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Quasisymmetric functions distinguishing trees

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Abstract A famous conjecture of Stanley states that his chromatic symmetric function distinguishes trees. As a quasisymmetric analogue, we conjecture that the chromatic quasisymmetric function of Shareshian and Wachs and of Ellzey distinguishes directed trees. This latter conjecture would be implied by an affirmative answer to a question of Hasebe and Tsujie about the P-partition enumerator distinguishing posets whose Hasse diagrams are trees. They proved the case of rooted trees and our results include a generalization of their result.

1. Introduction

As an extension of the chromatic polynomial $\chi_G(k)$ of a graph $G=(V,E)$, Stanley [29] introduced the chromatic symmetric function $X_G(\mathbf{x})$ defined by

$$X_G(\mathbf{x}) = \sum_{\kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots \tag{1}$$

where the sum is over all proper colorings $\kappa : V \to \{1,2,\ldots\}$. Observe that setting $x_i = 1$ for $1 \leqslant i \leqslant k$ and $x_i = 0$ otherwise yields $\chi_G(k)$. Two famous and unsolved conjectures appear in [29]. One of these, known as the Stanley–Stembridge conjecture [31, Conj. 5.5][29, Conj. 5.1] is about the e-positivity of $X_G(\mathbf{x})$ for incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free posets and does not concern us here. Of more interest to us is that Stanley gave a pair of non-isomorphic graphs on five vertices (see Figure 1(c) with the arrows removed) that have the same $X_G(\mathbf{x})$. As a result, we say that $X_G(\mathbf{x})$ does not *distinguish* graphs. Stanley stated "We do not know whether X_G distinguishes trees." Subsequent papers, such as [2, 3, 4, 10, 14, 15, 19, 22, 24, 26], have established that $X_G(\mathbf{x})$ distinguishing trees is certainly worthy of being called a conjecture; for example, [14] shows that $X_G(\mathbf{x})$ distinguishes trees with up to 29 vertices.

We focus on a generalization of $X_G(\mathbf{x})$ to labeled graphs introduced by Shareshian and Wachs [25], denoted $X_G(\mathbf{x},t)$, which has an extra parameter t and is now just a quasisymmetric function in general. In fact, we will use a further generalization of $X_G(\mathbf{x})$ to directed graphs (digraphs) $\overrightarrow{G}$ due to Ellzey [7, 8, 9] and denoted $X_{\overrightarrow{G}}(\mathbf{x},t)$.

Our original goal in this project was to study equality among $X_{\overrightarrow{G}}(\mathbf{x},t)$. It is not obvious if $X_{\overrightarrow{G}}(\mathbf{x},t)$ will be more or less successful at distinguishing digraphs compared to $X_G(\mathbf{x})$ distinguishing graphs: there are far more digraphs than graphs for a given

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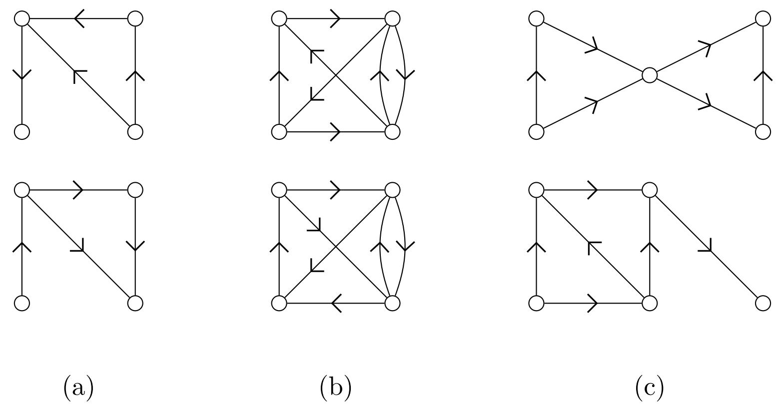

Figure 1. Pairs of digraphs with equal chromatic quasisymmetric functions

number of vertices, but $X_{\overrightarrow{G}}(\mathbf{x},t)$ contains more information than $X_G(\mathbf{x})$. It is not hard to find digraphs with the same $X_{\overrightarrow{G}}(\mathbf{x},t)$; three such equalities are given in Figure 1.

One way to bring Stanley's conjecture into the quasisymmetric setting would be by stating that $X_{\overrightarrow{G}}(\mathbf{x},t)$ distinguishes directed acyclic graphs, but (a) and (c) of Figure 1 show that such a statement is false. Instead, we offer the following conjecture, which is a natural extension of Stanley's conjecture to the quasisymmetric setting; it is stated as a question in [1].

Conjecture 1.1. *$X_{\overrightarrow{G}}(\mathbf{x},t)$ distinguishes directed trees.*

In other words, the conjecture states that if directed trees $\overrightarrow{G}$ and $\overrightarrow{H}$ are not isomorphic then $X_{\overrightarrow{G}}(\mathbf{x},t) \neq X_{\overrightarrow{H}}(\mathbf{x},t)$. Our approach to tackling Conjecture 1.1 will be to translate it into a question about posets. When $\overrightarrow{G}$ is a directed acyclic graph, we can represent $\overrightarrow{G}$ as a poset by saying $v_i \leqslant v_j$ if there is a directed path from v_i to v_j. In other words, draw $\overrightarrow{G}$ so that all the edges point upwards on the page; removing the arrows results in a Hasse diagram of a poset, possibly with some redundant edges; see Figure 2. Let P denote the resulting poset. As we will see, the coefficient of the highest power of t that appears in $X_{\overrightarrow{G}}(\mathbf{x},t)$ will be the well-known strict P-partition enumerator $\overline{K}_P(\mathbf{x})$, a fact previously mentioned in [1, Theorem 7.4] and [8, p. 11]. Obviously two chromatic quasisymmetric functions are different if their coefficients on the highest power of t are different. So to prove Conjecture 1.1 it suffices to prove the following conjecture. A poset being a tree simply means its Hasse diagram is a tree.

Conjecture 1.2. *$\overline{K}_P(\mathbf{x})$ distinguishes posets that are trees.*

We have verified this conjecture for all such posets with at most 11 elements. A main advantage of the poset setting is that Conjecture 1.2, and equality among $\overline{K}_P$ in general, has already been studied in the literature as we detail in Section 3. In fact, Conjecture 1.2 appears as a question in [12, Problem 6.1]. Section 2 will give the necessary background, including definitions of the quasisymmetric functions mentioned above. While we only needed the *strict* P-partition enumerator above, in Section 4, we consider the original (P,ω)-partition enumerator $K_{(P,\omega)}(\mathbf{x})$ which allows for a mixture of strict and weak relations. Such (P,ω) are called *labeled posets*. Interestingly, $K_{(P,\omega)}$ does *not* distinguish labeled posets that are trees, but we offer the following conjecture. A poset that is a tree is said to be a *rooted tree* if it has a unique minimal element.

Conjecture 1.3. $K_{(P,\omega)}(\mathbf{x})$ *distinguishes labeled posets that are rooted trees.*

Hasebe and Tsujie [12] have shown the case when all the relations are weak (or all strict), and we generalize their result by establishing Conjecture 1.3 for a class of labeled rooted trees that we call *fair trees*. The class of fair trees interpolates between rooted trees with all weak edges and those with all strict edges. We believe our result on fair trees (Theorem 4.4) is the first non-trivial result stating that $K_{(P,\omega)}(\mathbf{x})$ distinguishes a class of posets with a mixture of strict and weak edges. We conclude in Section 5 with several directions for further study.

2. Preliminaries

2.1. The chromatic symmetric function for directed graphs.

Let $\mathbb{P}$ denote the set of positive integers. For a positive integer n, we write $[n]$ to denote the set $\{1, \ldots, n\}$.

Definition 2.1 ([8]). *Let $\overrightarrow{G} = (V, E)$ be a directed graph. Given a proper coloring $\kappa : V \to \mathbb{P}$ of $\overrightarrow{G}$, we define an* ascent *of κ to be a directed edge $(v_i, v_j) \in E$ with $\kappa(v_i) < \kappa(v_j)$, and we let* $\operatorname{asc}(\kappa)$ *denote the number of ascents of κ. The chromatic quasisymmetric function of $\overrightarrow{G}$ is*

$$X_{\overrightarrow{G}}(\mathbf{x}, t) = \sum_{\kappa} t^{\operatorname{asc}(\kappa)} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots \tag{2}$$

where the sum is over all proper colorings of $\overrightarrow{G}$.

