ALGEBRAIC COMBINATORICS

Mike Cummings, Sergio Da Silva, Jenna Rajchgot & Adam Van Tuyl

Geometric vertex decomposition and liaison for toric ideals of graphs

Volume 6, issue 4 (2023), p. 965-997.
https://doi.org/10.5802/alco.295

Algebraic Combinatorics is published by The Combinatorics Consortium
and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

Algebraic Combinatorics
Volume 6, issue 4 (2023), p. 965–997
https://doi.org/10.5802/alco.295

Geometric vertex decomposition and liaison for toric ideals of graphs

Mike Cummings, Sergio Da Silva, Jenna Rajchgot & Adam Van Tuyl

Abstract Geometric vertex decomposability for polynomial ideals is an ideal-theoretic generalization of vertex decomposability for simplicial complexes. Indeed, a homogeneous geometrically vertex decomposable ideal is radical and Cohen-Macaulay, and is in the Gorenstein liaison class of a complete intersection (glicci).

In this paper, we initiate an investigation into when the toric ideal I_G of a finite simple graph G is geometrically vertex decomposable. We first show how geometric vertex decomposability behaves under tensor products, which allows us to restrict to connected graphs. We then describe a graph operation that preserves geometric vertex decomposability, thus allowing us to build many graphs whose corresponding toric ideals are geometrically vertex decomposable. Using work of Constantinescu and Gorla, we prove that toric ideals of bipartite graphs are geometrically vertex decomposable. We also propose a conjecture that all toric ideals of graphs with a square-free degeneration with respect to a lexicographic order are geometrically vertex decomposable. As evidence, we prove the conjecture in the case that the universal Gröbner basis of I_G is a set of quadratic binomials. We also prove that some other families of graphs have the property that I_G is glicci.

1. Introduction

Vertex decomposable simplicial complexes are recursively defined simplicial complexes that have been extensively studied in both combinatorial algebraic topology and combinatorial commutative algebra. This family of complexes, first defined by Provan and Billera [29] for pure simplicial complexes and later generalized to the non-pure case by Björner and Wachs [2], has many nice features. For example, they are shellable and hence Cohen-Macaulay in the pure case.

Because of the Stanley–Reisner correspondence between square-free monomial ideals and simplicial complexes, the definition and properties of vertex decomposable simplicial complexes can be translated into algebraic statements about square-free monomial ideals. For example, Moradi and Khosh-Ahang [26, Definition 2.1] introduced vertex splittable ideals, which are precisely the ideals of the Alexander duals of vertex decomposable simplicial complexes. As another example, which is directly relevant to this paper, Nagel and Römer [27] showed that if I_Δ is the square-free monomial ideal associated to a vertex decomposable simplicial complex Δ via the Stanley–Reisner correspondence, then the ideal I_Δ belongs to the Gorenstein liasion class of a complete intersection, i.e., the ideal I_Δ is *glicci*.

Knutson, Miller, and Yong [23] introduced the notion of a *geometric vertex decomposition*, which is an ideal-theoretic generalization (beyond the square-free monomial

Manuscript received 11th August 2022, revised 18th January 2023, accepted 21st January 2023.
Keywords. geometric vertex decomposition, toric ideals of graphs, liaison.

ISSN: 2589-5486

ideal setting) of a vertex decomposition of a simplicial complex. Building on this, Klein and Rajchgot [21] gave a recursive definition of a *geometrically vertex decomposable* ideal which is an ideal-theoretic generalization of a vertex decomposable simplicial complex. Indeed, when specialized to square-free monomial ideals, those ideals that are geometrically vertex decomposable are precisely those square-free monomial ideals whose associated simplicial complexes are vertex decomposable. As shown by Klein and Rajchgot [21, Theorem 4.4], this definition captures some of the properties of vertex decomposable simplicial complexes. For example, a more general version of Nagel and Römer's result holds; that is, a homogeneous ideal that is geometrically vertex decomposable is also glicci. Because geometrically vertex decomposable ideals are glicci, identifying such families allows us to give further evidence to an important open question in liaison theory: is every arithmetically Cohen-Macaulay subscheme of P^n glicci (see [22, Question 1.6])?

Since the definition of geometrically vertex decomposable ideals is recent, there is a need to not only develop the corresponding theory (e.g., which properties of Stanley–Reisner ideals of vertex decomposable simplicial complexes also hold for geometrically vertex decomposable ideals?), but also a need to find families of concrete examples. There has already been some work in these two directions. Klein and Rajchgot [21] showed that Schubert determinantal ideals, (homogeneous) ideals coming from lower bound cluster algebras, and ideals defining equioriented type A quiver loci are all geometrically vertex decomposable. Klein [20] used geometric vertex decomposability to prove a conjecture of Hamaker, Pechenik, and Weigandt [16] on Gröbner bases of Schubert determinantal ideals. Da Silva and Harada have investigated the geometric vertex decomposability of certain Hessenberg patch ideals which locally define regular nilpotent Hessenberg varieties [6].

We contribute to this program by further developing the theory of geometric vertex decomposibility, and show that many families of toric ideals of graphs are geometrically vertex decomposable. Let K be an algebraically closed field of characteristic 0. If $G = (V, E)$ is a finite simple graph with vertex set $V = \{x_1, \ldots, x_m\}$ and edge set $E = \{e_1, \ldots, e_n\}$, we can define a ring homomorphism $\varphi : \mathsf{K}[e_1, \ldots, e_n] \to \mathsf{K}[x_1, \ldots, x_m]$ by letting $\varphi(e_i) = x_k x_l$ where the edge $e_i = \{x_k, x_l\}$. The *toric ideal of G* is the ideal $I_G = \ker(\varphi)$. The study of toric ideals of graphs is an active area of research (e.g., see [1, 3, 9, 10, 14, 28, 31, 32]), so our work also complements the recent developments in this area. What makes toric ideals of graphs amenable to our investigation of geometric vertex decomposability is that their (universal) Gröbner bases are fairly well-understood (see Theorem 3.1) and can be related to the graph's structure.

Our first main result describes how geometric vertex decomposability behaves over tensor products:

Theorem 1.1 (Theorem 2.9). *Let $I \subsetneq R = \mathsf{K}[x_1, \ldots, x_n]$ and $J \subsetneq S = \mathsf{K}[y_1, \ldots, y_m]$ be proper ideals. Then I and J are geometrically vertex decomposable if and only if $I + J$ is geometrically vertex decomposable in $R \otimes S = \mathsf{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$.*

Our result can be viewed as the ideal-theoretic version of the fact that two simplicial complexes Δ_1 and Δ_2 are vertex decomposable if and only if their join $\Delta_1 \star \Delta_2$ is vertex decomposable [29, Proposition 2.4]. Moreover, this result allows us to reduce our study of toric ideals of graphs to the case that the graph G is connected (Theorem 3.3).

When we restrict to toric ideals of graphs, we show that the graph operation of "gluing" an even length cycle onto a graph preserves the geometric vertex decomposability property:

Theorem 1.2 (Theorem 3.11). *Let G be a finite simple graph with toric ideal I_G. Let H be obtained from G by gluing a cycle of even length to G along a single edge. If I_G is geometrically vertex decomposable, then I_H is also geometrically vertex decomposable.*

This gluing operation and its connection to toric ideals of graphs appears in work of Favacchio, Hofscheier, Keiper and Van Tuyl [9], while a similar construction of using H-paths is employed by Gitler, Reyes, and Villarreal [11] to characterize the toric ideals of bipartite graphs that are complete intersections. By repeatedly applying this operation, we can construct many toric ideals of graphs that are geometrically vertex decomposable and glicci.

Our gluing operation requires one to start with a graph whose corresponding toric ideal is geometrically vertex decomposable. It is therefore desirable to identify families of graphs whose toric ideals have this property. Towards this end, we prove:

Theorem 1.3 (Theorem 5.8). *Let G be a finite simple graph with toric ideal I_G. If G is bipartite, then I_G is geometrically vertex decomposable.*

Our proof of Theorem 1.3 relies on work of Constantinescu and Gorla [3]. For some families of bipartite graphs, we give alternative proofs for the geometric vertex decomposable property that exploit the additional structure of the graph (see Theorem 5.10). These families are also used to illustrate that in certain cases, the recursive definition of geometric vertex decomposability easily lends itself to induction.

Based on our results and computer experimentation in Macaulay2 [13], we propose the following conjecture:

Conjecture 1.4 (Conjecture 6.1). *Let G be a finite simple graph with toric ideal $I_G \subseteq \mathsf{K}[e_1, \ldots, e_n]$. If $\mathrm{in}_<(I_G)$ is square-free with respect to a lexicographic monomial order $<$, then I_G is geometrically vertex decomposable, and thus glicci.*

We provide a framework to prove this conjecture. In fact, we show that the conjecture is true if one can prove that a particular family of ideals is equidimensional (see Theorem 6.6). As further evidence for Conjecture 1.4, we prove the following special case:

Theorem 1.5 (Theorem 6.11). *Let I_G be the toric ideal of a finite simple graph G. Assume that I_G has a universal Gröbner basis consisting entirely of quadratic binomials. Then I_G is geometrically vertex decomposable.*

Finally, we prove that additional collections of toric ideals of graphs are glicci (though not necessarily geometrically vertex decomposable). Our first result in this direction relies on a very general result of Migliore and Nagel [25, Lemma 2.1] from the liaison literature.

Theorem 1.6 (Corollary 4.10). *Let G be a finite simple graph and let $I_G \subseteq R = \mathsf{K}[e_1, \ldots, e_n]$ be its toric ideal. Let H be obtained from G by gluing a cycle of even length to G along a single edge. If R/I_G is Cohen-Macaulay, then I_H is glicci.*

We also show that many toric ideals of graphs which contain 4-cycles are glicci. The following is a slightly weaker version of Corollary 4.13.

Theorem 1.7 (Corollary 4.13). *Let G be a finite simple graph and suppose there is an edge $y \in E(G)$ contained in a 4-cycle. If the initial ideal $\mathrm{in}_< I_G$ is a square-free monomial ideal for some lexicographic monomial order with $y > e$ for all $e \in E(G)$ with $e \neq y$, then I_G is glicci.*

As a corollary to this theorem, we show that the toric ideal of any *gap-free* graph which contains a 4-cycle is glicci. For the definition of gap-free graph and this result, see the end of Section 4.2.

1.1. Outline of the paper. In the next section we formally introduce geometrically vertex decomposable ideals, along with the required background and notation about Gröbner bases. We also explain how geometrically vertex decomposable ideals behave with respect to tensor products. In Section 3 we provide the needed background on toric ideals of graphs, and we explain how a particular graph operation preserves the geometric vertex decomposability property. In Section 4, we focus on the glicci property for toric ideals of graphs that can be deduced from the results of Section 3 together with general results from the liaison theory literature. In Section 5 we prove that toric ideals of bipartite graphs are geometrically vertex decomposable. In Section 6 we propose a conjecture on toric ideals with a square-free initial ideal, describe a framework to prove this conjecture, and illustrate this framework by proving that toric ideals of graphs which have quadratic universal Gröbner bases are geometrically vertex decomposable.

1.2. Remark on the field K. Many of the arguments in this paper are valid over any infinite field. Indeed, the liaison-theoretic setup in Sections 2 and 4 requires an infinite field but is characteristic-free. Similarly, toric ideals of graphs can be defined combinatorially, and since the coefficients of their generators are ± 1, defining such ideals in positive characteristic does not pose any issues. Nevertheless, we assume that K throughout this paper is algebraically closed of characteristic zero since some of the references that we use make this assumption (e.g., [30, Proposition 13.15], which is needed in the proof of Theorem 3.4).

Acknowledgements. We thank Patricia Klein for some helpful conversations. Cummings was partially supported by an NSERC USRA. Da Silva was partially supported by an NSERC postdoctoral fellowship. Rajchgot's research is supported by NSERC Discovery Grant 2017-05732. Van Tuyl's research is supported by NSERC Discovery Grant 2019-05412.

2. Geometrically vertex decomposable ideals

In this paper K denotes an algebraically closed field of characteristic zero and $R = \mathsf{K}[x_1, \ldots, x_n]$ is the polynomial ring in n variables. This section gives the required background on geometrically vertex decomposable ideals, following [21]. We also examine how geometric vertex decomposability behaves over tensor products.

Fix a variable $y = x_i$ in R. For any $f \in R$, we can write f as $f = \sum_i \alpha_i y^i$, where α_i is a polynomial only in the variables $\{x_1, \ldots, \hat{x}_i, \ldots, x_n\}$. For $f \neq 0$, the *initial y-form* of f, denoted $\mathrm{in}_y(f)$, is the non-zero coefficient of the highest power of y appearing in $\sum_i \alpha_i y^i$. That is, if $\alpha_d \neq 0$, but $\alpha_t = 0$ for all $t > d$, then $\mathrm{in}_y(f) = \alpha_d y^d$. Note that if y does not appear in any term of f, then $\mathrm{in}_y(f) = f$. For any ideal I of R, we set $\mathrm{in}_y(I) = \langle \mathrm{in}_y(f) \mid f \in I \rangle$ to be the ideal generated by all the initial y-forms in I. A monomial order $<$ on R is said to be *y-compatible* if the initial term of f satisfies $\mathrm{in}_<(f) = \mathrm{in}_<(\mathrm{in}_y(f))$ for all $f \in R$. For such an order, we have $\mathrm{in}_<(I) = \mathrm{in}_<(\mathrm{in}_y(I))$, where $\mathrm{in}_<(I)$ is the *initial ideal* of I with respect to the order $<$.

Given an ideal I and a y-compatible monomial order $<$, let $\mathcal{G}(I) = \{g_1, \ldots, g_m\}$ be a Gröbner basis of I with respect to this monomial order. For $i = 1, \ldots, m$, write g_i as $g_i = y^{d_i} q_i + r_i$, where y does not divide any term of q_i; that is, $\mathrm{in}_y(g_i) = y^{d_i} q_i$. It can then be shown that $\mathrm{in}_y(I) = \langle y^{d_1} q_1, \ldots, y^{d_m} q_m \rangle$ (see [23, Theorem 2.1(a)]).

Given this setup, we define two ideals:

$$C_{y,I} = \langle q_1, \ldots, q_m \rangle \text{ and } N_{y,I} = \langle q_i \mid d_i = 0 \rangle.$$

Recall that an ideal I is *unmixed* if the ideal I satisfies $\dim(R/I) = \dim(R/P)$ for all associated primes $P \in \mathrm{Ass}_R(R/I)$. We come to our main definition:

Definition 2.1. *An ideal I of $R = \mathsf{K}[x_1, \ldots, x_n]$ is geometrically vertex decomposable if I is unmixed and*

(1) *$I = \langle 1 \rangle$, or I is generated by a (possibly empty) subset of variables of R, or*
(2) *there is a variable $y = x_i$ in R and a y-compatible monomial order $<$ such that*
$$\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle),$$
and the contractions of the ideals $C_{y,I}$ and $N_{y,I}$ to $\mathsf{K}[x_1, \ldots, \hat{x}_i, \ldots, x_n]$ are geometrically vertex decomposable.

We make the convention that the two ideals $\langle 0 \rangle$ and $\langle 1 \rangle$ of the ring K are also geometrically vertex decomposable.

Remark 2.2. For any ideal I of R, if there exists a variable $y = x_i$ in R and a y-compatible monomial order $<$ such that $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$, then this decomposition is called a *geometric vertex decomposition of I with respect to y*. This decomposition was first defined in [23]. Consequently, Definition 2.1 (2) says that there is a variable y such that I has a geometric vertex decomposition with respect to this variable.

We say that a geometric vertex decomposition is *degenerate* if either $C_{y,I} = \langle 1 \rangle$ or $\sqrt{C_{y,I}} = \sqrt{N_{y,I}}$ (see [21, Section 2.2] for further details and results). Otherwise, we call a geometric vertex decomposition *nondegenerate*.

If elements in our Gröbner basis are square-free in y, i.e., if $\mathrm{in}_y(g_i) = y^{d_i} q_i$ with $d_i = 0$ or 1 for all $g_i \in \mathcal{G}(I)$, then Knutson, Miller, and Yong note that we get the geometric vertex decomposition of I with respect to y for "free":

Lemma 2.3 ([23, Theorem 2.1 (a), (b)]). *Let I be an ideal of R and let $<$ be a y-compatible monomial order. Suppose that $\mathcal{G}(I) = \{g_1, \ldots, g_m\}$ is a Gröbner basis of I with respect to $<$, and also suppose that $\mathrm{in}_y(g_i) = y^{d_i} q_i$ with $d_i = 0$ or 1 for all i. Then*

(1) *$\{q_1, \ldots, q_m\}$ is a Gröbner basis of $C_{y,I}$ and $\{q_i \mid d_i = 0\}$ is a Gröbner basis of $N_{y,I}$.*
(2) *$\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$, i.e., I has a geometric vertex decomposition with respect to y.*

Remark 2.4. If I is a square-free monomial ideal in R, then I is geometrically vertex decomposable if and only if the simplicial complex Δ associated with I via the Stanley–Reisner correspondence is a vertex decomposable simplicial complex; see [21, Proposition 2.8] for more details. As a consequence, we can view Definition 2.1 as a generalization of the notion of vertex decomposability. When I is a square-free monomial ideal with associated simplicial complex Δ, then $C_{y,I}$ is the Stanley–Reisner ideal of the star of y, i.e., $\mathrm{star}_\Delta(y) = \{F \in \Delta \mid F \cup \{y\} \in \Delta\}$ and $N_{y,I} + \langle y \rangle$ corresponds to the deletion of y from Δ, that is, $\mathrm{del}_\Delta(y) = \{F \in \Delta \mid y \notin F\}$ (see [21, Remark 2.5]).

If I has a geometric vertex decomposition with respect to a variable y, we can determine some additional information about a reduced Gröbner basis of I with respect to any y-compatible monomial order. In the following statement, I is *square-free in y* if there is a generating set $\{g_1, \ldots, g_s\}$ of I such that no term of $g_1, \ldots, g_s$ is divisible by y^2.

