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Analysing flag-transitive point-imprimitive 2-designs

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Abstract In this paper we develop several general methods for analysing flag-transitive point-imprimitive 2-designs, which give restrictions on both the automorphisms and parameters of such designs. These constitute a tool-kit for analysing these designs and their groups. We apply these methods to complete the classification of flag-transitive, point-imprimitive 2-(v, k, λ) designs with λ at most 4.

1. Introduction

A 2-(v, k, λ) *design* $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ consists of a set $\mathcal{P}$ of v *points* and a set $\mathcal{B}$ of *blocks* such that each block is a k-subset of $\mathcal{P}$ and each pair of distinct points is contained in λ blocks. To avoid degenerate cases we assume that $2 < k < v$; such designs are called *nontrivial*. In general, the number of blocks $b := |\mathcal{B}|$ is at least v by Fisher's inequality (see [5, 1.3.8]) and $\mathcal{D}$ is said to be *symmetric* if $b = v$. We study (not necessarily symmetric) 2-designs $\mathcal{D}$ possessing a high degree of symmetry, namely they admit a subgroup G of automorphisms (permutations of $\mathcal{P}$ preserving $\mathcal{B}$) that acts transitively on the set of *flags* (incident point-block pairs), and moreover leaves invariant a nontrivial partition of the point set $\mathcal{P}$, that is to say, G is flag-transitive and point-imprimitive. We develop several general purpose methods for analysing flag-transitive point-imprimitive 2-designs, and then we test their effectiveness by applying them to complete the classification of the flag-transitive, point-imprimitive 2-(v, k, λ) designs with $\lambda \leqslant 4$.

For a flag-transitive, point-imprimitive 2-(v, k, λ) design, the parameter $\lambda \neq 1$ by the celebrated work of Higman and McLaughlin [9, Proposition 3]. Moreover the classification of such designs with $\lambda = 2$ was completed in [6, Theorem 1.1], showing that there are exactly two examples up to isomorphism. The classification for $\lambda \in \{3, 4\}$ with $v < 100$ was given in [7, Theorem 2], identifying nine designs up to isomorphism, and here we complete that work (solving [7, Problem 3]) and proving that there are no examples with 100 points or more.

Theorem 1.1. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a 2-(v, k, λ) design with $\lambda \leqslant 4$, which admits a flag-transitive, point-imprimitive subgroup of automorphisms. Then $v \in \{15, 16, 36, 45, 96\}$, and $\mathcal{D}$ is one of the eleven designs listed in* [7, Theorem 2].

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A number of papers have appeared recently providing classifications of flag-transitive, point-imprimitive 2-designs under various parameter constraints. We give a brief survey of such results in Subsection 1.1, where we also introduce the relevant parameters we will use to describe the point-imprimitivity system, notably Hypothesis 1.2.

1.1. Survey and parameters. As we mentioned above, if $G \leqslant \mathrm{Aut}(\mathcal{D})$ is flag-transitive and point-imprimitive on a nontrivial $2-(v,k,\lambda)$ design $\mathcal{D}=(\mathcal{P},\mathcal{B})$, then $\lambda \geqslant 2$ by [9, Proposition 3]. This condition was sharpened by Dembowski [5, 2.3.7(a)] in his 1968 book, where he proved that each of the following conditions must hold, where r is the (constant) number of blocks containing a given point, and (s,t) denotes the greatest common divisor $\gcd(s,t)$ for integers s,t.

(1) $(\lambda,r)\geqslant 2,\ \lambda\leqslant(\lambda,r)((\lambda,r)-1),\ (r-\lambda,k)\geqslant 2,\ r\leqslant\lambda(k-3)$ and $(v-1,k-1)\geqslant 3$.

We mention in passing that, very recently, the fourth inequality was strengthened by Zhao and Zhou [19, Lemma 1.3] to $r \leqslant (r,\lambda)(k-3)$. After Dembowski's work the next significant breakthrough was due to Davies [4] in 1987. Davies showed by example that there are flag-transitive point-imprimitive designs with arbitrarily large λ, and also showed that, for a given λ, both the block-size k and the number v of points are bounded in terms of λ, and hence there are only finitely many flag-transitive point-imprimitive designs for each λ. Unfortunately Davies did not give upper bonds for k,v as explicit functions of λ.

The first such explicit bounds were due to O'Reilly–Regueiro [18] in 2005 in the case of symmetric designs, for example, she showed that $k\leqslant\lambda(\lambda+1)$. These bounds were improved by Zhou and Praeger in [17, Theorem 1.1], and were expressed in terms of additional parameters, namely the number d of classes of a nontrivial point-partition $\mathcal{C}$, the size c of each class $\Delta\in\mathcal{C}$, and the (constant) size ℓ of each non-empty intersection $\Delta\cap B$ of a block B and a class Δ. The bounds were sufficiently good to show that for $\lambda\leqslant 10$, there are exactly 22 feasible parameter tuples (v,k,λ,c,d,ℓ) meeting these bounds, [17, Corollary 1.3 and Table 1]. In 2022, Mandić and Šubašić [12, Proposition 12] made small improvements in the bounds for symmetric designs in [17] (see also Proposition 2.8) and, building on classifications in [11, 15, 17] of the examples with $\lambda\leqslant 4$, they were able to identify all examples with parameter tuples in [17, Table 1] except for two parameter sequences. Even more recently, Montinaro [13, 14] classified all flag-transitive, point-imprimitive symmetric designs for which $k>\lambda(\lambda-3)/2$, and thereby ruled out these two parameter sequences. This completes the classification of flag-transitive, point-imprimitive symmetric designs with $\lambda\leqslant 10$ [14, Theorem 2.3]: there are eight such designs up to isomorphism (corresponding to four parameter sequences) all with $\lambda\leqslant 4$ and $v<100$. These examples are also the eight designs from [7, Theorem 2] that are symmetric.

For general (not necessarily symmetric) flag-transitive point-imprimitive designs with a given λ, explicit upper bounds for k and v were obtained in [7, Theorem 1]. In this general case the upper bound on k is cubic, namely $k\leqslant 2\lambda^2(\lambda-1)$; the parametric restrictions obtained were sufficiently strong to list in [7, Proposition 8] all parameter sequences (v,k,λ,c,d,ℓ) meeting these restrictions with $\lambda\in\{3,4\}$, noting that the examples with $\lambda=2$ were classified in [6, Theorem 1.1]. In this paper, after discussing a number of general methods in Section 2, we complete the classification of all examples arising from these parameter tuples in Section 3.

We formalise the conditions we have been discussing in Hypothesis 1.2. We will use these assumptions throughout the paper. A summary reference for the conditions in Hypothesis 1.2 may be found in [7, Section 2]. Note that we make no restrictions on the parameter λ in Hypothesis 1.2.

Hypothesis 1.2. (a) *$\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a 2-(v, k, λ) design, with point-set $\mathcal{P}$ of size v, block-set $\mathcal{B}$ of size b, each block a k-subset of $\mathcal{P}$, and each point-pair lying in λ blocks. Let $b = |\mathcal{B}|$, the number of blocks, and let r be the number of blocks containing a given point.*

(b) *$\mathcal{D}$ admits a group G of automorphisms such that G is transitive on flags, and is imprimitive on points, preserving a point-partition $\mathcal{C} = \{\Delta_1, \ldots, \Delta_d\}$ of size $d \geqslant 2$ with classes Δ_i of size $c \geqslant 2$. Let $D = G^{\mathcal{C}}$ and $L = (G_\Delta)^\Delta$ denote the induced permutation groups on $\mathcal{C}$ and Δ, for $\Delta \in \mathcal{C}$. We may (see* [16, Theorem 5.5]*), and will, assume that $G \leqslant L \wr D$. Let $K \coloneqq G_{(\mathcal{C})}$, the kernel of the G-action on $\mathcal{C}$.*

(c) *For $B \in \mathcal{B}$ intersecting a class $\Delta \in \mathcal{C}$ nontrivially, the size $\ell = |B \cap \Delta|$ is independent of the choices of B, Δ; and $\ell \geqslant 2$.*

The general results presented in Section 2 about flag-transitive, point-imprimitive designs, include also various properties of the groups L and D. Together these constitute a tool-kit of general methods for analysing these designs and their groups. We are in particular interested in the case $\lambda = 3$ or 4 with $v \geqslant 100$ and in Section 3 we apply these methods to prove Theorem 1.1.

