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Diagonal degenerations of matrix Schubert varieties

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Abstract Knutson and Miller (2005) established a connection between the anti-diagonal Gröbner degenerations of matrix Schubert varieties and the pre-existing combinatorics of pipe dreams. They used this correspondence to give a geometrically-natural explanation for the appearance of the combinatorially-defined Schubert polynomials as representatives of Schubert classes. Recently, Hamaker, Pechenik, and Weigandt (2022) proposed a similar connection between diagonal degenerations of matrix Schubert varieties and bumpless pipe dreams, newer combinatorial objects introduced by Lam, Lee, and Shimozono (2021). Hamaker, Pechenik, and Weigandt described new generating sets of the defining ideals of matrix Schubert varieties and conjectured a characterization of permutations for which these generating sets form diagonal Gröbner bases. They proved special cases of this conjecture and described diagonal degenerations of matrix Schubert varieties in terms of bumpless pipe dreams in these cases. The purpose of this paper is to prove the conjecture in full generality. The proof uses a connection between liaison and geometric vertex decomposition established in earlier work with Rajchgot (2021).

1. Introduction

Schubert polynomials, introduced by Lascoux and Schützenberger [33] based on the work of Bernstein, Gel'fand, and Gel'fand [4], give combinatorially-natural representatives of Schubert classes in the cohomology ring of the complete flag variety. Matrix Schubert varieties, defined by Fulton [18], are generalized determinantal (affine) varieties corresponding to a permutation $w \in S_n$. A key insight of Knutson and Miller [28] is that the combinatorics of Schubert polynomials is naturally reflected in the geometry of the initial ideals of the defining ideals I_w of matrix Schubert varieties X_w under anti-diagonal degeneration (i.e. Gröbner degeneration under a term order in which the leading term of the determinant of a generic matrix is the product of the entries along the anti-diagonal). In particular, Knutson and Miller were able to identify irreducible components of anti-diagonal initial schemes with the *pipe dreams* that had arisen in earlier combinatorial study of Schubert polynomials [3, 17]. They were also able to give a geometric explanation for the positivity of coefficients of Schubert polynomials, a fact not obvious from their recursive definition, and show that the multidegrees of X_w give the torus-equivariant cohomology classes of Schubert varieties.

Later, Knutson, Miller, and Yong [29] connected the geometry of diagonal degenerations (defined analogously to anti-diagonal degenerations) of matrix Schubert varieties corresponding to *vexillary* permutations to the combinatorics of flagged tableaux. More recently, Hamaker, Pechenik, and Weigandt [24] proposed a diagonal Gröbner

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basis, which they call the set of CDG generators (see Subsection 2.2), for a wider class of permutations that includes the vexillary permutations. They proved that CDG generators form a diagonal Gröbner basis when w is what they called *banner* and used that result to connect the geometry of the diagonal degenerations of X_w to the *bumpless pipe dreams* introduced by Lam, Lee, and Shimozono [31] (closely related to the 6-vertex ice model used by Lascoux [32, 35, 9]). With an eye towards extending their main theorem [24, Theorem 6.4], Hamaker, Pechenik, and Weigandt made the following conjecture:

Conjecture 1.1. [24, Conjecture 7.1] *Let $w \in S_n$ be a permutation. The CDG generators are a diagonal Gröbner basis for I_w if and only if w avoids all eight of the patterns*

$$13254, 21543, 214635, 215364, 215634, 241635, 315264, 4261735.$$

The purpose of the present paper is to prove Conjecture 1.1, which we do as Corollaries 3.20 and 4.3. An important step in our proof is an application of the author's work with Rajchgot [27, Corollary 4.13], which uses the connection between *liaison*, whose use in studying Gröbner bases is described in [21], and *geometric vertex decomposition*, introduced in [29], to essentially reduce the requirements of the liaison-theoretic approach to a check on one ideal containment.

The liaison-theoretic approach to studying Gröbner bases is the subject of [21]. In it, Gorla, Migliore, and Nagel show that the module isomorphisms making up a particular kind of pair of Gorenstein links, called an *elementary G-biliaison*, naturally encode information about the Hilbert function that facilitates the establishment of Gröbner bases. The examples in [21] are of various generalized determinantal ideals, and the framework they introduce is the foundation of the present paper.

Gorla, Migliore, and Nagel's [21] work builds on a long history of commutative algebraic inquiry into determinantal ideals, especially since Hochster and Eagon's [26] results on determinantal ideals and Cohen–Macaulayness. Abhyankar [2] studied special cases of what are now known as one-sided ladder determinantal varieties in connection to Young tableaux and Schubert varieties in flag manifolds and showed those special cases to be irreducible. Narasimhan [34] established Gröbner bases for a more general class of one-sided and two-sided ladder determinantal ideals and used this result to show that all of the varieties in the class he studied are irreducible, extending Abhyankar's result. Gonciuliea and Miller [20] extended the Gröbner basis results to allow sizes of minors to vary in various regions within a ladder, and Gorla [22] showed in full generality that the natural generators of a two-sided mixed ladder determinantal ideal form a Gröbner basis. Herzog and Trung [25] gave analogous results for cogenerated and Pfaffian ideals and used them to give an elegant formula for the multiplicities of the corresponding quotient rings. See also [14] for related results. Bruns and Conca [7] gave Gröbner bases for powers of determinantal ideals, which they used to show the Cohen–Macaulayness of Rees algebras associated to determinantal ideals. In [23], Gorla gave a quite substantial generalization of Gaeta's theorem. See also, for example, [6, 13, 11, 8].

The appearance of pattern avoidance in determining when CDG generators form a Gröbner basis is natural in light of similar results in the Schubert literature. For example, pattern avoidance has previously been seen to govern the singularity [30] and Gorenstein property [36] of Schubert varieties as well as when the *Fulton generators* (see Subsection 2.1) of I_w constitute a diagonal Gröbner basis [29]. (Again, we refer to [22] for the first proof of one direction of this last result in the language of mixed ladder determinantal varieties.) For a survey of results in this vein, see [1].

In [24], the authors note that Conjecture 1.1 implies the following conjecture by the work of [16]:

Conjecture 1.2. [24, Conjecture 7.2] *If the (single) Schubert polynomial of $w \in S_n$ is a multiplicity-free sum of monomials, then the CDG generators of I_w are a diagonal Gröbner basis.*

We refer the reader to [24, 35] for more information on Schubert polynomials and bumpless pipe dreams.

The structure of this paper: Section 2 is devoted to preliminaries on matrix Schubert varieties and CDG generators. In Section 3, we prove the backward direction of Conjecture 1.1, and, in Section 4, we prove the forward direction. Finally, in Section 5, we use geometric vertex decomposition to give some intuition on what unifies the eight non-CDG permutations listed in Conjecture 1.1.

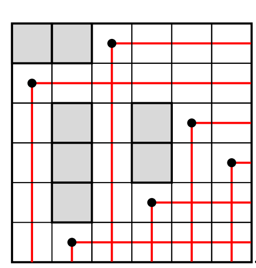
2. Preliminaries

In this section, we review the basics of matrix Schubert varieties as well as the CDG generators introduced in [24]. For a more detailed introduction to matrix Schubert varieties, we refer the reader to [19, Chapter 10]. For basic properties of and standard terminology for Gröbner bases, we refer the reader to [15].

2.1. Matrix Schubert varieties. We begin by describing how each permutation is associated to the affine variety called a matrix Schubert variety. Throughout this paper, we will take $[n] = \{1, 2, \ldots, n\}$ for any $n \geqslant 1$ and let S_n denote the symmetric group on $[n]$. Each permutation $w \in S_n$ is a bijection $w : [n] \to [n]$, which we will record in its one-line notation $w = w_1 w_2 \ldots w_n$ where $w_i = w(i)$. To every $w \in S_n$, we associate a *Rothe diagram* D_w, defined as follows:

$$D_w = \{(i,j) \in [n] \times [n] : w(i) > j, w^{-1}(j) > i\}.$$

A Rothe diagram has the following visualization: In an $n \times n$ grid, place a $\bullet$ in position (i, w_i) for each $i \in [n]$, and draw a line down from each $\bullet$ to the bottom of the grid and a line to the right from each $\bullet$ to the side of the grid. Then D_w is the set of boxes in the grid without a $\bullet$ in them or a line through them. For example, D_{315642} is the set $\{(1,1),(1,2),(3,2),(3,4),(4,2),(4,4),(5,2)\}$ and corresponds to the visualization below, in which the elements of D_{315642} appear in gray and will be referred to as the *boxes* of w:



The *Coxeter length* of the permutation w is equal to its inversion number, i.e.

$$|\{(i,j) \mid i < j, w_i > w_j\}|,$$

which is in turn equal to $|D_w|$. For example, the Coxeter length of 315642 is 7, easily read off as the number of gray boxes in the diagram above.

Definition 2.1. *Fix a permutation $w = w_1 \cdots w_n \in S_n$ and a permutation $v = v_1 \ldots v_k \in S_k$ with $k \leqslant n$. If there is some substring $w_{i_1} \cdots w_{i_k}$ of w satisfying $w_{i_j} < w_{i_\ell}$ exactly when $v_j < v_\ell$, then we say that w* contains *v. Otherwise, we say that w* avoids *v.*

For example, $w = 13254$ contains $v = 2143$ with 3254 the substring of w realizing the containment, but w does not contain $v' = 3214$. Notice that if $w_{i_1} \cdots w_{i_k}$ satisfies $w_{i_j} < w_{i_\ell}$ exactly when $v_j < v_\ell$, then the Rothe diagram of v can be obtained from that of w by restricting to the rows $i_1, \ldots, i_k$ and columns $w_{i_1}, \ldots, w_{i_k}$ in the $[n] \times [n]$ grid giving the visualization of D_w.

By restricting to the maximally southeast boxes of connected components of D_w, we define the *essential set* of w:

$$\mathrm{Ess}(w) = \{(i,j) \in D_w \mid (i+1,j),(i,j+1) \notin D_w\}.$$

In the example above, $\mathrm{Ess}(315642) = \{(1,2),(4,4),(5,2)\}$. Borrowing a term from the literature on ladder determinantal varieties, if $(i,j) \in \mathrm{Ess}(w)$ and there is no $(i'j') \in \mathrm{Ess}(w)$ with $i \leqslant i'$, $j \leqslant j'$, and $(i,j) \neq (i',j')$, we will say that (i,j) is a *lower outside corner* of D_w. In the case of $w = 315642$, $(4,4)$ and $(5,2)$ are lower outside corners, but $(1,2)$ is not.

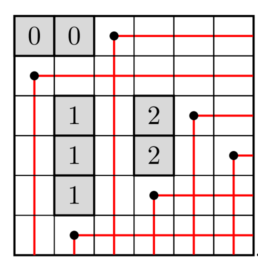
To every permutation $w \in S_n$, we associate a *rank function* $\mathrm{rank}_w : [n] \times [n] \to \mathbb{Z}$, where

$$\mathrm{rank}_w(i,j) = |\{k \leqslant i \mid w(k) \leqslant j\}|,$$

and the rank matrix M_w whose $(i,j)^{th}$ entry is $\mathrm{rank}_w(i,j)$. Visually, we assign to every square (i,j) in the $[n] \times [n]$ grid underlying the Rothe diagram of w the number of $\bullet$s weakly northwest of (i,j). In our running example $w = 315642$, one may find it helpful to record the information as

$$M_w = \begin{bmatrix} \boxed{0} & \boxed{0} & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & \boxed{1} & 2 & \boxed{2} & 3 & 3 \\ 1 & \boxed{1} & 2 & \boxed{2} & 3 & 4 \\ 1 & \boxed{1} & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

or as



.