Lemma 2.5 ([21, Lemma 2.6]). *Suppose that the ideal I of R has a geometric vertex decomposition with respect to the variable $y = x_i$. Then I is square-free in y. Moreover, for any y-compatible term order, the reduced Gröbner basis of I with respect to this order has the form $\{yq_1 + r_1, \ldots, yq_k + r_k, h_1, \ldots, h_t\}$ where y does not divide any term of q_i, r_i, h_j for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, t\}$.*

The following lemma and its proof helps to illustrate some of the above ideas. Furthermore, since the definition of geometrically vertex decomposable lends itself to proof by induction, the following facts are sometimes useful for the base cases of our induction.

Lemma 2.6. (1) *An an ideal I of $R = \mathsf{K}[x]$ is geometrically vertex decomposable if and only if $I = \langle ax + b\rangle$ for some $a, b \in \mathsf{K}$.*

(2) *Let $f = c_1m_1 + \cdots + c_sm_s$ be any polynomial in $R = \mathsf{K}[x_1, \ldots, x_n]$ with $c_i \in \mathsf{K}$ and m_i a monomial. If each m_i is square-free, then $I = \langle f\rangle$ is geometrically vertex decomposable. In particular, if m is a square-free monomial, then $\langle m\rangle$ is geometrically vertex decomposable.*

Proof. (1) ($\Leftarrow$) If $a = 0$, or $b = 0$, or both $a = b = 0$, the ideal $I = \langle ax + b\rangle$ satisfies Definition 2.1 (1). So, suppose $a, b \neq 0$. The ideal I is prime, so it is unmixed. Since x is the only variable of R, and because there is only one monomial order on this ring, it is easy to see that this monomial order is x-compatible, and that $\{ax+b\}$ is a Gröbner basis of I. So, $C_{x,I} = \langle a\rangle = \langle 1\rangle$ and $N_{x,I} = \langle 0\rangle$. It is straightforward to check that we have a geometric vertex decomposition of I with respect to x. Furthermore, as ideals in $\mathsf{K}[\hat{x}] = \mathsf{K}$, $C_{x,I} = \langle 1\rangle$ and $N_{x,I} = \langle 0\rangle$ are geometrically vertex decomposable by definition. So, I is geometrically vertex decomposable.

($\Rightarrow$) Since $R = \mathsf{K}[x]$ is a principal ideal domain, $I = \langle f\rangle$ for some $f \in R$, i.e., $f = a_dx^d + \cdots + a_1x + a_0$ with $a_i \in \mathsf{K}$. Since I is geometrically vertex decomposable, and because x is the only variable of R, by Lemma 2.5, the ideal I is square-free in x. This fact then forces $d \leqslant 1$, and thus $I = \langle a_1x + a_0\rangle$ as desired.

(2) We proceed by induction on the number of variables in $R = \mathsf{K}[x_1, \ldots, x_n]$. The base case $n = 1$ follows from statement (1). Because $I = \langle f\rangle$ is principal, f is a Gröbner basis with respect to any monomial order. In particular, let $>$ be the lexicographic order on R with $x_1 > \cdots > x_n$, and assume $m_1 > \cdots > m_s$. Let y be the largest variable dividing m_1. Then we can write f as $f = y(c_1m_1' + \cdots + c_im_i') + c_{i+1}m_{i+1} + \cdots + c_sm_s$ for some i such that y does not divide $m_{i+1}, \ldots, m_s$. Note that $>$ is a y-compatible monomial order, and so by Lemma 2.3 we have $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$ with $C_{y,I} = \langle c_1m_1' + \cdots + c_im_i'\rangle$ and $N_{y,I} = \langle 0\rangle$. The ideal $N_{y,I}$ is geometrically vertex decomposable in $\mathsf{K}[x_1, \ldots, \hat{y}, \ldots, x_n]$ by definition, and $C_{y,I}$ is geometrically vertex decomposable in the same ring by induction. Observe that $I, C_{y,I}$ and $N_{y,I}$ are also unmixed since they are principal.

Theorem 2.9, which is of independent interest, shows how we can treat ideals whose generators lie in different sets of variables. We require a lemma about Gröbner bases in tensor products. For completeness, we give a proof, although it follows easily from standard facts about Gröbner bases.

We first need to recall a characterization of Gröbner bases using standard representations. Fix a monomial order $<$ on $R = \mathsf{K}[x_1, \ldots, x_n]$. Given $G = \{g_1, \ldots, g_s\}$ in R, we say f *reduces to zero modulo G* if f has a *standard representation*

$$f = f_1g_1 + \cdots + f_sg_s \text{ with } f_i \in R$$

with multidegree$(f) \geqslant$ multidegree(f_ig_i) for all i with $f_ig_i \neq 0$. Here

$$\text{multidegree}(h) = \max\{\alpha \in \mathsf{N}^n \mid x^\alpha \text{ is a term of } h\},$$

where we use the monomial order $<$ to order N^n. We then have the following result.

Theorem 2.7 ([5, Chapter 2.9, Theorem 3]). *Let $R = \mathsf{K}[x_1, \ldots, x_n]$ with fixed monomial order $<$. A basis $G = \{g_1, \ldots, g_s\}$ of an ideal I in R is a Gröbner basis for I if and only if each S-polynomial $S(g_i, g_j)$ reduces to zero modulo G.*

For the lemma below, note that if $R = \mathsf{K}[x_1, \ldots, x_n]$ and $S = \mathsf{K}[y_1, \ldots, y_m]$, and if $<$ is a monomial order on $R \otimes S := R \otimes_{\mathsf{K}} S$, then $<$ induces a monomial order $<_R$ on R where $m_1 <_R m_2$ if and only if $m_1 < m_2$, where we view m_1, m_2 as monomials of both R and $R \otimes S$. Here,"viewing $f \in R$ as an element of $R \otimes S$" means writing $\varphi_R(f)$ as f where $\varphi_R : R \to R \otimes S$ is the natural inclusion $f \mapsto f \otimes 1$. Similarly, we let $<_S$ denote the induced monomial order on S.

Lemma 2.8. *Let $I \subseteq R = \mathsf{K}[x_1, \ldots, x_n]$ and $J \subseteq S = \mathsf{K}[y_1, \ldots, y_m]$ be ideals. For any monomial order $<$ on $R \otimes S$, there exists a Gröbner basis of $I + J$ in $R \otimes S$ which has the form $\mathcal{G}(I + J) = \mathcal{G}_1 \cup \mathcal{G}_2$, where $\mathcal{G}_1$ is a Gröbner basis of I in R with respect to $<_R$ but viewed as elements of $R \otimes S$, and $\mathcal{G}_2$ is a Gröbner basis of J in S with respect to $<_S$ but viewed as elements of $R \otimes S$.*

Proof. Given $<$, select a Gröbner basis $\mathcal{G}_1$ of I and $\mathcal{G}_2$ of J with respect to the induced monomial orders $<_R$ and $<_S$ on R and S respectively. Since $\mathcal{G}_1$ generates I and $\mathcal{G}_2$ generates J, the set $\mathcal{G}_1 \cup \mathcal{G}_2$ generates $I + J$ as an ideal of $R \otimes S$. To prove that $\mathcal{G}_1 \cup \mathcal{G}_2$ is a Gröbner basis of $I + J$, by Theorem 2.7 it suffices to show that for any $g_i, g_j \in \mathcal{G}_1 \cup \mathcal{G}_2$, the S-polynomial $S(g_i, g_j)$ reduces to zero modulo this set.

If $g_i, g_j \in \mathcal{G}_1$, then since $g_i, g_j \in R$, and since $\mathcal{G}_1$ is a Gröbner basis of I in R, by Theorem 2.7, the S-polynomial $S(g_i, g_j)$ reduces to zero modulo $\mathcal{G}_1$. But then in the larger ring $R \otimes S$, the S-polynomial $S(g_i, g_j)$ also reduces to zero modulo $\mathcal{G}_1 \cup \mathcal{G}_2$. A similar result holds if $g_i, g_j \in \mathcal{G}_2$.

So, suppose $g_i \in \mathcal{G}_1$ and $g_j \in \mathcal{G}_2$. Note that the leading monomial of g_i is only in the variables $\{x_1, \ldots, x_n\}$, while the leading monomial of g_j is only in the variables $\{y_1, \ldots, y_m\}$. Consequently, their leading monomials are relatively prime. Thus, according to [5, Chapter 2.9, Proposition 4], the S-polynomial $S(g_i, g_j)$ reduces to zero modulo $\mathcal{G}_1 \cup \mathcal{G}_2$.

Theorem 2.9. *Let $I \subsetneq R = \mathsf{K}[x_1, \ldots, x_n]$ and $J \subsetneq S = \mathsf{K}[y_1, \ldots, y_m]$ be proper ideals. Then I and J are geometrically vertex decomposable if and only if $(I + J)$ is geometrically vertex decomposable in $R \otimes S = \mathsf{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$.*

Proof. First suppose that $I \subsetneq R$ and $J \subsetneq S$ are geometrically vertex decomposable. Since neither ideal contains 1, we have $I + J \neq \langle 1 \rangle$. By [15, Corollary 2.8], the set of associated primes of $(R \otimes S)/(I + J) \cong R/I \otimes S/J$ satisfies

$$\operatorname{Ass}_{R\otimes S}(R/I \otimes S/J) = \{P + Q \mid P \in \operatorname{Ass}_R(R/I) \text{ and } Q \in \operatorname{Ass}_S(S/J)\}. \tag{1}$$

Thus any associated prime $P + Q$ of $(R \otimes S)/(I + J)$ satisfies

$$\begin{aligned} \dim((R \otimes S)/(P + Q)) &= \dim(R/P) + \dim(S/Q) \\ &= \dim(R/I) + \dim(S/J) \\ &= \dim((R \otimes S)/(I + J)) \end{aligned}$$

where we are using the fact that I and J are unmixed for the second equality. So, $I + J$ is also unmixed.

To see that $I + J \subseteq R \otimes S$ is geometrically vertex decomposable, we proceed by induction on the number of variables $\ell = n + m$ in $R \otimes S$. The base case $\ell = 0$ is trivial. Assume now that $\ell > 0$. If both I and J are generated by indeterminates, then $I + J$ is too and so is geometrically vertex decomposable. Thus, without loss of generality, suppose that I is not generated by indeterminates (note that $I \neq \langle 1 \rangle$ by assumption).

Because I is geometrically vertex decomposable in R, there is a variable $y = x_i$ in R such that $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is a geometric vertex decomposition and the contractions of $C_{y,I}$ and $N_{y,I}$ to $R' = \mathsf{K}[x_1, \ldots, \hat{y}, \ldots, x_n]$ are geometrically vertex decomposable. Extend the y-compatible monomial order $<$ on R to a y-compatible monomial order on $R \otimes S$ by taking any monomial order on S, and let our new monomial order $\prec$ be the product order of these two monomial orders (where $x_i \succ y_j$ for all i, j).

If we write K^e to denote the extension of an ideal K in R into the ring $R \otimes S$, then one checks that with respect to this new y-compatible order

$$\begin{aligned}\mathrm{in}_y(I+J) &= (\mathrm{in}_y(I))^e + J = [C_{y,I} \cap (N_{y,I} + \langle y \rangle)]^e + J \\ &= ((C_{y,I})^e + J) \cap ((N_{y,I})^e + J + \langle y \rangle).\end{aligned}$$

Using the identities

$$(C_{y,I})^e + J = C_{y,I+J} \text{ and } (N_{y,I})^e + J = N_{y,I+J}$$

(note that $\prec$ is being used to define $C_{y,I+J}$ and $N_{y,I+J}$ and $<$ is being used to define $C_{y,I}$ and $N_{y,I}$), we have a geometric vertex decomposition of $I+J$ with respect to y in $R \otimes S$:

$$\mathrm{in}_y(I+J) = C_{y,I+J} \cap (N_{y,I+J} + \langle y \rangle).$$

Now let C' and N' denote the contractions of $C_{y,I}$ and $N_{y,I}$ to R'. First assume that C' and N' are both proper ideals. Then, since C' and N' are geometrically vertex decomposable, we may apply induction to see that $C' + J$ and $N' + J$ in $R' \otimes S$ are geometrically vertex decomposable. In particular, as $C' + J$ and $N' + J$ are the contractions of $(C_{y,I})^e + J$ and $(N_{y,I})^e + J$ to $R' \otimes S$, we have that $I+J$ is geometrically vertex decomposable by induction. If either C' or N' is the ideal $\langle 1 \rangle$, the same would be true for the contractions of $(C_{y,I})^e + J$ or $(N_{y,I})^e + J$ because the contraction of $(C_{y,I})^e + J$, respectively $(N_{y,I})^e + J$, contains C', respectively N'. So $I + J$ is geometrically vertex decomposable.

For the converse, we proceed by induction on the number of variables ℓ in $R \otimes S$. The base case is $\ell = 0$, which is trivial. So suppose $\ell > 0$. We first show that I is unmixed. Suppose that I is not unmixed; that is, there are associated primes P_1 and P_2 of $\mathrm{Ass}(R/I)$ such that $\dim(R/P_1) \neq \dim(R/P_2)$. For any associated prime Q of S/J, we know by (1) that $P_1 + Q$ and $P_2 + Q$ are associated primes of $(R \otimes S)/(I+J)$. Since $I + J$ is unmixed, we can derive the contradiction

$$\begin{aligned}\dim((R \otimes S)/(I+J)) &= \dim((R \otimes S)/(P_1 + Q)) \\ &= \dim(R/P_1) + \dim(S/Q) \\ &\neq \dim(R/P_2) + \dim(S/Q) \\ &= \dim((R \otimes S)/(P_2 + Q)) = \dim((R \otimes S)/(I+J)).\end{aligned}$$

So, I is unmixed (the proof for J is similar).

If $I + J$ is generated by indeterminates, then so are I and J, hence they are geometrically vertex decomposable. So, suppose that there is a variable y in $R \otimes S$ and a y-compatible monomial order $<$ such that

$$\mathrm{in}_y(I+J) = C_{y,I+J} \cap (N_{y,I+J} + \langle y \rangle).$$

Without loss of generality, assume that $y \in \{x_1, \ldots, x_n\}$. So $C_{y,I+J}$ and $N_{y,I+J}$ are geometrically vertex decomposable in $\mathsf{K}[x_1, \ldots, \hat{y}, \ldots, x_n, y_1, \ldots, y_m]$.

By Lemma 2.8, we can construct a Gröbner basis $\mathcal{G}$ of $I+J$ with respect to $<$ such that

$$\mathcal{G} = \{g_1, \ldots, g_s\} \cup \{h_1, \ldots, h_t\}$$

where $\{g_1, \ldots, g_s\}$ is a Gröbner basis of I with respect to the order $<_R$ in R, and $\{h_1, \ldots, h_t\}$ is a Gröbner basis of J with respect to $<_S$ in S. Since y can only appear among the g_i's, we have

$$C_{y,I+J} = (C_{y,I}) + J \text{ and } N_{y,I+J} = (N_{y,I}) + J$$

where $C_{y,I}$, respectively $N_{y,I}$, denote the ideals constructed from the Gröbner basis $\{g_1, \ldots, g_s\}$ of I in R using the monomial order $<_R$. Note that in R, $<_R$ is still y-compatible.

Since the ideals $(C_{y,I}) + J$ and $(N_{y,I}) + J$ are geometrically vertex decomposable in the ring $\mathsf{K}[x_1, \ldots, \hat{y}, \ldots, x_n, y_1, \ldots, y_m]$, by induction, $C_{y,I}$ and $N_{y,I}$ are geometrically vertex decomposable in $\mathsf{K}[x_1, \ldots, \hat{y}, \ldots, x_n]$ and J is geometrically vertex decomposable in S. To complete the proof, note that in R, we have $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$. Thus I is also geometrically vertex decomposable in R.

Remark 2.10. If we weaken the hypotheses in Theorem 2.9 to allow I or J to be $\langle 1 \rangle$, then only one direction remains true. In particular, if I and J are geometrically vertex decomposable, then so is $I + J$. However, the converse statement would no longer be true. To see why, let $I = \langle 1 \rangle$ and let J to be any ideal which is not geometrically vertex decomposable. Then $I + J = \langle 1 \rangle$ is geometrically vertex decomposable in $R \otimes S$, but we do not have that both I and J are geometrically vertex decomposable.

Remark 2.11. Theorem 2.9 is an algebraic generalization of [29, Proposition 2.4] which showed that if Δ_1 and Δ_2 were simplicial complexes on different sets of variables, then the join $\Delta_1 \star \Delta_2$ is vertex decomposable if and only if Δ_1 and Δ_2 are vertex decomposable.

Corollary 2.12. *Let $I \subseteq R = \mathsf{K}[x_1, \ldots, x_n]$ be a square-free monomial ideal. If I is a complete intersection, then I is geometrically vertex decomposable.*

Proof. Suppose $I = \langle m_1, \ldots, m_t \rangle$, where $m_1, \ldots, m_t$ are the minimal square-free monomial generators. Because I is a complete intersection, the ideal is unmixed. Furthermore, because I is a complete intersection, the support of each monomial is pairwise disjoint. So, after a relabelling, we can assume that $m_1 = x_1 x_2 \cdots x_{a_1}$, $m_2 = x_{a_1+1} \cdots x_{a_2}, \ldots, m_t = x_{a_{t-1}+1} \cdots x_{a_t}$. Then

$$R/I \cong \mathsf{K}[x_1, \ldots, x_{a_1}]/\langle m_1 \rangle \otimes \cdots \otimes \mathsf{K}[x_{a_{t-1}+1}, \ldots, x_{a_t}]/\langle m_t \rangle \otimes \mathsf{K}[x_{a_{t+1}}, \ldots, x_n].$$

By Lemma 2.6, the ideals $\langle m_i \rangle$ are geometrically vertex decomposable for $i = 1, \ldots, t$. Now repeatedly apply Theorem 2.9.