We present the methods in Section 2 to draw attention to the strategies we employ in the analysis in Section 3. We would be glad if this encouraged others to strengthen the methods in Section 2, or obtain more extensive classifications of flag-transitive designs. In particular we wonder if these considerations might lead to improvements in the general upper bounds derived in [7]. With these potential later applications in mind we offer, in Table 1, a summary of the constraints arising from the general results in Section 2.

Constraints on	Reference
The parameter tuple $(v, k, \lambda, c, d, \ell)$	Lemma 2.1
The order of $G^{\mathcal{C}}$	Proposition 2.2
Some $(G_B)^{\mathcal{C}}$-orbit lengths and on $\lvert G^{\mathcal{C}} : (G_B)^{\mathcal{C}}\rvert$	Proposition 2.3
The multiple transitivity of G_Δ on Δ	Proposition 2.4
Existence of a complete subdesign of $\mathcal{D}$ admitting $N_G(P)$ as an automorphism group, for certain Sylow subgroups P of $G_{(\mathcal{C})}$	Propositions 2.5 - 2.7
Generalisation of some results from [12, 13, 17] on related designs	Proposition 2.8

Table 1. Summary of the constraints provided by results in Section 2

2. General methods

In this section we assume that Hypothesis 1.2 holds and we use the notation introduced there. We begin by giving in Lemma 2.1 a summary of basic restrictions on the parameters in Hypothesis 1.2, which are in addition to the conditions in (1). Lemma 2.1 is a simplification of the statements of [7, Lemmas 4 and 5], together with an extra part (iii) which follows from an argument to be found in the proof of [12, Proposition 9] (see also Proposition 2.8 below). Our proof of part (iii) is essentially the first part of the proof of [12, Proposition 9].

Lemma 2.1. *The following equalities, inequalities and divisibility conditions hold:*

(i) $bk = vr$ *and* $r(k-1) = \lambda(v-1)$
(ii) $\ell \mid k$ *and* $1 < \ell < k$;
(iii) $\ell^2 \mid c^2\lambda$
(iv) $\ell - 1 \leqslant (k - 1 - d(\ell-1))\,(\ell-1) \leqslant \lambda - 1$;
(v) $\lambda(c-1) = r(\ell-1)$;
(vi) $k \mid \lambda\ell(\ell-1)^2 d(d-1)$.

Proof. Part (i) is standard, see for example [7, Lemma 4]. Parts (ii) and (v) are [7, Lemma 5(ii) and (iii)], respectively. For part (iv), the first inequality follows from the assertion in [7, Lemma 5] that the quantity $x = k - 1 - d(\ell - 1)$ is a positive integer, and the second inequality follows from [7, Lemma 5(viii)] on substituting for x. Part (vi) follows from [7, Lemma 5(vii)] on substituting for x.

Finally for part (iii), we choose distinct classes $\Delta, \Delta' \in \mathcal{C}$, and determine the cardinality of the set $\Pi \coloneqq \{(B, \alpha, \alpha') \mid B \in \mathcal{B}, \alpha \in B \cap \Delta, \alpha' \in B \cap \Delta'\}$. On the one hand there are c^2 choices for a pair $(\alpha, \alpha') \in \Delta \times \Delta'$ and each pair lies in exactly λ blocks B, so $|\Pi| = c^2\lambda$. On the other hand each of the, say n, blocks B meeting both Δ and Δ' nontrivially meets each of these classes in exactly ℓ points, so $|\Pi| = n\ell^2$. It follows that $c^2\lambda = n\ell^2$, and hence the number of these blocks $n = c^2\lambda/\ell^2$. Part (iii) follows since n is an integer. □

Next we derive restrictions on the group D in Hypothesis 1.2.

Proposition 2.2. *Assume that Hypothesis* 1.2 *holds, and suppose that p is a prime such that:*

(i) $0 \leqslant d - p < \frac{k}{\ell} < p$, *and*
(ii) *p does not divide* $b = \frac{vr}{k}$.

Then p does not divide $|D|$.

Proof. Suppose p divides $|D|$ and p satisfies the two stated conditions. Note the first condition implies $d < 2p$. Let P be a Sylow p-subgroup of G. Since p divides $|D|$ and hence $|G|$, P is not trivial. Moreover, the index of G_B in G is b, which is not divisible by p. Therefore there exists a block B such that $P \leqslant G_B$. Since p divides $|D|$, P acts non-trivially on $\mathcal{C}$. The condition $d < 2p$ implies that $P^{\mathcal{C}}$ must have one orbit of size p and $d - p$ fixed classes. If B contains a point in a class in the orbit of size p, then $k \geqslant \ell p$ since P fixes B, contradicting $\frac{k}{\ell} < p$. Thus each point in B lies in one of the $d - p$ fixed classes. This implies $k \leqslant \ell(d - p)$, contradicting $d - p < \frac{k}{\ell}$. This final contradiction proves that p does not divide $|D|$. □

Proposition 2.3. *Assume that Hypothesis* 1.2 *holds. Let $B \in \mathcal{B}$ and $X \coloneqq (G_B)^{\mathcal{C}}$, a subgroup of D. Also let c_0 be the (constant) length of the K-orbits in $\mathcal{P}$, and let $x \coloneqq c_0/\gcd(c_0, \ell)$. Then X has an orbit of length k/ℓ in $\mathcal{C}$ and*

(a) *if $c_0 = 1$ (or equivalently, if $K = 1$), then $|D : X| = b$; while*
(b) *if $c_0 > 1$, then c_0 divides c, x divides b, and $|D : X|$ divides b/x.*

Proof. Since G_B is transitive on B, X is transitive on the set of k/ℓ classes intersecting B non-trivially, proving the first assertion. Also, since $X = G_BK/K$ and $D = G/K$, it follows that $|D : X| = |G : G_BK|$. In particular, if $K = 1$, or equivalently, $c_0 = 1$, then $|D : X| = |G : G_B| = b$ and part (a) holds.

Assume now that $K \neq 1$, or equivalently that $c_0 > 1$. Since $K^\Delta \lhd G_\Delta^\Delta = L$, it follows that c_0 divides c. Since G is block-transitive, its normal subgroup K has orbits of equal length, say b_0, in $\mathcal{B}$, so $b_0 = |K : K_B|$ divides b. Let $\Delta \in \mathcal{C}$ be such that $B \cap \Delta \neq \varnothing$, and let $\alpha \in B \cap \Delta$. Since G_B is transitive on B it follows that $B \cap \Delta$ is a $G_{B,\Delta}$-orbit of size ℓ, and as $K_B \trianglelefteq G_{B,\Delta}$, the K_B-orbits in $B \cap \Delta$ have equal length, say ℓ_0, Thus $\ell_0 \mid \ell$, and $|K : K_{B,\alpha}| = b_0\ell_0$ is divisible by $|K : K_\alpha| = c_0$. Let $\ell' \coloneqq \gcd(c_0, \ell_0)$. Then $\ell' \mid \gcd(c_0, \ell)$ and c_0/ℓ' divides b_0. Since

$$\frac{c_0}{\ell'} = \frac{c_0}{\gcd(c_0, \ell)} \cdot \frac{\gcd(c_0, \ell)}{\ell'} = x \cdot \frac{\gcd(c_0, \ell)}{\ell'}$$

it follows that x divides b_0 which, in turn, divides b. Finally

$$|D : X| = |G : G_B K| = \frac{|G : G_B|}{|G_B K : G_B|} = \frac{|G : G_B|}{|K : K_B|} = \frac{b}{b_0} = \frac{b}{x} \cdot \frac{1}{b_0/x},$$

and it follows that $|D : X|$ divides b/x, and part (b) holds, completing the proof. ☐

Now we turn our attention to restrictions on the group L in Hypothesis 1.2.

