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Automorphisms of the double cover of a circulant graph of valency at most 7

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ABSTRACT A graph X is said to be *unstable* if the direct product $X \times K_2$ (also called the canonical double cover of X) has automorphisms that do not come from automorphisms of its factors X and K_2 . It is *nontrivially unstable* if it is unstable, connected, and non-bipartite, and no two distinct vertices of X have exactly the same neighbors.

We find all of the nontrivially unstable circulant graphs of valency at most 7. (They come in several infinite families.) We also show that the instability of each of these graphs is explained by theorems of Steve Wilson. This is best possible, because there is a nontrivially unstable circulant graph of valency 8 that does not satisfy the hypotheses of any of Wilson’s four instability theorems for circulant graphs.

1. INTRODUCTION

Let X be a circulant graph. (All graphs in this paper are finite, simple, and undirected.)

DEFINITION 1.1 ([15]). *The canonical bipartite double cover of X is the bipartite graph BX with $V(BX) = V(X) \times \{0, 1\}$, where*

$$(v, 0) \text{ is adjacent to } (w, 1) \text{ in } BX \iff v \text{ is adjacent to } w \text{ in } X.$$

Letting S_2 be the symmetric group on the 2-element set $\{0, 1\}$, it is clear that the direct product $\text{Aut } X \times S_2$ is a subgroup of $\text{Aut } BX$. We are interested in cases where this subgroup is proper:

DEFINITION 1.2 ([12, p. 160]). *If $\text{Aut } BX \neq \text{Aut } X \times S_2$, then X is unstable.*

It is easy to see (and well known) that if X is disconnected, or is bipartite, or has “twin” vertices (see Definition 2.5 below), then X is unstable (unless X is a bipartite graph with trivial automorphism group). The following definition rules out these trivial examples:

DEFINITION 1.3 (cf. [16, p. 360]). *If X is connected, nonbipartite, twin-free, and unstable, then X is nontrivially unstable.*

S. Wilson found the following interesting conditions that force a circulant graph to be unstable. (See Definition 2.1 for the definition of the “Cayley graph” notation $\text{Cay}(G, S)$.)

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THEOREM 1.4 (Wilson [16, Appendix A.1] (and [14, p. 156])). *Let $X = \text{Cay}(\mathbb{Z}_n, S)$ be a circulant graph, such that n is even. Let $S_e = S \cap 2\mathbb{Z}_n$ and $S_o = S \setminus S_e$. If any of the following conditions is true, then X is unstable.*

- (C.1) *There is a nonzero element h of $2\mathbb{Z}_n$, such that $h + S_e = S_e$.*
- (C.2') *n is divisible by 4, and there exists $h \in 1 + 2\mathbb{Z}_n$, such that*
 - (a) *$2h + S_o = S_o$, and*
 - (b) *for each $s \in S$, such that $s \equiv 0$ or $-h \pmod{4}$, we have $s + h \in S$.*
- (C.3') *There is a subgroup H of \mathbb{Z}_n , such that the set*

$$R = \{s \in S \mid s + H \not\subseteq S\},$$

is nonempty and has the property that if we let $d = \gcd(R \cup \{n\})$, then n/d is even, r/d is odd for every $r \in R$, and either $H \not\subseteq d\mathbb{Z}_n$ or $H \subseteq 2d\mathbb{Z}_n$.

- (C.4) *There exists $m \in \mathbb{Z}_n^\times$, such that $(n/2) + mS = S$.*

REMARK 1.5. As explained in [7, Rem. 3.14], the two statements (C.2') and (C.3') are slightly corrected versions of the original statements of Theorems C.2 and C.3 that appear in [16]. The correction (C.2') is due to Qin-Xia-Zhou [14, p. 156].

DEFINITION 1.6. *We say that X has Wilson type (C.1), (C.2'), (C.3'), or (C.4), respectively, if it satisfies the corresponding condition of Theorem 1.4.*

REMARK 1.7. The Wilson type of a graph need not be unique; i.e., a graph may satisfy more than one condition from Theorem 1.4. For example, for every odd integer k with $\gcd(k, 3) = 1$, the graph

$$\text{Cay}(\mathbb{Z}_{8k}, \{\pm 2k, \pm 3k\})$$

has Wilson type (C.1) (with $h = 4k$) as well as Wilson types (C.3') (with $H = \{0, 4k\}$, $R = \{\pm 3k\}$ and $d = k$) and (C.4) (with $m = 3$).

Additionally, the graph $\text{Cay}(\mathbb{Z}_{12}, \{\pm 1, \pm 2, \pm 4, \pm 5, 6\})$ has Wilson type (C.2') (with $h = 3$) and Wilson type (C.3') (with $H = \langle 3 \rangle$, $R = \{6\}$ and $d = 6$).

In this terminology (modulo the corrections mentioned in Remark 1.5), Wilson [16, p. 377] conjectured that every nontrivially unstable circulant graph has a Wilson type. Unfortunately, this is not true: other conditions that force a circulant graph to be unstable are described in [7, §3] (and these produce infinitely many counterexamples). Prior to the work in [7], the following counterexample (which is the smallest) had been published:

EXAMPLE 1.8 (Qin-Xia-Zhou [14, p. 156]). The circulant graph

$$\text{Cay}(\mathbb{Z}_{24}, \{\pm 2, \pm 3, \pm 8, \pm 9, \pm 10\})$$

is nontrivially unstable, but does not have a Wilson type.

The main result of this paper establishes that Wilson's conjecture is true for graphs of valency at most 7:

THEOREM 1.9. *Every nontrivially unstable circulant graph of valency at most 7 has Wilson type (C.1), (C.2'), (C.3'), or (C.4).*

We actually prove more precise (but more complicated) results, which show that all of the graphs in Theorem 1.9 belong to certain explicit families (and it is easy to see that the graphs in each family have a specific Wilson type).

The following example shows that the constant 7 in Theorem 1.9 cannot be increased:

EXAMPLE 1.10 ([7, Example 3.10]). Let $n := 3 \cdot 2^\ell$, where $\ell \geq 4$ is even, and let

$$S := \left\{ \pm 3, \pm 6, \pm \frac{n}{12}, \frac{n}{2} \pm 3 \right\}.$$

Then the circulant graph $X := \text{Cay}(\mathbb{Z}_n, S)$ has valency 8 and is nontrivially unstable, but does not have a Wilson type.

Here is an outline of the paper. After this introduction come two sections of preliminaries: Section 2 presents material from the theory of normal Cayley graphs and some other miscellaneous information that will be used; Section 3 lists some conditions that imply X is not unstable. The remaining sections each find the nontrivially unstable circulant graphs of a particular valency (or valencies). Namely, Section 4 considers valencies ≤ 4 , whereas Sections 5 to 7 each consider a single valency (5, 6, or 7, respectively). The main results are Proposition 4.2 (valency ≤ 3), Theorem 4.3 (valency 4), Theorem 5.1 (valency 5), Theorem 6.1 and Corollary 6.2 (valency 6), and Theorem 7.1 (valency 7).

2. PRELIMINARIES

For ease of reference, we repeat a basic assumption from the first paragraph of the introduction:

ASSUMPTION 1. *All graphs in this paper are finite, undirected, and simple (no loops or multiple edges).*

2.1. BASIC DEFINITIONS AND NOTATION. For simplicity (and because it is the only case we need), the following definition is restricted to abelian groups, even though the notions easily generalize to nonabelian groups.

DEFINITION 2.1. *Let S be a subset of an abelian group G , such that $-s \in S$ for all $s \in S$.*

- (1) *The Cayley graph $\text{Cay}(G, S)$ is the graph whose vertices are the elements of G , and with an edge from v to w if and only if $w = v + s$ for some $s \in S$ (cf. [10, §1]).*
- (2) *For $(g, 1) \in G \times \mathbb{Z}_2$, we let $\tilde{g} = (g, 1)$.*
- (3) *Note that if $X = \text{Cay}(G, S)$, and we let $\tilde{S} = \{\tilde{s} \mid s \in S\}$, then*

$$BX = \text{Cay}(G \times \mathbb{Z}_2, \tilde{S}).$$

- (4) *For $g \in G$, we say that an edge $\{u, v\}$ of the complete graph on $G \times \mathbb{Z}_2$ is a g -edge if $v = u \pm \tilde{g}$. Note that $\{u, v\}$ is an edge of BX if and only if it is an s -edge for some $s \in S$.*

NOTATION 2.2. For convenience, proofs will sometimes use the following abbreviation:

$$\# = n/2.$$

Besides the fairly standard notation from graph theory, we will employ the following:

NOTATION 2.3.

- (1) For $a \in \mathbb{Z}_n$, we use $|a|$ to denote the *order* of a as an element of the cyclic group \mathbb{Z}_n . So

$$|a| = \frac{n}{\gcd(n, a)}.$$

It does *not* denote the absolute value of a .

- (2) ϕ denotes the Euler's totient function.

Throughout the paper various notions of graph products will be used. We now recall their definitions and notation.

DEFINITION 2.4 ([5, pp. 35, 36, and 43]). *Let X and Y be graphs.*

- (1) *The direct product $X \times Y$ is the graph with $V(X \times Y) = V(X) \times V(Y)$, such that (x_1, y_1) is adjacent to (x_2, y_2) if and only if*

$$(x_1, x_2) \in E(X) \text{ and } (y_1, y_2) \in E(Y).$$

- (2) *The Cartesian product $X \square Y$ is the graph with $V(X \times Y) = V(X) \times V(Y)$, such that (x_1, y_1) is adjacent to (x_2, y_2) if and only if either*

- $x_1 = x_2$ and $(y_1, y_2) \in E(Y)$, or
- $y_1 = y_2$ and $(x_1, x_2) \in E(X)$.

- (3) *The wreath product $X \wr Y$ is the graph that is obtained by replacing each vertex of X with a copy of Y . (Vertices in two different copies of Y are adjacent in $X \wr Y$ if and only if the corresponding vertices of X are adjacent in X .) This is called the lexicographic product in [5, p. 43] (and denoted $X \circ Y$).*

DEFINITION 2.5 (Kotlov-Lovász [8]). *A graph X is twin-free if there do not exist two distinct vertices v and w , such that $N_X(v) = N_X(w)$, where $N_X(v)$ denotes the set of neighbors of v in X .*

The notion of a “block” (or “block of imprimitivity”) is a fundamental concept in the theory of permutation groups, but we need only the following special case:

DEFINITION 2.6 (cf. [2, pp. 12–13]). *Let G be a finite abelian group. Let $X = \text{Cay}(G, S)$ be a Cayley graph. A nonempty subset \mathcal{B} of $V(X)$ is a block for the action of $\text{Aut } X$ if, for every $\alpha \in \text{Aut } X$, we have*

$$\text{either } \alpha(\mathcal{B}) = \mathcal{B} \text{ or } \alpha(\mathcal{B}) \cap \mathcal{B} = \emptyset.$$

It is easy to see that this implies \mathcal{B} is a coset of some subgroup H of G . Then every coset of H is a block. Indeed, the action of $\text{Aut } X$ permutes these cosets, so there is a natural action of $\text{Aut } X$ on the set of cosets.

REMARK 2.7. The most important instance of Definition 2.6 for us will be the case of canonical bipartite double covers. Indeed, if $X = \text{Cay}(\mathbb{Z}_n, S)$ is a circulant graph then its canonical bipartite double cover $BX = \text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \tilde{S})$, defined in Definition 2.1(3), is a Cayley graph. Therefore, every block for the action of $\text{Aut } BX$ is a coset of some subgroup H of $\mathbb{Z}_n \times \mathbb{Z}_2$.

2.2. NORMAL CAYLEY GRAPHS.

DEFINITION 2.8. (M.-Y. Xu [17, Defn. 1.4]) *For each $g \in G$, it is easy to see that the translation g^* , defined by $g^*(x) = g + x$, is an automorphism of $\text{Cay}(G, S)$. The set*

$$G^* = \{ g^* \mid g \in G \}$$

is a subgroup of $\text{Aut } \text{Cay}(G, S)$. (It is often called the regular representation of G .) We say that $\text{Cay}(G, S)$ is normal if the subgroup G^ is normal in $\text{Aut } \text{Cay}(G, S)$. This means that if φ is an automorphism of the graph $\text{Cay}(G, S)$, and $\varphi(0) = 0$, then φ is an automorphism of the group G .*

LEMMA 2.9. *Assume $X = \text{Cay}(\mathbb{Z}_n, S)$ is connected and unstable. If the Cayley graph BX is normal, then X has Wilson type (C.4).*

Proof. Since X is unstable, we know that $(0, 1)$ is not central in $\text{Aut } BX$. So it is conjugate to some other element of order 2. However, since BX is normal, we know that $\mathbb{Z}_n \times \mathbb{Z}_2$ is a normal subgroup of $\text{Aut } BX$. Therefore, $(0, 1)$ cannot be the only element of order 2 in $\mathbb{Z}_n \times \mathbb{Z}_2$; so n is even. (This is also immediate from Theorem 3.1.)

Also note that $\mathbb{Z}_n \times \{0\}$ is normal in $\text{Aut } BX$, because it consists of the elements of the normal subgroup $\mathbb{Z}_n \times \mathbb{Z}_2$ that preserve each bipartition set. Since $(\# , 0)$ is the unique element of order 2 in this normal subgroup, it must be central in $\text{Aut } BX$.

Now, since $(0, 1)$, $(\# , 0)$, and $(\# , 1)$ are the only elements of order 2 in $\mathbb{Z}_n \times \mathbb{Z}_2$, and $(\# , 0)$ is not conjugate to any other elements, we see that $(0, 1)$ must be conjugate to $(\# , 1)$, by some $\alpha \in \text{Aut } BX$.

Since BX is normal, α must be a group automorphism. Hence, there is some $m \in \mathbb{Z}_n^\times$, such that $\alpha(s, 0) = (ms, 0)$ for all s . Since $\alpha(0, 1) = (\# , 1)$, this implies $\alpha(s, 1) = (ms + \# , 1)$. Since $S \times \{1\}$ is α -invariant, this implies that S is invariant under the map $s \mapsto ms + \#$, which is precisely the condition of Wilson type (C.4). \square

COROLLARY 2.10. *If $X = \text{Cay}(\mathbb{Z}_n, S)$ is a nontrivially unstable circulant graph of odd valency, then the Cayley graph BX is not normal.*

Proof. If BX is normal, then Lemma 2.9 implies $\# + gS = S$, for some $g \in \mathbb{Z}_n^\times$. Also, since X has odd valency, we know that $\# \in S$. Also, we know that g is odd (because n is even). Therefore

$$0 = \# + \# = \# + g\# \in \# + gS = S.$$

This contradicts our standing assumption that all graphs are simple (no loops). \square

LEMMA 2.11. *Let $X = \text{Cay}(\mathbb{Z}_n, S)$ be a nontrivially unstable circulant graph of odd valency, and let $X_0 = \text{Cay}(\mathbb{Z}_n, S \setminus \{n/2\})$, so*

$$BX_0 = \text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, (S \setminus \{n/2\}) \times \{1\}).$$

If every automorphism of BX maps $n/2$ -edges to $n/2$ -edges, then BX_0 is not a normal Cayley graph. Moreover, if X_0 is bipartite, then X_0 is not a normal Cayley graph.

Proof. By the assumption on $\#$ -edges, it follows that every automorphism of BX induces an automorphism of BX_0 . If BX_0 is normal, it follows that BX is also normal, contradiction with Corollary 2.10. We conclude that BX_0 is non-normal.

Suppose now that X_0 is bipartite. It is not difficult to see that X_0 is connected, since X is connected. It follows that every element of $S \setminus \{\#\}$ is odd. Since X is nonbipartite, it follows that $\#$ is even. Suppose that X_0 is normal Cayley graph. Observe that BX_0 is isomorphic to the disjoint union of two copies of X_0 , and that the connected component containing the vertex $(0, 0)$ is $X_1 = \text{Cay}(H, (S \setminus \{\#\}) \times \{1\})$, where $H = \langle (1, 1) \rangle \leq \mathbb{Z}_n \times \mathbb{Z}_2$. The map $\theta : \mathbb{Z}_n \rightarrow H$ defined by $\theta(k) = (k, k \pmod 2)$ is an isomorphism between X_0 and X_1 . Then X_1 is a normal Cayley graph on H . Let φ be an automorphism of BX that fixes $(0, 0)$. Then φ is also an automorphism of BX_0 , and consequently of its connected component X_1 . Since X_1 is normal, it follows that the action of φ on H is an automorphism of the group H , hence φ fixes the unique element of order 2 in H , which is $(\# , 0)$. Observe that the $\#$ -edge in BX incident with $(\# , 0)$, must also be fixed, hence $(0, 1)$ is fixed. Now Lemma 3.2 implies that X is stable, a contradiction. The obtained contradiction shows that X_0 is non-normal. \square

PROPOSITION 2.12 (Baik-Feng-Sim-Xu [1, Thm. 1.1]). *Let $\text{Cay}(G, S)$ be a connected Cayley graph on an abelian group G . Assume, for all $s, t, u, v \in S$:*

$$s + t = u + v \neq 0 \implies \{s, t\} = \{u, v\}.$$

Then the Cayley graph $\text{Cay}(G, S)$ is normal.

2.3. MISCELLANY. The following result is a very special case of the known results on automorphism groups of Cartesian products. (Note that C_4 is isomorphic to $K_2 \square K_2$.)

PROPOSITION 2.13 (cf. [5, Thm. 6.10, p. 69]). *Let X be a connected graph. If there does not exist a graph Y , such that $X \cong K_2 \square Y$, then*

$$\text{Aut}(K_2 \square X) = S_2 \times \text{Aut } X \quad \text{and} \quad \text{Aut}(C_4 \square X) = \text{Aut } C_4 \times \text{Aut } X.$$

LEMMA 2.14. *Let $X = \text{Cay}(\mathbb{Z}_n, S)$ be a connected circulant graph of order n such that X is not twin-free, and let d be the valency of X .*

- (1) *There is a connected circulant graph Y and some $m \geq 2$, such that $X \cong Y \wr \overline{K_m}$ and $d = \delta m$, where δ is the valency of Y .*
- (2) *If d is prime, then $X \cong K_{d,d}$.*
- (3) *If $d = 4$, then X is isomorphic either to $K_{4,4}$ or to $C_\ell \wr \overline{K_2}$ with $\ell = |V(X)|/2$. Moreover, the unique twin of 0 in the second case is $n/2$.*

Proof. (1) Let \sim be the relation of being twins on $V(X)$ defined in Definition 2.5, i.e., write $x \sim y$ if and only if $N_X(x) = N_X(y)$ for $x, y \in V(X)$. Note that since X is assumed to have no loops, equivalence classes of \sim are independent sets. Furthermore, they are clearly blocks for the action of $\text{Aut}(X)$ since $x \sim y$ if and only if $\alpha(x) \sim \alpha(y)$ for all $\alpha \in \text{Aut}(X)$. Since X is a circulant (so in particular, a Cayley graph), by Definition 2.6 the blocks are cosets of some subgroup H . Clearly, each block is of size $m := |H|$. From the assumption that X is not twin-free, we obtain that $m \geq 2$. It is easy to see that if x and y are adjacent, then x' is adjacent to y' for all $x' \sim x$ and $y' \sim y$. It is now clear that $X \cong Y \wr \overline{K_m}$ with $Y := \text{Cay}(\mathbb{Z}_n/H, \hat{S})$, where \mathbb{Z}_n/H is the quotient group and $\hat{S} := \{s + H : s \in S\}$.

