identity (see Stanley [9, equation (7.13)]): for $n \geqslant 1$,

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0.$$

The above identity contains the complete homogeneous symmetric function h_{n-i} and the elementary symmetric function e_i, which is the Frobenius characteristic of the representation of $\mathcal{S}_i$ on the top homology of the subset lattice B_i. Our analogue, equation (5), involves $h_{n-i}(x)h_{n-i}(y)$ and the representation of $\mathcal{S}_n \times \mathcal{S}_n$ on the top homology of the Segre product poset $B_n \circ B_n$. The product $\mathcal{S}_n \times \mathcal{S}_n$ acts on $B_n \circ B_n$ using the usual action of $\mathcal{S}_n$ on B_n in each component separately. For instance, given a pair of permutations $(123, 213) \in \mathcal{S}_3 \times \mathcal{S}_3$ written in one-line notation and an element $(\{1,2\}, \{2,3\}) \in B_3 \circ B_3$, the first permutation 123 fixes the subset $\{1,2\}$ and the second permutation 213 takes $\{2,3\}$ to $\{1,3\}$ by permuting the numbers in the subset. In the proof of our analogue, we use the Whitney homology technique, which was introduced by Sundaram [12] for pure posets and then generalized by Wachs [13] for semipure posets.

Let Q be a poset with a bottom element $\hat{0}$ and G an automorphism group of Q. Suppose Q is a Cohen-Macaulay G-poset, for each integer r, the r-th *Whitney homology* of Q is defined as

$$WH_r(Q) = \bigoplus_{x \in Q_r} \widetilde{H}_{r-2}(\hat{0}, x),$$

where $Q_r \coloneqq \{x \in Q \mid \mathrm{rank}(x) = r\}$.

For the subset lattice B_n, let P_n be the proper part of the Segre product poset $B_n \circ B_n$. The action of $\mathcal{S}_n \times \mathcal{S}_n$ on P_n induces a representation on the reduced top homology of P_n.

Theorem 4.1. *Let* $\mathrm{ch}(\widetilde{H}_{n-2}(P_n))$ *be the product Frobenius characteristic of this representation. Then*

$$\sum_{i=0}^{n}(-1)^i h_{n-i}(x)h_{n-i}(y)\mathrm{ch}(\widetilde{H}_{i-2}(P_i)) = 0. \tag{5}$$

Proof. Let Q be $P_n \cup \hat{0}$, which is Cohen-Macaulay. We consider the Whitney homology of Q. From the work of Sundaram on Whitney homology (Sundaram [11, 12], Wachs [14, Theorem 4.4.1]), we know that

$$\widetilde{H}_{n-2}(P_n) \cong_{\mathcal{S}_n\times\mathcal{S}_n} \bigoplus_{r=0}^{n-1}(-1)^{n-1+r}\mathrm{WH}_r(Q).$$

The action of $\mathcal{S}_n \times \mathcal{S}_n$ on Q induces a representation of $\mathcal{S}_n \times \mathcal{S}_n$ on the reduced top homology of Q and its Whitney homology groups. An interval $[\hat{0}, x]$ is taken to $[\hat{0}, (\sigma,\tau)x]$ for $(\sigma,\tau) \in \mathcal{S}_n \times \mathcal{S}_n$. Both $\widetilde{H}_{n-2}(P_n)$ and $\mathrm{WH}_r(Q)$ are $\mathcal{S}_n \times \mathcal{S}_n$-modules. Let x be a rank r element of Q. The stabilizer of x is then the Young subgroup $(\mathcal{S}_r \times \mathcal{S}_{n-r}) \times (\mathcal{S}_r \times \mathcal{S}_{n-r})$. Viewing the Whitney homology groups as $\mathcal{S}_n \times \mathcal{S}_n$-modules,

$$\mathrm{WH}_r(Q) = \bigoplus_{x\in Q_r/(\mathcal{S}_n\times\mathcal{S}_n)} \widetilde{H}_{r-2}(\hat{0},x)\uparrow^{\mathcal{S}_n\times\mathcal{S}_n}_{(\mathcal{S}_r\times\mathcal{S}_{n-r})\times(\mathcal{S}_r\times\mathcal{S}_{n-r})},$$