Proposition 2.4. *Assume that Hypothesis* 1.2 *holds, and let* $\Delta \in \mathcal{C}$ *and* $\alpha, \beta \in \Delta$ *be distinct points.*

(a) *If* $\ell = 2$*, then* L *is* 2*-transitive on* Δ *of degree* c*.*
(b) *If* $\ell \geqslant 3$*, then* $L_{\{\alpha,\beta\}}$ *has an orbit in* $\Delta \setminus \{\alpha, \beta\}$ *of size at most* $(\ell - 2)\lambda$*. In particular, if* $c > 2 + (\ell - 2)\lambda$ *then* L *is not* 3*-transitive.*

Proof. (a) Assume that $\ell = 2$ and let (α', β') be a second pair of distinct points of Δ. Let $B, B' \in \mathcal{B}$ be such that $\alpha, \beta \in B$ and $\alpha', \beta' \in B'$. Since G is flag-transitive, there exists $g \in G$ mapping (α, B) to (α', B'). Then since $\alpha' = \alpha^g \in \Delta \cap \Delta^g$ it follows that g fixes Δ, so $g^\Delta \in L$. Also, since $B^g = B'$ and $\ell = 2$, the element g must map β to β'. It follows that L is 2-transitive.

(b) Let $\mathcal{B}(\alpha, \beta) = \{B_1, \ldots, B_\lambda\}$ be the set of λ blocks containing $\{\alpha, \beta\}$. Then $G_{\{\alpha,\beta\}}$ leaves $\mathcal{B}(\alpha, \beta)$ invariant, and hence leaves invariant the union $X \coloneqq \cup_{i=1}^{\lambda} B_i \cap \Delta$. Thus $X \setminus \{\alpha, \beta\}$ is a subset of $\Delta \setminus \{\alpha, \beta\}$ of size at most $(\ell - 2)\lambda$, it is non-empty since $\ell \geqslant 3$, and it is preserved by $G_{\{\alpha,\beta\}}$. Any $G_{\{\alpha,\beta\}}$-orbit in $X \setminus \{\alpha, \beta\}$ (which is also an $L_{\{\alpha,\beta\}}$-orbit) has size at most $(\ell - 2)\lambda$. If L were 3-transitive, then $L_{\{\alpha,\beta\}}$ would be transitive on $\Delta \setminus \{\alpha, \beta\}$, and hence $c \leqslant 2 + (\ell - 2)\lambda$. ☐

We now explore properties of certain p-subgroups of G.

Proposition 2.5. *Assume that Hypothesis* 1.2 *holds, and suppose that* p *is a prime dividing* $|G|$ *such that* p *does not divide* λ*. Suppose also that a nontrivial* p*-subgroup* P *of* G *fixes at least two distinct points* α, β*, and let* $\Delta \in \mathcal{C}$ *such that* $\alpha \in \Delta$*.*

(a) *Then* P *fixes at least one block* B *containing* $\{\alpha, \beta\}$ *and hence leaves invariant the* ℓ*-subset* $B \cap \Delta$ *of* Δ *and the* k/ℓ*-subset* $\mathcal{C}(B) \coloneqq \{\Delta' \in \mathcal{C} \mid B \cap \Delta' \neq \varnothing\}$ *of* $\mathcal{C}$*.*
(b) *Moreover, if* $p > \lambda$ *then* P *fixes setwise each block containing* $\{\alpha, \beta\}$*.*

Proof. Let $\mathcal{B}(\alpha, \beta)$ be the set of λ blocks containing $\{\alpha, \beta\}$. Then P leaves $\mathcal{B}(\alpha, \beta)$ invariant. Since P is a p-group, all its orbits on $\mathcal{B}(\alpha, \beta)$ are p-powers. Since p does not divide λ, it follows that P fixes at least one block $B \in \mathcal{B}(\alpha, \beta)$. Part (a) follows. It also follows that if $p > \lambda$ then P must fix setwise each block in $\mathcal{B}\{\alpha, \beta\}$. ☐

Our next results concern a *subdesign* of a design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$. This is a pair $\mathcal{D}_0 = (\mathcal{P}_0, \mathcal{B}_0)$ where $\mathcal{P}_0 \subset \mathcal{P}$ and $\mathcal{B}_0$ is a collection of k_0-subsets of $\mathcal{P}_0$ such that each $B_0 \in \mathcal{B}_0$ is contained in some block of $\mathcal{B}$; and such that $\mathcal{D}_0$ is a 2-design (or sometimes only a 1-design). A subdesign is called *complete* if $k_0 = k$, and in this case $\mathcal{B}_0 \subset \mathcal{B}$, see for example [10, p. 31]. Subdesigns provide useful information about the structure of a design. For example the design of points and lines of a projective space $\mathrm{PG}_n(q)$ is a flag-transitive $2 - (\frac{q^n-1}{q-1}, q + 1, 1)$ design and for each proper subspace, the set of lines it contains forms a complete subdesign. Similar complete subdesigns arise for the designs of points and lines of a Desarguesian affine space, and in both cases these subdesigns inherit a flag-transitive action from the automorphism group of the original design. In these examples the designs are point-primitive.

Question 2.6. *Do there exist* 2*-designs admitting a flag-transitive, point-imprimitive group of automorphisms and containing a complete subdesign?*

In Proposition 2.7 we show that under additional assumptions, there is a point-transitive complete subdesign associated with a Sylow subgroup of K. This gives a strong restriction on the parameters which we exploit in Section 3 to exclude certain parameter tuples. However we do not know of any examples satisfying Hypothesis 1.2 where conditions (a)–(c) of Proposition 2.7 hold.

Proposition 2.7. *Assume that Hypothesis* 1.2 *holds and that there exists a prime* p *satisfying the following conditions:*

$$\text{(a) } p \text{ divides } |K|, \quad \text{(b) } p \text{ does not divide } c, \quad \text{(c) } p > \lambda.$$

Then $\mathcal{D}$ *has a complete point-transitive subdesign which is a* $2-(df, k, \lambda)$ *design, where* f *satisfies*

$$\ell < f < c \quad \text{and} \quad c \equiv f \pmod{p}.$$

In particular, $k-1$ *divides* $\lambda(df-1)$.

Proof. Let P be a Sylow p-subgroup of K, and let $\mathcal{P}_0 = \mathrm{fix}_{\mathcal{P}}(P)$ and $\mathcal{B}_0 = \{B \in \mathcal{B} \mid B \subseteq \mathcal{P}_0\}$. We show that $(\mathcal{P}_0, \mathcal{B}_0)$ is a complete subdesign of $\mathcal{D}$, and is a $2-(df, k, \lambda)$ design for some integer f as in the statement, and that $N_G(P)$ induces a point-transitive automorphism group.

By condition (a), P is non-trivial. Since P is a Sylow p-subgroup of K, each G-conjugate of P is also a Sylow p-subgroup of K and hence is K-conjugate to P. It follows from this that $G = N_G(P)K$. Thus, as K is the kernel of the G-action on $\mathcal{C}$, $N_G(P)^{\mathcal{C}} = G^{\mathcal{C}} = D$, and in particular $N_G(P)$ is transitive on $\mathcal{C}$.

