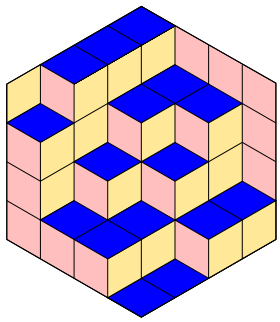


# ALGEBRAIC COMBINATORICS



Henning Krause


**The category of finite strings**


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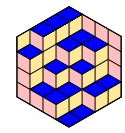

# The category of finite strings

Henning Krause

Abstract We introduce the category of finite strings and study its basic properties. The category is closely related to the augmented simplex category, and it models categories of linear representations. Each lattice of non-crossing partitions arises naturally as a lattice of subobjects.

## 1. Introduction

Strings are considered to be one of the most basic combinatorial structures arising in representation theory of associative algebras. In fact, many of the interactions with neighbouring fields involve strings and their corresponding representations, which are also known as string modules.

In this note we introduce a category of finite strings and establish some connections. First of all, we notice that the category of connected strings is equivalent to the augmented simplex category $\Delta$ (cf. [8, 17]), once the initial and terminal objects in $\Delta$ are identified. Then we show that the category of finite strings models categories of linear representations. More precisely, we provide an equivalence between finite strings and certain abelian categories (hereditary and uniserial length categories with only finitely many simple objects and split over a fixed field, cf. [1]), where morphisms between strings correspond to certain exact functors. In this context it is appropriate to include cyclic strings which correspond to abelian categories of infinite representation type. This is somewhat parallel to the cyclic category of Connes and others [3, 4]; however we add new objects (cyclic strings) while the cyclic category keeps the objects of $\Delta$ and only morphisms are added.

Any morphism in the category of finite strings admits an epi-mono factorisation. Thus it is of interest to study the subobjects of a given object, at least for any connected string. We show for a linear $n$-string that the lattice of subobjects is isomorphic to the lattice $\mathrm{NC}(n+1)$ of non-crossing partitions [16], while the lattice $\mathrm{NC}^B(n)$ of type $B$ non-crossing partitions [19] arises for a cyclic $n$-string.

The correspondence between strings and categories of linear representations identifies subobjects of strings with thick subcategories of abelian categories. In this way we recover the beautiful classification of thick subcategories for quiver representations of type $A$ due to Ingalls and Thomas [13], and we add a classification for nilpotent representations of cyclic quivers which seems to be new. The cyclic case can be used to complete the classification of all thick subcategories for representations of any

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tame hereditary algebra, including the ones that are not generated by exceptional sequences, and therefore complementing the work in [11, 12, 13]. We point out that the category of strings can be extended to include all Dynkin types, beyond the type $A$ in this work, and analogous to the categorification of non-crossing partitions for all Dynkin types in [11].

Finally, let us mention the connection with some recent work which is concerned with Iyama's higher Auslander algebras of type $A$ [14]. These algebras form a natural generalisation of the hereditary algebras of type $A$ arising in the present work. In [7] the authors point out the simplicial structure of the representations for these higher Auslander algebras, using some advanced categorical formalism. Wide subcategories of representations generalise thick subcategories and these are studied for type $A$ higher Auslander algebras in [10].

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## 2. Connected strings

In this work we introduce the category of finite strings. For any natural number $n$ the connected $n$-string is denoted by $\Sigma_n$. Each string comes equipped with its set of (connected) substrings, together with a multiplication on the set of substrings given by concatenation. The objects of the category are finite coproducts of connected strings, and the morphisms are maps that preserve substrings and their multiplication.

A *basic string* is a pair $s = (s', s'')$ of integers $s' \leqslant s''$. We write $\ell(s) = s'' - s' + 1$ for the *length* of $s$ and add a *zero string* $*$ satisfying $\ell(*) = 0$. Strings of length one are called *simple* and we set $s_i := (i, i)$, $i \in \mathbb{Z}$. For a string $s = (i, j)$ we call the simple strings $s_i, s_{i+1}, \dots, s_j$ its *composition factors*.

A multiplication of basic strings is given by concatenation. For $s, t$ set

$$st := \begin{cases} (s', t'') & \text{for } s = (s', s''),\ t = (t', t''),\ s'' + 1 = t', \\ t & \text{for } s = *, \\ s & \text{for } t = *, \end{cases}$$

and the product $st$ is undefined otherwise. This multiplication is associative.

For $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ the *connected $n$-string* is by definition the set of basic strings

$$\Sigma_n := \{s = (s', s'') \mid 0 \leqslant s' \leqslant s'' < n\} \cup \{*\}.$$

A morphism $\phi\colon \Sigma_m \to \Sigma_n$ is by definition a map such that for all $s, t \in \Sigma_m$

$$\phi(st) = \phi(s)\phi(t) \qquad \text{and} \qquad \phi(*) = *. \tag{1}$$

The string $*$ plays the role of a base point, and morphism are base point preserving. Any morphism is determined by the images of the simple strings but it need not preserve the length of basic strings.

We define *standard morphisms* as follows. Let $n \geqslant 1$. The morphism

$$\delta_n^i\colon \Sigma_{n-1} \longrightarrow \Sigma_n \qquad (0 \leqslant i \leqslant n)$$

is given by the unique injective map such that $s_{i-1}$ and $s_i$ are not in its image. Note that $\delta_1^0 = \delta_1^1$. The morphism

$$\sigma_n^i\colon \Sigma_n \longrightarrow \Sigma_{n-1} \qquad (0 \leqslant i < n)$$

is given by the unique surjective map that sends $s_i$ to the zero string.

The standard morphisms are analogues of the face and degeneracy maps for simplices. In fact, they satisfy the following *simplicial identities* [17, VII.5].

LEMMA 2.1. *The standard morphisms satisfy the following identities:*

$$\begin{aligned} \delta^i_{n+1}\circ\delta^j_n &= \delta^{j+1}_{n+1}\circ\delta^i_n && i\leqslant j\\ \sigma^j_n\circ\sigma^i_{n+1} &= \sigma^i_n\circ\sigma^{j+1}_{n+1} && i\leqslant j\\ \sigma^j_n\circ\delta^i_n &= \begin{cases}\delta^i_n\circ\sigma^{j-1}_n & i<j\\ \mathrm{id} & i=j \text{ or } i=j+1\\ \delta^{i-1}_n\circ\sigma^j_n & i>j+1\end{cases}\end{aligned} \tag{2}$$

*Proof.* This is easily checked, for instance by tracing the images of the simple strings. We have

$$\delta^i_n(s_j)=\begin{cases}s_j & j<i-1\\ (j-1,j) & j=i-1\\ s_{j+1} & j>i-1\end{cases}\qquad\text{and}\qquad \sigma^i_n(s_j)=\begin{cases}s_j & j<i\\ * & j=i\\ s_{j-1} & j>i.\end{cases} \tag{3}$$

It remains to note that any morphism $\phi$ is determined by the images $\phi(s_j)$. □

Small examples are easily computed. We have

$$\Sigma_0=\{*\}\qquad \Sigma_1=\{*,(0,0)\}\qquad \Sigma_0=\{*,(0,0),(1,1),(0,1)\}.$$

Note that for any $n\in\mathbb{N}$ the maps $\delta^0_n$ and $\delta^n_n$ are special; they preserve the length of basic strings since

$$\delta^0_n(s_j)=s_{j+1}\quad\text{and}\quad \delta^n_n(s_j)=s_j\qquad (0\leqslant j<n).$$

We denote by $\Sigma$ the *category of connected strings* with objects given by the strings $\Sigma_n$, $n\in\mathbb{N}$.

Any morphism can be written in some canonical form. First observe that there is a canonical epi-mono factorisation.

