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Abstract Let G be a (finite or infinite) group such that $G/Z(G)$ is not simple. The non-commuting, non-generating graph $\Xi(G)$ of G has vertex set $G \setminus Z(G)$, with vertices x and y adjacent whenever $[x,y] \neq 1$ and $\langle x,y \rangle \neq G$. We investigate the relationship between the structure of G and the connectedness and diameter of $\Xi(G)$. In particular, we prove that the graph either: (i) is connected with diameter at most 4; (ii) consists of isolated vertices and a connected component of diameter at most 4; or (iii) is the union of two connected components of diameter 2. We also describe in detail the finite groups with graphs of type (iii). In the companion paper [17], we consider the case where $G/Z(G)$ is finite and simple.

1. Introduction

Given a group G, knowledge about the generating pairs of G and their statistics have found a vast number of applications throughout abstract and computational group theory. Similarly, it is useful to gain information about the pairs of elements of G that *do not* generate the group. If G is non-abelian, then these clearly include all commuting pairs, and so our interest lies in the remaining non-generating pairs.

Information about these pairs of elements is encoded in the *non-commuting, non-generating* graph $\Xi(G)$ of G, which has vertex set $G \setminus Z(G)$, with two vertices x and y adjacent if and only if $[x,y] \neq 1$ and $\langle x,y \rangle \neq G$. Note that the central elements of G are excluded from the graph's vertex set for convenience, as otherwise they would always be isolated. Aside from this redefinition of the vertex set, the graph $\Xi(G)$ is the difference between two consecutive graphs in Cameron's [10, §2.6] hierarchy of graphs defined on the elements of a group, namely, the *non-generating graph* and the *commuting graph*. The same is true for the *generating graph*, which was introduced in [21], and for certain graphs studied in [1, 6].

Many authors (e.g. [8, 9, 15]) have studied the generating graph, and in particular its connectedness and diameter. For example, Burness, Guralnick and Harper [9] recently showed that the generating graph of a finite group is connected if and only if its diameter is at most 2, and that this occurs precisely when every proper quotient of the group is cyclic.

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In [11], we showed that the graph $\Xi^+(G)$ induced by the non-isolated vertices of $\Xi(G)$ has a similarly small diameter whenever the (finite or infinite) group G is nilpotent, and more generally, when every maximal subgroup of G is normal. In particular, we proved the following theorem, together with detailed structural relationships between G and $\Xi(G)$ in the finite case. Note that, in general, $\Xi(G)$ has no edge precisely when every proper subgroup of G is abelian.

Theorem 1.1 ([11, Theorem 13]). *Let G be a group with every maximal subgroup normal. If $\Xi(G)$ has an edge, then $\Xi^+(G)$ is connected with diameter 2 or 3. Moreover, if $\Xi^+(G)$ has diameter 3, then $\Xi(G) = \Xi^+(G)$.*

In this paper, we extend the results of [11] to the case where $G/Z(G)$ is an arbitrary non-simple group, via the following theorem. Here, an abstract group is *primitive* if it has a core-free maximal subgroup H (so that G acts faithfully and primitively on the right cosets of H, which is a point stabiliser for the action). Finally, we denote the diameter of a graph Γ by $\mathrm{diam}(\Gamma)$.

Theorem 1.2. *Let G be a group such that $\overline{G} := G/Z(G)$ is not simple and $\Xi(G)$ has an edge. Then (at least) one of the following holds.*

(i) *$\Xi(G)$ has an isolated vertex, and $\mathrm{diam}(\Xi^+(G)) = 2$. If $\overline{G}$ has a proper non-cyclic quotient, then G is soluble.*
(ii) *$\Xi(\overline{G})$ has an isolated vertex, and $\mathrm{diam}(\Xi^+(G)) \in \{2,3,4\}$. Additionally, $\overline{G}$ is an infinite, insoluble primitive group with every proper quotient cyclic.*
(iii) *$\Xi(G)$ is connected with diameter 2 or 3.*
(iv) *$\Xi(G)$ is connected with diameter 4, G is infinite, and $\overline{G}$ has a proper non-cyclic quotient.*
(v) *$\Xi(G)$ is the union of two connected components of diameter 2.*