Setting $t = 1$ yields Stanley's chromatic symmetric function $X_G(\mathbf{x})$ of (1), where G denotes the undirected version of $\overrightarrow{G}$. When $\overrightarrow{G}$ is a directed acyclic graph, which is the case of most interest to us, $X_{\overrightarrow{G}}(\mathbf{x}, t)$ coincides with the chromatic quasisymmetric function of Shareshian and Wachs. We use the digraph setting because in the labeled graph setting of Shareshian–Wachs, there are lots of trivial equalities among $X_G(\mathbf{x}, t)$ that result just from relabeling.

Example 2.2. Let $\overrightarrow{G}$ be the 3-element path with the directions as shown below. With colors $a < b < c$, the proper colorings κ of $\overrightarrow{G}$ fall into the 8 classes given by the following table.

$$v_1 \rightarrow v_2 \leftarrow v_3$$

$\kappa(v_1)$	$\kappa(v_2)$	$\kappa(v_3)$	$\operatorname{asc}(\kappa)$
a	b	c	1
a	c	b	2
b	a	c	0
b	c	a	2
c	a	b	0
c	b	a	1
a	b	a	2
b	a	b	0

Thus

$$X_{\overrightarrow{G}}(\mathbf{x}, t) = (2 + 2t + 2t^2) M_{111} + t^2 M_{21} + M_{12}\,,$$

where M denotes the basis of monomial quasisymmetric functions (see Subsection 2.3 for the necessary background on quasisymmetric functions). Setting $t = 1$ gives $X_G(\mathbf{x}) = 6m_{111} + m_{21}$ from which we get $\chi_G(k) = 6\binom{k}{3} + k(k-1) = k(k-1)^2$, as expected.

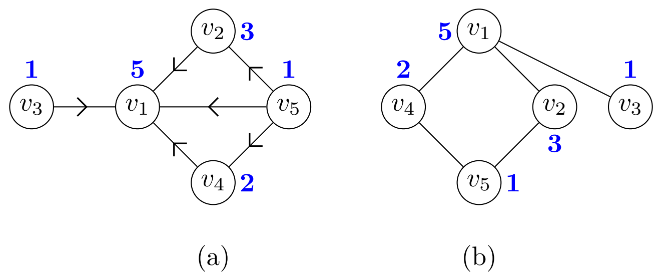

Figure 2. Converting from a proper coloring of a digraph to a strict P-partition. The numbers next to each node correspond to a coloring in the digraph on the left and the corresponding (P,ω)-partition of the labeled poset on the right.

Let us make a couple of observations about types of $X_{\overrightarrow{G}}(\mathbf{x},t)$-equality that arise. By setting $t=1$, we know equal $X_{\overrightarrow{G}}(\mathbf{x},t)$ means the underlying undirected graphs must have equal $X_G(\mathbf{x})$ and the examples in Figure 1 show two scenarios: either the underlying undirected graphs are isomorphic, or they are not isomorphic but have equal $X_G(\mathbf{x})$. The $X_G(\mathbf{x})$-equality implied by Figure 1(c) is the one given by Stanley in [29].

Figure 1(a) shows an example of $X_{\overrightarrow{G}}(\mathbf{x},t)$ being invariant under reversal of all the edge directions. Letting α^{rev} denote the reversal of the composition α, this invariance will hold whenever the coefficients of M_α and $M_{\alpha^{\mathrm{rev}}}$ in $X_{\overrightarrow{G}}(\mathbf{x},t)$ are equal for all α, so in particular when $X_{\overrightarrow{G}}(\mathbf{x},t)$ is symmetric [25, Cor. 2.7] [8, Prop. 2.6]. But not all equalities among $X_{\overrightarrow{G}}(\mathbf{x},t)$ with isomorphic underlying graphs arise from reversal of all edges, as shown by Figure 1(b).

2.2. The poset perspective. As mentioned in the Introduction, when $\overrightarrow{G}$ is a directed acyclic graph we can view it as a poset; see Figure 2 for an example, with a coloring given by numbers next to each vertex.

Now consider the coefficient of the highest power of t that appears in $X_{\overrightarrow{G}}(\mathbf{x},t)$. This coefficient enumerates colorings that strictly increase along every directed edge, as in Figure 2(a). We now compare this to the definition of Stanley's (P,ω)-partitions.

Let P be a poset with n elements; we write $|P|=n$. Denote the order relation on P by $\leqslant_P$, while $\leqslant$ denotes the usual order on the positive integers. A *labeling* of P is a bijection $\omega : P \to [n]$. A *labeled poset* (P,ω) is then a poset P with an associated labeling ω.

Definition 2.3. *For a labeled poset (P,ω), a* (P,ω)-partition *is a map f from P to the positive integers satisfying the following two conditions:*

- *if $a <_P b$, then $f(a) \leqslant f(b)$, i.e. f is order-preserving;*
- *if $a <_P b$ and $\omega(a) > \omega(b)$, then $f(a) < f(b)$.*

In other words, a (P,ω)-partition is an order-preserving map from P to the positive integers with certain strictness conditions determined by ω. Examples of (P,ω)-partitions f are given in Figure 3, where the images under f are written in bold and blue next to the nodes.

The meaning of the double edges in the figure follows from the following observation about Definition 2.3. For $a,b \in P$, we say that a is *covered* by b in P, denoted $a \prec_P b$, if $a <_P b$ and there does not exist c in P such that $a <_P c <_P b$. Note that a definition equivalent to Definition 2.3 is obtained by replacing both appearances of the relation

$a <_P b$ with the relation $a \prec_P b$. In other words, we require that f be order-preserving along the edges of the Hasse diagram of P, with $f(a) < f(b)$ when the edge $a \prec_P b$ satisfies $\omega(a) > \omega(b)$. With this in mind, we will consider those edges $a \prec_P b$ with $\omega(a) > \omega(b)$ as *strict edges* and we will represent them in Hasse diagrams by double lines. Similarly, edges $a \prec_P b$ with $\omega(a) < \omega(b)$ will be called *weak edges* and will be represented by single lines.

From the point-of-view of (P, ω)-partitions, the labeling ω only determines which edges are strict and which are weak. Therefore, we say that two labeled posets (P, ω) and (Q, ω') are *isomorphic* if P and Q are isomorphic as posets and they have equivalent sets of strict and weak edges according to a poset isomorphism. Thus many of our figures from this point on will not show the labeling ω, but instead show some collection of strict and weak edges determined by an underlying ω.

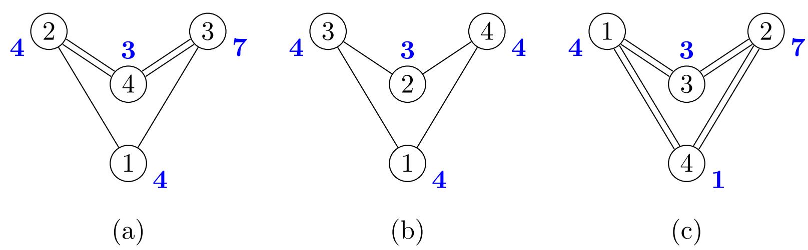

Figure 3. Examples of (P, ω)-partitions

If ω is order-preserving, as in Figure 3(b), then P is said to be *naturally labeled* and all the edges are weak. In this case, we typically omit reference to the labeling and so a (P, ω)-partition is then traditionally called a P-partition. At the other extreme, when ω is order-reversing, as in Figure 3(c), f must strictly increase along each edge. Such (P, ω)-partitions that are required to strictly increase along each edge will be called *strict P-partitions*.

Returning to Figure 2, the key observation is now clear: the proper colorings of a directed acyclic graph $\overrightarrow{G}$ that contribute to the coefficient of the highest power of t in $X_{\overrightarrow{G}}(\mathbf{x}, t)$ are in bijection with strict P-partitions of the corresponding poset P.