where Q_r is the set of rank r elements in Q and $Q_r/(\mathcal{S}_n \times \mathcal{S}_n)$ is a set of orbit representatives in Q_r (see Wachs [14, Lecture 4.4]). The action of $\mathcal{S}_n \times \mathcal{S}_n$ on Q_r is transitive. So the contribution of the r-th Whitney homology to $\widetilde{H}_{n-2}(P_n)$ is the induced representation $\widetilde{H}_{r-2}(\hat{0},x)\uparrow^{\mathcal{S}_n\times\mathcal{S}_n}_{(\mathcal{S}_r\times\mathcal{S}_{n-r})\times(\mathcal{S}_r\times\mathcal{S}_{n-r})}$ for any x in Q_r. The open interval $(\hat{0}, x)$ is isomorphic to the poset P_r. We then have

$$\mathrm{WH}_r(Q) = \widetilde{H}_{r-2}(P_r)\uparrow^{\mathcal{S}_n\times\mathcal{S}_n}_{(\mathcal{S}_r\times\mathcal{S}_{n-r})\times(\mathcal{S}_r\times\mathcal{S}_{n-r})},$$

and

$$\widetilde{H}_{n-2}(P_n) \cong_{\mathcal{S}_n\times\mathcal{S}_n} \bigoplus_{r=0}^{n-1}(-1)^{n-1+r}\widetilde{H}_{r-2}(P_r)\uparrow^{\mathcal{S}_n\times\mathcal{S}_n}_{(\mathcal{S}_r\times\mathcal{S}_{n-r})\times(\mathcal{S}_r\times\mathcal{S}_{n-r})}.$$

Taking the product Frobenius characteristic of both sides of the above equation, we get

$$\mathrm{ch}(\widetilde{H}_{n-2}(P_n)) = \sum_{r=0}^{n-1}(-1)^{n-1+r}ch\big(\widetilde{H}_{r-2}(P_r)\uparrow^{\mathcal{S}_n\times\mathcal{S}_n}_{(\mathcal{S}_r\times\mathcal{S}_{n-r})\times(\mathcal{S}_r\times\mathcal{S}_{n-r})}\big). \tag{6}$$

Now let ψ_r be the character of the $(\mathcal{S}_r \times \mathcal{S}_r)$-module $\widetilde{H}_{r-2}(P_r)$. Write $1_{\mathcal{S}_{n-r}\times\mathcal{S}_{n-r}}$ for the character of the trivial representation of $S_{n-r} \times S_{n-r}$. When viewing $\widetilde{H}_{r-2}(P_r)$ as a $(\mathcal{S}_r \times \mathcal{S}_{n-r}) \times (\mathcal{S}_r \times \mathcal{S}_{n-r})$-module, its character equals $\psi_r \otimes 1_{\mathcal{S}_{n-r}\times\mathcal{S}_{n-r}}$ (Sagan [7, Theorem 1.11.2]). Then

$$\begin{aligned}\widetilde{H}_{r-2}(P_r)\uparrow^{\mathcal{S}_n\times\mathcal{S}_n}_{(\mathcal{S}_r\times\mathcal{S}_{n-r})\times(\mathcal{S}_r\times\mathcal{S}_{n-r})} &= \psi_r \otimes 1_{\mathcal{S}_{n-r}\times\mathcal{S}_{n-r}}\uparrow^{\mathcal{S}_n\times\mathcal{S}_n}_{(\mathcal{S}_r\times\mathcal{S}_{n-r})\times(\mathcal{S}_r\times\mathcal{S}_{n-r})}\\ &= \psi_r \circ 1_{\mathcal{S}_{n-r}\times\mathcal{S}_{n-r}}.\end{aligned}$$

It follows from Proposition 3.7 that the product Frobenius characteristic

$$\mathrm{ch}(\psi_r \circ 1_{\mathcal{S}_{n-r}\times\mathcal{S}_{n-r}}) = \mathrm{ch}(\psi_r)\mathrm{ch}(1_{\mathcal{S}_{n-r}\times\mathcal{S}_{n-r}}).$$

