ALGEBRAIC COMBINATORICS

Anna E. Weigandt

Schubert polynomials, 132-patterns, and Stanley's conjecture

Volume 1, issue 4 (2018), p. 415-423.

<http://alco.centre-mersenne.org/item/ALCO_2018__1_4_415_0>

Algebraic Combinatorics
Volume 1, issue 4 (2018), p. 415–423
https://doi.org/10.5802/alco.27

Schubert polynomials, 132-patterns, and Stanley's conjecture

Anna E. Weigandt

Abstract Motivated by a recent conjecture of R. P. Stanley we offer a lower bound for the sum of the coefficients of a Schubert polynomial in terms of 132-pattern containment.

1. Introduction

This paper is motivated by a conjecture of R. P. Stanley [8, Conjecture 4.1] concerning the *Schubert polynomials* of A. Lascoux and M.-P. Schützenberger [5]. A *permutation* is a bijection from the set $\{1, 2, \dots, n\}$ to itself. We typically represent a permutation in one-line notation. For instance, $w = 25143$ is the permutation which maps 1 to 2, 2 to 5, 3 to 1, and so on. The *symmetric group* S_n consists of the set of permutations.

If $w_0 = n\, n-1\, \dots\, 1$ is the longest permutation in S_n, define

$$\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

For any other $w \in S_n$, there is some i so that $w(i) < w(i+1)$. Then $\mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i}$, where $\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}$ and $s_i = (i, i+1)$ acts on f by exchanging the variables x_i and x_{i+1}. The ∂_i's satisfy the same braid and commutativity relations as the simple transpositions and so $\mathfrak{S}_w$ is well defined. The polynomial $\mathfrak{S}_w$ is called a *Schubert polynomial*. We will use an equivalent definition for Schubert polynomials as a weighted sum over *pipe dreams*. See Section 2 for these definitions.

We are interested in the following specialization: $\nu_w := \mathfrak{S}_w(1, 1, \dots, 1)$. Let

$$P_{132}(w) := \{(i, j, k) : i < j < k \text{ and } w(i) < w(k) < w(j)\}. \tag{1}$$

Write $\eta_w := \# P_{132}(w)$. If $\eta_w \geqslant 1$ then w *contains* the pattern 132.

Example 1.1. Let $w = 25143$. Below, we list the elements of $P_{132}(w)$ by marking in bold the positions $i < j < k$ for which $(i, j, k) \in P_{132}(w)$.

251**4**3 **25**14**3** **2**51**43** 25**143**

As such, $\eta_w = 4$.

We prove that η_w provides a lower bound for ν_w.

Theorem 1.2 (The 132-bound). *For any $w \in S_n$, $\nu_w \geqslant \eta_w + 1$.*

Manuscript received 28th July 2017, revised 28th May 2018, accepted 12th June 2018.

Keywords. Schubert polynomials, permutation patterns.

Acknowledgements. The author was supported by the Ruth V. Shaff and Genevie I. Andrews Fellowship.

As a corollary, we obtain the following conjecture of R. P. Stanley [8, Conjecture 4.1].

Corollary 1.3. *$\nu_w = 2$ if and only if $\eta_w = 1$.*

Proof. Let $w \in S_n$. If $\eta_w = 0$ then $\nu_w = 1$ [6, Chapter 4]. If $\eta_w = 1$ then $\nu_w = 2$ [8, Section 4]. Otherwise, $\eta_w \geqslant 2$. Then we apply Theorem 1.2 and obtain

$$\nu_w \geqslant \eta_w + 1 \geqslant 3.$$

As such, $\nu_w = 2$ if and only if $\eta_w = 1$. □

2. Background on Permutations and Pipe Dreams

We will recall the necessary background on permutations and Schubert polynomials; our references are [7, Chapter 2] and [1] respectively. Each permutation has an associated *rank function* r_w, where

$$r_w(i,j) := \#\{k : 1 \leqslant k \leqslant i \text{ and } w(k) \leqslant j\}. \tag{2}$$