To make the algebraic connection, the well-known (P, ω)-partition enumerator is defined by

$$K_{(P,\omega)}(\mathbf{x}) = \sum_f x_1^{\#f^{-1}(1)} x_2^{\#f^{-1}(2)} \cdots \tag{3}$$

where the sum is over all (P, ω)-partitions $f : P \to \mathbb{P}$. When all the edges of (P, ω) are weak, so the sum is over P-partitions, we will denote the P-partition enumerator $K_{(P,\omega)}(\mathbf{x})$ simply by $K_P(\mathbf{x})$ or just K_P. Similarly, we will use $\overline{K}_P$ to denote $K_{(P,\omega)}(\mathbf{x})$ when all the edges are strict, thus enumerating strict P-partitions. Comparing (2) and (3) when P is the poset corresponding to a directed acyclic graph $\overrightarrow{G}$, we see that $\overline{K}_P(\mathbf{x})$ is exactly the coefficient of the highest power of t in $X_{\overrightarrow{G}}(\mathbf{x}, t)$. This connection between $\overline{K}_P$ and $X_{\overrightarrow{G}}(\mathbf{x}, t)$ has previously been mentioned in [1, Theorem 7.4] [8, p. 11]. As a corollary, if Conjecture 1.2 is true, then so is Conjecture 1.1.

With this implication now established, we will work almost entirely in the poset setting. Although the direct connection between $X_{\overrightarrow{G}}(\mathbf{x}, t)$ and $K_{(P,\omega)}$ uses the setting of strict P-partitions and $\overline{K}_P$, most of the results in the literature work with P-partitions and K_P. However, for equality questions, the two settings are equivalent

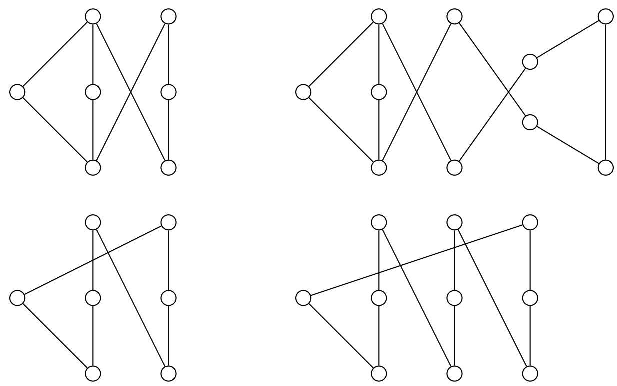

Figure 4. Pairs of posets with equal P-partition enumerators

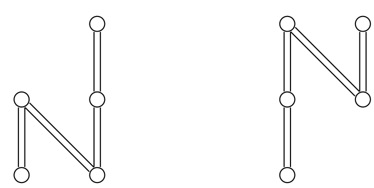

Figure 5. The strict P-partition enumerators of these two posets have the same F-support

since $\overline{K}_P$ can be obtained from K_P and vice-versa; see [23, §3] for the full details of this equivalence and involutions on (P, ω)-partition enumerators. In particular, Conjecture 1.2 can be restated as the assertion that $K_P(\mathbf{x})$ distinguishes posets that are trees.

Remark 2.4. In addition to its simple statement, Conjecture 1.2 has the virtue that some natural more general statements are false. The first example of non-isomorphic posets with the same K_P was given in [23] and appears in Figure 4(a). A *bowtie* is the poset consisting of elements a_1, a_2, b_1, b_2 with cover relations $a_i < b_j$ for all i, j. Notice that each poset in Figure 4(a) has a bowtie as an induced subposet. Otherwise, we say the poset is *bowtie-free*. Weakening the tree hypothesis of Conjecture 1.2 to bowtie-free results in a false statement, with Figure 4(b) being the smallest counterexample.

Conjecture 1.2 equivalently states that for posets P and Q that are trees, if $\overline{K}_P = \overline{K}_Q$ then P and Q are isomorphic. We can consider weakenings of the equality hypothesis in this statement. The *F-support* of a quasisymmetric function f is the set of compositions α for which the coefficient of F_α is non-zero when f is expanded in the F-basis. But the F-support of $\overline{K}_P$ and $\overline{K}_Q$ being equal does not imply that P and Q are isomorphic, with a counterexample given by the posets in Figure 5, which both have F-support $\{221, 212, 122, 2111, 1211, 1121, 1112, 11111\}$.

In contrast, see Subsection 5.3 for versions of Conjecture 1.2 that use much less information than in $\overline{K}_P(\mathbf{x})$ to distinguish posets that are trees.

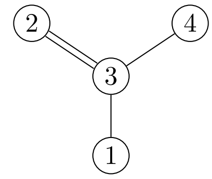

Figure 6. The labeled poset of Example 2.6

2.3. Quasisymmetric functions. It follows directly from its definition that $K_{(P,\omega)}$ is a quasisymmetric function. In fact, $K_{(P,\omega)}$ served as a motivating example for Gessel's original definition [11] of quasisymmetric functions.

For our purposes, quasisymmetric functions are elements of $\mathbb{Z}[[x_1, x_2, \ldots]]$ and we denote the ring of quasisymmetric functions by *QSym*. We will make use of both of the classical bases for *QSym*. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a composition of n, then we define the *monomial quasisymmetric function* M_α by

$$M_\alpha = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

As we know, compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n are in bijection with subsets of $[n-1]$, and let $S(\alpha)$ denote the set $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$. Thus we also denote M_α by $M_{S(\alpha),n}$. Notice that these two notations are distinguished by the latter one including the subscript n; this subscript is helpful since $S(\alpha)$ does not uniquely determine n.

The second classical basis is composed of the *fundamental quasisymmetric functions* F_α defined by

$$F_\alpha = F_{S(\alpha),n} = \sum_{S(\alpha) \subseteq T \subseteq [n-1]} M_{T,n}\,. \tag{4}$$

The relevance of this latter basis to $K_{(P,\omega)}$ is due to Theorem 2.5 below, which first appeared in [27, 28] and, in the language of quasisymmetric functions, in [11].

Every permutation $\pi \in S_n$ has a descent set $\mathrm{des}(\pi)$ given by $\{i \in [n-1] : \pi(i) > \pi(i+1)\}$, and we will call the corresponding composition of n the *descent composition* of π, denoted $\mathrm{co}(\pi)$. For example, if $\pi = 243561$, then $\mathrm{des}(\pi) = \{2, 5\}$ and $\mathrm{co}(\pi) = 231$. Let $\mathcal{L}(P,\omega)$ denote the set of all linear extensions of P, regarded as permutations of the ω-labels of P. For example, for the labeled poset in Figure 3(a), $\mathcal{L}(P,\omega) = \{1423, 1432, 4123, 4132\}$.

Theorem 2.5 ([11, 27, 28]). *Let (P,ω) be a labeled poset with $|P| = n$. Then*

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\mathrm{des}(\pi),n} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\mathrm{co}(\pi)}\,.$$

Example 2.6. The labeled poset (P,ω) of Figure 6 has $\mathcal{L}(P,\omega) = \{1324, 1342\}$ and hence

$$\begin{aligned} K_{(P,\omega)} &= F_{\{2\},4} + F_{\{3\},4} \\ &= F_{22} + F_{31} \\ &= \begin{array}{l}(M_{\{2\},4} + M_{\{1,2\},4} + M_{\{2,3\},4} + M_{\{1,2,3\},4}) \\ \;+ (M_{\{3\},4} + M_{\{1,3\},4} + M_{\{2,3\},4} + M_{\{1,2,3\},4})\end{array} \\ &= M_{22} + M_{31} + M_{112} + 2M_{211} + M_{121} + 2M_{1111}. \end{aligned}$$

3. Consequences of the poset viewpoint

As mentioned in the Introduction, an advantage of the poset viewpoint is that equality among K_P has already been studied in the literature. In this largely expository section, we gather together these results, especially those relevant to Conjecture 1.2. As the K_P notation indicates, these results are confined to posets with all weak edges.

Results from the literature mostly fall into three classes: irreducibility, classes of posets within which we know that the P-partition enumerator distinguishes the posets, and necessary conditions on P and Q for $K_P = K_Q$.

3.1. Irreducibility. If P disconnects into two posets P_1 and P_2, then it follows from the definition of K_P that $K_P = K_{P_1} K_{P_2}$. On the other hand, a key result from [18] is that if P is connected, then K_P is irreducible in *QSym*. Moreover, *QSym* is known to be a unique factorization domain [13, 16, 21]. As a consequence, Liu and Weselcouch deduce the following result.

