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Bijecting hidden symmetries for skew staircase shapes

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Abstract We present a bijection between the set $\mathrm{SYT}(\lambda/\mu)$ of standard Young tableaux of *staircase minus rectangle shape* $\lambda = \delta_k$, $\mu = (b^a)$, and the set $\mathrm{ShSYT}'(\eta)$ of *marked shifted standard Young tableaux* of a certain shifted shape $\eta = \eta(k, a, b)$. Numerically, this result is due to DeWitt (2012). Combined with other known bijections this gives a bijective proof of the product formula for $|\mathrm{SYT}(\lambda/\mu)|$. This resolves an open problem by Morales, Pak and Panova (2019), and allows an efficient random sampling from $\mathrm{SYT}(\lambda/\mu)$. Other applications include a bijection for semistandard Young tableaux, and a bijective proof of Stembridge's symmetry of LR–coefficients of the staircase shape. We also extend these results to *set-valued standard Young tableaux* in the *combinatorics of K-theory*, leading to new proofs of results by Lewis and Marberg (2019) and Abney-McPeek, An and Ng (2020).

1. Introduction

The phrase 'hidden symmetries' in the title refers to coincidences between the numbers of seemingly different (yet similar) sets of combinatorial objects. When such coincidences are discovered, they tend to be fascinating because they reflect underlying algebraic symmetries — even when the combinatorial objects themselves appear to possess no such symmetries.

It is always a relief to find a simple combinatorial explanation of hidden symmetries. A direct bijection is the most natural approach, even if sometimes such a bijection is both hard to find and to prove (cf. §5.5). Such a bijection restores order to a small corner of an otherwise disordered universe, suggesting we are on the right path in our understanding. It is also an opportunity to learn more about our combinatorial objects.

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Keywords. tableau, shifted tableau, Schur P-function, Worley–Sagan insertion, mixed shifted insertion, shifted Hecke insertion, Knuth class, shifted Knuth class, K-Knuth class, queer Lie superalgebra.

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1.1. The results. We start with the following unusual product formula. Denote by $\delta_k = (k-1, \ldots, 2, 1)$ the *staircase shape*.

Theorem 1.1 (Staircase minus rectangle, see below). *For all $a, b, c \in \mathbb{N}$, let $\lambda = \delta_{a+b+2c}$, and $\mu = (b^a)$. Then the number $f^{\lambda/\mu} = \left|\mathrm{SYT}(\lambda/\mu)\right|$ is equal to*

$$n! \, \frac{\mathrm{F}(a)\,\mathrm{F}(b)\,\mathrm{F}(c)\,\mathrm{F}(a+b+c) \cdot \mathrm{G}(c)\,\mathrm{G}(a+b+c)}{\mathrm{F}(a+b)\,\mathrm{F}(b+c)\,\mathrm{F}(a+c) \cdot \mathrm{G}(a+c)\,\mathrm{G}(b+c)\,\mathrm{G}(a+b+2c)},$$

where $n = |\lambda/\mu| = \binom{a+b+2c}{2} - ab$, $\mathrm{F}(m) := 1! \cdot 2! \cdots (m-1)!$, and $\mathrm{G}(m) := 1!! \cdot 3!! \cdots (2m-3)!!$

This curious formula was first derived by DeWitt [15] in a somewhat different form (see discussion after Theorem 1.7). The q-version was given in [41], and further generalizations were obtained in [50] for a more general class of skew shapes. To understand the product formula in the theorem, consider the following:

Theorem 1.2 (DeWitt). *Let $\lambda = \delta_k$ be a staircase and $\mu = (b^a)$ be a rectangle, such that $a + b < k$. Then:*

$$\left|\mathrm{SYT}(\lambda/\mu)\right| \, = \, 2^N \left|\mathrm{ShSYT}(\eta)\right|, \tag{1.1}$$

where $\mathrm{ShSYT}(\eta)$ *is the set of* shifted standard Young tableaux *of shifted shape $\eta = \eta(k, a, b)$ defined in 1, and $N := |\eta| - k + d = \binom{k}{2} - ab - k + d$ where $d = \min\{a, b\}$.*

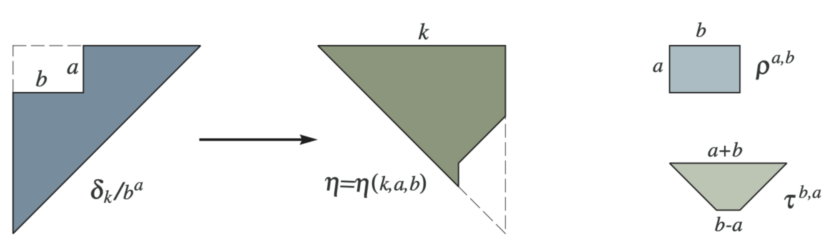


Figure 1. Main bijection $\varphi : \mathrm{SYT}(\delta_k/b^a) \longrightarrow \mathrm{ShSYT}'(\eta)$.

Our main result is a bijective proof of (1.1), where we interpret the RHS as the number $\left|\mathrm{ShSYT}'(\eta)\right|$ of certain *marked shifted standard Young tableaux* (see below). For the case of straight shapes, i.e. for $a = b = 0$ and $\mu = \varnothing$, an equivalent bijection was given by Purbhoo [59] (see §5.1). Our bijection has additional properties, as it extends to the proof of a symmetric function identity (see Theorem 1.7). Theorem 1.1 follows from Theorem 1.2 and the *hook-length formula* (HLF) for shifted shapes due to Thrall [76]. Combined with the bijection in [19], this gives the first direct bijective proof of Theorem 1.1, resolving an open problem in [50, §9.2].

Theorem 1.3. *Let $\lambda = \delta_k$ be a staircase, $\mu = (b^a)$ be a rectangle, s.t. $a + b < k$. Then there is a $O(k^3 \log k)$ time algorithm for uniform random generation of standard Young tableaux in* $\mathrm{SYT}(\lambda/\mu)$.

For the proof, we combine the bijection for (1.1) and the known uniform random generation algorithms for shifted shapes: either the NPS–style *two-dimensional bubble sorting*[(1)] in [19], or the GNW–style *hook walk* in [62].

The uniform random generation of combinatorial objects is a classical problem that is well understood for many planar structures (see §5.6). For standard Young

(1)This type of insertion is a close relative to *jeu-de-taquin* and *promotion*, all heavily studied in this context.

tableaux of skew shapes, the iterative application of the *Aitken–Feit determinant formula* [2, 18], gives an easy $O\big(k^{2+\omega}\big)$ time algorithm. Here $\omega \geqslant 2$ is the *matrix multiplication constant* with currently best known upper bound $\omega < 2.3729$ [3]. Our algorithm is thus a substantial improvement over this approach.

Our final application is a bijective proof of the following unusual symmetry of *Littlewood–Richardson* (LR–) *coefficients* $c^{\lambda}_{\mu\nu} = \big|\mathrm{LR}(\lambda/\mu,\nu)\big|$, where $\mathrm{LR}(\lambda/\mu,\nu) \subseteq \mathrm{SSYT}(\lambda/\mu,\nu)$ denotes the set of *LR–tableaux* (see e.g. [64, §4.9]).

Corollary 1.4 (of Theorem 1.5). *Let $\mu,\nu \subseteq \delta_k$, such that $|\mu| + |\nu| = |\delta_k|$. Then:*

$$c^{\delta_k}_{\mu\nu} = c^{\delta_k}_{\mu'\nu}\,, \tag{1.2}$$

where μ' denotes the conjugate partition *of μ.*

Since the LR–coefficients $c^{\lambda}_{\mu\nu}$ determine the intersection numbers of three Schubert varieties in a suitable Grassmannian, general LR–coefficients have an S_3–symmetry [74]. Combining this general symmetry with equation (1.2) implies that

$$c^{\delta_k}_{\mu\nu} = c^{\delta_k}_{\mu'\nu} = c^{\delta_k}_{\mu\nu'} = c^{\delta_k}_{\mu'\nu'} = c^{\delta_k}_{\nu'\mu'} = c^{\delta_k}_{\nu\mu'} = c^{\delta_k}_{\nu'\mu} = c^{\delta_k}_{\nu\mu}\,. \tag{1.3}$$

Note that LR coefficients satisfy two more symmetries (see e.g. [6]), hence

$$c^{\delta_k}_{\mu\nu} = c^{\mu^*}_{\delta_k\nu} = c^{\nu^*}_{\mu\delta_k}\,,$$

where λ^* denotes the complement to λ in the $(k-1)\times k$ rectangle, so that $\delta_k^* = \delta_k$. This triples the number of equal LR–coefficients given by (1.3), with the staircase δ_k as one of the partitions. In the next section we explain the algebra behind both Theorem 1.2 and Corollary 1.4.

1.2. Algebraic interpretation. The ring of symmetric functions Λ has many bases indexed by integer partitions, including the Schur functions s_λ. The subring $\widetilde{\Lambda}$ generated by power sums $p_k = x_1^k + x_2^k + \cdots$ with k odd has bases indexed by strict integer partitions, one being the *Schur P–functions*. The hidden symmetry in Theorem 1.2 was first proved using relations between skew staircase Schur functions and certain Schur P–functions given in Theorem 1.7. We give the first bijective explanations for these relations.

To state the identities in question, we introduce some notation. Let $\mu = (\mu_1,\ldots,\mu_k)$ and $\lambda = (\lambda_1,\ldots,\lambda_k)$ be integer partitions, and let μ' be the transpose of μ. We write $\mu \subseteq \lambda$ if $\mu_i \leqslant \lambda_i$ for all i, and let $s_{\lambda/\mu}$ denote the associated *skew Schur function* defined in §2.3.

Theorem 1.5 (J. Stembridge, 2004, see §5.2). *Let $\mu \subseteq \delta_n$. Then $s_{\delta_n/\mu} = s_{\delta_n/\mu'}$.*

Stembridge's theorem immediately implies Corollary 1.4; it is in fact equivalent to the corollary.

A partition λ with $\lambda_1 > \cdots > \lambda_k$ is called *strict*. When $\mu \subseteq \lambda$ are both strict, let $P_{\lambda/\mu}$ denote the associated *skew Schur P–function* defined in §2.3. As before, let $\delta_n = (n-1, n-2, \ldots, 1)$ be the staircase partition and ϱ_n be the same partition, viewed as a shifted shape. Since $\widetilde{\Lambda}$ is fixed under the ω involution that maps $s_{\lambda\mu}$ to $s_{\lambda'/\mu'}$, the next result is an algebraic explanation of Theorem 1.5:

Theorem 1.6 ([5, Thm 4.10]). *Let $\mu \subseteq \delta_n$. Then $s_{\delta_n/\mu} \in \widetilde{\Lambda}$. Moreover, in the expansion $s_{\delta_n/\mu} = \sum_\nu a_\nu P_\nu$, all a_ν are non-negative integers.*

This theorem was originally conjectured by Stanley in 2001, and proved by Ardila and Serrano using algebraic methods. The bijection in (3.2) gives a bijective proof of both Theorems 1.5 and 1.6.