Remark 2.13. Corollary 2.12 can also be deduced via results from Stanley–Reisner theory, which we sketch out. One proceeds by induction on the number of generators of the complete intersection I. If $I = \langle x_1 \cdots x_k \rangle$ has one generator, then one can prove directly from the definition of a vertex decomposable simplicial complex (e.g., see [29]) that the simplicial complex associated with I, denoted by $\Delta = \Delta(I)$, is vertex decomposable. For the induction step, note that if $I = \langle m_1, \ldots, m_t \rangle$, then $I = I_1 + I_2 = \langle m_1, \ldots, m_{t-1} \rangle + \langle m_t \rangle$. If $\{w_1, \ldots, w_m\}$ are variables that appear in the generator m_t and $\{x_1, \ldots, x_\ell\}$ are the other variables, then we have

$$R/I \cong \mathsf{K}[x_1, \ldots, x_\ell]/I_1 \otimes \mathsf{K}[w_1, \ldots, w_m]/I_2.$$

By induction, the simplicial complexes Δ_1 and Δ_2 defined by I_1 and I_2 are vertex decomposable. As noted in Remark 2.11, the join $\Delta_1 \star \Delta_2$ is also vertex decomposable. So, the ideal I is a square-free monomial ideal whose associated simplicial complex is vertex decomposable. The result now follows from [21, Theorem 4.4] which implies that the ideal I is also geometrically vertex decomposable.

3. Toric ideals of graphs

This section initiates a study of the geometric vertex decomposability of toric ideals of graphs. We have subdivided this section into three parts: (a) a review of the needed background on toric ideals, (b) an analysis of the ideals $C_{y,I}$ and $N_{y,I}$ when I is the toric ideal of a graph, and (c) an explanation of how the graph operation of "gluing" a cycle to a graph preserves geometric vertex decomposability.

We will study some specific families of graphs whose toric ideals are geometrically vertex decomposable in Sections 5 and 6.

3.1. Toric ideals of graphs. We review the relevant background on toric ideals of graphs. Our main references for this material are [30, 33].

Let $G = (V(G), E(G))$ be a finite simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G) = \{e_1, \ldots, e_t\}$ where each $e_i = \{x_j, x_k\}$. Let $\mathsf{K}[E(G)] = \mathsf{K}[e_1, \ldots, e_t]$ be a polynomial ring, where we treat the e_i's as indeterminates. Similarly, let $\mathsf{K}[V(G)] = \mathsf{K}[x_1, \ldots, x_n]$. Consider the K-algebra homomorphism $\varphi_G : \mathsf{K}[E(G)] \to \mathsf{K}[V(G)]$ given by

$$\varphi_G(e_i) = x_j x_k \text{ where } e_i = \{x_j, x_k\} \text{ for all } i \in \{1, \ldots, t\}.$$

The *toric ideal of the graph* G, denoted I_G, is the kernel of the homomorphism φ_G.

While the generators of I_G are defined implicitly, these generators (and a Gröbner basis) of I_G can be described in terms of the graph G, specifically, the walks in G. A *walk* of length ℓ is an alternating sequence of vertices and edges

$$\{x_{i_0}, e_{i_1}, x_{i_1}, e_{i_2}, \cdots, e_{i_\ell}, x_{i_\ell}\}$$

such that $e_{i_j} = \{x_{i_{j-1}}, x_{i_j}\}$. The walk is *closed* if $x_{i_\ell} = x_{i_0}$. When the vertices are clear, we simply write the walk as $\{e_{i_1}, \ldots, e_{i_\ell}\}$. It straightforward to check that every closed walk of even length, say $\{e_{i_1}, \ldots, e_{i_{2\ell}}\}$, results in an element of I_G; indeed

$$\varphi_G(e_{i_1} e_{i_3} \cdots e_{i_{2\ell-1}} - e_{i_2} e_{i_4} \cdots e_{2\ell}) = x_{i_0} x_{i_1} \cdots x_{2\ell-1} - x_{i_1} x_{i_2} \cdots x_{i_{2\ell}} = 0$$

since $x_{i_{2\ell}} = x_{i_0}$. Note that $e_{i_1} e_{i_3} \cdots e_{i_{2\ell-1}} - e_{i_2} e_{i_4} \cdots e_{i_{2\ell}}$ is a binomial. For any $\alpha = (a_1, \ldots, a_t) \in \mathsf{N}^t$, let $e^\alpha = e_1^{a_1} e_2^{a_2} \cdots e_t^{a_t}$. A binomial $e^\alpha - e^\beta \in I_G$ is *primitive* if there is no other binomial $e^\gamma - e^\delta \in I_G$ such that $e^\gamma | e^\alpha$ and $e^\delta | e^\beta$. We can now describe generators and a universal Gröbner basis of I_G.

Theorem 3.1. *Let G be a finite simple graph.*

(1) ([33, Proposition 10.1.5]) *The ideal I_G is generated by the set of binomials*

$$\{e_{i_1} e_{i_3} \cdots e_{i_{2\ell-1}} - e_{i_2} e_{i_4} \cdots e_{i_{2\ell}} \mid \{e_{i_1}, \ldots, e_{i_{2\ell}}\} \text{ is a closed even walk of } G\}.$$

(2) ([33, Proposition 10.1.9]) *The set of all primitive binomials that also correspond to closed even walks in G is a universal Gröbner basis of I_G.*

Going forward, we will write $\mathcal{U}(I_G)$ to denote this universal Gröbner basis of I_G.

The next two results allow us to make some additional assumptions on G when studying I_G. First, we can ignore leaves in G when studying I_G. Recall that the degree of a vertex $x \in V(G)$ is the number of edges $e \in E(G)$ that contain x. An edge $e = \{x, y\}$ is a *leaf* of G if either x or y has degree one. In the statement below, if $e \in E(G)$, then by $G \setminus e$ we mean the graph formed by removing the edge e from G; note $V(G \setminus e) = V(G)$. We include a proof for completeness.

Lemma 3.2. *Let G be a finite simple graph. If e is a leaf of G, then $I_G = I_{G \setminus e}$.*

Proof. For the containment $I_{G \setminus e} \subseteq I_G$, observe that any closed even walk in $G \setminus e$ is also a closed even walk in G. For the reverse containment, if a closed even walk $\{e_{i_1}, \ldots, e, \ldots, e_{i_{2\ell}}\}$ contains the leaf e, then e must be repeated, i.e., $\{e_{i_1}, \ldots, e, e, \ldots, e_{i_{2\ell}}\}$. The corresponding binomial $b_1 - b_2$ is divisible by e, i.e., $b_1 - b_2 = e(b_1' - b_2') \in I_G$. But since I_G is a prime binomial ideal, this forces $b_1' - b_2' \in I_G$. Thus every minimal generator of I_G corresponds to a closed even walk that does not go through e, and thus is an element of $I_{G \setminus e}$.

A graph G is *connected* if for any two pairs of vertices in G, there is a walk in G between these two vertices. A connected component of G is a subgraph of G that is connected, but it is not contained in any larger connected subgraph. To study the geometric vertex decomposability of I_G, we may always assume that G is connected.

Theorem 3.3. *Suppose that $G = H \sqcup K$ is the disjoint union of two finite simple graphs. Then I_G is geometrically vertex decomposable in $\mathsf{K}[E(G)]$ if and only if I_H, and respectively I_K, is geometrically vertex decomposable in $\mathsf{K}[E(H)]$, and respectively $\mathsf{K}[E(K)]$.*

Proof. Apply Theorem 2.9 to $I_G = I_H + I_K$ in $\mathsf{K}[E(G)] = \mathsf{K}[E(H)] \otimes \mathsf{K}[E(K)]$.

The well-known result below gives a condition for $\mathsf{K}[E(G)]/I_G$ to be Cohen-Macaulay.

Theorem 3.4. *Let G be a finite simple graph with toric ideal $I_G \subseteq \mathsf{K}[E(G)]$. Suppose that there is a monomial order $<$ such that $\mathrm{in}_<(I_G)$ is a square-free monomial ideal. Then $\mathsf{K}[E(G)]/I_G$ is Cohen-Macaulay.*

Proof. If $\mathrm{in}_<(I_G)$ is a square-free monomial ideal, then I_G is normal by [30, Proposition 13.15]. Thus, by Hochster [19], $\mathsf{K}[E(G)]/I_G$ is Cohen-Macaulay.

3.2. Structure results about $N_{y,I}$ and $C_{y,I}$. To study the geometric vertex decomposability of I_G, we need access to both N_{y,I_G} and C_{y,I_G}. While determining C_{y,I_G} in terms of G will prove to be subtle, the ideal N_{y,I_G} has a straightforward description.

Lemma 3.5. *Let G be a finite simple graph with toric ideal $I_G \subseteq \mathsf{K}[E(G)]$. Let $<$ by any y-compatible monomial order with $y = e$ for some edge e of G. Then*

$$N_{y,I_G} = I_{G \setminus e}.$$

In particular, a universal Gröbner basis of N_{y,I_G} consists of all the binomials in the universal Gröbner basis $\mathcal{U}(I_G)$ of I_G where neither term is divisible by y.

Proof. By Theorem 3.1 (2), I_G has a universal Gröbner basis $\mathcal{U}(I_G)$ of primitive binomials associated to closed even walks of G. Write this basis as $\mathcal{U}(I_G) = \{y^{d_1}q_1 + r_1, \ldots, y^{d_k}q_k + r_k, g_1, \ldots, g_r\}$, where $d_i > 0$ and where y does not divide any term of g_i and q_i. By definition

$$N_{y,I_G} = \langle g_1, \ldots, g_r \rangle.$$

In particular, N_{y,I_G} is generated by primitive binomials in $\mathcal{U}(I_G)$ which do not include the variable y. These primitive binomials correspond to closed even walks in G which do not pass through the edge e. In particular, they are also closed even walks in $G \setminus e$, so $\{g_1, \ldots, g_r\} \subset \mathcal{U}(I_{G \setminus e})$, the universal Gröbner basis of $I_{G \setminus e}$ from Theorem 3.1 (2).

To show the reverse containment $\mathcal{U}(I_{G \setminus e}) \subseteq \{g_1, \ldots, g_r\}$, suppose that there is some binomial $u - v \in \mathcal{U}(I_{G \setminus e})$ which is not in $\mathcal{U}(I_G)$. Then there would be some closed even walk of G which is not primitive, but becomes primitive after deleting the edge e. For $u - v$ to not be primitive means that there is some primitive binomial $u' - v' \in \mathcal{U}(I_G)$ such that $u'|u$ and $v'|v$. Since y does not divide u or v, we must have $u' - v' \in \mathcal{U}(I_{G \setminus e})$, a contradiction to $u - v$ being primitive. Therefore $\mathcal{U}(I_{G \setminus e}) = \{g_1, \ldots, g_r\}$. Since $\{g_1, \ldots, g_r\}$ generates $I_{G \setminus e}$, we have $I_{G \setminus e} = \langle g_1, \ldots, g_r \rangle = N_{y,I_G}$, thus proving the result.

It is more difficult to give a similar description for C_{y,I_G}. For example, C_{y,I_G} may not be prime, and thus, it may not be the toric ideal of any graph. If we make the extra assumption that the binomial generators in $\mathcal{U}(I_G)$ are *doubly square-free* (i.e., each binomial is the difference of two square-free monomials), then it is possible to give a slightly more concrete description of C_{y,I_G}. We work out these details below.

Fix a variable y in $\mathsf{K}[E(G)]$, and write the elements of $\mathcal{U}(I_G)$ as $\{y^{d_1}q_1 + r_1, \dots, y^{d_k}q_k + r_k, g_1, \dots, g_r\}$, where $d_i > 0$ and where y does not divide q_i or any term of g_i. Since we are assuming the elements in $\mathcal{U}(I_G)$ are doubly square-free, we have $d_i = 1$ for $i = 1, \dots, k$ and $q_1, \dots, q_k$ are square-free monomials. Consequently

$$\mathrm{in}_y(I_G) = \langle yq_1, \dots, yq_k, g_1, \dots, g_r\rangle$$

is generated by doubly square-free binomials and square-free monomials. Let $\bigcap_j Q_j$ be the primary decomposition of $\langle yq_1, \dots, yq_k\rangle$. Each Q_j is an ideal generated by variables since $\langle yq_1, \dots, yq_k\rangle$ is a square-free monomial ideal. Thus

$$\mathrm{in}_y(I_G) = \left(\bigcap_j Q_j\right) + \langle g_1, \dots, g_r\rangle = \bigcap_j (Q_j + \langle g_1, \dots, g_r\rangle).$$

If there is a $g_l = u_l - v_l$ with either u_l or $v_l \in Q_j$, then $Q_j + \langle g_1, \dots, g_r\rangle$ can be further decomposed into an intersection of ideals generated by variables and square-free binomials.

Continuing this process, we can write $\mathrm{in}_y(I_G) = \bigcap_i P_i$, where each $P_i = M_i + T_i$, with M_i an ideal generated by a subset of indeterminates in $\{e_1, \dots, e_t\}$, and $T_i \subseteq \mathcal{U}(I_G)$ is an ideal of binomials generated by $g_l = u_l - v_l$ where $u_l, v_l \notin M_i$. Again, we point out that each binomial is a doubly square-free binomial by our assumption on $\mathcal{U}(I_G)$. As the next result shows, the binomial ideal T_i is a toric ideal corresponding to a subgraph of G.

Theorem 3.6. *Let G be a finite simple graph with toric ideal $I_G \subseteq \mathsf{K}[E(G)]$, and suppose that the elements of $\mathcal{U}(I_G)$ are doubly square-free. For a fixed variable y in $\mathsf{K}[E(G)]$, suppose that*

$$\mathrm{in}_y(I_G) = \bigcap_i P_i \text{ with } P_i = M_i + T_i,$$

using the notation as above. Let $E_i \subseteq E(G)$ be the set of edges that correspond to the variables in $M_i + \langle y\rangle$, and let $G \smallsetminus E_i$ be the graph G with all the edges of E_i removed. Then $T_i = I_{G \smallsetminus E_i}$.

Proof. The generators of T_i are those elements of $\mathcal{U}(I_G)$ whose terms are not divisible by any variable contained in $M_i + \langle y\rangle$. So a generator of T_i corresponds to a primitive closed even walk that does not contain any of the edges in E_i. Therefore, each generator of T_i is a closed even walk in $G \smallsetminus E_i$, and thus $T_i \subset I_{G \smallsetminus E_i}$ by Theorem 3.1 (1). Conversely, suppose that $\Gamma \in \mathcal{U}(I_{G \smallsetminus E_i})$. Then by Theorem 3.1 (2), Γ corresponds to some primitive closed even walk of G not passing through any edge of E_i. These are exactly the generators in T_i.

We now arrive at a primary decomposition of $\mathrm{in}_y(I_G)$.

Corollary 3.7. *Let G be a finite simple graph with toric ideal $I_G \subseteq \mathsf{K}[E(G)]$, and suppose that the elements of $\mathcal{U}(I_G)$ are doubly square-free. For a fixed variable y in $\mathsf{K}[E(G)]$, suppose that*

$$\mathrm{in}_y(I_G) = \bigcap_i P_i,$$

using the notation as above. Then each P_i is a prime ideal, and after removing redundant components, this intersection defines a primary decomposition of $\mathrm{in}_y(I_G)$.

Proof. By the previous result, $P_i = M_i + I_{G \smallsetminus E_i}$ for every i. So the fact that P_i is a prime ideal immediately follows from the fact that any toric ideal is prime, and that no cancellation occurs between variables in M_i and elements of $T_i = I_{G \smallsetminus E_i}$.

If I_G is generated by a doubly square-free universal Gröbner basis, choosing any $y = e_i$ defines a geometric vertex decomposition of I_G with respect to y by Lemma 2.3. Note that $\langle y \rangle$ appears in the primary decomposition of $\langle yq_1, \dots, yq_k \rangle$, so one prime ideal in the decomposition given in Corollary 3.7 $\mathrm{in}_y(I_G)$ will always be $\langle y \rangle + \langle g_1, \dots, g_r \rangle$. But this is exactly $\langle y \rangle + N_{y,I_G} = \langle y \rangle + I_{G \restriction e}$, by Theorem 3.5. As the next theorem shows, if we omit this prime ideal, the remaining prime ideals form a primary decomposition of C_{y,I_G}.

Theorem 3.8. *Let G be a finite simple graph with toric ideal $I_G \subseteq \mathsf{K}[E(G)]$, and suppose that the elements of $\mathcal{U}(I_G)$ are doubly square-free. Fix any variable $y = e_i$. Suppose that after relabelling the primary decomposition* $\mathrm{in}_y(I_G)$ *of Corollary* 3.7 *we have*

$$\mathrm{in}_y(I_G) = \bigcap_{i=0}^{d} (M_i + I_{G \restriction E_i}) = (\langle y \rangle + I_{G \restriction e_i}) \cap \bigcap_{i=1}^{d} (M_i + I_{G \restriction E_i}). \tag{2}$$

Then

$$C_{y,I_G} = \bigcap_{i=1}^{d} (M_i + I_{G \restriction E_i})$$

is a primary decomposition for C_{y,I_G}. Furthermore, if $<$ is a y-compatible monomial order, then (2) *is a geometric vertex decomposition for I_G with respect to y.*

Proof. The fact about the geometric vertex decomposition follows from Lemma 2.3.

Since $\mathcal{U}(I_G)$ contains doubly square-free binomials, we can write

$$\mathrm{in}_y(I_G) = \langle ym_1, \dots, ym_k, g_1, \dots, g_r \rangle = \langle y, g_1 \dots, g_r \rangle \cap \langle m_1, \dots, m_k, g_1, \dots, g_r \rangle$$

where y does not divide any m_i or any term of any g_i. By definition,

$$N_{y,I_G} = \langle g_1, \dots, g_r \rangle \text{ and } C_{y,I_G} = \langle m_1, \dots, m_k, g_1, \dots, g_r \rangle.$$

Applying the process described before Theorem 3.6 to $\langle m_1, \dots, m_k, g_1, \dots, g_r \rangle$ proves the first claim.

Remark 3.9. Let M be a square-free monomial ideal and I_H a toric ideal of a graph H where elements of $\mathcal{U}(H)$ are doubly square-free. The arguments presented above can be adapted to prove that $M + I_H$ has a primary decomposition into prime ideals of the form $M_i + T_i$ as in Theorem 3.6.

3.3. Geometric vertex decomposability under graph operations. Given a graph G whose toric ideal I_G is geometrically vertex decomposable, it is natural to ask if there are any graph operations we can perform on G to make a new graph H so that the associated toric ideal I_H is also geometrically vertex decomposable. We show that the operation of "gluing" an even cycle onto a graph G is one such operation.