Hence each component of $\Xi(G)$ has diameter at most 4, and if the graph has more than one nontrivial component (i.e. containing at least two vertices), then $\Xi(G)$ is the union of two components of diameter 2. The former restriction distinguishes $\Xi(G)$ from the generating graph of a non-simple finite group, where a component can have arbitrarily large diameter [15, Theorem 1.3]. On the other hand, if G is finite and soluble, then the subgraph of the generating graph induced by its non-isolated vertices has diameter at most 3 [22, Theorem 1]. The same holds for $\Xi(G)$, unless Theorem 1.2(v) applies. We also show in [16, Proposition 5.9.9] that if $\overline{G}$ is finite, G has an abelian maximal subgroup, and $\Xi(G)$ has an edge, then $\mathrm{diam}(\Xi^+(G)) = 2$.

Note that it is an open problem to determine whether cases (ii) and (iv) can occur; see Questions 4.5, 4.10 and 6.6, and Remark 4.11. If G does satisfy case (ii), then Lemma 6.5 yields further information about the structures of $\Xi(G)$ and $\Xi(\overline{G})$.

In our companion paper [17], we explore the diameter of $\Xi(G)$ when $G/Z(G)$ is finite and simple; in particular, we prove that $\Xi(G)$ is always connected in this case. Hence each finite group G with $\Xi^+(G)$ not connected satisfies Theorem 1.2(v). Our second main theorem precisely describes these finite groups. We write $\Phi(H)$ to denote the Frattini subgroup of a group H.

Theorem 1.3. *Let G be a finite group. Then $\Xi(G)$ is the union of two connected components of diameter 2 if and only if the following all hold:*

(i) *$G = P{:}Q$, where P and Q are nontrivial Sylow subgroups;*
(ii) *Q is cyclic and acts irreducibly on $P/\Phi(P)$;*
(iii) *$\Phi(P) = Z(P) \nleqslant Z(G)$; and*
(iv) *the unique maximal subgroup of Q is normal in G.*

We will observe in Theorem 3.2 below (see also [3]) that conditions (i)–(ii) of Theorem 1.3 hold if and only if the finite group G has exactly two conjugacy classes of maximal subgroups.

The paper is organised as follows. In §2, we present preliminary results on maximal subgroups of G and on $\Xi(G)$. Next, §3 focuses on groups whose maximal subgroups satisfy certain conditions, including finite groups with exactly two conjugacy classes of maximal subgroups. In §4, we bound distances in $\Xi(G)$ when G has a normal, non-abelian maximal subgroup M with $Z(G) < Z(M)$. These bounds are applied in §5, where we prove Theorem 1.2 and 1.3 when $G/Z(G)$ has a proper non-cyclic quotient. We then complete the proof of our main theorems in §6 by considering the remaining groups. In §5–6, we also exhibit structural relationships between G and $\Xi(G)$ using concrete examples, many of which involve Magma [7] computations.

2. Preliminaries

In this section, we state several preliminary results related to maximal subgroups of a group G and its non-commuting, non-generating graph $\Xi(G)$. Given vertices x and y of a graph, $d(x, y)$ denotes their distance in the graph, and we write $x \sim y$ if the vertices are adjacent.

Throughout this paper, we will implicitly use the following proposition.

Proposition 2.1 ([12, Proposition 2.1.1]). *Suppose that G is finitely generated, and let H be a proper subgroup of G. Then H is contained in a maximal subgroup of G.*

Theorem 2.2 ([24, Theorems I.4, IV.11, and IV.14]). *Suppose that G is finite and soluble, and let L and M be distinct maximal subgroups of G. Then the following are equivalent:*

(i) *L and M are conjugate in G;*
(ii) $\mathrm{Core}_G(L) = \mathrm{Core}_G(M)$*; and*
(iii) *$LM \neq G$.*

Lemma 2.3. *Let (X, Y) be a pair of proper subgroups of G, with X maximal and $Y \nleqslant X$. If $Z(X) \cap Y \nleqslant Z(G)$, then $Z(Y) \leqslant X \cap Y$. If, in addition, X is abelian, then $Z(Y) \leqslant Z(G)$.*

Proof. Let $z \in (Z(X) \cap Y)\backslash Z(G)$. Then $X = C_G(z)$, and hence $Z(Y) \leqslant C_Y(z) = X \cap Y$. If X is abelian, then each element of $Z(Y)$ is centralised by $\langle X, Y\rangle = G$, and hence $Z(Y) \leqslant Z(G)$. □

Our next result generalises an argument used in the proof of [11, Proposition 10].