(2) By (1), we then represent X as $Y \wr \overline{K_m}$, where Y is m -regular and connected, and $m \geq 2$. As $d = \delta m$, and d is prime, it follows that $m = d$ and $\delta = 1$. In particular, $Y = K_2$ and $X = K_2 \wr \overline{K_d} \cong K_{d,d}$. (Conversely, it is clear that $K_{d,d}$ is a connected circulant graph, but is not twin-free.)

(3) By (1), we then represent X as $Y \wr \overline{K_m}$, where Y is m -regular and connected, and $m \geq 2$. As $4 = \delta m$ and $m \geq 2$, it follows that $m \in \{2, 4\}$. If $m = 4$, then $\delta = 1$ and consequently $X \cong K_2 \wr \overline{K_4} \cong K_{4,4}$. If $m = 2$, then Y is connected and 2-regular, so it is isomorphic to the cycle C_ℓ with $\ell = |V(X)|/2$. It follows that $X \cong C_\ell \wr \overline{K_2}$. Note that in this case, two vertices are twins in X if and only if they are in the same copy of $\overline{K_2}$ (see Definition 2.4(3)). It is clear that these 2-element sets of twins form blocks for the action of $\text{Aut } X$ by Definition 2.6. From Definition 2.6, it also follows that they are cosets of the subgroup of \mathbb{Z}_n of order 2. As this subgroup is $\{0, n/2\}$, the conclusion follows. \square

PROPOSITION 2.15 ([7, Cor. 4.6]). *Let α be an automorphism of BX , where X is a circulant graph $\text{Cay}(\mathbb{Z}_n, S)$, and let $s, t \in S$. If α maps some s -edge to a t -edge, and either $\gcd(|s|, |t|) = 1$, or S contains every element that generates $\langle s \rangle$ (e.g., if $|s| \in \{2, 3, 4, 6\}$), then S contains every element that generates $\langle t \rangle$.*

THEOREM 2.16 (Kovács [9], Li [11]). *Let X be a connected, arc-transitive, circulant graph of order n . Then one of the following holds:*

- (1) $X = K_n$,
- (2) X is a normal circulant graph,
- (3) $X = Y \wr \overline{K_d}$, where $n = md$, $d \geq 2$, and Y is a connected arc-transitive circulant graph of order m ,
- (4) $X = Y \wr \overline{K_d} - dY$, where $n = md$, $d > 3$, $\gcd(d, m) = 1$, and Y is a connected arc-transitive circulant graph of order m .

The statement of the following result in [7] requires the graph to have even order (because the statement refers to $n/2$), but the same proof applies to graphs of odd order. Although the proofs in this paper will only apply Lemma 2.17 to graphs of even order, we omit this unnecessary hypothesis.

LEMMA 2.17 ([7, Cor. 4.3]). *Let $X = \text{Cay}(\mathbb{Z}_n, S)$ be a circulant graph, let φ be an automorphism of BX , and let*

$$S' = \{s' \in S \mid 2t \neq 2s' \text{ for all } t \in S, \text{ such that } t \neq s'\}.$$

Then φ is an automorphism of $\text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, 2S' \times \{0\})$.

PROPOSITION 2.18 ([7, Cor. 5.6(4)]). *Let $X = \text{Cay}(\mathbb{Z}_n, S)$ be a nontrivially unstable, circulant graph, such that $n \equiv 2 \pmod{4}$, and such that $2\mathbb{Z}_n \times \{0\}$ is a block for the action of $\text{Aut } BX$. Define $X_e := \text{Cay}(2\mathbb{Z}_n, S \cap 2\mathbb{Z}_n)$. If the valency of X_e is ≤ 5 , then X has Wilson type (C.1) or (C.4).*

PROPOSITION 2.19 (cf. [4, Prop. 3.4]). *If X is a connected, cubic, arc-transitive multi-graph, and the girth of X is ≤ 5 , then X is one of the following graphs: the theta graph Θ_2 (which has multiple edges), K_4 , $K_{3,3}$, the cube $Q_3 = K_2 \square K_2 \square K_2$, the Petersen graph $GP(5, 2)$, or the dodecahedron graph $GP(10, 2)$.*

COROLLARY 2.20. *The only connected, cubic, arc-transitive circulant graphs are K_4 and $K_{3,3}$.*

3. SOME CONDITIONS THAT IMPLY STABILITY

THEOREM 3.1 (Fernandez-Hujdurović [3] (or [13])). *There are no nontrivially unstable, circulant graphs of odd order.*

LEMMA 3.2 (cf. [3, Lem. 2.4]). *A circulant graph $X = \text{Cay}(\mathbb{Z}_n, S)$ is stable if and only if, for every $\alpha \in \text{Aut } BX$, such that $\alpha(0, 0) = (0, 0)$, we have $\alpha(0, 1) = (0, 1)$.*

The complete graph on 2 vertices is bipartite, and therefore unstable. It is not difficult to see that all of the larger complete graphs are stable:

EXAMPLE 3.3 ([14, Example 2.2]). If $n \geq 3$, then K_n is stable.

LEMMA 3.4. *Let $X = \text{Cay}(\mathbb{Z}_n, S)$ be a connected, nonbipartite circulant graph, and let S_0 be a nonempty subset of $\mathbb{Z}_n \setminus \{0\}$ such that $S_0 = -S_0$. If every automorphism of BX maps S_0 -edges to S_0 -edges, and some (or, equivalently, every) connected component of $\text{Cay}(\mathbb{Z}_n, S_0)$ is a stable graph, then X is stable.*

Proof. Let α be an automorphism of BX that fixes $(0, 0)$, and let X_0 be the connected component of $\text{Cay}(\mathbb{Z}_n, S_0)$ that contains 0. Since α maps S_0 -edges to S_0 -edges, it restricts to an automorphism of BX_0 . Since X_0 is a stable graph, this implies that $\alpha(0, 1) = (0, 1)$. \square

LEMMA 3.5. *Let $X = \text{Cay}(G, S)$ be a Cayley graph on an abelian group, and let $k \in \mathbb{Z}^+$, such that k is odd. Suppose there exists $c \in S$, such that*

- (1) $|c| = k$,
- (2) $2c \neq s + t$, for all $s, t \in S \setminus \{c\}$, and
- (3) for all $a \in S$ of order $2k$, there exist $s, t \in S \setminus \{a\}$, such that $2a = s + t$.

Then X is stable.

Proof. Let us say that a cycle in BX is *exceptional* if, for every pair v_i, v_{i+2} of vertices at distance 2 on the cycle, the unique path of length 2 from v_i to v_{i+2} is v_i, v_{i+1}, v_{i+2} . It is clear that every automorphism of BX must map each exceptional cycle of length k to an exceptional cycle of length k .

Let α be an automorphism of BX fixing $(0, 0)$. If c is any element satisfying the conditions, then $(c, 1)^{2k}$ is an exceptional cycle. Furthermore, every exceptional cycle of length $2k$ is of this form. Since $k(c, 1) = (0, 1)$, for every such exceptional cycle, this implies that α fixes $(0, 1)$. So X is stable by Lemma 3.2. \square

LEMMA 3.6. *Let $X := \text{Cay}(\mathbb{Z}_n, S)$ be a circulant graph of even order and odd valency. Let $S_0 \subset S$ be non-empty such that $n/2 \in \langle S_0 \rangle$. Assume that the set of S_0 -edges is invariant under the elements of $\text{Aut } BX$ (and $S_0 = -S_0$). If some (equivalently every) connected component X'_0 of $X_0 := \text{Cay}(\mathbb{Z}_n, S_0)$ is not bipartite and has the property that BX'_0 is normal, then X is stable.*

Proof. We can take $X'_0 := \text{Cay}(\langle S_0 \rangle, S_0)$. Because X is of odd valency, we know $\# \in S$. By our assumptions, $\#$ is a vertex of X'_0 . As X'_0 is connected and assumed to be non-bipartite, BX'_0 is connected. Let $\alpha \in \text{Aut}(BX)_{(0,0)}$. Then by our assumptions, $\alpha \in \text{Aut}(BX_0)$. Because α fixes $(0, 0)$, it also fixes the connected component of BX_0 containing it, which is BX'_0 . As BX'_0 is normal, the restriction of α onto BX'_0 is a group automorphism of $\langle S_0 \rangle \times \mathbb{Z}_2$. Note that as $(0, 1), (\#, 0), (\#, 1)$ are the only elements of order 2 in $\langle S_0 \rangle \times \mathbb{Z}_2$, it follows that α must permute them among themselves. As α fixes the colors of BX'_0 , it follows that α fixes $(\#, 0)$ because this is the unique element of order 2 in the set $\langle S_0 \rangle \times 0$. Because X is loopless, $0 \notin S$ and the only element of order 2 in the connection set of BX , which is $S \times 1$, is $(\#, 1)$. Since α fixes $S \times 1$ set-wise, it must hold that α fixes $(\#, 1)$ and consequently it also fixes $(0, 1)$. It follows that X is stable. \square

Proofs in later sections assume that the following circulant graphs are known to be stable.

LEMMA 3.7. *Each of the following circulant graphs is stable:*

- (1) *valency 3:*
 - (a) $\text{Cay}(\mathbb{Z}_6, \{\pm 2, 3\})$,
- (2) *valency 4:*
 - (a) $\text{Cay}(\mathbb{Z}_{12}, \{\pm 2, \pm 3\})$,
 - (b) $\text{Cay}(\mathbb{Z}_{12}, \{\pm 3, \pm 4\})$,
- (3) *valency 5:*
 - (a) $\text{Cay}(\mathbb{Z}_{10}, \{\pm 2, \pm 4, 5\})$,
 - (b) $\text{Cay}(\mathbb{Z}_{12}, \{\pm 1, \pm 5, 6\})$,
 - (c) $\text{Cay}(\mathbb{Z}_{12}, \{\pm 3, \pm 4, 6\})$,
- (4) *valency 6:*
 - (a) $\text{Cay}(\mathbb{Z}_{20}, \{\pm 4, \pm 5, \pm 8\})$,
 - (b) $\text{Cay}(\mathbb{Z}_{20}, \{\pm 2, \pm 5, \pm 6\})$,
- (5) *valency 7:*
 - (a) $\text{Cay}(\mathbb{Z}_{12}, \{\pm 2, \pm 3, \pm 4, 6\})$,
 - (b) $\text{Cay}(\mathbb{Z}_{14}, \{\pm 2, \pm 4, \pm 6, 7\})$,
 - (c) $\text{Cay}(\mathbb{Z}_{20}, \{\pm 4, \pm 5, \pm 8, 10\})$,
 - (d) $\text{Cay}(\mathbb{Z}_{24}, \{\pm 3, \pm 8, \pm 9, 12\})$,
 - (e) $\text{Cay}(\mathbb{Z}_{30}, \{\pm 5, \pm 6, \pm 12, 15\})$,
 - (f) $\text{Cay}(\mathbb{Z}_{30}, \{\pm 3, \pm 9, \pm 10, 15\})$,
 - (g) $\text{Cay}(\mathbb{Z}_{30}, \{\pm 6, \pm 10, \pm 12, 15\})$.

Proof. This is easily verified by computer (in less than a second). For example, one can use the MAGMA or sagemath programs available at <https://arxiv.org/src/2108.05164/anc/>. \square

LEMMA 3.8. *Let $k \geq 3$ be an integer. The graph $X_k := C_k \wr K_2$ is unstable if and only if $k = 4$.*

Proof. We note that $X_k \cong \text{Cay}(\mathbb{Z}_{2k}, \{\pm 1, k \pm 1, k\})$ for all $k \geq 3$.

- Note that $X_3 \cong K_6$ is stable by Example 3.3.
- The graph X_4 is unstable with Wilson type (C.3').

Assume that $k \geq 5$. We observe the ball \mathcal{B} of radius 2 in BX_k centered at $(0, 0)$. An easy computation then shows that \mathcal{B} consists of vertices $(\pm 1, 1), (k \pm 1, 1), (k, 1)$, which are neighbours of $(0, 0)$, and vertices $(\pm 1, 0), (\pm 2, 0), (k \pm 1, 0), (k \pm 2, 0)$ and $(k, 0)$, all of which are at distance 2 from $(0, 0)$. Since $k \geq 5$, all of these vertices are distinct.

Note that $(k, 0)$ is the only element of \mathcal{B} sharing 4 neighbours with $(0, 0)$. Its only remaining neighbour $(0, 1)$ is then at distance 3 from $(0, 0)$. So an automorphism of BX_k fixing $(0, 0)$ must fix $(k, 0)$ and consequently, it also fixes $(0, 1)$. It follows by Lemma 3.2 that X_k is stable. \square

4. UNSTABLE CIRCULANTS OF VALENCY ≤ 4

THEOREM 4.1. *Every nontrivially unstable circulant graph of valency ≤ 4 has Wilson type (C.4).*

The union of the following two results provides a more precise formulation of the above Theorem 4.1.

PROPOSITION 4.2. *There are no nontrivially unstable circulant graphs of valency ≤ 3 .*

Proof. We consider twin-free, connected, nonbipartite, circulant graphs of each valency ≤ 3 .

(valency 0) The one-vertex trivial graph K_1 is stable, because $BK_1 = \overline{K_2}$, and $|\text{Aut } \overline{K_2}| = 2 = 2|\text{Aut } K_1|$.

(valency 1) K_2 is bipartite.

(valency 2) If C_n is a nonbipartite cycle, then n is odd, so $BC_n \cong C_{2n}$, so

$$|\text{Aut } BC_n| = |\text{Aut } C_{2n}| = 2 \cdot 2n = 2 \cdot |\text{Aut } C_n|,$$

so C_n is stable.

(valency 3) A connected, nonbipartite, circulant graph X of valency 3 is either an odd prism or a nonbipartite Möbius ladder. In either case, the canonical double cover is the even prism $K_2 \square C_n$, where $n = |V(X)|$. It is easy to check that K_4 is stable (see Example 3.3). And the following calculation (which uses Proposition 2.13) shows that X is also stable when $n > 4$:

$$|\text{Aut } BX| = |\text{Aut}(K_2 \square C_n)| = |S_2 \times \text{Aut } C_n| = 2 \cdot 2n \leq 2|\text{Aut } X|. \quad \square$$

THEOREM 4.3. *A circulant graph $X = \text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b\})$ of valency 4 is unstable if and only if either it is trivially unstable, or one of the following conditions is satisfied (perhaps after interchanging a and b):*

- (1) $n \equiv 2 \pmod{4}$, $\gcd(a, n) = 1$, and $b = ma + (n/2)$, for some $m \in \mathbb{Z}_n^\times$, such that $m^2 \equiv \pm 1 \pmod{n}$, or
- (2) n is divisible by 8 and $\gcd(|a|, |b|) = 4$.

In both of these cases, X has Wilson type (C.4).

Proof. For convenience, let $\mathfrak{n} = n/2$ (see Notation 2.2).

(\Leftarrow) Condition (1) clearly implies that X has Wilson type (C.4), so X is unstable.

We may now assume condition (2) holds, which means $\gcd(|a|, |b|) = 4$. We may also assume that X is connected (for otherwise it is trivially unstable). This implies

that we may assume a is odd (perhaps after interchanging a and b), which means $n/|a|$ is odd. Since $\gcd(|a|, |b|) = 4$, it also implies that $|b| = 4n/|a|$. Write $|a| = 2^r \ell$, where ℓ is odd.

Now, since ℓ and $n/|a|$ are odd, we have $\gcd(2^r, \ell) = \gcd(2^r, n/|a|) = 1$. Also note that

$$4 = \gcd(|a|, |b|) = \gcd(2^r \ell, 4n/|a|) = 4 \gcd(2^{r-2} \ell, n/|a|).$$

Therefore 2^r , ℓ , and $n/|a|$ are pairwise relatively prime, so we may choose $m \in \mathbb{Z}_n^\times$, such that

$$m \equiv 2^{r-1} + 1 \pmod{2^r}, \quad m \equiv 1 \pmod{\ell}, \quad \text{and} \quad m \equiv -1 \pmod{n/|a|}.$$

Then:

- (a) $2(m-1)$ is divisible by $2^r \ell$, but $m-1$ is not divisible by $2^r \ell$, so $2(m-1)a = 0$, but $(m-1)a \neq 0$. Therefore $(m-1)a = \mathfrak{n}$, so $ma + \mathfrak{n} = a \in S$, and
- (b) $2(m+1)$ is divisible by $4n/|a|$, but $m+1$ is not divisible by $4n/|a|$ (because $m+1 \equiv 2 \pmod{4}$). Since $|b| = 4n/|a|$, this implies that $2(m+1)b = 0$, but $(m+1)b \neq 0$. Therefore $(m+1)b = \mathfrak{n}$, so $mb + \mathfrak{n} = -b \in S$.

So $mS + \mathfrak{n} = S$, which means that X has Wilson type (C.4) (and is therefore unstable).

(\Rightarrow) Assuming that X is nontrivially unstable, we will show that it satisfies the conditions of (1) or (2). Note that n must be even (see Theorem 3.1). Since X is connected, and not bipartite, the subgroup $2\mathbb{Z}_n$ must contain exactly one of the elements of $\{a, b\}$.

Let α be an automorphism of BX that fixes $(0, 0)$, and is not in $\text{Aut } X \times S_2$. Since $\mathbb{Z}_n \times \{0\}$ and $\mathbb{Z}_n \times \{1\}$ are the bipartition sets of BX , we know that each of these sets is α -invariant.

Case 1. Assume α is a group automorphism. Since $\mathbb{Z}_n \times \{0\}$ is α -invariant, this implies there is some $m \in \mathbb{Z}_n^\times$, such that $\alpha(x, 0) = (mx, 0)$ for all $x \in \mathbb{Z}_n$. Since $\alpha(0, 1)$ is an element of order 2 (and $(\mathfrak{n}, 0)$ is fixed by α), we must have $\alpha(0, 1) \in \{(0, 1), (\mathfrak{n}, 1)\}$. If $\alpha(0, 1) = (0, 1)$, then $\alpha(x, i) = (mx, i)$, which contradicts the assumption that $\alpha \notin \text{Aut } X \times S_2$. Therefore, we have $\alpha(0, 1) = (\mathfrak{n}, 1)$. So

$$\alpha(x, i) = (mx + i\mathfrak{n}, i) \quad \text{for all } (x, i) \in BX.$$

Since $S \times \{1\}$ is α -invariant, this implies that $mS + \mathfrak{n} = S$, so X is of Wilson type (C.4).