Note that the transpose of the rank matrix of w and transpose of the Rothe diagram of w correspond to the rank matrix and Rothe diagram of w^{-1}, respectively, because the inverse of a permutation matrix is its transpose. We have recorded elements of D_w in the rank matrix M_w as boxes and colored the boxes of the essential set orange in anticipation of our discussion of Fulton generators, below.

Let $\mathrm{Mat}_{n,n}$ denote the affine n^2-space of complex $n \times n$ matrices. Given $N \in \mathrm{Mat}_{n,n}$ and subsets $A, B \subseteq [n]$, let $N_{A,B}$ be the submatrix of N determined by the rows whose indices are elements of A and the columns whose indices are elements of B. Then the *matrix Schubert variety* of $w \in S_n$ is the affine variety

$$X_w = \left\{ Z \in \mathrm{Mat}_{n,n} \mid \mathrm{rank}\, Z_{[i],[j]} \leqslant \mathrm{rank}_w(i,j) \text{ for all } (i,j) \in [n] \times [n] \right\}.$$

Let $Z = (z_{i,j})_{(i,j) \in [n] \times [n]}$ be a matrix of distinct indeterminates and $R = \mathbb{C}[Z]$ so that $\mathrm{Spec}(R)$ is identified with $\mathrm{Mat}_{n,n}$. The *Schubert determinantal ideal* of w is

$$I_w = ((\mathrm{rank}_w(i,j)+1)\text{-minors in } Z_{[i],[j]} \mid (i,j) \in [n] \times [n]) \subseteq R.$$

This naive generating set will typically include a good deal of redundancy, and so we will more often consider the smaller set of *Fulton generators* of I_w:

$$\{(\mathrm{rank}_w(i,j)+1)\text{-minors in } Z_{[i],[j]} \mid (i,j) \in \mathrm{Ess}(w)\}.$$

Fulton showed that the Fulton generators indeed generate I_w [18, Lemma 3.10], that I_w is prime, and, in particular, that $X_w \cong \mathrm{Spec}(R/I_w)$ as reduced schemes [18, Proposition 3.3]. The height of the ideal I_w (equivalently, codimension of $\mathrm{Spec}(R/I_w)$ in $\mathrm{Spec}(R)$) is equal to the Coxeter length of w [18, Proposition 3.3].

In the example $w = 315642$, the Fulton generators of I_w are $z_{1,1}$, $z_{1,2}$, the 2-minors of

$$\begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \\ z_{3,1} & z_{3,2} \\ z_{4,1} & z_{4,2} \\ z_{5,1} & z_{5,2} \end{bmatrix}, \text{ and the 3-minors of } \begin{bmatrix} z_{1,1} & z_{1,2} & z_{1,3} & z_{1,4} \\ z_{2,1} & z_{2,2} & z_{2,3} & z_{2,4} \\ z_{3,1} & z_{3,2} & z_{3,3} & z_{3,4} \\ z_{4,1} & z_{4,2} & z_{4,3} & z_{4,4} \end{bmatrix}.$$

2.2. CDG generators. In [24], the authors introduce *CDG generators* of defining ideals of matrix Schubert varieties. These generators are named after Conca, De Negri, and Gorla, whose result [12, Theorem 4.2] served as inspiration for the generating set used in [24] and, in particular, for Conjecture 1.1.

Definition 2.2. *Fix a permutation $w \in S_n$ and an $n \times n$ matrix $Z = (z_{i,j})_{(i,j)\in[n]\times[n]}$ of distinct indeterminates. Let* $\mathrm{Dom}(w) = \{(i,j) \in D_w \mid \mathrm{rank}_w(i,j) = 0\}$*, and call* $\mathrm{Dom}(w)$ *the* dominant part *of the Rothe diagram D_w. From Z, form the matrix Z' by replacing $z_{i,j}$ by 0 whenever $(i,j) \in \mathrm{Dom}(w)$. Set*

$$G'_w = \{(\mathrm{rank}_w(i,j)+1)\text{-minors in } Z'_{[i],[j]} \mid (i,j) \in \mathrm{Ess}(w) \smallsetminus \mathrm{Dom}(w)\},$$

and $G_w = G'_w \cup \{z_{i,j} \mid (i,j) \in \mathrm{Dom}(w)\}$. We call G_w the set of CDG generators *of I_w.*

Example 2.3. If $w = 315642$ the CDG generators of I_w are $z_{1,1}$, $z_{1,2}$, the 2-minors of

$$\begin{bmatrix} 0 & 0 \\ z_{2,1} & z_{2,2} \\ z_{3,1} & z_{3,2} \\ z_{4,1} & z_{4,2} \\ z_{5,1} & z_{5,2} \end{bmatrix}, \text{ and the 3-minors of } \begin{bmatrix} 0 & 0 & z_{1,3} & z_{1,4} \\ z_{2,1} & z_{2,2} & z_{2,3} & z_{2,4} \\ z_{3,1} & z_{3,2} & z_{3,3} & z_{3,4} \\ z_{4,1} & z_{4,2} & z_{4,3} & z_{4,4} \end{bmatrix}.$$

Notice that $\mathrm{Dom}(w) = \varnothing$ if and only if $w_1 = 1$, in which case the CDG generators and the Fulton generators coincide.

2.3. Gröbner bases. Let $S = \mathbb{C}[x_1, \ldots, x_d]$. A *term order* $<$ on S is a total order on the monic monomials of S so that $1 \leqslant \mu$ for every monomial μ of S and so that, for all monomials μ_1, μ_2, and ν, $\mu_1 < \mu_2$ implies $\mu_1\nu < \mu_2\nu$. Let $f = \sum_{i=1}^k c_i\mu_i \in S$ where $c_i \in \mathbb{C}$ and the μ_i are monic monomials (written in the usual way so that $\mu_i \neq \mu_j$ whenever $i \neq j$ and no c_i is 0). Fix i so that $c_i\mu_i > c_j\mu_j$ whenever $i \neq j$, and define the *leading term* of f to be $LT(f) = c_i\mu_i$. For an ideal I of S, define the *initial ideal* of I to be $LT(I) = (LT(f) \mid f \in I)$. A generating set $\mathcal{G}$ of I is called a *Gröbner basis* if $LT(I) = (LT(g) \mid g \in \mathcal{G})$. For a detailed introduction to the theory of Gröbner bases, including Buchberger's algorithm, we refer the reader to [15, Chapter 15].

Definition 2.4. *When the set of CDG generators forms a Gröbner basis for the Schubert determinantal ideal I_w under every diagonal term order, we will say that the permutation w* is CDG*.*

3. Rothe diagrams of CDG permutations

3.1. Obstructions to being CDG. We begin this section by describing in terms of the Rothe diagram D_w conditions that prevent w from being CDG. In Subsection 3.2, we will show that when D_w does not satisfy these conditions, w is necessarily CDG.

Before we begin, we note that the visualization of the Rothe diagram of 214635 is obtained from that of 215364 by transposition. The same is true of 315264 and 241635. The visualizations of the Rothe diagrams of the remaining permutations listed in Conjecture 1.1 are self transpose. This symmetry will allow us to consolidate some of our case work below. We understand the cardinal directions in reference to D_w in terms of its visualization. We say, for example, that (i',j') is “strictly southeast" of (i,j) to mean that both $i'>i$ and also $j'>j$, or that (i',j') is “strictly south and weakly east" of (i,j) to mean $i'>i$ and also $j'\geqslant j$.

Definition 3.1. *The permutation w has an* obstruction *of*

- Type 1 *if there is some $(r,s)\in \mathrm{Dom}(w)\cap \mathrm{Ess}(w)$ and two distinct entries (i,j) and (i',j') of D_w strictly southeast of (r,s) with $i'\neq i$ and $j'\neq j$,*
- Type 2 *if there is some $(r,s)\in \mathrm{Dom}(w)\cap \mathrm{Ess}(w)$ and two distinct entries (i,j) and (i,j') of $\mathrm{Ess}(w)$ strictly southeast of (r,s) with*

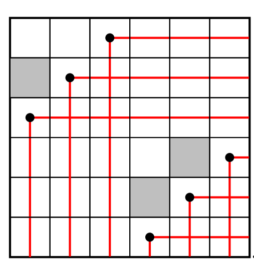
$$\max_k\{(k,j)\in \mathrm{Dom}(w)\} = \max_k\{(k,j')\in \mathrm{Dom}(w)\}$$

or, symmetrically, two distinct entries (i,j) and (i',j) of $\mathrm{Ess}(w)$ strictly southeast of (r,s) with

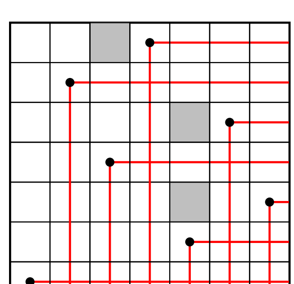
$$\max_\ell\{(i,\ell)\in \mathrm{Dom}(w)\} = \max_\ell\{(i',\ell)\in \mathrm{Dom}(w)\},$$

- Type 3 *if there are two distinct entries (i,j) and (i',j') of $\mathrm{Ess}(w)\smallsetminus \mathrm{Dom}(w)$ with (i',j') strictly southeast of (i,j).*

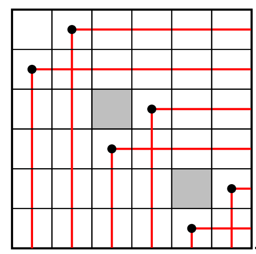
Example 3.2. The permutation 321654 has an obstruction of Type 1 with $(r,s)=(2,1)$, $(i,j)=(4,5)$ and $(i',j')=(5,4)$. These three essential boxes are shaded in light grey.



The permutation $w=4263751$ has an obstruction of Type 2 with $(r,s)=(1,3)$, $(i,j)=(3,5)$, and $(i,j')=(5,5)$. These three essential boxes are shaded in light grey. Here $\max_\ell\{(3,\ell)\in \mathrm{Dom}(w)\}=1=\max_\ell\{(5,\ell)\in \mathrm{Dom}(w)\}$.



The permutation 214365 has an obstruction of Type 3 with $(i,j)=(3,3)$ and $(i',j')=(5,5)$. These two essential boxes are shaded in light grey.



Lemma 3.3. *If the permutation $w \in S_n$ has an obstruction of Type* 1*, then w contains* 21543*,* 215634*,* 214635*,* 215364*, or* 13254*.*

An example illustrating some of the cases involved in the proof of Lemma 3.3 appears below the proof itself as Example 3.4.

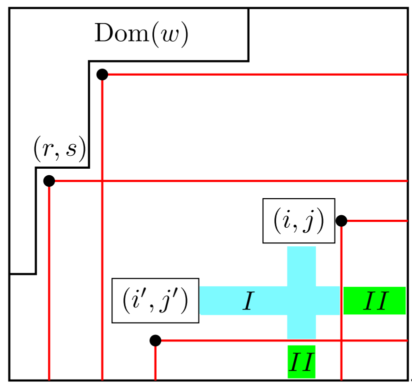
Proof. Fix a permutation $w \in S_n$ that has an obstruction of Type 1, and fix entries (r, s), (i, j), and (i', j') as in the definition of an obstruction of Type 1. Consider the visualization of the Rothe diagram D_w.