For $\mu \subseteq \varrho_n$ strict, write $\varrho_n - \mu$ for the partition whose diagram is obtained by reflecting the diagram of ϱ_n/μ across the line $y = x$. Let $\rho^{\ell,m} = (m^\ell)$ be the $\ell \times m$ rectangle partition, and when $\ell > m$ let $\tau^{\ell,m} = (\ell{+}m{-}1, \ell{+}m{-}3, \ldots, \ell{-}m{+}1)$ be the shifted trapezoid, see 1.

Theorem 1.7 ([15, Thm V.3]). *Let a, b and k be integers so that $a + b < k$. Then:*

$$s_{\delta_k/\rho^{a,b}} \;=\; P_{\varrho_k/\tau^{a,b}} \;=\; P_{\varrho_k - \tau^{a,b}}$$

Note that $\delta_k/\rho^{a,b}$ and $\varrho_k - \tau^{a,b}$ are the shapes λ/μ and η appearing in Theorem 1.2. Schur functions are weighted generating functions of semistandard Young tableaux (see e.g. [44, 69, 68]). Similarly, Schur P–functions are weighted generating functions of semistandard Young tableaux of shifted shape with marked entries [71]. As discussed in §2.3, semistandard tableaux can be associated to standard tableaux via Gessel's *fundamental quasisymmetric functions* [22]. Using this correspondence, Theorem 1.2 becomes an immediate corollary of Theorem 1.7.

Our bijection for (1.1) uses *Worley–Sagan insertion* [63, 78], and the combinatorics of reduced words for *fully commutative permutations* [7, 72].(2) The map is easily seen to be surjective, hence bijectivity follows from Theorem 1.2. However, our goal is to prove Theorem 1.7 and hence Theorem 1.2 using our bijection. To do this, we show the map reverses descent sets and prove injectivity via a related map constructed using RSK and mixed shifted insertion [25]. Note that our proof of injectivity gives a *different* bijective proof of Theorem 1.2. A third bijection is introduced in §5.4.

In §5.9, we recall an elegant geometric interpretation of Theorem 1.7. This geometric interpretation suggests that Theorems 1.5,1.6, and 1.7 should extend to *stable Grothendieck polynomials* and K-theoretic analogues of Schur P-functions. We prove these extensions in §4 as Theorem 4.7, Theorem 4.9 and Corollary 4.8, the first appearing in [8] and the last being a main result in [1]. Our proofs of Theorem 4.7 and Corollary 4.8 are combinatorial as opposed to the previous algebraic proofs, but our proof of Theorem 4.9 uses algebraic identities. As a consequence, in Theorem 4.10 we extend our original bijective proof of Theorem 1.2 to a bijection between certain *set-valued tableaux*. Note the other bijections do not extend easily to this more general context.

Paper structure. In the lengthy and detailed §2, we recall much of the background on the usual and shifted Young tableaux, reduced words and insertion algorithms. Although our notation is self-contained, our arguments are technical and rely on many pieces of Young tableau technology. This appears to be unavoidable for a self-contained proof that our map is bijective, and we disperse references to the literature throughout the paper. A reader willing to assume Theorem 1.7 can find a concise description of our bijection at the beginning of §3, with proof in Proposition 3.7, which does not depend on earlier results in §3 (see also the equivalent maps discussed after Lemma 3.8 and in §5.4).

We prove our main results in §3. Our arguments are both dense and concise, so examples are added for clarity. In a short §4, we give generalizations of Theorems 1.5 and 1.7 to *Grothendieck polynomials* and extend our bijections to work in this case. These sections will be of greatest interest to experts on shifted tableaux, as they are quite technical. We conclude with final remarks in §5, including some large simulations.

(2)These are also known as 321-*avoiding permutations* [7]. We will not use this characterization.

2. Tableaux, reduced words and insertion algorithms

2.1. Basic notation. We write $\mathbb{N} = \{0, 1, \ldots\}$, $[n] = \{1, \ldots, n\}$ and 2^S for the power set of S. We also fix the linear ordering $1' < 1 < 2' < 2 < \cdots < n' < n$ of *marked integers*.

2.2. Young diagrams and Young tableaux. Recall an integer partition is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of non-negative integers $\lambda_1 \geqslant \ldots \geqslant \lambda_k \geqslant 0$. We identify λ with its *Young diagram*

$$D_\lambda = \{(i,j) \in \mathbb{N}^2 : 1 \leqslant j \leqslant \lambda_i\},$$

which we view as a poset via pointwise comparison, so $(i,j) \leqslant (k,\ell)$ if $i \leqslant k$ and $j \leqslant \ell$. Given $\mu \subseteq \lambda$, i.e. μ, λ such that $D_\mu \subseteq D_\lambda$, the *skew partition* λ/μ corresponds to the diagram $D_\lambda \smallsetminus D_\mu$. A *standard Young tableau* of shape λ/μ is a linear extension of $D_\lambda \smallsetminus D_\mu$, the set of which is denoted $\mathrm{SYT}(\lambda/\mu)$.

There is a parallel theory for *strict partitions* λ, which satisfy $\lambda = (\lambda_1 > \cdots > \lambda_k)$. The *shifted Young diagram* of a strict partition λ is

$$D^\triangledown_\lambda = \{(i,j) \in \mathbb{N}^2 : i \leqslant j \leqslant \lambda_i + i - 1\},$$

which we again view as a poset via pointwise comparison. The *diagonal* of D_λ is the subset $\{(i,i) : i \in \mathbb{N}\} \cap D_\lambda$. For μ, λ strict partitions with $\mu \subseteq \lambda$, a *shifted standard Young tableau* of shifted shape λ/μ is a linear extension of $D^\triangledown_\lambda \smallsetminus D^\triangledown_\mu$. Let $\mathrm{ShSYT}(\lambda/\mu)$ denote the set of shifted standard Young tableaux of shape λ/μ. A *shifted standard Young tableau with marked entries* of shape λ/μ is a pair (T,S) where $T \in \mathrm{ShSYT}(\lambda/\mu)$ and $S \subseteq D_\lambda \smallsetminus D_\mu$ such that S contains no diagonal entries. Let $\mathrm{ShSYT}'(\lambda/\mu)$ be the set of shifted standard Young tableaux with marked entries of shape λ/μ. The entries in S are *marked* with symbol $'$, see 2. For T a tableau, let $T_{ij} = T((i,j))$ denote its (i,j)th entry.

Note for any skew partition λ/μ that $D_{\lambda/\mu}$ and $D^\triangledown_{\lambda+\delta_n/\mu+\delta_n}$ are isomorphic as posets, so the theory of (unshifted) Young tableaux can be viewed as a special case of theory of shifted Young tableaux. Lastly, we recall that a *semistandard Young tableau* of shape λ/μ is a tableau with positive integer entries whose rows are weakly increasing and columns are strictly increasing. Let $\mathrm{SSYT}(\lambda/\mu)$ denote the set of semistandard Young tableaux of shape λ/μ. For shifted shapes, instead use $\mathrm{ShSSYT}(\lambda/\mu)$. The set $\mathrm{ShSSYT}'(\lambda/\mu)$ of *marked shifted semistandard Young tableaux* allows the off-diagonal entries of a shifted semistandard Young tableaux to be marked by the symbol $'$, with at most one i in each column and at most one i' in each row.

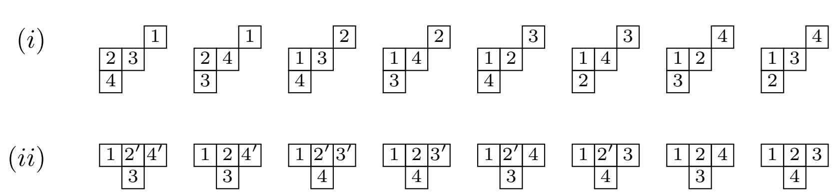


Figure 2. The sets (i) $\mathrm{SYT}(\delta_4/(2))$ and (ii) $\mathrm{ShSYT}'(\varrho_4 - (2))$, with columns corresponding to our bijection.

The reading word of an unmarked tableau T is the word $r(T)$ obtained by reading its rows from left to right, bottom to top. For example, the reading word of the third tableau in $2(i)$ is 4132.

2.3. Descents and symmetric functions. For T a tableau of shape λ, let $x^T = \prod_{(i,j)\in D_\lambda} x_{T_{ij}}$, where T_{ij} is the (i,j)-th entry in T, and $x_{i'} = x_i$ for marked entries. The *Schur functions* and *Schur P-functions* are defined as

$$s_\lambda = \sum_{T\in \mathrm{SSYT}(\lambda)} x^T \quad \text{and} \quad P_\mu = \sum_{T\in \mathrm{ShSSYT}'(\mu)} x^T \quad (\mu \text{ strict}). \tag{2.1}$$

As we will see, there is a natural way to partition semistandard tableaux into sets indexed by standard tableaux. This will give a formula for s_λ and P_μ.

For $\mathbf{a} = (a_1,\ldots,a_p)$ a word in $\mathbb{Z}$, the *descent set* of $\mathbf{a}$ is $\mathrm{Des}(\mathbf{a}) = \{i \in [p-1] : a_i > a_{i+1}\}$. Similarly, for $T \in \mathrm{SYT}(\lambda)$ with λ a partition of n, the *descent set* of T is defined as

$$\mathrm{Des}(T) \coloneqq \big\{ i \in [n-1] \ : \ i \text{ is strictly above } i+1 \big\}.$$

Here by *strictly above* we mean that the row number with entry i is strictly smaller than that with $(i+1)$. For $T \in \mathrm{ShSYT}'(\mu)$ with μ a strict partition of size n, the *descent set* of T is defined as

$$\mathrm{Des}(T) \coloneqq \left\{ i \in [n-1] : \begin{array}{c} i, (i+1)' \in T\,, \ \underline{\text{or}} \\ i \text{ is strictly above } i+1\,, \ \underline{\text{or}} \\ i' \text{ is strictly to the left of } (i+1)' \end{array} \right\}.$$

These sets are closely related to each other, as we will see in Theorem 2.6.

For $T \in \mathrm{ShSYT}'(\lambda)$, let T° be the tableau obtained by marking all unmarked off-diagonal entries and unmarking all marked entries. For example, if T is the first tableau in 2 (ii), then T° is the seventh. Given $S \subseteq [n-1]$, its *complement* is $S^c = [n-1] \smallsetminus S$. The next result follows from the definition of $\mathrm{Des}(T)$.