LEMMA 2.2. *Let $\phi\colon\Sigma_m\to\Sigma_n$ be a morphisms. Let $0\leqslant j_v<\cdots<j_0<m$ be the indices $j$ such that $\phi(s_j)=*$ and set $\phi'=\sigma^{j_v}_{m-v}\circ\cdots\circ\sigma^{j_1}_{m-1}\circ\sigma^{j_0}_m$. Then there is a factorisation $\phi=\phi''\circ\phi'$ such that $\phi''$ is injective.*

*Proof.* When $\phi(s_i)=*$ for some simple string $s_i$, then $\phi(ss_i)=\phi(s)$ and $\phi(s_it)=\phi(t)$ for all $s,t\in\Sigma_m$. This yields the factorisation $\phi=\phi''\circ\phi'$. Given strings $s,t$ such that $\phi''(s)=\phi''(t)$, an induction on their length shows that $s=t$. □

LEMMA 2.3. *Every morphism $\phi\colon\Sigma_m\to\Sigma_n$ can be written uniquely as composite*

$$\phi=\delta^{i_u}_n\circ\delta^{i_{u-1}}_{n-1}\circ\cdots\circ\delta^{i_0}_{n-u}\circ\sigma^{j_v}_{m-v}\circ\cdots\circ\sigma^{j_1}_{m-1}\circ\sigma^{j_0}_m \tag{4}$$

*with $0\leqslant i_0<\cdots<i_u\leqslant n$, $0\leqslant j_v<\cdots<j_0<m$, and $n-u=m-v$.*[(1)]

We call (4) the *canonical decomposition* of $\phi$ in $\Sigma$.

*Proof.* Let $0\leqslant j_v<\cdots<j_0<m$ be the indices $j$ such that $\phi(s_j)=*$. And let $0\leqslant i_0<\cdots<i_u\leqslant n$ be the indices such that for all $(s',s'')\in\Sigma_n$ in the image of $\phi$ we have $s',s''+1\notin\{i_0,\dots,i_u\}$. Then it is clear that $\phi$ satisfies (4). Conversely, if $\phi$ is written as in (4), then the indices $i_0,\cdots,i_u$ and $j_0,\cdots,j_v$ are characterised as above. □

LEMMA 2.4. *The category $\Sigma$ is generated by the objects $\Sigma_n$, the morphisms $\delta^i_n$, $\sigma^j_m$, plus the relations $\delta^0_1=\delta^1_1$ and (2).*

[(1)] We need to exclude $\delta^0_1$ as a factor and choose instead $\delta^1_1$ in order to achieve uniqueness.

*Proof.* Let us denote by $\Sigma'$ the category generated by the objects $\Sigma_n$, the morphisms $\delta_n^i$, $\sigma_m^j$, plus the relations $\delta_1^0 = \delta_1^1$ and (2). Since the relations are satisfied in $\Sigma$, there is a unique functor $\Sigma' \to \Sigma$ which induces the identity on the objects and on the morphisms $\delta_n^i$ and $\sigma_m^j$. Since every morphism in $\Sigma$ is a composite of morphisms $\delta_n^i$ and $\sigma_m^j$, the induced map $\mathrm{Hom}_{\Sigma'}(\Sigma_m, \Sigma_n) \to \mathrm{Hom}_{\Sigma}(\Sigma_m, \Sigma_n)$ is surjective for all $m, n$. To show injectivity, choose $\phi, \psi$ in $\mathrm{Hom}_{\Sigma'}(\Sigma_m, \Sigma_n)$ with same image in $\Sigma$. Since the relations $\delta_1^0 = \delta_1^1$ and (2) in $\Sigma'$ are satisfied, there are decompositions (4) in $\Sigma'$ for $\phi$ and $\psi$. These decompositions coincide in $\Sigma$ by Lemma 2.3, and therefore $\phi = \psi$. □

Example 2.5. For any $n \geqslant 3$ we have the following pullback in $\Sigma$.

$$\begin{array}{ccc} \Sigma_n & \xrightarrow{\sigma_n^0} & \Sigma_{n-1} \\ \downarrow{\scriptstyle \sigma_n^{n-1}} & & \downarrow{\scriptstyle \sigma_{n-1}^{n-2}} \\ \Sigma_{n-1} & \xrightarrow{\sigma_{n-1}^0} & \Sigma_{n-2} \end{array}$$

## 3. The simplicial category

Let $\Delta$ denote the *simplicial category*, which is also known as *augmented simplex category* (terminology and notation follows [17, §VII.5]). The objects are given by the finite ordinals $[n] = \{0, 1, \dots, n-1\}$, $n \in \mathbb{N}$, and the morphisms $\phi\colon [m] \to [n]$ are given by maps satisfying $\phi(i) \leqslant \phi(j)$ for all $0 \leqslant i \leqslant j < m$. For $n \geqslant 0$ there is the *face map*

$$\bar{\delta}_n^i\colon [n] \longrightarrow [n+1] \qquad (0 \leqslant i \leqslant n)$$

(the unique injective map not taking the value $i$) and for $n \geqslant 1$ the *degeneracy map*

$$\bar{\sigma}_n^i\colon [n+1] \longrightarrow [n] \qquad (0 \leqslant i < n)$$

(the unique surjective map taking twice the value $i$) which are known to satisfy the simplicial identities (2). In fact, the category $\Delta$ is generated by the objects $[n]$, the morphisms $\bar{\delta}_n^i$, $\bar{\sigma}_m^j$, and the identities (2); see [8, §II.2] or [17, §VII.5].

We write $\bar{\Delta} = \Delta[\alpha^{-1}]$ for the category which is obtained by formally inverting the morphism $\alpha = \bar{\delta}_0^0$. This amounts to identifying the initial and the terminal object in $\Delta$.

Proposition 3.1. *The assignments*

$$[n] \mapsto \begin{cases} \Sigma_0 & n = 0 \\ \Sigma_{n-1} & n > 0, \end{cases} \qquad \bar{\delta}_n^i \mapsto \delta_n^i, \qquad \bar{\delta}_0^0 \mapsto \mathrm{id}, \qquad \bar{\sigma}_n^i \mapsto \sigma_n^i$$

*provide a functor $p\colon \Delta \to \Sigma$ which induces an equivalence $\bar{\Delta} \xrightarrow{\sim} \Sigma$.*

*Proof.* The functor $p$ is well defined since it maps generators to generators and the simplicial identities are satisfied in both categories. The functor $p$ inverts $\bar{\delta}_0^0$ and induces therefore a functor $\bar{\Delta} \to \Sigma$, which yields a bijection between the isomorphism classes of objects. Also for the morphisms we obtain bijections since the functor matches generators and relations. Note that $\bar{\delta}_1^0 = \bar{\delta}_1^1$ in $\bar{\Delta}$ since $\bar{\delta}_0^0$ is invertible. □

The functor $p\colon \Delta \to \Sigma$ admits two sections $s_0$ and $s_1$ that are given by

$$\Sigma_n \mapsto [n+1], \qquad \delta_n^i \mapsto \bar{\delta}_n^i, \qquad \sigma_n^i \mapsto \bar{\sigma}_n^i,$$

except that $s_i(\delta_1^0) = \bar{\delta}_1^i = s_i(\delta_1^1)$ for $i = 0, 1$.

Let us summarise. We have for all $n \geqslant 1$ diagrams

$$\Sigma_{n-1} \;\begin{array}{c} \xrightarrow{\delta_n^{i+1}} \\ \xleftarrow{\sigma_n^i} \\ \xrightarrow[\delta_n^i]{} \end{array}\; \Sigma_n \qquad (0 \leqslant i < n)$$

satisfying the simplicial identities, but a difference from the usual simplex category arises because of the extra identity $\delta_1^0 = \delta_1^1$.

The equivalence in Proposition 3.1 can be explained in terms of linear representations of posets. We refer to Theorem 4.9 and the appendix for further details.

## 4. Representations

The category of finite strings models certain categories of linear representations. In the following we specify the relevant class of abelian categories and the exact functors between them.

Let $P$ be a poset and $k$ a field. A $k$-linear representation of $P$ is by definition a functor $P \to \operatorname{mod} k$ into the category of finite dimensional $k$-spaces, where $P$ is viewed as a category (with objects the elements in $P$ and a unique morphism $x \to y$ if and only if $x \leqslant y$). Morphisms between representations are the natural transformations, and we denote by $\operatorname{Rep}(P, k)$ the category of all finite dimensional $k$-linear representations.

For a $k$-linear abelian category $\mathcal{A}$ over a field $k$ we consider the following conditions.