Subcase 1.1. Assume that \mathfrak{n} is odd. Since X is connected, we may assume, without loss of generality, that $a \notin 2\mathbb{Z}_n$. Then $ma + \mathfrak{n} \in 2\mathbb{Z}_n$, so we must have $\alpha(a, 1) \in \{(\pm b, 1)\}$. Therefore $b = ma + \mathfrak{n}$ (perhaps after composing α with the group automorphism $x \mapsto -x$, which replaces m with $-m$).

Now, we have $\alpha(a, 1) = (ma + \mathfrak{n}, 1) = (b, 1)$, so $\alpha(\pm a, 1) = (\pm b, 1)$. Since α is a group automorphism that preserves the set $S \times \{1\}$, this implies $\alpha(\pm b, 1) = (\pm a, 1)$, so we may write $\alpha(b, 1) = (\epsilon a, 1)$ with $\epsilon \in \{\pm 1\}$. Then we have $m^2(a, 1) = \alpha^2(a, 1) = \epsilon(a, 1)$ and $m^2(b, 1) = \alpha^2(b, 1) = \epsilon(b, 1)$, so $m^2x = \epsilon x$ for all $x \in \mathbb{Z}_n \times \mathbb{Z}_2$. This implies $m^2 \equiv \epsilon \equiv \pm 1 \pmod{n}$. So X is as described in (1).

Subcase 1.2. Assume that \mathfrak{n} is even. Then $ma + a$ has the same parity as a (and $mb + b$ has the same parity as b), so we must have $\alpha(a, 1) \in \{(\pm a, 1)\}$ and $\alpha(b, 1) \in \{(\pm b, 1)\}$. There is no harm in assuming $\alpha(a, 1) = (a, 1)$ (by replacing m with $-m$ if necessary). Then, since α is not the identity map, we must have $\alpha(b, 1) = (-b, 1)$. Therefore

$$(ma + \mathfrak{n}, 1) = \alpha(a, 1) = (a, 1), \quad \text{so } (m-1)a = \mathfrak{n},$$

and

$$(mb + \mathfrak{n}, 1) = \alpha(b, 1) = (-b, 1), \quad \text{so } (m+1)b = \mathfrak{n}.$$

Since $m - 1$ and $m + 1$ are even (and π has order 2), this implies that $|a|$ and $|b|$ are divisible by 4.

This also implies that $2(m - 1)a = 0$ and $2(m + 1)b = 0$, so $|a|$ is a divisor of $2(m - 1)$ and $|b|$ is a divisor of $2(m + 1)$. Therefore $\gcd(|a|, |b|)$ is a divisor of $\gcd(2(m - 1), 2(m + 1)) \leq 4$. By combining this with the conclusion of the preceding paragraph, we conclude that $\gcd(|a|, |b|) = 4$. Then, since X is not bipartite, we must have $n \equiv 0 \pmod{8}$. This establishes that conclusion (2) holds.

Case 2. Assume $2s \neq 2t$, for all $s, t \in S$, such that $s \neq t$. We may assume α is not a group automorphism, for otherwise Case 1 applies. So BX is not normal. Therefore, Proposition 2.12 implies there exist $s, t, u, v \in S$ such that $s + t = u + v \neq 0$ and $\{s, t\} \neq \{u, v\}$. From the assumption of this Case 2, we see that this implies $3a = \pm b$ (perhaps after interchanging a with b). This implies that a and b have the same parity, which contradicts the assumption that X is connected and nonbipartite.

Case 3. *The remaining case.* Since Case 2 does not apply, we have $2s = 2t$, for some $s, t \in S$, such that $s \neq t$. We may assume $s = a$.

Subcase 3.1. Assume that $t = -s = -a$. Then $|a| = 4$. If n is divisible by 8, then condition (2) is satisfied. (If $|a| = 4$, and n is divisible by 8, then $|b|$ must be divisible by 8, so $\gcd(|a|, |b|) = 4$.) So we may assume $n = 4k$, where k is odd. Since X is nonbipartite, we know that $|b|$ is not divisible by 4, so the fact that $|a| = 4$ implies $|\langle a \rangle \cap \langle b \rangle| \leq 2$. Hence, there is an automorphism of \mathbb{Z}_n that fixes a , but inverts b , so:

$$|\text{Aut } X| \geq 4n.$$

Also, since k is odd and X is nonbipartite, we must have $kb \neq \pm a$. Since $4kb = nb = 0$, this implies $k(b, 1) \in \{(0, 1), (2a, 1)\}$. Since $(2a, 1) \notin \langle (a, 1) \rangle$, this implies that $\langle (a, 1) \rangle \cap \langle (b, 1) \rangle = \{(0, 0)\}$, so $BX \cong C_4 \square C_{n/2}$. Therefore (using Proposition 2.13) we have

$$\begin{aligned} |\text{Aut } BX| &= |\text{Aut}(C_4 \square C_{n/2})| = |\text{Aut } C_4 \times \text{Aut } C_{n/2}| \\ &= |\text{Aut } C_4| \cdot |\text{Aut } C_{n/2}| = 8 \cdot n = 2 \cdot 4n \leq 2 \cdot |\text{Aut } X|. \end{aligned}$$

This contradicts the assumption that X is unstable.

Subcase 3.2. Assume that $t \neq -s$. Therefore, we may assume $s = a$ and $t = b$, so $2a = 2b$. This means that $a - b$ has order 2, and must therefore be equal to π , so $S + \pi = S$. This contradicts the fact that $\text{Cay}(\mathbb{Z}_n, S)$ is twin-free. \square

The following observation can be verified by inspecting the list [1] of connected, non-normal Cayley graphs of valency 4 on abelian groups, and confirming that none of them are the canonical double cover of a nontrivially unstable circulant graph. (Recall that “normal” is defined in Definition 2.8.) For the reader’s convenience, we provide a proof that avoids reliance on the entire classification, by extracting the relevant part of the proof in [1] (and by using Theorem 4.3 to reduce the number of cases).

COROLLARY 4.4. *If X is a nontrivially unstable circulant graph of valency 4, then BX is normal.*

Proof. Write $X = \text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b\})$, and suppose BX is not normal. (This will lead to a contradiction.) By using Proposition 2.12 (and the fact that X is not bipartite), as in Case 2 of the proof of Theorem 4.3, we see that $2s = 2t$, for some $s, t \in S$, such that $s \neq t$. Therefore, we may assume that either $2a = 2b$ or $|a| = 4$. However, we cannot have $2a = 2b$, since X is twin-free. (If $b = a + \pi$, then $S = S + \pi$.)

So we have $|a| = 4$. Then, since X is nontrivially unstable, we see from Theorem 4.3 that n is divisible by 8 and $\langle b \rangle = \mathbb{Z}_n$. (We are now in the situation of [1, Lem. 3.4],

but will briefly sketch the proof.) Let $\alpha \in \text{Aut } BX$, such that $\alpha(0, 0) = (0, 0)$. The subgraph induced by the ball of radius 2 around $(0, 0)$ has only two automorphisms, so the restriction of α to this ball is the same as the restriction of a group automorphism β (such that $\beta(a) \in \{\pm a\}$ and $\beta(b) \in \{\pm b\}$). It is then easy to see that $\alpha(x) = \beta(x)$ for all x , so α is a group automorphism. This means that BX is normal. \square

The following technical result will be used in Sections 5 and 7.

COROLLARY 4.5. *Let $X = \text{Cay}(\mathbb{Z}_n, S)$ be a circulant graph of even order and odd valency, and let $S_0 \subset S$ with $|S_0| = 4$. Let $X_0 := \text{Cay}(\mathbb{Z}_n, S_0)$ and let X'_0 be a connected component of X_0 . Assume that the set of S_0 -edges is invariant under $\text{Aut}(BX)$. If either*

- (1) $|V(X'_0)|$ is odd, or
- (2) X_0 is twin-free and nonbipartite,

then X is stable.

Proof. Let us first assume that $|\langle S_0 \rangle| = |V(X'_0)|$ is odd. Note that as X'_0 is 4-valent, it is twin-free. (Otherwise, Lemma 2.14 tells us that $X'_0 \cong Y \wr \overline{K_m}$, where $m \geq 2$ and Y is δ -regular. Then $|V(X'_0)| = m |V(Y)|$, so m is odd. But also $4 = \delta m$, so m is even, a contradiction.) Therefore X'_0 is a connected, twin-free, circulant graph of odd order, so, by Theorem 3.1, it must be stable. It follows by Lemma 3.4 that X is stable.

Let us now suppose that $|\langle S_0 \rangle| = |V(X'_0)|$ is even. It then follows that $\# \in \langle S_0 \rangle$. By assumption (2), X_0 must be twin-free and nonbipartite. As all of its connected components are isomorphic to X'_0 , it follows that X'_0 is twin-free and nonbipartite. In particular, X'_0 is not trivially unstable. If it is stable, we conclude that X is stable by Lemma 3.4. If it is not stable, it is nontrivially unstable so by Corollary 4.4 it follows that BX'_0 is normal. Applying Lemma 3.6, we conclude that X is stable. \square

5. UNSTABLE CIRCULANTS OF VALENCY 5

THEOREM 5.1. *A circulant graph $\text{Cay}(\mathbb{Z}_n, S)$ of valency 5 is unstable if and only if either it is trivially unstable, or it is one of the following:*

- (1) $\text{Cay}(\mathbb{Z}_{12k}, \{\pm s, \pm 2k, 6k\})$ with s odd, which has Wilson type (C.1),
- (2) $\text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3, 4\})$, which has Wilson type (C.3').

REMARK 5.2. It is easy to see that each connection set listed in Theorem 5.1 contains both even elements and odd elements, so none of the graphs are bipartite. Then it follows from Lemma 2.14(2) that the graphs are also twin-free. Therefore, a graph listed in Theorem 5.1 is nontrivially unstable if and only if it is connected. And this is easy to check:

- the graph in (2) is connected (since 1 is in the connection set);
- a graph in (1) is connected if and only if s is relatively prime to k .

The proof of Theorem 5.1 will use the following lemmas.

The first lemma can be obtained by inspecting the list [1, Cor. 1.3] of connected, non-normal, circulant graphs of valency 4: K_5 , $C_m \wr \overline{K_2}$ (with $m \geq 3$) and $K_2 \wr \overline{K_5} - 5K_2$. For the reader's convenience, we reproduce the relevant parts of the proof in [1].

LEMMA 5.3 (cf. [1, Cor. 1.3]). *If $X_0 = \text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b\})$ is a connected, bipartite, twin-free, non-normal, circulant graph of valency 4, then $X_0 = \text{Cay}(\mathbb{Z}_{10}, \{\pm 1, \pm 3\})$.*

Proof (cf. [1, proof of Thm. 1.2]). From Proposition 2.12, we see that (perhaps after permuting the generators), we have either $2a = 2b$ or $b = 3a$ or $4a = 0$. However, we cannot have $2a = 2b$, since X is twin-free. So there are two cases to consider.

Case 1. Assume $b = 3a$. Since $|b| \neq 2$, we have $n > 6$.

For $n = 8$, we have $2b = 6a = -2a$, which contradicts the assumption that X is twin-free.

For $n = 10$, we have the Cayley graph that is specified in the statement of Lemma 5.3.

For $n \geq 12$, we have $X \cong \text{Cay}(\mathbb{Z}_n, \{\pm 1, \pm 3\})$. This is the situation of [1, Lem. 3.5], but, for completeness, we sketch the proof. The only nontrivial automorphism of the ball of radius 2 centered at 0 is $x \mapsto -x$. From this, it easily follows that $x \mapsto -x$ is the only nontrivial automorphism of X , so X is normal. This contradicts our hypothesis.

Case 2. Assume $4a = 0$. This means $|a| = 4$. Since X is bipartite, we know that $|b|$ is even, so $\langle b \rangle$ contains the unique element of order 2; this means $2a = \ell b$ for some $\ell \in \mathbb{Z}$. Since X is bipartite, we know that ℓ is even. So $|b|$ is divisible by 4. Therefore, $\langle b \rangle$ contains a . So $X \cong \text{Cay}(n, \{\pm 1, \pm n/4\})$. If we consider the subgraph induced by the ball of radius 2, we note that only the vertices ± 1 have a pendant edge. In particular, every automorphism of X maps 1-edges to 1-edges and is therefore an automorphism of the graph $X_0 := \text{Cay}(\mathbb{Z}_n, \{\pm 1\}) \cong C_n$. Since this Cayley graph is normal, we can conclude the same about X , which is a contradiction. \square

Recall that an edge $\{u, v\}$ of the complete graph on $G \times \mathbb{Z}_2$ is called a g -edge if $v = u \pm (g, 1)$ for $g \in G, 1 \in \mathbb{Z}_2$ (see Definition 2.1(4)).

LEMMA 5.4. *Let $X = \text{Cay}(\mathbb{Z}_n, S)$ be a nontrivially unstable, circulant graph of valency 5. If every automorphism of BX preserves the set of $n/2$ -edges, then $X = \text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3, 4\})$.*

Proof. Recall that Notation 2.2 introduced $\#$ as an abbreviation for $n/2$. Since X has odd valency, we know that n is even; indeed, we may write

$$X = \text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b, \#\}).$$

Since every automorphism of BX maps $\#$ -edges to $\#$ -edges, we know that every automorphism of BX is an automorphism of BX_0 , where

$$X_0 = \text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b\}).$$

If X_0 is stable, then by Lemma 3.4 it follows that X is stable, a contradiction. So we may assume now that X_0 is unstable.

Case 1. Assume X_0 is nontrivially unstable. As X_0 is 4-valent, by applying Corollary 4.4 we conclude that BX_0 is a normal Cayley graph. Because every automorphism of BX is an automorphism of BX_0 , it then follows that BX is normal as well. However, since X is nontrivially unstable and of valency 5, Corollary 2.10 implies that BX is not normal, a contradiction.

Case 2. Assume X_0 is trivially unstable. There are three possibilities to consider:

Subcase 2.1. Assume X_0 is not connected. Then a and b generate a proper subgroup of \mathbb{Z}_n , while a, b and $\#$ generate the whole group. From here, $n = 2k$, where k is odd, and $\langle a, b \rangle = 2\mathbb{Z}_n$ has order k . The connected components of X_0 then have order k , and therefore have odd order. By applying Corollary 4.5(1) we conclude that X is stable, a contradiction.

Subcase 2.2. Assume X_0 is connected, but is not twin-free. Then (by Lemma 2.14(1)) we can represent X_0 as a wreath product $Y \wr \overline{K_m}$, where Y is a δ -regular connected graph and $m > 1$ is an integer such that $\delta m = 4$.

Subsubcase 2.2.1. Assume $m = 4$. Then $\delta = 1$, so we get that $X_0 = K_2 \wr \overline{K_4} = K_{4,4}$. Hence, X_0 is a connected, 4-valent Cayley graph on \mathbb{Z}_8 and its connection set can only contain odd numbers, because it is also bipartite. This uniquely determines X_0

as $\text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3\})$. From here, $X = \text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3, 4\})$, so X is the graph in the statement of Lemma 5.4.

Subsubcase 2.2.2. Assume $m = 2$. Then $\delta = 2$ and

$$|V(Y)| = |V(X)|/m = 2k/2 = k,$$

so Y is a k -cycle, so $X_0 \cong C_k \wr \overline{K_2}$. Consequently, $X \cong C_k \wr K_2$. From Lemma 3.8, and the fact that X is not stable, we conclude that $k = 4$, i.e., $X \cong C_4 \wr K_2$. It is easy to see that this implies $X = \text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3, 4\})$. So X is again the graph in the statement of Lemma 5.4.

Subcase 2.3. Assume X_0 is bipartite, connected and twin-free. By applying Lemma 2.11, we conclude that the Cayley graph X_0 is not normal. Then Lemma 5.3 tells us that $X_0 = \text{Cay}(\mathbb{Z}_{10}, \{\pm 1, \pm 3\})$, meaning that $X = \text{Cay}(\mathbb{Z}_{10}, \{\pm 1, \pm 3, 5\})$. But then X is bipartite, a contradiction. \square

The following simple observation provides a converse to Lemma 5.4.

LEMMA 5.5. Let $X = \text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3, 4\})$. Then:

- (1) X is nontrivially unstable, and has Wilson type (C.3'),
- (2) every automorphism of BX maps $n/2$ -edges to $n/2$ -edges, and
- (3) all other edges of BX are in a single orbit of $\text{Aut } BX$.

Proof. (1) X has Wilson type (C.3') with parameters $H = \langle 2 \rangle = \{0, 2, 4, 6\}$, $R = \{4\}$, and $d = 4$. (Then $n/d = 2$ is even, $r/d = 1$ for the unique element r of R , and $H = 2\mathbb{Z}_8 \not\subseteq 4\mathbb{Z}_8 = d\mathbb{Z}_8$.) See Remark 5.2 for an explanation that X is therefore nontrivially unstable.

(2) and (3) Let

$$X_0 = \text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3\}) \cong K_{4,4}.$$

Since X_0 is bipartite, we know that BX_0 is isomorphic to the disjoint union of two copies of X_0 . But BX is connected, and the element $(\# , 1)$ of order 2 is the only element of its connection set that is not in the connection set of BX_0 . It follows that

$$BX \cong X_0 \square K_2 \cong K_{4,4} \square K_2.$$

So we see from Proposition 2.13 that the set of $\#$ -edges is invariant under all automorphisms of BX . On the other hand, $K_{4,4}$ is edge-transitive, so the other edges are all in a single orbit. \square

LEMMA 5.6. Let $X = \text{Cay}(\mathbb{Z}_n, S)$ be a nontrivially unstable, circulant graph of valency 5. If the set of $n/2$ -edges is not invariant under $\text{Aut } BX$, then:

- (1) $X = \text{Cay}(\mathbb{Z}_{12k}, \{\pm s, \pm 2k, 6k\})$, for some $s, k \in \mathbb{Z}^+$, with s odd, and
- (2) $\text{Aut } BX$ has exactly two orbits on the set of edges of BX .

Proof. Write

$$X = \text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b, \#\}).$$

By assumption, some automorphism of BX maps an $\#$ -edge to an a -edge (perhaps after interchanging a and b). Then Proposition 2.15 shows that every generator of $\langle a \rangle$ is in $S \setminus \{\#\}$. So the number of generators of $\langle a \rangle$ is ≤ 4 (and $|a| > 2$), and therefore $|a| \in \{3, 4, 5, 6, 8, 10, 12\}$.

Case 1. Assume $|a| \in \{5, 8, 10, 12\}$. The four generators of $\langle a \rangle$ are in S , so they must coincide with $\pm a$ and $\pm b$. Therefore, $\langle a, \# \rangle = \langle a, b, \# \rangle = \mathbb{Z}_n$. Therefore, X is one of the following Cayley graphs:

$$\begin{aligned} & \text{Cay}(\mathbb{Z}_{10}, \{\pm 2, \pm 4, 5\}), \text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3, 4\}), \\ & \text{Cay}(\mathbb{Z}_{10}, \{\pm 1, \pm 3, 5\}) \text{ or } \text{Cay}(\mathbb{Z}_{12}, \{\pm 1, \pm 5, 6\}). \end{aligned}$$

Note that the first and fourth graph appear in Lemma 3.7 under (3a) and (3b) respectively, and are therefore both stable. The third graph is bipartite, so it is trivially unstable. Lemma 5.5(2) implies that the second graph is also not permissible (since the statement of Lemma 5.6 requires that the set of $\#$ -edges is not invariant under $\text{Aut } BX$). So this case cannot occur.