Now $P^{\Delta} \neq 1$ for some $\Delta \in \mathcal{C}$, since $P \neq 1$. By condition (b), the set $\mathrm{fix}_{\Delta}(P)$ of fixed points of P in Δ is non-empty, and is a proper subset of Δ since $P^{\Delta} \neq 1$. Hence $f := |\mathrm{fix}_{\Delta}(P)|$ satisfies $0 < f < c$ and $c \equiv f \pmod{p}$. Let $\alpha \in \mathrm{fix}_{\Delta}(P)$ and note that P is a Sylow p-subgroup of K_{α}. If $g \in G$ is such that $P^g \leqslant G_{\alpha}$, then P^g is contained in $K^g \cap G_{\alpha} = K_{\alpha}$ so P^g is also a Sylow p-subgroup of K_{α} and hence $P^g = P^x$ for some $x \in K_{\alpha}$. Thus P^g is conjugate to P in G_{α} and hence, by [16, Corollary 2.24], $N_G(P)$ is transitive on $\mathrm{fix}_{\mathcal{P}}(P)$. It follows that P fixes the same number f of points in each class of $\mathcal{C}$, and hence $|\mathcal{P}_0| = |\mathrm{fix}_{\mathcal{P}}(P)| = df$.

Let $\alpha, \beta \in \mathrm{fix}_{\mathcal{P}}(P)$ with $\alpha \neq \beta$, and let B be a block containing $\{\alpha, \beta\}$. By Proposition 2.5(b), P fixes B setwise. Hence P fixes setwise each non-trivial intersection $B \cap \Delta'$, for $\Delta' \in \mathcal{C}$, and each such intersection has size ℓ. Now $\lambda \geqslant \ell$ by Lemma 2.1(iv), and therefore, by condition (c), $p > \ell$. Thus each non-trivial intersection $B \cap \Delta'$ is fixed pointwise by P, and hence $f \geqslant \ell$ and P fixes B pointwise, that is, $B \subseteq \mathrm{fix}_{\mathcal{P}}(P)$. This implies that each of the λ blocks containing $\{\alpha, \beta\}$ lies in $\mathcal{B}_0$, and it follows that $(\mathcal{P}_0, \mathcal{B}_0)$ is a $2-(df, k, \lambda)$ design, and hence a complete subdesign of $\mathcal{D}$, admitting $N_G(P)$ acting as a point-transitive automorphism group. In particular $k-1$ divides $\lambda(df-1)$, see for example [7, Lemma 4(i)].

It remains to prove that $f > \ell$. Suppose to the contrary that $f = \ell$. Then whenever $B \cap \Delta \neq \varnothing$ we have $B \cap \Delta = \mathrm{fix}_{\Delta}(P)$ of size ℓ. Thus each block B containing at least two points of $\mathrm{fix}_{\mathcal{P}}(P)$ is the disjoint union of the subsets $\mathrm{fix}_{\Delta}(P)$ over all classes $\Delta \in \mathcal{C}$ such that $B \cap \Delta \neq \varnothing$. Fix $\Delta \in \mathcal{C}$, choose distinct points $\alpha, \beta \in \mathrm{fix}_{\Delta}(P)$, and let N be the number of pairs (B, Δ') such that $B \in \mathcal{B}$, $\Delta' \in \mathcal{C} \setminus \{\Delta\}$, and $B \cap \Delta$, $B \cap \Delta'$ are both non-empty. The blocks B occurring are precisely the λ blocks containing $\{\alpha, \beta\}$ and each such block B occurs in pairs (B, Δ') for exactly $k/\ell - 1$ classes Δ', so $N = \lambda(k/\ell - 1)$. On the other hand, for each of the $d-1$ classes $\Delta' \neq \Delta$, choosing $\gamma \in \mathrm{fix}_{\Delta'}(P)$, we see that Δ' occurs in pairs (B, Δ') for precisely the λ blocks B containing $\{\alpha, \gamma\}$, so $N = \lambda(d-1)$. We conclude that $k/\ell = d$, However, this implies that each of the λ blocks containing $\{\alpha, \beta\}$ is equal to $\mathrm{fix}_{\mathcal{P}}(P)$ which, in turn, implies that $\lambda = 1$, contradicting [9]. Thus $f > \ell$. ☐

There are also flag-transitive subdesigns (not complete ones) and 'quotient designs' arising from flag-transitive, point-imprimitive designs, see Proposition 2.8 below and the design construction in [2, Construction 3.1]. Their natural definition involves the possibility of 'repeated blocks', that is distinct blocks incident with exactly the same subset of points, so we use a more formal incidence structure in their definition. Given the assumptions in Hypothesis 1.2, let $\Delta \in \mathcal{C}$, and define $\mathcal{D}(\Delta) = (\Delta, \mathcal{B}(\Delta), \mathcal{I}(\Delta))$, where

$$\mathcal{B}(\Delta) = \{B \cap \Delta \mid B \in \mathcal{B}, B \cap \Delta \neq \varnothing\}$$
$$\mathcal{I}(\Delta) = \{(\alpha, B \cap \Delta) \mid \alpha \in B \cap \Delta, B \cap \Delta \in \mathcal{B}(\Delta)\},$$

and $\mathcal{D}(\mathcal{C}) = (\mathcal{C}, \mathcal{C}(\mathcal{B}), \mathcal{I}(\mathcal{C}))$, with $\mathcal{C}(\mathcal{B})$ the set of all $\mathcal{C}(B)$ for $B \in \mathcal{B}$, where

$$\mathcal{C}(B) = \{\Delta \in \mathcal{C} \mid B \cap \Delta \neq \varnothing\}, \text{ and } \mathcal{I}(\mathcal{C}) = \{(\Delta, \mathcal{C}(B)) \mid \Delta \in \mathcal{C}(B)\}.$$

The notion of a 2-design carries over to these point-block incidence structures (each pair of points incident with the same number of blocks), and automorphisms are permutations of the point set and the block set which preserve the incidence relation. The group G_Δ or G acts as a flag-transitive group of automorphisms on $\mathcal{D}(\Delta)$ or $\mathcal{D}(\mathcal{C})$, respectively. This means in particular that, for each design, each block occurs with the same *block-multiplicity* (the number of blocks incident with the same subset of points). In the following proposition, part (a) was proved in both [12, Propositions 6 and 8] and [13, Corollary 2.2 and Theorem 2.3]; and part (b) is proved in [12, Propositions 7 and 9] (and in both parts the block multiplicities may be greater than 1). In both papers these observations were applied to strengthen the results of [17] (see [12, Propositions 12 and 9] and [13, Theorem 2.4]). Note that although the applications in [12, 13] are to symmetric designs, Proposition 2.8 is valid for all flag-transitive point-imprimitive designs.

Proposition 2.8. *Assume that Hypothesis* 1.2 *holds, let* $\Delta \in \mathcal{C}$*, and consider the incidence structures* $\mathcal{D}(\Delta) = (\Delta, \mathcal{B}(\Delta), \mathcal{I}(\Delta))$ *and* $\mathcal{D}(\mathcal{C}) = (\mathcal{C}, \mathcal{C}(\mathcal{B}), \mathcal{I}(\mathcal{C}))$ *as defined above. Then*

(a) $\mathcal{D}(\Delta)$ *is a* $2-(c, \ell, \lambda)$ *design with* $L = G_\Delta^\Delta$ *acting flag-transitively, and the block multiplicity* θ *equals* $\theta = |G_{\alpha, B\cap\Delta} : G_{\alpha,B}|$ *and divides* λ*, where* $B \cap \Delta \in \mathcal{B}(\Delta)$ *and* $\alpha \in B \cap \Delta$*;*

(b) $\mathcal{D}(\mathcal{C})$ *is a* $2 - (d, k/\ell, c^2\lambda/\ell^2)$ *design with* $D = G^{\mathcal{C}}$ *acting flag-transitively.*

3. Ruling out designs with λ small

In this section we assume that Hypothesis 1.2 holds with $\lambda = 3$ or 4 and $v \geqslant 100$. As mentioned in the introduction, a list of all feasible parameters satisfying these conditions was compiled by the authors in [7]. Unfortunately two cells in the table of [7, Proposition 8] which recorded these parameters contained errors – they were typos which did not match the (correct) proof given for the result. We record formally the details in Remark 3.1.