Label the $\bullet$ in the column $s + 1$ with (a, w_a) and the $\bullet$ in row $r + 1$ with (b, w_b). Notice $a < b$ and $w_a > w_b$. Because (i, j) and (i', j') are strictly southeast of (r, s), both (i, j) and (i', j') must be south of row b and east of column w_a. We consider two orientations of (i, j) and (i', j'). Without loss of generality, assume $i < i'$.

First, if (i, j) is strictly northeast of (i', j'), then we label the $\bullet$ in row i with (c, w_c) and the $\bullet$ in column j' with (d, w_d). Now $a < b < c < d$ and $w_b < w_a < w_d < w_c$. (Here $c = i$ and $w_d = j'$. Similar renamings will occur below.) If there is any $\bullet$ in any row strictly between c and d whose column index is strictly between w_d and w_c, choose one and name it (e, w_e). Then we will have $a < b < c < e < d$ and $w_b < w_a < w_d < w_e < w_c$, which is to say that w contains 21543. Otherwise, the $\bullet$ in row i', which we call (f, w_f), must be east of column w_c, and the $\bullet$ in column j, which we call (g, w_g), must be south of row d, and so $a < b < c < f < d < g$ and $w_b < w_a < w_d < w_f < w_c < w_e$, which is to say that w contains 215634.

Alternatively, if (i, j) is strictly northwest of (i', j'), we label the $\bullet$ in row i' with (c, w_c), the $\bullet$ in column j' with (d, w_d), the $\bullet$ in row i with (e, w_e), and the $\bullet$ in column j with (f, w_f). If $w_e > w_c$, then $a < b < e < c < d$ while $w_b < w_a < w_d < w_c < w_e$, so w contains 21543. Similarly, if $f > d$, then w contains 21543. Hence, we may now assume that either $w_e < w_d < w_c$ or $w_d < w_e < w_c$ and either $f < c < d$ or $c < f < d$. Each of these four possibilities require the containment of 215634, 215364, 214635, or 13254 (ignoring (b, w_b) in case $w_e < w_d < w_c$ and $f < c < d$ and by using all six dots in the other three cases). □

Example 3.4. We give an illustration of the case of (i, j) strictly northeast of (i', j') to demonstrate the process of considering allowable regions of the visualization of D_w for $\bullet$s we know must exist but whose locations are unknown. Either at least one the $\bullet$s in column j or row i' falls in Region I (in blue), or both fall in Region II (in green). If the former, then w contains 21543, and, if the latter, then w contains 215634.



Lemma 3.5. *If the permutation $w \in S_n$ contains an obstruction of Type* 2*, then w contains* 21543*,* 214635*,* 241635*,* 215364*, or* 315264*.*

Proof. Fix a permutation $w \in S_n$ that has an obstruction of Type 2. We will first assume that we have fixed (i,j) and (i,j') as in the definition of a Type 2 obstruction with $j' < j$ and (r,s) assumed to be the easternmost element of $\mathrm{Dom}(w) \cap \mathrm{Ess}(w)$ northwest of both (i,j) and (i,j'). As before, we consider the visualization of the Rothe diagram D_w.

Label the $\bullet$ in column $s+1$ with (a, w_a) and the $\bullet$ in the row $r+1$ with (b, w_b). Notice $a < b$ and $w_a > w_b$. Label the $\bullet$ in row i with (c, w_c), the $\bullet$ in column j with (d, w_d), and the $\bullet$ in column j' with (e, w_e). If $e > d$, then we have $a < b < c < d < e$ and $w_b < w_a < w_e < w_d < w_c$, which is to say that w contains 21543. Now assume $e < d$. Because $(i', j'), (i,j) \in \mathrm{Ess}(w)$, there must be some $\bullet$ in column $j'+1$, which we label (f, w_f), north of row i. Because of the easternmost assumption on (r,s) and because $\max_k\{(k,j) \in \mathrm{Dom}(w)\} = \max_k\{(k,j') \in \mathrm{Dom}(w)\}$, we must have that $f > a$. If $f > b > a$, then w contains 214635 and, if $b > f > a$, then w contains 241635.

A parallel argument shows that if w has an obstruction of Type 2 with $i' < i$, $j' = j$, and $\max_\ell\{(i,\ell) \in \mathrm{Dom}(w)\} = \max_\ell\{(i',\ell) \in \mathrm{Dom}(w)\}$, then w contains 21543, 215364, or 315264. □

Lemma 3.6. *If the permutation $w \in S_n$ has an obstruction of Type* 3*, then w contains* 13254*,* 21543*,* 214635*,* 215364*,* 215634*,* 241635*,* 315264*, or* 4261735*.*

Proof. Fix a permutation $w \in S_n$ that has an obstruction of Type 3. We fix (i,j), (i',j') as in the definition of an obstruction of Type 3, and consider the visualization of the Rothe diagram of w. If there is some $(r,s) \in \mathrm{Dom}(w) \cap \mathrm{Ess}(w)$ strictly northwest of (i,j), then w has an obstruction of Type 1, and so it follows from Lemma 3.3 that w contains 21543, 215634, 215364, 214635, or 13254. Hence, we assume no such (r,s) exists. Without loss of generality, we assume that the $\bullet$ in row 1 is west of column j' and that the $\bullet$ in column 1 is north of row i'.

First suppose that the $\bullet$ in row 1 is west of column j. Then the assumption that there is no $(r,s) \in \mathrm{Dom}(w) \cap \mathrm{Ess}(w)$ strictly northwest of (i,j) implies that the $\bullet$ in column 1 is south of row i. Because $(i,j) \in \mathrm{Ess}(w)$, there must be a $\bullet$ in column $j+1$ weakly north of row i. If the $\bullet$ in column j is north of row i', then the $\bullet$s in row 1 and columns j and $j+1$ combine with the $\bullet$s in row i' and column j' to form 13254. If the $\bullet$ in column j is south of the $\bullet$ in column j', then they combine with the $\bullet$s in row 1, column 1, and row i' to form 21543. And if the $\bullet$ in column j is north of the $\bullet$ in column j' but south of row i', then all $\bullet$s described in this paragraph form 241635.

Alternatively, assume that the $\bullet$ in row 1 is between columns j and j'. If the $\bullet$ in column 1 is north of row i, then taking transposes in the argument in the previous paragraph shows that w must contain 13254, 21543, or 315264.

If the $\bullet$ in column 1 is south of row i, label the $\bullet$ in row 1 with (a, w_a), any fixed $\bullet$ northwest of (i, j) with (b, w_b), and the $\bullet$ in column 1 with (c, w_c). We know that there is some $\bullet$ northwest of (i, j) because $(i, j) \notin \mathrm{Dom}(w)$. As before, if the $\bullet$ in row i is west of column j' and the $\bullet$ in column j is north of row i', then w contains 13254.

Suppose that the $\bullet$ in row i is east of column j', and label that $\bullet$ with (d, w_d). Label the $\bullet$ in row i' with (e, w_e), and the $\bullet$ in column j' with (f, w_f). If $w_e < w_d$, then (a, w_a), (b, w_b), (d, w_d), (e, w_e), and (f, w_f) form 21543. If $w_e > w_d$, then we consider the placement of the $\bullet$ in column j, which we label (g, w_g). If $g < e$, then (a, w_a), (b, w_b), (d, w_d), (e, w_e), (f, w_f), and (g, w_g) form 315264. If $e > g > f$, then all $\bullet$s (a, w_a) to (g, w_g), form 4261735. And if $f < g$, then (b, w_b), (c, w_c), (e, w_e), (f, w_f), and (g, w_g) form 21543.

Finally, the cases in which $w_d < w_f$ (equivalently, the $\bullet$ in row i west of column j') and $g < e$ are achieved by taking the transpose of the configurations in the preceding paragraph. In these cases, w contains 21543, 241635, or 4261735. $\square$

3.2. Permutations avoiding the specified patterns are CDG. The remainder of this section is devoted to the backward direction of Conjecture 1.1. We will build up to a use of [27, Corollary 4.13]. We begin with some notation.

If I_w is the Schubert determinantal ideal of the permutation $w \in S_n$, we will use Z_w to denote the matrix obtained from an $n \times n$ matrix of indeterminates by setting $z_{i,j}$ to 0 whenever $(i, j) \in \mathrm{Dom}(w)$. If (i, j) is a lower outside corner of D_w and $y = z_{i,j}$, we write the CDG generators of I_w as $\{yq_1 + r_1, \ldots, yq_k + r_k, h_1, \ldots, h_\ell\}$ where y does not divide any term of any q_i, r_i or h_j. Define $N_{y,I_w} = (h_1, \ldots, h_\ell)$ and $C_{y,I_w} = (q_1, \ldots, q_k, h_1, \ldots, h_\ell)$. This notation mimics that in [27]. When the CDG generators are a Gröbner basis of I_w, C_{y,I_w} will be the ideal corresponding to the star and $N_{y,I_w} + (y)$ the ideal corresponding to the deletion in a geometric vertex decomposition in the sense of [29].

We will call $\{q_1, \ldots, q_k, h_1, \ldots, h_\ell\}$ the *CDG generators* of C_{y,I_w}, which is itself not typically a Schubert determinantal ideal. With notation as above, we begin by showing that N_{y,I_w} is the Schubert determinantal ideal of a permutation whose Coxeter length is smaller than that of w, which will be an essential component of an inductive argument.

Let $t_{a,b}$ denote the transposition $(ab) \in S_n$.

Lemma 3.7. *Suppose that I_w is the Schubert determinantal ideal of the permutation $w \in S_n$ and that (i, j) is a lower outside corner of D_w corresponding to the variable $y = z_{i,j}$. The ideal N_{y,I_w} is the Schubert determinantal ideal of a permutation $w' \in S_n$ whose Coxeter length is strictly smaller than that of w. Specifically, $w' = wt_{i,w^{-1}(j)}$, and $D_w = D_{w'} \sqcup \{(i, j)\}$.*

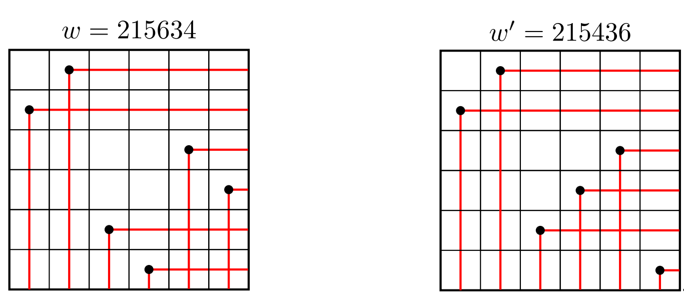
Proof. We claim that whenever there is some $(\mathrm{rank}_w(i, j)+1)$-minor with $yq+r = r$, that $r \in N_{y,I_w}$, i.e. that all of the CDG generators of I_w determined only by the rank condition at (i, j) involve y. Fix a $(\mathrm{rank}_w(i, j) + 1) \times (\mathrm{rank}_w(i, j) + 1)$ submatrix M of Z_w so that $\det(M) = yq + r$, and suppose that $q = 0$. Recall that the entry in row a and column b of M is 0 if and only if $(a, b) \in \mathrm{Dom}(w)$. Then q is the determinant of a $\mathrm{rank}_w(i, j) \times \mathrm{rank}_w(i, j)$ submatrix of M whose anti-diagonal has an entry in row a and column b for some $(a, b) \in \mathrm{Dom}(w)$. Assume without loss of generality that $i \geqslant j$.