(Ab1) $\mathcal{A}$ is *connected*, that is, $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ implies $\mathcal{A}_1 = 0$ or $\mathcal{A}_2 = 0$.
(Ab2) $\mathcal{A}$ is a *length category*, that is, every object has a finite composition series, and there are only finitely many isomorphism classes of simple objects.
(Ab3) $\mathcal{A}$ is *hereditary*, that is, $\operatorname{Ext}^2$ vanishes.
(Ab4) $\mathcal{A}$ is *uniserial*, that is, every indecomposable object has a unique composition series.
(Ab5) $\mathcal{A}$ is *split*, that is, $\operatorname{End}(S) \cong k$ for every simple object $S$.
(Ab6) $\mathcal{A}$ is of *finite type*, that is, there are only finitely many isomorphism classes of indecomposable objects.

Lemma 4.1. *Let $\mathcal{A}$ be a $k$-linear abelian category satisfying* (Ab1)–(Ab6)*. Then there is an equivalence $\mathcal{A} \xrightarrow{\sim} \operatorname{Rep}([n]^{\mathrm{op}}, k)$, where $n$ equals the number of isomorphism classes of simple objects in $\mathcal{A}$.*

*Proof.* See for example the description of uniserial categories in [1]. □

From now on we fix a field $k$ and set $\mathcal{A}_n := \operatorname{Rep}([n]^{\mathrm{op}}, k)$ for $n \in \mathbb{N}$. An object $M \in \mathcal{A}_n$ is a diagram

$$M(n-1) \longrightarrow \cdots \longrightarrow M(1) \longrightarrow M(0)$$

of $k$-spaces. For any string $s \in \Sigma_n$ we define a representation $M_s \in \mathcal{A}_n$ as follows. Set $M_* = 0$. For $s = (s', s'')$ let $M_s$ be the representation[(2)]

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow k \xrightarrow{1} \cdots \xrightarrow{1} k \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

such that $M_s(i) = k$ if and only if $s' \leqslant i \leqslant s''$. Set $M_i := M_{s_i}$ for $0 \leqslant i < n$. Observe that the composition length of $M_s$ equals $\ell(s)$, and the composition factors of $M_s$ correspond bijectively to the composition factors of $s$.

Lemma 4.2. *Let $n \in \mathbb{N}$.*

(1) *The assignment $s \mapsto M_s$ induces a bijection between $\Sigma_n \setminus \{*\}$ and the isomorphism classes of indecomposable objects in $\mathcal{A}_n$.*
(2) *For $s, t, u \in \Sigma_n$ there is an exact sequence $0 \to M_s \to M_t \to M_u \to 0$ if and only if $t = su$.*

*Proof.* Straightforward. □

[(2)] This is also known as *string module* in the terminology of [2].

Next we specify the class of exact functors which arises naturally in our context. For each exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories we denote by $\operatorname{Ker} F$ the full subcategory of $\mathcal{A}$ given by the objects $X \in \mathcal{A}$ such that $FX = 0$. This is a Serre subcategory and we denote by $\mathcal{A}/(\operatorname{Ker} F)$ the corresponding quotient, cf. [9].

We say that an exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories admits a *homological factorisation* if the induced functor $\mathcal{A}/\mathcal{A}' \to \mathcal{B}$ with $\mathcal{A}' = \operatorname{Ker} F$ induces for all objects $X, Y \in \mathcal{A}$ bijections

$$\operatorname{Ext}^i_{\mathcal{A}/\mathcal{A}'}(X,Y) \longrightarrow \operatorname{Ext}^i_{\mathcal{B}}(FX,FY) \qquad (i \geqslant 0).$$

A full subcategory of an abelian category is *thick* if it is closed under direct summands and the *two out of three property* holds for any short exact sequence (that is, if two terms belong to the subcategory, then also the third).

Lemma 4.3. *An exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between hereditary abelian categories admits a* homological factorisation *if and only if $\mathcal{A}/(\operatorname{Ker} F)$ identifies with a thick subcategory of $\mathcal{B}$.*

*Proof.* Set $\mathcal{A}' = \operatorname{Ker} F$ and suppose $F$ identifies $\mathcal{A}/\mathcal{A}'$ with a full subcategory $\mathcal{B}' \subseteq \mathcal{B}$. Clearly, $\mathcal{B}'$ is closed under kernels and cokernels of morphisms since $F$ is exact. The subcategory $\mathcal{B}'$ is extension closed if and only if the induced map $\operatorname{Ext}^1_{\mathcal{A}/\mathcal{A}'}(X,Y) \to \operatorname{Ext}^1_{\mathcal{B}}(FX,FY)$ is a bijection for all $X, Y \in \mathcal{A}$. □

Not all exact functors admit a homological factorisation. A simple example is for any field $k$ the exact functor $\operatorname{mod} k \to \operatorname{mod} k$ given by $X \mapsto X \otimes_k k^2$.

For $m, n \in \mathbb{N}$ we denote by $\operatorname{Hom}(\mathcal{A}_m, \mathcal{A}_n)$ the set of $k$-linear exact functors $\mathcal{A}_m \to \mathcal{A}_n$, up to natural isomorphism, that admit a homological factorisation. We define natural maps

$$\operatorname{Hom}(\mathcal{A}_m, \mathcal{A}_n) \xrightarrow{\alpha_{mn}} \operatorname{Hom}(\Sigma_m, \Sigma_n) \quad \text{and} \quad \operatorname{Hom}(\Sigma_m, \Sigma_n) \xrightarrow{\beta_{mn}} \operatorname{Hom}(\mathcal{A}_m, \mathcal{A}_n)$$

as follows.

Any morphism $\phi\colon [m] \to [n]$ induces an exact functor $\phi^*\colon \mathcal{A}_n \to \mathcal{A}_m$ via precomposition. Let us set $s^i_n := (\bar{\delta}^i_{n-1})^*$ for $0 \leqslant i < n$.

Lemma 4.4. *Let $n \geqslant 1$. There are canonical recollements of abelian categories*

$$\mathcal{A}_1 \;\; \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} \;\; \mathcal{A}_n \;\; \substack{\overset{d^{i+1}_n}{\longleftarrow}\\ \overset{s^i_n}{\longrightarrow}\\ \underset{d^i_n}{\longleftarrow}} \;\; \mathcal{A}_{n-1} \qquad (0 \leqslant i < n)$$

*such that*

$$\operatorname{Ker} s^i_n = \operatorname{add} M_i, \qquad \operatorname{Im} d^i_n = {M_i}^{\perp}, \qquad \operatorname{Im} d^{i+1}_n = {}^{\perp}M_i.$$

*The functors $s^i_n$, $d^i_n$, $d^{i+1}_n$ are exact. Moreover, they send indecomposable objects to indecomposable objects or to zero.*

*Proof.* Let $\mathcal{C} = \operatorname{Ker} s^i_n$ denote the full subcategory of objects in $\mathcal{A}_n$ that are annihilated by $s^i_n$. It is clear that $M_i$ is the unique simple object in $\mathcal{C}$. Thus $\mathcal{C}$ equals the full subcategory given by the finite direct sums of copies of $M_i$. Then the right adjoint of the quotient functor $\mathcal{A}_n \to \mathcal{A}_n/\mathcal{C}$ identifies $\mathcal{A}_n/\mathcal{C}$ with $\mathcal{C}^{\perp}$, while the left adjoint identifies $\mathcal{A}_n/\mathcal{C}$ with ${}^{\perp}\mathcal{C}$. Here, we consider the perpendicular categories defined with respect to Hom and $\operatorname{Ext}^1$; cf. [9, §III.2]. This yields the descriptions of $d^i_n$ and $d^{i+1}_n$. The embedding of any perpendicular category into $\mathcal{A}_n$ is exact since $\operatorname{Ext}^2$ vanishes. The functor $s^i_n$ annihilates $M_i$ and sends all other indecomposables to indecomposable objects. □

For $n \geqslant 1$ we set

$$\beta_{n,n-1}(\sigma_n^i) \coloneqq s_n^i \qquad \text{and} \qquad \beta_{n-1,n}(\delta_n^i) \coloneqq d_n^i.$$

One checks that these functors satisfy the identities (2). Thus the assignment extends uniquely to maps $\beta_{mn}\colon \operatorname{Hom}(\Sigma_m, \Sigma_n) \to \operatorname{Hom}(\mathcal{A}_m, \mathcal{A}_n)$ for all $m, n \in \mathbb{N}$, using Lemma 2.4.

REMARK 4.5. We have $d_n^i = (\bar{\sigma}_{n-1}^{i-1})^*$ for $0 < i < n$. Thus $d_n^0$ and $d_n^n$ are not obtained from morphisms $[n-1] \to [n]$.