Remark 3.1. In [7, Proposition 8], it was shown that the parameter tuple $(\lambda, v, k, r, b, c, d, \ell)$ must belong to a list of 18 possibilities. Unfortunately there were two mistakes in that table: the parameter tuple $(4, 435, 32, 42, 15, 29, 2)$ should have read $(4, 435, 32, 56, 15, 29, 2)$ and should have been excluded because $b = v \cdot r/k$ is not an integer (see Lemma 2.1(i)). Secondly the tuple $(\lambda, v, k, r, c, d, \ell) = (4, 196, 16, 42, 14, 14, 2)$ should have read $(4, 196, 16, 52, 14, 14, 2)$. In addition, one of the remaining tuples does not satisfy Lemma 2.1(iii), namely the tuple $(\lambda, v, k, r, c, d, \ell) = (3, 561, 36, 48, 17, 33, 2)$.

Line	λ	v	k	r	b	c	d	ℓ	Result
1	3	100	12	27	$3^2 \cdot 5^2$	10	10	2	Lemma 3.10
2	3	120	18	21	$2^2 \cdot 5 \cdot 7$	8	15	2	Lemma 3.7
3	3	120	18	21	$2^2 \cdot 5 \cdot 7$	15	8	3	Lemma 3.7
4	3	256	18	45	$2^7 \cdot 5$	16	16	2	Lemma 3.5
5	3	561	36	48	$2^2 \cdot 11 \cdot 17$	33	17	3	Lemma 3.6
6	3	1156	36	99	$11 \cdot 17^2$	34	34	2	Lemma 3.4
7	4	100	12	36	$2^2 \cdot 3 \cdot 5^2$	10	10	2	Lemma 3.10
8	4	196	16	52	$7^2 \cdot 13$	14	14	2	Lemma 3.5
9	4	231	24	40	$5 \cdot 7 \cdot 11$	11	21	2	Lemma 3.9
10	4	231	24	40	$5 \cdot 7 \cdot 11$	21	11	3	Lemma 3.5
11	4	280	32	36	$3^2 \cdot 5 \cdot 7$	10	28	2	Lemma 3.10
12	4	280	32	36	$3^2 \cdot 5 \cdot 7$	28	10	4	Lemma 3.11
13	4	484	24	84	$2 \cdot 7 \cdot 11^2$	22	22	2	Lemma 3.5
14	4	1976	80	100	$2 \cdot 5 \cdot 13 \cdot 19$	26	76	2	Lemma 3.4
15	4	1976	80	100	$2 \cdot 5 \cdot 13 \cdot 19$	76	26	4	Lemma 3.6
16	4	2116	48	180	$3 \cdot 5 \cdot 23^2$	46	46	2	Lemma 3.4

Table 2. Remaining parameter sets for flag-transitive imprimitive designs with $\lambda \leqslant 4$

Thus we have 16 parameter tuples to consider, and we list these in Table 2. We will apply the results from Section 2 to show that there are no designs for any of these parameter tuples. In the last column of Table 2 we record the results in this section which rule out each of these parameter tuples. In our statements, Line numbers refer to Table 2. Moreover, in our proofs we frequently mention use of the computational algebra package Magma [1], and we comment briefly in Remark 3.2 on the main ways in which this is used.

Remark 3.2. We make some comments here on our use of computation in analysing the possibilities for flag-transitive $2-(v,k,\lambda)$ designs, for some specified parameters v,k,λ, noting that these parameters determine the number b of blocks since $bk(k-1) = v(v-1)\lambda$.

(a) At many stages in our proofs we have information about a primitive permutation group of a given degree (usually c or d where $v = cd$), and we use the library of primitive groups in Magma [3] to determine a list of possibilities for the group with the required properties. Much of this information could equally be obtained by consulting [8, Table B.4].

(b) Occasionally in our proofs we have specified the point set $\mathcal{P}$ of size v, and a transitive permutation group $G \leqslant \mathrm{Sym}(\mathcal{P})$ as a candidate for the flag-transitive group G. Moreover we have specified a candidate subgroup H of index b in G for the stabiliser $H = G_B$ of a block B. Since G should be flag-transitive, the block B should be an H-orbit in $\mathcal{P}$ of length k. We use Magma [1] to identify all H-orbits of length k in $\mathcal{P}$. These will be the possibilities for the block B left invariant by H. If the set $\mathcal{B}$ of G-images of such a subset B forms the block set of a 2-design, then the group G will act flag-transitively on this design, giving an example. To determine whether this is the case for a given H-orbit B of length k, we need to confirm whether or not each pair of points is contained in exactly λ blocks in $\mathcal{B}$. Since G is transitive on $\mathcal{P}$ it is sufficient to test this for point-pairs $\{\alpha,\beta\}$, where $\alpha \in \mathcal{P}$ is a fixed chosen point, and β runs over a set of representative points, one from each G_α-orbit in $\mathcal{P} \setminus \{\alpha\}$.This check is carried out computationally using [1].

Line	$d-p$	k/ℓ	p	b
6	3	18	31	$11 \cdot 17^2$
14	3	40	73	$2 \cdot 5 \cdot 13 \cdot 19$
16	3	24	43	$3 \cdot 5 \cdot 23^2$

TABLE 3. Table for the proof of Lemma 3.4

Line	d	k/ℓ	b
10	11	8	$5 \cdot 7 \cdot 11$
13	22	12	$2 \cdot 7 \cdot 11^2$

TABLE 4. Table for the proof of Lemma 3.5

First we observe that in each Line the groups L, D are primitive permutation groups.

LEMMA 3.3. *If Hypothesis* 1.2 *is satisfied with* $\lambda \in \{3, 4\}$*, then* D *and* L *are primitive.*

Proof. Note that, in Table 2, there is no pair of parameter tuples with the same (λ, v, k) and with the class size c in one tuple a multiple of the class size in the other. It follows that L is primitive of degree c, and also D is primitive of degree d. ☐

Proposition 2.2 directly allows us to rule out three Lines of Table 2.

LEMMA 3.4. *There are no examples with parameter tuple in Line* 6, 14 *or* 16 *of Table* 2.

Proof. Assume that Hypothesis 1.2 holds with parameter tuple as in one of Lines 6, 14 or 16 of Table 2. By Lemma 3.3, D is primitive of degree $d = 34, 76, 46$ respectively. Checking the possibilities for such groups using Magma [3], we find that $D = A_d$ or S_d. Let p be the largest prime less than d. Then p divides $|D|$ and also p satisfies conditions (i) and (ii) of Proposition 2.2 (see details in Table 3). This contradicts Proposition 2.2. Thus there are no examples.

☐

Next we use Proposition 2.3 to rule out three more Lines of Table 2.

LEMMA 3.5. *There are no examples with parameter tuple in Line 4, 8, 10 or 13 of Table* 2.

Proof. The argument for Lines 10 and 13 requires only the first assertion of Proposition 2.3, so we deal with these Lines first. The parameters $d, k/\ell, b$ in Lines 10 and 13 are given in Table 4. There are eight primitive groups of degree 11 (Line 10), and four primitive groups of degree 22 (Line 13). An exhaustive search (with Magma [3]) of all subgroups of index dividing b for each of these primitive groups shows that none has an orbit of size k/ℓ. This contradicts Proposition 2.3.