We make this more precise. Given a graph $G = (V(G), E(G))$ and a vertex subset $W \subseteq V(G)$, the *induced graph* of G on W, denoted G_W, is the graph $G_W = (W, E(G_W))$ where $E(G_W) = \{e \in E(G) \mid e \subseteq W\}$. A graph G is a *cycle* (of length n) if $V(G) = \{x_1, \dots, x_n\}$ and $E(G) = \{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\}$.

Following [9, Construction 4.1], we define the *gluing* of two graphs as follows. Let G_1 and G_2 be two graphs, and suppose that $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ are induced subgraphs of G_1 and G_2 that are isomorphic. If $\varphi : H_1 \to H_2$ is the corresponding graph isomorphism, we let $G_1 \cup_\varphi G_2$ denote the disjoint union $G_1 \sqcup G_2$ with the associated edges and vertices of $H_1 \cong H_2$ being identified. We may say G_1 and G_2 are *glued along* H if both the induced subgraphs $H_1 \cong H_2 \cong H$ and φ are clear.

Example 3.10. Figure 1 (which is adapted from [9]) shows the gluing of a cycle C of even length onto a graph G to make a new graph H. The labelling is included to help illuminate the proof of the next theorem. In this figure, the cycle C has edges

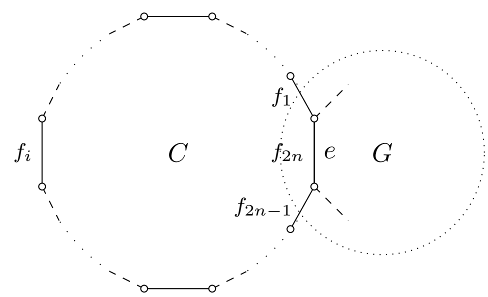

Figure 1. Gluing an even cycle C to a graph G along an edge.

$f_1, f_2, \ldots, f_{2n}$. The edge e is part of the graph G. We have glued C and G along the edge $e \cong f_{2n}$.

The geometric vertex decomposability property is preserved when an even cycle is glued along an edge of a graph whose toric ideal is geometrically vertex decomposable.

Theorem 3.11. *Suppose that G is a graph such that I_G is geometrically vertex decomposable in $\mathsf{K}[E(G)]$. Let H be the graph obtained from G by gluing a cycle of even length onto an edge of G (as in Figure 1). Then I_H is geometrically vertex decomposable in $\mathsf{K}[E(H)]$.*

Proof. The ideal I_H is clearly unmixed since I_H is a prime ideal. Now let $E(G) = \{e_1, \ldots, e_s\}$ denote the edges of G and let $E(C) = \{f_1, \ldots, f_{2n}\}$ denote the edges of the even cycle C. Let e be any edge of G, and after relabelling the f_i's we can assume that C is glued to G along f_{2n} and e (see Figure 1). Consequently,

$$E(H) = E(G) \cup \{f_1, \ldots, f_{2n-1}\}.$$

Let $e = f_{2n} = \{a, b\}$, and suppose that $a \in f_1$ and $b \in f_{2n-1}$, i.e., a is the vertex that f_1 shares with f_{2n}, and b is the vertex of f_{2n-1} shared with f_{2n}. By Theorem 3.1 (2), a universal Gröbner basis of I_H is given by the primitive binomials that correspond to even closed walks. Consider any even closed walk that passes through f_1. It will have the form

$$(f_1, f_2, \ldots, f_{2n-1}, e) \text{ or } (f_1, f_2, \ldots, f_{2n-1}, e_{j_1}, \ldots, e_{j_{2k-1}})$$

for some odd walk $(e_{j_1}, \ldots, e_{j_{2k-1}})$ in G that connects the vertex a of f_1 with the vertex b of f_{2n-1}. Thus, any primitive binomial involving the variable f_1 has the form

$$f_1 f_3 \cdots f_{2n-1} - e f_2 \cdots f_{2n-2}$$

or

$$f_1 f_3 \cdots f_{2n-1} e_{j_2} e_{j_4} \cdots e_{j_{2k-2}} - f_2 f_4 \cdots f_{2n-2} e_{j_1} \cdots e_{j_{2k-1}}.$$

Let $y = f_1$ and let $<$ be a y-compatible monomial order, and consider the universal Gröbner basis of I_H written as $\mathcal{U}(I_H) = \{y^{d_1} q_1 + r_1, \ldots, y^{d_k} q_k + r_k, g_1, \ldots, g_r\}$, where $d_i > 0$ and where y does not divide any term of g_i and q_i. Each $g_1, \ldots, g_r$ corresponds to a primitive closed even walk that does not pass through f_1. Consequently, each g_i corresponds to a primitive closed even walk in G. Thus $\langle g_1, \ldots, g_r \rangle = I_G$ (we abuse notation and write I_G for the induced ideal $I_G \mathsf{K}[E(H)]$).

Additionally, by Lemma 3.5 we have $N_{y, I_H} = \langle g_1, \ldots, g_r \rangle = I_{H \smallsetminus f_1}$. But note that in $H \smallsetminus f_1$, the edge f_2 is a leaf. Removing f_2 from $(H \smallsetminus f_1)$ makes f_3 a leaf, and so on. So, by repeatedly applying Lemma 3.2, we have

$$N_{y, I_H} = \langle g_1, \ldots, g_r \rangle = I_{H \smallsetminus f_1} = I_{(H \smallsetminus f_1) \smallsetminus f_2} = \cdots = I_{(\cdots (H \smallsetminus f_1) \cdots) \smallsetminus f_{2n-1}} = I_G.$$

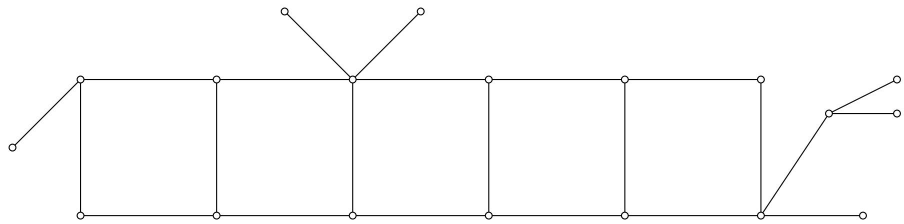

Figure 2. A graph whose toric ideal is geometrically vertex decomposable

Since $f_1f_3\cdots f_{2n-1}-ef_2\cdots f_{2n-2}$ is a primitive binomial $f_3\cdots f_{2n-1}\in C_{y,I_H}$. Furthermore, by our discussion above, any other primitive binomial containing a term divisible by $y=f_1$ has the form

$$f_1f_3\cdots f_{2n-1}e_{j_2}e_{j_4}\cdots e_{j_{2k-2}}-f_2f_4\cdots f_{2n-2}e_{j_1}\cdots e_{j_{2k-1}},$$

and consequently $f_3\cdots f_{2n-1}e_{j_2}\cdots e_{j_{2k-2}}\in C_{y,I_H}$. But this form is divisible by $f_3\cdots f_{2n-1}$, so

$$C_{y,I_H}=\langle f_3\cdots f_{2n-1},g_1,\ldots,g_r\rangle=\langle f_3\cdots f_{2n-1}\rangle+I_G.$$

It is now straightforward to check that

$$\mathrm{in}_y(I_H)=\langle f_1f_3\cdots f_{2n-1}\rangle+I_G=C_{y,I_H}\cap(N_{y,I_H}+\langle y\rangle),$$

thus giving a geometric vertex decomposition of I_H with respect to y. (We could also deduce this from Lemma 2.3 since each $d_i=1$ in our description of $\mathcal{U}(I_H)$ above.)

To complete the proof, the contraction of N_{y,I_H} to $\mathsf{K}[f_2,\ldots,f_{2n},e_1,\ldots,e_s]$ satisfies

$$N_{y,I_H}=\langle 0\rangle+I_G\subseteq\mathsf{K}[f_2,\ldots,f_{2n}]\otimes\mathsf{K}[E(G)].$$

So N_{y,I_H} is geometrically vertex decomposable by Theorem 2.9 since I_G is geometrically vertex decomposable in $\mathsf{K}[E(G)]$, and similarly for $\langle 0\rangle$ in $\mathsf{K}[f_2,\ldots,f_n]$. The ideal C_{y,I_H} contracts to

$$C_{y,I_H}=\langle f_3\cdots f_{2n-1}\rangle+I_G\subseteq\mathsf{K}[f_2,\ldots,f_{2n}]\otimes\mathsf{K}[E(G)].$$

Since $\langle f_3f_5\cdots f_{2n-1}\rangle\subseteq\mathsf{K}[f_2,\ldots,f_{2n}]$ is geometrically vertex decomposable by Lemma 2.6 (2), and I_G is geometrically vertex decomposable in $\mathsf{K}[E(G)]$ by hypothesis, the ideal C_{y,I_H} is geometrically vertex decomposable by again appealing to Theorem 2.9. Thus I_H is geometrically vertex decomposable, as desired.

Example 3.12. Let G be a cycle of even length, i.e., G has edge set $e_1,\ldots,e_{2n}$ with $(e_1,\ldots,e_{2n})$ a closed even walk. The ideal $I_G=\langle e_1e_3\cdots e_{2n-1}-e_2e_4\cdots e_{2n}\rangle$ is geometrically vertex decomposable by Lemma 2.6 (2). By repeatedly applying Theorem 3.11, we can glue on even cycles to make new graphs whose toric ideals are geometrically vertex decomposable. Note that by Lemma 3.2, we can also add leaves (and leaves to leaves, and so on) and not destroy the geometrically vertex decomposability property. These constructions allow us to build many graphs whose toric ideal is geometrically vertex decomposable.

As a specific example, the graph in Figure 2 is geometrically vertex decomposable since we have repeatedly glued cycles of length four along edges, and then added some leaves. This bipartite graph is also an example of what Gitler, Reyes, and Villarreal call a *ring graph* [11, Definition 2.5].

4. Toric ideals of graphs and the glicci property

In this section we recall some of the basics of Gorenstein liaison (Section 4.1) and then show that some large classes of toric ideals of graphs are glicci (Section 4.2). This section is partly motivated by a result of Klein and Rajchgot [21, Theorem 4.4], which says that geometrically vertex decomposable ideals are glicci. We note that while geometrically vertex decomposable ideals are glicci, glicci ideals need not be geometrically vertex decomposable. Indeed, we do not know if the toric ideals of graphs proven to be glicci in this section are also geometrically vertex decomposable. However, the results of this section make use of the geometric vertex decomposition language of Remark 2.2. For the remainder of the section, we will let $S = \mathsf{K}[x_0, \ldots, x_n]$ denote the graded polynomial ring with respect to the standard grading.

4.1. Gorenstein liaison preliminaries.

We provide a quick review of the basics of Gorenstein liaison; our main references for this material are [24, 25].

Definition 4.1. *Suppose that V_1, V_2, X are subschemes of P^n defined by saturated ideals I_{V_1}, I_{V_2} and I_X of $S = \mathsf{K}[x_0, \ldots, x_n]$, respectively. Suppose also that $I_X \subset I_{V_1} \cap I_{V_2}$ and $I_{V_1} = I_X : I_{V_2}$ and $I_{V_2} = I_X : I_{V_1}$. We say that V_1 and V_2 are directly algebraically G-linked if X is Gorenstein. In this case we write $V_1 \overset{X}{\sim} V_2$.*

We can now define an equivalence relation using the notion of algebraically G-linked.

Definition 4.2. *Let $V_1, \ldots, V_k$ be subschemes of P^n defined by the saturated ideals $I_{V_1}, \ldots, I_{V_k}$. If there are Gorenstein varieties $X_1, \ldots, X_{k-1}$ such that $V_1 \overset{X_1}{\sim} V_2 \overset{X_2}{\sim} \cdots \overset{X_{k-1}}{\sim} V_k$, then we say V_1 and V_k are in the same Gorenstein liaison class (or G-liaison class) and V_1 and V_k are G-linked in $k-1$ steps. If V_k is a complete intersection, then we say V_1 is in the Gorenstein liaison class of a complete intersection or glicci.*

In what follows, we say a homogeneous saturated ideal I is glicci if the subscheme defined by I is glicci.

Example 4.3. Consider the twisted cubic $V_1 \subset \mathsf{P}^3$ with

$$I_{V_1} = \langle xz - y^2, xw - z^2, xw - yz \rangle \subseteq \mathsf{K}[x, y, z].$$

Choose two of these quadrics, and let X be subscheme defined by their intersection. It is an exercise to check that X is the union of V_1 and a line, which we denote by V_2. Therefore, $V_1 \overset{X}{\sim} V_2$. Furthermore, since X is a complete intersection, and thus Gorenstein, the twisted cubic and a line are directly G-linked.

Remark 4.4. One of the main open questions in liaison theory asks if every arithmetically Cohen-Macaulay subscheme of P^n is glicci (see [22, Question 1.6]).

While it can be difficult in general to find a sequence of G-links between two varieties, there is a tool called an elementary G-biliaison which simplifies the process when it exists.

Definition 4.5. *Let $S = \mathsf{K}[x_0, \ldots, x_n]$ with the standard grading. Let C and I be homogeneous, saturated, and unmixed ideals of S such that* $\mathrm{ht}(C) = \mathrm{ht}(I)$. *Suppose that there is some $d \in \mathsf{Z}$ and Cohen-Macaulay homogeneous ideal $N \subset C \cap I$ with* $\mathrm{ht}(N) = \mathrm{ht}(I) - 1$ *such that I/N is isomorphic to $[C/N](-d)$ as an R/N-module. If N is generically Gorenstein, then I is obtained from C via an elementary G-biliaison of height d.*

In fact, suppose that V and W are two subschemes of P^n such that I_V and I_W are homogeneous, saturated and unmixed ideals. If I_V is obtained from I_W by an elementary G-biliaison, then V and W are G-linked in two steps [17, Theorem 3.5]. Moreover, elementary G-biliaisons preserve the Cohen-Macaulay property. This and other properties of G-linked varieties can be found in [24]. Indeed, we will use the following:

Lemma 4.6 ([24, Corollary 5.13]). *Let I and J be homogeneous, saturated ideals in S and assume that I and J are directly G-linked. Then S/I is Cohen-Macaulay if and only S/J is Cohen-Macaulay.*

Migliore and Nagel have given a criterion for an ideal to be glicci.

Theorem 4.7 ([25, Lemma 2.1]). *Let $I \subset S$ be a homogeneous ideal such that S/I is Cohen-Macaulay and generically Gorenstein. If $f \in S$ is a homogeneous non-zero-divisor of S/I, then the ideal $I + \langle f \rangle \subset S$ is glicci.*

Another criterion for an ideal to be glicci is geometric vertex decomposability. In fact a geometric vertex decomposition gives rise to an elementary G-biliaison of height 1.

Lemma 4.8 ([21, Corollary 4.3]). *Let I be a homogeneous, saturated, unmixed ideal of S and $\mathrm{in}_y I = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ a nondegenerate geometric vertex decomposition with respect to some variable $y = x_i$ of S. Assume that $N_{y,I}$ is Cohen–Macaulay and generically Gorenstein and that $C_{y,I}$ is also unmixed. Then I is obtained from $C_{y,I}$ by an elementary G-biliaison of height* 1.

Theorem 4.9 ([21, Theorem 4.4]). *If the saturated homogeneous ideal $I \subseteq S$ is geometrically vertex decomposable, then I is glicci.*

As noted in the introduction of the paper, the previous result partially motivates our interest in developing a deeper understanding of geometrically vertex decomposable ideals.

4.2. Some toric ideals of graphs which are glicci. In this section we use Migliore and Nagel's result [25, Lemma 2.1] (see Theorem 4.7 above) to show that some classes of toric ideals of graphs are glicci. We begin with a straightforward consequence of this theorem together with [9, Theorem 3.7].

Theorem 4.10. *Let G be a finite simple graph such that $\mathsf{K}[E(G)]/I_G$ is Cohen-Macaulay. Let H be the graph obtained by gluing an even cycle C to G along any edge. Then I_H is glicci.*

Proof. As in the proof of Theorem 3.11, let $E(G) = \{e_1, \ldots, e_s\}$ denote the edges of G and $E(C) = \{f_1, \ldots, f_{2n}\}$ denote the (consecutive) edges of the even cycle C. Assume that C is glued to G along f_{2n} and e. Then $\mathsf{K}[E(H)] = \mathsf{K}[E(G)] \otimes \mathsf{K}[f_1, \ldots, f_{2n-1}]$. For convenience, write I_G for the induced ideal $I_G\mathsf{K}[E(H)]$.

Let $F = f_1 f_3 \cdots f_{2n-1} - f_2 f_4 \cdots f_{2n-2} e$ be the primitive binomial associated to the even cycle C. By [9, Theorem 3.7], $I_H = I_G + \langle F \rangle$. As I_G is prime, we have that F is a homogeneous non-zero-divisor on $\mathsf{K}[E(H)]/I_G$ and $\mathsf{K}[E(H)]/I_G$ is generically Gorenstein. As $\mathsf{K}[E(H)]/I_G$ is Cohen-Macaulay by assumption, Theorem 4.7 implies that I_H is glicci.

We can combine a one step geometric vertex decomposition with Theorem 4.7 to see that many toric ideals of graphs which contain 4-cycles are glicci. Our main theorem in this direction is Theorem 4.14, which says that the toric ideal of a *gap-free* graph containing a 4-cycle is glicci. We begin with a general lemma which is not necessarily about toric ideals of graphs.

Lemma 4.11. *Let $S = \mathsf{K}[x_0, \ldots, x_n]$ with the standard grading, and let $I \subset S$ be a homogeneous, saturated ideal such that S/I is Cohen-Macaulay. Assume the following conditions are satisfied:*

(1) *I is square-free in y with a nondegenerate geometric vertex decomposition*

$$\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle);$$

(2) *I contains a homogeneous polynomial Q of degree 2 such that y divides some term of Q; and*

(3) *$S/N_{y,I}$ is Cohen-Macaulay and generically Gorenstein, and $C_{y,I}$ is radical.*

Then I is glicci.