LEMMA 4.6. *Let $m, n \in \mathbb{N}$. An exact functor $F\colon \mathcal{A}_m \to \mathcal{A}_n$ that admits a homological factorisation induces a morphism $\phi\colon \Sigma_m \to \Sigma_n$ which is given by $F(M_s) = M_{\phi(s)}$.*

*Proof.* The functor $F$ identifies $\mathcal{A}_m/(\operatorname{Ker} F)$ with a full subcategory of $\mathcal{A}_n$. The canonical functor $\mathcal{A}_m \to \mathcal{A}_m/(\operatorname{Ker} F)$ can be written as composite of functors of the form $s_p^i\colon \mathcal{A}_p \to \mathcal{A}_{p-1}$, which map indecomposables either to indecomposables or to zero. Thus for any $s \in \Sigma_m$ we have $F(M_s) = M_t$ for some $t \in \Sigma_n$, using Lemma 4.2. This yields a morphism $\phi\colon \Sigma_m \to \Sigma_n$ by setting $\phi(s) = t$. □

The above lemma provides maps $\alpha_{mn}\colon \operatorname{Hom}(\mathcal{A}_m, \mathcal{A}_n) \to \operatorname{Hom}(\Sigma_m, \Sigma_n)$ satisfying $\alpha_{nn}(\mathrm{id}) = \mathrm{id}$ and $\alpha_{mp}(G \circ F) = \alpha_{np}(G) \circ \alpha_{mn}(F)$ for any pair of composable functors $F, G$.

LEMMA 4.7. *A $k$-linear equivalence $\mathcal{A}_n \xrightarrow{\sim} \mathcal{A}_n$ is naturally isomorphic to the identity.*

*Proof.* The category $\mathcal{A}_n$ is *standard*, that is, equivalent to the mesh category given by its Auslander–Reiten quiver [20, §2.4]. Clearly, an equivalence induces the identity on the Auslander–Reiten quiver and preserves the mesh ideal. From this the assertion follows. □

LEMMA 4.8. *Let $m, n \in \mathbb{N}$. Then $\beta_{mn} \circ \alpha_{mn} = \mathrm{id}$ and $\alpha_{mn} \circ \beta_{mn} = \mathrm{id}$.*

*Proof.* The identity $\alpha_{mn} \circ \beta_{mn} = \mathrm{id}$ is clear since this can be checked on the standard morphisms, thanks to Lemma 2.4. We consider only $k$-linear exact functors $F\colon \mathcal{A}_m \to \mathcal{A}_n$ that admit a homological factorisation. Such functors are determined, up to natural isomorphism, by the values $F(M_s)$ of the indecomposable objects; see Lemma 4.7. Thus $\alpha_{mn}$ is injective and $\beta_{mn} \circ \alpha_{mn} = \mathrm{id}$ follows. □

Combining the above lemmas yields a combinatorial description of the abelian categories that are specified in Lemma 4.1.

THEOREM 4.9. *Let $k$ be a field. The assignment $\Sigma_n \mapsto \mathcal{A}_n$ provides an equivalence between the category of connected strings and the category of $k$-linear abelian categories satisfying* (Ab1)–(Ab6) *(with morphisms given by $k$-linear exact functors admitting a homological factorisation).*

We refer to the appendix for some further explanation of this result. In view of Proposition 3.1 we have the following consequence.

COROLLARY 4.10. *Let $k$ be a field. The assignment $[n] \mapsto \mathcal{A}_{n-1}$ for $n > 0$ provides an equivalence between the augmented simplex category $\bar{\Delta}$ (with the morphism $[0] \to [1]$ inverted) and the category of $k$-linear abelian categories satisfying* (Ab1)–(Ab6).

## 5. FINITE COPRODUCTS

For a finite set of natural numbers $n_\alpha \in \mathbb{N}$ we define the coproduct $\coprod_\alpha \Sigma_{n_\alpha}$ of strings by taking from the product of the underlying sets all elements $s = (s_\alpha)$ such that

$s_\alpha \neq *$ for at most one index $\alpha$ (that is, the coproduct of the pointed sets $\Sigma_{n_\alpha}$). For $s = (s_\alpha)$ and $t = (t_\alpha)$ in $\coprod_\alpha \Sigma_{n_\alpha}$ set

$$st \coloneqq (s_\alpha t_\alpha).$$

For each index $\alpha$ and $0 \leqslant i < n_\alpha$ we denote by $s_{\alpha,i}$ the *simple string* $s$ given by $s_\alpha = s_i$.

Each coproduct $\coprod_\alpha \Sigma_{n_\alpha}$ comes with canonical inclusions $i_\alpha \colon \Sigma_{n_\alpha} \to \coprod_\alpha \Sigma_{n_\alpha}$ and projections $p_\alpha \colon \coprod_\alpha \Sigma_{n_\alpha} \to \Sigma_{n_\alpha}$ satisfying $p_\alpha \circ i_\alpha = \mathrm{id}$.

Morphisms $\coprod_\alpha \Sigma_{m_\alpha} \to \coprod_\beta \Sigma_{n_\beta}$ are by definition maps $\phi$ between the underlying sets such that the composite $p_\beta \circ \phi \circ i_\alpha$ is a morphism $\Sigma_{m_\alpha} \to \Sigma_{n_\beta}$ for all $\alpha, \beta$.

Lemma 5.1. *There are canonical isomorphisms of pointed sets*

$$\mathrm{Hom}\Big(\coprod_\alpha \Sigma_{m_\alpha}, \coprod_\beta \Sigma_{n_\beta}\Big) \xrightarrow{\sim} \prod_\alpha \mathrm{Hom}\Big(\Sigma_{m_\alpha}, \coprod_\beta \Sigma_{n_\beta}\Big) \xleftarrow{\sim} \prod_\alpha \coprod_\beta \mathrm{Hom}(\Sigma_{m_\alpha}, \Sigma_{n_\beta}).$$

*Proof.* Isomorphisms of pointed sets are nothing but bijections, but it is important to take (co)products of pointed sets. The first bijection is induced by the canonical inclusions $\Sigma_{m_\alpha} \to \coprod_\alpha \Sigma_{m_\alpha}$. The second bijection uses the fact that each morphism $\Sigma_{m_\alpha} \to \coprod_\beta \Sigma_{n_\beta}$ factors through the inclusion $\Sigma_{n_\beta} \to \coprod_\beta \Sigma_{n_\beta}$ for one index $\beta$. □

We obtain the *category of finite strings* which has as objects the finite coproducts of connected strings.[(3)]

## 6. Non-crossing partitions

We wish to describe the subobjects of $\Sigma_n$ in the category of finite strings. This requires some preparations.

Let $S \subseteq \Sigma_n$. We call $S$ *thick* if $* \in S$ and for any pair $s, t \in S$ of non-zero strings we have $st \in S$, and moreover $(s', t'-1), (t', s''), (s''+1, t'') \in S$ provided that $s' \leqslant t' \leqslant s'' \leqslant t''$. We denote by $\mathrm{Thick}(S)$ the smallest thick subset of $\Sigma_n$ containing $S$.

A set $S \subseteq \Sigma_n$ of non-zero strings is called *non-crossing* provided that $s, t \in S$ and $s' \leqslant t' \leqslant s'' \leqslant t''$ implies $s = t$.

Lemma 6.1. *The assignment $S \mapsto \mathrm{Thick}(S)$ gives a bijection between the non-crossing subsets and the thick subsets of $\Sigma_n$.*

*Proof.* The inverse map takes a thick subset $T \subseteq \Sigma_n$ to the unique non-crossing subset $S \subseteq T$ with $\mathrm{Thick}(S) = T$. □

For non-crossing subsets $S, S'$ of $\Sigma_n$ we set

$$S \leqslant S' \quad :\Longleftrightarrow \quad \mathrm{Thick}(S) \subseteq \mathrm{Thick}(S').$$

This yields the structure of a poset. In fact, the non-crossing subsets form a lattice since the thick subsets of $\Sigma_n$ are closed under intersections. We denote this lattice by $\mathrm{NC}(\Sigma_n)$.