Case 2. Assume $|a| \in \{3, 4, 6\}$. Note that then $\pm a$ are the only generators of $\langle a \rangle$. Because all elements of S are pairwise distinct, it follows that $\langle a \rangle \neq \langle b \rangle$. Therefore, $|a| \neq |b|$.

Subcase 2.1. Assume $|b| \in \{3, 4, 6\}$. We consider each of the three possibilities for $\{|a|, |b|\}$:

Subsubcase 2.1.1. Assume $\{|a|, |b|\} = \{3, 4\}$. Then

$$X = \text{Cay}(\mathbb{Z}_{12}, \{\pm 3, \pm 4, 6\}).$$

This graph is stable by Lemma 3.7(3c).

Subsubcase 2.1.2. Assume $\{|a|, |b|\} = \{3, 6\}$. Then

$$X = \text{Cay}(\mathbb{Z}_6, \{\pm 1, \pm 2, 3\}) \cong K_6.$$

This graph is stable by Example 3.3.

Subsubcase 2.1.3. Assume $\{|a|, |b|\} = \{4, 6\}$. Then

$$X = \text{Cay}(\mathbb{Z}_{12}, \{\pm 2, \pm 3, 6\}).$$

This graph appears in part (1) of the statement of Lemma 5.6 with parameters $s = 3$ and $k = 1$.

Also note that

$$BX = \text{Cay}(\mathbb{Z}_{12} \times \mathbb{Z}_2, \{\pm(2, 1), \pm(3, 1), (6, 1)\}).$$

Since $\langle(3, 1)\rangle \cap \langle(2, 1), (6, 1)\rangle = \{0, 0\}$, this implies

$$BX \cong C_4 \square \text{Cay}(2\mathbb{Z}_{12} \times \mathbb{Z}_2, \{\pm(2, 1), (6, 1)\}) \cong C_4 \square M_6,$$

where M_6 is the Möbius ladder with 6 vertices. Then Proposition 2.13 implies that BX is not edge-transitive. Since $\#$ -edges are in the same orbit as a -edges, this establishes part (2) of the statement of Lemma 5.6 for this graph.

Subcase 2.2. Assume $|b| \notin \{3, 4, 6\}$. From here, we see from Proposition 2.15 that no automorphism of BX can map an $\#$ -edge to a b -edge (because S cannot contain more than 2 generators of $\langle b \rangle$, in addition to $\pm a$). Hence, the set of b -edges is invariant under all automorphisms of BX . (Note that this establishes part (2) of the statement of Lemma 5.6 for this subcase.) Now, we see that every automorphism of BX is also an automorphism of the graphs

$$BX_1 := \text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \{(\pm a, 1), (\#, 1)\})$$

and

$$BX_2 := \text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \{(\pm b, 1)\}),$$

which are the canonical double covers of

$$X_1 := \text{Cay}(\mathbb{Z}_n, \{\pm a, \#\}) \quad \text{and} \quad X_2 := \text{Cay}(\mathbb{Z}_n, \{\pm b\}),$$

respectively. Note that BX_1 is arc-transitive (because a -edges and $\#$ -edges are in the same orbit of $\text{Aut } BX$).

If $|a| = 3$, then every connected component of BX_1 is isomorphic to a 6-prism, which is not arc-transitive. If $|a| = 4$, then every connected component of X_1 is isomorphic to K_4 , which is a stable graph by Example 3.3, so it follows from Lemma 3.4 that X is stable, a contradiction. Therefore, we must have $|a| = 6$.

The connected components of X_2 are $|b|$ -cycles. If $|b|$ is odd, then these are stable (by Theorem 3.1, for example). By another application of Lemma 3.4, it follows that X is stable, a contradiction.

So we can now assume that $|a| = 6$ and $|b|$ is even. Write $n = 6\ell$. From here $\# = 3\ell$ and $\{\pm a\} = \{\ell, 5\ell\}$.

Note that if ℓ is odd, then a and b must both be odd (since $|a|$ and $|b|$ are even). Since $\# = 3\ell$ is also odd, this means that all elements of S are odd, so X is bipartite, a contradiction.

Therefore, we know that ℓ is even, so we may write $\ell = 2k$. Then $n = 12k$, $\# = 6k$ and $\{\pm a\} = \{\pm 2k\}$. In particular, $\pm a$ and $\#$ are all even. So b must be odd (since X is connected). This means that X appears in part (1) of the statement of Lemma 5.6 with parameter $s = b$. □

Proof of Theorem 5.1. (\Leftarrow) It suffices to show that each of the graphs in (1) and (2) has the specified Wilson type. For any member of (1), it holds that $S_e = \{2k, 6k, 10k\}$; therefore $4k + S_e = S_e$, so the graph has Wilson type (C.1). For the graph in (2), see Lemma 5.5(1).

(\Rightarrow) Assume $X = \text{Cay}(\mathbb{Z}_n, S)$ has valency 5, and is nontrivially unstable. Then either Lemma 5.4 or Lemma 5.6 must apply (depending on whether the set of $\#$ -edges is invariant or not). So X is one of the graphs listed in these two lemmas, and is therefore listed in the statement of Theorem 5.1. □

Combining Lemmas 5.4 to 5.6 also yields the following observation that will be used in Section 7:

COROLLARY 5.7. *If X is a nontrivially unstable, 5-valent, circulant graph, then $\text{Aut } BX$ has exactly two orbits on the edges of BX .*

6. UNSTABLE CIRCULANTS OF VALENCY 6

See Corollary 6.2 for a more explicit formulation of the following Theorem 6.1.

THEOREM 6.1. *Every nontrivially unstable, circulant graph of valency 6 has Wilson type (C.1), (C.2'), (C.3'), or (C.4).*

Proof. Let $X = \text{Cay}(\mathbb{Z}_n, S)$ be a nontrivially unstable, circulant graph of valency 6, and write $S = \{\pm a, \pm b, \pm c\}$. The proof is by contradiction, so assume that X does not have any of the four listed Wilson types. As usual, we let $\# = n/2$, for convenience. The proof considers several cases.

Case 1. *Assume $2a = 2b$.* This means $b = a + \#$ (and $-b = -a + \#$). Since X does not have Wilson type (C.1), we also know that

$$(S \cap 2\mathbb{Z}_n) \neq \# + (S \cap 2\mathbb{Z}_n).$$

Therefore, we must have $S \cap 2\mathbb{Z}_n \neq \{\pm a, \pm b\}$. Since $S \not\subseteq 2\mathbb{Z}_n$, this implies

$$(2) \quad \{a, b\} \not\subseteq 2\mathbb{Z}_n.$$

We claim that $|c|$ is not divisible by 4. To see this, note that otherwise $n/\text{gcd}(c, n) = |c|$ is even, so $c/\text{gcd}(c, n)$ is odd, and we also know that $|2c|$ is even, so $\# \in \langle 2c \rangle$. This contradicts the fact that X does not have Wilson type (C.3') (with $H = \langle \# \rangle$, $R = \{\pm c\}$, and $d = \text{gcd}(c, n)$). This completes the proof of the claim.

Also note that $2c \notin \{\pm 2a, \pm 2b\}$. (For example, if $2c = 2a$, then, since $c \neq a$, we must have $c = a + \varkappa = b$, which contradicts the fact that a, b , and c must be distinct, because $|S| = 6$.) Furthermore, since $|c|$ is not divisible by 4, we also know that $|c| \neq 4$, so $2c \neq -2c$. Thus, we have

$$(3) \quad 2c \notin \{\pm 2a, \pm 2b, -2c\}.$$

Therefore, we see from Lemma 2.17 (with $S' = \{\pm c\}$) that

$$(4) \quad \text{every automorphism of } BX \text{ is also an automorphism of } \text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \{(\pm 2c, 0)\}).$$

We now consider two subcases.

Subcase 1.1. Assume $(c, 1)$ is the only common neighbor of $(0, 0)$ and $(2c, 0)$ in BX . Let α be an automorphism of BX that fixes $(0, 0)$, but does not fix $(0, 1)$. Combining (4) with the assumption of this subcase implies that α must preserve the set of c -edges.

If $|c|$ is odd, then letting $S_0 = \{\pm c\}$ in Lemma 3.4 implies that X is stable (because cycles of odd length are stable), which is a contradiction.

Therefore, since $|c|$ is not divisible by 4, we must have $|c| \equiv 2 \pmod{4}$, so $(|c|/2) \cdot (c, 1) = (\varkappa, 1)$, so this implies that

$$\alpha(v + (\varkappa, 1)) = \alpha(v) + (\varkappa, 1) \text{ for every vertex } v \text{ of } BX.$$

Also note that, since α preserves the set of c -edges in BX , it must also preserve the complement, which consists of the a -edges and b -edges. Hence, α is an automorphism of the canonical double cover of the 5-valent circulant graph $X' = \text{Cay}(\mathbb{Z}_n, S')$, where $S' = \{\pm a, \pm b, \varkappa\}$.

Let X'_0 be the connected component of X' that contains 0. Note that X'_0 is not stable (since α does not fix $(0, 1)$). Also note that X'_0 is connected, by definition. Furthermore, it is not bipartite, because X is not bipartite and $\varkappa = kc$ where k is odd. We therefore see from Lemma 2.14 that it is also twin-free. So X'_0 is nontrivially unstable, and must therefore be one of the graphs listed in Theorem 5.1 (after identifying the cyclic group $V(X'_0)$ with some \mathbb{Z}_m by a group isomorphism). Since $2a = 2b$, it follows that X'_0 is the graph in part (2) of Theorem 5.1, so

$$X' = \text{Cay}(\mathbb{Z}_n, \{\pm n/8, \pm 3n/8, \varkappa\}).$$

Therefore, if we write $|c| = 2k$, where k is odd, then

$$X = \text{Cay}(\mathbb{Z}_{8k}, \{\pm k, \pm 3k, \pm c\}).$$

So X has Wilson type (C.3'), with $H = \langle 2k \rangle$, $R = \{\pm c\}$,

$$d = \gcd(c, 8k) = \frac{8k}{|c|} = \frac{8k}{2k} = 4,$$

and $H \not\subseteq d\mathbb{Z}_{8k}$.

Subcase 1.2. Assume $(c, 1)$ is not the only common neighbor of $(0, 0)$ and $(2c, 0)$ in BX . This implies that

$$2c \text{ is equal to either } -2c \text{ or } a - c \text{ or } 2a \text{ or } a + b \text{ or } a - b$$

(perhaps after interchanging a with b and/or replacing both of them with their negatives).

However, we know from (3) that $2c \neq -2c$ and $2c \neq 2a$. Also, if $2c = a - b = \varkappa$, then $|c| = 4$, which contradicts the fact that $|c|$ is not divisible by 4. Thus, only two possibilities need to be considered.

Subsubcase 1.2.1. Assume $2c = a + b = 2a + \#$. Since $2a$ and $2c$ are even, this implies $\#$ is even, which means $n \equiv 0 \pmod{4}$. It also implies that a and b have the same parity, so we conclude from (2) that a and b are odd. Since X is not bipartite, then c must be even. Therefore,

$$0 \equiv 2c = 2a + \# \equiv 2 + \# \pmod{4},$$

so $\# \equiv 2 \pmod{4}$, which means $n/4$ is odd. Also note that $4c = 4a = 4b$. Therefore, setting $k := n/4$, we get that

$$X = \text{Cay}(\mathbb{Z}_{4k}, \{\pm c, \pm(c+k), \pm(c-k)\}).$$

If $c \equiv 0 \pmod{4}$, this graph has Wilson type (C.2') (with $h = k$).

So we may assume $c \equiv 2 \pmod{4}$. We will show that this implies X is stable (which is a contradiction). Any two vertices of BX that are in the same coset of the subgroup $\langle(k, 0)\rangle$ have 4 common neighbors, but no two vertices of BX that are in different cosets of $\langle(k, 0)\rangle$ have more than 3 common neighbors. Therefore, each coset of $\langle(k, 0)\rangle$ is a block for $\text{Aut } BX$. Also note that $(c + 2k, 1)$ is the only element of the coset $(c, 1) + \langle(k, 0)\rangle$ that is not adjacent to $(0, 0)$. So every automorphism of BX must preserve the set of $(c + 2k)$ -edges. Since $c + 2k$ is an element of odd order (because $c + 2k \equiv 2 + 2 \equiv 0 \pmod{4}$) and $n = 4k$ is not divisible by 8), we conclude from Lemma 3.4 that X is stable.

Subsubcase 1.2.2. Assume $2c = a - c$. This implies $a = 3c$, so

$$\mathbb{Z}_n = \langle a, b, c \rangle = \langle 3c, 3c + \#, c \rangle = \langle c, \# \rangle.$$

If $|c|$ is even, we know that $\# \in \langle c \rangle$, so $\langle c \rangle = \mathbb{Z}_n$. This implies that c is odd, so $\{a, b\} \cap 2\mathbb{Z}_n \neq \emptyset$ (because X is not bipartite). Since $\{a, b\} \not\subseteq 2\mathbb{Z}_n$, this implies $\#$ is odd (i.e., $n \equiv 2 \pmod{4}$). And Lemma 2.17 (together with (3)) implies that $\langle 2(c, 1) \rangle = 2\mathbb{Z}_n \times \{0\}$ is a block for the action of $\text{Aut } BX$. Then, since $|S \cap 2\mathbb{Z}_n| = 2 \leq 5$, we see from Proposition 2.18 that X has Wilson type (C.1) or (C.4).

We may now assume that $|c|$ is odd. Since $2a = 2b$ and we may assume that Subsubcase 1.2.1 does not apply, it is easy to see that X is stable by Lemma 3.5, which is a contradiction.

Case 2. Assume $|a| = 4$, and the previous case does not apply.

Subcase 2.1. Assume some automorphism of BX maps an a -edge to a b -edge. Then Proposition 2.15 tells us that S contains every generator of $\langle b \rangle$. Since $|S| = 6$ (and the only generators of $\langle a \rangle$ are $\pm a$), it follows that $\phi(|b|) \leq 4$. We also know $|b| \geq 3$, so we conclude that

$$|b| \in \{3, 4, 5, 6, 8, 10, 12\}.$$

Also note that if $\phi(|b|) = 4$, then $\{\pm b, \pm c\}$ consists of the 4 generators of $\langle b \rangle$, so $n = \text{lcm}(|a|, |b|) = \text{lcm}(4, |b|)$.

- If $|b| = 4$, then a and b are generators of the same cyclic subgroup of order 4, but then $\{\pm a\} = \{\pm b\}$, a contradiction.

- If $|b| \in \{5, 10\}$, then $\phi(|b|) = 4$, so $\{\pm b, \pm c\}$ consists of the 4 generators of $\langle b \rangle$, and we have $n = \text{lcm}(4, |b|) = 20$. If $|b| = 5$, then X is isomorphic to $\text{Cay}(\mathbb{Z}_{20}, \{\pm 4, \pm 5, \pm 8\})$; if $|b| = 10$, then X is isomorphic to $\text{Cay}(\mathbb{Z}_{20}, \{\pm 2, \pm 5, \pm 6\})$. Both of these graphs are stable by Lemma 3.7(4).

- If $|b| \in \{8, 12\}$, then $\{\pm a\} \subseteq \langle b \rangle$, and a is not a generator of this subgroup. So the other two generators of $\langle b \rangle$ must be $\pm c$. This implies that b and c each generate the whole group \mathbb{Z}_n . If $|b| = 8$, then X is isomorphic to $\text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 2, \pm 3\})$, which is not twin-free since $4 + S = S$. If $|b| = 12$, then X is isomorphic to $\text{Cay}(\mathbb{Z}_{12}, \{\pm 1, \pm 3, \pm 5\})$, which is bipartite.

- We may now assume $|b| \in \{3, 6\}$. We consider two subsubcases.

Subsubcase 2.1.1. Assume $|b| \in \{3, 6\}$ and $|c| \in \{3, 6\}$. If $|c| = |b|$, then b and c generate the same cyclic subgroup, which has only two generators, which contradicts the fact that $\{\pm b\} \neq \{\pm c\}$. So we may assume $|b| = 3$ and $|c| = 6$. Then X is isomorphic to $\text{Cay}(\mathbb{Z}_{12}, \{\pm 2, \pm 3, \pm 4\})$, which is not twin-free since $6 + S = S$.

Subsubcase 2.1.2. Assume $|b| \in \{3, 6\}$ and $|c| \notin \{3, 6\}$. Then we can assume that no isomorphism of BX maps an a -edge to a c -edge (for otherwise an earlier argument would apply after interchanging b and c). Since a -edges can be mapped to b -edges, this implies that no b -edge can be mapped to a c -edge. Therefore, every automorphism of BX is an automorphism of BX_0 , where $X_0 = \text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b\})$. Let X'_0 be the connected component of X_0 that contains 0. Recalling that $|a| = 4$, we see that if $|b| = 3$ then X'_0 is isomorphic to $\text{Cay}(\mathbb{Z}_{12}, \{\pm 3, \pm 4\})$, and if $|b| = 6$, then X'_0 is isomorphic to $\text{Cay}(\mathbb{Z}_{12}, \{\pm 2, \pm 3\})$. Both of these graphs are stable (by Lemma 3.7(2) or Theorem 4.3). So Lemma 3.4 tells us that X is stable as well, a contradiction.

Subcase 2.2. Assume every automorphism of BX maps a -edges to a -edges. Then every automorphism of BX is also an automorphism of BX_0 , where $X_0 = \text{Cay}(\mathbb{Z}_n, \{\pm b, \pm c\})$. As usual, let X'_0 be the connected component of X_0 that contains 0.

Subsubcase 2.2.1. Assume X_0 is not twin-free. Then $b = c + \#$ (perhaps after replacing c with $-c$). Since $a + \# = -a$, this implies $S + \# = S$, which contradicts the fact that X is twin-free.

Subsubcase 2.2.2. Assume X_0 is not connected (but is twin-free). We know that X'_0 is not stable (for otherwise Lemma 3.4 contradicts the fact that X is not stable). Since X'_0 is connected (by definition) and twin-free (by assumption), this implies that it has even order (see Theorem 3.1). Since $\langle a, b, c \rangle = \mathbb{Z}_n$ and $|a| = 4$, we conclude that $\langle b, c \rangle = 2\mathbb{Z}_n$ and $n \equiv 4 \pmod{8}$.