Proof. By assumption (1), we have a nondegenerate geometric vertex decomposition $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$. Since I is Cohen-Macaulay and hence unmixed, we can conclude that $C_{y,I}$ is equidimensional by [21, Lemma 2.8]. Since $C_{y,I}$ is also radical by assumption (3), $C_{y,I}$ must be unmixed. Furthermore, because $S/N_{y,I}$ is Cohen-Macaulay and generically Gorenstein by assumption (3), we may use Lemma 4.8 to see that the geometric vertex decomposition gives rise to an elementary G-biliaison of height 1 from I to $C_{y,I}$. Hence S/I being Cohen-Macaulay implies that $S/C_{y,I}$ is too by Lemma 4.6.

Let $<$ be a y-compatible monomial order. By assumptions (1) and (2), I contains a degree 2 form which can be written as $Q = yf + R$ where y does not divide any term in f or R. Thus, $f \in C_{y,I}$. Let $z = \operatorname{in}_<(f)$. Since the geometric vertex decomposition $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is nondegenerate, we have that $C_{y,I} \neq \langle 1 \rangle$. Hence $C_{y,I}$ has a reduced Gröbner basis of the form $\{f', t_1, \ldots, t_s\}$ where $\operatorname{in}_<(f') = z$ and z does not divide any term of any t_i, $1 \leqslant i \leqslant s$. Let $C' = \langle t_1, \ldots, t_s \rangle$ so that $C_{y,I} = \langle f' \rangle + C'$. With this set-up, we see that $f' \neq 0$ is a non-zero-divisor on S/C'.

Let $S_{\hat{z}} = \mathsf{K}[x_1, \ldots, \hat{z}, \ldots, x_n]$. Then $S/C_{y,I} \cong S_{\hat{z}}/C'$. Thus, $S_{\hat{z}}/C'$ (and hence S/C' after extending C' to S) is Cohen-Macaulay because $S/C_{y,I}$ is Cohen-Macaulay. Similarly, $C_{y,I}$ being radical implies that C' (viewed in $S_{\hat{z}}$ or S) is radical. Thus, by [25, Lemma 2.1] (see Theorem 4.7), we conclude that $C_{y,I}$ is glicci.

By applying the elementary G-biliaison between I and $C_{y,I}$ once more, we conclude that I is also glicci.

We will now apply Lemma 4.11 to see that certain classes of toric ideals of graphs are glicci. In what follows, let $y = x_i$ be an indeterminate in $S = \mathsf{K}[x_1, \ldots, x_n]$ and let $<$ be a y-compatible monomial order. Let M_y^G be the ideal generated by all monomials $m \in S$ such that $ym - r \in \mathcal{U}(I_G)$ and $\operatorname{in}_<(ym - r) = ym$. Observe that M_y^G does not depend on the choice of y-compatible monomial order. Furthermore, since I_G is prime and $ym - r$ is primitive, y cannot appear in both terms of the binomial. We will consider generalizations of M_y^G in Section 6.

Theorem 4.12. *Let G be a finite simple graph where $\mathsf{K}[E(G)]/I_G$ is Cohen-Macaulay. Suppose that there exists an edge $y \in E(G)$ such that y is contained in a 4-cycle of G, and a y-compatible monomial order $<_y$ such that $\operatorname{in}_y(I_G)$ is square-free in y. Suppose also that $I_{G \setminus y}$ is Cohen-Macaulay and $I_{G \setminus y} + M_y^G$ is radical. Then I_G is glicci.*

Proof. We will show that the three assumptions of Lemma 4.11 hold. Let $<$ be a y-compatible monomial order.

Since I_G is square-free in y, there exists a geometric vertex decomposition

$$\operatorname{in}_y(I_G) = C_{y,I_G} \cap (N_{y,I_G} + \langle y \rangle)$$

by Lemma 2.3. Then $N_{y,I_G} = I_{G \setminus y}$ and $C_{y,I_G} = I_{G \setminus y} + M_y^G$. Since I_G is a toric ideal of a graph, and hence generated in degree 2 or higher, we do not have that

$C_{y,I} = \langle 1 \rangle$. Furthermore, I_G and N_{y,I_G} are each the toric ideal of a graph, hence radical (and therefore saturated since I_G is not the irrelevant ideal), and C_{y,I_G} is radical by assumption. Thus, by [21, Proposition 2.4], we conclude that the geometric vertex decomposition $\mathrm{in}_y(I_G) = C_{y,I_G} \cap (N_{y,I_G} + \langle y \rangle)$ is nondegenerate since the reduced Gröbner basis of I_G involves y by assumption. Thus, assumption (1) of Lemma 4.11 holds.

Assumption (2) of Lemma 4.11 holds because there exists an edge $y \in E(G)$ such that y is contained in a 4-cycle of G. Assumption (3) of Lemma 4.11 holds by the assumption that $I_{G \setminus y}$ is Cohen-Macaulay and $I_{G \setminus y} + M_y^G$ is radical.

Recall from Theorem 3.4 that if $I_G \subseteq \mathsf{K}[E(G)]$ is a toric ideal of a graph which has a square-free degeneration, then $\mathsf{K}[E(G)]/I_G$ is Cohen-Macaulay. We can use Theorem 4.12 to show that many toric ideals of graphs which have both square-free degenerations and 4-cycles are glicci. Specifically, we have the following:

Corollary 4.13. *Let G be a finite simple graph and suppose that there exists an edge $y \in E(G)$ such that y is contained in a 4-cycle of G. Suppose also that there exists some y-compatible monomial order $<$ such that $\mathrm{in}_<(I_G)$ is a square-free monomial ideal. Then I_G is glicci.*

Proof. Since $\mathrm{in}_<(I_G)$ is a square-free monomial ideal, we have that $\mathsf{K}[E(G)]/I_G$ is Cohen-Macaulay. Furthermore, I_G is square-free in y.

Let $\{yq_1 + r_1, \dots, yq_s + r_s, h_1, \dots, h_t\}$ be a reduced Gröbner basis for I_G so that each $\mathrm{in}_<(yq_i)$, $1 \leqslant i \leqslant s$, and each $\mathrm{in}_<(h_j)$, $1 \leqslant j \leqslant t$ are square-free monomials. Consider the geometric vertex decomposition

$$\mathrm{in}_y(I_G) = C_{y,I_G} \cap (N_{y,I_G} + \langle y \rangle).$$

By [23, Theorem 2.1], $\{h_1, \dots, h_t\}$ and $\{q_1, \dots, q_s, h_1, \dots, h_t\}$ are a Gröbner bases for N_{y,I_G} and C_{y,I_G} respectively. Thus, $\mathrm{in}_<(N_{y,I_G})$ and $\mathrm{in}_<(C_{y,I_G})$ are square-free monomial ideals. Since $N_{y,I_G} = I_{G \setminus y}$ is a toric ideal of a graph, it follows that $I_{G \setminus y}$ is Cohen-Macaulay. Since $C_{y,I_G} = I_{G \setminus y} + M_y^G$, it follows that $I_{G \setminus y} + M_y^G$ is radical. Thus, the assumptions of Theorem 4.12 hold and we conclude that I_G is glicci.

We end by proving that the toric ideal of a gap-free graph containing a 4-cycle is glicci. A graph G is *gap-free* if for any two edges $e_1 = \{u, v\}$ and $e_2 = \{w, x\}$ with $\{u, v\} \cap \{w, x\} = \emptyset$, there is an edge $e \in E(G)$ that is is adjacent to both e_1 and e_2, i.e., one of the edges $\{u, w\}, \{u, x\}, \{v, w\}, \{v, x\}$ is also in G. Note that the name for this family is not standardized; these graphs are sometimes called $2K_2$-free, or C_4-free, among other names (see D'Alì [7] for more). Note that G has a 4-cycle if and only if the graph complement $\bar{G}$ is not gap-free.

Theorem 4.14. *Let G be a gap-free graph such that the graph complement $\bar{G}$ is not gap-free. Then I_G is glicci.*

Proof. Since $\bar{G}$ is not gap-free, G must contain a 4-cycle. Pick any variable y belonging to this cycle. By [7, Theorem 3.9], since G is gap-free, there exists a y-compatible order $<_y$ such that $\mathrm{in}_{<_y}(I_G)$ is square-free (we can ensure this by choosing $<_\sigma$ in [7, Theorem 3.9] so that the vertices defining y have the smallest weight). The result now follows from Corollary 4.13.

5. Toric ideals of bipartite graphs

In this section, we show that toric ideals of bipartite graphs are geometrically vertex decomposable. In Section 5.1, we treat the general case, making use of results of Constantinescu and Gorla from [3]. Then, in Section 5.2 we give alternate proofs of geometric vertex decomposibility in special cases.

5.1. Toric ideals of bipartite graphs are geometrically vertex decomposable. Recall that a simple graph G is *bipartite* if its vertex set $V(G) = V_1 \sqcup V_2$ is a disjoint union of two sets V_1 and V_2, such that every edge in G has one of its endpoints in V_1 and the other endpoint in V_2. The purpose of this subsection is to prove Theorem 5.8 below, which says that the toric ideal of a bipartite graph is geometrically vertex decomposable. We will make use of the results and ideas in Constantinescu and Gorla's paper [3] on Gorenstein liaison of toric ideals of bipartite graphs.

Let G be a bipartite graph. Following [3, Definition 2.2], we say that a subset $\mathbf{e} = \{e_1, \ldots, e_r\} \subseteq E(G)$ is a *path ordered matching* of length r if the vertices of G can be relabelled such that $e_i = \{i, i+r\}$ and

(1) $f_i = \{i, i+r+1\} \in E(G)$, for each $1 \leqslant i \leqslant r-1$,
(2) if $\{i, j+r\} \in E(G)$ and $1 \leqslant i, j \leqslant r$, then $i \leqslant j$.

The following is straightforward. It will be referenced later in the subsection.

Lemma 5.1. *Let $\mathbf{e} = \{e_1, \ldots, e_r\}$ be a path ordered matching. Then $\{e_1, \ldots, e_{r-1}\}$ is a path ordered matching on $G \setminus e_r$.*

Given a subset $\mathbf{e} \subseteq E(G)$, let $M^G_{\mathbf{e}}$ be the set of all monomials m such that there is some non-empty subset $\tilde{\mathbf{e}} \subseteq \mathbf{e}$ where $m\left(\prod_{e_i \in \tilde{\mathbf{e}}} e_i\right) - n$ is the binomial associated to a cycle in G. Let

$$I^G_{\mathbf{e}} = I_{G \setminus \mathbf{e}} + \langle M^G_{\mathbf{e}} \rangle, \tag{3}$$

and observe that when $\mathbf{e} = \emptyset$, $I^G_{\mathbf{e}} = I_G$.

Let G be a bipartite graph and $\mathbf{e} = \{e_1, \ldots, e_r\}$ a path ordered matching. Let $\prec$ be a lexicographic monomial order on $\mathsf{K}[E(G)]$ with $e_r > e_{r-1} > \cdots > e_1$ and $e_1 > f$ for all $f \in E(G) \setminus \mathbf{e}$. Let $\mathcal{C}(G)$ denote the set of binomials associated to cycles in G. By [3, Lemma 2.6], $\mathcal{C}(G \setminus \mathbf{e}) \cup M^G_{\mathbf{e}}$ is a Gröbner basis for $I^G_{\mathbf{e}}$ with respect to the term order $\prec$, and $\mathrm{in}_{\prec}(I^G_{\mathbf{e}})$ is a square-free monomial ideal.

Remark 5.2. Let $\widetilde{M}^G_{\mathbf{e}}$ be the set of monomials m such that there is some non-empty subset $\tilde{\mathbf{e}} \subseteq \mathbf{e}$ where $m\left(\prod_{e_i \in \tilde{\mathbf{e}}} e_i\right) - n$ is the binomial associated to a cycle in G and n is not divisible by any $e_i \in \mathbf{e}$. By [3, Remark 2.7], $\mathcal{C}(G \setminus \mathbf{e}) \cup \widetilde{M}^G_{\mathbf{e}}$ is also a Gröbner basis for $I^G_{\mathbf{e}}$ with respect to $\prec$. Furthermore, observe that if $me_i \in \widetilde{M}^G_{\mathbf{e}}$ for some $e_i \in \mathbf{e}$, then m is also an element of $\widetilde{M}^G_{\mathbf{e}}$. Hence, if we let $L^G_{\mathbf{e}}$ be the set of monomials in $\widetilde{M}^G_{\mathbf{e}}$ which are not divisible by any $e_i \in \mathbf{e}$, then $\mathcal{C}(G \setminus \mathbf{e}) \cup L^G_{\mathbf{e}}$ is a Gröbner basis for $I^G_{\mathbf{e}}$ with respect to $\prec$.

Using Remark 5.2, we obtain the following lemma, which we will need when proving geometric vertex decomposability of toric ideals of bipartite graphs.

Lemma 5.3. *Let G be a bipartite graph and let $\mathbf{e} = \{e_1, \ldots, e_r\}$, $r \geqslant 1$, be a path ordered matching on G, and let $\mathbf{e}' = \{e_1, \ldots, e_{r-1}\}$. Let $\prec$ be a lexicographic monomial order on $\mathsf{K}[E(G)]$ with $e_r > e_{r-1} > \cdots > e_1$ and $e_1 > f$ for all $f \in E(G) \setminus \mathbf{e}$. The set $\mathcal{C}(G \setminus \mathbf{e}') \cup L^G_{\mathbf{e}'}$ is a Gröbner basis for $I^G_{\mathbf{e}'}$ with respect to $\prec$ and $\mathrm{in}_{\prec}(I^G_{\mathbf{e}'})$ is a square-free monomial ideal.*

Proof. By Remark 5.2, $\mathcal{G} := \mathcal{C}(G \setminus \mathbf{e}') \cup L^G_{\mathbf{e}'}$ is a Gröbner basis for $I^G_{\mathbf{e}'}$ with respect to the lexicographic term order $e_{r-1} > e_{r-2} > \cdots > e_1 > e_r$ and $e_r > f$ for all $f \in E(G) \setminus \mathbf{e}$. Since none of $e_1, \ldots, e_{r-1}$ appear in $\mathcal{G}$, we have that $\mathcal{G}$ is also a Gröbner basis for the lexicographic monomial order $\prec$. Furthermore, all terms of all elements in $\mathcal{G}$ are square-free, so $\mathrm{in}_{\prec}(I^G_{\mathbf{e}'})$ is a square-free monomial ideal.

We now use Lemma 5.3 to obtain a geometric vertex decomposition of $I^G_{\mathbf{e}'}$:

Proposition 5.4. *Let G be a bipartite graph and let $\mathbf{e} = \{e_1, \ldots, e_r\}$ be a path ordered matching. Let $\mathbf{e}' = \{e_1, \ldots, e_{r-1}\}$. Then there is a geometric vertex decomposition*

$$\mathrm{in}_{e_r}(I^G_{\mathbf{e}'}) = (I^{G \smallsetminus e_r}_{\mathbf{e}'} + \langle e_r \rangle) \cap I^G_{\mathbf{e}}. \tag{4}$$

Proof. Let $\prec$ be a lexicographic monomial order on $\mathsf{K}[E(G)]$ with $e_r > e_{r-1} > \cdots > e_1$ and $e_1 > f$ for all $f \in E(G) \smallsetminus \mathbf{e}$. This is an e_r-compatible monomial order. By Lemma 5.3, $\mathcal{C}(G \smallsetminus \mathbf{e}') \cup L^G_{\mathbf{e}'}$ is a Gröbner basis for $I^G_{\mathbf{e}'}$ with respect to $\prec$, and $\mathcal{C}(G \smallsetminus \mathbf{e}') \cup L^G_{\mathbf{e}'}$ are square-free in e_r. We can write:

$$\mathcal{C}(G \smallsetminus \mathbf{e}') = \{e_r m_1 - n_1, e_r m_2 - n_2, \ldots, e_r m_q - n_q, h_1, \ldots, h_t\}, \text{ and}$$

$$L^G_{\mathbf{e}'} = \{e_r a_1, \ldots, e_r a_u, b_1, \ldots, b_v\}$$

where e_r does divide any m_ℓ, n_ℓ, $1 \leq \ell \leq q$, nor any term of h_k, $1 \leq k \leq t$, nor any of the monomials $a_1, \ldots, a_u, b_1, \ldots, b_v$. We thus have the geometric vertex decomposition

$$\begin{aligned}\mathrm{in}_{e_r}(I^G_{\mathbf{e}'}) &= (\langle h_1, \ldots, h_t, b_1, \ldots, b_v \rangle + \langle e_r \rangle) \cap \langle m_1, \ldots, m_q, h_1, \ldots, h_t, a_1, \ldots, a_u, b_1, \ldots, b_v \rangle \\ &= (\langle h_1, \ldots, h_t, b_1, \ldots, b_v \rangle + \langle e_r \rangle) \cap I^G_{\mathbf{e}}.\end{aligned}$$

It remains to show that $\langle h_1, \ldots, h_t, b_1, \ldots, b_v \rangle = I^{G \smallsetminus e_r}_{\mathbf{e}'}$.

By Lemma 5.1, $\mathbf{e}'$ is a path ordered matching on $G \smallsetminus e_r$. Thus, $I^{G \smallsetminus e_r}_{\mathbf{e}'}$ is generated by

$$\mathcal{C}((G \smallsetminus e_r) \smallsetminus \mathbf{e}') \cup L^{G \smallsetminus e_r}_{\mathbf{e}'} = \mathcal{C}(G \smallsetminus \mathbf{e}) \cup L^{G \smallsetminus e_r}_{\mathbf{e}'}.$$

Observe that $\{h_1, \ldots, h_t\} = \mathcal{C}(G \smallsetminus \mathbf{e})$. Also, it follows from the definitions that $L^{G \smallsetminus e_r}_{\mathbf{e}'} \subseteq \{b_1, \ldots, b_v\}$. Thus, we have the inclusion $I^{G \smallsetminus e_r}_{\mathbf{e}'} \subseteq \langle h_1, \ldots, h_t, b_1, \ldots, b_v \rangle$.