Let $n \in \mathbb{N}$. A *partition* $P = (P_\alpha)$ of $[n]$ is given by pairwise disjoint non-empty subsets $P_\alpha$ of $[n]$ such that $\bigcup_\alpha P_\alpha = [n]$. Each partition is determined by the corresponding set of strings $S(P) \subseteq \Sigma_{n-1}$, where by definition $s = (s', s'') \in S(P)$ if for some $\alpha$ we have $s', s'' \in P_\alpha$ and $i \notin P_\alpha$ for all $s' < i \leqslant s''$. This is clear since any part $P_\alpha = \{a_1 < a_2 < \cdots < a_r\}$ is determined by the corresponding set of strings $S_\alpha = \{(a_1, a_2 - 1), \dots, (a_{r-1}, a_r - 1)\}$.

[(3)] We may consider the category $\mathcal{H}om(\Sigma^{\mathrm{op}}, \mathrm{Set}_*)$ of functors $\Sigma^{\mathrm{op}} \to \mathrm{Set}_*$ into the category of pointed sets, which is the analogue of the category $\mathcal{H}om(\Delta^{\mathrm{op}}, \mathrm{Set})$ of simplicial sets. Then the category of finite strings identifies via the embedding $X \mapsto \mathrm{Hom}(-, X)|_\Sigma$ with the full subcategory of finite coproducts of representable functors in $\mathcal{H}om(\Sigma^{\mathrm{op}}, \mathrm{Set}_*)$.

Call a subset $S \subseteq \Sigma_{n-1}$ of non-zero strings *partitioning* when for any $s, t \in S$ we have $s' = t'$ if and only if $s'' = t''$. In that case there is a unique partition $P = P(S)$ such that $S(P) = S$. This yields a bijective correspondence between partitions of $[n]$ and partitioning sets of strings in $\Sigma_{n-1}$.

A partition $P$ is *non-crossing* provided given elements $i < j < i' < j'$ with $i, i'$ in the same part and $j, j'$ in the same part, then all elements belong to the same part. The partitions of $[n]$ are partially ordered via *refinement*, so $P \leqslant P'$ if any part of $P$ is contained in a part of $P'$. The non-crossing partitions then form a lattice which is denoted by $\mathrm{NC}(n)$; cf. [16, 22].

LEMMA 6.2. *There is a lattice isomorphism* $\mathrm{NC}(\Sigma_{n-1}) \xrightarrow{\sim} \mathrm{NC}(n)$ *which is given by* $S \mapsto P(S)$.

*Proof.* It is clear that $S \subseteq \Sigma_{n-1}$ is non-crossing if and only if $P(S)$ is non-crossing. Let $S \leqslant S'$. This means any $s \in S$ can be written as $s = s_1 s_2 \cdots s_r$ with $s_1, \ldots, s_r$ in $S'$. On the other hand, $P \leqslant P'$ means that for any part $P_\alpha = \{a_1 < a_2 < \cdots < a_u\}$ of $P$ and $t = (a_i, a_{i+1} - 1) \in S(P)$, there is a part of $P'$ containing $a_i, a_{i+1}$ and therefore $t = t_1 t_2 \cdots t_r$ with $t_1, \ldots, t_r$ in $S(P')$. Thus we have $S \leqslant S'$ if and only if $P(S) \leqslant P(S')$. □

We say that two monomorphisms $X_1 \rightarrowtail X$ and $X_2 \rightarrowtail X$ are *equivalent* if there exists an isomorphism $X_1 \to X_2$ making the following diagram commutative.

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

An equivalence class of monomorphisms into $X$ is called a *subobject* of $X$. Given subobjects $X_1 \rightarrowtail X$ and $X_2 \rightarrowtail X$, we write $X_1 \leqslant X_2$ if there is a morphism $X_1 \to X_2$ making the above diagram commutative; this yields a partial order.

For a monomorphism $\phi\colon X \to \Sigma_n$ in the category of finite strings we set

$$S(\phi) \coloneqq \{\phi(s) \mid s \in X \text{ simple}\}.$$

LEMMA 6.3. *Let* $S \subseteq \Sigma_n$ *be non-crossing. Then there exists a monomorphism* $\phi\colon X \to \Sigma_n$ *such that* $S(\phi) = S$.

*Proof.* Consider the equivalence relation on $S$ generated by $s \sim t$ when $st \neq *$. This yields a partition $S = \bigcup_\alpha S_\alpha$ and we set $n_\alpha \coloneqq \operatorname{card} S_\alpha$. Using the fact that $S$ is non-crossing, there is a unique morphism $\phi\colon \coprod_\alpha \Sigma_{n_\alpha} \to \Sigma_n$ which identifies the simple strings $s_{\alpha,i}$ with the elements in $S_\alpha$. Thus $S = S(\phi)$. □

LEMMA 6.4. *A morphism* $\phi\colon X \to \Sigma_n$ *is a monomorphism if and only if it is given by an injective map.*

*Proof.* Clearly, any injective map yields a monomorphism. Thus we suppose that $\phi$ is a monomorphism and need to show that $\phi$ is given by an injective map.

Let $X = \coprod_{i=1}^r \Sigma_{n_i}$. The canonical decomposition of a morphism $\Sigma_{n_i} \to \Sigma_n$ from Lemma 2.3 yields the case $r = 1$. For the general case we may assume that $r = 2$. For each index $i$ the restricted morphism $\phi_i\colon \Sigma_{n_i} \to \Sigma_n$ is given by an injective map by the first case. Then each subset $\operatorname{Im}\phi_i$ is thick, and $\operatorname{Im}\phi_1 \cap \operatorname{Im}\phi_2 = \mathrm{Thick}(S)$ for some non-crossing $S \subseteq \Sigma_n$. Let $\psi\colon Y \to \Sigma_n$ be the corresponding morphism with $S(\psi) = S$ which exists by Lemma 6.3. Clearly, $\psi$ factors through each $\phi_i$ via a morphism $\psi_i\colon Y \to \Sigma_{n_i}$. We obtain a diagram

$$Y \underset{\psi_2}{\overset{\psi_1}{\rightrightarrows}} X \xrightarrow{\ \phi\ } \Sigma_n$$

where both composites equal $\psi$. Thus $\psi_1 = \psi_2$ and therefore $\operatorname{Im}\phi_1 \cap \operatorname{Im}\phi_2 = \{*\}$. We conclude that $\phi$ is given by an injective map. □

Lemma 6.5. *Let $\phi\colon X \to \Sigma_n$ be a monomorphism. Then the set $S(\phi)$ is non-crossing and we have* $\operatorname{Thick}(S(\phi)) = \operatorname{Im}\phi$. *Moreover, $\phi$ factors through a monomorphism $\phi'\colon X' \to \Sigma_n$ if and only if $S(\phi) \leqslant S(\phi')$.*

*Proof.* Let $X = \coprod_{i=1}^{r} \Sigma_{n_i}$ and $m \in \mathbb{N}$. The set $S$ of simple strings in $\Sigma_m$ is non-crossing and we have $\operatorname{Thick}(S) = \Sigma_m$. This property is preserved under a monomorphism $\Sigma_m \to \Sigma_n$ and yields the case $r = 1$. The general case follows since the restrictions $\phi_i\colon \Sigma_{n_i} \to \Sigma_n$ satisfy $\operatorname{Im}\phi_i \cap \operatorname{Im}\phi_j = \{*\}$ for $i \neq j$, by Lemma 6.4.

For a monomorphism $\phi'\colon X' \to \Sigma_n$ we have

$$\begin{aligned} S(\phi) \leqslant S(\phi') &\iff \operatorname{Thick}(S(\phi)) \subseteq \operatorname{Thick}(S(\phi')) \\ &\iff \operatorname{Im}\phi \subseteq \operatorname{Im}\phi' \\ &\iff \phi \text{ factors through } \phi'. \end{aligned}$$

□

Theorem 6.6. *Let $n \in \mathbb{N}$. The subobjects of $\Sigma_n$ in the category of finite strings form a lattice canonically isomorphic to the lattice of non-crossing partitions* $\mathrm{NC}(n+1)$. *The isomorphism sends a monomorphism $\phi\colon X \to \Sigma_n$ to $P(S(\phi))$.*

This result could be deduced from [11, 13], using the correspondence between strings and representations from Theorem 4.9, which identifies thick subsets of $\Sigma_n$ with thick subcategories of $\mathcal{A}_n$. We refer to [21, §4] for a detailed exposition. The following is a direct proof.