Corollary 3.1 ([18, Corollary 4.20]). *For a poset P, the irreducible factorization of K_P is given by $K_P = \prod_i K_{P_i}$, where the P_i are the connected components of P.*

Therefore, when studying $K_P = K_Q$, it suffices to consider the case when both P and Q are connected (see [18, Corollary 4.21]). Additionally, a proof of Conjecture 1.2 would also mean that K_P distinguishes forests.

Returning briefly to the setting of the chromatic quasisymmetric function $X_{\overrightarrow{G}}$ we get the following consequence.

Proposition 3.2. *For directed acyclic graphs $\overrightarrow{G}$ and $\overrightarrow{H}$ with $X_{\overrightarrow{G}}(\mathbf{x}, t) = X_{\overrightarrow{H}}(\mathbf{x}, t)$, if $\overrightarrow{G}$ is connected then so is $\overrightarrow{H}$.*

Proof. If $\overrightarrow{H}$ has connected components $\overrightarrow{H}_1, \ldots, \overrightarrow{H}_r$ with $r \geqslant 2$, then it follows from Definition 2.1 that $X_{\overrightarrow{H}}(\mathbf{x}, t)$ factors as

$$X_{\overrightarrow{H}}(\mathbf{x}, t) = X_{\overrightarrow{H}_1}(\mathbf{x}, t) \cdots X_{\overrightarrow{H}_r}(\mathbf{x}, t).$$

Thus

$$X_{\overrightarrow{G}}(\mathbf{x}, t) = X_{\overrightarrow{H}_1}(\mathbf{x}, t) \cdots X_{\overrightarrow{H}_r}(\mathbf{x}, t). \tag{5}$$

Now consider the irreducibility of the coefficient of the highest power of t on both sides of (5). On the left-hand side, this is K_P for the poset P corresponding to $\overrightarrow{G}$, and K_P is irreducible since $\overrightarrow{G}$ and hence P is connected. On the right-hand side, this coefficient is the product of the coefficients of the highest powers of t in each $X_{\overrightarrow{H}_i}(\mathbf{x}, t)$, which is a contradiction since we know these coefficients are not constants. ☐

3.2. Distinguishing within classes of posets. The P-partition enumerator K_P is known to distinguish posets within each of the following classes.

- Rooted trees [12, 33], i.e. posets that are trees with a single minimal element.
- More generally, posets that are both bowtie-free and N-free [12]. As one would expect, N is the poset consisting of elements a_1, a_2, b_1, b_2 whose cover relations are $a_1 < b_1 > a_2 < b_2$, and a poset is N-free if it does not contain N as an induced subposet.
- Series-parallel posets [18, Theorem 5.2]. These can be defined in two ways. They are the posets that can be formed by repeated operations of disjoint union and ordinal sum. Equivalently, they are the N-free posets.
- Posets of width two [17].

- Posets whose Greene shape is a hook [17]. The *Greene shape* of a poset P is the partition $(c_1 - c_0, c_2 - c_1, \ldots)$ where c_i is the maximum cardinality of a union of k chains of P. So a poset whose Greene shape is the hook $(j, 1^i)$ has a maximal chain with j elements and i additional elements which form an antichain.
- Posets with Greene shape $(k, 2, 1, 1, \ldots, 1)$ for some $k \geqslant 2$ [17].

3.3. Necessary conditions for equality. If $K_P = K_Q$, then all the statements in the next list hold. In the same way that knowledge of K_P is equivalent to knowledge of $\overline{K}_P$, both are equivalent to knowledge of K_{P^*} where P^* is the dual of P (see, for example, [23, §3]). Thus all the statements below have dual versions.

- P and Q obviously have the same number of vertices; if they are not trees, they need not have the same number of edges, as shown by Figure 4(a).
- By Theorem 2.5, P and Q have the same number of linear extensions.
- The *jump* of an element p of P is defined to be the length (number of edges) of the longest chain from p down to a minimal element of P. Then P and Q have the same number of elements of jump i for all i [23]. This can sometimes be a quick way to show that $K_P \neq K_Q$.
- $K_{P_i} = K_{Q_i}$, where P_i denotes the induced subposet of P consisting of elements of jump at least i [23]. For example, with $i = 1$, we get that $K_{P^-} = K_{Q^-}$, where P^- denotes the result of deleting all the minimal elements from P.
- The *up-jump* of p denotes the length of the longest chain from p to a maximal element, and define the *jump-pair* of P to be (jump of p, up-jump of p). Then for all i and j, the number of elements with jump pair (i, j) is the same for P as for Q [17].
- Let $\mathrm{anti}_{k,i,j}(P)$ denote the number of k-element order ideals I of P such that I has i maximal elements and $P \setminus I$ has j minimal elements. Then $\mathrm{anti}_{k,i,j}(P) = \mathrm{anti}_{k,i,j}(Q)$ for all i, j, k [17].
- Summing over j and k in the previous item, we get that P and Q have the same number of antichains of each size, as conjectured in [23] and shown in [17].
- Suppose that for some k and i, P has a unique order ideal I_P of size k with i maximal elements, and similarly for an order ideal I_Q of Q. Then $K_{P \setminus I_P}$ can be determined from K_P [17, Corollary 3.6] and hence $K_{P \setminus I_P} = K_{Q \setminus I_Q}$.
- Summing over j in $\mathrm{anti}_{k,i,j}(P)$ shows that the number of order ideals of size k with i maximal elements has to be the same for P as for Q. Then there are various ways we can combine the results above. For example, the number of elements with principal order ideal of size k and up-jump j is the same for P as for Q [17].
- The Greene shape of P equals that of Q [17].
- A P-partition f is *pointed* if f is surjective onto $[k]$ for some k and $f^{-1}(i)$ has a unique minimal element for all i with $1 \leqslant i \leqslant k$. The weight of f is the composition $\mathrm{wt}(f) = (\#f^{-1}(1), \#f^{-1}(2), \ldots)$. The number of pointed P-partitions of any given weight is the same for P and Q. This follows immediately from the result of [1] that if we expand K_P in the (unnormalized) power sum basis ψ_α of type I, then the coefficient of ψ_α equals the number of pointed P-partitions of weight α .
- [17, Lemma 4.10] shows that any finite poset P has a unique antichain A of maximum size such that any other antichain of maximum size is contained in the order ideal $I(A)$ generated by A. Let P^- be the subposet consisting of

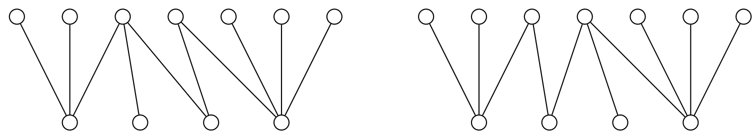

Figure 7. Two trees of rank one with the same degree sequences but different K_P

elements less than A in P. Then since K_{P^-} is determined by K_P (also shown in [17]), we must have $K_{P^-} = K_{Q^-}$. Similarly for the subposet P^+ above A.

If one is given non-isomorphic trees T_1 and T_2, it is typically straightforward to find a result on the list above that will show that they have unequal P-partition enumerators. However, the problem is that the result used will depend on T_1 and T_2, i.e. we don't have a systematic way.

Trees of rank one are difficult enough that they are a good test case for techniques; see Figure 7 for a non-isomorphic pair. We can use $\mathrm{anti}_{k,1,j}(P)$ and $\mathrm{anti}_{k,1,j}(P^*)$ to determine the degree sequences for the maximal and minimal elements, respectively; these match up in the figure. To distinguish the pair in the figure, we can use pointed P-partitions: the tree on the right has a pointed P-partition of weight $(4,1,4,2)$ but the tree on the left does not.

4. Adding strict edges

This section considers extending Conjecture 1.2 in the following way, as inspired by [12, Problem 6.2]: does $K_{(P,\omega)}$ distinguish labeled trees when we allow any mixture of strict and weak edges? The answer is "no" in general as shown, for example, by the labeled trees in Figure 8.