Lemma 2.4. *Let (W, X, Y) be a triple of distinct proper subgroups of G, with X and $W \cap X$ normal in G, X and Y maximal in G, and $W \cap X \nleqslant Y$. Then $X \cap Y \nleqslant W$.*

Proof. Assume for a contradiction that $X \cap Y \leqslant W$. Then

$$G/(W \cap X) = (W \cap X)Y/(W \cap X) \cong Y/(W \cap X \cap Y) = Y/(X \cap Y) \cong XY/X = G/X.$$

This contradicts the fact that G/X is simple, while $G/(W \cap X)$ is not. □

Proposition 2.5 ([11, §2]). *Suppose that G is non-abelian.*

(i) *No connected component of $\Xi(G)$ has diameter 1.*
(ii) *If G is not 2-generated, then $\Xi(G)$ is connected with diameter 2.*
(iii) *Suppose that G is 2-generated, and let $g \in G\backslash Z(G)$. Then g is an isolated vertex of $\Xi(G)$ if and only if g lies in a unique maximal subgroup M of G and in $Z(M)$. Moreover, if g is not isolated, then $g \in L\backslash Z(L)$ for some maximal subgroup L of G.*

(iv) *Let H be a proper non-abelian subgroup of G. Then the induced subgraph of $\Xi(G)$ corresponding to $H\backslash Z(H)$ is connected with diameter* 2.

Proposition 2.6. *Suppose that G is* 2*-generated, and that an element $g \in G$ lies in a unique maximal subgroup M of G and in $Z(M)$. If G is finite or $M \trianglelefteq G$, then M is abelian.*

Proof. If $M \trianglelefteq G$, then M is abelian by [11, Proposition 11]. Assume therefore that G is finite and $M \ntrianglelefteq G$. Then $g \notin Z(G)$ and $M = C_G(g)$. Moreover, $g^h \in M^h\backslash M$ for each $h \in G\backslash M$, yielding $\langle g, g^h\rangle = G$ and $Z(G) = C_G(g) \cap C_G(g^h) = M \cap M^h$. Thus $G/Z(G)$ is a Frobenius group with Frobenius complement $M/Z(G)$, and so G has a nontrivial normal subgroup N with $G = NM$ and $N \cap M = Z(G)$. Observe that $G = \langle N, g\rangle = N\langle g\rangle$, and hence M is the abelian group $\langle Z(G), g\rangle$. □

Our next two results involve the *non-commuting graph* of G, which has vertex set $G\backslash Z(G)$, with two vertices adjacent if and only if they do not commute.

Proposition 2.7 ([2, Proposition 2.1]). *The non-commuting graph of a non-abelian group is connected with diameter* 2.

Lemma 2.8. *Let N be a normal subgroup of G, and suppose that G/N is not cyclic. Additionally, let $n \in N$ and $g \in G$. Then $\{n, g\}$ is an edge of $\Xi(G)$ if and only if $[n, g] \neq 1$, i.e. if and only if $\{n, g\}$ is an edge of the non-commuting graph of G.*

Proof. Since G/N is not cyclic, it is clear that $\langle N, g\rangle < G$. The result now follows from the definitions of $\Xi(G)$ and the non-commuting graph of G. □

Lemma 2.9. *Let N be a normal subgroup of G.*

(i) *Let $x, y \in G$. Then $\{Nx, Ny\}$ is an edge of $\Xi(G/N)$ if and only if $[x, y] \notin N$ and $\langle x, y, N\rangle < G$.*
(ii) *Suppose that $\Xi(G/N)$ has a connected component C containing at least two vertices. Then the subgraph of $\Xi(G)$ induced by the vertices in the set $\{x \in G\backslash Z(G) \mid Nx \in C\}$ is connected with diameter at most* $\operatorname{diam}(C)$.