The pair (i,j) is an *inversion* of w if $i < j$ and $w(i) > w(j)$. Equivalently, each inversion corresponds to a 21-pattern in w. The *length* of a permutation is the number inversions,

$$\ell(w) := \#\{(i,j) : i < j \text{ and } w(i) > w(j)\}. \tag{3}$$

The *Rothe diagram* of $w \in S_n$ is the set

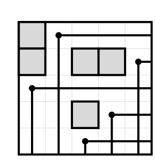
$$D(w) := \{(i,j) : 1 \leqslant i,j \leqslant n, w(i) > j, \text{ and } w^{-1}(j) > i\}. \tag{4}$$

Notice immediately from (4), we have

$$D(w^{-1}) = D(w)^t. \tag{5}$$

The diagram $D(w)$ is in bijection with the set of inversions of w by the map

$$(i,j) \mapsto (i, w^{-1}(j)). \tag{6}$$



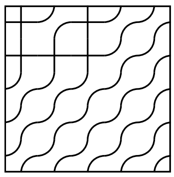
We may visualize $D(w)$ as follows. For each $i = 1, \ldots, n$, plot $(i, w(i))$. Then, strike out all boxes to the right and below each of the plotted points. The boxes which remain form $D(w)$. For example, $D(25143)$ is pictured to the right. Notice that we use matrix conventions; cell (i,j) sits in the ith row from the top and the jth column from the left.

Schubert polynomials can be written as a sum over *pipe dreams*. Pipe dreams appear in the literature under various names; they are the *pseudo-line configurations* of S. Fomin and A. N. Kirillov [3] and the *RC-graphs* of N. Bergeron and S. C. Billey [1]. They were studied from a geometric perspective by A. Knuston and E. Miller [4].

Let $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ be the semi-infinite grid, starting from the northwest corner. A *pipe dream* is a tiling of this grid with $\lrcorner\ulcorner$'s (elbows) and a finite number of $+$'s (pluses). For simplicity, we will often draw the elbows as dots. We freely identify each pipe dream with a subset of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ by recording the coordinates of the pluses. Associate a weight monomial to $\mathcal{P}$:

$$\texttt{wt}(\mathcal{P}) = \prod_{(i,j) \in \mathcal{P}} x_i.$$

Equivalently, the exponent of x_i counts the number of pluses which appear in row i of $\mathcal{P}$.



We may interpret $\mathcal{P}$ as a collection of overlapping strands, using the rule that a strand never bends at a right angle. The $+$'s indicate the positions where two strands cross. Each row on the left edge of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is connected by some strand to a unique column along the top, and vice versa. If the ith row is connected to the jth column, let $w_{\mathcal{P}}(i) := j$. There exists some n so that $w_{\mathcal{P}}(i) = i$ for all $i > n$, so $w_{\mathcal{P}} \in S_\infty$. In practice, we identify $w_{\mathcal{P}}$ with its representative in some finite symmetric group. For example, if $\mathcal{P}$ is the pipe dream pictured above, then we write $w_{\mathcal{P}} = 25143$.

If $\#\mathcal{P} = \ell(w_{\mathcal{P}})$ then $\mathcal{P}$ is *reduced*. Let

$$\mathtt{RP}(w) := \{\mathcal{P} : w_{\mathcal{P}} = w \text{ and } \mathcal{P} \text{ is reduced}\}.$$

Theorem 2.1 ([1, 3]).

$$\mathfrak{S}_w = \sum_{\mathcal{P} \in \mathtt{RP}(w)} \mathtt{wt}(\mathcal{P}). \tag{7}$$

Recall, $\nu_w := \mathfrak{S}_w(1,1,\ldots,1)$. Immediately from (7), $\nu_w = \#\mathtt{RP}(w)$.