Now we consider Line 4 of Table 2, where we have $(c, d, \ell, k/\ell) = (16, 16, 2, 9)$, and $b = 2^7 \cdot 5 = 640$. In particular 5 divides $|G|$ since G is block-transitive. Note that $p = 5 > \lambda = 3$, and that p does not divide c. Suppose first that 5 divides $|K|$, so that Proposition 2.7 applies with $p = 5$. Then there exists an integer f such that $\ell < f < c$, $c \equiv f \pmod{p}$, and $k - 1$ divides $\lambda(df - 1)$, which is a contradiction. Thus $|K|$ is not divisible by 5. Since 5 divides $|G| = |D| \cdot |K|$, it follows that 5 divides $|D|$. There are 18 primitive groups D of degree $d = 16$ with order divisible by 5, so D must be one of these by Lemma 3.3. Let $X = (G_B)^{\mathcal{C}} \leqslant D$. By Proposition 2.3, X has an orbit of size 9.

If $K \neq 1$ then, since L is primitive, the K-orbits in $\mathcal{P}$ have size $c = 16$, so $|D : X|$ divides $b/8 = 2^4 \cdot 5 = 80$ by Proposition 2.3. However, an exhaustive search using Magma [3] shows that none of the 18 possibilities for D has a subgroup of index dividing 80 with an orbit of size 9. Thus $K = 1$, so $G \cong D$ has order divisible by the number $f = bk = 640 \cdot 18$ of flags, and the index $|D : X| = 640$. Again a computer search using Magma [3] shows that the only primitive group D of degree $d = 16$ such that $|D|$ is divisible by f, and D has a subgroup of index 640 with an orbit of size 9, is the affine group $[2^4] : S_6$. Thus $G = [2^4] : S_6$ and $G_\Delta = S_6$. However G_α is a subgroup of G_Δ with index $c = 16$, and S_6 has no such subgroup, which is a contradiction.

Finally we consider Line 8 of Table 2, where we have $(c, d, \ell, k/\ell) = (14, 14, 2, 8)$, and $b = 7^2 \cdot 13 = 637$. Suppose that 13 divides $|K|$. Since 13 does not divide $c = 14$ and $13 > \lambda = 4$, we may apply Proposition 2.7 with $p = 13$. So there exists an integer f such that $2 = \ell < f < c$ and $c = 14 \equiv f \pmod{13}$, which is a contradiction. Thus $|K|$ is not divisible by 13. Now L is primitive of degree $c = 14$ by Lemma 3.3, and so L is one of

$$\mathrm{PSL}(2, 13), \mathrm{PGL}(2, 13), A_{14}, S_{14}. \tag{2}$$

Since $K^\Delta \lhd L$ and all non-trivial normal subgroups of all these possible groups L have order divisible by 13, we conclude that $K^\Delta = 1$. Thus the length of the K-orbits in $\mathcal{P}$ is $c_0 = 1$, and hence, by Proposition 2.3, D has a subgroup X with index $b = 7^2 \cdot 13$. Now D is primitive of degree $d = 14$, by Lemma 3.3, and so D is also one of the groups in (2). However, none of these groups has a subgroup of index b, which is a contradiction. □

We rule out two more Lines of Table 2 with Proposition 2.4.

Lemma 3.6. *There are no examples with parameter tuple in Line 5 or 15 of Table* 2.

Proof. In Line 5, $\ell = 3$, and L is primitive of degree $c = 33$ by Lemma 3.3, but L is not 3-transitive by Proposition 2.4, since $c = 33 > 2+(\ell-2)\lambda = 5$. However each primitive group of degree $c = 33$, namely $\mathrm{PSL}(2, 32), \mathrm{P\Gamma L}(2, 32), A_{33}$, or S_{33}, is 3-transitive. In Line 15, $\ell = 4$, $c = 76$, and the only primitive groups of degree c are A_{76} and S_{76}, so L is 3-transitive, contradicting Proposition 2.4, since $c = 76 > 2 + (\ell - 2)\lambda = 10$. □

In the next lemma we rule out Lines 2 and 3 of Table 2 using a combination of Propositions 2.3 and 2.7.

Lemma 3.7. *There are no examples with parameter tuple in Line 2 or 3 of Table* 2.

Proof. In Line 2 or 3 we have $(c, d, \ell, k/\ell) = (8, 15, 2, 9)$ or $(15, 8, 3, 6)$, respectively, and $b = 2^2 \cdot 5 \cdot 7 = 140$. In particular 7 divides $|G|$. Note that $p = 7 > \lambda = 3$, and that p does not divide c. If 7 divides $|K|$, then Proposition 2.7 applies with $p = 7$, so there exists an integer f such that $\ell < f < c$, $c \equiv f \pmod{p}$, and $k - 1$ divides $\lambda(df - 1)$. However there is no such integer in either case. Hence $|K|$ is not divisible by 7. Since 7 divides $|G| = |D| \cdot |K|$, it follows that 7 divides $|D|$.

First consider Line 2. By Lemma 3.3, D is a primitive group of degree $d = 15$, and also $|D|$ is divisible by 7. There are four such groups, namely A_7, $\mathrm{PSL}(4, 2)$, A_{15} and S_{15}, so D is one of these. Let $X = (G_B)^{\mathcal{C}} \leqslant D$. By Proposition 2.3, X has an orbit of size 9. If $K \neq 1$ then, since L is primitive, the K-orbits in $\mathcal{P}$ have size $c = 8$, so $|D : X|$ divides $140/4 = 35$ by Proposition 2.3. However, an exhaustive search using Magma [3] shows that none of the four possibilities for D has a subgroup of index dividing 35 with an orbit of size 9. Thus $K = 1$ and $|D : X| = 140$. Again a computer search shows that the only primitive group of degree $d = 15$ that has a subgroup of index 140 with an orbit of size 9 is A_7, so $D = A_7$.

Thus $G \cong D = A_7$, $G_\Delta = \mathrm{PSL}(2, 7)$ and, for $\alpha \in \Delta$, G_α is a Frobenius subgroup F_{21} of order 21. Now A_7 has a unique conjugacy class of subgroups F_{21}, and we may identify the point-set $\mathcal{P}$ with the set of right cosets of such a subgroup G_α. A block stabiliser G_B is a subgroup of index 140 with an orbit of size $k = 18$ on points. Using

Line	λ	k	b	c	d	ℓ	possible groups D
1	3	12	$3^2 \cdot 5^2$	10	10	2	$A_5, S_5, \mathrm{PSL}(2,9), S_6$ (out of 9)
7	4	12	$2^2 \cdot 3 \cdot 5^2$	10	10	2	$A_5, S_5, \mathrm{PSL}(2,9), S_6, \mathrm{PGL}(2,9), M_{10}, \mathrm{P\Gamma L}(2,9)$ (out of 9)
9	4	24	$5 \cdot 7 \cdot 11$	11	21	2	A_7, S_7 (out of 9)
11	4	32	$3^2 \cdot 5 \cdot 7$	10	28	2	$\mathrm{PSU}(3,3), \mathrm{P\Gamma U}(3,3), \mathrm{PSp}(6,2)$ (out of 14)
12	4	32	$3^2 \cdot 5 \cdot 7$	28	10	4	$\mathrm{PGL}(2,9), M_{10}, \mathrm{P\Gamma L}(2,9), A_{10}, S_{10}$ (out of 9)

TABLE 5. Possibilities for the primitive group D

[1], we checked that there is a unique conjugacy class of subgroups of $G = A_7$ of index 140 and that a subgroup G_B in this class has six orbits of length 18 on points. However, a further computation with [1] showed that, for each of these 18-subsets B, the set of G-images of B is not the block-set of a 2-design.

Consider now Line 3. As we showed above, $|D|$ is divisible by 7 while $|K|$ is not divisible by 7. Thus a Sylow 7-subgroup P of G is non-trivial and $P \cong P^{\mathcal{C}}$, a Sylow 7-subgroup of D. Since $d = 8$, P fixes one class $\Delta \in \mathcal{C}$ and acts transitively on the other 7 classes. Suppose that 7 does not divide $|L|$. Then P fixes Δ pointwise and, by Proposition 2.5(b), P fixes setwise each block intersecting Δ nontrivially. Let B be such a block. Then P fixes setwise the 6-subset $\mathcal{C}(B)$ of $\mathcal{C}$ (defined in Proposition 2.5), a contradiction. Hence 7 divides $|L|$.