- If X'_0 is bipartite, then BX_0 is isomorphic to the union of four disjoint copies of X'_0 . Since $|a| = 4$ and BX is connected, this implies that $BX \cong C_4 \square X'_0$. Since X'_0 has even valency, and $|V(X'_0)|/2$ is odd, it is easy to see that X'_0 does not have K_2 as a Cartesian factor. (If $X'_0 \cong K_2 \square Y$, then Y is a regular graph of odd valency and odd order, which is impossible.) This implies (by Proposition 2.13) that

$$|\text{Aut } BX| = |\text{Aut}(C_4 \square X'_0)| = |\text{Aut } C_4| \cdot |\text{Aut } X'_0| = 8 |\text{Aut } X'_0|.$$

Also note that, since X'_0 is a bipartite circulant graph whose order is congruent to 2 modulo 4, it is (isomorphic to) the canonical double cover of a circulant graph of odd order. Since connected, twin-free, circulant graphs of odd order are stable (see Theorem 3.1) and $2a$ is the element of order 2 in \mathbb{Z}_n , we conclude that if β is any automorphism of X'_0 , then $\beta(v + 2a) = \beta(v) + 2a$ for every $v \in V(X'_0)$. This implies that we can extend β to an automorphism β' of X by defining $\beta'(a + v) = a + \beta(v)$ for $v \in V(X'_0)$; so $\text{Aut } X$ contains a copy of $\text{Aut } X'_0$. Since $\text{Aut } X$ also contains the translation $v \mapsto v + a$ and the negation automorphism $v \mapsto -v$, we conclude that $|\text{Aut } X| \geq 4 |\text{Aut } X'_0|$. Combining this with the above calculation of $|\text{Aut } BX|$ contradicts the fact that X is not stable.

- If X'_0 is not bipartite, then X'_0 is nontrivially unstable, and therefore must be described by Theorem 4.3. Since $|2\mathbb{Z}_n| \equiv 2 \pmod{4}$, we must be in the situation of 4.3(1): $\langle c \rangle = 2\mathbb{Z}_n$, and $b = mc + \#$, for some $m \in \mathbb{Z}_n^\times$, such that $m^2 \equiv \pm 1 \pmod{n}$. (Technically, Theorem 4.3 only tells us that $m^2 \equiv 1 \pmod{n/2}$. However, we know that m is odd, so $m^2 \equiv 1 \pmod{4}$. Therefore $m^2 \equiv 1 \pmod{n}$.) We also have $ma + \# = \pm a$ (since $|a| = 4$). Therefore $mS + \# = S$, which means that X has Wilson type (C.4).

Subsubcase 2.2.3. Assume X_0 is connected and bipartite. This means that b and c are odd, so $S \cap 2\mathbb{Z}_n = \{\pm a\}$. Since $|a| = 4$, we conclude that $(S \cap 2\mathbb{Z}_n) + \# = (S \cap 2\mathbb{Z}_n)$, so X has Wilson type (C.1).

Subsubcase 2.2.4. Assume X_0 is nontrivially unstable. Then Theorem 4.3 tells us that X_0 has Wilson type (C.4). Then X also has Wilson type (C.4), with the same value of m .

Case 3. Assume neither of the previous cases apply (even after permuting and/or negating some of the generators). Let α be an automorphism of BX that fixes $(0, 0)$. The assumption of this case implies that $2s \neq 2t$ for all $s, t \in S$, such that $s \neq t$. Therefore, Lemma 2.17 implies that the cosets of $2\mathbb{Z}_n \times \{0\}$ are blocks for the action of $\text{Aut } BX$. So α must fix the two cosets that are in $\mathbb{Z}_n \times \{0\}$, and either fixes or interchanges the other two. However, also note that S is the disjoint union of

$$S_e := S \cap 2\mathbb{Z}_n \text{ and } S_o := S \cap (1 + 2\mathbb{Z}_n).$$

Each of these two sets has even cardinality (since it is closed under inverses), and $|S_e| + |S_o| = 6$, so it is easy to see that $|S_e| \neq |S_o|$. Therefore, α cannot interchange S_e and S_o , which means that α must fix all four cosets of $2\mathbb{Z}_n \times \{0\}$. So

$$(5) \quad \alpha \text{ maps } S_e\text{-edges to } S_e\text{-edges, and maps } S_o\text{-edges to } S_o\text{-edges.}$$

Hence, by Lemma 3.4, we know that

$$\text{the connected components of } \text{Cay}(\mathbb{Z}_n, S_e) \text{ are unstable.}$$

(The connected components of $\text{Cay}(\mathbb{Z}_n, S_o)$ are always unstable, since they are bipartite.) We also know that $\text{Cay}(\mathbb{Z}_n, S_e)$ is twin-free (since X does not have Wilson type (C.1)). Therefore, we see from Theorem 3.1 that $|\langle S_e \rangle|$ is even, so

$$n \equiv 0 \pmod{4}.$$

From Lemma 2.9, we see that BX is not normal. So applying Proposition 2.12 to BX implies that either $|a| = 4$ or $3a = b$ or $2a = 2b$ or $2a = b + c$ (perhaps after permuting and/or negating some of the generators). Since neither of the previous cases apply, this implies that

$$\text{either } 3a = b \text{ or } 2a = b + c.$$

We will consider each of these two possibilities as a separate subcase.

Recall that Definition 2.1(2) introduced \tilde{s} as an abbreviation for $(s, 1)$ with $s \in S$.

Subcase 3.1. Assume $3a = b$. Then also $3\tilde{a} = \tilde{b}$.

We claim that $|\tilde{a}| \geq 10$. First of all, we have $|\tilde{a}| \neq 2$, because $|a| \notin \{1, 2\}$. We also have $|\tilde{a}| \neq 4$, because $|a| \neq 4$ (since Case 2 does not apply). Now, note that if $|\tilde{a}| = 6$, then $|a| = 3$ or $|a| = 6$; however, the fact that $b = 3a$ would then imply that $|a| = 3|b|$, so $|b| \in \{1, 2\}$, which is a contradiction. Finally, note that if $|a| = 8$, then $2(-a) = -2a = 6a = 2(3a) = 2b$, which contradicts the assumption that Case 1 does not apply. This completes the proof of the claim.

Subsubcase 3.1.1. Assume $|\tilde{a}| = 10$. Then $|a| = 5$ or $|a| = 10$. However, if $|a| = 5$, then $|b| = 5$ as well and they generate the same cyclic subgroup of \mathbb{Z}_n . In particular, the connected components of $\text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b\})$ are isomorphic to K_5 , which is stable, so by Lemma 3.4, we get that X is stable, a contradiction.

Therefore, we must have $|a| = 10$. (Then $|b| = 10$ and they generate the same cyclic subgroup.) Since $n \equiv 0 \pmod{4}$, we may write

$$n = 20k \text{ for some } k \in \mathbb{Z}.$$

Since $|a| = |b| = 10 \not\equiv 0 \pmod{4}$ and $n \equiv 0 \pmod{4}$, it is clear that $|c|$ is even (in fact, it is divisible by 4). It follows that $|c|$ is either n (if $|c|$ is divisible by 5) or $n/5$ (otherwise). So we see that (up to isomorphism) X is

$$\text{Cay}(\mathbb{Z}_{20k}, \{\pm c, 2k, 6k, 14k, 18k\}) \text{ with } c = 1 \text{ or } c = 5.$$

From (5), we know that α is also an automorphism of the graphs

$$Y_1 = \text{Cay}(\mathbb{Z}_{20k} \times \mathbb{Z}_2, \{(\pm c, 1)\})$$

and

$$Y_2 = \text{Cay}(\mathbb{Z}_{20k} \times \mathbb{Z}_2, \{(2k, 1), (6k, 1), (14k, 1), (18k, 1)\}).$$

Note that $(0, 0)$ and $(10k, 1)$ lie in the same connected component of Y_2 , which is isomorphic to $K_{5,5} - 5K_2$. In this component, $(10k, 1)$ is the unique vertex at distance 3 from $(0, 0)$, so α fixes $(10k, 1)$.

Also note that the vertices $(0, 1)$ and $(10k, 1)$ lie in the same connected component of Y_1 , which is a cycle (of length $20k$ or $4k$), and that these two vertices are diametrically opposite on this cycle. Since we already know that α fixes $(10k, 1)$, it must also fix $(0, 1)$. We conclude that X is stable, a contradiction.

Subsubcase 3.1.2. Assume $|\tilde{a}| = 12$. Then $|a| = 12$. Since $b = 3a$, we have $|b| = |3a| = 4$, which contradicts the fact that Case 2 does not apply.

Subsubcase 3.1.3. Assume $|\tilde{a}| \geq 14$. Let B_2 be the subgraph induced by the ball of radius 2 centered at 0 in $\text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \{\pm\tilde{a}, \pm\tilde{b}\})$. This graph is drawn in Figure 1, under the assumption of this subcase that $|\tilde{a}| \geq 14$. From this drawing, it can be seen that $\pm\tilde{b}$ are the only vertices in B_2 that have a pendant edge. (These edges are colored white in the figure.) So $\{\pm\tilde{b}\}$ is α -invariant. This means that α maps b -edges to b -edges. Since we already know that α maps c -edges to c -edges, it must also map a -edges to a -edges.

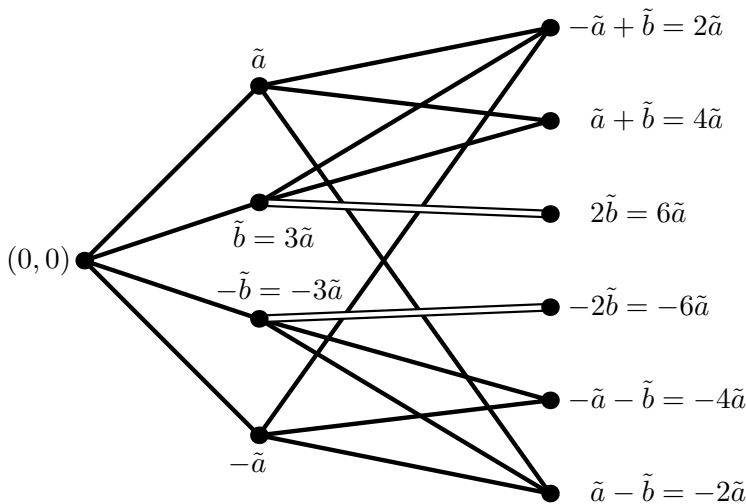


FIGURE 1. The subgraph B_2 induced by the ball of radius 2 centered at $(0, 0)$ in $\text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \pm\tilde{a}, \pm\tilde{b})$.

In the terminology of [6], this means that α is a *color-preserving* graph automorphism. We will now use a simple argument from [6, §4] to establish that α is a group automorphism (so BX is normal, and then it follows from Lemma 2.9 that X has Wilson type (C.4)).

We provide only a sketch of the proof. By composing with negation, if necessary, we may assume that $\alpha(\tilde{a}) = \tilde{a}$. This implies $\alpha(k\tilde{a}) = k\tilde{a}$ for all $k \in \mathbb{Z}$. Let $m \in \mathbb{Z}^+$, such that $m\tilde{c} \in \langle \tilde{a} \rangle$. Then $\alpha(km\tilde{c}) = km\tilde{c}$ for all $k \in \mathbb{Z}$. If $2m\tilde{c} \neq 0$, this implies that $\alpha(\ell\tilde{c}) = \ell\tilde{c}$ for all $\ell \in \mathbb{Z}$; in fact, $\alpha(k\tilde{a} + \ell\tilde{c}) = k\tilde{a} + \ell\tilde{c}$, for all $k, \ell \in \mathbb{Z}$, which means that α is the identity map, contradicting the fact that $\alpha \notin \text{Aut } X \times \mathbb{Z}_2$.

Therefore, we may assume $2m\tilde{c} = 0$, for all $m \in \mathbb{Z}$, such that $m\tilde{c} \in \langle \tilde{a} \rangle$. This means that $|\langle \tilde{c} \rangle \cap \langle \tilde{a} \rangle| \leq 2$, so there is a group automorphism of \mathbb{Z}_n that fixes \tilde{a} and negates \tilde{c} . So we may assume that $\alpha(\tilde{c}) = \tilde{c}$. Since $\tilde{a} + \tilde{c}$ is the only common neighbor of \tilde{a} and \tilde{c} , we must have $\alpha(\tilde{a} + \tilde{c}) = \tilde{a} + \tilde{c}$. Similarly, we must have $\alpha(2\tilde{a} + \tilde{c}) = 2\tilde{a} + \tilde{c}$ and $\alpha(\tilde{a} + 2\tilde{c}) = \tilde{a} + 2\tilde{c}$. Repeating the argument shows that $\alpha(k\tilde{a} + \ell\tilde{c}) = k\tilde{a} + \ell\tilde{c}$, for all $k, \ell \in \mathbb{Z}$, so, once again, α is the identity map.

Subcase 3.2. Assume $2a = b + c$. (Note that this implies $2\tilde{a} = \tilde{b} + \tilde{c}$.) We will show that $\alpha(k\tilde{b} + \ell\tilde{a}) = k\alpha(\tilde{b}) + \ell\alpha(\tilde{a})$, for all $k, \ell \in \mathbb{Z}^{\geq 0}$. (This implies that α is a group automorphism of \mathbb{Z}_n , so BX is normal, so Lemma 2.9 implies that X has Wilson type (C.4).)

Since b and c have the same parity, we see from (5) that α maps $\{b, c\}$ -edges to $\{b, c\}$ -edges, and maps a -edges to a -edges. In particular, we may assume (by composing with negation if necessary) that

$$(6) \quad \alpha \text{ fixes every element of } \langle \tilde{a} \rangle.$$

Since α maps $\{b, c\}$ -edges to $\{b, c\}$ -edges, α is an automorphism of the graph $\text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \{\pm\tilde{b}, \pm\tilde{c}\})$, which has at most two connected components. Let X'_0 be its connected component containing $(0, 0)$ with the vertex set $\langle \tilde{b}, \tilde{c} \rangle$. Since α fixes $(0, 0)$, it restricts to an automorphism of X'_0 . Because we are assuming Case 1 and Case 2 do not hold and we can additionally assume Subcase 3.1 does not hold either, Proposition 2.12 applies to X'_0 . It follows that the restriction of α to $V(X'_0)$ is a group automorphism of $\langle \tilde{b}, \tilde{c} \rangle$. We let $b', c' \in \{\pm b, \pm c\}$, such that $\alpha(\tilde{b}) = \tilde{b}'$ and $\alpha(\tilde{c}) = \tilde{c}'$. It follows that:

$$(7) \quad \alpha(k\tilde{b} + \ell\tilde{c}) = k\tilde{b}' + \ell\tilde{c}' \quad \forall k, \ell \in \mathbb{Z}.$$

Notice that:

$$\begin{aligned} \tilde{b} + \tilde{c} &= 2\tilde{a} && \text{(assumption of Subcase 3.2)} \\ &= \alpha(2\tilde{a}) && (6) \\ &= \alpha(\tilde{b} + \tilde{c}) && (2\tilde{a} = \tilde{b} + \tilde{c}) \\ &= \tilde{b}' + \tilde{c}' && (7). \end{aligned}$$

As we have already established that the hypothesis of Proposition 2.12 holds, we obtain that $\{b', c'\} = \{b, c\}$.

To complete the proof of this Subcase 3.2, we now prove by induction on k that, for all $k, \ell \in \mathbb{Z}^{\geq 0}$, we have

$$\alpha(k\tilde{b} + \ell\tilde{a}) = k\tilde{b}' + \ell\tilde{a}.$$

The base case is provided by (6), so assume $k > 0$. Since α maps a -edges to a -edges, there exists $\epsilon \in \{\pm 1\}$, such that $\alpha(k\tilde{b} + \ell\tilde{a}) = k\tilde{b}' + \epsilon\ell\tilde{a}$ for all $\ell \in \mathbb{Z}$. We wish to show that $\epsilon = 1$, so suppose $\epsilon = -1$. (This will lead to a contradiction.) Letting $\ell = -2$

tells us

$$\begin{aligned}
 (k-1)\tilde{b}' - \tilde{c}' &= \alpha((k-1)\tilde{b} - \tilde{c}) & (7) \\
 &= \alpha(k\tilde{b} - 2\tilde{a}) & (2\tilde{a} = \tilde{b} + \tilde{c}) \\
 &= k\tilde{b}' + 2\tilde{a} & (\epsilon = -1) \\
 &= k\tilde{b}' + (\tilde{b} + \tilde{c}) & (2\tilde{a} = \tilde{b} + \tilde{c}) \\
 &= k\tilde{b}' + (\tilde{b}' + \tilde{c}') & (\{b, c\} = \{b', c'\}).
 \end{aligned}$$

This implies $-2\tilde{b}' = 2\tilde{c}'$, which contradicts the fact that Case 1 does not apply. \square

The following result provides a more explicit version of Theorem 6.1. Remark 6.3 explains which of these graphs are nontrivially unstable.

COROLLARY 6.2. *A circulant graph $X = \text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b, \pm c\})$ of valency 6 is unstable if and only if either it is trivially unstable, or it is one of the following:*

- (1) $\text{Cay}(\mathbb{Z}_{8k}, \{\pm a, \pm b, \pm 2k\})$, where a and b are odd, which is of Wilson type (C.1).
- (2) $\text{Cay}(\mathbb{Z}_{4k}, \{\pm a, \pm b, \pm b + 2k\})$, where a is odd and b is even, which is of Wilson type (C.1).
- (3) $\text{Cay}(\mathbb{Z}_{4k}, \{\pm a, \pm(a+k), \pm(a-k)\})$, where $a \equiv 0 \pmod{4}$ and k is odd, which is of Wilson type (C.2').
- (4) $\text{Cay}(\mathbb{Z}_{8k}, \{\pm a, \pm b, \pm b + 4k\})$, where a is even and $|a|$ is divisible by 4, which is of Wilson type (C.3').
- (5) $\text{Cay}(\mathbb{Z}_{8k}, \{\pm a, \pm k, \pm 3k\})$, where $a \equiv 0 \pmod{4}$ and k is odd, which is of Wilson type (C.3').
- (6) $\text{Cay}(\mathbb{Z}_{4k}, \{\pm a, \pm b, \pm mb + 2k\})$, where

$$\gcd(m, 4k) = 1, \quad (m-1)a \equiv 2k \pmod{4k}, \quad \text{and}$$

$$\text{either } m^2 \equiv 1 \pmod{4k} \text{ or } (m^2 + 1)b \equiv 0 \pmod{4k},$$

which is of Wilson type (C.4).

- (7) $\text{Cay}(\mathbb{Z}_{8k}, \{\pm a, \pm b, \pm c\})$, where there exists $m \in \mathbb{Z}$, such that

$$\gcd(m, 8k) = 1, \quad m^2 \equiv 1 \pmod{8k}, \quad \text{and}$$

$$(m-1)a \equiv (m+1)b \equiv (m+1)c \equiv 4k \pmod{8k},$$

which is of Wilson type (C.4).

Proof. (\Leftarrow) It is easy to see that each graph has the specified Wilson type, and is therefore unstable.