Example 2.2. The reduced pipe dreams for $w = 25143$ are pictured below.

$$\begin{bmatrix} + & \cdot & + & + & \cdot \\ + & \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} + & \cdot & + & \cdot & \cdot \\ + & + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} + & \cdot & + & + & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} + & \cdot & \cdot & + & \cdot \\ + & + & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\begin{bmatrix} + & \cdot & \cdot & \cdot & \cdot \\ + & + & + & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} + & \cdot & + & + & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} + & \cdot & \cdot & + & \cdot \\ + & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} + & \cdot & \cdot & \cdot & \cdot \\ + & + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Therefore,

$$\mathfrak{S}_w = x_1^3x_2^2 + x_1^2x_2^3 + x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1x_2^3x_3 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1x_2^3x_4$$

and $\nu_w = 8$.

There are two pipe dreams which have an explicit description in terms of w. Let

$$m_i(w) = \#\{j : (i,j) \in D(w)\}. \tag{8}$$

Then the *bottom pipe dream* is

$$\mathcal{B}_w = \{(i,j) : j \leqslant m_i(w)\}. \tag{9}$$

Graphically, $\mathcal{B}_w$ is obtained from $D(w)$ by replacing each box with a plus and then left justifying within each row. We define the *top pipe dream* as the transpose of the bottom pipe dream of w^{-1}:

$$\mathcal{T}_w := \mathcal{B}^t_{w^{-1}}.$$

By (5), $\mathcal{T}_w$ is obtained from $D(w)$ by top justifying pluses within columns.

Example 2.3. Let $w = 25143$.

$$\mathcal{B}_w = \begin{bmatrix} + & \cdot & \cdot & \cdot & \cdot \\ + & + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad \mathcal{T}_w = \begin{bmatrix} + & \cdot & + & + & \cdot \\ + & \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Pictured above are the bottom and top pipe dreams for w.

N. Bergeron and S. C. Billey gave a procedure to obtain any pipe dream in $\texttt{RP}(w)$ algorithmically, starting from $\mathcal{B}_w$. A *ladder move* is an operation on pipe dreams which produces a new pipe dream by a replacement of the following type.

$$\begin{array}{cc} \cdot & \cdot \\ + & + \\ + & + \\ \vdots & \vdots \\ + & + \\ + & \cdot \end{array} \quad \mapsto \quad \begin{array}{cc} \cdot & + \\ + & + \\ + & + \\ \vdots & \vdots \\ + & + \\ \cdot & \cdot \end{array}$$

In the above picture, the columns and rows are consecutive. If $\mathcal{P} \mapsto \mathcal{P}'$ is a ladder move, then $\mathcal{P} \in \texttt{RP}(w)$ if and only if $\mathcal{P}' \in \texttt{RP}(w)$. In other words, $\texttt{RP}(w)$ is closed under ladder moves [1]. Furthermore, any element of $\texttt{RP}(w)$ can be reached by some sequence of ladder moves from the bottom pipe dream.

Theorem 2.4 ([1, Theorem 3.7]). *If $\mathcal{P} \in \texttt{RP}(w)$, then $\mathcal{P}$ can be obtained by a sequence of ladder moves from $\mathcal{B}_w$.*

We will mostly focus on a special type of ladder move. A *simple ladder move* is a replacement of the following form.

$$\begin{array}{cc} \cdot & \cdot \\ + & \cdot \end{array} \quad \mapsto \quad \begin{array}{cc} \cdot & + \\ \cdot & \cdot \end{array}$$

The outline of the proof is as follows. In Lemma 3.6, we show that any sequence of ladder moves connecting $\mathcal{B}_w$ to $\mathcal{T}_w$ must use only simple ladder moves. The exact number of pipe dreams in any such sequence, is $\eta_w + 1$. Since each pipe dream in the sequence is distinct, this provides a lower bound for $\nu_w = \#\texttt{RP}(w)$.

3. Proof of Theorem 1.2

We start by interpreting η_w as a weighted sum over $D(w)$. The "32" in each 132-pattern of w corresponds to a box $(i,j) \in D(w)$. The "1" contributes to the rank function $r_w(i,j)$.