Thus, equation (6) becomes

$$\tag{7} \begin{aligned} \mathrm{ch}(\widetilde{H}_{n-2}(P_n)) &= \sum_{r=0}^{n-1}(-1)^{n-1+r}\mathrm{ch}(\widetilde{H}_{r-2}(P_r))\mathrm{ch}(1_{\mathcal{S}_{n-r}\times\mathcal{S}_{n-r}}) \\ &= \sum_{r=0}^{n-1}(-1)^{n-1+r}\mathrm{ch}(\widetilde{H}_{r-2}(P_r))\mathrm{ch}(1_{\mathcal{S}_{n-r}})(x)\mathrm{ch}(1_{\mathcal{S}_{n-r}})(y). \end{aligned}$$

It is known that the Frobenius characteristic of the trivial representation of $\mathcal{S}_n$ is h_n (Stanley [9]). Multiplying both sides of equation (7) by $(-1)^{n-1}$, we get

$$(-1)^{n-1}\mathrm{ch}(\widetilde{H}_{n-2}(P_n)) = \sum_{r=0}^{n-1}(-1)^{r}\mathrm{ch}(\widetilde{H}_{r-2}(P_r))h_{n-r}(x)h_{n-r}(y).$$

Finally, we conclude that

$$\sum_{i=0}^{n}(-1)^{i}h_{n-i}(x)h_{n-i}(y)\mathrm{ch}(\widetilde{H}_{i-2}(P_i)) = 0.$$

□

Let $ps : \Lambda \to \mathbb{Q}[q]$ be the stable principal specialization, that is, for a symmetric function $f(x_1, x_2, x_3, \dots)$, $\mathrm{ps}(f)$ is defined to be $f(1, q, q^2, ...)$. A summary of the specializations of different bases for the symmetric functions can be found in Stanley [9, Proposition 7.8.3]. Consider a symmetric function f in two sets of variables $(x_1, x_2, ...)$ and $(y_1, y_2, ...)$. We take the stable principal specialization of f in each set of variables, i.e. substitute $(1, q, q^2, ...)$ for both $(x_1, x_2, ...)$ and $(y_1, y_2, ...)$. The product Frobenius characteristic of the $\mathcal{S}_n \times \mathcal{S}_n$-modules $\widetilde{H}_{n-2}(P_n)$ is a symmetric function in two sets of variables. Then it is natural to ask what we can say about its specialization.

Recall that P_n is the proper part of the Segre product of the subset lattice B_n with itself. The product Frobenius characteristic of the $\mathcal{S}_n \times \mathcal{S}_n$-module $\widetilde{H}_{n-2}(P_n)$ has an innate connection with the Euler characteristic of $B_n(q) \circ B_n(q)$. From Corollary 2.10, $W_n(q)$ is the signless Euler characteristic of $B_n(q) \circ B_n(q)$. The following theorem gives us a connection between the stable principal specialization of $\mathrm{ch}(\widetilde{H}_{n-2}(P_n))$ and the Euler characteristic $W_n(q)$.

Theorem 4.2. *Let $W_n(q)$ be the signless Euler characteristic of $B_n(q) \circ B_n(q)$. For a symmetric function f in two sets of variables $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$, the stable principal specialization $\mathrm{ps}(f)$ specializes both x_i and y_i to q^{i-1}. Then*

$$\mathrm{ps}(\mathrm{ch}(\widetilde{H}_{n-2}(P_n))) = \frac{W_n(q)}{\prod_{i=1}^{n}(1-q^i)^2},$$

where $\mathrm{ch}(V)$ is the product Frobenius characteristic of V.