If $r \neq 0$, then there is some (i', j) with nonvanishing $z_{i',j} \cdot q'$ summand of r so that q' is the determinant of a $\mathrm{rank}_w(i, j) \times \mathrm{rank}_w(i, j)$ submatrix of M without a 0 along its anti-diagonal. Because $\mathrm{Dom}(w)$ forms a partition shape, if $z_{i',j} \cdot q' \neq 0$, then $z_{i'',j} \cdot q'' \neq 0$ whenever $i'' < i'$ and q'' is the cofactor corresponding to $z_{i'',j}$ in an expansion of $yq + r$ along column j. In particular, there is a unique t so that no $\mathrm{rank}_w(i,j) \times \mathrm{rank}_w(i,j)$ submatrix of M that excludes column j and involves the final t rows has a 0 along its anti-diagonal and every $\mathrm{rank}_w(i,j) \times \mathrm{rank}_w(i,j)$ submatrix of M that excludes column j and also excludes one of the final t rows has a 0 along its anti-diagonal.

The same argument can be applied to columns, and, because vanishing is determined by 0's along the anti-diagonal, will select the final $\mathrm{rank}_w(i.j) + 1 - t$ columns. Hence, we may write r as the product of one t-minor determined by the final t rows and initial t columns of M and one $(\mathrm{rank}_w(i.j) + 1 - t)$-minor consisting of the initial $\mathrm{rank}_w(i.j) + 1 - t$ rows and final $\mathrm{rank}_w(i.j) + 1 - t$ columns of M.

Call the southeast corner of the lower block z and the southeast corner of the upper block z'. If there are fewer than t dots northwest of z, then the t-minor that is one factor of r is an element of N_{y,I_w}, and so $r \in N_{y,I_w}$. If there are t or more dots northwest of z, then there are at most $\mathrm{rank}_w(i.j) - t$ dots northwest of z', and so the factor of r corresponding to the upper block is an element of N_{y,I_w}, and so $r \in N_{y,I_w}$. Hence, N_{y,I_w} is generated by the CDG generators of I_w determined by the diagram boxes other than (i,j). The permutation with exactly those diagram boxes and rank conditions at those diagram boxes is $w' = wt_{i,w^{-1}(j)}$. Then $N_{y,I_w} = I_{w'}$, the Coxeter length of w' is one less than the Coxeter length of w, and $D_w = D_{w'} \sqcup \{(i,j)\}$. $\square$

Example 3.8. With notation as in Lemma 3.7, let $w = 215634$ and $(i,j) = (4,4)$. If $t_{i,w^{-1}(j)} = t_{4,6}$, set $w' = wt_{4,6} = 215436$. The Rothe diagrams of w and w' appear below. We may view the Rothe diagram of w' as arising from the Rothe diagram of w by swapping rows $i = 4$ and $w^{-1}(j) = 6$. This exchange eliminates the diagram box in position $(i,j) = (4,4)$ and leaves the rest of the diagram boxes undisturbed.



Remark 3.9. With notation as in Lemma 3.7, it is possible that $\mathrm{Ess}(w') \not\subseteq \mathrm{Ess}(w)$, as is the case if $w = 215634$ and $(i,j) = (4,4)$. In that case, $(3,4), (4,3) \in \mathrm{Ess}(w') \smallsetminus \mathrm{Ess}(w)$.

In general, the only possible elements of $\mathrm{Ess}(w') \smallsetminus \mathrm{Ess}(w)$ are $(i-1,j)$ and $(i,j-1)$. Indeed, suppose $(a,b) \in \mathrm{Ess}(w') \smallsetminus \mathrm{Ess}(w)$. By the definition of $\mathrm{Ess}(w')$, $(a,b) \in D_{w'}$ and $(a+1,b), (a,b+1) \notin D_{w'}$. Because $D_{w'} \subset D_w$, the assumption $(a,b) \notin \mathrm{Ess}(w)$ implies $(a+1,b) \in \mathrm{Ess}(w)$ or $(a,b+1) \in \mathrm{Ess}(w)$. The fact that $D_w \smallsetminus D_{w'} = \{(i,j)\}$ then implies that $(a+1,b) = (i,j)$ or $(a,b+1) = (i,j)$.

Corollary 3.10. *If w has no obstruction of Type 1, 2, or 3, (i,j) is a lower outside corner of D_w corresponding to the variable $y = z_{i,j}$, and $I_{w'} = N_{y,I_w}$, then w' has no obstruction of Type 1, 2, or 3.*

Proof. If $(i,j) \in \mathrm{Dom}(w)$, then, because there are no diagram boxes southeast of (i,j) by assumption, (i,j) cannot be one of the boxes involved in any of the obstructions. In that case, any set of diagram boxes realizing an obstruction in w' would also realize an obstruction in w. Hence we assume that $(i,j) \notin \mathrm{Dom}(w)$, in which case $\mathrm{Dom}(w) = \mathrm{Dom}(w')$.

Suppose that w' has an obstruction of Type 1 with $(r,s) \in \mathrm{Dom}(w') \cap \mathrm{Ess}(w')$ and (a,b), $(a'b') \in D_{w'}$ strictly southeast of (r,s) with $a' \neq a$ and $b' \neq b$. Because $\mathrm{Dom}(w') = \mathrm{Dom}(w)$ and $D_{w'} \subset D_w$, the diagram boxes (r,s), (a,b), (a',b') also constitute an obstruction of Type 1 in w as well.

Next suppose that w' has an obstruction of Type 2. Assume without loss of generality that the Type 2 obstruction has the form $(r,s) \in \mathrm{Dom}(w') \cap \mathrm{Ess}(w')$ and $(a,b),(a,b') \in \mathrm{Ess}(w')$ strictly southeast of (r,s) with $\max_k\{(k,b) \in \mathrm{Dom}(w')\} = \max_k\{(k,b') \in \mathrm{Dom}(w')\}$ and $b < b'$. In this case neither (a,b) nor (r,s) may be a lower outside corner of w'. In particular, neither (r,s) nor (a,b) is equal to (i,j), and so $(r,s) \in \mathrm{Dom}(w) \cap \mathrm{Ess}(w)$ and $(a,b) \in \mathrm{Ess}(w)$. If $(a,b') \in \mathrm{Ess}(w)$ also, then (r,s), (a,b), and (a,b') also constitute a Type 2 obstruction in w. If $(a,b') \notin \mathrm{Ess}(w)$, then either $(a+1,b') = (i,j) \in \mathrm{Ess}_w$, in which case (r,s), (a,b), and $(a+1,b')$ together constitute a Type 1 obstruction, or $(a,b'+1) = (i,j) \in \mathrm{Ess}(w)$.

If $(a,b'+1) = (i,j) \in \mathrm{Ess}(w)$, set $m = \max_k\{(k,b) \in \mathrm{Dom}(w')\} = \max_k\{(k,b') \in \mathrm{Dom}(w')\}$. Because $\mathrm{Dom}(w) = \mathrm{Dom}(w')$, we have $m = \max_k\{(k,b) \in \mathrm{Dom}(w)\}$. We claim that $m = \max_k\{(k,b'+1) \in \mathrm{Dom}(w)\}$. Because $b'+1 > b'$ and $\mathrm{Dom}(w)$ forms a partition shape, $m \geqslant \max_k\{(k,b'+1) \in \mathrm{Dom}(w)\}$. If $m > \max_k\{(k,b'+1) \in \mathrm{Dom}(w)\}$, then $(m,b'+1) \notin \mathrm{Dom}(w)$. But $(m,b') \in \mathrm{Dom}(w') = \mathrm{Dom}(w)$. Therefore, if $(m,b'+1) \notin \mathrm{Dom}(w)$, the visualization of D_w must have a $\bullet$ in column $b'+1$ weakly north of row m. Clearly $m < a$ since $(a,b) \notin \mathrm{Dom}(w')$. But the $\bullet$ in column $b'+1 = j$ must be south of row $a = i$ because $(i,j) \in D_w$. Hence, $m = \max_k\{(k,b'+1) \in \mathrm{Dom}(w)\}$ and (r,s), (a,b), and $(a,b'+1)$ together constitute a Type 3 obstruction in w.

Finally suppose that w' has an obstruction of Types 3. Suppose that $(a,b),(a',b') \in \mathrm{Ess}(w') \smallsetminus \mathrm{Dom}(w')$ with (a',b') strictly southeast of (a,b). If $(a',b') \in \mathrm{Ess}(w)$, then (a,b) and (a',b') constitute a Type 3 obstruction in w also. Otherwise, it must be that $(i,j) = (a'+1,b')$ or $(i,j) = (a',b+1)$. In either case, (a,b) and (i,j) constitute a Type 3 obstruction in w. □

Next, we will show that the CDG generators of C_{y,I_w} form a Gröbner basis whenever w has no obstruction of Type 1, 2, or 3. Before proceeding, we review some standard notation and make one new definition to help with bookkeeping during this subsection. We will use $\deg(f)$ to denote the degree of the homogeneous polynomial f and $LCM(\mu_1,\mu_2)$ to denote the least common multiple of two monomials (which will arise for us as the monic leading terms of ideal generators).

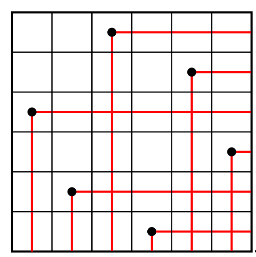
Definition 3.11. *Fix a permutation $w \in S_n$, lower outside corner (i,j) of D_w corresponding to the variable $y = z_{i,j}$ in Z_w, and CDG generators $\{yq_1 + r_1, \ldots, yq_k + r_k, h_1, \ldots, h_\ell\}$ of I_w, where y does not divide any q_a, r_a, or h_b. Assume also that there is some $0 \leqslant \ell' \leqslant \ell$ so that all variables appearing in h_b are northwest of some $z_{i',j}$ for $(i',j) \in \mathrm{Ess}(w)$ with $\mathrm{rank}_w(i',j)+1 = \deg(h_b)$ or of some $z_{i,j'}$ for $(i,j') \in \mathrm{Ess}(w)$ with $\mathrm{rank}_w(i,j')+1 = \deg(h_b)$ if and only if $b \leqslant \ell'$. If $(1,j) \in \mathrm{Dom}(w)$, set $m_1 = \min\{i-p \mid (p,j) \in \mathrm{Dom}(w)\}$, and set $m_1 = i$ if $(1,j) \notin \mathrm{Dom}(w)$. Similarly, if $(i,1) \in \mathrm{Dom}(w)$, set $m_2 = \min\{j-q \mid (i,q) \in \mathrm{Dom}(w)\}$, and set $m_2 = j$ if $(1,j) \notin \mathrm{Dom}(w)$.*

We form the ideal

$$Q_{y,I_w} = \begin{cases} (q_1,\ldots,q_k) & \mathrm{rank}_w(i,j)+1 = \min\{m_1,m_2\} \\ (q_1,\ldots,q_k,h_1,\ldots,h_{\ell'}) & \textit{otherwise.} \end{cases}$$

Less formally, we are taking $Q_{y,I_w} = (q_1, \dots, q_k)$ when the rank condition on (i,j) is determining maximal minors in the submatrix of Z_w obtained from the submatrix northwest of $z_{i,j}$ by deleting any full rows or columns of 0's. Otherwise, we include also as generators of Q_{y,I_w} the CDG generators determined by essential boxes in the same row or column as $z_{i,j}$.