(\Rightarrow) If X is unstable, then we know from Theorem 6.1 that X has Wilson type (C.1), (C.2'), (C.3'), or (C.4). We treat each of these possibilities as a separate case.

Case 1. Assume X has Wilson type (C.1).

Subcase 1.1. Assume $|S_e| = 2$. Then we may assume $S_e = \{\pm c\}$. Since X has Wilson type (C.1), we must have $-c = c + \#$, so $|c| = 4$. (Since $c \in S_e$, this implies that n is divisible by 8.) Therefore X is as described in part (1) of Corollary 6.2.

Subcase 1.2. Assume $|S_e| = 4$. Then we may assume $S_e = \{\pm b, \pm c\}$. Since X has Wilson type (C.1) (and $|S_e|$ is a power of 2), we must have $S_e + \# = S_e$. Therefore, we may assume $c = b + \#$ (because we cannot have $b + \# = -b$ and $c + \# = -c$). Since b and c are even, this implies $\#$ is even, so n is divisible by 4. Hence, X is as described in part (2) of Corollary 6.2.

Case 2. Assume X has Wilson type (C.2'). Let $h \in 1 + 2\mathbb{Z}_n$, such that (a) and (b) of condition (C.2') hold.

Subcase 2.1. Assume $|S_o| = 2$. Then we may assume $S_o = \{\pm c\}$, so part (a) of (C.2') tells us $-c = c + 2h$ and $c = -c + 2h$. This implies that $|c| = 4$ and $|h| = 4$. So we may assume $c = h = n/4$. Then it is obvious that $-c \equiv -h \pmod{4}$, so part (b) of (C.2') implies that $0 = -c + h \in S$. This contradicts the fact that graphs in this paper do not have loops (see Assumption 1).

Subcase 2.2. Assume $|S_o| = 4$. Then we may assume $S_o = \{\pm b, \pm c\}$, so part (a) of (C.2') implies that we may assume $c = b + 2h$ (by replacing b with $-b$ if necessary), and we must have $|2h| \in \{2, 4\}$.

Subsubcase 2.2.1. Assume $|2h| = 2$. Then we have $c = b + \#$. We may assume $b \equiv -h \pmod{4}$ (by interchanging b and c if necessary). So part (a) of condition (C.2') implies $b + h \in S$, which means $a = b + h$ (perhaps after replacing a with $-a$). In other words, we have $b = a - h$; then $c = b + 2h = a + h$. Therefore, since $|h| = 4$, we have

$$S = \{\pm a, \pm(a + (n/4)), \pm(a - (n/4))\}.$$

We see from part (b) of condition (C.2') that S contains an element that is divisible by 4, so we must have $a \equiv 0 \pmod{4}$. Also, since $a + (n/4) = b$ is odd, we know that $n/4$ is odd. Hence, X is as described in part (3) of Corollary 6.2.

Subsubcase 2.2.2. Assume $|2h| = 4$. Then

$$\{\pm b, \pm c\} = \{b, b + 2h, b + 4h, b + 6h\},$$

so $-c = b + 4h$ (and $-b = b + 6h$). Since $c = b + 2h$, this implies $2c = 6h$ has order 4, so $|b| = 8$. So $S_o = \{\pm n/8, \pm 3n/8\}$ consists of all of the elements of order 8. Since $-h$ has order 8, we conclude that $-h \in S_o$, so part (b) of condition (C.2') implies $0 \in S$, which (again) contradicts the fact that graphs in this paper do not have loops.

Case 3. Assume X has Wilson type (C.3'). Let H , R , and d be as in the definition of Wilson type (C.3'). We may assume that X does not have Wilson type (C.1), so R contains at least one element of S_e ; for definiteness, let us say that a is in $R \cap S_e$. Since r/d is odd for every $r \in R$, we know that all elements of R have the same parity, so this implies $R \subseteq S_e$.

Subcase 3.1. Assume $|R| = 2$. This means $R = \{\pm a\}$, $d = \gcd(n, a)$, and $\{\pm b, \pm c\} + H = \{\pm b, \pm c\}$. Let h be a generator of H , so $b + h = c$ (perhaps after replacing c with $-c$).

Subsubcase 3.1.1. Assume $|h| = 2$ (so $h = \#$). Since n/d is even, we know that $\# \in d\mathbb{Z}_n$. Therefore, the last condition in (C.3') implies that $|a|$ is divisible by 4. (Since a is even, this implies that n is divisible by 8.) Hence, X is as described in part (4) of Corollary 6.2.

Subsubcase 3.1.2. Assume $|h| = 4$. The argument of Subsubcase 2.2.2 shows $\{\pm b, \pm c\} = \{\pm n/8, \pm 3n/8\}$. Since the elements of S cannot all be even, we know that $n/8$ is odd. Then, since a is even, we know that $2a\mathbb{Z}_n$ does not contain an element of order 4, so we conclude from the last sentence of condition (C.3') that $a\mathbb{Z}_n$ does not contain H . This means that a is divisible by 4. Hence, X is as described in part (5) of Corollary 6.2.

Subcase 3.2. Assume $|R| = 4$. This means we may assume $R = \{\pm a, \pm b\}$, $H = \langle \# \rangle$, and $|c| = 4$. Since n/d is even, we know that the elements of R have even order, so $\# \in \langle R \rangle = d\mathbb{Z}_n$. Therefore, the last sentence of condition (C.3') implies $\# \in 2d\mathbb{Z}_n$, which means n/d is divisible by 4. Since r/d is odd for every $r \in R$, this implies $|a| \equiv |b| \equiv 0 \pmod{4}$. Since $a \in R \subseteq S_e$, we conclude that n is divisible by 8. So $c = \pm n/4$ is even. This contradicts the fact that at least one element of S must be odd (since S generates \mathbb{Z}_n).

Case 4. Assume X has Wilson type (C.4). This means there exists $m \in \mathbb{Z}$ such that $\# + mS = S$ and $\gcd(m, n) = 1$. Let $\alpha(x) := \# + mx$, so $\alpha(S) = S$. Since $|S_e| \neq |S_o|$, we know that $\alpha(S_e) \neq S_o$. Therefore, we must have $\alpha(S_e) = S_e$ and $\alpha(S_o) = S_o$. This means that $\#$ is even (so n is divisible by 4).

Assume, without loss of generality, that b and c have the same parity (and a has the opposite parity).

We must have $\# + ma \in \{\pm a\}$. So we may assume $\# + ma = a$ (by replacing m with $-m$ if necessary), so $(m - 1)a = \#$.

We also have $\alpha(\{\pm b, \pm c\}) = \{\pm b, \pm c\}$. Therefore, either α fixes $\{\pm b\}$ and $\{\pm c\}$, or α interchanges these two sets.

Subcase 4.1. Assume $\alpha(b) \in \{\pm b\}$ and $\alpha(c) \in \{\pm c\}$. This means $\# + mb = \pm b$ and $\# + mc = \pm c$, so $(m \pm 1)b = \#$ and $(m \pm 1)c = \#$. Since $(m - 1)a = \#$, and a has the opposite parity from b and c , we cannot have $(m - 1)b = \#$ or $(m - 1)c = \#$. Therefore, we must have $(m + 1)b = \#$ and $(m + 1)c = \#$.

We know that m is odd, so either $m - 1$ or $m + 1$ is divisible by 4. Since $(m - 1)a = \#$ and $(m + 1)b = \#$, this implies that $\#$ is divisible by 4. So n is divisible by 8

Note that, since $m + 1$ is even, we have

$$(m^2 - 1)a = (m + 1)(m - 1)a = (m + 1)\# = 0.$$

Similarly, we also have $(m^2 - 1)b = 0$ and $(m^2 - 1)c = 0$. Since $\langle a, b, c \rangle = \mathbb{Z}_n$, this implies that $m^2 \equiv 1 \pmod{n}$. Hence, X is as described in part (7) of Corollary 6.2.

Subcase 4.2. Assume $\alpha(b) \in \{\pm c\}$ and $\alpha(c) \in \{\pm b\}$. We have $\alpha^2(b) \in \{\pm b\}$, which means $\pm b = \# + m(\# + mb) = m^2b$, so there exists $\epsilon \in \{\pm 1\}$, such that $(m^2 + \epsilon)b = 0$.

Subsubcase 4.2.1. Assume $\epsilon = -1$. This means $(m^2 - 1)b = 0$. Since m must be odd, we also have

$$(m^2 - 1)a = (m + 1) \cdot (m - 1)a = (m + 1)\# = 0$$

and $(m^2 - 1)\# = 0$. Since

$$\mathbb{Z}_n = \langle a, b, c \rangle = \langle a, b, \# + mb \rangle = \langle a, b, \# \rangle,$$

we conclude that $(m^2 - 1)\mathbb{Z}_n = \{0\}$, so $m^2 \equiv 1 \pmod{n}$. Hence, X is as described in part (6) of Corollary 6.2 (with $m^2 \equiv 1 \pmod{n}$).

Subsubcase 4.2.2. Assume $\epsilon = 1$. This means $(m^2 + 1)b = 0$. Hence, X is as described in part (6) of Corollary 6.2 (with $(m^2 + 1)b = 0$). \square

REMARK 6.3. It is easy to determine whether a graph in Corollary 6.2 is nontrivially unstable. Indeed, here are quite simple necessary and sufficient conditions for each of the lists in the statement of Corollary 6.2:

6.2(1): $\gcd(a, b, k) = 1$ and $b \notin \{\pm a + 4k\}$.

6.2(2): $\gcd(a, b, k) = 1$ and $a \notin \{\pm k\}$.

6.2(3): $\gcd(a, k) = 1$.

6.2(4): $\gcd(a, b, 4k) = 1$ and $a \notin \{\pm 2k\}$.

6.2(5): $\gcd(a, k) = 1$.

6.2(6): $\gcd(a, b, 2k) = 1$, either a or b is even, and either $a \notin \{\pm k\}$ or $mb \notin \{\pm b\}$.

6.2(7): $\gcd(a, b, c, 4k) = 1$, either a or b is even, and either $a \notin \{\pm 2k\}$ or $c \notin \{\pm b + 4k\}$.

Proof. For convenience, let $S = \{\pm a, \pm b, \pm c\}$.

It is clear that X is connected if and only if $\gcd(S \cup \{n\}) = 1$. Therefore, the first condition in each part of Remark 6.3 is precisely the condition for X to be connected.

Knowing that $\gcd(S \cup \{n\}) = 1$ implies that at least one element of S is odd (since n is even). Therefore, X is nonbipartite if and only if at least one element of S is even.

It is obvious that S has an even element in all parts of Corollary 6.2 other than (6) and (7), so the statement of Remark 6.3 only adds this as an explicit condition for parts (6) and (7). (In part (7), the fact that $(m + 1)b \equiv (m + 1)c \equiv 4k \pmod{8k}$ implies that b and c have the same parity, so there is no need to mention the possibility that c is even.)

Now, let us suppose that X is not twin-free (but is connected and nonbipartite). Then by Lemma 2.14(1), it follows that $X \cong Y \wr \overline{K_m}$ with Y being a connected circulant of valency δ and $m \geq 2$ an integer. Clearly, $6 = \delta m$, so $m \in \{2, 3, 6\}$.

- If $m = 6$, then $\delta = 1$ and $Y \cong K_2$. Consequently, $X \cong K_2 \wr \overline{K_6} \cong K_{6,6}$, which contradicts the fact that X is not bipartite.
- If $m = 3$, then $\delta = 2$ and Y is a cycle of even length, which again contradicts the fact that X is not bipartite.
- If $m = 2$, then from the proof of Lemma 2.14(3), it can be concluded that the unique twin of 0 is \mathfrak{n} . Therefore, it must hold that $\mathfrak{n} + S = S$.

Thus, we see that X is twin-free if and only if $\mathfrak{n} + S \neq S$.

In parts (3) and (5), it is clear that if $\mathfrak{n} + S = S$, then $a + \mathfrak{n} = -a$, which means $a = \pm n/4$. Since $a \equiv 0 \pmod{4}$, this implies n is divisible by 16, which contradicts the fact that k is odd. So all of the graphs of these two types are twin-free. All other parts of Remark 6.3 add a final condition that specifically rules out the possibility that $\mathfrak{n} + S = S$. \square

7. UNSTABLE CIRCULANTS OF VALENCY 7

THEOREM 7.1. *A circulant graph $\text{Cay}(\mathbb{Z}_n, S)$ of valency 7 is unstable if and only if either it is trivially unstable, or it is one of the following:*

- (1) $\text{Cay}(\mathbb{Z}_{6k}, \{\pm 2t, \pm 2(k - t), \pm 2(k + t), 3k\})$, with k odd, which has Wilson type (C.1).
- (2) $\text{Cay}(\mathbb{Z}_{12k}, \{\pm 2k, \pm b, \pm c, 6k\})$, with b and c odd, which has Wilson type (C.1).
- (3) $\text{Cay}(\mathbb{Z}_{20k}, \{\pm t, \pm 2k, \pm 6k, 10k\})$, with t odd, which has Wilson type (C.1).
- (4) $\text{Cay}(\mathbb{Z}_{4k}, \{\pm t, \pm(k - t), 2k \pm t, 2k\})$, with k odd and $t \equiv k \pmod{4}$, which has Wilson type (C.2').
- (5) $\text{Cay}(\mathbb{Z}_{8k}, \{\pm 4t, \pm k, \pm 3k, 4k\})$, with k and t odd, which has Wilson type (C.3').
- (6) $\text{Cay}(\mathbb{Z}_{12k}, \{\pm t, \pm(4k - t), \pm(4k + t), 6k\})$, with t odd, which has Wilson type (C.3').

REMARK 7.2. It is easy to see that each connection set listed in Theorem 7.1 contains both even elements and odd elements, so none of the graphs are bipartite. Then it follows from Lemma 2.14(2) that the graphs are also twin-free. Therefore, a graph in the list is nontrivially unstable if and only if it is connected. And this is easy to check: a graph listed in Theorem 7.1 is connected if and only if $\gcd(t, k) = 1$ (except that the condition for part (2) is $\gcd(b, c, k) = 1$).

REMARK 7.3. In the statement of Theorem 7.1, it is implicitly assumed that $k, t \in \mathbb{Z}^+$. In order for the graphs to have valency 7, the parameters k and t (or k, b , and c) must be chosen so that all of the listed elements of the connection set are distinct in the cyclic group. (Note that this implies $k > 1$ in parts (4), (1), and (5).)

Proof of Theorem 7.1. (\Leftarrow) This is the easy direction. For each family of graphs in the statement of Theorem 7.1, we briefly justify the specified Wilson type (which implies that the graphs are unstable):

- (1) Type (C.1) with $S_e = \{\pm 2t, \pm 2(k - t), \pm 2(k + t)\}$ and $2k + S_e = S_e$.
- (2) Type (C.1) with $S_e = \{\pm 2k, 6k\}$ and $4k + S_e = S_e$.
- (3) Type (C.1) with $S_e = \{\pm 2k, \pm 6k, \pm 10k\}$ and $4k + S_e = S_e$.

- (4) Type (C.2') with $h = k$ and $S_o = \{\pm t, 2k \pm t\}$. Note that $s \equiv 0$ or $-k \pmod{4}$ if and only if

$$s \in \{-t, k - t, -(k - t), 2k + t\},$$

in which case

$$s + k \in \{k - t, 2k - t, t, -k + t\}.$$

(Note that $-k + t = -(k - t)$ is in the connection set.)

- (5) Type (C.3') with $H = \langle 2k \rangle$, $R = \{\pm 4t, 4k\}$, $d = 4$, $n/d = 2k$, $\{r/d\} = \{t, k\}$, and $H \not\subseteq d\mathbb{Z}_{8k}$.
 (6) Type (C.3') with $H = \langle 4k \rangle$, $R = \{6k\}$, $d = 6k$, $n/d = 2$, $\{r/d\} = \{1\}$, and $H \not\subseteq d\mathbb{Z}_{12k}$.

(\Rightarrow) Let $X = \text{Cay}(\mathbb{Z}_n, S)$ be a nontrivially unstable circulant graph of valency 7. Since the graph has odd valency (or by Theorem 3.1), n must be even. We will write $S = \{\pm a, \pm b, \pm c, \mp\}$ for its connection set.

Case 1. Assume there exists an automorphism of BX that maps an \mp -edge to an a -edge. By Proposition 2.15, S must contain every generator of $\langle a \rangle$. It follows that $\phi(|a|) \leq 6$ (and we also know $|a| \neq 2$). In particular, we obtain that

$$|a| \text{ is an element of the set } \{3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18\}.$$

We consider the following cases.

Subcase 1.1. Assume $|a| \in \{7, 9\}$. Then S must contain the 6 generators of $\langle a \rangle$. Since $|a|$ is odd, we know $\mp \notin \langle a \rangle$, so we conclude that $\mathbb{Z}_n = \langle a \rangle \times \langle \mp \rangle$. In particular, X is one of the following:

$$\text{Cay}(\mathbb{Z}_{14}, \{\pm 2, \pm 4, \pm 6, 7\}) \text{ or } \text{Cay}(\mathbb{Z}_{18}, \{\pm 2, \pm 4, \pm 8, 9\}).$$

Note that the first graph appears in Lemma 3.7(5b), which contradicts the assumption that X is (nontrivially) unstable. The second graph is listed in part (1) of the statement of Theorem 7.1 (with parameters $k = 3$ and $t = 1$).

Subcase 1.2. Assume $|a| \in \{14, 18\}$. Then S again contains the 6 generators of $\langle a \rangle$. Additionally, since $|a|$ is even, we have $\mp \in \langle a \rangle$. We conclude that a generates \mathbb{Z}_n and it follows that X is one of the following:

$$\text{Cay}(\mathbb{Z}_{14}, \{\pm 1, \pm 3, \pm 5, 7\}) \text{ or } \text{Cay}(\mathbb{Z}_{18}, \{\pm 1, \pm 5, \pm 7, 9\}).$$

It is clear that both of these graphs are bipartite, which contradicts the assumption that X is nontrivially unstable.

Subcase 1.3. Assume $|a| = 5$. Then S contains the 4 generators of $\langle a \rangle$. We may suppose without loss of generality that besides $\pm a$, the remaining two generators of $\langle a \rangle$ are $\pm b$.

Subsubcase 1.3.1. Assume $|c| \in \{3, 4, 6\}$. Then X is one of the following graphs, all of which are stable by Lemma 3.7.

- (1) $|c| = 3 \implies X = \text{Cay}(\mathbb{Z}_{30}, \{\pm 6, \pm 10, \pm 12, 15\})$. See Lemma 3.7(5g).
- (2) $|c| = 4 \implies X = \text{Cay}(\mathbb{Z}_{20}, \{\pm 4, \pm 5, \pm 8, 10\})$. See Lemma 3.7(5c).
- (3) $|c| = 6 \implies X = \text{Cay}(\mathbb{Z}_{30}, \{\pm 5, \pm 6, \pm 12, 15\})$. See Lemma 3.7(5e).