For the reverse inclusion, fix some b_j, $1 \leq j \leq v$. Then there is some non-empty subset $\widetilde{\mathbf{e}} \subseteq \mathbf{e}'$ and a binomial $b_j(\prod_{e_i \in \widetilde{\mathbf{e}}} e_i) - n$ associated to a cycle in G. If e_r does not divide n then $b_j \in M^{G \smallsetminus e_r}_{\mathbf{e}'}$, and hence $b_j \in I^{G \smallsetminus e_r}_{\mathbf{e}'}$. Otherwise, since $\mathbf{e}$ is also a path ordered matching, one can apply the proof of [3, Remark 2.7] to find another cycle in G which does not pass through e_r and which gives rise to an element $c_j \in M^G_{\mathbf{e}}$ which divides b_j. Since the cycle does not pass through e_r, we have $c_j \in M^{G \smallsetminus e_r}_{\mathbf{e}'}$. As $\mathcal{C}((G \smallsetminus e_r) \smallsetminus \mathbf{e}') \cup M^{G \smallsetminus e_r}_{\mathbf{e}'}$ is a Gröbner basis for $I^{G \smallsetminus e_r}_{\mathbf{e}'}$, we see that c_j, and hence b_j, is an element of $I^{G \smallsetminus e_r}_{\mathbf{e}'}$. Thus, $\langle h_1, \ldots, h_t, b_1, \ldots, b_v \rangle \subseteq I^{G \smallsetminus e_r}_{\mathbf{e}'}$.

We say that a path ordered matching $\mathbf{e} = \{e_1, \ldots, e_r\}$ is *right-extendable* if there is some edge $e_{r+1} \in E(G)$ such that $\{e_1, \ldots, e_r, e_{r+1}\}$ is also a path ordered matching.

Lemma 5.5. *Let G be a bipartite graph with no leaves and let $\mathbf{e} = \{e_1, \ldots, e_r\}$ be a path ordered matching which is not right-extendable. Then, $M^G_{\mathbf{e}}$ contains an indeterminate $x \in E(G)$ and $\mathbf{e}$ is a path ordered matching on $G \smallsetminus x$. Furthermore, $I^G_{\mathbf{e}} = I^{G \smallsetminus x}_{\mathbf{e}} + \langle x \rangle$.*

Proof. The proof is identical to the proof of [3, Lemmas 2.12 and 2.13] upon replacing maximal path ordered matchings in [3, Lemmas 2.12 and 2.13] with right-extendable path ordered matchings.

Lemma 5.6. *Let G be a bipartite graph and let $\mathbf{e} = \{e_1, \ldots, e_r\}$ be a path ordered matching. Suppose that G has a leaf y. Then:*

1. *if $y \notin \mathbf{e}$, then $\mathbf{e}$ is a path ordered matching in $G \smallsetminus y$ and $I^G_{\mathbf{e}} = I^{G \smallsetminus y}_{\mathbf{e}}$;*
2. *if $y \in \mathbf{e}$, then $y = e_1$ or e_r and $\mathbf{e} \smallsetminus y$ is a path ordered matching in $G \smallsetminus y$ and $I^G_{\mathbf{e}} = I^{G \smallsetminus y}_{\mathbf{e} \smallsetminus y}$.*

Proof. Since $\mathbf{e}$ is a path ordered matching, the vertices of G can be labelled such that $e_i = \{i, i+r\}$, $1 \leq i \leq r$. Let $f_i = \{i, i+r+1\}$, $1 \leq i \leq r-1$ so that

$$e_1, f_1, e_2, f_2, \ldots, e_{r-1}, f_{r-1}, e_r$$

is a path of consecutive edges in G. Since y is a leaf, we see that $y \notin \{f_1, \dots, f_{r-1}\}$. If $y \notin \mathbf{e}$, then each e_i, f_i remains and $\mathbf{e}$ is a path ordered matching in $G \setminus y$. Furthermore no cycle in G passes through y, hence $I_{\mathbf{e}}^G = I_{\mathbf{e}}^{G \setminus y}$.

If $y \in \mathbf{e}$, then either $y = e_1$ or $y = e_r$. In either case, since each f_i remains in $G \setminus y$, $\mathbf{e} \setminus y$ is still a path ordered matching in $G \setminus y$. Since there is no cycle in G which passes through y, we have $I_{\mathbf{e}}^G = I_{\mathbf{e} \setminus y}^G = I_{\mathbf{e} \setminus y}^{G \setminus y}$.

We will need one more result from [3]:

Theorem 5.7 ([3, Theorem 2.8]). *Let G be a bipartite graph and $\mathbf{e} = \{e_1, \dots, e_r\}$ a path ordered matching. Then $\mathsf{K}[E(G)]/I_{\mathbf{e}}^G$ is Cohen-Macaulay.*

We now adapt the proof of [3, Corollary 2.15] on vertex decomposability of the simplicial complex associated to an initial ideal of $I_{\mathbf{e}}^G$ to prove the main theorem of this subsection.

Theorem 5.8. *Let G be a bipartite graph and $\mathbf{e} = \{e_1, \dots, e_r\}$ a path ordered matching. Then the ideal $I_{\mathbf{e}}^G$ is geometrically vertex decomposable. In particular, the toric ideal I_G is geometrically vertex decomposable.*

Proof. Let $R = \mathsf{K}[E(G)]$. By Theorem 5.7, each $R/I_{\mathbf{e}}^G$ is Cohen-Macaulay, hence unmixed.

We proceed by double induction on $|E(G)|$ and $s - r$ where $\tilde{\mathbf{e}} = \{\tilde{e}_1, \dots, \tilde{e}_s\}$ is a path ordered matching that is not right-extendable and is such that $\tilde{e}_1 = e_1, \dots, \tilde{e}_r = e_r$.

If $|E(G)| \leqslant 3$, then $I_G = \langle 0 \rangle$ as there are no primitive closed even walks in G, so the result holds trivially.

If G has a leaf, then by Lemma 5.6, there is an edge y and a path ordered matching $\mathbf{e}'$ in $G \setminus y$ such that $I_{\mathbf{e}}^G = I_{\mathbf{e}'}^{G \setminus y}$. By induction on the number of edges in the graph, $I_{\mathbf{e}'}^{G \setminus y}$ is geometrically vertex decomposable, hence so is $I_{\mathbf{e}}^G$.

So, assume that G has no leaves. If $s - r = 0$, then $\mathbf{e}$ is not right extendable. Then, by Lemma 5.5, there is an indeterminate $z \in M_{\mathbf{e}}^G$ such that

$$I_{\mathbf{e}}^G = I_{\mathbf{e}}^{G \setminus z} + \langle z \rangle.$$

By Lemma 5.5, $\mathbf{e}$ is a path ordered matching on $G \setminus z$, so again by induction on the number of edges in the graph, we have the $I_{\mathbf{e}}^{G \setminus z}$ is geometrically vertex decomposable, hence so is $I_{\mathbf{e}}^G$.

Now suppose that $\mathbf{e}$ is right extendable, so that $s - r > 0$ and $\mathbf{e}^* = \{e_1, \dots, e_{r+1}\}$ is a path ordered matching. By Lemma 5.4, we have the geometric vertex decomposition

$$\mathrm{in}_{e_{r+1}}(I_{\mathbf{e}}^G) = (I_{\mathbf{e}}^{G \setminus e_{r+1}} + \langle e_{r+1} \rangle) \cap I_{\mathbf{e}^*}^G.$$

By Lemma 5.1, $\mathbf{e}$ is a path ordered matching on $G \setminus e_{r+1}$. So, by induction on the number of edges, $I_{\mathbf{e}}^{G \setminus e_{r+1}}$ is geometrically vertex decomposable. By induction on $s - r$, $I_{\mathbf{e}^*}^G$ is geometrically vertex decomposable. Hence, $I_{\mathbf{e}}^G$ is geometrically vertex decomposable.

The final conclusion now follows from the fact that $I_G = I_{\mathbf{e}}^G$ when $\mathbf{e} = \emptyset$.

5.2. Alternate proofs in special cases. In this section, we apply results from the literature to give alternate proofs of geometric vertex decomposability for some well-studied families of bipartite graphs. These proofs illustrate that in some cases, we can prove that a family of ideals is geometrically vertex decomposable directly from the definition. Moreover, these examples do not require the full strength of the machinery of Section 5.1; in particular, these families of examples have the property that the ideals $C_{y,I}$ and $N_{y,I}$ usually do not leave the family of ideals we are considering, thus giving us nice inductive proofs.

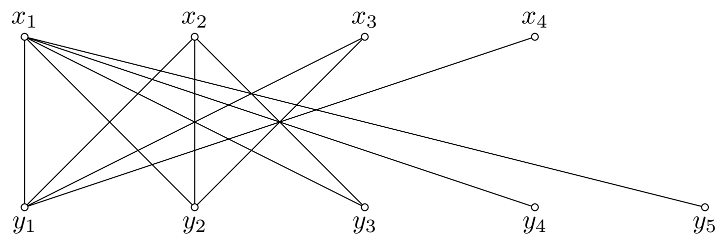

Figure 3. The graph T_λ for $\lambda = (5,3,2,1)$

We define the relevant families of graphs. A *Ferrers graph* is a bipartite graph on the vertex set $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ such that $\{x_n, y_1\}$ and $\{x_1, y_m\}$ are edges, and if $\{x_i, y_j\}$ is an edge, then so are all the edges $\{x_k, y_l\}$ with $1 \leqslant k \leqslant i$ and $1 \leqslant l \leqslant j$. We associate a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ to a Ferrers graph where $\lambda_i = \deg x_i$. Some of the properties of the toric ideals of these graphs were studied by Corso and Nagel [4]. Following Corso and Nagel, we denote a Ferrers graph as T_λ where λ denotes the associated partition.

As an example, consider the partition $\lambda = (5,3,2,1)$ which can be visualized as

$$\begin{array}{cccccc} & y_1 & y_2 & y_3 & y_4 & y_5 \\ x_1 & \bullet & \bullet & \bullet & \bullet & \bullet \\ x_2 & \bullet & \bullet & \bullet & & \\ x_3 & \bullet & \bullet & & & \\ x_4 & \bullet & & & & \end{array}$$

We have labelled the rows with the x_i vertices and the columns with the y_j vertices. From this representation, the graph T_λ is the graph on the vertex set $\{x_1, \ldots, x_4, y_1, \ldots, y_5\}$ where $\{x_i, y_j\}$ is an edge if and only if there is dot in the row and column indexed by x_i and y_j respectively. Figure 3 gives the corresponding bipartite graph T_λ for λ.

Next we consider the graphs studied in Galetto, *et al.* [10] as our second family of graphs. For integers $r \geqslant 3$ and $d \geqslant 2$, we let $G_{r,d}$ be the graph with vertex set

$$V(G_{r,d}) = \{x_1, x_2, y_1, \ldots, y_d, z_1, \ldots, z_{2r-3}\}$$

and edge set

$$\begin{aligned} E(G_{r,d}) = \{\{x_i, y_j\} \mid 1 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant d\} \cup \\ \{\{x_1, z_1\}, \{z_1, z_2\}, \{z_2, z_3\}, \ldots, \{z_{2r-4}, z_{2r-3}\}, \{z_{2r-3}, x_2\}\}. \end{aligned}$$

Observe that $G_{r,d}$ is the graph formed by taking the complete bipartite graph $K_{2,d}$ (defined below), and then joining the two vertices of degree d by a path of length $2r-2$. As an example, see Figure 4 for the graph $G_{3,5}$. We label the edges so that $a_i = \{x_1, y_i\}$ and $b_i = \{x_2, y_i\}$ for $i = 1, \ldots, d$, and $e_1 = \{x_1, z_1\}$, $e_{2r-2} = \{z_{2r-3}, x_2\}$ and $e_{i+1} = \{z_i, z_{i+1}\}$ for $1 \leqslant i \leqslant 2r-4$.

Using the above labelling, we can describe the universal Gröbner basis of $I_{G_{r,d}}$.

Theorem 5.9 ([10, Corollary 3.3]). *Fix integers $r \geqslant 3$ and $d \geqslant 2$. A universal Gröbner basis for $I_{G_{r,d}}$ is given by*

$$\{a_i b_j - b_i a_j \mid 1 \leqslant i < j \leqslant d\} \cup \{a_i e_2 e_4 \cdots e_{2r-2} - b_i e_1 e_3 e_5 \cdots e_{2r-3} \mid 1 \leqslant i \leqslant d\}.$$

The next result provides many examples of toric ideals which are geometrically vertex decomposable. Recall that the *complete bipartite graph $K_{n,m}$* is the graph with vertex set $V = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ and edge set $\{\{x_i, y_j\} \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$.

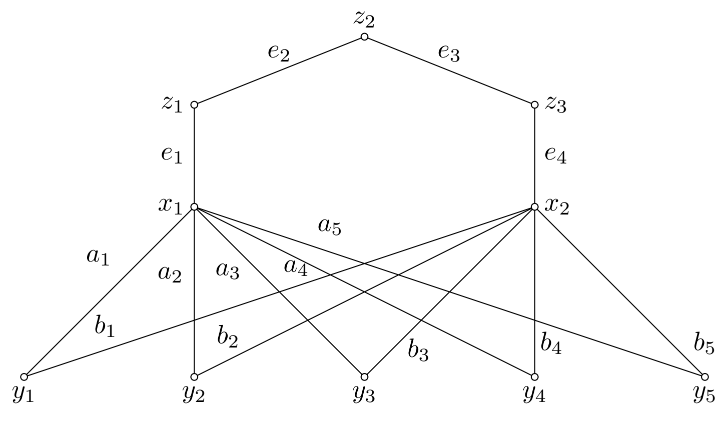

Figure 4. The graph $G_{3,5}$

Theorem 5.10. *The toric ideals of the following families of graphs are geometrically vertex decomposable:*

(1) *G is a cycle;*
(2) *G is a Ferrers graph T_λ for any partition λ;*
(3) *G is a complete bipartite graph $K_{n,m}$; and*
(4) *G is the graph $G_{r,d}$ for any $r \geqslant 3, d \geqslant 2$.*

Proof. (1) Suppose that G is a cycle with $2n$ edges. Then $I_G = \langle e_1e_3\cdots e_{2n-1} - e_2e_4\cdots e_{2n}\rangle$, so the result follows from Lemma 2.6 (2). If G is an odd cycle, then $I_G = \langle 0\rangle$, and so it is geometrically vertex decomposable by definition.
(2) As shown in the proof of [4, Proposition 5.1], the toric ideal of T_λ is generated by the 2×2 minors of a one-sided ladder. The ideal generated by the 2×2 minors of a one-sided ladder is an example of Schubert determinantal ideal (e.g., see [23]). The conclusion now follows from [21, Proposition 5.2] which showed that all Schubert determinantal ideals are geometrically vertex decomposable.[(1)]
(3) Apply the previous result using the partition $\lambda = \underbrace{(m, m, \ldots, m)}_{n}$.

(4) Let $I = I_{G_{r,d}}$. Since it is a prime ideal, it is unmixed. We first show that the statement holds if $d = 2$ and for any $r \geqslant 3$. Let $y = a_2$, and consider the lexicographic order on $\mathsf{K}[E(G_{r,d})] = \mathsf{K}[a_1, a_2, b_1, b_2, e_1, \ldots, e_{2r-2}]$ with $a_2 > a_1 > b_2 > b_1 > e_{2r-2} > \cdots > e_1$. This monomial order is y-compatible.

By using the universal Gröbner basis of Theorem 5.9, we have

$$C_{y,I} = \langle b_1, e_2e_4\cdots e_{2r-2}, a_1e_2\cdots e_{2r-2} - b_1e_1e_3\cdots e_{2r-3}\rangle = \langle b_1, e_2e_4\cdots e_{2r-2}\rangle$$

and $N_{y,I} = \langle a_1e_2\cdots e_{2r-2} - b_1e_1\cdots e_{2r-3}\rangle$. Note that each binomial in $\mathcal{U}(I)$ is doubly square-free, so we can use Lemma 2.3 to deduce that

$$\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$$

is a geometric vertex decomposition. To complete this case, note that $C_{y,I}$ is a monomial complete intersection in $\mathsf{K}[a_1, b_1, b_2, e_1, \ldots, e_{2r-2}]$, so this ideal is geometrically

(1) It is not necessary to use the connection to Schubert determinantal ideals. Indeed, it is known from the ladder determinantal ideal literature that (mixed) ladder determinantal ideals from (two-sided) ladders possess initial ideals which are Stanley–Reisner ideals of vertex decomposable simplicial complexes (see [12] and references therein). Then, an analogous proof to our proof of Theorem 5.8 can be given to show that these ideals are geometrically vertex decomposable.

vertex decomposable by Corollary 2.12. The ideal $N_{y,I}$ is a principal ideal generated by $a_1e_2\cdots e_{2r-2} - b_1e_1\cdots e_{2r-3}$, so it is geometrically vertex decomposable by Lemma 2.6 (2). So, for all $r \geqslant 3$, the toric ideal $I_{G_{r,2}}$ is geometrically vertex decomposable.

We proceed by induction on d. Assume $d > 2$ and let $r \geqslant 3$. Let $y = a_d$, and consider the lexicographic order on $\mathsf{K}[E(G_{r,d})] = \mathsf{K}[a_1,\ldots,a_d,b_1,\ldots,b_d,e_1,\ldots,e_{2r-2}]$ with $a_d > \cdots > a_1 > b_d > \cdots > b_1 > e_{2r-2} > \cdots > e_1$. This monomial order is y-compatible.

By again appealing to Theorem 5.9, we have

$$\begin{aligned} C_{y,I} &= \langle b_1,\ldots,b_{d-1},e_2e_4\cdots e_{2r-2}\rangle + \langle a_ib_j - b_ia_j \mid 1 \leqslant i < j \leqslant d-1\rangle + \\ &\quad\ \langle a_ie_2e_4\cdots e_{2r-2} - b_ie_1e_3e_5\cdots e_{2r-3} \mid 1 \leqslant i \leqslant d-1\rangle \\ &= \langle b_1,\ldots,b_{d-1},e_2e_4\ldots e_{2r-2}\rangle, \end{aligned}$$

where the last equality comes from removing redundant generators. On the other hand, by Lemma 3.5, $N_{y,I} = I_K$ where $K = G_{r,d} \smallsetminus a_d$. Note that in this graph, the edge b_d is a leaf, and consequently, $N_{y,I} = I_{G_{r,d-1}}$ since $K \smallsetminus b_d = G_{r,d-1}$.

We can again use Lemma 2.3 to deduce that

$$\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$$

is a geometric vertex decomposition.