Lemma 3.1.

$$\eta_w = \sum_{(i,j) \in D(w)} r_w(i,j).$$

Proof. Suppose $(i,j,k) \in P_{132}(w)$. Then $w(j) > w(k)$ and $w^{-1}(w(k)) = k > j$. By (4), we have $(j, w(k)) \in D(w)$. Furthermore, $i \leqslant j$ and $w(i) \leqslant w(k)$. Then by (2),

$$\#\{\ell : (\ell, j, k) \in P_{132}(w)\} \leqslant \#\{\ell : \ell \leqslant j \text{ and } w(\ell) \leqslant w(k)\} = r_w(j, w(k)).$$

Then

$$\eta_w \leqslant \sum_{(i,j) \in D(w)} r_w(i,j). \tag{10}$$

On the other hand, suppose $(i,j) \in D(w)$. Then

$$w(i) > j = w(w^{-1}(j)) \text{ and } w^{-1}(j) > i.$$

Take

$$k \in \{k : k \leqslant i \text{ and } w(k) \leqslant j\}.$$

Since $(i,j) \in D(w)$, we must have $k < i$ and $w(k) < j$. Then

$$k < i < w^{-1}(j) \text{ and } w(k) < w(w^{-1}(j)) < w(i)$$

and so

$$(k, i, w^{-1}(j)) \in P_{132}(w).$$

As such, if $(i, j) \in D(w)$,

$$\#\{\ell : (\ell, i, w^{-1}(j)) \in P_{132}(w)\} \geqslant r_w(i, j).$$

Therefore,

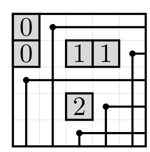
$$\eta_w \geqslant \sum_{(i,j)\in D(w)} r_w(i, j). \tag{11}$$

As such,

$$\eta_w = \sum_{(i,j)\in D(w)} r_w(i, j).$$

□

Example 3.2. Again, let $w = 25143$. Below, we label each box $(i, j) \in D(w)$ with $r_w(i, j)$.



As such,

$$\sum_{(i,j)\in D(w)} r_w(i, j) = 4.$$

In Example 1.1, we found that $\eta_w = 4$. This agrees with Lemma 3.1.

If $\mathcal{P} \in \mathtt{RP}(w)$, let $\mathbf{a}_{\mathcal{P}} := (a_{\mathcal{P}}(1), \ldots, a_{\mathcal{P}}(n))$ where

$$a_{\mathcal{P}}(k) = \#\{(i, j) \in \mathcal{P} : i + j - 1 = k\}. \tag{12}$$

Equivalently, $a_{\mathcal{P}}(k)$ is the number of pluses that occur in the kth antidiagonal of $\mathcal{P}$. Notice if $\mathcal{P} \mapsto \mathcal{P}'$ is a simple ladder move, then $\mathbf{a}_{\mathcal{P}} = \mathbf{a}_{\mathcal{P}'}$. If $\mathcal{P} \mapsto \mathcal{P}'$ is a ladder move which is not simple, then $\mathbf{a}_{\mathcal{P}} \neq \mathbf{a}_{\mathcal{P}'}$.

Example 3.3. Let $\mathcal{P}, \mathcal{P}' \in \mathtt{RP}(25143)$ be as pictured below.

$$\mathcal{P} = \begin{bmatrix} + & \cdot & \cdot & \cdot & \cdot \\ + & + & + & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad \mathcal{P}' = \begin{bmatrix} + & \cdot & + & \cdot & \cdot \\ + & + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Although $\mathcal{P}'$ can be obtained from $\mathcal{P}$ by a ladder move, it is not a simple ladder move. Indeed, $\mathbf{a}_{\mathcal{P}} = (1, 1, 1, 2, 0)$ and $\mathbf{a}_{\mathcal{P}'} = (1, 1, 2, 1, 0)$. Therefore, $\mathbf{a}_{\mathcal{P}} \neq \mathbf{a}_{\mathcal{P}'}$.