Proof. Observe that $[Nx, Ny]$ is the identity in G/N if and only if $[x, y] \in N$, and $\langle Nx, Ny\rangle < G/N$ if and only if $\langle x, y, N\rangle < G$. Thus we obtain (i).

To prove (ii), let $x, y \in G\backslash Z(G)$ with $Nx, Ny \in C$. As $k := \operatorname{diam}(C) \geqslant 2$ by Proposition 2.5(i), there exist $g_1, \ldots, g_n \in G\backslash Z(G)$, with $n \leqslant k$ and $g_n = y$, such that $\Xi(G/N)$ contains the path $(Nx, Ng_1, \ldots, Ng_n)$. By (i), $(x, g_1, \ldots, g_n)$ is a path in $\Xi(G)$ of length $n \leqslant k$. □

Lemma 2.10. *Let (x, J, K) be such that J and K are proper subgroups of G, with $x \in J\backslash Z(J)$ and $x \notin K$. In addition, suppose that $H := J \cap K$ is a maximal subgroup of J, or that K is a normal maximal subgroup of G.*

(i) *There exists $h \in H$ such that $\{x, h\}$ is an edge of $\Xi(G)$, and in particular $C_H(x) < H$.*
(ii) *Suppose that there exists $y \in K\backslash Z(K)$ with $y \notin J$. If H is a maximal subgroup of K, or if J is a normal maximal subgroup of G, then there exists an element $g \in H$ such that (x, g, y) is a path in $\Xi(G)$.*

Proof. Note that if K is a normal maximal subgroup of G, then H is maximal in J. Thus we may assume in general that H is maximal in J, and similarly, that H is maximal in K in (ii).

Observe that $C_H(x) < H$, as otherwise $\langle H, x\rangle = J$ would centralise x. For each $h \in H\backslash C_H(x)$, the subgroup $\langle x, h\rangle$ lies in $J < G$. Hence $x \sim h$, and we obtain (i).

Now suppose that H is maximal in K, and let y be as in (ii). Arguing as above, $C_H(y) < H$. There exists $g \in H\backslash(C_H(x) \cup C_H(y))$, as the union of two proper subgroups of H is a proper subset. Furthermore, $\langle x, g\rangle \leqslant J < G$ and $\langle g, y\rangle \leqslant K < G$, so that $x \sim g \sim y$, yielding (ii). ☐

LEMMA 2.11. *Suppose that G is 2-generated, and let (x, L, y, M) be such that L and M are non-abelian maximal subgroups of G, with $L \trianglelefteq G$, $x \in L\backslash Z(L)$ and $y \in M\backslash Z(M)$. Suppose also that $C_L(x) \trianglelefteq G$ or $M \trianglelefteq G$. Then $d(x, y) \leqslant 3$. Moreover, $d(x, y) = 3$ if and only if either:*

- (i) *$x \in Z(M)$, $y \notin L$, and M is the only maximal subgroup of G containing but not centralising y; or*
- (ii) *$y \in Z(L)$, $x \notin M$, and L is the only maximal subgroup of G containing but not centralising x.*

Proof. If $M \trianglelefteq G$, then the result is precisely [11, Lemma 12]. Assume therefore that $C_L(x) \trianglelefteq G$ and $M \ntrianglelefteq G$. Additionally, let $\{(f, A), (g, B)\} = \{(x, L), (y, M)\}$. We claim that if $f \in Z(B)$, then $G/\langle f\rangle^G$ is not cyclic. This is clear if $B = L$. If instead $B = M$, then $C_L(x) = L \cap C_G(x) = L \cap M$, and so $\langle x\rangle^G \leqslant L \cap M$. Furthermore, $M/(L\cap M) \ntrianglelefteq G/(L\cap M)$, since $M \ntrianglelefteq G$. Hence $G/(L\cap M)$ is not cyclic, and it follows that $G/\langle x\rangle^G$ is also not cyclic, as claimed.

We split the remainder of the proof into four cases, corresponding to where x lies with respect to M and $Z(M)$ and where y lies with respect to L and $Z(L)$.

Case (a): $x \in M\backslash Z(M)$ or $y \in L\backslash Z(L)$. Here, we obtain $d(x, y) \leqslant 2$ from Proposition 2.5(iv).