*Proof.* The assignment $\phi \mapsto P(S(\phi))$ gives a well defined map from the poset of subobjects of $\Sigma_n$ to $\mathrm{NC}(n+1)$ by Lemmas 6.2 and 6.5. In fact, the map is injective and $\phi$ factors through a monomorphism $\phi'$ if and only if $P(S(\phi)) \leqslant P(S(\phi'))$. Thus it remains to show surjectivity. Let $P \in \mathrm{NC}(n+1)$ and set $S = S(P)$. Then there is a morphism $\phi\colon \coprod_\alpha \Sigma_{n_\alpha} \to \Sigma_n$ satisfying $S = S(\phi)$ by Lemma 6.3. Thus we have that $P = P(S(\phi))$. □

## 7. Cyclic strings

We enlarge the category of finite strings and add cyclic strings as follows. Let $\Sigma_{\mathbb{Z}}$ denote the set of all basic strings. There is a natural action of the group of integers given by

$$*^z := * \qquad \text{and} \qquad s^z := (s' + z, s'' + z) \qquad \text{for } s = (s', s''),\ z \in \mathbb{Z}.$$

For $n > 0$ the *cyclic $n$-string* is the set of orbits with respect to the action of the subgroup $(n) = n\mathbb{Z}$. Thus

$$\tilde{\Sigma}_n := \{s^{(n)} \mid s \in \Sigma_{\mathbb{Z}}\} \quad \text{where} \quad s^{(n)} = \{s^{ni} \mid i \in \mathbb{Z}\},$$

with multiplication given by

$$s^{(n)} t^{(n)} := u^{(n)}$$

provided that there is a pair of integers $i, j$ such that $s^{ni} t^{nj} = u$. We set $\tilde{\Sigma}_0 := \{*\}$.

A morphism $\phi\colon \tilde{\Sigma}_m \to \tilde{\Sigma}_n$ is by definition a map satisfying (1). We define *standard morphisms* which are given by their values on simple strings as in (3). Let $n \geqslant 1$. Then the morphism

$$\tilde{\delta}_n^i\colon \tilde{\Sigma}_{n-1} \longrightarrow \tilde{\Sigma}_n \qquad (0 \leqslant i < n)$$

is given by the injective map such that $s_{i-1}^{(n)}$ and $s_i^{(n)}$ are not in its image, and the morphism

$$\tilde{\sigma}_n^i\colon \tilde{\Sigma}_n \longrightarrow \tilde{\Sigma}_{n-1} \qquad (0 \leqslant i < n)$$

is given by the surjective map such that $s_i^{(n)}$ is sent to the zero string. The *cyclic permutation*

$$\tau_n^i \colon \tilde{\Sigma}_n \longrightarrow \tilde{\Sigma}_n \qquad (0 \leqslant i < n)$$

is given by $s^{(n)} \mapsto (s^i)^{(n)}$.

LEMMA 7.1. *The standard morphisms satisfy the identities* (2)*, and every morphism* $\phi \colon \tilde{\Sigma}_m \to \tilde{\Sigma}_n$ *admits a unique decomposition*

$$\phi = \delta_n^{i_u} \circ \delta_{n-1}^{i_{u-1}} \circ \cdots \circ \delta_{n-u}^{i_0} \circ \sigma_{m-v}^{j_v} \circ \cdots \circ \sigma_{m-1}^{j_1} \circ \sigma_m^{j_0} \circ \tau_m^k$$

*with* $0 \leqslant i_0 < \cdots < i_u < n$, $0 \leqslant j_v < \cdots < j_0 < m$, $0 \leqslant k < m$, *and* $n - u = m - v$.

*Proof.* Adapt the proof of Lemmas 2.1 and 2.3. The only difference arises from cyclic permutations. □

Next we consider morphisms $\Sigma_m \to \tilde{\Sigma}_n$ and $\tilde{\Sigma}_m \to \Sigma_n$, which are by definition maps satisfying (1). Let $n \geqslant 1$. The standard morphism

$$\varepsilon_n^i \colon \Sigma_{n-1} \longrightarrow \tilde{\Sigma}_n \qquad (0 \leqslant i < n)$$

is given by the unique injective map such that $s_i^{(n)}$ is not in its image. We consider the morphism

$$\tilde{\Sigma}_n \xrightarrow{\tau_n^{-i}} \tilde{\Sigma}_n \xrightarrow{\tilde{\sigma}_n^{n-1}} \tilde{\Sigma}_{n-1} \xrightarrow{\tilde{\sigma}_{n-1}^{n-2}} \cdots \xrightarrow{\tilde{\sigma}_2^1} \tilde{\Sigma}_1$$

and note that its *kernel* (that is, the set of elements sent to the zero string) equals the image of $\varepsilon_n^i$.

LEMMA 7.2. *Let* $m, n \in \mathbb{N}$. *Every morphism* $\Sigma_m \to \tilde{\Sigma}_n$ *factors through* $\varepsilon_n^i$ *for some* $0 \leqslant i < n$, *and every morphism* $\tilde{\Sigma}_m \to \Sigma_n$ *factors through* $\tilde{\Sigma}_0 = \Sigma_0$.

*Proof.* First consider a morphism $\phi \colon \Sigma_m \to \tilde{\Sigma}_n$. It is easily checked that the longest string $s^{(n)}$ in the image of $\phi$ has length at most $n-1$, because the image of $\phi$ is finite. Choose an index $i$ such that the simple $s_i$ does not arise as a composition factor of $s$. It follows that $\phi$ factors through $\varepsilon_n^i$, since all composition factors of strings in the image of $\phi$ are composition factors of $s$.

Now consider for $\psi \colon \tilde{\Sigma}_m \to \Sigma_n$ its epi-mono factorisation $\psi = \psi'' \circ \psi'$. The image is of the form $\tilde{\Sigma}_p$ for some $p \leqslant m$. Because $\psi''$ is injective and $\tilde{\Sigma}_p$ is infinite for $p > 0$ we conclude that $p = 0$. □

Let us consider the category of all connected strings (linear and cyclic). The objects are of the form $\Sigma_n$ or $\tilde{\Sigma}_n$ with $n \in \mathbb{N}$. As before, we add finite coproducts and obtain the *enlarged category of finite strings*. The objects are of the form

$$\Big( \coprod_\alpha \Sigma_{m_\alpha} \Big) \amalg \Big( \coprod_\beta \tilde{\Sigma}_{n_\beta} \Big)$$

given by a finite set of natural numbers $m_\alpha$ and $n_\beta$.

Let $k$ be a field. For the quiver

$$n-1 \rightleftarrows n-2 \longrightarrow \cdots \longrightarrow 1 \longrightarrow 0 \qquad (n \geqslant 1)$$

we denote by $\tilde{\mathcal{A}}_n$ the category of all finite dimensional and nilpotent $k$-linear representations. Then we have the following analogue of Lemma 4.1.

LEMMA 7.3. *Let* $k$ *be a field and* $\mathcal{A}$ *a* $k$*-linear abelian category. Suppose that* $\mathcal{A}$ *satisfies* (Ab1)–(Ab5) *but not* (Ab6). *Then there is an equivalence* $\mathcal{A} \xrightarrow{\sim} \tilde{\mathcal{A}}_n$, *where* $n$ *equals the number of isomorphism classes of simple objects in* $\mathcal{A}$.