This idea extends to sequences of ladder moves.

Lemma 3.4. *Suppose there is a path of ladder moves from $\mathcal{P}$ to $\mathcal{Q}$:*

$$\mathcal{P} = \mathcal{P}_0 \mapsto \mathcal{P}_1 \mapsto \cdots \mapsto \mathcal{P}_N = \mathcal{Q}. \tag{13}$$

Each ladder move in (13) *is simple if and only if* $\mathbf{a}_{\mathcal{P}} = \mathbf{a}_{\mathcal{Q}}$.

Proof.
($\Rightarrow$) Assume each $\mathcal{P}_i \mapsto \mathcal{P}_{i+1}$ is a simple ladder move. Then $\mathcal{P}_{i+1}$ is obtained from $\mathcal{P}_i$ by moving a single plus to a new position in the same antidiagonal. As such, $\mathbf{a}_{\mathcal{P}_i} = \mathbf{a}_{\mathcal{P}_{i+1}}$ for each i. Therefore $\mathbf{a}_{\mathcal{P}} = \mathbf{a}_{\mathcal{Q}}$.

($\Leftarrow$) We prove the contrapositive. Suppose there is a nonsimple ladder move in the sequence (13). It acts by removing a plus from the ith antidiagonal and replacing it in the jth antidiagonal with $i < j$. In particular, we may pick j to be the maximum such label. By the maximality, no plus moves into the jth antidiagonal from a different antidiagonal. Then $a_{\mathcal{P}}(j) > a_{\mathcal{Q}}(j)$ and so $\mathbf{a}_{\mathcal{P}} \neq \mathbf{a}_{\mathcal{Q}}$. □

Fix an indexing set I. A labeling of a pipe dream is an injective map $\mathcal{L}_{\mathcal{P}} : \mathcal{P} \to I$. Suppose $\mathcal{P} \mapsto \mathcal{P}'$ is a simple ladder move. Then $\mathcal{P}'$ inherits a labeling from $\mathcal{P}$ as follows:

$$\mathcal{L}_{\mathcal{P}'}(i,j) = \begin{cases} \mathcal{L}_{\mathcal{P}}(i,j) & \text{if } (i,j) \in \mathcal{P} \\ \mathcal{L}_{\mathcal{P}}(i+1,j-1) & \text{otherwise.} \end{cases}$$

Since $\mathcal{P} \mapsto \mathcal{P}'$ is a simple ladder move, $\mathcal{P}'$ is obtained from $\mathcal{P}$ by adding some (i,j) to $\mathcal{P}$ and removing $(i+1, j-1)$. Therefore $\mathcal{L}_{\mathcal{P}'}$ is well defined. If there is a path of simple ladder moves from $\mathcal{P}$ to $\mathcal{Q}$, then $\mathcal{Q}$ inherits the labeling $\mathcal{L}_{\mathcal{Q}}$ from $\mathcal{L}_{\mathcal{P}}$ inductively.

Lemma 3.5. *Let $L_{\mathcal{P}}$ be a labeling. Suppose $\mathcal{Q}$ can be reached from $\mathcal{P}$ by simple ladder moves. Then $\mathcal{Q}$ inherits the same labeling from $\mathcal{P}$ regardless of the choice of sequence.*

Proof. Suppose $\mathcal{P} \mapsto \mathcal{P}'$ is a simple ladder move. Then within any antidiagonal, both pipe dreams have the same set of labels in the same relative order. Iterate this argument along a path of simple ladder moves from $\mathcal{P}$ to $\mathcal{Q}$. Then, in each antidiagonal, $\mathcal{P}$ and $\mathcal{Q}$ have the same set of labels, still in the same relative order. As such, the labeling is uniquely determined and independent of the choice of path. □

Lemma 3.6.