Case (b): $x \notin M$ and $y \notin L$. Since x lies in L and in $C_G(x)$ but not in M, we see that $C_G(x) \cap L \nleqslant M$. Applying Lemma 2.4 to $(C_G(x), L, M)$ therefore gives $L \cap M \nleqslant C_G(x)$. Hence $C_{L\cap M}(x) < L \cap M$, and applying Lemma 2.10 to (y, M, L) yields $C_{L\cap M}(y) < L \cap M$. Thus there exists $h \in L \cap M$ that centralises neither x nor y. It follows that $x \sim h \sim y$, and so $d(x, y) \leqslant 2$.

Case (c): $x \in Z(M)$ and $y \in Z(L)$. Here, $[x, y] = 1$. As the non-commuting graph of G is connected with diameter 2 by Proposition 2.7, this graph contains the path (x, r, y) for some $r \in G\backslash Z(G)$. In addition, $G/\langle x\rangle^G$ and $G/\langle y\rangle^G$ are non-cyclic, by the first paragraph of the proof. It follows from Lemma 2.8 that (x, r, y) is also a path in $\Xi(G)$, and hence $d(x, y) = 2$.

Case (d): $x \in Z(M)$ and $y \notin L$, or $y \in Z(L)$ and $x \notin M$. Here, $f \in Z(B)$ and $g \notin A$, where $\{(f, A), (g, B)\} = \{(x, L), (y, M)\}$ as above. We claim that $C_{A\cap B}(g) < H := A \cap B$. Indeed, if $B = M$, then applying Lemma 2.10(i) to (g, B, A) yields the claim. Otherwise, the claim follows by applying Lemma 2.4 to $(C_G(g), B, A)$, as in the proof of Case (b). In general, as $f \in H\backslash Z(A)$, we see that $Z(A) \cap B < H$. Thus there exists $k \in H\backslash(C_H(g) \cup C_H(A))$. Observe that $g \sim k$, while $d(f, k) \leqslant 2$ by Proposition 2.5(iv). Hence $d(f, g) \leqslant 3$.

It remains to show that $d(f, g) = 3$ if and only if B is the unique maximal subgroup of G that contains but does not centralise g. If B is the unique such maximal subgroup, then B contains the neighbourhood of g in $\Xi(G)$, while no element of B is a neighbour of $f \in Z(B)$. Thus $d(f, g) > 2$, and so $d(f, g) = 3$ by the previous paragraph.

If instead $g \in K\backslash Z(K)$ for some maximal subgroup $K \neq B$ of G, then $K\cap B$ and $C_K(g)$ are proper subgroups of K. Hence there exists an element $s \in K\backslash(B \cup C_K(g))$, and in particular, $s \sim g$. Additionally, since $f \in Z(B)$, the quotient $G/\langle f\rangle^G$ is non-cyclic, by the first paragraph of the proof. As $s \notin B = C_G(f)$, Lemma 2.8 implies that $f \sim s$, and thus $d(f, g) \leqslant 2$. ☐

3. Groups with two conjugacy classes of maximal subgroups

Here, we consider groups whose maximal subgroups satisfy certain conditions, and in particular, finite groups with exactly two conjugacy classes of maximal subgroups.

Lemma 3.1. *Suppose that G is finitely generated and non-cyclic. Moreover, assume that G contains a normal maximal subgroup M, and that $K \cap M = L \cap M$ for all maximal subgroups K and L of G distinct from M. Then $K \cap M = K \cap L = \Phi(G)$. Moreover, if G is finite, then G is soluble. If, in addition, M is the unique normal maximal subgroup of G, then G contains exactly two conjugacy classes of maximal subgroups.*

Proof. Let K and L be distinct maximal subgroups of G that are not equal to M (these exist as G is finitely generated and not cyclic). As $K \cap M = L \cap M$, we observe that $K \cap M \leqslant K \cap L < K$. Moreover, $K \cap M$ is maximal in K (by the normality of M), and thus $K \cap M = K \cap L$. Hence $K \cap M$ is the intersection of each pair of distinct maximal subgroups of G, and so $K \cap M = \Phi(G)$.