Example 3.12. Let $w = 351624$ and $y = z_{4,4}$. The Rothe diagram of w is



Then

$$I_w = (z_{1,1}, z_{1,2}, z_{2,1}, z_{2,2}, \text{2-minors in } (Z_w)_{[4],[2]}, \text{2-minors in } (Z_w)_{[2],[4]}, \text{3-minors in } (Z_w)_{[4],[4]})$$

and

$$Q_{y,I_w} = (\text{2-minors in } (Z_w)_{[4],[2]}, \text{2-minors in } (Z_w)_{[2],[4]}, \text{2-minors in } (Z_w)_{[3],[3]}).$$

The 2-minors in $(Z_w)_{[2],[4]}$, for example, are included as generators of Q_{y,I_w} because

$$\operatorname{rank}_w(4,4) + 1 = 3 < 4 = \min\{4,4\} = \min\{m_1, m_2\}$$

and $(2,4) \in \operatorname{Ess}(w)$ is in the same column as $(4,4)$.

We make this definition purely for technical convenience below and not out of an independent interest in $\operatorname{Spec}(R/Q_{y,I_w})$. We will say that Q_{y,I_w} is CDG if the generators given above form a Gröbner basis under any diagonal term order.

Lemma 3.13. *If $w \in S_n$ has no obstruction of Type 1, 2, or 3 and Conjecture 1.1 holds for all permutations of smaller Coxeter length than that of w, then either $D_w = \operatorname{Dom}(w)$ or there is some lower outside corner (i,j) of $D_w \smallsetminus \operatorname{Dom}(w)$ corresponding to the variable y in Z_w so that Q_{y,I_w} is CDG.*

Proof. Fix a permutation $w \in S_n$ that has no obstruction of Type 1, 2, or 3, and assume $D_w \neq \operatorname{Dom}(w)$. First suppose that D_w has some lower outside corner $(i,j) \notin \operatorname{Dom}(w)$ corresponding to the variable y in Z_w satisfying $\operatorname{rank}_w(i,j) + 1 = \min\{m_1, m_2\}$, with notation as in Definition 3.11.

Write the CDG generators of I as $\{yq_1 + r_1, \dots, yq_k + r_k, h_1, \dots, h_\ell\}$, where y does not divide any q_i, r_i, or h_j. As discussed in Lemma 3.7, every $(\operatorname{rank}_w(i,j)+1)$-minor involving row i and column j has a term divisible by y, and so $\{q_1, \dots, q_k\}$ generates the ideal of $\operatorname{rank}_w(i,j)$-minors in the submatrix of Z_w strictly northwest of y. Because the $Q_{y,I}$ is an ideal of maximal minors (after possible removing full rows or columns of 0's), the result follows from [12, Theorem 4.2] or [5, Proposition 5.4].

Alternatively, suppose that D_w has no such lower outside corner, and fix any lower outside corner of D_w. Because Q_{y,I_w} depends only on D_w weakly northwest of (i,j), we may assume that (i,j) is the only lower outside corner of D_w and that $(1,j), (i,1) \notin \operatorname{Dom}(w)$. With this assumption, $Q_{y,I_w} + (z_{a,b} \mid (a,b) \in \operatorname{Dom}(w)) = C_{y,I_w}$. Because the $z_{a,b}$ with $(a,b) \in \operatorname{Dom}(w)$ are indeterminates that do not divide any term of any CDG generator of Q_{y,I_w}, it follows that Q_{y,I_w} is CDG if and only if C_{y,I_w} is CDG.

If $\operatorname{Dom}(w) = \varnothing$, then, because w has no obstruction of Type 1, w must be vexillary and so I_w is CDG by [29, Theorem 3.8]. It then follows from [29, Theorem 2.1(a)] that C_{y,I_w} is CDG as well.

Now suppose $\mathrm{Dom}(w) \neq \varnothing$. The assumptions that $(1,j),(i,1) \notin \mathrm{Dom}(w)$ imply that there is at least one element of $\mathrm{Dom}(w) \cap \mathrm{Ess}(w)$ northwest of (i,j). Choose the southernmost such element and label it (r,s) and the easternmost element and label it (r',s'). Because w has no obstruction of Type 1, all essential boxes of $D_w \smallsetminus \mathrm{Dom}(w)$ must be in either row i or column j. Because $\mathrm{rank}_w(i,j)+1 < \min\{i,j\}$, there must be at least one essential box in row i and at least one in column j aside from (i,j). Then because w has no obstruction of Type 2, there are no elements of $\mathrm{Ess}(w)$ north of row i and south of row r or west of column j and east of column s'.

We will argue directly in this case that for each pair q_a and h_b, their s-polynomial has a Gröbner reduction by the CDG generators of Q_{y,I_w}. (For an example illustrating this process, see Example 3.14 below.) Choose such a q_a and h_b. Because the CDG generators of N_{y,I_w} form a Gröbner basis by induction on the Coxeter length of w and Corollary 3.10, we may assume that $q_a \notin (h_1,\dots,h_\ell)$.

Because Q_{y,I_w} involves only indeterminates of Z_w northwest of $z_{i,j}$, we will work within the submatrix of Z_w northwest of $z_{i,j}$, which we will call Y_w. Suppose that h_b is determined by the essential box at (i',j) for some $i'<i$. (The case of (i,j') with $j'<j$ will follow by symmetry.) We consider two cases.

Case 1: Suppose that $(i',j-1) \in D_w$. Then, because w has no obstruction of Type 1, there can be no element of $\mathrm{Dom}(w) \cap \mathrm{Ess}(w)$ northwest of (i',j). In particular, $r' \geqslant i'$, and h_b is a $(\mathrm{rank}_w(i',j)+1)$-minor in the submatrix of Y_w formed of its final $j-s'$ columns, which is a generic matrix and which we will call Y'_w. Suppose that there are t columns determining q_a strictly east of column s' and $\mathrm{rank}_w(i,j)-t$ columns determining q_a weakly west of column s'. Express q_a as a sum of products of t-minors and $(\mathrm{rank}_w(i,j)-t)$-minors corresponding to this subdivision. For any fixed set of rows of Y'_w, the set of $(\mathrm{rank}_w(i',j)+1)$-minors weakly northwest of (i',j) together with the t-minors strictly northwest of (i,j) in the submatrix of Y'_w including only those specified rows forms a Gröbner basis for the ideal they generate because it is a mixed ladder determinantal ideal [22, Theorem 1.10].

Choose the t-minor ε_1 from the eastern t columns determining q_a and the $(\mathrm{rank}_w(i,j)-t)$-minor ε_2 from the remaining columns satisfying $LT(\varepsilon_1)\cdot LT(\varepsilon_2) = LT(q_a)$. Because ε_1 and h_b belong to the ideal of t-minors weakly north of their southernmost entries together with the $(\mathrm{rank}_w(i',j)+1)$-minors northwest of (i',j) in Y'_w, their s-polynomial $s(\varepsilon_1,h_b)$ has a Gröbner reduction $s(\varepsilon_1,h_b) = \sum \alpha_c\delta_c$ by the natural generators of that ideal. For each $\delta_c \in N_{y,I_w}$, set $\widehat{\delta_c} = LT(\varepsilon_2)\cdot\delta_c$, and for each δ_c involving some row south of i', set $\widehat{\delta_c}$ to be (up to sign) the determinant of the augmentation of the matrix determining δ_c by the rows and columns determining ε_2 (with sign chosen so that $LT(\delta_c)$ and $LT(\widehat{\delta_c})$ share a sign). Let $s(q_a,h_b)$ denote the s-polynomial of q_a and h_b.

We claim that $s(q_a,h_b)-\sum\alpha_c\widehat{\delta_c} \in N_{y,I}$. It is clear that $s(q_a,h_b)-\sum\alpha_c\widehat{\delta_c}$ contains a $LT(\varepsilon_2)$-multiple of $s(\varepsilon_1,h_b)-\sum\alpha_c\delta_c$, which is 0 because $s(\varepsilon_1,h_b)-\sum\alpha_c\delta_c$ is. Fix any non-leading term μ of ε_2, and write $s(q_a,h_b)-\sum\alpha_c\widehat{\delta_c} = \mu\tilde{s}+\tilde{\tilde{s}}$ where μ does not divide any term of $\tilde{\tilde{s}}$. For each column involved in ε_2, whenever a different variable from that column divides μ and $LT(\varepsilon_2)$, replace in $\sum\alpha_c\widehat{\delta_c}$ the variables in the row of the divisor of $LT(\varepsilon_2)$ with the variables in the same columns from the row of the corresponding divisor of μ, which we note gives an expression of $\tilde{s}$ in terms of the natural generators of the ideal of $(\mathrm{rank}_w(i',j)+1)$-minors northwest of (i',j), each of which is a CDG generator of $N_{y,I}$.

Now because $LT(\alpha_c\delta_c) \leqslant LT(s(\varepsilon_1,h_b))$ and $LT(\alpha_c\widehat{\delta_c}) = LT(\varepsilon_2)\cdot LT(\alpha_c\delta_c)$ for each c and because $LT(s(q_a,h_b)) = LT(\varepsilon_2)\cdot LT(s(\varepsilon_1,h_b))$, subtracting each $\alpha_c\widehat{\delta_c}$ is a valid step in a Gröbner reduction of $s(q_a,h_b)$ by the generators of Q_{y,I_w}. The fact that the CDG generators of N_{y,I_w}, each of which is a CDG generator of Q_{y,I_w}, form a Gröbner basis for the ideal they generate implies that $s(q_a, h_b) - \sum \alpha_c \widehat{\delta_c} \in N_{y,I_w}$ has a reduction in terms of those generators and so that $s(q_a, h_b)$ has a reduction by the CDG generators of Q_{y,I_w}.