Lemma 2.1. *For λ a shifted shape with $|\lambda| = n$ and $T \in \mathrm{ShSYT}'(\lambda)$, we have:*

$$\mathrm{Des}(T^\circ) \,=\, \mathrm{Des}(T)^c.$$

Let $S \subseteq [n-1]$, and define $I_S = \{(i_1 \leqslant \cdots \leqslant i_n) : k \in S \Rightarrow i_k < i_{k+1}\}$. For $\mathbf{i} = (i_1,\ldots,i_n)$ and T a shifted marked standard tableau with n entries, let $T(\mathbf{i})$ be the tableau obtained from T by replacing entries labeled k with i_k and entries labeled k' with i'_k for each $k \in [n]$.

Theorem 2.2 ([22]). *Let λ/ν be a skew partition of size n and μ/ρ be a strict skew partition of n. Then $(T,\mathbf{i}) \mapsto T(\mathbf{i})$ is a bijection from*

$$(a) \quad \bigcup_{T\in\mathrm{SYT}(\lambda/\nu)} \{T\} \times I_{\mathrm{Des}(T)} \to \mathrm{SSYT}(\lambda/\nu)$$

$$(b) \quad \bigcup_{T\in\mathrm{ShSYT}'(\mu/\rho)} \{T\} \times I_{\mathrm{Des}(T)} \to \mathrm{ShSSYT}'(\mu/\rho).$$

Part (a) of Theorem 2.2 is essentially equivalent to Gessel's formula for Schur functions in terms of fundamental quasisymmetric functions, while part (b) is equivalent to Stembridge's analogous formula for Schur P–functions [71], both of which we now state:

Corollary 2.3. *Let λ,ν be partitions and μ,ρ be strict partitions with $\nu \subseteq \lambda$ and $\rho \subseteq \mu$. Then*

$$s_{\lambda/\nu} = \sum_{T\in\mathrm{SYT}(\lambda/\nu)} \sum_{\mathbf{i}\in I_{\mathrm{Des}(T)}} x^{T(\mathbf{i})} \quad \text{and} \quad P_{\mu/\rho} = \sum_{T\in\mathrm{ShSYT}'(\mu/\rho)} \sum_{\mathbf{i}\in I_{\mathrm{Des}(T)}} x^{T(\mathbf{i})}.$$

For $S \subseteq [n-1]$, its *reverse* is $S^r = \{n-k : k \in S\}$. The following is a corollary of [68, A1.2.11], and can be proved bijectively using Schützenberger's evacuation involution.

Lemma 2.4. *Let λ be a partition.*

$$s_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} \sum_{\mathbf{i} \in I_{\mathrm{Des}(T)^r}} x^{T(\mathbf{i})}.$$

As a consequence, we see that Theorem 1.7 follows from the existence of a bijection between $\mathrm{SYT}(\delta_n/\rho_{r,\ell})$ and $\mathrm{ShSYT}'(\varrho_n/\tau_{r,\ell})$ that reverses descent sets.

2.4. Insertion algorithms and jeu de taquin. To construct our bijection, we use RSK as well as three closely related bijections for shifted tableaux: Worley–Sagan insertion, mixed shifted insertion and jeu de taquin. We assume the reader is familiar with ordinary RSK (see e.g. [64, 69, 68]), which we denote $\mathbf{a} \xmapsto{\mathrm{RSK}} (P(\mathbf{a}), Q(\mathbf{a}))$.

Weak and strict insertion: For x a letter and $\mathbf{r} = (r_1, \dots, r_n)$ a word, let i be the smallest index so that $r_i > x$ or $n+1$ if $r_n \leqslant x$. We *weak insert* x into $\mathbf{r}$ by replacing r_i with x, *bumping* r_i if $i \neq n+1$. Similarly, if j is the smallest index so that $r_j \geqslant x$ or $n+1$ if $r_n < x$, we *strict insert* x into r by replacing r_j with x, bumping r_j if $j \neq n+1$.

Worley–Sagan bumping: For T a shifted tableau and $x \coloneqq x^1$ a letter, insert x^1 into T by:

(1) Weak insert x^i into the i-th row of T (viewing this row as a word); if no entry is bumped, then the process terminates.
(2) If x^i bumps a letter x^{i+1} that isn't the first entry of the i-th row of T, return to step (1) and continue to weak insert into rows.
(3) Otherwise, if x^i bumps the first entry of the i-th row of T, begin to strict insert x^j into the j-th column of T for $j = i+1, i+2, \dots$ (again, viewing the column as a word) until no entry is bumped, then terminate.

Worley–Sagan insertion: Given a word $\mathbf{a} = (a_1, \dots, a_p)$, initialize P_{SW} and Q_{SW} as empty tableaux. For $i = 1, 2, \dots, p$, insert a_i into P_{SW} using Worley–Sagan bumping, resulting in a new cell c. In Q_{SW}, we label c with i if the final insertion was a row insertion and i' if the final insertion was a column insertion.

The *Knuth relations* are the transformations:[(3)]

$$acb \leftrightarrow cab \quad \text{with} \quad a \leqslant b < c \quad \text{and} \quad bac \leftrightarrow bca \quad \text{with} \quad a < b \leqslant c.$$

For a word $\mathbf{a} = (a_1, \dots, a_p)$, a *Knuth move* is the application of a Knuth relation to a consecutive triple $a_i a_{i+1} a_{i+2}$, and a *shifted Knuth move* is a Knuth move or the exchange of the first two entries a_1 and a_2.

We say $\mathbf{a}$ and $\mathbf{b}$ are *Knuth equivalent*, denoted $\mathbf{a} \equiv \mathbf{b}$, if they differ by a sequence of Knuth moves. Similarly, $\mathbf{a}$ and $\mathbf{b}$ are *shifted Knuth equivalent*, denoted $\mathbf{a} \doteq \mathbf{b}$, if they differ by a sequence of shifted Knuth moves. Note that $\mathbf{a} \equiv \mathbf{b}$ implies $\mathbf{a} \doteq \mathbf{b}$, but the converse need not hold. The equivalence classes under the relations $\equiv$ and $\doteq$ are called *Knuth classes* and *shifted Knuth classes*, respectively.

Example 2.5. Applying Worley–Sagan insertion to the words $\mathbf{a} = (1,3,2,5,4,3)$ and $\mathbf{b} = (3,1,5,2,4,3)$, we see

$$P_{\mathrm{SW}}(\mathbf{a}) = P_{\mathrm{SW}}(\mathbf{b}) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 3 & 4 \\ \cline{2-3} \multicolumn{2}{c|}{} & 5 \\ \cline{3-3} \end{array}, \quad Q_{\mathrm{SW}}(\mathbf{a}) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 \\ \cline{2-3} \multicolumn{2}{c|}{} & 6 \\ \cline{3-3} \end{array}, \quad Q_{\mathrm{SW}}(\mathbf{b}) = \begin{array}{|c|c|c|} \hline 1 & 2' & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 \\ \cline{2-3} \multicolumn{2}{c|}{} & 6 \\ \cline{3-3} \end{array}.$$

(3) See e.g. [68, §A1.1], where these are called *Knuth equivalences* and *Knuth transformations*.

Note that $\mathbf{a} \equiv \mathbf{b}$ (so that $\mathbf{a} \doteq \mathbf{b}$) via $\mathbf{a} = (\underline{1,3,2},5,4,3) \leftrightarrow (3,1,\underline{2,5,4},3) \leftrightarrow (3,1,5,2,4,3) = \mathbf{b}$.

We summarize some key properties of Worley–Sagan insertion.

Theorem 2.6 ([63, 78]).

(1) *Worley–Sagan insertion is a bijection from words to pairs* $(P_{\mathrm{SW}}, Q_{\mathrm{SW}})$ *of tableaux, where* $P_{\mathrm{SW}} \in \mathrm{ShSSYT}(\lambda)$ *and* $Q_{\mathrm{SW}} \in \mathrm{ShSYT}'(\lambda)$ *for some shifted shape* λ*.*
(2) *The words* $\mathbf{a}$ *and* $\mathbf{b}$ *are shifted Knuth equivalent if and only if* $P_{\mathrm{SW}}(\mathbf{a}) = P_{\mathrm{SW}}(\mathbf{b})$*.*
(3) $\mathrm{Des}(\mathbf{a}) = \mathrm{Des}(Q_{\mathrm{SW}}(\mathbf{a}))$.

For $\mathbf{a} = (a_1, \ldots, a_p)$, denote $-\mathbf{a} \coloneqq coloneqq(-a_1, \ldots, -a_p)$.

Theorem 2.7 ([25, Cor. 8.9]). *In the notation above, we have:* $Q_{\mathrm{SW}}(-\mathbf{a}) = Q_{\mathrm{SW}}(\mathbf{a})^{\circ}$.

Next, we introduce *mixed shifted insertion*.

Mixed shifted bumping: For T a shifted marked tableau and $x \coloneqq x^1$ an unmarked letter, set the index of the initial cell to be $(y_1, z_1) = (1,1)$.

(1) If x^i is unmarked, weak insert x^i into the (y_i+1)st row of T; if x^i is marked, weak insert x^i into the (z_i+1)-st column of T; if no entry is bumped, then the process terminates.
(2) If x^i bumps a letter, define (y_{i+1}, z_{i+1}) to be the cell of the bumped entry and x^{i+1} to be the bumped entry if $y_{i+1} < z_{i+1}$ and $(x^{i+1})'$ if $y_{i+1} = z_{i+1}$. Return to step (1).

In other words, when a letter is bumped from the diagonal (which is unmarked, by definition), the letter becomes marked and gets weak inserted into the next column.

Mixed shifted insertion: Given a word $\mathbf{a} = (a_1, \ldots, a_p)$, initialize P_{MS} and Q_{MS} as empty tableaux. For $i \in [p]$, insert a_i into P_{MS} using mixed shifted bumping. This process terminates with a new cell c, which is added to Q_{MS} with label i.

Theorem 2.8 ([25]).

(1) *Mixed shifted insertion is a bijection from permutations to pairs* $(P_{\mathrm{MS}}, Q_{\mathrm{MS}})$ *of tableaux, where* $P_{\mathrm{MS}} \in \mathrm{ShSYT}'(\lambda)$ *and* $Q_{\mathrm{MS}} \in \mathrm{SYT}(\lambda)$ *for some shifted shape* λ*.*
(2) *For* w *a permutation,* $P_{\mathrm{MS}}(w) = Q_{\mathrm{SW}}(w^{-1})$ *and* $Q_{\mathrm{MS}}(w) = P_{\mathrm{SW}}(w^{-1})$*.*

We now define jeu de taquin. An *inner corner* of the skew partition λ/μ is a maximal element of D_μ and an *outer corner* is a minimal entry of $\mathbb{N}^2 \smallsetminus D_\lambda$. When λ and μ are strict partitions, the definition of inner and outer corners extends immediately using D^{SH}_μ and $D^{SH}_{\lambda/\mu}$.