Thus L is a primitive subgroup of S_{15} with order divisible by 7, and L is not 3-transitive by Proposition 2.4(b), since $c = 15 > 2 + (\ell - 2)\lambda = 5$. Checking with [3], we see that L must be either A_7 or $\mathrm{PSL}(4,2)$, and in particular L is simple. Since $K^\Delta \lhd L$ and 7 does not divide $|K|$, it follows that $K^\Delta = 1$ and hence $K = 1$. Therefore $G \cong D \leqslant S_8$. Thus $8! \geqslant |G| = 8 \cdot |G_\Delta| \geqslant 8 \cdot |L|$ and it follows that $L = A_7$ and $G \cong D = A_8$ or S_8. By Proposition 2.3, D has subgroup X of index $b = 140$ such that X has an orbit of size 6 in $\mathcal{C}$. Thus X is contained in the setwise stabiliser in D of a 6-element subset (which is isomorphic to S_6 or $S_6 \times S_2$), and hence $|D : X|$ is divisible by $|S_8 : S_6 \times S_2| = |A_8 : S_6| = 28$, which is a contradiction. □

There are just five lines of Table 2 still to be resolved, namely Lines 1, 7, 9, 11, 12. We consider these together to obtain restrictions on the groups D and K. For a group H and prime p, $O_p(H)$ denotes the largest normal p-subgroup of H.

LEMMA 3.8. *Assume that Hypothesis* 1.2 *holds, and that the parameters are as in one of the Lines 1, 7, 9, 11, 12 of Table* 2. *Then*

- (a) *the kernel K of the G-action on $\mathcal{C}$ is nontrivial;*
- (b) *the group $D = G^{\mathcal{C}}$ is one of those listed in Table* 5*;*
- (c) *either K^Δ is primitive, or Line 12 holds with $L = \mathrm{PGL}(2,7)$ and $K^\Delta = \mathrm{PSL}(2,7)$;*
- (d) *K acts faithfully on each $\Delta \in \mathcal{C}$.*

Proof. (a) In order to prove that $K \neq 1$, we assume to the contrary that $K = 1$. Then G is isomorphic to D, a primitive group of degree d. The fact that the number of flags $vr = bk$ must divide $|D|$ implies that $D = A_d$ or S_d for Lines 1, 7, 9. Applying Proposition 2.2 with the prime $p = 7$ for Lines 1 and 7, or $p = 13$ for Line 9 leads to contradictions. We deal with Lines 11, 12 separately, considering the stabilisers G_Δ and G_α, for $\alpha \in \Delta$.

In Line 12, $G \cong D$, a primitive subgroup of S_{10} of order divisible by $bk = 2^5.3^2.5.7$, and this implies that $G = A_{10}$ or S_{10}. Thus G_Δ is A_9 or S_9, respectively, and G_Δ has a subgroup G_α of index $c = 28$, but neither A_9 nor S_9 has such a subgroup.

Finally, in Line 11, $G \cong D$ is a primitive subgroup of S_{28} of order divisible by $2^5.3^2.5.7$ but not by $p = 17$ (by Proposition 2.2). Hence, using [3] we see that $G = A_8$,

S_8 or $\mathrm{PSp}(6,2)$, and so $G_\Delta = S_6$, $S_6\times 2$, or $\mathrm{PSU}(4,2):2$, respectively (in each case there is a unique conjugacy class of these subgroups of index $d=28$), and G_Δ has a subgroup G_α of index $c=10$. However in the third case there is no such subgroup, so $G = A_8$ or S_8, and $G_\alpha = 3^2 : D_8$ or $(3^2 : D_8)\times 2$, respectively, (again unique up to conjugacy), and we identify the point set with the set of right cosets of G_α in G. Also a block stabiliser G_B lies in a unique conjugacy class of subgroups of index $b=315$, and a computation with [1] shows that G_B has 3 or 2 orbits of length $k=32$ on points, according as $D=A_8$ or S_8 respectively. However a further computation with [1] (as described in Remark 3.2) shows that, for each of these 32-subsets X, the set of G-images of X is not the block-set of a 2-design. Thus in all cases we conclude that $K\neq 1$.

(b) Since L is primitive, the K-orbits on points have size c. We computationally exploit Proposition 2.3(b) with the appropriate $x := c/\gcd(c,\ell)$ to find the possibilities for D listed in Table 5; in each line of Table 5 we record the total number of primitive groups of degree d (to indicate how many have been eliminated by these restrictions).

(c) By Lemma 3.3, L is primitive. Since $K^\Delta\neq 1$, it follows that K^Δ is transitive and contains $\mathrm{Soc}(L)$. For $c=10$ or $c=11$, all primitive groups of degree c have a primitive socle, so K^Δ is primitive in Lines 1, 7, 9, 11. In Line 12, we have $c=28$, and there is a single primitive group of degree c with an imprimitive socle, namely $\mathrm{PGL}(2,7)$. So K^Δ is primitive also in Line 12, except if $L=\mathrm{PGL}(2,7)$ and $K^\Delta=\mathrm{PSL}(2,7)$.

(d) Note that the group G permutes the set $\{K_{(\Delta)} \mid \Delta\in\mathcal{C}\}$ by conjugation. For $\Delta'\neq\Delta$, since $K^{\Delta'}$ is transitive and $(K_{(\Delta)})^{\Delta'}\trianglelefteq K^{\Delta'}$, it follows that either $(K_{(\Delta)})^{\Delta'}\neq 1$ with equal length orbits, or $K_{(\Delta)}$ fixes Δ' pointwise, and in the latter case $K_{(\Delta)} = K_{(\Delta')}$ (since these subgroups are G-conjugate). For a given Δ, the subset $\{\Delta'\in\mathcal{C} \mid K_{(\Delta)} = K_{(\Delta')}\}$ is a block of imprimitivity for $G^{\mathcal{C}} = D$ containing Δ, and since D is primitive, it follows that either the subgroups $K_{(\Delta)}$ are all equal and hence are trivial $K_{(\Delta)}=1$, or the subgroups $K_{(\Delta)}$ are pairwise distinct and the set of fixed points of $K_{(\Delta)}$ is precisely Δ. Thus it is sufficient to prove that $K_{(\Delta)}=1$ for some $\Delta\in\mathcal{C}$.

Suppose, in order to derive a contradiction, that $K_{(\Delta)}\neq 1$. Then as explained above, for all $\Delta'\neq\Delta$, the normal subgroup $K^{\Delta'}_{(\Delta)}$ of $K^{\Delta'}$ is nontrivial. Since $K^{\Delta'}$ is either primitive or a nonabelian simple group, by part (c), this implies that $K^{\Delta'}_{(\Delta)}$ is transitive. This leads to a contradiction as follows: each pair $\{\alpha,\beta\}\subseteq\Delta$ is fixed by $K_{(\Delta)}$, and hence $K_{(\Delta)}$ leaves invariant the set of λ blocks containing $\{\alpha,\beta\}$. Let $\Delta'\neq\Delta$ be a class intersecting non-trivially at least one of these blocks. Then $K_{(\Delta)}$ leaves invariant the (non-empty) union of the subsets $B\cap\Delta'$ over the λ blocks B containing $\{\alpha,\beta\}$. This non-empty union is a subset of size at most $\lambda\ell < c = |\Delta'|$, and this contradicts the fact that $K^{\Delta'}_{(\Delta)}$ is transitive. Hence $K_{(\Delta)}=1$, and part (d) is proved. □

We now give ad hoc arguments, involving many of the results from Section 2, to deal with the five remaining Lines of Table 2, namely the Lines of Table 5.