Case 2: Suppose $(i', j-1) \notin D_w$. Then there can be no $j'' < j$ with $(i', j'') \in D_w \smallsetminus \mathrm{Dom}(w)$ because w has no obstruction of Type 3. Hence, $\mathrm{rank}_w(i', j) + 1 = \min\{j - k \mid (i', k) \in \mathrm{Dom}(w)\}$, and so the $(\mathrm{rank}_w(i', j)+1)$-minors northwest of (i', j) are the maximal minors of the submatrix of Z_w northwest of (i', j) after removing complete rows or columns of 0's. Then the argument is similar to the first case but uses, instead of results on ladder determinantal ideals in a generic matrix, the fact that the maximal minors of matrices of indeterminates and 0's form a Gröbner basis by [12, Theorem 4.2] or [5, Proposition 5.4]. □

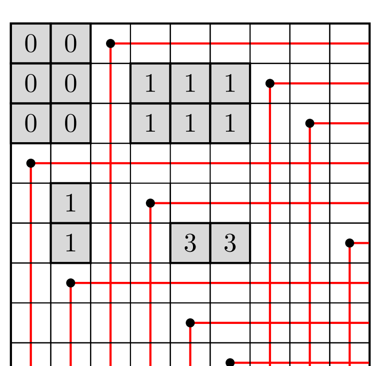
Example 3.14. Observe that $w = 378149256$, whose annotated Rothe diagram appears below, is an example of a CDG permutation with a unique lower outside corner $(i, j) = (6, 6)$ and $4 = \mathrm{rank}_w(i, j) + 1 \neq \min\{m_1, m_2\} = 6$. We illustrate how the reduction of the s-polynomial of $h_b = \begin{vmatrix} z_{2,3} & z_{2,4} \\ z_{3,3} & z_{3,4} \end{vmatrix}$ and $\begin{vmatrix} z_{2,3} & z_{2,5} \\ z_{5,3} & z_{5,5} \end{vmatrix}$ gives rise to that of $\begin{vmatrix} z_{2,3} & z_{2,4} \\ z_{3,3} & z_{3,4} \end{vmatrix}$ and $q_a = \begin{vmatrix} 0 & z_{2,3} & z_{2,5} \\ z_{4,2} & z_{4,3} & z_{4,5} \\ z_{5,2} & z_{5,3} & z_{5,5} \end{vmatrix}$ by the process described in Lemma 3.13. Using that

$$z_{5,5}\begin{vmatrix} z_{2,3} & z_{2,4} \\ z_{3,3} & z_{3,4} \end{vmatrix} - z_{3,4}\begin{vmatrix} z_{2,3} & z_{2,5} \\ z_{5,3} & z_{5,5} \end{vmatrix} + z_{2,4}\begin{vmatrix} z_{3,3} & z_{3,5} \\ z_{5,3} & z_{5,5} \end{vmatrix} + z_{5,3}\begin{vmatrix} z_{2,4} & z_{2,5} \\ z_{3,4} & z_{3,5} \end{vmatrix} = 0,$$

we construct the equation

$$\underline{z_{4,2}z_{5,5}\begin{vmatrix} z_{2,3} & z_{2,4} \\ z_{3,3} & z_{3,4} \end{vmatrix}} + \underline{z_{3,4}}\begin{vmatrix} 0 & \underline{z_{2,3}} & \underline{z_{2,5}} \\ \underline{z_{4,2}} & z_{4,3} & z_{4,5} \\ z_{5,2} & \underline{z_{5,3}} & \underline{z_{5,5}} \end{vmatrix} - \underline{z_{2,4}}\begin{vmatrix} 0 & \underline{z_{3,3}} & \underline{z_{3,5}} \\ \underline{z_{4,2}} & z_{4,3} & z_{4,5} \\ z_{5,2} & \underline{z_{5,3}} & \underline{z_{5,5}} \end{vmatrix} + \underline{z_{4,2}z_{5,3}\begin{vmatrix} z_{2,4} & z_{2,5} \\ z_{3,4} & z_{3,5} \end{vmatrix}}$$
$$= z_{5,2}\left(z_{4,5}\begin{vmatrix} z_{2,3} & z_{2,4} \\ z_{3,3} & z_{3,4} \end{vmatrix} + z_{4,3}\begin{vmatrix} z_{2,4} & z_{2,5} \\ z_{3,4} & z_{3,5} \end{vmatrix}\right).$$

In the latter equation, we find an $z_{4,2}$-multiple of the first equation (whose terms appear in blue and also are underlined) and an $z_{5,2}$-multiple of an element easily seen to be in N_{y,I_w}. We record in orange the generators of N_{y,I_w} that give the inclusion of the $z_{5,2}$-multiple summand of $s(q_a, h_b)$ in N_{y,I_w} and see that relation arising from the first equation. The new relation is obtained from the first by exchanging rows 4 and 5.



Before proceeding, we recall one very useful lemma.

LEMMA 3.15. [10, Lemma 1.3.14] *Let I and J be homogeneous ideals of a polynomial ring over a field, and fix a term order σ. With respect to σ, let $\mathcal{F}$ be a Gröbner basis of I and $\mathcal{G}$ a Gröbner basis of J. Then $\mathcal{F}\cup\mathcal{G}$ is a Gröbner basis of $I+J$ if and only if for all $f\in\mathcal{F}$ and $g\in\mathcal{G}$ there exists $e\in I\cap J$ such that $LT(e)=LCM(LT(f),LT(g))$.* □

We are now prepared to show that, under a suitable inductive hypothesis, there is a lower outside corner so that the CDG generators of C_{y,I_w} are Gröbner.

LEMMA 3.16. *With notation as above, if $w\in S_n$ avoids obstructions of Types 1, 2, and 3 and Conjecture 1.1 holds for all permutations of smaller Coxeter length than that of w, then either $D_w=\mathrm{Dom}(w)$ or there is some lower outside corner (i,j) of $D_w\smallsetminus\mathrm{Dom}(w)$ corresponding to the variable $y=z_{i,j}$ so that the generators $\{q_1,\dots,q_k,h_1,\dots,h_\ell\}$ of C_{y,I_w} form a Gröbner basis under any diagonal term order.*

Proof. Fix a diagonal term order σ, and assume $D_w\neq\mathrm{Dom}(w)$. By Lemma 3.13, there is some lower outside corner of D_w so that, with notation as above, there exists $\ell'\geqslant 0$ so that the generators $\{q_1,\dots,q_k,h_1,\dots,h_{\ell'}\}$ form a Gröbner basis for Q_{y,I_w}.

By Corollary 3.10 and the inductive hypothesis, $\{h_1,\dots,h_\ell\}$ is a diagonal Gröbner basis for N_{y,I_w}. Then by Lemma 3.15, the generators $\{q_1,\dots,q_k,h_1,\dots,h_\ell\}$ form a diagonal Gröbner basis for C_{y,I_w} if and only if for every q_a and h_b there exists some $f\in(q_1,\dots,q_k,h_1,\dots,h_{\ell'})\cap(h_1,\dots,h_\ell)$ satisfying $LT(f)=LCM(LT(q_a),LT(h_b))$.

If $h_b\in(q_1,\dots,q_k,h_1,\dots,h_{\ell'})$ or $q_a\in(h_1,\dots,h_\ell)$, the result follows from Lemma 3.15, and if $LCM(LT(q_a),LT(h_b))=LT(q_a)\cdot LT(h_b)$, then we take $f=q_a\cdot h_b$. Otherwise, there is some $(r,s)\in\mathrm{Ess}(w)\smallsetminus\mathrm{Dom}(m)$ with $\mathrm{rank}_w(r,s)=\deg(h_b)-1$ corresponding to the variable $e=z_{r,s}$ weakly southeast of all of the variables involved in h_b. Because w has no Type 3 obstruction, e is not strictly northwest of y. Suppose first that y is strictly east and weakly north of e.

In this case, let M' be the matrix consisting of the union of the columns determining q_a and h_b and the union of the rows determining q_a and h_b. We will next describe an auxiliary matrix M formed from M'. (For an example of the construction, see Example 3.17 below.) Form a matrix M from M' as follows: First set to 0 any entry whose row index is not one of the rows determining q_a and whose column index is not one of the columns determining h_b. Next, whenever a column of M' contains a variable dividing the leading term of q_a and a distinct variable dividing the leading term of h_b, duplicate that column and replace in one copy of the column the variables coming only from h_b by 0. Whenever a row of M' contains a variable dividing the leading term of q_a and a distinct variable dividing the leading term of h_b, duplicate that row and replace in one copy of the row the variables coming only from q_a by 0.

Now M will be a $d\times d$ matrix where $d=\deg(LCM(LT(q_a),LT(h_b)))$ because it will have one row and one column for each monomial dividing $LT(q_a)$ and one each for every monomial dividing $LT(h_b)$ but not $LT(q_a)$.

By expressing $\det(M)$ as a sum of products of the $\deg(q_a)$-minors from the rows of M originating from the submatrix of Z_w determining q_a and the $(d-\deg(q_a))$-minors in the remaining rows, we see that $\det(M)\in(q_1,\dots,q_k)$. Similarly, by expressing $\det(M)$ as a sum of products of $\deg(h_b)$-minors in the column originating from the submatrix of Z_w determining from h_b and $(d-\deg(h_b))$-minors in the remaining columns, we have $\det(M)\in(h_1,\dots,h_\ell)$. It is because y is strictly east and weakly north of e that the rows determining q_a that every $\deg(q_a)$-minor in the specified rows is an element of $(q_1,\dots,q_k)$ and that every $\deg(h_b)$-minor in the specified column is an element of $(h_1,\dots,h_\ell)$.

Next we will see that $LT(\det(M))=LCM(LT(q_a),LT(h_b))$. Call $\tilde{M}$ the submatrix of M' whose entries are northwest of both e and y. Set μ_1 to be the product of the terms of $\tilde{M}$ that divide either $LT(q_a)$ or $LT(h_b)$. Call μ_2 the leading term of the determinant of the submatrix of M consisting of the rows and columns used to determine h_b excluding those involving a divisor of μ_1. Similarly, define μ_3 to be the leading term of the determinant of the submatrix of M consisting of the rows and columns determining q_a excluding those involving a divisor of μ_1.

Notice that $\mu_1 \cdot \mu_2 \cdot \mu_3 = LCM(LT(q_a), LT(h_b))$ and that this product is a term of $\det(M)$. To see that every other term of $\det(M)$ is smaller under σ, notice that because w has no obstruction of Type 1, the submatrix of M' whose entries are both northwest of e and northwest of y must have 0's only in full rows and full columns along the north and west sides.

It will now be more convenient for us to work with $\widehat{M}$, obtained from M by adding the doubled copies of rows and columns obtained in the transition from M' to M so that no variable appears more than once. Note that $\det(\widehat{M}) = \det(M)$. If some other term of $\det(\widehat{M})$ is larger than $\mu_1 \cdot \mu_2 \cdot \mu_3$, there must be some entries of $\widehat{M}$ dividing $\mu_1 \cdot \mu_2 \cdot \mu_3$ whose row indices we may permute to obtain a larger monomial. We may assume that this permutation consists of one cycle. If all entries divide either $\mu_1 \cdot \mu_2$ or $\mu_1 \cdot \mu_3$, we would obtain a term of q_a or h_b, respectively, that is strictly larger than its leading term, which also cannot be. But the permutation cannot send any divisor of μ_3 to the row of a divisor of μ_1, all of which are 0 in that column, or vice versa. Hence, $\mu_1 \cdot \mu_2 \cdot \mu_3 = LT(\det(\widehat{M})) = LT(\det(M))$, as desired.