(i) *The map*

$$(i,j) \mapsto (i, j - r_w(i,j))$$

is a bijection between $D(w)$ and $\mathcal{B}_w$.

(ii) *The map*

$$(i,j) \mapsto (i - r_w(i,j), j)$$

is a bijection between $D(w)$ and $\mathcal{T}_w$.

(iii) *$\mathcal{B}_w$ and $\mathcal{T}_w$ are connected by simple ladder moves.*

Proof.

(i). Suppose $\ell > i$ and $w(\ell) < w(i)$. Since $w^{-1}(w(\ell)) = \ell > i$ and $w(i) > w(\ell)$, by (4), we have $(i, w(\ell)) \in D(w)$. Therefore,

$$w(\ell) \in \{j : (i, w(j)) \in D(w)\}.$$

By (8), the ith row of $D(w)$ has as many boxes as there are pluses in the ith row of $\mathcal{B}_w$. Let

$$j_1 < j_2 < \cdots < j_{m_i(w)}$$

be the sequence obtained by sorting the set $\{j : (i,j) \in D(w)\}$. Then

$$\begin{aligned} j_\ell - r_w(i, j_\ell) &= j_\ell - \#\{k : k \leqslant i \text{ and } w(k) \leqslant j_\ell\} \\ &= \#\{k : k > i \text{ and } w(k) \leqslant j_\ell\} \\ &= \#\{j : (i,j) \in D(w) \text{ and } j \leqslant j_\ell\} \\ &= \ell. \end{aligned}$$

Therefore $(i, j_\ell) \mapsto (i, \ell)$. Since $1 \leqslant \ell \leqslant m_i(w)$ the map is well defined. This holds for any

$$\ell \in \{1, \ldots, m_i(w)\}$$

so the map is surjective. By definition, $j_\ell = j_{\ell'}$ if and only if $\ell = \ell'$, giving injectivity. As such, this is a bijection.

(ii). Let ϕ be the map defined by $(i,j) \mapsto (j,i)$. Restricted to $D(w)$, ϕ is a bijection between $D(w)$ and $D(w^{-1})$. By the definition of $\mathcal{T}_w$, the restriction

$$\phi : \mathcal{B}_{w^{-1}} \to \mathcal{T}_w$$

is also a bijection.

Let

$$\psi : \mathcal{P}(w^{-1}) \to \mathcal{B}_w$$

be the map in (i). Then the composition

$$D(w) \xrightarrow{\phi} D(w^{-1}) \xrightarrow{\psi} \mathcal{B}_{w^{-1}} \xrightarrow{\phi} \mathcal{T}_w$$

is a bijection. Computing directly,

$$\begin{aligned} \phi(\psi(\phi(i,j))) &= \phi(\psi(j,i)) \\ &= \phi(j, i - r_{w^{-1}}(j,i)) \\ &= (i - r_{w^{-1}}(j,i), j). \end{aligned}$$

Applying (2),

$$\begin{aligned} r_{w^{-1}}(j,i) &= \#\{k : k \leqslant j \text{ and } w^{-1}(k) \leqslant i\} \\ &= \#\{\ell : w(\ell) \leqslant j \text{ and } w^{-1}(w(\ell)) \leqslant i\} \\ &= \#\{\ell : \ell \leqslant i \text{ and } w(\ell) \leqslant j\} \\ &= r_w(i,j). \end{aligned}$$

Therefore,

$$\phi(\psi(\phi(i,j))) = (i - r_w(i,j), j).$$

(iii). By Theorem 2.4, there is a path of ladder moves from $\mathcal{B}_w$ to $\mathcal{T}_w$. Applying (12) and the bijections in parts (i) and (ii),

$$\begin{aligned} \mathbf{a}_{\mathcal{B}_w}(k) &= \#\{(i,j) \in D(w) : i + (j - r_w(i,j)) - 1 = k\} \\ &= \#\{(i,j) \in D(w) : (i - r_w(i,j)) + j - 1 = k\} \\ &= \mathbf{a}_{\mathcal{T}_w}(k). \end{aligned}$$

By Lemma 3.4, the path uses only simple ladder moves. □

In light of the previous lemma, we may label the pluses of $\mathcal{B}_w$ using the map $(i,j) \mapsto (i, j - r_w(i,j))$, i.e. we refer to the plus which is the image of (i,j) as $+_{(i,j)}$. Likewise we label $\mathcal{T}_w$ using the map $(i,j) \mapsto (i - r_w(i,j), j)$.