We assume from now on that G is finite. Then $|K| = |L|$, and so the set S of orders of maximal subgroups of G has size at most 2. Suppose first that G is insoluble. Since $|S| \leqslant 2$, the quotient $G/\Phi(G)$ is isomorphic to $H \coloneqq (C_2^{3i}{:}\mathrm{PSL}(2,7)) \times C_7^j$, where i and j are non-negative integers [25]. As $\mathrm{PSL}(2,7)$ contains maximal subgroups of index 7 and 8, we deduce from the simplicity of that group that G contains non-normal maximal subgroups A and B of index 7 and 8, respectively, contradicting the requirement $|A| = |B|$.

Hence G is soluble. Assume now that M is the unique normal maximal subgroup of G. Then $K \ntrianglelefteq G$, and $\mathrm{Core}_G(K) = K \cap M = \Phi(G)$, for each maximal subgroup $K \neq M$. Thus Theorem 2.2 shows that G has exactly two conjugacy classes of maximal subgroups. □

We now examine the finite groups satisfying the final conclusion of the previous lemma.

Theorem 3.2. *Suppose that G is finite. Then the following statements hold.*

- (i) *G contains exactly two conjugacy classes of maximal subgroups if and only if:*
  - (a) *$G = P{:}Q$, where P and Q are nontrivial Sylow subgroups; and*
  - (b) *Q is cyclic and acts irreducibly on $P/\Phi(P)$.*
- (ii) *Suppose that* (i)(a) *and* (b) *hold, and let R be the unique maximal subgroup of Q. Then:*
  - (a) *the maximal subgroups of G are $M \coloneqq PR$ and the conjugates of $\Phi(P)Q$;*
  - (b) *$R \trianglelefteq G$ if and only if $\Phi(G) = M \cap \Phi(P)Q$;*
  - (c) *if $R \trianglelefteq G$, then $M = P \times R$, and $\Phi(G) = \Phi(P) \times R$ is the intersection of each pair of distinct maximal subgroups of G; and*
  - (d) *if $R \trianglelefteq G$, $C_M(\Phi(G)) \nleqslant \Phi(G)$, and M is non-abelian, then $\Phi(G) = Z(M)$, i.e. $\Phi(P) = Z(P)$.*

Proof. We will begin by proving (i) and (ii)(a). Adnan [3] proved that if G contains exactly two conjugacy classes of maximal subgroups, then G satisfies (i)(a) and (b). We will therefore assume that (i)(a) and (b) hold. The irreducibility of the action of Q on $P/\Phi(P)$ implies that $\langle \Phi(P), Q, x\rangle = G$ for each $x \in P \backslash \Phi(P)$, and so $\Phi(P)Q$ and its G-conjugates are maximal subgroups of G. Since R is maximal in Q and $P \trianglelefteq G$, we deduce that $M \coloneqq PR$ is also a maximal subgroup of G. As G/P is cyclic, its subgroup M/P is normal, and hence $M \trianglelefteq G$.

To complete the proofs of (i) and (ii)(a), it suffices to show that we have described all maximal subgroups of G. Suppose, for a contradiction, that G contains a maximal subgroup T that is neither equal to M nor conjugate to $\Phi(P)Q$. Then T contains an element xy, where $x \in P$ and (without loss of generality) y is a generator for Q. For each integer k, the projection of $(xy)^k$ onto Q is equal to y^k. Thus $|Q|$ divides $|xy|$, and it follows that T contains an element of order $|Q|$. Hence T contains a Sylow subgroup of G of order $|Q|$, and we may assume that $Q \leqslant T$.

Let S be the projection of T onto P, i.e. the set of elements $v \in P$ such that there exists $w \in Q$ with $vw \in T$. By the previous paragraph, $S = T \cap P$. Additionally, Theorem 2.2 implies that $G = T\Phi(P)Q = TQ\Phi(P) = SQ\Phi(P) = S\Phi(P)Q$. As $G = P{:}Q$, we deduce that $\langle S, \Phi(P)\rangle = P$, hence $S = P$ and $P \leqslant T$. Thus T contains $\langle P, Q\rangle = G$. This contradicts the maximality of T, and we obtain (i) and (ii)(a).