*Proof.* See for example the description of uniserial categories in [1]. □

We continue with analogues of Lemmas 4.2 and 4.4. Let $n \geqslant 1$. The indecomposable objects of $\tilde{\mathcal{A}}_n$ are parameterised by the elements of $\tilde{\Sigma}_n \smallsetminus \{*\}$. There are canonical recollements of abelian categories

$$\mathcal{A}_1 \;\overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}}\; \tilde{\mathcal{A}}_n \;\overset{\tilde{d}_n^{i+1}}{\underset{\tilde{d}_n^{i}}{\overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}}}}\; \tilde{\mathcal{A}}_{n-1} \qquad (0 \leqslant i < n)$$

(with $\tilde{d}_n^0 = \tilde{d}_n^n$) and composing them yields a recollement

$$\mathcal{A}_{n-1} \;\overset{\longleftarrow}{\underset{\longleftarrow}{\overset{e_n^0}{\longrightarrow}}}\; \tilde{\mathcal{A}}_n \;\overset{\longleftarrow}{\underset{\longleftarrow}{\overset{\tilde{s}_2^1 \cdots \tilde{s}_n^{n-1}}{\longrightarrow}}}\; \tilde{\mathcal{A}}_1.$$

Furthermore, there are equivalences

$$t_n^i \colon \tilde{\mathcal{A}}_n \xrightarrow{\sim} \tilde{\mathcal{A}}_n \qquad (0 \leqslant i < n)$$

which are given by a cyclic permutation $S_j \mapsto S_{j+i}$ of the simple representations. We obtain a correspondence between standard morphisms in $\tilde{\Sigma}_n$ and exact functors:

$$\tilde{\delta}_n^i \longleftrightarrow \tilde{d}_n^i \qquad \tilde{\sigma}_n^i \longleftrightarrow \tilde{s}_n^i \qquad \varepsilon_n^i \longleftrightarrow e_n^i \qquad \tau_n^i \longleftrightarrow t_n^i \qquad (0 \leqslant i < n).$$

The following result generalises Theorem 4.9. As before, we consider $k$-linear abelian categories together with $k$-linear exact functors, up to natural isomorphism, that admit a homological factorisation.

Theorem 7.4. *Let $k$ be a field. The assignments $\Sigma_n \mapsto \mathcal{A}_n$ and $\tilde{\Sigma}_n \mapsto \tilde{\mathcal{A}}_n$ provide an equivalence between the enlarged category of finite strings and the category of $k$-linear abelian categories satisfying* (Ab2)–(Ab5).

*Proof.* We adapt the proof of Theorem 4.9. Any $k$-linear abelian category satisfying (Ab2)–(Ab5) decomposes into a finite coproduct of connected abelian categories, which are (up to an equivalence) of the form $\mathcal{A}_n$ or $\tilde{\mathcal{A}}_n$, respectively, by Lemmas 4.1 and 7.3. Thus we obtain a bijection between the isomorphism classes of objects. It remains to consider the morphisms, and we may restrict ourselves to connected categories. An exact functor $F \colon \mathcal{A} \to \mathcal{B}$ which admits a homological factorisation sends indecomposable objects either to indecomposables or to zero; see Lemma 4.6. This yields a morphism $\phi \colon \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{B}}$ between the corresponding strings, given by $M_{\phi(s)} = F(M_s)$ for each $s \in \Sigma_{\mathcal{A}}$. The assignment $F \mapsto \phi$ is injective since $k$-linear exact functors which admit a homological factorisation are naturally isomorphic when they coincide on indecomposable objects; see Lemma 4.7. The assignment is surjective, by Lemmas 2.3 and 7.1, in combination with Lemma 7.2. □

## 8. Non-crossing partitions of type $B$

We wish to describe the subobjects of $\tilde{\Sigma}_n$ in the enlarged category of finite strings. This description is parallel to that for $\Sigma_n$ and involves the non-crossing partitions of type $B$.

Let $S \subseteq \Sigma_{\mathbb{Z}}$. We call $S$ *thick* if $* \in S$ and for any pair $s, t \in S$ of non-zero strings we have $st \in S$, and moreover $(s', t'-1), (t', s''), (s''+1, t'') \in S$ provided that $s' \leqslant t' \leqslant s'' \leqslant t''$. We denote by $\mathrm{Thick}(S)$ the smallest thick subset of $\Sigma_{\mathbb{Z}}$ containing $S$.

A set $S \subseteq \Sigma_{\mathbb{Z}}$ of non-zero strings is called *non-crossing* provided that $s, t \in S$ and $s' \leqslant t' \leqslant s'' \leqslant t''$ implies $s = t$.

For $n > 0$ consider the canonical projection $p \colon \Sigma_{\mathbb{Z}} \to \tilde{\Sigma}_n$. Then a subset $S \subseteq \tilde{\Sigma}_n$ is *thick* if $p^{-1}(S)$ is thick, and $S$ is *non-crossing* if $p^{-1}(S)$ is non-crossing. Note that $\ell(s) \leqslant n$ for any $s^{(n)}$ when $S$ is non-crossing.

LEMMA 8.1. *The assignment* $S \mapsto \mathrm{Thick}(S)$ *gives a bijection between the non-crossing subsets and the thick subsets of* $\tilde{\Sigma}_n$.

*Proof.* The inverse map takes a thick subset $T \subseteq \tilde{\Sigma}_n$ to the unique non-crossing subset $S \subseteq T$ with $\mathrm{Thick}(S) = T$. □

For non-crossing subsets $S, S'$ of $\tilde{\Sigma}_n$ we set

$$S \leqslant S' \quad :\Longleftrightarrow \quad \mathrm{Thick}(S) \subseteq \mathrm{Thick}(S').$$

This yields the structure of a poset. In fact, the non-crossing subsets form a lattice since the thick subsets of $\tilde{\Sigma}_n$ are closed under intersections. We denote this lattice by $\mathrm{NC}(\tilde{\Sigma}_n)$.

Let $n \in \mathbb{N}$. We consider the set

$$[2n] = \{0, 1, \cdots, n-1, \bar{0}, \bar{1}, \cdots, \overline{n-1}\},$$

where $\bar{x}$ is identified with $x + n$ for $0 \leqslant x < n$, and $\bar{\bar{x}} \coloneqq x$. For a partition $P = (P_\alpha)$ of $[2n]$ we require that $\overline{P_\alpha}$ is a part of $P$ for each $\alpha$. Each partition is determined by the corresponding set of strings $S(P) \subseteq \tilde{\Sigma}_n$, where by definition $s^{(n)} \in \tilde{\Sigma}_n$ with $0 \leqslant s' < n$ belongs to $S(P)$ if for some $\alpha$ we have $s', s'' \in P_\alpha$ and $i \notin P_\alpha$ for all $s' < i \leqslant s''$. The partitions of $[2n]$ are partially ordered via refinement, and the non-crossing partitions then form a lattice which is denoted by $\mathrm{NC}^B(n)$; cf. [19, 22].

LEMMA 8.2. *There is a lattice isomorphism* $\mathrm{NC}(\tilde{\Sigma}_n) \xrightarrow{\sim} \mathrm{NC}^B(n)$ *which is given by* $S \mapsto P(S)$.

*Proof.* Adapt the proof of Lemma 6.2. □

REMARK 8.3. Let $P = (P_\alpha)$ be a non-crossing partition of $[2n]$ and denote by $S = (S_\alpha)$ the corresponding partition of $S = S(P)$. Then $P$ has at most one part $P_\alpha$ satisfying $\overline{P_\alpha} = P_\alpha$. In fact, $\overline{P_\alpha} = P_\alpha$ holds if and only if $\mathrm{Thick}(S_\alpha)$ is infinite.

As before, we write $P(S)$ for the partition of $[2n]$ corresponding to a non-crossing set $S \subseteq \tilde{\Sigma}_n$. For a monomorphism $\phi\colon X \to \tilde{\Sigma}_n$ in the category of finite strings we set

$$S(\phi) \coloneqq \{\phi(s) \mid s \in X \text{ simple}\}.$$

THEOREM 8.4. *Let* $n \in \mathbb{N}$. *The subobjects of* $\tilde{\Sigma}_n$ *in the enlarged category of finite strings form a lattice which is canonically isomorphic to the lattice of non-crossing partitions* $\mathrm{NC}^B(n)$. *The isomorphism sends a monomorphism* $\phi\colon X \to \tilde{\Sigma}_n$ *to* $P(S(\phi))$.

*Proof.* Adapt the proof of Theorem 6.6. □

## 9. THICK SUBCATEGORIES

Results about subobjects in categories of strings correspond to statements about thick subcategories of abelian categories, because of the correspondence from Theorem 7.4.

Recall that a full subcategory of an abelian category is *thick* if it is closed under direct summands and the two out of three property holds for any short exact sequence.

LEMMA 9.1. *Let* $k$ *be a field and let* $\mathcal{A}$ *be a* $k$*-linear abelian category satisfying* (Ab2)–(Ab5). *Then every thick subcategory of* $\mathcal{A}$ *satisfies again* (Ab2)–(Ab5).