Proof. We will use induction. The base cases $n = 2$ and $n = 3$ can be verified by hand. We can compute that

$$\mathrm{ps}(\mathrm{ch}(\widetilde{H}_0(P_2))) = \frac{q^2+2q}{(1-q)^2(1-q^2)^2} = \frac{W_2(q)}{(1-q)^2(1-q^2)^2},$$

and

$$\mathrm{ps}(\mathrm{ch}(\widetilde{H}_1(P_3))) = \frac{q^6+4q^5+6q^4+6q^3+2q^2}{(1-q)^2(1-q^2)^2(1-q^3)^2} = \frac{W_3(q)}{(1-q)^2(1-q^2)^2(1-q^3)^2}.$$

Assume that the statement is true for $P_i, i = 1, ..., n-1$. Now let us consider the reduced top homology of P_n. Equation (5) gives us a way to express $\mathrm{ch}(\widetilde{H}_{n-2}(P_n))$ in terms of the product Frobenius characteristic of smaller posets. We get

$$\mathrm{ch}(\widetilde{H}_{n-2}(P_n)) = \sum_{i=0}^{n-1} (-1)^{n-1+i} h_{n-i}(x) h_{n-i}(y) \mathrm{ch}(\widetilde{H}_{i-2}(P_i)). \tag{8}$$

Then we take the stable principal specialization of both sides of equation (8). We know from Stanley [9] that $\mathrm{ps}(h_n) = \prod_{i=1}^{n} \frac{1}{1-q^i}$. It follows from our induction hypothesis that

$$\begin{aligned}
\mathrm{ps}(\mathrm{ch}(\widetilde{H}_{n-2}(P_n))) &= \sum_{i=0}^{n-1} (-1)^{n-1+i} \mathrm{ps}(\mathrm{ch}(\widetilde{H}_{i-2}(P_i))) \prod_{j=1}^{n-i} \frac{1}{(1-q^j)^2} \\
&= \sum_{i=0}^{n-1} (-1)^{n-1+i} \frac{W_i(q)}{\prod_{k=1}^{i} (1-q^k)^2} \prod_{j=1}^{n-i} \frac{1}{(1-q^j)^2} \\
&= \frac{1}{\prod_{k=1}^{n} (1-q^k)^2} \cdot \sum_{i=0}^{n-1} (-1)^{n-1+i} W_i(q) \frac{\prod_{j=i+1}^{n} (1-q^j)^2}{\prod_{j=1}^{n-i} (1-q^j)^2} \\
&= \frac{1}{\prod_{k=1}^{n} (1-q^k)^2} \cdot \sum_{i=0}^{n-1} (-1)^{n-1+i} W_i(q) \begin{bmatrix} n \\ i \end{bmatrix}_q^2 .
\end{aligned}$$

Finally, using the identity involving the signless Euler characteristic $W_n(q)$ given in Theorem 1.2, we obtain

$$\mathrm{ps}(\mathrm{ch}(\widetilde{H}_{n-2}(P_n))) = \frac{W_n(q)}{\prod_{j=1}^{n} (1-q^j)^2}.$$

□

Theorem 4.1 was motivated by our initial findings regarding the q-analogue of equation (1). Once we formulated the specialization of $\mathrm{ch}(\widetilde{H}_{i-2}(P_i))$, the q-analogue can be retrieved by taking the stable principal specialization of equation (5).

5. Alternative proof of the result of Carlitz–Scoville–Vaughan

Carlitz, Scoville and Vaughan's result, Theorem 1.1, provides a combinatorial explanation for the coefficients ω_k in the reciprocal Bessel function. They showed that ω_k is the number of pairs of k-permutations with no common ascent. When letting $q = 1$ in our q-analogue (2), the subspaces of $\mathbb{F}_q^n$ become subsets of $\{1, 2, ..., n\}$. The value $W_n(1) = \sum_{(\sigma,\tau)\in\mathcal{D}_n} 1^{\mathrm{inv}(\sigma)+\mathrm{inv}(\tau)}$ simply counts the number of pairs of permutations of $[n]$ with no common ascent, i.e. ω_n. The proof of Theorem 1.2 is then easily adapted into an alternative proof of Carlitz, Scoville and Vaughan's result (1). Carlitz, Scoville and Vaughan's proof in [5] includes general cases where occurrences of common ascent are allowed. Our proof does not account for those general cases, but it gives a less technical approach by utilizing Björner and Wachs' work on shellability and poset homology [3].

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