In fact, this question connects to a studied one in the realm of symmetric functions. Semistandard Young tableaux of a skew shape λ/μ can be considered as (P,ω)-partitions of a particular labeled poset; see [30, §7.19], for example. In this case, $K_{(P,\omega)}$ equals the skew Schur function $s_{\lambda/\mu}$. When λ/μ is a *ribbon*, meaning it is connected and has no 2-by-2 block of cells, the corresponding (P,ω) will be a tree. It is well known that $s_{\lambda/\mu}$ is invariant under rotation of λ/μ by 180° thus yielding an infinite class of pairs of trees with the same $K_{(P,\omega)}$; the simplest example is in Figure 8(b). It is natural to ask if they are other pairs of ribbons, unequal under 180° rotation, that give rise to the same skew Schur function. The answer is "yes" and the full classification of such pairs is given in [6].

On the other hand, Hasebe and Tsujie [12] have shown that $K_{(P,\omega)}$ distinguishes *rooted* trees with all strict (equivalently all weak) edges. In Conjecture 1.3, we propose that their result also holds for arbitrary labeled posets that are rooted trees. In other words, if we want $K_{(P,\omega)}$ to distinguish trees with a mixture of strict and weak edges, restricting to rooted trees works. We have verified Conjecture 1.3 for $n \leqslant 10$.

Although we have not succeeded in resolving Conjecture 1.3, the rest of this section focuses on results that still significantly generalize those in [12], as we next explain.

4.1. Fair trees.

We consider rooted trees as being rooted at the bottom, and thus a child is above its parent.

Definition 4.1. *A* fair tree *is a labeled poset (P,ω) such that:*

- *the underlying poset P is a rooted tree,*
- *for each vertex v in (P,ω), its outgoing edges (to its children) are either all strict or all weak.*

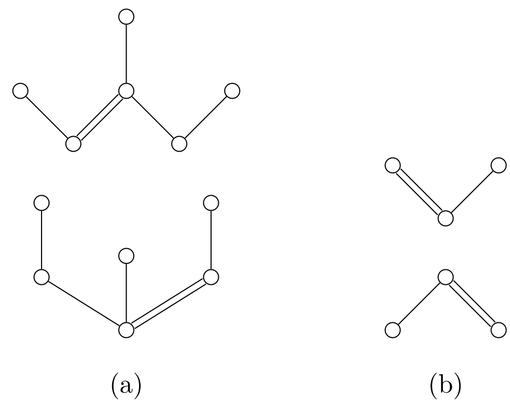

FIGURE 8. Pairs of labeled trees with the same $K_{(P,\omega)}$

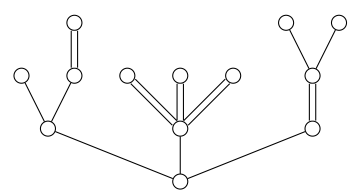

FIGURE 9. Example of a fair tree of size 13

Figure 9 shows an example of a fair tree. The "fair" terminology comes from the idea that each parent is equally strict with all its children.

We will prove that the fair trees are distinguished by the (P,ω)-partition enumerator $K_{(P,\omega)}$. In fact, we shall consider a wider class $\mathcal{C}$ which we introduce next.

Let us define the following (noncommutative) operations on labeled posets. For (P,ω) and (Q,ω') two labeled posets (considered as posets with assignments of strict and weak edges), the *weak (resp. strict) ordinal sum* $(P,\omega) \uparrow (Q,\omega')$ (resp. $(P,\omega) \Uparrow (Q,\omega')$) is the labeled poset obtained by placing (P,ω) below (Q,ω') and adding a weak (resp. strict) edge from each maximal element of P to each minimal element of Q.

It will be helpful in what follows to pick an explicit labeling of the elements of these ordinal sums that is consistent with the strictness of the edges. We will label $(P,\omega) \uparrow (Q,\omega')$ by copying over the labels from (P,ω) and (Q,ω') but increasing each of the ω'-labels by $|P|$ so that the labeling is a bijection to $[|P|+|Q|]$. We can label the disjoint union $(P,\omega) \sqcup (Q,\omega')$ in the same way. Similarly, $(P,\omega) \Uparrow (Q,\omega')$ will be labeled by instead increasing each of the ω-labels by $|Q|$.

DEFINITION 4.2. *We define the set $\mathcal{C}$ of labeled posets recursively by:*

(a) *the one-element labeled poset* $[1]$ *is in* $\mathcal{C}$*;*
(b) *for any* (P,ω) *and* (Q,ω') *in* $\mathcal{C}$*, their disjoint union* $(P,\omega) \sqcup (Q,\omega')$ *is in* $\mathcal{C}$*;*
(c) *for any* (P,ω) *in* $\mathcal{C}$*, the ordinal sums* $[1] \uparrow (P,\omega)$ *and* $[1] \Uparrow (P,\omega)$ *are in* $\mathcal{C}$*;*
(d) *for any* (P,ω) *in* $\mathcal{C}$*, the ordinal sums* $(P,\omega) \uparrow [1]$ *and* $(P,\omega) \Uparrow [1]$ *are in* $\mathcal{C}$*.*

Figure 10 shows an example. See Subsection 4.5 for a characterization of $\mathcal{C}$ in terms of forbidden subposets.

REMARK 4.3. We can use Definition 4.2 to give an alternative and recursive definition of fair trees. Define *fair forests* as the class defined recursively by (a)–(c) in

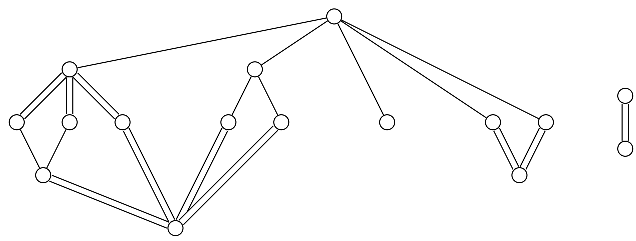

FIGURE 10. Example of an element of $\mathcal{C}$ of size 16.

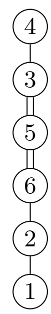

FIGURE 11. This labeled poset has (P,ω)-partition enumerator F_{312}.

Definition 4.2. Fair trees are nothing but connected fair forests, thus are elements of $\mathcal{C}$.

The main result in this section is the following. It is a generalization of Theorems 1.3 and 5.1 in [12]; see Proposition 4.12 below for the analogue of Hasebe and Tsujie's bowtie- and N-free characterization.

THEOREM 4.4. *The (P,ω)-partition enumerator $K_{(P,\omega)}$ distinguishes elements in $\mathcal{C}$, thus in particular fair trees. More formally, for labeled posets (P,ω) and (Q,ω') in $\mathcal{C}$, the following assertions are equivalent:*

(a) *(P,ω) and (Q,ω') are isomorphic;*
(b) *$K_{(P,\omega)} = K_{(Q,\omega')}$.*

The crux of the proof is the irreducibility result Proposition 4.8, in the same way that irreducibility played a key role in [12, 18]. We first need more background on *QSym*.

4.2. PRODUCTS OF QUASISYMMETRIC FUNCTIONS. It will help our intuition to recall how to interpret F_α as a (P,ω)-partition enumerator. If α is a composition of n, we let P be the chain with n elements, labeled from top to bottom by any permutation π of $[n]$ such that $\mathrm{co}(\pi) = \alpha$. That the resulting $K_{(P,\omega)} = F_\alpha$ follows directly from Theorem 2.5; see Figure 11 for an example. Thinking just in terms of strict and weak edges, for a general α, simply insert the strict edges so that the numbers of elements in the chains of contiguous weak edges, from bottom to top, match the parts of α.

From (3), we know that labeled posets (P,ω) and (Q,ω') satisfy

$$K_{(P,\omega)}K_{(Q,\omega')} = K_{(P,\omega)\sqcup(Q,\omega')}.$$

Thus we can interpret the product $F_\alpha F_\beta$ as a (P,ω)-partition enumerator for a disjoint union of two appropriately labeled chains. One fact that we need later from this interpretation is summarized in the following lemma.