Lemma 3.7. *The above labeling of $\mathcal{T}_w$ is the same as the labeling it inherits from $\mathcal{B}_w$.*

Proof. It is enough to show that within any given antidiagonal the labels in $\mathcal{B}_w$ and $\mathcal{T}_w$ are the same and have the same relative order. If $(i,j) \in D(w)$, then $+_{(i,j)}$ is in position $(i, j - r_w(i,j))$ in $\mathcal{B}_w$ and in position $(i - r_w(i,j), j)$ in $\mathcal{T}_w$. Since

$$i + j - r_w(i,j) = i - r_w(i,j) + j,$$

they are in the same antidiagonal.

Now consider the rth antidiagonal in $\mathcal{B}_w$. Suppose the sorted list of pluses from top to bottom is

$$+_{(i_1,j_1)}, +_{(i_2,j_2)}, \cdots, +_{(i_k,j_k)}.$$

Since the map from $D(w)$ is by left justification, we must have $i_1 < i_2 < \cdots < i_k$. As $i_\ell + j_\ell - 1 = r$ for all ℓ, it follows that $j_1 > j_2 > \cdots > j_k$. Since the map from $D(w)$ to $\mathcal{T}_w$ is by top justification, the sorted list of pluses from top to bottom must also be

$$+_{(i_1,j_1)}, +_{(i_2,j_2)}, \cdots, +_{(i_k,j_k)}.$$

Therefore, the labeling which $\mathcal{T}_w$ inherits from $\mathcal{B}_w$ coincides with the labeling determined by the map $(i,j) \mapsto (i - r_w(i,j), j)$. $\square$

We conclude with the proof of the 132-bound.

Proof of Theorem 1.2. By Lemma 3.6, there is a path of simple ladder moves connecting $\mathcal{B}_w$ to $\mathcal{T}_w$, say

$$\mathcal{B}_w = \mathcal{P}_0 \mapsto \mathcal{P}_1 \mapsto \cdots \mapsto \mathcal{P}_N = \mathcal{T}_w. \tag{14}$$

Let $n_{i,j} = \#\{k : \mathcal{P}_k \mapsto \mathcal{P}_{k+1} \text{ moves } +_{(i,j)}\}$. By definition, $\mathcal{P}_k \mapsto \mathcal{P}_{k+1}$ moves exactly one plus, labeled by an element of $D(w)$. Therefore,

$$N = \sum_{(i,j) \in D(w)} n_{i,j}. \tag{15}$$

Claim 3.8. *If $(i,j) \in D(w)$ then $n_{i,j} = r_w(i,j)$.*

Proof. By Lemma 3.7, $+_{(i,j)}$ must move from position $(i, j - r_w(i,j))$ in $\mathcal{B}_w$ to position $(i - r_w(i,j), j)$ in $\mathcal{T}_w$. At each step $+_{(i,j)}$ remains stationary or it moves up a row and one column to the right. As such, $+_{(i,j)}$ must move exactly $i - (i - r_w(i,j)) = r_w(i,j)$ times to go from row i to row $i - r_w(i,j)$. $\square$

Then

$$\begin{aligned}
\eta_w &= \sum_{(i,j) \in D(w)} r_w(i,j) && \text{(by Lemma 3.1)} \\
&= \sum_{(i,j) \in D(w)} n_{i,j} && \text{(by Claim 3.8)} \\
&= N && \text{(by (15))}.
\end{aligned}$$