Lemma 3.9. *There are no examples with parameter tuple in Line 9 of Table 2.*

Proof. In Line 9 we have $(c,d,\ell) = (11,21,2)$ and $\lambda=4$. Note that $p=5>\lambda$ and 5 does not divide c. If 5 divides $|K|$, then by Proposition 2.7, there exists an integer f such that $2<f<11$, $f\equiv 11\equiv 1 \pmod 5$, and $k-1=23$ divides $\lambda(df-1)$, which is a contradiction. Hence 5 does not divide $|K|$. Now by Lemma 3.8, $K\cong K^\Delta$ is a non-trivial normal subgroup of L, so K^Δ contains the socle of L. Moreover, by Proposition 2.4(a), L is 2-transitive of degree $c=11$. The socles of such groups are

C_{11}, $\mathrm{PSL}(2,11)$, M_{11}, or A_{11}, and the only one with order coprime to 5 is C_{11}. Thus $C_{11} \trianglelefteq K^{\Delta} \trianglelefteq L = \mathrm{AGL}(1,11)$. Since 5 does not divide $|K|$, we have $K \cong K^{\Delta} = C_{11}$ or $C_{11} \rtimes C_2$.

Let $K_0 := O_{11}(K) \cong C_{11}$, and let $C := C_G(K_0)$. By Lemma 3.8(b), $D = G/K$ is A_7 or S_7. Since the conjugation action of G on K_0 induces a subgroup of $\mathrm{Aut}(K_0) \cong C_{10}$, it follows that $C^{\mathcal{C}} = CK/K$ contains A_7. Thus $|G : CK| \leqslant 2$, and as $C \cap K = K_0$, we also have $|CK : C| = |K : C \cap K| \leqslant 2$. Hence $|G : C|$ divides 4. However this is a contradiction since, on the one hand, G/C is isomorphic to the subgroup of $\mathrm{Aut}(K_0)$ induced by G, and on the other hand, $G^{\Delta}_{\Delta} = \mathrm{AGL}(1,11)$ induces the whole group $\mathrm{Aut}(K_0) \cong C_{10}$ on K_0. □

Lemma 3.10. *There are no examples with parameter tuple in Line* 1, 7 *or* 11 *of Table* 2.

Proof. In these Lines, $(c,\ell) = (10,2)$, and the triple (λ,d,k) is $(3,10,12)$, $(4,10,12)$ or $(4,28,32)$ in Line 1, 7 or 11, respectively. By Lemma 3.8, for a class $\Delta \in \mathcal{C}$, $K^{\Delta} \cong K$ is primitive of degree $c = 10$, and the possibilities for $\mathrm{Soc}(K)$ are A_5, A_6 and A_{10}. Also $\mathrm{Soc}(L) \trianglelefteq K^{\Delta} \trianglelefteq L$, and L is 2-transitive by Proposition 2.4(a), so only A_6 and A_{10} are possible for $\mathrm{Soc}(K)$. In particular $\mathrm{Soc}(K^{\Delta})$ is 2-transitive. Let P be a Sylow 3-subgroup of K. Since K^{Δ} is 2-transitive, it follows that P^{Δ} has orbits of lengths $1, 9$. It follows that P fixes a unique point in each of the classes of $\mathcal{C}$. Let α, β be distinct points fixed by P with $\alpha \in \Delta$. Suppose first that $\lambda = 4$. Then by Proposition 2.5, since $\lambda \equiv 1 \pmod 3$, P fixes setwise at least one block containing $\{\alpha,\beta\}$, say P fixes B setwise. This implies that P fixes setwise the ℓ-subset $B \cap \Delta$, which is a contradiction since $\ell = 2$ and P fixes a unique point of Δ. Thus we are in Line 1 with $\lambda = 3$, and from the argument just given we may assume that P fixes none of the three blocks B_1, B_2, B_3 containing $\{\alpha,\beta\}$. Thus P permutes these three blocks cyclically. Noting that $\ell = 2$, let $B_i \cap \Delta = \{\alpha, \alpha_i\}$. Then P fixes $\{\alpha_1,\alpha_2,\alpha_3\}$ setwise, contradicting the fact that P^{Δ} has orbits of lengths $1, 9$. This contradiction completes the proof. □

Lemma 3.11. *There are no examples with parameter tuple in Line 12 of Table* 2.

Proof. Here $(c,d,\lambda,\ell,k) = (28,10,4,4,32)$ and the number of blocks is $b = 3^2 \cdot 5 \cdot 7$. By Lemma 3.3, the groups L, D are primitive of degree c, d respectively. By Proposition 2.4(b), for distinct $\alpha, \beta \in \Delta$, the setwise stabiliser $L_{\{\alpha,\beta\}}$ has an orbit in $\Delta \setminus \{\alpha,\beta\}$ of size at most $(\ell-2)\lambda = 8$. Checking with [3] we find that, of the 14 primitive groups L of degree 28, this property holds for only 7 of them. Thus L is one of

$$\mathrm{PGL}(2,7), \mathrm{PSL}(2,8), \mathrm{P\Gamma L}(2,8), \mathrm{PSU}(3,3), \mathrm{P\Gamma U}(3,3), A_8, S_8.$$

In all cases $\mathrm{Soc}(L)$ is a nonabelian simple group, and is not regular, and the centraliser $C_{\mathrm{Sym}(\Delta)}(\mathrm{Soc}(L)) = 1$. Also, by Lemma 3.8, $K^{\Delta} \cong K \neq 1$ and hence $\mathrm{Soc}(L) \leqslant K^{\Delta} \trianglelefteq L$.

Let $C := C_G(K)$. Then $C \cap K \leqslant \prod_{\Delta \in \mathcal{C}} C_{\mathrm{Sym}(\Delta)}(K^{\Delta})$, and it follows that $C \cap K = 1$. Now the conjugation action of G on K induces a subgroup of $\mathrm{Aut}(K)$ isomorphic to G/C, and the subgroup $CK/C \cong K$ induces the group of inner automorphisms of K. Hence G/CK is isomorphic to a section of the outer automorphism group of $\mathrm{Soc}(L)$, which in all cases is a group of order at most 3. Thus $C^{\mathcal{C}} = CK/K$ has index at most 3 in $D = G^{\mathcal{C}} = G/K$, and we note that $C^{\mathcal{C}} = CK/K \cong C$. It follows that $\mathrm{Soc}(D) \leqslant C^{\mathcal{C}} \trianglelefteq D$, and by Lemma 3.8(b), $\mathrm{Soc}(D) \in \{A_6, A_{10}\}$.

We claim that C has $c = 28$ orbits of length $d = 10$ in $\mathcal{P}$. Since $C \lhd G$, C has equal length orbits in $\mathcal{P}$, and since $C^{\mathcal{C}}$ contains $\mathrm{Soc}(D)$, $C^{\mathcal{C}}$ is transitive. Thus the length of the C-orbits is $10 \cdot c_0$, where c_0 is the (constant) length of the C_{Δ}-orbits in Δ. Now C^{Δ}_{Δ} is a normal subgroup of the primitive group G^{Δ}_{Δ} of degree $c = 28$. If $C^{\Delta}_{\Delta} \neq 1$ then C^{Δ}_{Δ} is transitive and in particular 7 divides $|C_{\Delta}|$, which divides $|D|$. Thus $\mathrm{Soc}(D) = A_{10}$, and hence $C_{\Delta} = A_9$ or S_9. Checking with Magma [1] we find that C_Δ has no subgroup of index 28. Thus $C_\Delta^\Delta = 1$, and this implies that the C-orbits have length $10 \cdot c_0 = 10$, proving the claim. The C-orbits form a G-invariant partition of $\mathcal{P}$ consisting of 28 classes of size 10. Therefore the design (if it existed) would also arise as a design with the parameters of Line 11. However there are no examples for Line 11 by Lemma 3.10. □

The collection of lemmas in this section together prove Theorem 1.1.

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