Lemma 4.5. *Let f_1 and f_2 be two non-constant elements of QSym. Then the F-support of $f_1 f_2$ contains at least one composition γ whose first part γ_1 is at least 2. The same is true when focusing on the last part of the compositions appearing in the F-support of $f_1 f_2$.*

Proof. First consider two non-empty compositions α and β. Construct two labeled chains (P,ω) and (Q,ω') such that $F_\alpha = K_{(P,\omega)}$ and $F_\beta = K_{(Q,\omega')}$. Thus $F_\alpha F_\beta = K_{(P,\omega)\sqcup(Q,\omega')}$. It is obvious that there is at least one element π in the set $\mathcal{L}((P,\omega)\sqcup(Q,\omega'))$ with $\pi_1 < \pi_2$. By Theorem 2.5, this gives rise to an element F_γ in the F-support of $F_\alpha F_\beta$ with $\gamma_1 \geqslant 2$.

Now, let us consider two non-constant elements f_1 and f_2 of *QSym*. Let us denote by α (resp. β) the lexicographically maximal composition in the F-support of f_1 (resp. f_2). It is clear that the lexicographically maximal element in the product $f_1 f_2$ comes from the product $F_\alpha F_\beta$, thus the case above applies.

The case of the last part is similar, requiring just two tweaks. First, if ℓ denotes the number of elements in $(P,\omega)\sqcup(Q,\omega')$, it is obvious that there is at least one element π in the set $\mathcal{L}((P,\omega)\sqcup(Q,\omega'))$ with $\pi_{\ell-1} < \pi_\ell$. This yields an element F_γ in the F-support of $F_\alpha F_\beta$ such that the last part of γ is at least 2. Secondly, instead of considering the lexicographically maximal composition, we use the composition whose reversal is lexicographically maximal. □

Let us introduce two operations on compositions (which are already known but we shall use notation relevant to our context):

$$(\alpha_1,\alpha_2,\dots,\alpha_k)\uparrow(\beta_1,\beta_2,\dots,\beta_\ell) = (\alpha_1,\alpha_2,\dots,\alpha_k+\beta_1,\beta_2,\dots,\beta_\ell), \tag{6}$$

and

$$(\alpha_1,\alpha_2,\dots,\alpha_k)\Uparrow(\beta_1,\beta_2,\dots,\beta_\ell) = (\alpha_1,\alpha_2,\dots,\alpha_k,\beta_1,\beta_2,\dots,\beta_\ell), \tag{7}$$

which give rise to two (noncommutative) products in *QSym*, defined on the F-basis:

$$F_\alpha \uparrow F_\beta = F_{\alpha\uparrow\beta}, \tag{8}$$

and

$$F_\alpha \Uparrow F_\beta = F_{\alpha\Uparrow\beta}. \tag{9}$$

The use of the same notation as for labeled posets is justified by the following statement.

Proposition 4.6. *For any two labeled posets (P,ω) and (Q,ω'),*

$$K_{(P,\omega)\uparrow(Q,\omega')} = K_{(P,\omega)} \uparrow K_{(Q,\omega')}$$

and

$$K_{(P,\omega)\Uparrow(Q,\omega')} = K_{(P,\omega)} \Uparrow K_{(Q,\omega')}.$$

Proof. Let us prove the first assertion. By Theorem 2.5,

$$K_{(P,\omega)\uparrow(Q,\omega')} = \sum_{\pi\in\mathcal{L}((P,\omega)\uparrow(Q,\omega'))} F_{\mathrm{co}(\pi)}.$$

By definition of $\uparrow$ for labeled posets, those $\pi \in \mathcal{L}((P,\omega)\uparrow(Q,\omega'))$ are concatenations of $\sigma \in \mathcal{L}(P,\omega)$ and $\tau \in \mathcal{L}(Q,\omega')$ but where each entry of τ is increased by $|P|$.

Consequently, $\mathrm{co}(\pi) = \mathrm{co}(\sigma) \uparrow \mathrm{co}(\tau)$. Thus, using (8), we have

$$\begin{aligned} K_{(P,\omega)\uparrow(Q,\omega')} &= \sum_{\sigma\in\mathcal{L}(P,\omega),\ \tau\in\mathcal{L}(Q,\omega')} F_{\mathrm{co}(\sigma)\uparrow\mathrm{co}(\tau)} \\ &= \sum_{\sigma\in\mathcal{L}(P,\omega),\ \tau\in\mathcal{L}(Q,\omega')} F_{\mathrm{co}(\sigma)} \uparrow F_{\mathrm{co}(\tau)} \\ &= K_{(P,\omega)} \uparrow K_{(Q,\omega')}\ . \end{aligned}$$

The second assertion is proved similarly. □

4.3. An involution on labeled posets. Given a labeled poset (P, ω), we can switch the strictness of the edges to obtain a new labeled poset $\overline{(P, \omega)}$. We will follow [23] by referring to this operation on labeled posets as the *bar operation*. In the setting of quasisymmetric functions, we may define

$$\overline{F_\alpha} = \overline{F_{S(\alpha),n}} = F_{\overline{S(\alpha)},n}$$

where for any subset S of $[n-1]$ we let $\overline{S} = [n-1]\backslash S$, and we extend it to *QSym* by linearity. In the example of Figure 11, we get $\overline{F_{312}} = \overline{F_{\{3,4\},6}} = F_{\{1,2,5\},6} = F_{1131}$. The following lemma states that these two bar operations are compatible, and that the latter operation commutes with the product in *QSym*.

Lemma 4.7. *We have:*

(a) $\overline{K_{(P,\omega)}} = K_{\overline{(P,\omega)}}$ *for any labeled poset* (P, ω)*;*
(b) $\overline{fg} = \overline{f}\,\overline{g}$ *for any* f *and* g *in QSym.*

Proof. For the first assertion, observe that the bar operation that sends (P, ω) to $\overline{(P, \omega)}$ on general labeled posets can be done at the level of the labeling ω by simply replacing each label $\omega(i)$ by $|P|+1-\omega(i)$. Then in the notation of the F-expansion of $K_{(P,\omega)}$ in Theorem 2.5, each $\mathrm{des}(\pi)$ is sent to its complement $[n-1]\backslash\mathrm{des}(\pi)$, resulting in $\overline{K_{(P,\omega)}}$, as required.

For the second assertion, recall the interpretation of F_α as $K_{(P,\omega)}$ for a labeled chain (P, ω) from the start of Subsection 4.2, along with the interpretation there of $F_\alpha F_\beta$. Thus (a) implies that $\overline{F_\alpha F_\beta} = \overline{F_\alpha}\ \overline{F_\beta}$, from which (b) follows by linearity. □

4.4. Irreducibility. A crucial fact towards proving Theorem 4.4 is the irreducibility of $K_{(P,\omega)}$ for elements of $\mathcal{C}$.

Proposition 4.8. *If* (P, ω) *is a connected element of* $\mathcal{C}$ *then* $K_{(P,\omega)}$ *is irreducible in QSym.*

To prove this, we first recall the following general property. A polynomial with integer coefficients is said to be *primitive* if whenever an integer k divides all its coefficients we have $k = \pm 1$. In the same vein, since we consider $QSym \subseteq \mathbb{Z}[[x_1, x_2, \ldots]]$, we say $f \in QSym$ is *primitive* if whenever an integer k divides f in *QSym* we have $k = \pm 1$. We are interested in whether $K_{(P,\omega)}$ is primitive.

To answer this question, let us define the *leading term* of a formal power series f expanded in terms of monomials as the term $c_\alpha x^\alpha = c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ with lexicographically largest α such that $c_\alpha \neq 0$; naturally, we call this c_α the *leading coefficient*. For a labeled poset (P, ω), we define $x^{\mathrm{jump}(P,\omega)} = x_1^{j_1} x_2^{j_2} \cdots$ where j_i is the number of elements of (P, ω) of jump $i-1$.

Proposition 4.9 ([23, proof of Proposition 4.2]). *For any labeled poset* (P, ω)*, the leading term of* $K_{(P,\omega)}$ *is* $x^{\mathrm{jump}(P,\omega)}$*. In particular, the leading coefficient of* $K_{(P,\omega)}$ *is* 1 *and* $K_{(P,\omega)}$ *is primitive.*

The next lemma is relevant to the recursive construction of $\mathcal{C}$.