Finally, if y is strictly south and weakly west of e, a parallel argument gives the result. $\square$

Example 3.17. Below we give an example of the construction of the matrices M' and M. Let $w = 5237164$, $y = z_{4,6}$,

$$q_a = \det \begin{bmatrix} 0 & 0 & \underline{z_{1,5}} \\ \underline{z_{2,2}} & z_{2,3} & z_{2,5} \\ z_{3,2} & \underline{z_{3,3}} & z_{3,5} \end{bmatrix} \in Q_{y,I_w}, \text{ and } h_b = \det \begin{bmatrix} 0 & \underline{z_{2,2}} & z_{2,3} & z_{2,4} \\ 0 & z_{4,2} & \underline{z_{4,3}} & z_{4,4} \\ \underline{z_{5,1}} & z_{5,2} & z_{5,3} & z_{5,4} \\ z_{6,1} & z_{6,2} & z_{6,3} & \underline{z_{6,4}} \end{bmatrix} \in N_{y,I_w},$$

in which case $LT(q_a) = z_{1,5}z_{2,2}z_{3,3}$, $LT(h_b) = z_{2,2}z_{4,3}z_{5,1}z_{6,4}$, and

$$LCM(LT(q_a), LT(h_b)) = z_{1,5}z_{2,2}z_{3,3}z_{4,3}z_{5,1}z_{6,4} = LT(\det(M)).$$

The variables dividing the leading terms appearing throughout this example are noted in blue and also underlined. Then

$$M' = \begin{bmatrix} 0 & 0 & 0 & 0 & \underline{z_{1,5}} \\ 0 & \underline{z_{2,2}} & z_{2,3} & z_{2,4} & z_{2,5} \\ 0 & z_{3,2} & \underline{z_{3,3}} & z_{3,4} & z_{3,5} \\ 0 & z_{4,2} & \underline{z_{4,3}} & z_{4,4} & z_{4,5} \\ \underline{z_{5,1}} & z_{5,2} & z_{5,3} & z_{5,4} & z_{5,5} \\ z_{6,1} & z_{6,2} & z_{6,3} & \underline{z_{6,4}} & z_{6,5} \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \underline{z_{1,5}} \\ 0 & \underline{z_{2,2}} & z_{2,3} & z_{2,3} & z_{2,4} & z_{2,5} \\ 0 & z_{3,2} & \underline{z_{3,3}} & \underline{z_{3,3}} & z_{3,4} & z_{3,5} \\ 0 & z_{4,2} & 0 & \underline{z_{4,3}} & z_{4,4} & 0 \\ \underline{z_{5,1}} & z_{5,2} & 0 & z_{5,3} & z_{5,4} & 0 \\ z_{6,1} & z_{6,2} & 0 & z_{6,3} & \underline{z_{6,4}} & 0 \end{bmatrix}.$$

By expressing $\det(M)$ as a sum of products of 3-minors in the first 3 rows of M with 3-minors in the final 3 rows, we see $\det(M) \in Q_{y,I_w}$, and, by expressing $\det(M)$ as the sum of products of 4-minors in columns 1, 2, 4, and 5 with 2-minors in columns 3 and 6, we see $\det(M) \in N_{y,I_w}$. Observe that

$$\begin{aligned} LT(\det(M)) &= LT\left(\det \begin{bmatrix} z_{2,2} & z_{2,3} \\ z_{3,2} & z_{3,3} \end{bmatrix}\right) \cdot LT\left(\det \begin{bmatrix} 0 & z_{4,3} & z_{4,4} \\ z_{5,1} & z_{5,3} & z_{5,4} \\ z_{6,1} & z_{6,3} & z_{6,4} \end{bmatrix}\right) \cdot z_{1,5}. \\ &= (z_{2,2}z_{3,3})(z_{4,3}z_{5,1}z_{6,4})(z_{1,5}). \end{aligned}$$

This expression corresponds to the product $\mu_1 \cdot \mu_2 \cdot \mu_3$ in the proof of Lemma 3.16. One may prefer to use $\widehat{M}$, as in the lemma, obtained from M by subtracting column 3 from column 4, which has the effect of setting the copies of $z_{2,3}$ and $z_{3,3}$ in column 4 to 0.

One aspect of this example that is typical of the general case is that the intersection of the submatrix of of M' northwest of $z_{6,4}$ (playing the role of e) and that northwest of $z_{3,5}$ (from $z_{4,6}$ playing the role of y) is a rectangular matrix of indeterminates with 0's appearing only in full rows along the top and full columns along the western side of the submatrix. This arrangement follows from the fact that 5237164 has no obstruction of Type 1 and gives rise to the decomposition of $LT(\det(M))$ described above into a product of the leading term of entries along the main diagonal of a matrix of indeterminates together with entries coming only from q_a and entries coming only from h_b.

We are now prepared to use our lemmas above that establish Gröbner bases for N_{y,I_w} and C_{y,I_w} by induction to give the backward direction of Conjecture 1.1. We recall a lemma that structures our proof below. This lemma gives an implementation of the strategy employed throughout [21].

Lemma 3.18. [27, Corollary 4.13]. *Let $I = (yq_1 + r_1, \dots, yq_k + r_k, h_1, \dots, h_\ell)$ be a homogenous ideal of the polynomial ring R with y some variable of R and y not dividing any term of any q_i nor any term of any h_j. Fix a term order σ satisfying $LT(yq_i + r_i) = yLT(q_i)$ for each i. Suppose that $\mathcal{G}_C = \{q_1, \dots, q_k, h_1, \dots, h_\ell\}$ and $\mathcal{G}_N = \{h_1, \dots, h_\ell\}$ are Gröbner bases for the ideals they generate, which we call C and N, respectively, and that $\mathrm{ht}(I)$, $\mathrm{ht}(C) > \mathrm{ht}(N)$. Assume that N has no embedded primes, and let $M = \begin{pmatrix} q_1 & \cdots & q_k \\ r_1 & \cdots & r_k \end{pmatrix}$. If the ideal of 2-minors of M is contained in N, then the given generators of I are a Gröbner basis.*

Theorem 3.19. *If $w \in S_n$ is a permutation that has no obstruction of Type 1, Type 2, or Type 3, then w is CDG.*

Proof. Fix a diagonal term order σ. We proceed by induction on the Coxeter length of w, with the case of length 0 trivial. Fix a permutation $w \in S_n$ with n arbitrary and assume that w has no obstruction of Type 1, Type 2, or Type 3. If $D_w = \mathrm{Dom}(w)$, then I_w is generated by variables, and so the result is immediate. Hence, we assume $\mathrm{Dom}(w) \neq D_w$.

According to Lemma 3.16, there is some lower outside corner (i,j) of $D_w \smallsetminus \mathrm{Dom}(w)$ corresponding to the variable $y = z_{i,j}$ so that, with our usual notation, $\{yq_1 + r_1, \dots, yq_k + r_k, h_1, \dots, h_\ell\}$ are the CDG generators of I_w, the generators $\{q_1, \dots, q_k, h_1, \dots, h_\ell\}$ of C_{y,I_w} form a Gröbner basis, and $LT(yq_a + r_a) = yLT(q_a)$ for each a. By Corollary 3.10 and the inductive hypothesis, $\{h_1, \dots, h_\ell\}$ is a Gröbner basis for N_{y,I_w}.

We will show that $I_2\begin{pmatrix} q_1 & \dots & q_k \\ r_1 & \dots & r_k \end{pmatrix} \subseteq N_{y,I}$. For each CDG generator, $yq_a + r_a$, let $yq'_a + r'_a$ be the corresponding natural generator of I_w, i.e. the generator taken in a matrix of indeterminates in which the variables corresponding to $\mathrm{Dom}(w)$ have not been set to 0. Let $J = (z_{i,j} \mid (i,j) \in \mathrm{Dom}(w))$. Then $I_2\begin{pmatrix} q_1 & \dots & q_k \\ r_1 & \dots & r_k \end{pmatrix} + J = I_2\begin{pmatrix} q'_1 & \dots & q'_k \\ r'_1 & \dots & r'_k \end{pmatrix} + J$. Hence, because $J \subseteq N_{y,I_w}$, it suffices to show $I_2\begin{pmatrix} q'_1 & \dots & q'_k \\ r'_1 & \dots & r'_k \end{pmatrix} \subseteq N_{y,I_w}$.

In order to show this last containment, we will temporarily consider a possibly different diagonal term order σ', which will be a lexicographic term order in which

y is largest. It is because y is a lower outside corner that there must exist a term order that is both diagonal and is also lexicographic with y largest. Now because each $q'_a r'_b - q'_b r'_a = (yq'_a + r'_a)q'_b - (yq'_b + r'_b)q'_a$ for $1 \leqslant a < b \leqslant k$ is an element of the ideal of $(\text{rank}_w(i,j)+1)$-minors in a matrix of indeterminates weakly northwest of y, an ideal for which the natural generators form a Gröbner basis under σ' because it is a diagonal order, we know that $q'_a r'_b - q'_b r'_a$ has a Gröbner reduction in terms of those generators. Because $q'_a r'_b - q'_b r'_a$ does not involve y and y is lexicographically largest under σ', that reduction must be in terms of $(\text{rank}_w(i,j)+1)$-minors weakly northwest of y that do not involve y. Each such minor is an element of N_{y,I_w}. Hence,

$$I_2\begin{pmatrix} q_1 & \dots & q_k \\ r_1 & \dots & r_k \end{pmatrix} + J = I_2\begin{pmatrix} q'_1 & \dots & q'_k \\ r'_1 & \dots & r'_k \end{pmatrix} + J \subseteq N_{y,I_w},$$

as desired.

The height requirements $\text{ht}I_w, \text{ht}C_{y,I_w} > \text{ht}N_{y,I_w}$ are immediate from the fact that N_{y,I_w} is prime [18, Proposition 3.3] together with the proper containment of N_{y,I_w} in each of I_w and C_{y,I_w}. The result now follows from Lemma 3.18. □

Notice that we do not claim that the Gröbner reduction of $q'_a r'_b - q'_b r'_a$ with respect to σ' gives rise to a Gröbner reduction of $a_a r_b - q_b r_a$ in terms of the CDG generators with respect to σ. Lemma 3.18 requires only that we demonstrate an ideal containment.

Corollary 3.20. *If $w \in S_n$ avoids all eight of the following patterns, then w is CDG:*

$$13254, 21543, 214635, 215364, 215634, 241635, 315264, 4261735.$$

Proof. If $w \in S_n$ avoids the patterns above, then it does not have an obstruction of Type 1, Type 2, or Type 3 by Lemmas 3.3, 3.5, and 3.6, and so the result follows from Theorem 3.19. □

4. The non-CDG Permutations

In this section, we show that a permutation $w \in S_n$ that contains one of the eight permutations listed in Conjecture 1.1 is not CDG. We will show that slightly stronger claim that if $w \in S_n$ contains one of the eight listed patterns, then the CDG generators do not form a Gröbner basis under *any* diagonal term order.

Note that a generating set for an ideal I in a polynomial ring R forms a Gröbner basis for I if and only if that generating set forms a Gröbner basis in the larger polynomial ring $R[x]$. For that reason, we may view all ideals that arise in Theorem 4.1 as ideals of a polynomial ring in $(n+1)^2$ variables. An example illustrating the argument of Theorem 4.1 follows immediately after the proof.

Theorem 4.1. *Let Z be an $(n+1)\times(n+1)$ matrix of indeterminates, and let σ be a diagonal term order on $R = \mathbb{C}[Z]$. Let $d \leqslant n$, and suppose that there is some $w \in S_d$ so that the CDG generators of I_w do not form a Gröbner basis under σ. If $v \in S_{n+1}$ contains w, then the CDG generators of I_v do not form a Gröbner basis under σ.*

Proof. By induction, we may assume that $w = w_1 \dots w_n \in S_n$ and that $v = v_1 \dots v_{n+1}$ with $v_1 \dots v_{i-1} v_{i+1} \dots v_{n+1} = w$ for some $1 \leqslant i \leqslant n+1$.