We now prove (ii)(b)–(c). Assume first that $\Phi(G) = M \cap \Phi(P)Q$. Then this intersection, which is equal to $\Phi(P)R$, is normal in G. Since R is a Sylow q-subgroup of $\Phi(P)R$, the Frattini Argument yields $G = \Phi(P)RN_G(R) = \Phi(P)N_G(R)$. Thus $P = \Phi(P)N_G(R) \cap P = \Phi(P)(N_G(R) \cap P)$. Hence $P = N_G(R) \cap P$, i.e. $P \leqslant N_G(R)$. Therefore, $R \trianglelefteq P{:}Q = G$.

Conversely, assume that $R \trianglelefteq G$. Since $P \cap R = 1$, it is clear that $M = P \times R$. Additionally, as $G = P{:}Q^g$ for each $g \in G$, we see that

$$M \cap (\Phi(P)Q)^g = (P \times R) \cap \Phi(P)Q^g = (P \cap \Phi(P))(R \cap Q^g) = \Phi(P) \times R.$$

As $\Phi(P) \times R$ is the intersection of any two distinct G-conjugates of $\Phi(P)Q$, (ii)(b)–(c) hold.

To prove (ii)(d), assume again that $R \trianglelefteq G$. Observe that $C_M(\Phi(G)) = C_{P\times R}(\Phi(P) \times R) = C_P(\Phi(P)) \times R$. Note that $C_P(\Phi(P))Q$ is a subgroup of G, since $C_P(\Phi(P))$ is characteristic in P. As $\Phi(P)Q$ is the unique maximal subgroup of G containing Q, and as $P \cap Q = 1$, it follows that either $C_P(\Phi(P)) \leqslant \Phi(P)$ or $C_P(\Phi(P)) = P$. In the former case, $C_M(\Phi(G)) \leqslant \Phi(P) \times R = \Phi(G)$.

Assume now that M is non-abelian and $C_M(\Phi(G)) \not\leqslant \Phi(G)$. Then $C_M(\Phi(G)) = P \times R = M$ and $\Phi(G) \leqslant Z(M)$. Suppose for a contradiction that $\Phi(G) \neq Z(M)$, so that $Z(M) \not\leqslant \Phi(G) = M \cap \Phi(P)Q$. Then $Z(M) \not\leqslant \Phi(P)Q$, and applying Lemma 2.4 to $(Z(M), M, \Phi(P)Q)$ yields $\Phi(G) = M \cap \Phi(P)Q \not\leqslant Z(M)$, a contradiction. Thus $\Phi(G) = Z(M)$, and (ii)(d) follows. ☐

Note that the above theorem is closely related to Theorem 1.3. For convenience, we will collect conditions (i)–(iv) of the latter theorem in the following assumption, together with the condition on maximal subgroups that Theorem 3.2(i) shows is equivalent to (i)–(ii).

Assumption 3.3. *Assume that G is finite and contains exactly two conjugacy classes of maximal subgroups, i.e. that $G = P{:}Q$, where P and Q are nontrivial Sylow subgroups such that Q is cyclic and acts irreducibly on $P/\Phi(P)$. In addition, assume that $\Phi(P) = Z(P) \not\leqslant Z(G)$, and that the unique maximal subgroup of Q is normal in G.*

4. Normal, non-abelian maximal subgroups with large centres

We now focus on the case where G contains a normal, non-abelian maximal subgroup M satisfying $Z(G) < Z(M)$ (equivalently, $Z(M) \not\leqslant Z(G)$). In particular, we determine upper bounds for the distance in $\Xi(G)$ between an element of $M \backslash Z(M)$ and an element of $G\backslash M$ or $Z(M)\backslash Z(G)$. We will apply these results in §5 in order to bound $\mathrm{diam}(\Xi(G))$.