To complete the proof, note that in the ring $\mathsf{K}[a_1,\ldots,a_{d-1},b_1,\ldots,b_d,e_1,\ldots,e_{2r-2}]$, the ideal $C_{y,I}$ is geometrically vertex decomposable by Corollary 2.12 since this ideal is a complete intersection monomial ideal. Also, the ideal $N_{y,I} = I_{G_{r,d-1}}$ is geometrically vertex decomposable by induction. Thus, $I_{G_{r,d}}$ is geometrically vertex decomposable for all $d \geqslant 2$ and $r \geqslant 3$.

As we will see in the remainder of the paper, there are many non-bipartite graphs which have geometrically vertex decomposable toric ideals.

6. Toric ideals with a square-free degeneration

As mentioned in the introduction, an important question in liaison theory asks if every arithmetically Cohen-Macaulay subscheme of P^n is glicci (e.g., see [22, Question 1.6]). As shown by Klein and Rajchgot (see Theorem 4.9), if a homogeneous ideal I is a geometrically vertex decomposable ideal, then I defines an arithmetically Cohen-Macaulay subscheme, and furthermore, this scheme is glicci. It is therefore natural to ask if every toric ideal I_G of a finite graph G that has the property that $\mathsf{K}[E(G)]/I_G$ is Cohen-Macaulay is also geometrically vertex decomposable. If true, then this would imply that the scheme defined by I_G is glicci.

Instead of considering all toric ideals of graphs such that $\mathsf{K}[E(G)]/I_G$ is Cohen-Macaulay, we restrict ourselves to ideals I_G which possess a square-free Gröbner degeneration with respect to some monomial order $<$. By Theorem 3.4, $\mathsf{K}[E(G)]/I_G$ is Cohen-Macaulay. Furthermore, if $\mathrm{in}_<(I_G)$ defines a vertex decomposable simplicial complex via the Stanley–Reisner correspondence, then I_G would be geometrically vertex decomposable with respect to a *lexicographic* monomial order $<$ (see [21, Proposition 2.14]). We propose the conjecture below. Note that this conjecture would imply that any toric ideal of a graph with a square-free initial ideal is glicci.

Conjecture 6.1. *Let G be a finite simple graph with toric ideal $I_G \subseteq \mathsf{K}[E(G)]$. If $\mathrm{in}_<(I_G)$ is square-free with respect to a lexicographic monomial order $<$, then I_G is geometrically vertex decomposable.*

By Theorem 5.8, Conjecture 6.1 is true in the bipartite setting. In this section, we build a framework for proving Conjecture 6.1. In particular, we reduce Conjecture 6.1 to checking whether certain related ideals are equidimensional, and we prove Conjecture 6.1 for the case where the generators in the universal Gröbner basis $\mathcal{U}(I_G)$ are quadratic.

6.1. Framework for the conjecture. Suppose that G is a labelled graph with n edges $e_1,\dots,e_n$ and toric ideal $I_G \subseteq \mathsf{K}[E(G)]$. Let $<_G$ be the lexicographic monomial order induced from the ordering of the edges coming from the labelling. That is, $e_1 > e_2 > \cdots > e_n$.

We define a class of ideals of the form $I^G_{E,F}$ such that $E \cup F = E_k = \{e_1,\dots,e_k\}$ for some $0 \leqslant k \leqslant n$ with $E \cap F = \emptyset$. Here $E_0 = \emptyset$. Define

$$I^G_{E,F} := I_{G \smallsetminus (E\cup F)} + M^G_{E,F}$$

where $I_{G \smallsetminus (E\cup F)}$ is the toric ideal of the graph G with the edges $E \cup F$ removed, and where $M^G_{E,F}$ is the ideal of $\mathsf{K}[e_1,\dots,e_n]$ generated by those monomials m with $m\ell - p \in \mathcal{U}(I_G)$ such that:

(1) $\mathrm{in}_{<_G}(m\ell - p) = m\ell$,
(2) ℓ is a monomial only involving some non-empty subset of variables in E, and
(3) no $f \in F$ divides $m\ell$ and no $e \in E$ divides m.

If there are no monomials m which satisfy conditions (1), (2), and (3), we set $M^G_{E,F} = \langle 0\rangle$. Therefore $M^G_{\emptyset,F} = \langle 0\rangle$ and $I^G_{\emptyset,F} = I_{G\smallsetminus F}$ (which is generated by those primitive closed even walks in G which do not pass through any edge of $F = E_k$). On the other hand, if there is an $\ell - p \in \mathcal{U}(I_G)$ with $\mathrm{in}_{<_G}(\ell - p) = \ell$ where ℓ is a monomial only involving the variables in E, then we take $m = 1$, and so $M^G_{E,F} = \langle 1\rangle$.

There is a natural set of generators for $I^G_{E,F}$ using the primitive closed even walks of I_G. In particular, the ideal $I^G_{E,F}$ is generated by the set

$$\mathcal{U}(I_{G\smallsetminus(E\cup F)}) \cup \mathcal{U}(M^G_{E,F}),$$

where $\mathcal{U}(I_{G\smallsetminus(F\cup E)})$ is the set of binomials defined by primitive closed even walks of the graph $G \smallsetminus (E\cup F)$, and $\mathcal{U}(M^G_{E,F})$ are those monomials m appearing in a generator of $\mathcal{U}(I_G)$ and satisfying conditions (1), (2), and (3) above. Because $M^G_{E,F}$ is a monomial ideal, its minimal generators form a universal Gröbner basis, so our notation makes sense. Going forward, we restrict our attention to the case where $\mathrm{in}_{<_G}(I_G)$ is square-free (this setting includes families of graphs like gap-free graphs [7] for certain choices of $<_G$).

To illustrate some of the above ideas, we consider the case that $E\cup F = E_1 = \{e_1\}$. This example also highlights a connection to the geometric vertex decomposition of I_G with respect to e_1.

Example 6.2. Assume that $\mathrm{in}_{<_G}(I_G)$ is square-free. Then we can write

$$\mathcal{U}(I_G) = \{e_1m_1 - p_1,\dots,e_1m_r - p_r, t_1,\dots,t_s\}$$

where e_1 does not divide m_i, p_i or any term of t_i. This set defines a universal Gröbner basis for $I_G = I^G_{\emptyset,\emptyset}$. Since $I_{G\smallsetminus e_1} = \langle t_1,\dots,t_s\rangle$ (by Lemma 3.5), we can write

$$\begin{aligned}
\mathrm{in}_{e_1}(I^G_{\emptyset,\emptyset}) &= \langle e_1m_1,\dots,e_1m_r,t_1,\dots,t_s\rangle\\
&= \langle e_1,t_1,\dots,t_s\rangle \cap \langle m_1,\dots,m_r,t_1,\dots,t_s\rangle\\
&= (\langle e_1\rangle + I_{G\smallsetminus e_1}) \cap (M^G_{\{e_1\},\emptyset} + I_{G\smallsetminus e_1})\\
&= (\langle e_1\rangle + I_{G\smallsetminus e_1} + M^G_{\emptyset,\{e_1\}}) \cap I^G_{\{e_1\},\emptyset}\\
&= (\langle e_1\rangle + I^G_{\emptyset,\{e_1\}}) \cap I^G_{\{e_1\},\emptyset}.
\end{aligned}$$

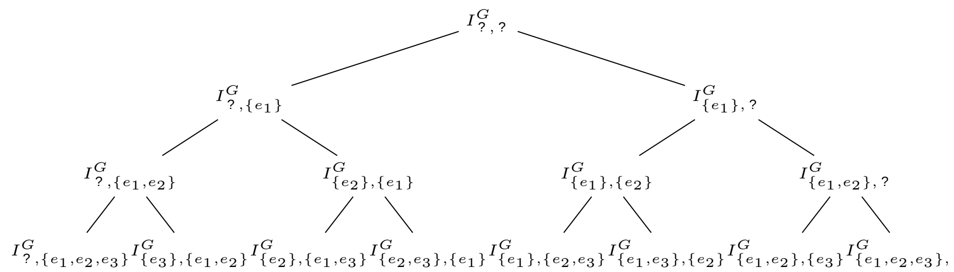

FIGURE 5. The relation between the ideals $I^G_{E,F}$

Note that $I_{G \,\mathsf{r}\, e_1} = I_{G \,\mathsf{r}\, e_1} + M^G_{?,\{e_1\}}$ since $M^G_{?,\{e_1\}} = \langle 0 \rangle$.

Note that if we take $y = e_1$ and $I = I^G_{?,?}$, then we get $C_{y,I} = I^G_{\{e_1\},?}$ and $N_{y,I} = I^G_{?,\{e_1\}}$. That is, $y = e_1$ defines a geometric vertex decomposition of I_G. Therefore, when $E \cup F = E_1 = \{e_1\}$, either $e_1 \in E$ or $e_1 \in F$, and each case appears in the geometric vertex decomposition.

If we continue the process by taking $\mathrm{in}_{e_2}(\cdot)$ of $I^G_{\{e_1\},?}$ and of $I^G_{?,\{e_1\}}$, we get one of four possible $C_{y,I}$ and $N_{y,I}$ ideals, each corresponding to a possible distribution of $\{e_1, e_2\}$ into the disjoint sets E and F such that $E \cup F = E_2$. Figure 5 shows the relationship between the ideals $I^G_{E,F}$ for the cases $E \cup F = E_i$ for $i = 0, \dots, 3$.

One strategy to verify Conjecture 6.1 is to prove the following three statements:

- (A) Given $I = I^G_{E,F}$ such that $E \cup F = E_{k-1}$ and $I \neq \langle 0 \rangle$ or $\langle 1 \rangle$, then $y = e_k$ defines a geometric vertex decomposition. Furthermore, $N_{y,I}$ and $C_{y,I}$ must also be of the form $I^G_{E',F'}$ where $E' \cup F' = E_k$.
- (B) If $E \cup F = E_n$, then $I^G_{E,F} = \langle 0 \rangle$ or $\langle 1 \rangle$.
- (C) For any $E \cup F = E_k$, the ideal $I^G_{E,F}$ must be unmixed.

Indeed, the next theorem verifies that proving $(A), (B)$, and (C) suffices to show that I_G is geometrically vertex decomposable.

THEOREM 6.3. *Let G be a finite simple graph with toric ideal $I_G \subseteq \mathsf{K}[E(G)]$, and suppose that* $\mathrm{in}_<(I_G)$ *is square-free with respect to a lexicographic monomial order $<$. If statements (A), (B), and (C) are true, then I_G is geometrically vertex decomposable.*

Proof. Let n be the number of edges of G. We show that for all sets E and F such that $E \cup F = E_k$, the ideal $I^G_{E,F}$ is geometrically vertex decomposable, and in particular, $I^G_{?,?} = I_G$ is geometrically vertex decomposable. We do descending induction on $|E \cup F|$. If $|E \cup F| = n$, then $E \cup F = E_n$, and so by statement (B), $I^G_{E,F} = \langle 0 \rangle$ or $\langle 1 \rangle$, both of which are geometrically vertex decomposable by definition.

For the induction step, assume that all ideals of the form $I^G_{E,F}$ with $E \cup F = E_\ell$ with $\ell \in \{k, \dots, n\}$ are geometrically vertex decomposable. Suppose that E and F are two sets such that $E \cup F = E_{k-1}$. The ideal $I^G_{E,F}$ is unmixed by statement (C). If $I^G_{E,F}$ is $\langle 0 \rangle$ or $\langle 1 \rangle$, then it is geometrically vertex decomposable by definition. Otherwise, by statement (A), the variable $y = e_k$ defines a geometric vertex decomposition of $I = I^G_{E,F}$, i.e.,

$$\mathrm{in}_y(I^G_{E,F}) = C_{y,I} \cap (N_{y,I} + \langle y \rangle).$$

Moreover, also by statement (A), the ideals $C_{y,I}$ and $N_{y,I}$ have the form $I^G_{E',F'}$ with $E' \cup F' = E_k$. By induction, these two ideals are geometrically vertex decomposable. So, $I^G_{E,F}$ is geometrically vertex decomposable.

We now show that (A) and (B) are always true. Thus, to prove Conjecture 6.1, one needs to verify (C). In fact, we will show that it is enough to show that $\mathsf{K}[E(G)]/I_{E,F}$ is equidimensional for all ideals of the form $I^G_{E,F}$.

We begin by proving that statement (A) holds if $\mathrm{in}_{<_G}(I_G)$ is a square-free monomial ideal. In fact, we prove some additional properties about the ideals $I^G_{E,F}$.

Theorem 6.4. *Let I_G be the toric ideal of a finite simple graph G such that $\mathrm{in}_{<_G}(I_G)$ is square-free. For each $k \in \{1,\dots,n\}$, let E, F be disjoint subsets of $\{e_1,\dots,e_n\}$ such that $E \cup F = E_{k-1} = \{e_1,\dots,e_{k-1}\}$. Then*

(1) *The natural generators $\mathcal{U}(I_{G \setminus (E\cup F)}) \cup \mathcal{U}(M^G_{E,F})$ of $I^G_{E,F}$ form a Gröbner basis for $I^G_{E,F}$ with respect to $<_G$. Furthermore, $\mathrm{in}_{<_G}(I^G_{E,F})$ is a square-free monomial ideal.*
(2) *$I^G_{E,F}$ is a radical ideal.*
(3) *The variable $y = e_k$ defines a geometric vertex decomposition of $I^G_{E,F}$.*
(4) *If $I = I^G_{E,F}$ and $y = e_k$, then $C_{y,I} = I^G_{E\cup\{e_k\},F}$ and $N_{y,I} = I^G_{E,F\cup\{e_k\}}$; in particular,*

$$\mathrm{in}_{e_k}(I^G_{E,F}) = I^G_{E\cup\{e_k\},F} \cap (I^G_{E,F\cup\{e_k\}} + \langle e_k \rangle).$$

Proof. (1) We will proceed by induction on $|E\cup F| = r = k-1$. If $r = 0$, then $E \cup F = \emptyset$ and $I^G_{E,F} = I_G$. In this case the natural generators are $\mathcal{U}(I_G) \cup \mathcal{U}(M^G_{\emptyset,\emptyset}) = \mathcal{U}(I_G)$, and this set defines a universal Gröbner basis consisting of primitive closed even walks of G. Its initial ideal is square-free by the assumption on $<_G$.

Now suppose that $|E \cup F| = r > 1$ and assume the result holds for $r-1$. There are two cases to consider:

<u>Case 1</u>: Assume that $e_r \in E$. By induction, the natural generators

$$\mathcal{U}(I_{G \setminus ((E \setminus \{e_r\})\cup F)}) \cup \mathcal{U}(M^G_{E \setminus \{e_r\},F})$$

of $I^G_{E\setminus\{e_r\},F}$ is a Gröbner basis with respect to $<_G$ and has a square-free initial ideal with respect to $<_G$. For the computations that follow, we can restrict to a minimal Gröbner basis by removing elements of this generating set which do not have a square-free lead term.

Since e_r cannot divide both terms of a binomial defined by a primitive closed even walk, we must have that this minimal Gröbner basis is square-free in $y = e_r$ (any e_r that appears in a binomial must appear in the lead term by definition of $<_G$, because none of the generators of $I^G_{E\setminus\{e_r\},F}$ involve $e_1,\dots,e_{r-1}$). Therefore, $I^G_{E\setminus\{e_r\},F}$ has a geometric vertex decomposition with respect to y by Lemma 2.3 (2).

The ideal $C_{y,I^G_{E\setminus\{e_r\},F}}$ is therefore generated by:

- Binomials corresponding to primitive closed even walks not passing through any edge of E_r. That is, elements of $\mathcal{U}(I_{G\setminus E_r})$.
- Monomials m which appear as the coefficient of e_r in $me_r - p \in \mathcal{U}(I_{G\setminus E_{r-1}})$.
- Monomials m which appear as the coefficient of e_r in $\mathcal{U}(M^G_{E\setminus\{e_r\},F})$. In this case, m is part of a binomial $me_r \prod_{i\in\mathcal{I}} e_i - p \in \mathcal{U}(I_G)$, where $\mathcal{I}$ indexes a subset of $E \setminus \{e_r\}$.

The last two types of monomials are exactly those monomials defining $\mathcal{U}(M^G_{E,F})$. Therefore

$$C_{y,I^G_{E\setminus\{e_r\},F}} = I^G_{E,F}.$$

Furthermore, the generators listed above for $C_{y,I^G_{E\setminus\{e_r\},F}}$ are a Gröbner basis with respect to $<_G$ by Lemma 2.3 (1) and are a subset of the natural generators of $I^G_{E,F}$.

Its initial ideal is also square-free since we restricted to a minimal Gröbner basis before computing $C_{y,I^G_{E\setminus\{e_r\},F}}$.

Case 2: Assume that $e_r \in F$. We argue similarly to Case 1 and omit the details. By induction $\mathcal{U}(I^G_{E,F\setminus\{e_r\}})$ is a Gröbner basis with respect to $<_G$ and defines a square-free initial ideal. We can once again restrict to a minimal Gröbner basis, both ensuring that all lead terms are square-free and that $y = e_r$ defines a geometric vertex decomposition. In this case, $N_{y,I^G_{E,F\setminus\{e_r\}}} = I^G_{E,F}$, and $\mathcal{U}(I^G_{E,F\setminus\{e_r\}})$ is a Gröbner basis by Lemma 2.3 (1) with respect to $<_G$. As in Case 1, the initial ideal of $I^G_{E,F}$ is square-free with respect to this monomial order since we restricted to a minimal Gröbner basis when computing $N_{y,I^G_{E,F\setminus\{e_r\}}}$.

For statement (2), the ideal $I^G_{E,F}$ is radical because it has a square-free degeneration. Statements (3) and (4) were shown as part of the proof of statement (1).

We now verify that statement (B) holds.

Theorem 6.5. *Let I_G be the toric ideal of a finite simple graph G such that* $\mathrm{in}_{<_G}(I_G)$ *is square-free. If $E \cup F = E_n$, then $I^G_{E,F} = \langle 0\rangle$ or $\langle 1\rangle$.*

Proof. Let $\mathcal{U}(I_G)$ be the universal Gröbner basis of I_G defined in Theorem 3.1. Since $\mathrm{in}_{<_G}(I_G)$ is square-free, we can take a minimal Gröbner basis where each lead term is square-free. We can write each element in our Gröbner basis as a binomial of the form $m\ell - p$ with $\mathrm{in}_{<_G}(m\ell - p) = m\ell$ where ℓ is a monomial only in the variables in E. Suppose that there is a binomial $m\ell - p \in \mathcal{U}(I_G)$ such that $m\ell = \ell$, i.e., the lead term only involves variables in E. Then $1 \in M^G_{E,F}$, and so $I^G_{E,F} = \langle 1\rangle$, since the monomials of $M^G_{E,F}$ form part of the generating set of $I^G_{E,F}$.