For $T \in \mathrm{ShSYT}(\lambda/\mu)$ (not marked!) and $c_1 = (i_1, j_1)$ an inner corner of λ/μ, we perform an *inner jeu de taquin slide* using the following algorithm.

While $c_k = (i_k, j_k)$ is not an outer corner:

- fill c_k with $\min T_{i+1\ j}, T_{i\ j+1}$,
- set $c_{k+1} \leftarrow (i+1, j)$ if $T_{i+1,j} \leqslant T_{i,j+1}$, and $c_{k+1} \leftarrow (i, j+1)$ otherwise.

When c_k is an outer corner, remove it from the resulting tableau, which we denote $J_{(i_1,j_1)}(T)$.

This definition applies equally well, mutatis mutandis, to shifted shapes.

For a tableau $T \in \mathrm{SSYT}(\lambda/\mu)$, choose $S \in \mathrm{SYT}(\mu)$. Let c^k be the cell in D_μ, so that $S(c^k) = k$ for all $1 \leqslant k \leqslant |\mu|$. The *rectification* of T is

$$\mathrm{rect}(T) = J_{c^1} \circ \cdots \circ J_{c^{|\mu|}}(T).$$

For $T \in \mathrm{ShSSYT}(\lambda/\mu)$, define rect_{Sh} analogously. It is known that $\mathrm{rect}(T)$ and $\mathrm{rect}_{Sh}(T)$ do not depend on the choice of S, see e.g. [26].

Example 2.9. We compute a jeu de taquin slide with inner corner $c = (1, 2)$:

$$T = \begin{array}{ccccc} & \bullet & \boxed{2} & \boxed{5} & \boxed{9} \\ & \boxed{2} & \boxed{4} & \boxed{7} & \boxed{10} \\ \boxed{1} & \boxed{6} & \boxed{8} & & \end{array} \to \begin{array}{ccccc} & \boxed{2} & \boxed{2} & \boxed{5} & \boxed{9} \\ & \bullet & \boxed{4} & \boxed{7} & \boxed{10} \\ \boxed{1} & \boxed{6} & \boxed{8} & & \end{array} \to \begin{array}{ccccc} & \boxed{2} & \boxed{2} & \boxed{5} & \boxed{9} \\ & \boxed{4} & \bullet & \boxed{7} & \boxed{10} \\ \boxed{1} & \boxed{6} & \boxed{8} & & \end{array} \to \begin{array}{ccccc} & \boxed{2} & \boxed{2} & \boxed{5} & \boxed{9} \\ & \boxed{4} & \boxed{7} & \bullet & \boxed{10} \\ \boxed{1} & \boxed{4} & \boxed{8} & & \end{array} \to \begin{array}{ccccc} & \boxed{2} & \boxed{2} & \boxed{5} & \boxed{9} \\ & \boxed{4} & \boxed{7} & \boxed{10} & \bullet \\ \boxed{1} & \boxed{6} & \boxed{8} & & \end{array} = J_{(1,2)}(T)$$

Here, the equalities ignore the sliding square which we denote by "$\bullet$". In this case, the rectification of T is given by

$$\mathrm{rect}(T) \;=\; \begin{array}{ccccc} \boxed{1} & \boxed{2} & \boxed{2} & \boxed{5} & \boxed{9} \\ \boxed{4} & \boxed{7} & \boxed{10} & & \\ \boxed{6} & \boxed{8} & & & \end{array}$$

Theorem 2.10 ([67, Thm 2.25]). *For every permutation w, we have $Q_{\mathrm{MS}}(w) = \mathrm{rect}_{SH}\big(Q(w)\big)$.*

2.5. Reduced words for fully commutative permutations. The symmetric group is generated by the simple transpositions $\{s_1, \ldots, s_n\}$ with relations

$$s_i s_j = s_j s_i \ \text{ for } |i-j| > 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{and} \quad s_i^2 = 1.$$

The first relation is called a *commutation relation*, while the second is called a *braid relation*. For $w \in W$, we say that $\mathbf{a} = (a_1, \ldots, a_p)$ is a *reduced word* of w if $w = s_{a_1} \ldots s_{a_p}$ and $p = \ell(w)$ is the number of inversions in w. Let $\mathcal{R}(w)$ denote the set of reduced words of w. In this setting, the *Matsumoto–Tits theorem* (see e.g. [11, Thm 25.2]), says that $\mathcal{R}(w)$ is connected by commutation relations and braid relations. A permutation w is called *fully commutative* if $\mathcal{R}(w)$ is connected only by commutation relations. We summarize a key result about fully commutative permutations from [7].

Theorem 2.11 ([7, §2]). *For each fully commutative permutation w, there is a skew shape $\sigma(w) = \lambda/\mu$ and a bijection $\Phi : \mathrm{SYT}(\sigma(w)) \to \mathcal{R}(w)$, so that*

$$\mathrm{Des}\big(\Phi(T)\big) \;=\; \mathrm{Des}(T)^r.$$

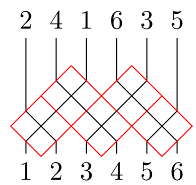
Our description of Φ follows [72]. Put a partial order $(I(w), <)$ on the *inversion set*

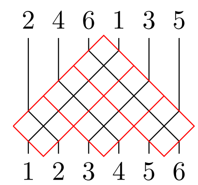
$$I(w) \coloneqq \big\{(i,j) \;:\; i < j\,,\ w^{-1}(i) > w^{-1}(j)\big\}.$$

We say that $(i,j) \gtrdot (k,l) \in I(w)$ if $k = i$ and $l = \min\{p > j : (j,p) \notin I(w)\}$, or $j = l$ and $i = \max\{p < k : (p,k) \notin I(w)\}$. The skew shape $\sigma(w)$ is now defined to be the poset $(I(w), <)$. Each cell of $\sigma(w)$ corresponds to an inversion of w, while each $T \in \mathrm{SYT}(\sigma(w))$ provides an ordering of inversions for w, which corresponds to a reduced word of w as follows. For L a linear extension of $(I(w), <)$, we define a word $\Phi(L) = (a_1, \ldots, a_p)$, where a_i is the index so that

$$\prod_{j=1}^{i-1} L(j)\, s_{a_i} \;=\; \prod_{j=1}^{i} L(j)\,.$$

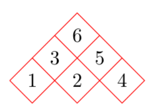
Example 2.12. The permutations (given in one-line notation) $v = 241635$ and $w = 246135$ are fully commutative. We can visualize their inversion sets $I(v) = \big\{(1,2),\, (1,4),\, (3,4),\, (3,6),\, (5,6)\big\}$ and $I(w) = I(v) \cup \big\{(1,6)\big\}$ using the diagrams





The corresponding shapes are $\sigma(241635) = (3,2,1)/(1)$ while $\sigma(246135) = (3,2,1)$.

The reduced word $(1,3,2,5,4,3) \in \mathcal{R}(w)$ with inversion sequence $(1,2)$, $(3,4)$, $(1,4)$, $(5,6)$, $(3,6)$, $(1,6)$ corresponds to the linear order shown below. Complementing the values, rotating 45° counter-clockwise and transposing, we obtain the desired $T \in \mathrm{SYT}(\sigma(w))$.



$$\mapsto \quad \begin{array}{|c|c|c|} \hline 1 & 4 & 3 \\ \hline 2 & 5 & \\ \cline{1-2} 6 & & \\ \cline{1-1} \end{array} = T.$$

One can check $\mathrm{Des}((1,3,2,5,4,3)) = \{2,4,5\}$ and $\mathrm{Des}(T) = \{1,2,4\} = \{2,4,5\}^r$.

Remark 2.13. The reason Φ reverses descent sets is that our conventions identify the skew shape λ/μ with the permutation w such that $I(w)$ is the reverse of $D_{\lambda/\mu}$ as a poset. This choice is necessary for our proof outline to work.

A permutation with exactly one descent is called *Grassmannian*.

Proposition 2.14. *For w Grassmannian, $\mathcal{R}(w)$ is a single Knuth class.*

This proposition is a consequence of results in [17] and [7]. Specifically, Edelman and Greene show for w Grassmannian that $\mathcal{R}(w)$ forms a single equivalence classes under relations called Coxeter-Knuth moves that generalize Knuth relations. Billey, Jockusch and Stanley show the Coxeter-Knuth moves coincide with Knuth moves for fully commutative permutations. One can also observe Proposition 2.14 directly from Φ by identifying the action of a Knuth move on $\mathbf{a} \in \mathcal{R}(w)$ with the corresponding action on $\Phi^{-1}(\mathbf{a})$, which is a dual equivalence move as introduced in [26]. In either case, the proof is bijective.

Fix a Grassmannian permutation $w^{\delta_n} := 24\ldots(2n)\,13\ldots(2n-1)$. By identifying its inversions, it is clear that $\sigma(w^{\delta_n}) = \delta_n$.

Corollary 2.15. *The map Φ is a bijection from $\mathrm{SYT}(\delta_n)$ to $\mathcal{R}(w^{\delta_n})$, and $\mathcal{R}(w^{\delta_n})$ is a single Knuth class.*

When w is Grassmannian, Proposition 2.14 shows that Φ^{-1} differs from the RSK recording tableau by a simple transformation (see also [17]). One can extend this relationship to the skew setting. However, our proofs in §4 require working with Φ. Additionally, note that Φ can be computed in $O(n \log n)$ operations when $|\lambda| = n$, while computing an insertion algorithm or its inverse requires $O\big(\max\{\lambda_1, \lambda_1'\} \cdot n \log n\big)$ operations, cf. [55].

3. Main results

Let $\mu \subset \delta_n$ and $T \in \mathrm{SYT}(\delta_n/\mu)$. The proofs in this section rely on the map $\varphi = Q_{\mathrm{SW}} \circ \Phi$:

$$T \xrightarrow{\Phi} \mathbf{a} \xrightarrow{Q_{\mathrm{SW}}} Q_{\mathrm{SW}}(\mathbf{a}). \tag{3.1}$$

By Theorem 2.6(3) and Theorem 2.11, φ reverses descent sets. When $\mu = \rho_{\ell,m}$, one checks for an appropriate choice of T that $\varphi(T) \in \mathrm{ShSYT}'(\varrho_n/\tau_{\ell,m})$. By applying properties of Knuth classes (see Proposition 3.5) and assuming Theorem 1.2, this implies φ is the desired bijection. In this section, we give an independent combinatorial proof that φ is a bijection.

3.1. The bijection for staircases. As a warmup, we show φ is a bijection between $\mathrm{SYT}(\delta_n)$ and $\mathrm{ShSYT}'(\varrho_n)$. Our proof relies on a simple observation that is used implicitly in [33, §10].