LEMMA 4.10. *If f is a primitive element of QSym then the polynomials $F_1 \uparrow f$, $F_1 \Uparrow f$, $f \uparrow F_1$, and $f \Uparrow F_1$ are irreducible in QSym.*

Proof. Let us first deal with $F_1 \Uparrow f$. Assume that $F_1 \Uparrow f$ is reducible. Note that if f expands as $\sum c_\alpha F_\alpha$ then $F_1 \Uparrow f = \sum c_\alpha F_{\alpha^+}$, where α^+ is obtained from α by appending an entry of 1 at the start. Thus since f is primitive, $F_1 \Uparrow f$ is also primitive, so there exist non-constants $g, g' \in QSym$, such that

$$F_1 \Uparrow f = gg'. \tag{10}$$

All α in the F-support of $F_1 \Uparrow f$ satisfy $\alpha_1 = 1$ but by Lemma 4.5, the F-support of the right-hand side of (10) contains at least one α with $\alpha_1 \geqslant 2$, a contradiction. The case of $f \Uparrow F_1$ is treated similarly, using the last part of α instead of the first part.

Let us now consider $F_1 \uparrow f$. We again assume $F_1 \uparrow f$ is reducible and now α^+ is obtained from α by increasing the first part by 1. The primitivity implies the existence of two non-constants $g, g' \in QSym$, such that $F_1 \uparrow f = gg'$. We apply the bar operator to this equality and get

$$\overline{F_1 \uparrow f} = \overline{gg'} = \overline{g}\overline{g'}.$$

by Lemma 4.7(b). Since the elements α of the F-support of $F_1 \uparrow f$ all satisfy $\alpha_1 \geqslant 2$, we know all α' in the F-support of $\overline{F_1 \uparrow f}$ satsify $\alpha'_1 = 1$. We are thus led to the same contradiction as in the first case. The case of $f \uparrow F_1$ is treated similarly. $\square$

Since we will only be considering labeled posets in $\mathcal{C}$ for the remainder of this section, we will abbreviate (P, ω) as P for easier reading.

Proof of Proposition 4.8. If P has just a single element, then $K_P = F_1$, which is irreducible. Otherwise, since P is connected and constructed recursively, it must be of one of the forms $[1] \uparrow P'$, $[1] \Uparrow P'$, $P' \uparrow [1]$, or $P' \Uparrow [1]$, where $P' \in \mathcal{C}$. The result now follows from Propositions 4.6 and 4.9, and Lemma 4.10. $\square$

REMARK 4.11. A special case of Proposition 4.8, or even just Lemma 4.10, is that F_α is irreducible in *QSym*, as previously shown in [16].

We are now in a position to prove the main result of this section.

Proof of Theorem 4.4. It is obvious that when P and Q are isomorphic, then $K_P = K_Q$. Let us prove the converse.

We shall use induction on the size of P, with the case of size 1 being trivial.

Assuming now that the size of P is at least 2, we may decompose P and Q uniquely into non-empty connected components: $P = \sqcup_{i=1}^r P_i$ and $Q = \sqcup_{i=1}^s Q_i$. We have

$$\prod_{i=1}^r K_{P_i} = \prod_{i=1}^s K_{Q_i}.$$

Since *QSym* is a unique factorization domain [13, 16, 21], the irreducibility of K_{P_i} and K_{Q_i} from Proposition 4.8 implies that $r = s$ and that for every i, we have $K_{P_i} = K_{Q_i}$ (up to a suitable renumbering).

When $r \geqslant 2$, the size of each P_i and Q_i is smaller than the size of P and thus P_i and Q_i are isomorphic for every i by the induction hypothesis. Thus P and Q are also isomorphic.

Suppose now that $r = 1$, i.e. P and Q are connected. Thus, since their size is greater than 1, they may be written as $P = [1] \uparrow P'$ or $P = [1] \Uparrow P'$ or $P = P' \uparrow [1]$ or $P = P' \Uparrow [1]$, and similarly for Q. By Theorem 2.5, we can distinguish among these four possibilities by observing whether the first part of the compositions α in the F-support of K_P (equivalently K_Q) are all equal to 1 or all greater than 1, and similarly for the last part of all such α. Specifically, $P = [1] \uparrow P'$ if and only if $\alpha_1 > 1$ for all α, implying $Q = [1] \uparrow Q'$. Similarly, $P = [1] \Uparrow P'$ if and only if $\alpha_1 = 1$ for all α, implying $Q = [1] \Uparrow Q'$. And $P = P' \uparrow [1]$ (resp. $P = P' \Uparrow [1]$) if and only if the last part of α is greater than 1 (resp. equals 1) for all α, implying $Q = Q' \uparrow [1]$ (resp. $Q = Q' \Uparrow [1]$). So for a given P and Q with $K_P = K_Q$, at least one of the following four properties holds:

- $P = [1] \uparrow P'$ and $Q = [1] \uparrow Q'$, or
- $P = [1] \Uparrow P'$ and $Q = [1] \Uparrow Q'$, or
- $P = P' \uparrow [1]$ and $Q = Q' \uparrow [1]$, or
- $P = P' \Uparrow [1]$ and $Q = Q' \Uparrow [1]$.

In the first case, applying Proposition 4.6 gives $F_1 \uparrow K_{P'} = F_1 \uparrow K_{Q'}$. Let us expand this in the F-basis as $\sum c_\alpha F_\alpha$. From (8) and (6), we see that by just subtracting 1 from the first part α_1 in each term F_α, we get $K_{P'} = K_{Q'}$, with P' of size smaller than P. By induction, P' and Q' are isomorphic, and hence so are P and Q.

In the second case, we have $F_1 \Uparrow K_{P'} = F_1 \Uparrow K_{Q'}$. Let us expand this in the F-basis as $\sum c_\alpha F_\alpha$. From (9) and (7), we see that by just removing the first part $\alpha_1 = 1$ in each term F_α, we get $K_{P'} = K_{Q'}$, with P' of size smaller than P, giving the same conclusion.

The last two cases are resolved similarly by focusing on the last entry of α. □

4.5. A different characterization of $\mathcal{C}$. To end this section, we shall give a characterization of the class $\mathcal{C}$ in terms of forbidden subposets.

We need to distinguish two notions of subposets of labeled posets. For the first, we ignore the labeling and consider the usual notion of induced subposets. A poset (labeled or not) that avoids a set of unlabeled subposets S in this sense is said to be (S)-free. The second notion is of convex labeled subposets, considered as convex subposets with specific assignments of strict and weak edges. A labeled poset that avoids a set of labeled convex subposets S' in this sense is said to be $[S']$-free. Of course these two notions can be used together.

The following result is the analogue in our context of [12, Theorem 4.3].

Proposition 4.12. *Labeled posets in $\mathcal{C}$ are exactly (,)-[,]-free posets.*

The proof is modeled on the proof of [12, Theorem 4.3]. We first recall [12, Lemma 4.4].

Lemma 4.13. *A finite connected (,)-free poset has a unique minimal or maximal element.*

Proof of Proposition 4.12. Let P be an element of $\mathcal{C}$. We shall prove that it is (,)-[,]-free by induction on its size. If P is disconnected then $P = P' \sqcup P''$ with P' and P'' in $\mathcal{C}$ and non-empty. By the induction hypothesis, P' and P'' are (,)-[,]-free, and thus so is P. If P is connected, then by definition of $\mathcal{C}$, P is of the form $[1] \uparrow P'$ or $[1] \Uparrow P'$ or $P' \uparrow [1]$ or $P' \Uparrow [1]$ for some P' in $\mathcal{C}$. We may assume without loss of generality that $P = [1] \uparrow P'$. The induction hypothesis shows that P' is (,)-[,]-free. This implies that P is (,)-free too; otherwise, we would find three elements a, b, c in P' such that the subposet $(\{1, a, b, c\}, \leqslant_P)$ is isomorphic to or , but this is absurd since and do not have a unique minimal element. A similar idea implies that P is [,]-free too; otherwise, we would find two elements a, b in P' such that the convex labeled subposet $(\{1, a, b\}, \leqslant_P)$ is equal to or , but this is impossible by construction.

Let us then prove the converse. The proof is again based on induction on the size of P. If a finite (,)-[,]-free poset P is disconnected, then $P = P' \sqcup P''$ for some non-empty (,)-[,]-free posets P' and P''. The induction hypothesis shows that P' and P'' are in $\mathcal{C}$, and thus so is P. So let us suppose that P is connected. Because of Lemma 4.13, together with the [,]-freeness, we may assume without loss of generality that P is of the form $[1] \uparrow P'$ for some poset P'. Since P' is a convex subposet of P, it is (,)-[,]-free, from which the induction hypothesis implies that P' is in $\mathcal{C}$, and thus so is P.