Each $\mathcal{P}_i$ in the sequence (14) is distinct. As such, $\#\mathtt{RP}(w) \geqslant N + 1$. Therefore

$$\nu_w = \#\mathtt{RP}(w) \geqslant N + 1 = \eta_w + 1. \qquad \square$$

Example 3.9. Let $w = 25143$. Below, we give a sequence of simple ladder moves connecting $\mathcal{B}_w$ to $\mathcal{T}_w$. The last row and column of each pipe dream has been omitted.

$$\begin{bmatrix} +_{(1,1)} & \cdot & \cdot & \cdot \\ +_{(2,1)} & +_{(2,3)} & +_{(2,4)} & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ +_{(4,3)} & \cdot & \cdot & \cdot \end{bmatrix} \mapsto \begin{bmatrix} +_{(1,1)} & \cdot & \cdot & +_{(2,4)} \\ +_{(2,1)} & +_{(2,3)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ +_{(4,3)} & \cdot & \cdot & \cdot \end{bmatrix} \mapsto \begin{bmatrix} +_{(1,1)} & \cdot & \cdot & +_{(2,4)} \\ +_{(2,1)} & +_{(2,3)} & \cdot & \cdot \\ \cdot & +_{(4,3)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\mapsto \begin{bmatrix} +_{(1,1)} & \cdot & +_{(2,3)} & +_{(2,4)} \\ +_{(2,1)} & \cdot & \cdot & \cdot \\ \cdot & +_{(4,3)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \mapsto \begin{bmatrix} +_{(1,1)} & \cdot & +_{(2,3)} & +_{(2,4)} \\ +_{(2,1)} & \cdot & +_{(4,3)} & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Notice for each $(i,j) \in D(w)$, the plus $+_{(i,j)}$ moves $r_w(i,j)$ times. For instance, $r_w(4,3) = 2$ and $+_{(4,3)}$ moves twice. This follows from Claim 3.8. The above sequence from $\mathcal{B}_w$ to $\mathcal{T}_w$ uses $\eta_w + 1 = 5$ pipe dreams in total. This agrees with the 132-bound, $\nu_w \geqslant \eta_w + 1$.

Acknowledgements. The author would like to thank Alexander Yong for helpful conversations and feedback. The author also thanks the anonymous referees for their comments and suggestions which improved the exposition of this work.

References

[1] Nantel Bergeron and Sara Billey, *RC-graphs and Schubert polynomials*, Exp. Math. **2** (1993), no. 4, 257–269.
[2] Sara Billey, William Jockusch, and Richard P. Stanley, *Some combinatorial properties of schubert polynomials*, J. Algebr. Comb. **2** (1993), no. 4, 345–374.
[3] Sergey Fomin and Anatol N. Kirillov, *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Discrete Math. **153** (1996), no. 1-3, 123–143.
[4] Allen Knutson and Ezra Miller, *Gröbner geometry of Schubert polynomials*, Ann. Math. (2005), 1245–1318.
[5] Alain Lascoux and Marcel-Paul Schützenberger, *Polynômes de Schubert*, C. R. Math. Acad. Sci. Paris **295** (1982), no. 3, 447–450.
[6] Ian G. Macdonald, *Notes on schubert polynomials*, Publications du Laboratoire de combinatoire et d'informatique mathématique, vol. 6, Université du Québec à Montréal, 1991.
[7] Laurent Manivel, *Symmetric functions, Schubert polynomials, and degeneracy loci*, SMF/AMS Texts and Monographs, vol. 6, American Mathematical Society, 2001.
[8] Richard P. Stanley, *Some Schubert shenanigans*, `https://arxiv.org/abs/1704.00851`, 2017.

Anna E. Weigandt, University of Illinois at Urbana-Champaign, Dept. of Mathematics, 1409 W. Green St., Urbana, IL 61801, USA
E-mail : `weigndt2@illinois.edu`