Proposition 3.1. *For any n, $\mathcal{R}(w^{\delta_n})$ is a single shifted Knuth class.*

Proof. By Corollary 2.15, the result will follow by confirming that exchanging the first two entries of $\mathbf{a} \in \mathcal{R}(w^{\delta_n})$ results in another element of $\mathcal{R}(w^{\delta_n})$. This follows from the definition of Φ, which implies in this case that the first two entries have the same parity so the exchange is a commutation relation. $\square$

For a shifted shape λ, recall M^{λ} is the minimal increasing tableau of shape λ in the alphabet $\{1, 2, \dots\}$. For example,

$$\begin{array}{cccc} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \\ & \boxed{3} & \boxed{4} & \boxed{5} \\ & & \boxed{5} & \boxed{6} \\ & & & \boxed{7} \end{array} = M^{\varrho_5}.$$

Lemma 3.2. *Let $\mathbf{a} \in \mathcal{R}(w^{\delta_n})$. Then $P_{\mathrm{SW}}(\mathbf{a}) = M^{\varrho_n}$.*

Proof. By Proposition 3.1, the result follows if it holds for some $\mathbf{a} \in \mathcal{R}(w^{\delta_n})$. This is easy to check for $\mathbf{a} = (1, 3, 5, \dots, 2n-1, 2, 4 \dots, 2n-2, 3, 5, \dots, 2n-3, \dots)$. $\square$

Proposition 3.3. *The map φ is a bijection from $\mathrm{SYT}(\delta_n)$ to $\mathrm{ShSYT}'(\varrho_n)$. Hence, $s_{\delta_n} = P_{\varrho_n}$.*

Proof. By Corollary 2.3 and Lemma 2.4, the result will follow from a bijection $\mathrm{SYT}(\delta_n) \to \mathrm{ShSYT}'(\varrho_n)$ that reverses descent sets. By Proposition 3.1 and Lemma 3.2, the map φ is a bijection. $\square$

3.2. Proof of Theorem 1.5. To prove Theorem 1.5, we will need to describe the fully commutative permutation corresponding to δ_n/μ and a small extension of Proposition 3.1.

Given $\mu = (\mu_1, \dots, \mu_\ell) \subseteq \delta_n$, let $w^{\delta_n/\mu}$ be the permutation obtained from w^{δ_n} by sequentially moving $(2i-1)$ to the left μ_i positions for $i \in [k]$. For example, for $n = 6$, we have $w^{\delta_6} = 2468\,10\,13579$ and $w^{\delta_6/(3,1)} = 241683\,10\,579$.

Lemma 3.4. *For all $\mu \subseteq \delta_n$, we have $\sigma(w^{\delta_n/\mu}) = \delta_n/\mu$, so $\Phi : \mathrm{SYT}(\delta_n/\mu) \to \mathcal{R}(w^{\delta_n/\mu})$.*

Proof. The result follows from direct inspection. $\square$

Proposition 3.5. *Let $\mu \subseteq \delta_n$. Then $\mathcal{R}\big(w^{\delta_n/\mu}\big)$ is a union of shifted Knuth classes.*

Proof. The proof is essentially identical to that of Proposition 3.1, except that $\mathcal{R}\big(w^{\delta_n/\mu}\big)$ is a union of Knuth classes, hence a union of (possibly fewer) shifted Knuth classes. $\square$

Proof of Theorem 1.5. Let $\mu \subseteq \delta_n$. By Proposition 3.5, $\mathcal{R}(w^{\delta_n/\mu})$ is a union of shifted Knuth classes. Hence, by Theorem 2.6(2), the map φ from (3.1) is a bijection

$$\mathrm{SYT}(\delta_n/\mu) \to \bigsqcup_{\lambda \in M_{\delta_n/\mu}} \mathrm{ShSYT}'(\lambda),$$

for some multiset of partitions determined by δ_n/μ. This gives a bijective proof that $s_{\delta_n/\mu} \in \widetilde{\Lambda}$.

To prove $s_{\delta_n/\mu} = s_{\delta_n/\mu'}$, we show that φ, as applied to both $\mathrm{SYT}(\delta_n/\mu)$ and $\mathrm{SYT}(\delta_n/\mu')$, results in the same multisets of shifted standard tableaux with marked entries. Indeed, let φ be the map for $\mathrm{SYT}(\delta_n/\mu)$, and let φ' be the map for $\mathrm{SYT}(\delta_n/\mu')$. Note that for $T \in \mathrm{SYT}(\delta_n/\mu)$ with $\Phi(T) = (a_1, \ldots, a_p)$, we have $\Phi(T') = (n - a_1, \ldots, n - a_p)$. Therefore, $\mathrm{Des}(\Phi(T')) = \mathrm{Des}(\Phi(T))^c$.

Since Q_{SW} is invariant under shifting each entry of a word, by Theorem 2.7 we have

$$T \xrightarrow{\varphi} \varphi(T) \xrightarrow{\circ} \varphi(T)^\circ \xrightarrow{(\varphi')^{-1}} (\varphi')^{-1}(\varphi(T)^\circ) \tag{3.2}$$

is a descent set preserving bijection from $\mathrm{SYT}(\delta_n/\mu)$ to $\mathrm{SYT}(\delta_n/\mu')$, so the result follows by Corollary 2.3. $\square$

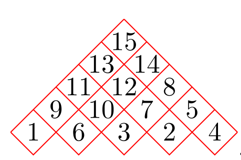
Proof of Corollary 1.4. Recall the RSK interpretation of LR–coefficients, see e.g. [68, Thm A1.3.1] and [80]. The desired bijection now follows combining our bijection in (3.2) with RSK and RSK^{-1}. The details are straightforward. $\square$

3.3. Proof of Theorem 1.7. We outline the strategy with an example.

Example 3.6. Consider the shifted tableaux $T \in \mathrm{ShSYT}'(\varrho_6 - \tau^{2,2}), \widetilde{T} \in \mathrm{ShSYT}'(\varrho_6)$ such that

$$T \;=\; \begin{array}{ccccc} 1 & 2 & 4 & 6' & 9' \\ & 3 & 5 & 8 & 11' \\ & & 7 & 10' & \end{array} \qquad \text{and} \qquad \widetilde{T} \;=\; \begin{array}{ccccc} 1 & 2 & 4 & 6' & 9' \\ & 3 & 5 & 8 & 11' \\ & & 7 & 10' & 13' \\ & & & 12 & 14 \\ & & & & 15 \end{array}\,.$$

The Worley–Sagan inverse of $(M^{\varrho_6}, \widetilde{T})$ is $\widetilde{\mathbf{a}} = (1, 7, 5, 9, 8, 3, 6, 7, 2, 4, 3, 5, 4, 6, 5)$. Removing the last four entries of $\widetilde{\mathbf{a}}$, we have $\mathbf{a} = (1, 7, 5, 9, 8, 3, 6, 7, 2, 4, 3)$ satisfying $Q_{\mathrm{SW}}(\mathbf{a}) = T$. Viewing $\widetilde{\mathbf{a}}$ as a linear extension of its heap, we obtain:



.

Now we can compute:

$$\Phi^{-1}(\widetilde{\mathbf{a}}) \;=\; \begin{array}{ccccc} 1 & 3 & 5 & 7 & 15 \\ 2 & 4 & 6 & 10 & \\ 8 & 9 & 13 & & \\ 11 & 14 & & & \\ 12 & & & & \end{array} \;\in\; \mathrm{SYT}(\delta_6/\rho^{2,2}) \text{ and } \Phi^{-1}(\mathbf{a}) \;=\; \begin{array}{ccccc} & & 1 & 3 & 1 \\ & & 2 & 6 & \\ 4 & 5 & 9 & & \\ 7 & 10 & & & \\ 8 & & & & \end{array} \;\in\; \mathrm{SYT}(\delta_6/\rho^{2,2})\,.$$

Therefore, $\widetilde{T} = \varphi\big(\Phi^{-1}(\widetilde{\mathbf{a}})\big)$, and our bijection maps $\Phi^{-1}(\mathbf{a})$ to T.

We generalize Example 3.6 as follows:

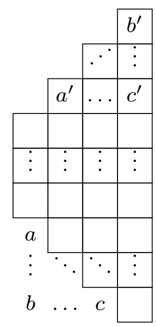
Proposition 3.7. *For positive integers $\ell \leqslant m < n$ with $\ell + m < n$, there is an injection*

$$\psi \;:\; \mathrm{ShSYT}'\big(\varrho_n - \tau^{\ell,m}\big) \;\longrightarrow\; \mathrm{SYT}\big(\delta_n/\rho^{\ell,m}\big)$$

that reverses descent sets.

Proof. Let $T \in \mathrm{ShSYT}'(\varrho_n - \tau^{\ell,m})$. Following Example 3.6, construct $\widetilde{T}$ by adding to T as follows.

(3.3)



Fill the entries of $\rho^{m,\ell}$ column by column from left to right, top to bottom, with values from $\left\{\binom{n}{2} - \ell\cdot m+1, \dots, \binom{n}{2}\right\}$, in increasing order. Reflect the cutout staircase $\rho^{\ell,m} \searrow \tau^{\ell,m}$, marking each entry and place the resulting tableau on top of $\rho^{\ell,m} \searrow \tau^{\ell,m}$. The values a, b, c in (3.3) show where corresponding entries are mapped by this operation. Computing the Worley–Sagan inverse of $(M^{\varrho_n}, \widetilde{T}^{\circ})$, we obtain some $\widetilde{\mathbf{a}} \in \mathcal{R}(w^{\delta_n})$. Removing the last $\ell \cdot m$ entries of $\widetilde{\mathbf{a}}$, we obtain $\mathbf{a}$. Now define $\psi(T) \coloneqq \mathbf{a}$. Since map ψ is a restriction of φ^{-1}, we see it reverses descent sets. What remains is to show the image of ψ lies in $\mathrm{SYT}(\delta_n/\rho^{\ell,m})$.