□

5. Open problems

5.1. Notes on the main conjectures. Resolving any one of Conjectures 1.1, 1.2 and 1.3 would represent a significant advance. Taking them in turn, we offer some further observations.

- Even though our starting point was Conjecture 1.1, the only approach taken here is to consider it in terms of Conjecture 1.2. Perhaps there is a more direct way to tackle the former.
- One difficulty we failed to surmount in tackling Conjecture 1.2 was how to use the fact that the posets under consideration are trees; any methods used cannot apply to the posets in Figure 4.
- A special case of Conjecture 1.3 worthy of consideration is that of binary trees, defined here as rooted trees where every element has exactly 0 or 2 children.

  Referring to Subsection 3.2, one could consider series-parallel posets where we allow a mixture of strict and weak edges. In particular, one could define *fair series-parallel posets* by replacing all appearances of "[1]" in parts (c) and (d) of Definition 4.2 with "(Q, ω')." Does the analogue of Theorem 4.4 hold?

  For any class of posets, labeled or not, we expect the crux of a proof would be the irreducibility, as in Proposition 4.8, [12, Lemma 3.13], and [18, Theorem 4.19]. The general question of the irreducibility of $K_{(P,\omega)}$ appears as [23, Questions 7.2 and 7.3]. The irreducibility of M_α and F_α is shown in [16].

5.2. Quasisymmetric power sum bases. We have given much less consideration to other open problems which we next describe. Liu and Weselcouch's impressive progress in the naturally labeled case [18] depends on the expansion of K_P in the (unnormalized) power sum basis ψ_α of type I and, in particular, the combinatorial interpretation of this expansion due to Alexandersson and Sulzgruber [1]. Can other bases of *QSym* give new insight? Even though the expansion of K_P in other bases might be neither integral nor positive, there could still be interesting combinatorics with appropriate normalization and interpretation of the signs. We did a careful study of the expansion of K_P in the (unnormalized) power sum basis ϕ_α of type II. Here, we are following the notation of Ballantine et al. [5], where extensive information about the bases ψ_α and ϕ_α can be found. In particular, they show that the power sum symmetric function p_λ expands as

$$p_\lambda = \sum_{\alpha\,:\,\widetilde{\alpha}=\lambda} z_\lambda \psi_\alpha = \sum_{\alpha\,:\,\widetilde{\alpha}=\lambda} z_\lambda \phi_\alpha \tag{11}$$

where both sums are over all compositions α whose weakly decreasing reordering is λ. As usual, $z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots k^{m_k} m_k!$ where m_i is the multiplicity of i in λ and where $\lambda_1 = k$. It follows immediately from (11) that when f is a symmetric function, the coefficient of ψ_α in f's ψ-expansion equals the coefficient of ϕ_α in f's ϕ-expansion. We offer the following question: is the converse true? In other words, does the coefficients being equal give a characterization of symmetric functions? We have only anecdotal evidence related to this question and have not given it significant thought.

5.3. Principal specialization. Given a quasisymmetric function $f(\mathbf{x})$ in infinitely many variables $x_1, x_2, \ldots$, we denote by $f(1, q, q^2, \ldots, q^{k-1})$ the element of $\mathbb{Z}[q]$ that results from setting $x_i = q^{i-1}$ for all $i \leqslant k$ and $x_i = 0$ otherwise. This is known as the *principal specialization of order* k of f [30, Sections 7.8, 7.19]. This specialization has a nice interpretation for $K_{(P,\omega)}$: if

$$K_{(P,\omega)}(1, q, q^2, \ldots, q^{k-1}) = \sum_{N \geqslant 0} a(N) q^N,$$

then we see that $a(N)$ counts the number of (P, ω)-partitions $f : P \to \{0, \ldots, k-1\}$ of N. Notice that f is now allowed to map to 0 but cannot map to integers larger than $k-1$, and the sum of $f(p)$ as p ranges over P is N. Similar interpretations apply to the principal specializations of X_G and $X_{\overrightarrow{G}}$.

Recall that Stanley's original conjecture states that $X_G(\mathbf{x})$ distinguishes trees. Clearly $X_G(1, q, q^2, \ldots, q^{k-1})$ contains far less information than $X_G(\mathbf{x})$, making the following recent conjecture of Loehr and Warrington surprising.

Conjecture 5.1 ([20]). $X_G(1, q, q^2, \ldots, q^{n-1})$ *distinguishes trees with at most* n *vertices.*

Certainly an affirmative answer to Loehr and Warrington's conjecture would prove Stanley's conjecture.

Inspired by this, we offer the following conjecture which should be compared with Conjecture 1.2.

Conjecture 5.2. $\overline{K}_P(1, q, q^2, \ldots, q^{n-1})$ *distinguishes posets with* n *elements that are trees.*

We have verified this conjecture for all such posets with at most 10 elements. In fact, it seems that $\overline{K}_P(1, q, q^2, \ldots, q^{n-2})$ suffices when the posets have n elements.

Conjecture 5.3. $\overline{K}_P(1, q, q^2, \ldots, q^{n-2})$ *distinguishes posets with* n *elements that are trees.*

It is not obvious to us whether Conjecture 5.3 being true would imply Conjecture 5.2.

Example 5.4. Referring to Figure 5, the labelel poset P on the left satisfies

$$\overline{K}_P(1, q, q^2, q^3) = q^4 + 2q^5 + 4q^6 + 4q^7 + 5q^8 + 4q^9 + 2q^{10} + q^{11}$$

whereas its dual P^* on the right satisfies

$$\overline{K}_{P^*}(1, q, q^2, q^3) = q^4 + 2q^5 + 4q^6 + 5q^7 + 4q^8 + 4q^9 + 2q^{10} + q^{11}.$$

So, even though their F-supports do not distinguish them, their principal specialization of order 4 does.

This example also exhibits how the principal specialization of $\overline{K}_P$ of order k compares to that of its dual P^*. One can check that for general P with n elements, the coefficient of q^N in $\overline{K}_P(1, q, q^2, \ldots, q^{k-1})$ equals the coefficient of $q^{n(k-1)-N}$ in $\overline{K}_{P^*}(1, q, q^2, \ldots, q^{k-1})$.

However, $\overline{K}_P(1, q, q^2, \ldots, q^{k-1})$ for $k - 1 < n - 2$ does not distinguish posets with n elements that are trees because it will evaluate to 0 for any posets containing a chain of length $n - 2$. This begs the question of what happens if we consider posets with all weak edges. Unlike in the case of $\overline{K}_P(\mathbf{x})$ where $\overline{K}_P(\mathbf{x})$ can be obtained from $K_P(\mathbf{x})$ and vice versa, the same is not true for their principal specializations. For example, $K_P(1, q, q^2)$ suffices to distinguish posets with 7 elements that are trees. It is not clear for general n which values of k are sufficient for $K_P(1, q, \ldots, q^{k-1})$ to distinguish posets with n elements that are trees. Our computations are consistent with Conjectures 5.2 and 5.3 remaining true when $\overline{K}_P$ is replaced by K_P.

Finally we note that the principal specialization version of Conjecture 1.3 is false: F_{131} and F_{212} correspond to $K_{(P,\omega)}(\mathbf{x})$ for different chains with 5 elements but

$$F_{131}(1, q, q^2, q^3, q^4) = F_{212}(1, q, q^2, q^3, q^4).$$

5.4. Another quasisymmetric version of Stanley's conjecture. We close with another question which is wide open and again brings us all the way back to Stanley's original consideration of $X_G(\mathbf{x})$ distinguishing (undirected) graphs and Conjecture 1.1. We ask the following ill-defined question: does $X_{\overrightarrow{G}}(\mathbf{x}, t)$ distinguish *undirected* trees? Here is one concrete way to make this question make sense.

Question 5.5. *Given an undirected tree G, construct the multiset $\{X_{\overrightarrow{G}}(\mathbf{x}, t)\}_{\overrightarrow{G}}$ as $\overrightarrow{G}$ varies over all orientations of G. Do the same for a different undirected tree H. Are the multisets for G and H always different?*

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