Subsubcase 1.3.2. Assume $|c| \notin \{3, 4, 6\}$. From here, $\langle c \rangle$ has more than two generators, while $\langle a \rangle = \langle b \rangle$ has no generators besides $\pm a, \pm b$. Hence, there cannot exist an automorphism of BX mapping an s -edge, with $s \in S \setminus \{\pm c\}$, onto a c -edge, since Proposition 2.15 would then imply that S contains all generators of $\langle c \rangle$. It follows the set of c -edges is invariant under $\text{Aut } BX$. Hence, every automorphism of BX maps

S_0 -edges to S_0 -edges, where we define $S_0 := \{\pm a, \pm b, \# \}$. The connected components of $\text{Cay}(\mathbb{Z}_n, S_0)$ are isomorphic to

$$\text{Cay}(\mathbb{Z}_{10}, \{2, 4, 5, 6, 8\}).$$

By Lemma 3.7(3a), this graph is stable. It then follows from Lemma 3.4 that X is stable as well, a contradiction.

Subcase 1.4. Assume $|a| = 8$. We may assume (as in Subcase 1.3) that $\pm a$ and $\pm b$ are the four generators of $\langle a \rangle$ (which contains $\#$, because $|a|$ is even).

Subsubcase 1.4.1. Assume $|c| \in \{3, 4, 6\}$. Then X is one of the following graphs

- (1) $|c| = 3 \implies X = \text{Cay}(\mathbb{Z}_{24}, \{\pm 3, \pm 8, \pm 9, 12\})$. This is stable, by Lemma 3.7(5d).
- (2) $|c| = 4 \implies X = \text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 2, \pm 3, 4\}) \cong K_8$. This is stable, by Example 3.3.
- (3) $|c| = 6 \implies X = \text{Cay}(\mathbb{Z}_{24}, \{\pm 3, \pm 4, \pm 9, 12\})$. This graph appears in part (5) of the statement of Theorem 7.1 (with parameters $k = 3$ and $t = 1$).

Subsubcase 1.4.2. Assume $|c| \notin \{3, 4, 6\}$. By the assumption of Case 1, there is an automorphism α of X that maps an $\#$ -edge to an a -edge. By composing with translations on the left and right, we may assume that α fixes the vertex $(0, 0)$, and maps an $\#$ -edge that is adjacent to $(0, 0)$ to an a -edge that is adjacent to $(0, 0)$.

We see from Proposition 2.15 (by the same argument as in Subsubcase 1.3.2) that the set of c -edges is invariant under all automorphisms of BX . So α restricts to an automorphism of BX'_0 , where X'_0 is the connected component of $\text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b, \# \})$ that contains 0. However, since $X'_0 \cong \text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3, 4\})$, we know from Lemma 5.5(2) that the set of $\#$ -edges is invariant under all automorphisms of BX'_0 . This contradicts the choice of α .

Subcase 1.5. Assume $|a| = 10$. Once again, we may assume that $\pm a$ and $\pm b$ are the four generators of $\langle a \rangle$ (which contains $\#$, because $|a|$ is even).

Subsubcase 1.5.1. Assume $|c| \in \{3, 4, 6\}$. Then X is one of the following graphs:

- (1) $|c| = 3 \implies X = \text{Cay}(\mathbb{Z}_{30}, \{\pm 3, \pm 9, \pm 10, 15\})$. By Lemma 3.7(5f), this graph is stable.
- (2) $|c| = 4 \implies X = \text{Cay}(\mathbb{Z}_{20}, \{\pm 2, \pm 5, \pm 6, 10\})$. This graph appears in part (3) of the statement of Theorem 7.1 (with parameters $k = 1$ and $t = 5$).
- (3) $|c| = 6 \implies X = \text{Cay}(\mathbb{Z}_{30}, \{\pm 3, \pm 5, \pm 9, 15\})$. This is a bipartite graph, so trivially unstable.

Subsubcase 1.5.2. Assume $|c| \notin \{3, 4, 6\}$. Since $|a| = 10$, we may write $n = 10m$.

As before, we see from Proposition 2.15 that the set of c -edges is invariant under all automorphisms of BX . Then Lemma 3.4 implies that $|c|$ is even (since cycles of odd length are stable).

If m is odd, then c must also be odd (since $|c|$ is even and $n = 10m$ is not divisible by 4), so all elements of S are odd. Then X is bipartite, which contradicts the assumption that X is nontrivially unstable.

So m must be even, which means we may write $m = 2k$. In this notation, we have $X = \text{Cay}(\mathbb{Z}_{20k}, \{\pm c, \pm 2k, \pm 6k, 10k\})$. Note that c must be odd, since X is not bipartite, so this is listed in part (3) of Theorem 7.1 (with parameter $t = c$).

Subcase 1.6. Assume $|a| = 12$. Then S contains the 4 generators of $\langle a \rangle$, which are without loss of generality $\pm a$ and $\pm b$.

Subsubcase 1.6.1. Assume $|c| \in \{3, 4, 6\}$. Then X is one of the following graphs:

- (1) $|c| = 3 \implies X = \text{Cay}(\mathbb{Z}_{12}, \{\pm 1, \pm 4, \pm 5, 6\})$. This is listed in part (4) of Theorem 7.1 (with parameters $k = 3$ and $t = -1$).
- (2) $|c| = 4 \implies X = \text{Cay}(\mathbb{Z}_{12}, \{\pm 1, \pm 3, \pm 5, 6\})$. This is listed in part (6) of Theorem 7.1 (with parameters $k = t = 1$).
- (3) $|c| = 6 \implies X = \text{Cay}(\mathbb{Z}_{12}, \{\pm 1, \pm 2, \pm 5, 6\})$. This is listed in part (2) of Theorem 7.1 (with parameters $k = 1$, $b = 1$, and $c = 5$).

Subsubcase 1.6.2. Assume $|c| \notin \{3, 4, 6\}$. As usual, we see from Proposition 2.15 (by the same argument as in Subsubcase 1.3.2) that the set of c -edges is invariant under all automorphisms of BX . Therefore, if we let $S_0 := \{\pm a, \pm b, \# \}$, then the set of S_0 -edges is also invariant under every automorphism of BX . Since the connected components of $\text{Cay}(\mathbb{Z}_n, S_0)$ are isomorphic to $\text{Cay}(\mathbb{Z}_{12}, \{\pm 1, \pm 5, 6\})$, we see from Lemma 3.7(3b) that these connected components are stable. It therefore follows from Lemma 3.4 that X is stable.

Subcase 1.7. Assume $|a| \in \{3, 4, 6\}$.

Subsubcase 1.7.1. Assume $|b| \in \{3, 4, 6\}$. Note that no two of a , b , and c can have the same order, as they cannot generate the same subgroup, since a cyclic group of order 3, 4, or 6 has only 2 generators.

If $|c| \in \{3, 4, 6\}$, then $\{|a|, |b|, |c|\} = \{3, 4, 6\}$, so

$$X = \text{Cay}(\mathbb{Z}_{12}, \{\pm 2, \pm 3, \pm 4, 6\}).$$

By Lemma 3.7(5a), this graph is stable.

So we must have $|c| \notin \{3, 4, 6\}$. Then, yet again, Proposition 2.15 implies that the set of c -edges is invariant under every automorphism of BX . Therefore, if we let $S_0 := \{\pm a, \pm b, \# \}$, then the set of S_0 -edges is also invariant.

- If $\{|a|, |b|\} = \{3, 4\}$, then the connected components of $\text{Cay}(\mathbb{Z}_n, S_0)$ are isomorphic to $\text{Cay}(\mathbb{Z}_{12}, \{\pm 3, \pm 4, 6\})$. By Lemma 3.7(3c), this graph is stable and consequently, by Lemma 3.4, so is X .
- If $\{|a|, |b|\} = \{3, 6\}$, then the connected components of $\text{Cay}(\mathbb{Z}_n, S_0)$ are isomorphic to $\text{Cay}(\mathbb{Z}_6, \{\pm 1, \pm 2, 3\})$, which, in turn, is isomorphic to K_6 . By Example 3.3, this is stable. Then, by Lemma 3.4, so is X .

Therefore, we must have $\{|a|, |b|\} = \{4, 6\}$.

In this situation, a connected component X'_0 of $\text{Cay}(\mathbb{Z}_n, S_0)$ is isomorphic to $\text{Cay}(\mathbb{Z}_{12}, \{\pm 2, \pm 3, 6\})$. This graph is nontrivially unstable, as it is listed in Theorem 5.1(1) with parameters $k = 1$ and $s = 3$ (and Remark 5.2 tells us that it is nontrivially unstable, not merely unstable). So Corollary 5.7 tells us that the group $\text{Aut } BX'_0$ has precisely two orbits on the edges of BX'_0 . We know from the assumption of Case 1 that there exists an automorphism of BX that maps an $\#$ -edge to an a -edge. Since the set of S_0 -edges is invariant, this implies there is an automorphism of BX'_0 that maps an $\#$ -edge to an a -edge. (So the $\#$ -edges are in the same orbit as the a -edges.) It follows that the set of b -edges is invariant.

If $|a| = 4$, then the invariant subgraph $\text{Cay}(\mathbb{Z}_n, \{\pm a, \# \})$ of X has connected components isomorphic to K_4 , which is stable by Example 3.3. It then follows from Lemma 3.4 that X is stable.

So we must have $|a| = 6$. Then $|b| = 4$. Write $n = 12k$. Then, since $|a|$ is not divisible by 4, we see that a must be even.

If b and c are of opposite parity, the connected components of $\text{Cay}(\mathbb{Z}_n, \{\pm b, \pm c\})$ are nonbipartite. It is not difficult to see that they are also twin-free (since $|b| = 4$ and the valency of the graph is so small). Since the set of b -edges is invariant under $\text{Aut } BX$ and the set of c -edges is also invariant, we know that the set of $\{b, c\}$ -edges is invariant. It therefore follows from Corollary 4.5(2) that X is stable.

So b and c must have the same parity. However, they cannot both be even, since a is known to be even, and X is not connected if every element of S is even. So b and c are odd. Since $|a| = 6$, we now see that X is listed in part (2) of Theorem 7.1.

Subsubcase 1.7.2. Assume $|b| \notin \{3, 4, 6\}$. By symmetry, we may additionally suppose that $|c| \notin \{3, 4, 6\}$. Then, by applying Proposition 2.15 one last time, we see that no automorphism of BX maps an $\#$ -edge to a b -edge or a c -edge. This implies that the set of $\{a, \#\}$ -edges is invariant under all automorphisms of BX , and the set of $\{b, c\}$ -edges is also invariant.

If $|a| = 3$, then the connected components of $\text{Cay}(\mathbb{Z}_n, S_0)$ are isomorphic to $\text{Cay}(\mathbb{Z}_6, \{\pm 2, 3\})$, which is stable by Lemma 3.7(1a). It follows by Lemma 3.4 that X is also stable.

If $|a| = 4$, then the connected components of $\text{Cay}(\mathbb{Z}_n, S_0)$ are isomorphic to $\text{Cay}(\mathbb{Z}_4, \{1, 2, 3\})$, which is further isomorphic to K_4 , which is stable by Example 3.3. It follows again by Lemma 3.4 that X is stable.

We now only have the case $|a| = 6$ to consider. Let $n = 6m$. Note that then $\{\pm a, \#\} = \{\pm m, 3m\}$. We focus on

$$X_0 := \text{Cay}(\mathbb{Z}_n, \{\pm b, \pm c\}).$$

Let X'_0 be the connected component of X_0 that contains 0. As X is assumed to be unstable and X_0 is 4-valent, we can conclude from Corollary 4.5 that $|V(X'_0)|$ is even and that either X'_0 is bipartite or X'_0 is not twin-free.

Suppose, first, that X'_0 is bipartite. This implies that b and c are of the same parity.

- (1) If b and c are even, then m must be odd. (Otherwise, S would contain only even integers, so X would not be connected.) Consequently, since $n = 6m$, it follows that b and c are both of odd order, so X'_0 contains an odd cycle. This contradicts the assumption that X'_0 is bipartite.
- (2) If b and c are odd, then m must be even. (Otherwise, every element of S is odd, so X is bipartite, which contradicts the fact that X is *nontrivially* unstable.). Hence, we may write $m = 2k$, and then we see that X is listed in part (2) of Theorem 7.1.

We can now assume that X'_0 is not bipartite and is not twin-free. By Lemma 2.14(3), it follows that X'_0 is isomorphic to $K_{4,4}$ or $C_\ell \wr \bar{K}_2$ with $\ell = |V(X'_0)|/2$. As X'_0 is assumed to be nonbipartite, the first case is not possible, and in the second case, we see that $|V(X'_0)|/2$ is odd.

From the fact that X'_0 is not twin-free (and the valency of X is small — only 4) we obtain that $c = b + \#$ (perhaps after replacing b with $-b$). For every $v \in \mathbb{Z}_n \times \mathbb{Z}_2$, we deduce that $v + (\#, 0)$ is the unique twin of v in the graph BX_0 . Since automorphisms must map twin vertices to twin vertices, and the set of $\{b, c\}$ -edges is invariant under $\text{Aut } BX$, we conclude that the cosets of the subgroup $\langle (\#, 0) \rangle$ are blocks for the action of $\text{Aut } BX$ (see Definition 2.6). Note that quotient graph of BX'_0 with respect to the partition induced by cosets of $\langle (\#, 0) \rangle$ is a cycle. Since $|V(X'_0)|/2$ is odd, the length of this cycle is $|V(X'_0)|$. Therefore, in this cycle, the vertices corresponding to the cosets $\{(0, 0), (\#, 0)\}$ and $\{(0, 1), (\#, 1)\}$ are at maximum distance.

It follows that if α is an automorphism of BX that fixes $(0, 0)$, then α must fix the coset $\{(0, 1), (\#, 1)\}$ set-wise. Also note that α must fix the set of neighbors of $(0, 0)$ in BX_1 with $X_1 := \text{Cay}(\mathbb{Z}_n, \{\pm a, \#\})$ (because $\{a, \#\}$ -edges are invariant); this set of neighbors is $\{\pm(a, 1), (\#, 1)\}$. Then α must fix the intersection of these two sets, which is $\{(\#, 1)\}$. The automorphism α must therefore also fix the twin of the vertex $(\#, 1)$, which is $(0, 1)$. We now conclude from Lemma 3.2 that X is stable.

This completes the proof of Case 1. For the remaining cases, we may assume that every automorphism of BX maps $\#$ -edges to $\#$ -edges, and is therefore an automorphism of the canonical double cover of the following subgraph X_0 of X :

NOTATION 7.4. For the remainder of the proof, we let

$$X_0 := \text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b, \pm c\})$$

be the graph that is obtained from X by removing all of the $\#$ -edges.

Case 2. Assume X_0 is not connected. (We also assume that every automorphism of BX maps $\#$ -edges to $\#$ -edges, for otherwise Case 1 applies.) Then $\langle a, b, c \rangle$ is a proper subgroup of \mathbb{Z}_n , but $\langle a, b, c, \# \rangle$ is the whole group \mathbb{Z}_n . It follows that $n = 2m$, where m is odd, and $|\langle a, b, c \rangle| = m$. Let X'_0 be the connected component of X_0 that contains 0. Note that X'_0 is connected by definition and also that it is not bipartite (since it is vertex-transitive and of odd order).

If X'_0 is twin-free, then it follows by Theorem 3.1 that X'_0 is stable. By Lemma 3.4, we conclude that X is stable, which is a contradiction.

Therefore, we know that X'_0 is not twin-free. Then by Lemma 2.14(1), $X'_0 \cong Y \wr \overline{K_d}$, where Y is a vertex-transitive, connected graph and $d \geq 2$. Let δ be the valency of Y . Since X'_0 is 6-valent, it follows that $6 = \delta d$ and therefore, $d \in \{2, 3, 6\}$. Because $d|V(Y)| = |V(X'_0)|$ is odd, it cannot happen that d is even. Hence, we conclude that $d = 3$. It follows that $\delta = 2$, so Y must be a cycle. Letting

$$k := |V(Y)| = m/3 = n/6,$$

we conclude that $X'_0 \cong C_k \wr \overline{K_3}$. Since X_0 is the disjoint union of two copies of X'_0 , we now see that

$$X_0 = \text{Cay}(\mathbb{Z}_{6k}, \{\pm s, 2k \pm s, 4k \pm s\}),$$

for some $s \in \mathbb{Z}_n$, with $|\langle s, 2k \rangle| = n/2$. This final condition means $\gcd(s, 2k, 6k) = 2$, so s must be even; write $s = 2t$. Then

$$X = \text{Cay}(\mathbb{Z}_{6k}, \{\pm 2t, 2k \pm 2t, 4k \pm 2t, 3k\}).$$

This graph is listed in part (1) of Theorem 7.1.

Case 3. Assume that X_0 is bipartite and that BX_0 is arc-transitive. (We also assume that every automorphism of BX maps $\#$ -edges to $\#$ -edges, and that X_0 is connected, for otherwise a previous case applies.) Since X_0 is bipartite, it follows that BX_0 is isomorphic to a disjoint union of two copies of X_0 . Since BX_0 is assumed to be arc-transitive, it follows that X_0 is a connected arc-transitive circulant graph. Consequently, it is one of the four types that are listed in Theorem 2.16. Type 2.16(1) is impossible because X_0 is bipartite (and has valency 6). By Lemma 2.11, it follows that X_0 is not a normal Cayley graph, so it does not have type 2.16(2) either.

Subcase 3.1. Assume X_0 has type 2.16(3). Then $X_0 = Y \wr \overline{K_d}$, where Y is a connected arc-transitive circulant graph, and $d \geq 2$. Let δ be the valency of Y . Since X_0 has valency 6, it follows that $\delta d = 6$, so $d \in \{2, 3, 6\}$.

Subsubcase 3.1.1. Assume $d = 6$. Then $X_0 \cong K_{6,6}$. It follows that $n = 12$ and we obtain $X = \text{Cay}(\mathbb{Z}_{12}, \{\pm 1, \pm 3, \pm 5, 6\})$. We have already seen this graph; it is listed in part (6) of Theorem 7.1 with parameters $k = t = 1$.

Subsubcase 3.1.2. Assume $d = 3$. Then Y is a cycle of even length $2m$. We obtain that

$$X_0 = \text{Cay}(\mathbb{Z}_{6m}, \{\pm t, 2m \pm t, 4m \pm t\}).$$

Then $X = \text{Cay}(\mathbb{Z}_{6m}, \{\pm t, 2m \pm t, 4m \pm t, 3m\})$. Since X_0 is connected, we have $\text{gcd}(t, 2m) = 1$. In particular, t is odd. Since X is not bipartite, this implies m is even. Writing $m = 2k$, we get that

$$X = \text{Cay}(\mathbb{Z}_{12k}, \{\pm t, 4k \pm t, 8k \pm t, 6k\}).$$

So X is listed in part (6) of Theorem 7.1.