Let $a \coloneqq m - \ell$. We claim the last $\ell \cdot m$ entries $\mathbf{a}$ are necessarily of the form

(3.4)
$$(n+\ell-1, n+\ell-2, \dots, n-a-1, n+\ell-2, n+\ell-3, \dots, n-a-2, \dots, n+m-1, \dots, n-1).$$

The proof is a straightforward induction using inverse Worley–Sagan insertion. Rather than give complete details, we demonstrate the result with an example with $\ell = 2$, $m = 3$, and $n = 6$. Here we have:

$$M^{\varrho_n} = \begin{array}{ccccc} 1&2&3&4&5\\ &3&4&5&6\\ &&5&6&7\\ &&&7&8\\ &&&&9 \end{array} \qquad \text{and} \qquad \widetilde{T}^{\circ} - T^{\circ} = \begin{array}{ccccc} \cdot&\cdot&\cdot&\cdot&\cdot\\ &\cdot&\cdot&\cdot&12'\\ &&\cdot&10&13\\ &&&11&14\\ &&&&15 \end{array}$$

When inverting the insertions that add the $15, 14$ and 13, we see they must come from row insertions with all bumps in the last column. After this, we obtain

$$M^{\varrho_n} = \begin{array}{ccccc} 1&2&3&4&8\\ &3&4&5&9\\ &&5&6&\\ &&&7& \end{array} \qquad \text{and} \qquad \widetilde{T}^{\circ} - T^{\circ} = \begin{array}{ccccc} \cdot&\cdot&\cdot&\cdot&\cdot\\ &\cdot&\cdot&\cdot&12'\\ &&\cdot&10&\\ &&&11& \end{array}$$

Inverting the insertion that added $12'$, we must use column insertion. Necessarily, the 9 was bumped by the 7, which was row bumped by the entries above it in the fourth column. This results in:

$$M^{\varrho_n} = \begin{array}{ccccc} 1&2&3&5&8\\ &3&4&6&\\ &&5&7&\\ &&&9& \end{array} \qquad \text{and} \qquad \widetilde{T}^{\circ} - T^{\circ} = \begin{array}{ccccc} \cdot&\cdot&\cdot&\cdot&\cdot\\ &\cdot&\cdot&\cdot&\\ &&\cdot&10&\\ &&&11& \end{array}$$

The remaining entries necessarily came from insertions occurring in the fourth column as well, resulting in

$$M^{\varrho_n} = \begin{array}{ccccc} 1&2&3&7&8\\ &3&4&9&\\ &&5&& \end{array} \qquad \text{and} \qquad \widetilde{T}^{\circ} - T^{\circ} = \begin{array}{ccccc} \cdot&\cdot&\cdot&\cdot&\cdot\\ &\cdot&\cdot&\cdot&\\ &&\cdot&& \end{array}$$

The last values of $\mathbf{a}$ in this example are $(6, 5, 4, 7, 6, 5)$, as claimed in (3.4). The reader should be able to extend this example to an inductive proof of our claim with little difficulty but some tedium. $\square$

Assuming Theorem 1.7, we see Proposition 3.7 gives a bijection from $\mathrm{ShSYT}'(\varrho_n - \tau^{\ell,m})$ to $\mathrm{SYT}(\delta_n/\rho^{\ell,m})$. Since $\psi = \varphi^{-1}$ when $\varphi(T) \in \mathrm{ShSYT}'(\varrho_n - \tau^{\ell,m})$, we can conclude φ is the desired bijection. However, it remains to prove this fact directly without resorting to Theorem 1.7. In principle, one could have some $\mathbf{a} \in \mathcal{R}(w^{\delta_n})$ that ends with the values in (3.4) such that $Q_{\mathrm{SW}}(\mathbf{a})$ does not contain $\widetilde{T} - T$ as a subtableau.

Rather than prove directly that φ has the desired image, we outline a map essentially equivalent to φ using RSK, mixed shifted insertion and jeu de taquin (this map could also be used to prove Theorems 1.2 and 1.3). For λ a partition, let $S^{\lambda} \in \mathrm{SYT}(\lambda)$ be the *superstandard tableau*, which is the unique standard tableau whose rows are consecutive. Computing slides row-by-row, we observe:

$$S^{\varrho_n} = \mathrm{rect}\big(S^{\delta_n}\big)\,. \tag{3.5}$$

The following lemma is implied by the proof of Proposition 3.7 and Theorem 2.8(2).

Lemma 3.8. *Let $T \in \mathrm{SYT}'(\tau_{r,\ell})$ be obtained from the tableau in (3.3) by reflecting across the line $y = x$ and complementing values. Then $T = Q_{\mathrm{SW}}\big(r(S^{\rho_{r,\ell}})\big)$.*

We now outline the alternate bijection to φ:[4]

1. For $P_0 \in \mathrm{SYT}(\delta_n/\rho_{r,\ell})$, complete it to $\widetilde{P_0} \in \mathrm{SYT}(\delta_n)$ with the same relative order
   in $D_{\delta_n/\rho_{r,\ell}}$ so that $\widetilde{T} \mid \rho_{r,\ell} = S^{\rho_{r,\ell}}$
2. Let $w \coloneqq \mathrm{RSK}^{-1}\big(\widetilde{T}, S^{\delta_n}\big)$
3. By Theorem 2.10 and Equation (3.5), the mixed shifted insertion maps w to $\big(P_{\mathrm{MS}}(w), S^{\varrho_n}\big)$
4. Flip $P_{\mathrm{MS}}(w)$ across the $y = x$ line and complement its values to obtain $\widetilde{P_1}$
5. Define $P_1 \coloneqq \widetilde{P_1} \mid D_{\varrho_n - \tau_{r,\ell}}$

By Lemma 3.8, we see the entries in $\widetilde{P_1} - P_1$ are the same as those in the tableau constructed in (3.3). Since each step outlined above is injective, we obtain an injection from $\mathrm{SYT}(\delta_n/\rho_{r,\ell})$ to $\mathrm{ShSYT}'(\varrho_n/-\tau_{r,\ell})$.

Note that Step (3) is not obviously bijective, since Theorem 2.10 only implies the forwards direction. This completes our bijective proof of Theorem 1.7, hence also of Theorem 1.2. $\square$

3.4. Proof of Theorem 1.3. Start by generating a random $T \in \mathrm{ShSYT}'\big(\varrho_k - \tau^{\ell,m}\big)$ using either [19] or [62]. Each algorithm involves $O(k^2)$ iterations of $O(k)$ steps, each step involving moving a single square, giving the total $O(k^3)$ cost. Use the algorithm above to compute $\varphi(T) \in \mathrm{SYT}\big(\delta_k/\rho^{\ell,m}\big)$. The cost of the Worley–Sagan insertion is equal to that of RSK, thus bounded by $O(k^3 \log k)$ in this case [55]. This implies the total bound as in the theorem. $\square$

4. K-theoretic extensions

4.1. K-theoretic objects. The objects and maps introduced in §2 have K-theoretic analogues. We give compact descriptions of these objects, referring the reader to our references for concrete examples.

For λ a partition, a *set-valued standard Young tableau* of shape λ and *size* n is a function $T : D_\lambda \to 2^{[n]}$, so that $T(i,j) \neq \varnothing$, $\max T(i,j) < \min T(i{+}1,j), \min T(i,j{+}1)$ and $\sqcup_{(i,j)\in D_\lambda} T(i,j) = [n]$. Note the latter condition requires the entries of T to be

[4] Strictly speaking, it follows from Theorem 1.7 that our map ψ in Proposition 3.7 is bijective. The argument here is employed to obtain a self-contained proof.

disjoint. Let $\overline{\mathrm{SYT}}(\lambda)$ be the set-valued tableaux of shape λ and $\overline{\mathrm{SYT}}_n(\lambda)$ be the subset of tableaux whose size is n.

Similarly, for μ a shifted shape, one defines *shifted set valued standard tableaux*, denoted by $\overline{\mathrm{ShSYT}}(\mu)$ and $\overline{\mathrm{ShSYT}}_n(\mu)$, as well as *marked shifted set valued standard tableaux*, denoted by $\overline{\mathrm{ShSYT}}'(\mu)$ and $\overline{\mathrm{ShSYT}}'_n(\mu)$. In the latter case, when $i > j$ each value in $T(i,j)$ is marked or unmarked individually. For $|\lambda| = n$, note that $\overline{\mathrm{SYT}}(\lambda) = \mathrm{SYT}(\lambda)$. Likewise, if $|\mu| = m$ then $\overline{\mathrm{ShSYT}}_m(\mu) = \mathrm{ShSYT}(\mu)$ and $\overline{\mathrm{ShSYT}}'(\mu) = \mathrm{ShSYT}^2(\mu)$.

For $T \in \overline{\mathrm{SYT}}_n(\lambda)$, the *descent set* is $\mathrm{Des}(T) \coloneqq \{i \in [n-1] : i \text{ is strictly above } i+1\}$ as before. Similarly, for $U \in \overline{\mathrm{ShSYT}}'_n(\mu)$, the *descent set* of U is defined the same as for marked shifted standard tableaux. Let U° be the tableau obtained from U by marking every unmarked value and unmarking every marked value in off-diagonal entries of U. It is easy to see that Lemma 2.1 extends to the set-valued setting.

Lemma 4.1. *For λ a shifted shape and $T \in \overline{\mathrm{ShSYT}}'_n(\lambda)$, we have:* $\mathrm{Des}(T^\circ) = \mathrm{Des}(T)^c$.

For $S \subset [n-1]$, recall the definition of I_S from §2.3. Define

$$G_\lambda \coloneqq \sum_{T \in \overline{\mathrm{SYT}}(\lambda)} \sum_{\mathbf{i} \in I_{\mathrm{Des}(T)}} x^{\mathbf{i}} \qquad \text{and} \qquad GP_\mu \coloneqq \sum_{T \in \overline{\mathrm{ShSYT}}'(\mu)} \sum_{\mathbf{i} \in \mathrm{Des}(T)} x^{\mathbf{i}} . \tag{4.1}$$

These definitions are non-standard, differing from the standard definitions by the invertible substitution of variables $x_i \mapsto \frac{-x_i}{1-x_i}$ and a factor of $(-1)^{|\lambda|}$, which is more or less an application of the ω involution for symmetric functions; see [56, Thm 6.11], [27, Thm 1.3] and [8, Cor. 6.6]. The tableaux T arising from the summations of (4.1) are multiset-valued semistandard tableaux, while the standard definition is in terms of set-valued semistandard tableaux.

The K-theoretic analogue of Worley–Sagan insertion is called *shifted Hecke insertion*, introduced in [57]. Rather than define this map, we refer the reader to [27, §2.2] for a verbose definition, or [30, §5.2] for a pseudocode description. An implementation of shifted Hecke insertion is available at [32].

The *K–Knuth relations* are the transformations:

$$acb \leftrightarrow cab, \quad bac \leftrightarrow bca, \quad aba \leftrightarrow bab, \quad aa \leftrightarrow a \quad \text{with} \quad a < b < c.$$

For a word $\mathbf{a} = (a_1, a_2, \ldots, a_p)$, a *$K$–Knuth move* is an application of a K–Knuth relation, while a *weak K–Knuth move* is a K–Knuth move or the exchange of the first two entries a_1 and a_2. Two words $\mathbf{a}$ and $\mathbf{b}$ are *K–Knuth equivalent* if they differ by a sequence of K–Knuth moves and *weak K–Knuth equivalent* if they differ by a sequence of weak K–Knuth moves. These notions of equivalence are introduced in [10] as K-theoretic analogues of Knuth and shifted Knuth equivalence.