Recall that D_w is obtained from D_v by deleting row i and column v_i. With Z_v an $(n+1)\times(n+1)$ matrix of indeterminates with $z_{i,j}$ set to 0 whenever $(i,j) \in \text{Dom}(v)$, identify Z_w with the $n \times n$ submatrix of Z_v obtained by the deletion of row i and column v_i. Consider that the rows of Z_w to be labeled $1, \dots, i-1, i+1, \dots n+1$ and the columns of to be labeled $1, \dots, v_{i-1}, v_{i+1}, \dots, n+1$.

Let $G_w = \{\delta_1, \dots, \delta_\ell\}$ for some $\ell \in \mathbb{N}$ be the set of CDG generators of w. Assume that G_w is ordered so that $\delta_1, \dots, \delta_k$ are determined by rank conditions in boxes

$(a,b) \in \mathrm{Ess}(w)$ with $a < i$ or $b < v_i$ and that $\delta_{k+1}, \ldots, \delta_\ell$ are determined by rank conditions in boxes (a,b) with $a > i$ and $b > v_i$.

Let f and g be two CDG generators of I_w whose s-polynomial $s = s(f,g)$ does not reduce to 0 by G_w under σ. Let r denote the remainder of s under the deterministic division algorithm with respect to G_w and the chosen ordering on G_w. Then we may write $r = s + \sum \alpha_j \delta_j$ where the leading term of $\alpha_j \delta_j$ is not in the ideal generated by the leading terms of the $\delta_{j'}$ with $j' < j$. By definition of remainder, no leading term of any element of G_w divides the leading term of r though $r \in I_w$.

Let G_v denote the set of CDG generators of I_v. We may write

$$G_v = \{\delta_1, \ldots, \delta_k, z_{i,v_i}\delta_{k+1} + \varepsilon_{k+1}, \ldots, z_{i,v_i}\delta_\ell + \varepsilon_\ell, \delta_{\ell+1}, \ldots, \delta_m\},$$

where the δ_j with $\ell < j \leqslant m$ are the elements of G_v involving at least one variable from row i or column v_i other than z_{i,v_i}, and the others are as expected. We will use r to construct an element of I_v whose leading term is not divisible by any leading term of G_v.

If the southeast corner of the submatrix of Z_w determining f is a box (a,b) satisfying $a < i$ or $b < v_i$, then $f \in G_v$. In that case, define $f' = f$. If $a > i$ and $b > v_i$, then take f' to be (up to sign) the element of G_v determined by the rows determining f together with row i and the columns determining f together with column v_i. After possibly multiplying by -1, $f' = z_{i,v_i} f + \varepsilon_f$, where every term of ε_f is divisible by exactly one variable from row i and exactly one variable from column v_i, neither of which is z_{i,v_i}. Define g' similarly, and take $s' = s(f', g')$ to be their s-polynomial.

If $f' = f$ and $g' = g$, then $s' = s$. Because no term of s is divisible by any variable in row i or in column v_i of Z_v, if s has a reduction by the elements of G_v, it must have a reduction by $\{\delta_1, \ldots, \delta_k\}$, which is known not to exist.

If $f' = z_{i,v_i} f + \varepsilon_f$ and $g' = g$, let $LT(f)$ denote the leading term of f, $LT(g)$ denote the leading term of g, and G the greatest common divisor of $LT(f)$ and $LT(g)$. Set

$$t = \frac{LT(g)}{G} f' - \frac{z_{i,v_i} LT(f)}{G} g = z_{i,v_i} s + \frac{LT(g)}{G} \varepsilon_f \in I_v.$$

(Notice that whenever $z_{i,v_i} LT(f)$ is the leading term of f', t will coincide with the s-polynomial of f' and g'.)

We claim that t cannot be reduced by G_v. We begin by modifying t by multiples of the δ_j for $1 \leqslant j \leqslant k$ and $z_{i,v_i}\delta_j + \varepsilon_j$ for $k < j \leqslant \ell$ following the deterministic division algorithm in G_w to obtain $t' = z_{i,v_i} r + \varepsilon' \in I_v$ where ε' does not involve z_{i,v_i}. Note that every element of G_v involving z_{i,v_i} involves it only as a multiple of some δ_j with $k < j \leqslant \ell$. Hence, the division algorithm will never call for the addition of any multiple of any $z_{i,v_i}\delta_j$. Therefore, no newly added polynomial could have any term that cancels with any term of $z_{i,v_i} r$, from which it follows that t' is an element of I_v with no reduction by G_v.

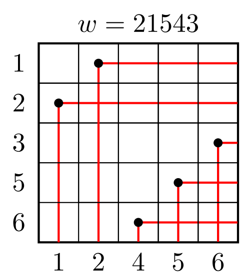
Finally, assume that $f' = z_{i,v_i} f + \varepsilon_f$ and $g' = z_{i,v_i} g + \varepsilon_g$. Then

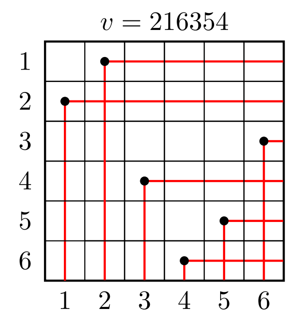
$$t = \frac{LT(g)}{G} f' - \frac{LT(f)}{G} g' = s + \frac{LT(g)}{G} \varepsilon_f - \frac{LT(f)}{G} \varepsilon_g \in I_v.$$

Again, we modify by multiples of the δ_j with $1 \leqslant j \leqslant k$ to obtain $t' = r + \frac{LT(g)}{G}\varepsilon_f - \frac{LT(f)}{G}\varepsilon_g$. Because no leading term of any δ_j with $1 \leqslant j \leqslant k$ divides any term of r and because every term of every other element of G_v involves a variable from row i or column v_i, which r does not, no further steps in the division algorithm can eliminate any term of r, and so t has no reduction by the elements of G_v.

It follows that in all cases, there is an element of I_v that has no Gröbner reduction by G_v, and so G_v is not a diagonal Gröbner basis of I_v. ☐

Example 4.2. Let $w = 21543$ and $v = 216354$, in which case $(i, v_i) = (4, 3)$. The visualizations of the Rothe diagrams of w and v are below with the modified row and column indexing for the diagram of w described in the proof of Theorem 4.1.





Let $f = \begin{vmatrix} z_{2,1} & z_{2,2} & z_{2,4} \\ z_{3,1} & z_{3,2} & z_{3,4} \\ z_{5,1} & z_{5,2} & z_{5,4} \end{vmatrix}$ and $g = \begin{vmatrix} 0 & z_{1,4} & z_{1,5} \\ z_{2,1} & z_{2,4} & z_{2,5} \\ z_{3,1} & z_{3,4} & z_{3,5} \end{vmatrix}$, in which case, under any diagonal term order, $s = s(f, g) = z_{1,4}z_{3,5}f - z_{3,2}z_{5,4}g$ and

$$\begin{aligned} r = &-z_{1,4}z_{2,2}z_{3,1}z_{3,5}z_{5,4} + z_{1,4}z_{2,2}z_{3,4}z_{3,5}z_{5,1} + z_{1,4}z_{2,4}z_{3,1}z_{3,5}z_{5,2} \\ &- z_{1,4}z_{2,4}z_{3,2}z_{3,5}z_{5,1} + z_{1,4}z_{2,5}z_{3,1}z_{3,2}z_{5,4} - z_{1,4}z_{2,5}z_{3,1}z_{3,4}z_{5,2} \\ &+ z_{1,5}z_{2,2}z_{3,1}z_{3,4}z_{5,4} - z_{1,5}z_{2,2}z_{3,4}^2z_{5,1} - z_{1,5}z_{2,4}z_{3,1}z_{3,2}z_{5,4} + z_{1,5}z_{2,4}z_{3,2}z_{3,4}z_{5,1}. \end{aligned}$$

Then $g' = g$ and $f' = \begin{vmatrix} z_{2,1} & z_{2,2} & z_{2,3} & z_{1,4} \\ z_{3,1} & z_{3,2} & z_{3,3} & z_{3,4} \\ z_{4,1} & z_{4,2} & z_{4,3} & z_{4,4} \\ z_{5,1} & z_{5,2} & z_{5,3} & z_{5,4} \end{vmatrix} = z_{4,3}f + \varepsilon_f$, where ε_f consists of the terms of f' arising as products of $z_{2,3}$, $z_{3,3}$, and $z_{5,3}$ with their respective cofactors (or, equivalently, products of $z_{4,1}$, $z_{4,2}$, and $z_{4,4}$ and their respective cofactors). Set $t = z_{4,3}s + z_{1,4}z_{3,5}\varepsilon_f$ and $t' = z_{4,3}r + (z_{1,4}z_{3,5} - z_{1,5}z_{3,4})\varepsilon_f$. The fact that $z_{4,3}r$ prevents the reduction of t' to 0 by G_v follows from the fact that no term of r is divisible by the leading term of any element of G_w.

Corollary 4.3. *Let w be a permutation. If there exists a diagonal term order σ so that the CDG generators of I_w form a Gröbner basis, then w avoids all eight of the patterns*

$$13254, 21543, 214635, 215364, 215634, 241635, 315264, 4261735.$$

Proof. This result is immediate from Theorem 4.1 together with explicit computations in the case of the eight permutations listed in Conjecture 1.1. □

5. Unifying characteristics of the non-CDG permutations

We conclude by describing briefly how [29, Theorem 2.1(a)] can be used to understand two properties that prevent the permutations listed in Conjecture 1.1 from being CDG. We note first that 13254 has no dominant part and so its failure to be CDG is due to the fact that it contains 2143 [29, Theorem 6.1]. For the remainder of this section, we consider the other seven permutations, all of which have nontrivial dominant parts.

For an arbitrary rank matrix, understand the CDG generators to be defined analogously to the case of defining ideals of matrix Schubert varieties. If any of the permutations listed in Conjecture 1.1 were CDG, [29, Theorem 2.1(a)] would require that either the ideal determined by the rank matrix N_1 or the ideal determined by the rank matrix N_2, below, have a CDG diagonal Gröbner basis, which they are easily seen not to:

$$N_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \qquad \text{and} \qquad N_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

The rank matrix N_1 encodes interference from $\mathrm{Dom}(w)$ that prevents I_w from being CDG, and N_2 encodes failures to be vexillary that are sufficiently far from $\mathrm{Dom}(w)$ that they are not handled by replacing Fulton generators by CDG generators.

Example 5.1. Consider the rank matrix M_w of the permutation $w = 21543$ with respect to any lexicographic term order in which $y = z_{3,4}$ is largest, with essential boxes marked by $\square$.

$$M_w = \begin{bmatrix} \boxed{0} & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & \boxed{2} & 3 \\ 1 & 2 & \boxed{2} & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

If the CDG generators of I_w were a diagonal Gröbner basis, then [29, Theorem 2.1(a)] would require the CDG generators of C_{y,I_w}, which is the ideal determined by N_1 and plays the role of the link in a geometric vertex decomposition at y, also to be a diagonal Gröbner basis.

We leave it to the reader to use (possibly repeated) application of [29, Theorem 2.1(a)] to obtain N_1 or N_2 from the rank matrices of the other six permutations listed in Conjecture 1.1.

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Patricia Klein, Texas A&M University, Department of mathematics, College Station, TX 77843 (USA)
E-mail : `pjklein@tamu.edu`