*Proof.* Let $\mathcal{C} \subseteq \mathcal{A}$ be a thick subcategory. Then $\mathcal{C}$ is closed under images of morphisms in $\mathcal{C}$ because $\mathcal{A}$ is hereditary. It follows that the category $\mathcal{C}$ is abelian and again hereditary. Also, $\mathcal{C}$ is necessarily a length category. If $X \in \mathcal{C}$ is simple, then $\mathrm{End}(X)$ is isomorphic to $k[t]/(t^p)$ for some $p \geqslant 1$, since $X$ is indecomposable in $\mathcal{A}$. Schur's lemma then implies $p = 1$. It remains to show that $\mathcal{C}$ is a uniserial category with finitely many simple objects. We may assume that either $\mathcal{A} = \mathcal{A}_n$ or $\mathcal{A} = \tilde{\mathcal{A}}_n$ for

some $n \in \mathbb{N}$. Then a representative set of simple objects in $\mathcal{C}$ identifies with a non-crossing subset $S$ in $\Sigma_n$ or $\tilde{\Sigma}_n$. The set $S$ is finite since the length of any string in $S$ is bounded by $n$. Let $M_s, M_t$ be simple objects in $\mathcal{C}$ corresponding to strings $s, t \in S$. Then $\operatorname{Ext}^1(M_t, M_s) \neq 0$ if and only if $st \neq *$. It is clear that for each $s \in S$ there is at most one $t \in S$ with $st \neq *$, and dually there is at most one $r \in S$ with $rs \neq *$. Then a criterion from [1] implies that $\mathcal{C}$ is uniserial. □

Now we can deduce classifications of thick subcategories from Theorems 6.6 and 8.4. The first part is due to Ingalls and Thomas [13] and only included for completeness; the second part seems to be new.

Corollary 9.2. *Let $k$ be a field and $n \in \mathbb{N}$.*

(1) *There is a canonical isomorphism between the lattice of thick subcategories of $\mathcal{A}_n$ and the lattice* $\mathrm{NC}(n+1)$.
(2) *There is a canonical isomorphism between the lattice of thick subcategories of $\tilde{\mathcal{A}}_n$ and the lattice* $\mathrm{NC}^B(n)$.

*Proof.* We apply Lemma 9.1. From the homological factorisation of an exact functor it follows that each subobject of $\mathcal{A}_n$ or $\tilde{\mathcal{A}}_n$ is given by the inclusion of a thick subcategory. On the other hand, all thick subcategories arise in this way. Theorem 7.4 provides the correspondence with subobjects of $\Sigma_n$ and $\tilde{\Sigma}_n$, respectively. Then the assertion follows for $\mathcal{A}_n$ from Theorem 6.6 and for $\tilde{\mathcal{A}}_n$ from Theorem 8.4. □

Remark 9.3. (1) The classification of thick subcategories for abelian categories of the form $\mathcal{A}_n$ or $\tilde{\mathcal{A}}_n$ given by a field and $n \in \mathbb{N}$ generalises to any connected hereditary and uniserial length category with finitely many isomorphism classes of simple objects. The proof is essentially the same, because indecomposable objects can be identified with strings which encode their composition series. Then thick subcategories correspond bijectively to non-crossing sets of strings.

(2) The category of regular modules over a tame hereditary algebra is an example of an hereditary and uniserial length category [6]. For the module category of a tame hereditary algebra one can show that any thick subcategory is contained in the thick subcategory of regular modules, provided it is not generated by an exceptional sequence [5]. This yields a classification of all thick subcategories, complementing the work in [11, 12, 13].

(3) For an hereditary abelian category $\mathcal{A}$, thick subcategories of the bounded derived category $\mathbf{D}^b(\mathcal{A})$ correspond bijectively to thick subcategories of $\mathcal{A}$ via

$$\mathbf{D}^b(\mathcal{A}) \supseteq \mathcal{C} \longmapsto \{H^0(X) \in \mathcal{A} \mid X \in \mathcal{C}\} \subseteq \mathcal{A},$$

see [15, Proposition 4.4.17].

## Appendix A. Basic strings

The properties of basic strings and the connection with linear representations become more transparent if we consider analogues of face and degeneracy maps for the poset of integers. We view this poset as a category, and this means that morphisms $\mathbb{Z} \to \mathbb{Z}$ are viewed as functors. For $i \in \mathbb{Z}$ we define morphisms

$$\delta^i \colon \mathbb{Z} \longrightarrow \mathbb{Z}, \qquad j \mapsto \begin{cases} j & j < i \\ j+1 & j \geqslant i \end{cases}$$

and

$$\sigma^i \colon \mathbb{Z} \longrightarrow \mathbb{Z}, \qquad j \mapsto \begin{cases} j & j \leqslant i \\ j-1 & j > i. \end{cases}$$

These satisfy the simplicial identities (2). Moreover, they are related via adjunctions:

$$\cdots \dashv \delta^{i+1} \dashv \sigma^i \dashv \delta^i \dashv \sigma^{i-1} \dashv \cdots$$

Let $k$ be a field. Then for each $i \in \mathbb{Z}$ precomposition with $\delta^i$ and $\sigma^i$ yields exact functors

$$\operatorname{Rep}(\mathbb{Z}^{\mathrm{op}}, k) \underset{(\delta^i)^*}{\overset{(\sigma^i)^*}{\leftleftarrows}} \operatorname{Rep}(\mathbb{Z}^{\mathrm{op}}, k).$$

This assignment is contravariant and therefore reverses the directions of functors. Thus the dual simplicial identities but the same adjunctions

$$\cdots \dashv (\delta^{i+1})^* \dashv (\sigma^i)^* \dashv (\delta^i)^* \dashv (\sigma^{i-1})^* \dashv \cdots$$

are satisfied.

For $i \in \mathbb{Z}$ let $S_i$ denote the simple representation concentrated in $i$, that is, $S_i(j) = 0$ for all $j \neq i$. Let $n \in \mathbb{N}$. Then precomposition with the inclusion $[n] \to \mathbb{Z}$ yields an exact functor

$$\operatorname{Rep}(\mathbb{Z}^{\mathrm{op}}, k) \longrightarrow \operatorname{Rep}([n]^{\mathrm{op}}, k) = \mathcal{A}_n$$

which becomes an equivalence when restricted to the full subcategory of objects in $\operatorname{Rep}(\mathbb{Z}^{\mathrm{op}}, k)$ with composition factors in $\{S_0, \ldots, S_{n-1}\}$. Viewing this as an identification, the functors $(\delta^i)^*$ and $(\sigma^{i-1})^*$ restrict to exact functors

$$\mathcal{A}_n \xrightarrow{(\delta^i)^*} \mathcal{A}_{n-1} \qquad (0 \leqslant i < n)$$

and

$$\mathcal{A}_{n-1} \xrightarrow{(\sigma^{i-1})^*} \mathcal{A}_n \qquad (0 \leqslant i \leqslant n).$$

Recall that $\Sigma_{\mathbb{Z}}$ denotes the set of basic strings. Then $\Sigma_{\mathbb{Z}} \smallsetminus \{*\}$ identifies with the indecomposable objects of finite length in $\operatorname{Rep}(\mathbb{Z}^{\mathrm{op}}, k)$ via $s \mapsto M_s$, as in Lemma 4.2. This identification yields maps $\Sigma_{\mathbb{Z}} \to \Sigma_{\mathbb{Z}}$ which are induced by $(\delta^i)^*$ and $(\sigma^i)^*$, respectively. Restricting these maps for any $n \in \mathbb{N}$ to the set $\Sigma_n$ of basic strings with composition factors in $\{s_0, \ldots, s_{n-1}\}$ gives

$$\sigma_n^i = (\delta^i)^*|_{\Sigma_n} \qquad \text{and} \qquad \delta_n^i = (\sigma^{i-1})^*|_{\Sigma_{n-1}}.$$

Then the following elementary observation (reflecting a duality for the simplicial category $\Delta$, cf. [18, VIII.7]) explains the simplicial relations (and any further properties) for $\delta_n^i \colon \Sigma_{n-1} \to \Sigma_n$ and $\sigma_n^i \colon \Sigma_n \to \Sigma_{n-1}$.

Lemma A.1. *Consider symbols $(\delta^i, \sigma^i)$ and $(d^i, s^i)$ for some integers $i \in \mathbb{Z}$. After substituting $\delta^i \mapsto s^i$ and $\sigma^i \mapsto d^{i+1}$ and reversing the order of composition, the identities* (2) *hold for $(d^i, s^i)$ if and only if they hold for $(\delta^i, \sigma^i)$.*

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