For μ a shifted shape, let $\mathrm{INC}(\mu) \subseteq \mathrm{ShSSYT}(\mu)$ be the subset of T whose rows are strictly increasing.

Theorem 4.2.

(1) [57, Thm 5.19] *Shifted Hecke insertion is a bijection from words of length n to pairs (P_{SH}, Q_{SH}) of tableaux where $P_{SH} \in \mathrm{INC}(\mu)$ and $Q_{SH} \in \overline{\mathrm{ShSYT}}'_n(\mu)$ for some shifted shape μ.*

(2) [27, Cor. 2.18] and [10, Cor. 7.2] *If $P_{SH}(\mathbf{a}) = P_{SH}(\mathbf{b})$, then $\mathbf{a}$ and $\mathbf{b}$ are shifted K–Knuth equivalent. The converse holds when $P_{SH}(\mathbf{a})$ is a minimal increasing tableau.*

(3) [27, Prop. 2.24] *For every word $\mathbf{a}$, we have $\mathrm{Des}(\mathbf{a}) = \mathrm{Des}(Q_{SH}(\mathbf{a}))$.*

Remark 4.3. Note Theorem 4.2(2) is a weaker statement than Theorem 2.6(2). An analogue of Theorem 2.7 should exist for shifted Hecke insertion, but does not appear in the literature. There is a K-theoretic analogue of jeu de taquin [75, 14] that we do not require. At present, there is no K-theoretic analogue of mixed shifted insertion.

Hecke expressions. The *0-Hecke monoid* $(S_n, \circ)$ of the symmetric group replaces the relation $s_i^2 = 1$ with the relation $s_i \circ s_i = s_i$. For w a permutation, we say $\mathbf{a} = (a_1, \ldots, a_p)$ is a *0-Hecke expression* for w if $w = s_{a_1} \circ \cdots \circ s_{a_p}$. Let $\mathcal{H}(w)$ be the set of 0-Hecke expressions for w and $\mathcal{H}_n(w)$ be the subset of expressions of length n. For $p = \ell(w)$, note $\mathcal{H}_p(w) = \mathcal{R}(w)$.

Theorem 4.4 ([51, Prop. 14], see also [79, §3]). *For each fully commutative permutation w, there is a bijection $\mathsf{res} : \overline{\mathrm{SYT}}_n(\sigma(w)) \to \mathcal{H}_n(w)$ so that*

$$\mathrm{Des}(\mathsf{res}(T)) = \mathrm{Des}(T)^r.$$

We should mention that the descent set relationship is not stated explicitly in [51], but follows immediately from their proof. The construction of res parallels that of Φ. Let w be fully commutative and let $\mathbf{a} = (a_1, \ldots, a_p) \in \mathcal{H}(w)$. For each $i \in [p]$, s.t.

$$s_{a_1} \circ \cdots \circ s_{a_{i-1}} \neq s_{a_1} \circ \cdots \circ s_{a_i} ,$$

place i in the same entry as would be done in Φ. Similarly, for each $i \in [p]$, s.t.

$$s_{a_1} \circ \cdots \circ s_{a_{i-1}} = s_{a_1} \circ \cdots \circ s_{a_i} ,$$

there exists maximal $h < i$ with $a_h = a_i$; place i in the same cell as h in this case.

Finally, there are K-theoretic analogues of Proposition 2.14 and Corollary 2.15.

Proposition 4.5. *For w Grassmannian, the set $\mathcal{H}(w)$ is a single K–Knuth class. In particular, the set $\mathcal{H}(w^{\delta_n})$ is a single K–Knuth class.*

A proof of the proposition is implicit in [10, Thm. 6.2], and follows directly by combining [56, Thm. 3.16] and [61, Lem. 5.4], since Grassmannian permutations are vexillary.

4.2. K-theoretic results. In this section, we explain how to extend Theorem 1.2 to set-valued tableaux. Along the way, we prove K-theoretic extensions of Theorems 1.5,1.6, and 1.7. Our construction is essentially the same as φ with the necessary K-theoretic substitutions:

$$\overline{\varphi} \ \coloneqq \quad T \xrightarrow{\mathsf{res}} \mathbf{a} \xrightarrow{Q_{SH}} Q_{SH}(\mathbf{a}).$$

The proofs from 3 will extend almost verbatim. By Theorem 4.4 and Theorem 4.2(3), we see that $\overline{\varphi}$ reverses descent sets.

To begin, we extend Proposition 3.3 and Theorems 1.5 and 1.6 to the K-theoretic setting.

Proposition 4.6. *The map $\overline{\varphi}$ is bijection: $\overline{\mathrm{SYT}}_m(\delta_n) \to \overline{\mathrm{ShSYT}}'_m(\varrho_n)$ that reverses descent sets.*

Proof. By Proposition 4.5, the set $\mathcal{H}(w^{\delta_n})$ is a single K–Knuth equivalence class. Moreover, the first two entries in each $\mathbf{a} \in \mathcal{H}(w^{\delta_n})$ are odd, so it is a single weak K–Knuth equivalence class. To see that $\overline{\varphi}$ is a bijection, apply Theorem 4.2(1) and (2). Finally, observe that $P_{SH}(\mathbf{a})$ is a minimal increasing tableau for some $\mathbf{a} \in \mathcal{H}(w^{\delta_n})$ (this was already shown in the proof of Proposition 3.3). ☐

Denote by $\overline{\Lambda}$ the ring generated by GP_ν's, see [37].

Theorem 4.7 ([8, Thm. 6.7]). *Let $\mu \subseteq \delta_n$. Then $G_{\delta_n/\mu} \in \overline{\Lambda}$. Moreover, we have $G_{\delta_n/\mu} = \sum_\nu b_\nu \, GP_\nu$ where each b_ν is a non-negative integer.*

Proof. By Theorem 4.4, we have:

$$G_{\delta_n/\mu} = \sum_{n\geqslant 1} \sum_{\mathbf{a}\in\mathcal{H}_n(w^{\delta_n/\mu})} \sum_{\mathbf{i}\in I_{(\mathrm{Des}(\mathbf{a})^c)^r}} x^{\mathbf{i}}\,.$$

Since $\mathcal{H}\big(w^{\delta_n/\mu}\big)$ is closed under K–Knuth moves, it is a finite union of K–Knuth classes. Additionally, the first two entries of $\mathbf{a}\in\mathcal{H}\big(w^{\delta_n/\mu}\big)$ must both be odd. This implies $\mathcal{H}\big(w^{\delta_n/\mu}\big)$ is a union of (possibly fewer) weak K–Knuth classes. Therefore by Theorem 4.2(2), we see $G_{\delta_n/\mu}$ is a sum of finitely many GP_μ's, and the result follows. □

Corollary 4.8 ([1, Thm 1.3]). *For all $\mu\subseteq\delta_n$, we have $G_{\delta_n/\mu} = G_{\delta_n/\mu'}$.*

Proof. By Theorem 4.7, both $G_{\delta_n/\mu}$ and $G_{\delta_n/\mu'}$ can be expressed as generating functions over marked shifted semi-standard set-valued tableaux, which have the descent-set complementing involution $\circ$. Therefore, the result follows by observing that $\mathbf{a}=(a_1,\dots,a_n)\mapsto(n-a_1,\dots,n-a_n)$ is descent-set complementing bijection $\mathcal{H}\big(w^{\delta_n/\mu}\big)\to\mathcal{H}\big(w^{\delta_n/\mu'}\big)$. □

Next, we give an algebraic proof for a K-theoretic analogue of Theorem 1.7, which we will use to prove the K-theoretic analogue of Theorem 1.2.

Theorem 4.9. *For $\ell+m<n$, $G_{\delta_n/\rho_{\ell,m}} = GP_{\varrho_n-\tau_{\ell,m}}$.*

Proof. Combining [29, Cor 6.22] and [46, Cor. 4.6], we see $G_{\delta_n/\mu}$ is a single symplectic stable Grothendieck polynomial for some fixed-point-free involution y. By Theorem 1.7 we see $s_{\delta_n/\rho_{\ell,m}} = P_{\varrho_n-\tau_{\ell,m}}$, so y is FPF-vexillary in the sense of [30]. The result now follows from [47, Cor. 3.11] and [31, Cor. 5.9], since with each application of Corollary 3.11 only one term occurs on the RHS after cancelling like terms. □

Theorem 4.10. *The map $\overline{\varphi}:\overline{\mathrm{SYT}}_m(\delta_n/\rho_{\ell,m})\longrightarrow\overline{\mathrm{ShSSYT}}'_m(\varrho_n-\tau_{\ell,m})$ is a bijection.*

Proof. We construct the inverse map as before. Fix $T\in\overline{\mathrm{ShSSYT}}'_m(\varrho_n-\tau_{\ell,m})$. Construct $\widetilde{T}$ by filling entries in $\varrho_n/(\varrho_n-\tau_{\ell,m})$ as described in (3.3), and let P_1 be the minimal increasing tableau of shape ϱ_n. One can easily check the first $\ell\cdot m$ steps to invert shifted Hecke insertion will coincide with those used in inverting Worley–Sagan insertion. Similarly, applying res^{-1} will then result in a tableau $\widetilde{T}'\in\overline{\mathrm{SYT}}(\delta_n)$ that, when restricted to $D_{\rho_{\ell,m}}$, gives the super standard tableau of that shape. The forward direction now follows by Theorems 1.2 and 4.9. □

For geometric reasons, it is easier to work with symplectic stable Grothendieck polynomials in our proof of Theorem 4.9. This is the same identification used by Lewis and Marberg in their proof of Theorem 4.7. However, we define $\overline{\varphi}$ using shifted Hecke insertion rather than its symplectic analogue [45], so that it is manifestly a generalization of φ.

5. Final remarks

5.1. In [59], Purbhoo constructs a bijection $\mathrm{SYT}(\delta_n)\to\mathrm{ShSYT}'(\varrho_n)$ via a jeu de taquin-like algorithm called *conversion*. By [25, Prop. 7.1], his map is equivalent to the alternate bijection we define after Lemma 3.8, which has a conversion formulation in full generality.

5.2. John Stembridge proved Theorem 1.5 in June 2004, but the proof was never published [70]. Two proofs of Theorem 1.5 were given in [60, Cor. 7.32], where the authors also attributed this result to Stembridge. A different algebraic proof was given in [24, Solution to Exc. 2.9.25]. Recently, a generalization of the theorem to stable Grothendieck polynomials is given in [1]

A different generalization to *Macdonald's ninth variation Schur functions* was given in [20]; the proof is based on the *Hamel–Goulden identities*.

5.3. The approach in [41] to the proof of Theorem 1.1 is based on explicit computation of determinants. This type of argument somewhat hides the role of the staircase shape which is crucial for the proof. In a forthcoming paper [42] the authors extend the determinant approach from $\delta_k/(b^a)$ to all δ_k/μ using certain new determinantal and q-determinantal identities.