Subsubcase 3.1.3. Assume $d = 2$. Then Y is a connected, cubic, arc-transitive, circulant graph, so Corollary 2.20 tells us that Y is either K_4 or $K_{3,3}$. Since X_0 is bipartite, it follows that Y is isomorphic to $K_{3,3}$. We obtain that $X_0 \cong K_{6,6}$, a case that has already been considered in Subsubcase 3.1.1.

Subcase 3.2. Assume X_0 has type 2.16(4). Letting $n = |V(X_0)|$, we have $X_0 = Y \wr \overline{K_d} - dY$, where $n = md$, $d > 3$, $\text{gcd}(d, m) = 1$ and Y is a connected arc-transitive circulant graph of order m . Let δ be the valency of Y . Since X_0 has valency 6, we must have $\delta(d - 1) = 6$. Since $d > 3$, this implies that either $d = 7$ and $\delta = 1$ or $d = 4$ and $\delta = 2$.

If $\delta = 1$, then $Y = K_2$ (so $m = 2$). This implies $n = md = 2 \cdot 7 = 14$. Then $X_0 = K_2 \wr \overline{K_7} - 7K_2$ and $X = K_2 \wr \overline{K_7} \cong K_{7,7}$. Hence, X is bipartite and trivially unstable.

Assume, now, that $\delta = 2$ and $d = 4$. Since $\delta = 2$, we have $Y = C_m$. Then m must be even, because X_0 is bipartite. This contradicts the fact that $\text{gcd}(d, m) = 1$.

Case 4. Assume that none of the preceding cases apply. This means that:

- (1) every automorphism of BX maps $\#$ -edges to $\#$ -edges,
- (2) X_0 is connected, and
- (3) either X_0 is not bipartite or BX_0 is not arc-transitive.

Subcase 4.1. Assume there exists $s \in \{a, b, c\}$, such that the set of s -edges is invariant under the action of $\text{Aut}(BX)$, and the graph $\text{Cay}(\mathbb{Z}_n, \{\pm s, \#\})$ is not bipartite. Let $X_1 := \text{Cay}(\mathbb{Z}_n, \{\pm s, \#\})$, and let X'_1 be a connected component of X_1 . Then X'_1 is connected, cubic, and nonbipartite. The graph X'_1 must also be twin-free, because otherwise Lemma 2.14(2) would imply that $X'_1 \cong K_{3,3}$, which contradicts the fact that X'_1 is not bipartite. We therefore conclude from Proposition 4.2 that X'_1 is stable. By our assumptions, the set of s -edges is invariant under the action of $\text{Aut}(BX)$, and the set of $\#$ -edges is also invariant. So it follows by Lemma 3.4 that X is stable, which is a contradiction.

Subcase 4.2. Assume there exists $s \in \{a, b, c\}$, such that the set of s -edges is invariant under $\text{Aut} BX$. Since X is not bipartite, we know that S contains two elements of opposite parity. Therefore, we may assume without loss of generality that $a + \#$ is odd. We may assume that the set of a -edges is not invariant under $\text{Aut} BX$, for otherwise Subcase 4.1 applies (with $s = a$). So $s \neq a$. Hence, we may assume, without loss of generality, that $s = c$, which means the set of c -edges is invariant under $\text{Aut} BX$. This implies that the a -edges and the b -edges are in the same orbit of $\text{Aut} BX$.

We consider the following two subgraphs of X :

$$X_1 := \text{Cay}(\mathbb{Z}_n, \{\pm c, \#\}) \text{ and } X_2 := \text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b\}).$$

Denote their connected components containing 0 by X'_1 and X'_2 , respectively.

We may assume that X'_1 is bipartite. (Otherwise, Subcase 4.1 applies with $s = c$.) This implies that $|c|$ is even, so $\# \in \langle c \rangle$. More precisely, we have $\# = (|c|/2)c$. Since X'_1 is bipartite, this implies that $|c|/2$ is odd.

Also, since X is unstable, it follows from Corollary 4.5 that $|V(X'_2)|$ is even and either X'_2 is not twin-free or X'_2 is bipartite.

Subsubcase 4.2.1. Assume X'_2 is not twin-free. We claim that $X'_2 \cong K_{4,4}$. From Lemma 2.14(3), we obtain that X'_2 is isomorphic to $K_{4,4}$ or $C_\ell \wr \overline{K_2}$ with $\ell := |\langle a, b \rangle|/2$. Let X_2^* be the connected component of $\text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b, \#\})$ containing 0. Note that X_2^* is obtained by adding $\#$ -edges to X'_2 . Since $X'_2 \cong C_\ell \wr \overline{K_2}$, this implies $X_2^* \cong C_\ell \wr K_2$. However, since $\#$ -edges are also invariant, X_2^* cannot be stable, since in this case Lemma 3.4 would imply that X is stable. By Lemma 3.8, we therefore conclude that $\ell = 4$. Finally, note that $C_4 \wr \overline{K_2} \cong K_{4,4}$. Hence, we may assume that $X'_2 \cong K_{4,4}$. This completes the proof of the claim.

It follows from the claim that $|V(X'_2)| = |\langle a, b \rangle| = 8$, so we may write $n = 8k$. The claim then implies that $\{\pm a, \pm b\} = \{\pm k, \pm 3k\}$. We also obtain:

$$8k = n = |\langle a, b, c, \# \rangle| = |\langle a, b, c \rangle| = \frac{|\langle a, b \rangle| \cdot |c|}{|\langle a, b \rangle \cap \langle c \rangle|} = \frac{8|c|}{|\langle a, b \rangle \cap \langle c \rangle|}$$

This immediately implies that k divides $|c|$. More precisely, since $|c|/2$ is odd, and the denominator of the right-most term is a divisor of $|\langle a, b \rangle| = 8$ (and is a multiple of $|\langle \# \rangle| = 2$), the only possibility is that $k = |c|/2$ (so k is odd). We conclude that:

$$X = \text{Cay}(\mathbb{Z}_{8k}, \{\pm c, \pm k, \pm 3k, 4k\}).$$

Since $|c| = 2k = 8k/4$, we know that $\gcd(c, 8k) = 4$; this means that we may write $c = 4t$ with t odd. So X is listed in part (5) of Theorem 7.1 (with parameter $4t = c$).

Subsubcase 4.2.2. Assume X'_2 is bipartite. Then a and b must be of the same parity, but as X is nonbipartite, it follows that their parity is opposite to that of c and $\#$. Therefore, if we let X_2^* be a connected component of $\text{Cay}(\mathbb{Z}_n, \{\pm a, \pm b, \#\})$, then X_2^* is a connected, nonbipartite, 5-valent circulant graph. By Lemma 2.14(2), it is also twin-free (since it is not bipartite, and therefore cannot be isomorphic to $K_{5,5}$). We conclude that X_2^* is not trivially unstable.

Due to Lemma 3.4 and the fact that X is unstable, we see that X_2^* is not stable. Hence, it is nontrivially unstable. From our assumptions, we already know that a -edges and b -edges are in the same orbit under the action of $\text{Aut} BX_2^*$. This implies that if the set of $\#$ -edges is not invariant, then all edges of BX_2^* are in the same orbit, which would contradict Corollary 5.7. Hence, the set of $\#$ -edges must be invariant, so Lemma 5.4 tells us that X_2^* is isomorphic to $\text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3, 4\})$. Therefore, $X'_2 \cong \text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3\}) \cong K_{4,4}$ is not twin-free, so Subsubcase 4.2.1 applies.

Subcase 4.3. Assume all edges of BX besides $\#$ -edges are in the same orbit of $\text{Aut} BX$. As every automorphism of BX is also an automorphism of BX_0 , it follows that BX_0 is arc-transitive. Then by the assumption of Case 4, X_0 must be nonbipartite.

Subsubcase 4.3.1. Assume $|a| = 4$. Since BX_0 is arc-transitive, there exist automorphisms of BX_0 mapping an a -edge to a b -edge and a c -edge. Since $|a| = 4$, by Proposition 2.15 it follows that all generators of the subgroups $\langle b \rangle$ and $\langle c \rangle$ are in S . As all elements of S are pairwise distinct, it is clear that either b and c generate distinct subgroups with exactly 2 generators each or b and c generate the same subgroup with exactly 4 generators.

In the first case, we get that $|b| \neq |c|$ and both lie in $\{3, 4, 6\}$. As $|a| = 4$, we conclude that $\{|a|, |b|, |c|\} = \{3, 4, 6\}$, so $X = \text{Cay}(\mathbb{Z}_{12}, \{\pm 2, \pm 3, \pm 4, 6\})$. By part (5a) of Lemma 3.7, this graph is stable.

In the second case, it follows that $|b| = |c| \in \{5, 8, 10, 12\}$. Then X is one of the following graphs:

- $|b| = |c| = 5 \implies X = \text{Cay}(\mathbb{Z}_{20}, \{\pm 4, \pm 5, \pm 8, 10\})$. This is stable by Lemma 3.7(5c).

- $|b| = |c| = 8 \implies X = \text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 2, \pm 3, 4\}) \cong K_8$. This is stable by Example 3.3.
- $|b| = |c| = 10 \implies X = \text{Cay}(\mathbb{Z}_{20}, \{\pm 2, \pm 5, \pm 6, 10\})$. We find this graph in part (3) of Theorem 7.1 (with parameters $k = 1$ and $t = 5$).
- $|b| = |c| = 12 \implies X = \text{Cay}(\mathbb{Z}_{12}, \{\pm 1, \pm 3, \pm 5, 6\})$. We find this graph in part (6) of Theorem 7.1 (with parameters $k = t = 1$).

Subsubcase 4.3.2. Assume $2s \neq 2t$, for all $s, t \in S$, such that $s \neq t$. Then from Lemma 2.17, it follows that any automorphism of BX_0 (and consequently of BX , as well) is an automorphism of

$$\text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \{(\pm 2a, 0), (\pm 2b, 0), (\pm 2c, 0)\}).$$

Therefore, (the vertex sets of) the connected components of this graph are blocks for the action of $\text{Aut } BX_0$. These blocks are the four cosets of the subgroup $2\mathbb{Z}_n \times \{0\}$. As BX_0 is 6-valent and arc-transitive (and all neighbors of $(0, 0)$ are in $\mathbb{Z}_n \times \{1\}$), it follows that either all 6 neighbors of $(0, 0)$ in BX_0 lie in the same coset of $2\mathbb{Z}_n \times \{0\}$ or three of its neighbors lie in $2\mathbb{Z}_n \times \{1\}$ and the other three lie in $(1 + 2\mathbb{Z}_n) \times \{1\}$. In the first case, it follows that a, b, c are all of the same parity, which contradicts the fact that X_0 is connected and nonbipartite. In the second case, it follows that exactly three elements of the set $\{\pm a, \pm b, \pm c\}$ are odd, which is impossible, since $-s$ has the same parity as s .

Subsubcase 4.3.3. Assume that neither of the two preceding cases apply. This means that:

- (1) S contains no element of order 4, and
- (2) we may assume, without loss of generality, that $2a = 2b$.

By Lemma 2.17, every automorphism of BX_0 is also an automorphism of $\text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \{(\pm 2c, 0)\})$. This implies that the cosets of $\langle 2c \rangle \times \{0\}$ are blocks for the action of $\text{Aut } BX_0$. The two c -neighbors of $(0, 0)$ are both in the coset $(c, 1) + (\langle 2c \rangle \times \{0\})$. Therefore, by arc-transitivity, either all neighbors of $(0, 0)$ in BX_0 are in this coset or there are three different cosets, each containing two neighbors of $(0, 0)$.

However, if all neighbors of $(0, 0)$ are in $(c, 1) + (\langle 2c \rangle \times \{0\})$, then a and b have the same parity as c . This contradicts the fact that X_0 is connected and nonbipartite.

So there are three different cosets that each contain two neighbors of $(0, 0)$. Consider the quotient graph of BX_0 with respect to the coset partition induced by $\langle 2c \rangle \times \{0\}$. This is a cubic, connected, bipartite, arc-transitive graph Q , which is a Cayley graph on $\mathbb{Z}_m \times \mathbb{Z}_2$, where m is the index of $\langle 2c \rangle$ in \mathbb{Z}_n . It follows from Theorem 2.16 that there are only three cubic, connected Cayley graphs on abelian groups (up to isomorphism): K_4 , $K_{3,3}$, and the cube Q_3 .

- K_4 is not bipartite.
- If $Q \cong K_{3,3}$, then $\langle 2c \rangle$ is of index 3 in \mathbb{Z}_n . But this means $\langle 2c \rangle$ cannot be of index 2 in $\langle c \rangle$, so it follows that $\langle 2c \rangle = \langle c \rangle$, so c is of odd order. Then $n = 3|c|$ is also odd. By Theorem 3.1, X is stable.

Therefore, Q must be the cube (and therefore has exactly 8 vertices).

It follows that $\langle 2c \rangle \times 0$ is of index 8 in $\mathbb{Z}_n \times \mathbb{Z}_2$, so the order of $2c$ is $n/4$. This means $n/\text{gcd}(n, 2c) = n/4$, so $\text{gcd}(n, 2c) = 4$. Therefore, c is an even integer, and n is also even. Since X is connected, then a and $b = a + n$ must be odd.

Note that the five non-zero neighbors of $(a, 1)$ in BX_0 are

$$(a \pm c, 0), (2a, 0) = (2b, 0), (a \pm b, 0).$$

There is more than one path of length 2 from $(0, 0)$ to each of these vertices. Due to arc-transitivity, this implies that the path $(0, 0), (c, 1), (2c, 0)$ is not the only

path of length 2 from $(0, 0)$ to $(2c, 0)$. Because the first coordinate of $(2c, 0)$ is an even integer, we conclude that $2c$ is a sum of two integers from $\{\pm a, \pm a + \mathfrak{n}, \pm c\}$ of the same parity, besides $c + c$. As we have assumed there is no element of order 4 in S , the case $2c = (-c) + (-c)$ is not possible. So $2c$ must be a sum of two (odd) integers from $\{\pm a, \pm a + \mathfrak{n}\}$. Recalling that $2c \neq 2s$ for $s \neq c$, we see that (up to relabeling $a, -a$ and $a + \mathfrak{n}$), we must have either

$$2c = a + (a + \mathfrak{n}) = 2a + \mathfrak{n} \quad \text{or} \quad 2c = a - (a + \mathfrak{n}) = \mathfrak{n}.$$

However, the case $2c = \mathfrak{n}$ is not possible, because no element of S has order 4. Therefore, we have $2c = 2a + \mathfrak{n}$. This means $2(c - a) = \mathfrak{n}$, so either $c = a + (n/4)$ or $c = a - (n/4)$. However, if $c = a + (n/4)$, then $-c = -a - (n/4)$; therefore, we may assume $c = a - (n/4)$, by replacing a, b , and c with their negatives, if necessary.

As a is an odd integer, we know that $n/|a| = \gcd(a, n)$ is odd, so $\langle a \rangle$ contains every element of \mathbb{Z}_n whose order is a power of 2. In particular, it contains \mathfrak{n} and $\pm n/4$. Since $b = a + \mathfrak{n}$ and $c = a - (n/4)$, we conclude that $\langle a \rangle = \mathbb{Z}_n$. Then, writing $n = 4k$, we have:

$$X = \text{Cay}(\mathbb{Z}_{4k}, \{\pm a, \pm(a + 2k), \pm(a - k), 2k\}).$$

If $a \equiv k \pmod{4}$, then this is listed in part (4) of Theorem 7.1 (with parameter $t = a$).

To complete the proof, we will show that if $a \not\equiv k \pmod{4}$, then the subgraph X_0 is stable. This then implies by Lemma 3.4 that X is stable as well, which is a contradiction.

Suppose, for a contradiction, that X_0 is unstable (and $a \equiv -k \pmod{4}$). To work around a conflict of notation, let us change our notation for X_0 , by writing α and κ instead of a and k :

$$X_0 = \text{Cay}(\mathbb{Z}_{4\kappa}, \{\pm\alpha, \pm(\alpha + 2\kappa), \pm(\alpha - \kappa)\}).$$

Recall that κ is odd, and $\alpha \equiv -\kappa \pmod{4}$.

Since X_0 is connected, nonbipartite, and twin-free, it is nontrivially unstable. So it must appear in the list of nontrivially unstable 6-valent graphs in Corollary 6.2. However:

- (1) Since κ is odd, we know that $|V(X_0)| = 4\kappa$ is not divisible by 8. So X_0 cannot appear under (1), (4), or (5) in Corollary 6.2.
- (2) Since the connection set of X_0 has 4 odd elements ($\pm\alpha$ and $\pm(\alpha + 2\kappa)$), we can also rule out the family 6.2(2).
- (3) If X_0 is a member of the family 6.2(3), then, since $\alpha - \kappa$ and its negative are the only even elements of the connection set for X_0 (and the four elements $\pm(a + k)$ and $\pm(a - k)$ all have the same parity), the element a of 6.2(3) must be $\alpha - \kappa$ (perhaps after replacing a with $-a$). Since $\alpha \equiv -\kappa \pmod{4}$, this implies

$$a = \alpha - \kappa \equiv 2\alpha \equiv 2 \pmod{4},$$

which contradicts the requirement of 6.2(3) that $a \equiv 0 \pmod{4}$.

- (4) Assume X_0 is a member of the family 6.2(7). Then we can find $m \in \mathbb{Z}$ with $\gcd(m, 4\kappa) = 1$ and $m^2 \equiv 1 \pmod{4\kappa}$ satisfying the identity listed in 6.2(7). Using the notation from Corollary 6.2, we observe the following cases:

- Assume $a = \alpha$. We then obtain that:

$$(m - 1)\alpha \equiv (m + 1)(\alpha + 2\kappa) \pmod{4\kappa},$$

so $2 \mid (\alpha + \kappa(m + 1))$. This is clearly a contradiction, since the right-most expression is odd, since α and m are odd.

- Assume $a = \alpha + 2\kappa$. We then obtain a contradiction of the same type:

$$(m - 1)(\alpha + 2\kappa) \equiv (m + 1)\alpha \pmod{4\kappa},$$

so $2 \mid (\kappa(m-1) - \alpha)$.

- Assume $a = \alpha - \kappa$. It then follows that:

$$(m-1)(\alpha - \kappa) \equiv (m+1)\alpha \pmod{4\kappa},$$

so $\kappa \mid \alpha$. This is a contradiction, since we have already established that α is a generator of the group $\mathbb{Z}_{4\kappa}$.

- (5) Finally, suppose X_0 is a member of the family 6.2(6). Since $\alpha - \kappa$ and its negative are the only even elements of the connection set of X_0 (and the four elements $\pm b$ and $\pm mb + 2k$ all have the same parity), the element a of 6.2(3) must be $\alpha - \kappa$ (or its negative). Then a and $m-1$ are even (and $k = n/4 = \kappa$ is odd), so we have

$$(m-1)a \equiv 0 \not\equiv 2 \equiv 2k \pmod{4}.$$

This contradicts the requirement that $(m-1)a \equiv 2k \pmod{n}$.

Therefore X_0 does not appear on any of the lists in Corollary 6.2. This contradiction completes this final case of the proof of Theorem 7.1. \square

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