Otherwise, for every $m\ell - p \in \mathcal{U}(I_G)$, there is a variable $e_j \notin E$ such that $e_j | m$. Since $E \cup F = E_n$, we must have $e_j \in F$. But then m is not in $M^G_{E,F}$ since it fails to satisfy condition (3) of being a monomial in $M^G_{E,F}$, and thus $M^G_{E,F} = \langle 0\rangle$. Since $G \setminus (E \cup F)$ is the graph G with all of its edges removed, $I_{G\setminus(E\cup F)} = \langle 0\rangle$. Thus $I^G_{E,F} = \langle 0\rangle$.

To prove Conjecture 6.1, it remains to verify statement (C); that is, each ideal $I^G_{E,F}$ must be unmixed. This has proven difficult to show in general without specific restrictions on G. Nonetheless, the framework presented above leads to the next theorem which reduces statement (C) to showing that $\mathsf{K}[E(G)]/I^G_{E,F}$ is equidimensional. Recall that a ring R/I is *equidimensional* if $\dim(R/I) = \dim(R/P)$ for all minimal primes P of $\mathrm{Ass}_R(R/I)$.

Theorem 6.6. *Let I_G be the toric ideal of a finite simple graph G such that* $\mathrm{in}_{<_G}(I_G)$ *is square-free. If $\mathsf{K}[E(G)]/I^G_{E,F}$ is equidimensional for every choice of E, F, ℓ such that $E \cup F = E_\ell$ and $0 \leqslant \ell \leqslant n$, then I_G is geometrically vertex decomposable.*

Proof. In light of Theorems 6.3, 6.4, and 6.5, we only need to check that each $I^G_{E,F}$ is unmixed. However, by Theorem 6.4 (3), each ideal $I^G_{E,F}$ is radical, so being unmixed is equivalent to being equidimensional.

Remark 6.7. The definition of $I^G_{E,F}$ is an extension of the setup of Constantinescu and Gorla in [3] and is also used in Section 5. It is designed to utilize known results about geometric vertex decomposition. In [3], G is a bipartite graph, and techniques from liaison theory are employed to prove that I_G is glicci. Using a similar argument for general G, we can use

$$\mathrm{in}_{<_G}(I^G_{E,F}) = e_k \mathrm{in}_{<_G}(I^G_{E\cup\{e_k\},F}) + \mathrm{in}_{<_G}(I^G_{E,F\cup\{e_k\}})$$

to show that $\mathrm{in}_{<_G}(I^G_{E,F})$ can be obtained from $\mathrm{in}_{<_G}(I^G_{E\cup\{e_k\},F})$ via a Basic Double G-link (see [3, Lemma 2.1 and Theorem 2.8]), and so $\mathrm{in}_{<_G}(I^G_{E,F})$ being Cohen-Macaulay implies that $\mathrm{in}_{<_G}(I^G_{E\cup\{e_k\},F})$ is too (see Lemma 4.6). Through induction, we could then prove that some (but not all) of the $I^G_{E,F}$ in the tree following Example 6.2 are Cohen-Macaulay.

On the other hand, to produce G-biliaisons as in [3, Theorem 2.11], we would need specialized information about the graph G, something which is not a straightforward extension of the bipartite case.

6.2. Proof of the conjecture in the quadratic case. In the case that $\mathcal{U}(I_G)$ contains only quadratic binomials, we are able to verify that Conjecture 6.1 is true, that is, I_G is geometrically vertex decomposable. We first show that when $\mathcal{U}(I_G)$ contains only quadratic binomials, it has the property that $\mathrm{in}_{<_G}(I_G)$ is a square-free monomial ideal for any monomial order. In the statement below, recall that a binomial $m_1 - m_2$ is doubly square-free if both monomials that make up the binomial are square-free.

Lemma 6.8. *Suppose that G is a graph such that I_G has a universal Gröbner basis $\mathcal{U}(I_G)$ of quadratic binomials. Then these generators are doubly square-free.*

Proof. By Theorem 3.1, a quadratic element of $\mathcal{U}(I_G)$ comes from a primitive closed walk of length four of G. Since consecutive edges cannot be equal, all primitive walks of length four are actually cycles, so no edge is repeated, or equivalently, the generator is doubly square-free.

As noted in the previous subsection, to verify the conjecture in this case, it suffices to show that $\mathsf{K}[E(G)]/I^G_{E,F}$ is equidimensional for all E, F, ℓ with $E \cup F = E_\ell$. In fact, we will show a stronger result and show that all of these rings are Cohen-Macaulay.

We start with the useful observation that the natural set of generators of $I^G_{E,F}$ actually defines a universal Gröbner basis for the ideal.

Lemma 6.9. *Under the assumptions of Theorem 6.4, $\mathcal{U}(I_{G \restriction E_\ell}) \cup \mathcal{U}(M^G_{E,F})$ is a universal Gröbner basis of $I^G_{E,F}$.*

Proof. We will proceed by induction on $|E \cup F|$. The result is clear when $|E \cup F| = 0$. For the induction step, observe that $I^G_{E,F}$ is either $N_{y,I^G_{E,F\restriction y}}$ or $C_{y,I^G_{E\restriction y,F}}$ for some variable $y = e_i$. Suppose towards a contradiction that there is some monomial order $<$ on $\mathsf{K}[e_1,\ldots,\hat{y},\ldots e_n]$ for which $\mathcal{U}(I^G_{E,F})$ is not a Gröbner basis. Extend $<$ to a monomial order $<_y$ on $\mathsf{K}[e_1,\ldots,e_n]$ which first chooses terms with the highest degree in y and breaks ties using $<$. Clearly $<_y$ is a y-compatible order. By [23, Theorem 2.1], $\mathcal{U}(I^G_{E,F})$ is a Gröbner basis with respect to $<_y$. But $<_y = <$ on $\mathsf{K}[e_1,\ldots,\hat{y},\ldots e_n]$, a contradiction.

Lemma 6.10. *Let $R = \mathsf{K}[E(G)]$, and suppose that G is finite simple graph such that I_G has a universal Gröbner basis $\mathcal{U}(I_G)$ of quadratic binomials. Then $R/I^G_{E,F}$ is Cohen-Macaulay for every choice of E, F and ℓ such that $E \cup F = E_\ell$.*

Proof. Fix some E and F such that $E \cup F = E_\ell$. By definition $I^G_{E,F} = I_{G\restriction E_\ell} + M^G_{E,F}$. Since $\mathcal{U}(I_G)$ consists of quadratic binomials, then $M^G_{E,F}$ is either $\langle 1\rangle, \langle 0\rangle$, or $\langle e_{i_1},\ldots,e_{i_s}\rangle$ with $s > 0$.

The statement of the theorem clearly holds if $M^G_{E,F} = \langle 1\rangle$. If $M^G_{E,F} = \langle 0\rangle$, then $I^G_{E,F} = I_{G\restriction E_\ell}$. Then $I_{G\restriction E_\ell}$ is generated by quadratic primitive binomials and therefore possesses a square-free degeneration. By Theorem 3.4 these are toric ideals of graphs that are Cohen-Macaulay. We are left with the case that $M^G_{E,F}$ is generated by s indeterminates.

We first show that each $I^G_{E,F}$ is actually equal to $\widetilde{I}^G_{E,F} := I_{G\upharpoonright(E_\ell\cup\{e_{i_1},\dots,e_{i_s}\})}+M^G_{E,F}$. We certainly have $\widetilde{I}^G_{E,F} \subset I^G_{E,F}$. Let $<_{E,F}$ be the monomial order $e_{i_1} > \cdots > e_{i_s}$ and $e_{i_s} > f$ for all $f \in E(G) \upharpoonright (E_\ell \cup \{e_{i_1},\dots,e_{i_s}\})$. By Lemma 6.9, $\mathcal{U}(I_{G\upharpoonright E_\ell})\cup\mathcal{U}(M^G_{E,F})$ is a universal Gröbner basis for $I^G_{E,F}$. A similar statement holds for $\widetilde{I}^G_{E,F}$ since no variable of $\mathcal{U}(M^G_{E,F})$ is used to define $I_{G\upharpoonright(E_\ell\cup\{e_{i_1},\dots,e_{i_s}\})}$.

Clearly $\mathrm{in}_{<_{E,F}}(\widetilde{I}^G_{E,F}) \subset \mathrm{in}_{<_{E,F}}(I^G_{E,F})$. On the other hand, if there is some $u - v \in \mathcal{U}(I_{G\upharpoonright E_\ell})$ where u or v is in the ideal $M^G_{E,F}$, then $\mathrm{in}_{<_{E,F}}(u-v)$ is a multiple of some e_{i_j} for $j \in \{1,\dots,s\}$. Therefore, $\mathrm{in}_{<_{E,F}}(\widetilde{I}^G_{E,F}) = \mathrm{in}_{<_{E,F}}(I^G_{E,F})$ which in turn implies that $\widetilde{I}^G_{E,F} = I^G_{E,F}$ (e.g., see [8, Problem 2.8]).

Therefore, we can show that $R/I^G_{E,F}$ is Cohen-Macaulay by proving that $R/\widetilde{I}^G_{E,F}$ is. Recall that if a ring S is Cohen-Macaulay and graded and x is a non-zero-divisor of S, then $S/\langle x\rangle$ is also Cohen-Macaulay.

Now it is easy to see that $e_{i_1} + I_{G\upharpoonright(E_\ell\cup\{e_{i_1},\dots,e_{i_s}\})},\dots,e_{i_s} + I_{G\upharpoonright(E_\ell\cup\{e_{i_1},\dots,e_{i_s}\})}$ is a regular sequence on $R/I_{G\upharpoonright(E_\ell\cup\{e_{i_1},\dots,e_{i_s}\})}$. This follows from the fact that $I_{G\upharpoonright(E_\ell\cup\{e_{i_1},\dots,e_{i_s}\})}$ is Cohen-Macaulay since it possesses a square-free degeneration, and from the fact that $\mathcal{U}(I_{G\upharpoonright(E_\ell\cup\{e_{i_1},\dots,e_{i_s}\})})$ is not defined using the variables $\{e_{i_1},\dots,e_{i_s}\}$.

The previous lemma provides the unmixed condition needed to use Theorem 6.6. In summary, we have the following result:

Theorem 6.11. *Let I_G be the toric ideal of a finite simple graph G such that $\mathcal{U}(I_G)$ consists of quadratic binomials. Then I_G is geometrically vertex decomposable and glicci.*

Proof. By Lemma 6.8, any lexicographic order on the variables will determine a square-free degeneration of I_G. By Lemma 6.10 the rings $\mathsf{K}[E(G)]/I^G_{E,F}$ are Cohen-Macaulay for all E, F, and ℓ such that $E \cup F = E_\ell$. In particular, all of these rings are equidimensional. Thus, by Theorem 6.6, I_G is geometrically vertex decomposable, and therefore glicci by Theorem 4.9.

Remark 6.12. Although the condition that $\mathcal{U}(I_G)$ consists of quadratic binomials is restrictive, it is worth noting that there are families of graphs for which this is true (e.g., certain bipartite graphs). See [28, Theorem 1.2] for a characterization of when I_G can be generated by quadratic binomials, and [18, Proposition 1.3] for the case where the Gröbner basis is quadratic.

References

[1] Jennifer Biermann, Augustine O'Keefe, and Adam Van Tuyl, *Bounds on the regularity of toric ideals of graphs*, Adv. in Appl. Math. **85** (2017), 84–102.

[2] Anders Björner and Michelle L. Wachs, *Shellable nonpure complexes and posets. II*, Trans. Amer. Math. Soc. **349** (1997), no. 10, 3945–3975.

[3] Alexandru Constantinescu and Elisa Gorla, *Gorenstein liaison for toric ideals of graphs*, J. Algebra **502** (2018), 249–261.

[4] Alberto Corso and Uwe Nagel, *Monomial and toric ideals associated to ferrers graphs*, Trans. Amer. Math. Soc **361** (2009), no. 3, 1371–1395.

[5] David A. Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms*, 4 ed., Springer Cham, 2015.

[6] Sergio Da Silva and Megumi Harada, *Geometric vertex decomposition, Gröbner bases, and Frobenius splittings for regular nilpotent Hessenberg varieties*, 2022, `https://arxiv.org/abs/2207.08573`, forthcoming, *Transform. Groups.*

[7] Alessio D'Alí, *Toric ideals associated with gap-free graphs*, J. Pure Appl. Algebra **219** (2015), no. 9, 3862–3872.

[8] Viviana Ene and Jürgen Herzog, *Gröbner bases in commutative algebra*, Graduate studies in mathematics, American Mathematical Soc., 2012.

[9] Giuseppe Favacchio, Johannes Hofscheier, Graham Keiper, and Adam Van Tuyl, *Splittings of toric ideals*, J. Algebra **574** (2021), 409–433.

[10] Federico Galetto, Johannes Hofscheier, Graham Keiper, Craig Kohne, Miguel Eduardo Uribe Paczka, and Adam Van Tuyl, *Betti numbers of toric ideals of graphs: A case study*, J. Algebra Appl. **18** (2019), no. 12, article no. 1950226 (12 pages).

[11] Isidoro Gitler, Enrique Reyes, and Rafael H. Villarreal, *Ring graphs and complete intersection toric ideals*, Discrete Math. **310** (2010), no. 3, 430–441.

[12] E. Gorla, J.C. Migliore, and U. Nagel, *Gröbner bases via linkage*, J. Algebra **384** (2013), 110–134.

[13] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, `https://math.uiuc.edu/Macaulay2/`.

[14] Zachary Greif and Jason McCullough, *Green-Lazarsfeld condition for toric edge ideals of bipartite graphs*, J. Algebra **562** (2020), 1–27.

[15] Huy Tài Hà, Hop Dang Nguyen, Ngo Viet Trung, and Tran Nam Trung, *Symbolic powers of sums of ideals*, Math. Z. **294** (2020), no. 3, 1499–1520.

[16] Zachary Hamaker, Oliver Pechenik, and Anna Weigandt, *Gröbner geometry of schubert polynomials through ice*, Adv. Math. **398** (2022), article no. 108228 (29 pages).

[17] Robin Hartshorne, *Generalized divisors and biliaison*, Illinois J. Math. **51** (2007), no. 1, 83–98.

[18] Takayuki Hibi, Kenta Nishiyama, Hidefumi Ohsugi, and Akihiro Shikama, *Many toric ideals generated by quadratic binomials possess no quadratic Gröbner bases*, J. Algebra **408** (2014), 138–146.

[19] Melvin Hochster, *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Ann. of Math. (2) **96** (1972), 318–337.

[20] Patricia Klein, *Diagonal degenerations of matrix Schubert varieties*, Algebr. Comb. **6** (2023), no. 4, 1073–1094.

[21] Patricia Klein and Jenna Rajchgot, *Geometric vertex decomposition and liaison*, Forum Math. Sigma **9** (2021), article no. e70 (23 pages).

[22] Jan O. Kleppe, Juan C. Migliore, Rosa Miró-Roig, Uwe Nagel, and Chris Peterson, *Complete intersection liaison invariants and unobstructedness*, Mem. Amer. Math. Soc. **154** (2001), no. 732, viii+116pp.

[23] Allen Knutson, Ezra Miller, and Alexander Yong, *Gröbner geometry of vertex decompositions and of flagged tableaux*, J. Reine Angew. Math. **630** (2009), 1–31.

[24] Juan C. Migliore and Uwe Nagel, *Liaison and related topics: Notes from the Torino workshop/school*, Rend. Semin. Mat. Univ. Politec. Torino **59** (2001), no. 2, 59–126.

[25] ———, *Glicci ideals*, Compos. Math. **149** (2013), no. 9, 1583–1591.

[26] Somayeh Moradi and Fahimeh Khosh-Ahang, *On vertex decomposable simplicial complexes and their Alexander duals*, Math. Scand. **118** (2016), no. 1, 43–56.

[27] Uwe Nagel and Tim Römer, *Glicci simplicial complexes*, J. Pure Appl. Algebra **212** (2008), no. 10, 2250–2258.

[28] Hidefumi Ohsugi and Takayuki Hibi, *Toric ideals generated by quadratic binomials*, J. Algebra **218** (1999), no. 2, 509–527.

[29] J. Scott Provan and Louis J. Billera, *Decompositions of simplicial complexes related to diameters of convex polyhedra*, Math. Oper. Res. **5** (1980), no. 4, 576–594.

[30] Bernd Sturmfels, *Gröbner bases and convex polytopes*, vol. 8, American Mathematical Soc., 1996.

[31] Christos Tatakis and Apostolos Thoma, *On the universal Gröbner bases of toric ideals of graphs*, J. Combin. Theory Ser. A **118** (2011), no. 5, 1540–1548.

[32] Rafael H. Villarreal, *Rees algebras of edge ideals*, Comm. Algebra **23** (1995), no. 9, 3513–3524.

[33] ———, *Monomial algebras*, second edition. ed., Monographs and research notes in mathematics, CRC Press, Taylor & Francis Group, Boca Raton, 2015.

Mike Cummings, McMaster University, Department of Mathematics and Statistics, 1280 Main St W, Hamilton, ON CAN L8S4L8
E-mail : cummim5@mcmaster.ca

Sergio Da Silva, Virginia State University, Department of Mathematics and Economics, 1 Hayden Dr, Petersburg, VA USA 23806
E-mail : sdasilva@vsu.edu

Jenna Rajchgot, McMaster University, Department of Mathematics and Statistics, 1280 Main St W, Hamilton, ON CAN L8S4L8
E-mail : rajchgot@math.mcmaster.ca

Adam Van Tuyl, McMaster University, Department of Mathematics and Statistics, 1280 Main St W, Hamilton, ON CAN L8S4L8
E-mail : vantuyl@math.mcmaster.ca