5.4. Let $\mathfrak{q}(n)$ be the *queer Lie superalgebra*. There is a $\mathfrak{q}(n)$-crystal structure on the set of words $[n]^m$ of length m on the alphabet $[n]$. The semistandard version of Theorem 2.8(1) [67, Def. 1.2] gives

$$\begin{aligned} \mathrm{MS} : [n]^m \ &\to \ \bigsqcup_{\lambda \vdash m} \mathrm{ShSSYT}'(\lambda) \times \mathrm{ShSYT}(\lambda)\,, \\ \text{where} \qquad w \ &\to \ \big(P_{\mathrm{MS}}(w), Q_{\mathrm{MS}}(w)\big). \end{aligned}$$

The connected component of the crystal in which a word $w \in [n]^m$ is found is specified by $Q_{\mathrm{MS}}(w)$, giving a $\mathfrak{q}(n)$-crystal structure on shifted marked semistandard tableaux [36, Thm 3.2] (see also [35]). Using the obvious correspondence between a partition without a staircase and the complementary shape in a larger square, by [12, Prop. 23], a $\mathfrak{q}(n)$-crystal structure on the set of semistandard skew tableaux with maximum entry n of shape staircase minus rectangle is obtained by sending a tableau T to a word $w_T \in [n]^m$ by scanning rows from top to bottom, with each row read from right to left and where a box with the entry i is recorded as $n+1-i$.

By uniqueness of crystals, it follows that a bijection from skew semistandard tableaux to shifted marked semistandard tableaux (each with entries bounded by n) is given by $P_{\mathrm{MS}}(w_T)$, where w_T is the reading word (as above) of a skew tableau T. The bijective correspondence of Theorem 1.2 follows by restricting to the zero-weight space when $n = m$ (containing tableaux with standard content).

Example 5.1. Continuing with Example 3.6, for

$$T \ = \ \begin{array}{ccccc} & & 1 & 3 & 11 \\ & & 2 & 6 & \\ 4 & 5 & 9 & & \\ 7 & 10 & & & \\ 8 & & & & \end{array}$$

we construct the reading word $w_T = (1, 9, 11, 6, 10, 3, 7, 8, 2, 5, 4)$, which mixed inserts to the pair

$$\big(P_{\mathrm{MS}}(w_T), Q_{\mathrm{MS}}(w_T)\big) \ = \ \left(\begin{array}{ccccc} 1 & 2 & 4 & 6' & 9' \\ & 3 & 5 & 8 & 11' \\ & & 7 & 10' & \end{array} \ , \ \begin{array}{ccccc} 1 & 2 & 3 & 7 & 8 \\ & 4 & 5 & 9 & 10 \\ & & 6 & 11 & \end{array} \right).$$

Restricting to $P_{\mathrm{MS}}(w_T)$ gives the desired bijection; it is possible to explicitly characterize $Q_{\mathrm{MS}}(w_T)$. Note that this bijection works equally well when starting with a skew semistandard tableau instead of a skew standard tableau.

5.5. For symmetries of LR–coefficients, see [6, 34]. See also a bijective proof in [54] relating the highly symmetric BZ–*triangles* and the (usual) LR–tableaux. In summary, all these hidden symmetries of LR–tableaux have now been established via a chain of bijections. See also an unusual construction in [74] which trades off effectiveness of a combinatorial interpretation for greater symmetry. Finally, we refer to [53] for a brief overview of further examples of hidden symmetry.

5.6. Random generation (sampling) of combinatorial objects from the (exactly) uniform distribution is a classical problem in both Combinatorics, see e.g. [52], and Theoretical Computer Science, see e.g. [38]. The approach of using determinantal formulas for uniform random generation of planar structures was introduced by Wilson [77].

For Young tableaux of staircase minus rectangle shape, our approach is also greatly superior to the MCMC approach for the *nearly uniform* generation of linear extensions of all posets. Indeed, the best known general bound is $O(n^3 \log n)$ time for n-element posets, due to Bubley and Dyer [9]. In our case, we have $n = \Theta(k^2)$, giving only a $O(k^6 \log k)$ time, which is much weaker compared with the $O(k^3 \log k)$ time in Theorem 1.3. It would be interesting to see if a nearly linear MCMC algorithm can be obtain for nearly uniform sampling from $\mathrm{SYT}(\lambda/\mu)$ in our case, or (even better) for general skew shapes.

Finally, we should mention a detailed complexity analysis of the *NPS algorithm* given in [66]; similar results likely hold for Fischer's algorithm [19]. Note that a rough $O(k^3)$ suffices for our purposes as the cost of the algorithm is (marginally) dominated by the Worley–Sagan insertion.

5.7. Theorem 4.7 does not have the same applications as Theorem 1.3, since there is no known probabilistic algorithm to sample from $\overline{\mathrm{ShSSYT}}'(\varrho_n - \tau_{\ell,m})$ uniformly at random, or even nearly-uniformly. In fact, we are only aware of a few incremental results for the number of certain set-valued tableaux in some nice special cases, see e.g. [16, 61]. In particular, no determinantal formula is known for the number of set-valued standard tableaux, cf. the discussion in [48, §5]. It would be interesting to show that the number of such tableaux is $\#\mathsf{P}$-complete.

5.8. Note that our bijection proving (1.2) is not computable in linear time in contrast with bijections in [54], nor is it easily comparable with bijections in [55] since the lengths of parts are not in binary. It would be interesting to show that in the terminology of [55], this bijection is linear time equivalent to the bijection in [34].

5.9. Theorem 1.7 has a geometric explanation in terms of certain *spherical orbit closures* on the type A flag varieties (see [30, Thm 4.58]). Here, the skew Schur functions are geometric representatives for *Schubert varieties* indexed by fully commutative permutations, while the Schur P–functions represent certain *involution Schubert varieties*. The use of $\mathcal{R}(w^{\delta_n})$ in our proof reflects the fact, shown in [28, Prop. 3.30], that the varieties in question have the same cohomology representatives.

From this perspective, the desired bijection follows from applying both Φ^{-1} and *involution Coxeter–Knuth insertion* [30, Thm 5.17] to $\mathcal{R}(w^{\delta_n/\mu})$. In this setting, the involution Coxeter–Knuth insertion restricts to the Worley–Sagan insertion, recovering φ. We use the parallel theory for fixed-point-free involution Schubert varieties in our proof of Theorem 4.9 since their K-theory is easier to understand.

The corresponding involutions are $\mathcal{I}$-Grassmannian in the sense of [30]. By analogy with the map Φ, it is an interesting open question to give a direct bijection between reduced involution words for $\mathcal{I}$-Grassmannian involutions and marked shifted tableaux of the appropriate shape. Here, direct can mean either without using an insertion algorithm or while using $O\big(|\lambda| \log |\lambda|\big)$ operations.

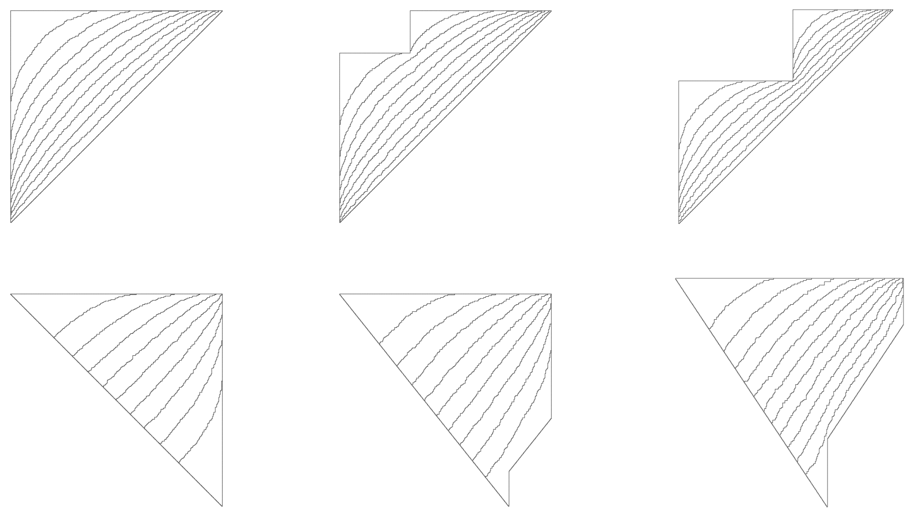


Figure 3. Simulations for $k = 300$. Left column: $\mathrm{SYT}(\delta_k)$ and $\mathrm{ShSYT}(\varrho_k)$. Middle column: $\mathrm{SYT}(\delta_k/b^a)$ and $\mathrm{ShSYT}(\varrho - \tau^{a,b})$ with $(a, b) = (60, 100)$. Right column: $\mathrm{SYT}(\delta_k/b^a)$ and $\mathrm{ShSYT}(\varrho - \tau^{a,b})$ with $(a, b) = (100, 160)$.

5.10. Our proof of Theorem 4.9 in fact proves a stronger statement. If y is a FPF-vexillary fixed-point-free involution in the sense of [31], we show its symplectic Grothendieck polynomial is a single GP_λ. The analogous statement for vexillary permutations appears in [61, Lem. 5.4]. As a consequence, the insertion tableau P_{SH} associated to y must be a unique rectification target in the sense of [10]. Identifying unique rectification targets is an interesting and challenging question that has received some attention [21]

5.11. The limit curves of random standard Young tableaux are of interest in integrable probability as in some cases they can be computed exactly. Most recently, their existence has been shown for a large class of skew shapes [23, 73]. For both the staircase [4] and the shifted staircase [43], these limit curves coincide with limit curves for the square [58] when restricted to either triangle.

We implemented our bijection for uniformly sampling from $\mathrm{SYT}(\delta_k/b^a)$ in Sage, see the proof of Theorem 1.3. Our code is available online on CoCalc [32]. It would be interesting to see if there are exact formulas for limit shapes in this case.

5.12. It follows from Lemma 2.1 and our bijective proof of Theorem 1.2, that this theorem has a q-analogue where the tableaux are weighted with $q^{\mathrm{maj}(T)}$, where $\mathrm{maj}(T)$ is the *major index* (see e.g. [68, §7.19]). It would be interesting to find a q-analogue of Theorem 1.3.

Let us note that the same (numerical) q-analogue of Theorem 1.1 is given in [41, 50]. Paper [49] gave a combinatorial interpretation for the GF over semistandard Young tableaux of skew and shifted skew shapes, which can be viewed as another natural q-analogue of $f^{\lambda/\mu} = |\mathrm{SYT}(\lambda/\mu)|$. Finally, Kerov defined a q-hook walk in [39]. His approach extends to the *weighted case* [13] and then can be modified to the *shifted weighted case* [40], which includes the *q-shifted case* as a special case.

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