Proposition 4.1 ([11, Proposition 10]). *Suppose that G contains a normal non-abelian maximal subgroup M, with $Z(G) < Z(M)$. Then each maximal subgroup of G is non-abelian.*

Lemma 4.2. *Suppose that G contains a normal, non-abelian maximal subgroup M with $Z(G) < Z(M)$, and let $z \in Z(M)\backslash Z(G)$. Then $G\backslash M$ is the set of neighbours of z in $\Xi(G)$.*

Proof. As M is non-abelian, $G/Z(M)$ is not cyclic. Thus Lemma 2.8 yields the result. □

Lemma 4.3. *Suppose that G contains a normal, non-abelian maximal subgroup M with $Z(G) < Z(M)$. In addition, let $x \in M\backslash Z(M)$ and $z \in Z(M)\backslash Z(G)$, and let $\mathcal{J}_M$ be the set of maximal subgroups of G distinct from M. If $\langle I \cap M \mid I \in \mathcal{I}\rangle = M$ for some $\mathcal{I} \subseteq \mathcal{J}_M$, then there exists $I \in \mathcal{J}_M$ such that $x \notin C_M(I \cap M)$. More generally, if such I exists, then $d(x, z) \leqslant 3$.*

Proof. First, if $\langle I \cap M \mid I \in \mathcal{I}\rangle = M$ for some $\mathcal{I} \subseteq \mathcal{J}_M$, then $\bigcap_{I\in\mathcal{I}} C_M(I \cap M) = Z(M)$, and so there exists $I \in \mathcal{I}$ such that $x \notin C_M(I \cap M)$.

We now assume, more generally, that there exists $I \in \mathcal{J}_M$ such that $x \notin C_M(I \cap M)$. Then $C_{I\cap M}(x) < I \cap M$. Additionally, by Proposition 4.1, I is non-abelian, and so there exists $s \in I\backslash(Z(I) \cup M)$. Applying Lemma 2.10(i) to (s, I, M) now yields $C_{I\cap M}(s) < I \cap M$. Therefore, there exists an element $t \in I \cap M$ that centralises neither x nor s. In addition, $s \sim z$ by Lemma 4.2. Thus (x, t, s, z) is a path in $\Xi(G)$, and $d(x, z) \leqslant 3$. □

We can now bound distances in $\Xi(G)$ between elements of $M\backslash Z(M)$ and elements of $Z(M)\backslash Z(G)$. Here, and in much of what follows, we will assume that G is 2-generated, as otherwise $\mathrm{diam}(\Xi(G)) = 2$ by Proposition 2.5(ii).

Proposition 4.4. *Suppose that G is 2-generated and contains a normal, non-abelian maximal subgroup M with $Z(G) < Z(M)$. In addition, let $x \in M\backslash Z(M)$ and $z \in Z(M)\backslash Z(G)$. Then the following statements hold.*

- (i) *x and z lie in distinct connected components of $\Xi(G)$ if and only if $K \cap M = Z(M)$ for every maximal subgroup K of G distinct from M. Otherwise, $d(x, z) \leqslant 4$.*
- (ii) *Suppose that $d(x, z) < \infty$. Then $d(x, z) = 4$ if and only if, for each maximal subgroup K of G distinct from M:*
  - (a) *$x \notin K$; and*
  - (b) *$x \in C_M(K \cap M)$.*
- (iii) *Suppose that* (ii)(a)–(b) *hold for each maximal subgroup K of G distinct from M. Then $K \cap M = \Phi(G)$ for all such K.*
- (iv) *Suppose that $d(x, z) < \infty$, and that G is finite. Then $d(x, z) \leqslant 3$.*

Proof. We first note that Proposition 4.1 implies that G contains no abelian maximal subgroups, while Lemma 4.2 shows that $G\backslash M$ is the set of neighbours of z in $\Xi(G)$.

(i) Suppose first that $K \cap M = Z(M)$ for every maximal subgroup K of G distinct from M, and let $y \in (G\backslash M) \cup Z(M)$. Then M is the unique maximal subgroup of G containing x, and so if $\langle x, y\rangle < G$, then $y \in Z(M)$, and hence $[x, y] = 1$. Thus there is no edge in $\Xi(G)$ between any element of $M\backslash Z(M)$ and any element of $(G\backslash M) \cup Z(M) = G\backslash(M\backslash Z(M))$. In particular, the connected component of $\Xi(G)$ containing x consists only of elements of $M\backslash Z(M)$, and so this component does not contain $z \in Z(M)$.

Conversely, suppose that there exists a maximal subgroup L of G distinct from M that satisfies $L \cap M \neq Z(M)$. We claim that $L \cap M \nleqslant Z(M)$. Indeed, either $Z(M) \nleqslant L \cap M$ or $L \cap M \nleqslant Z(M)$, and if the former holds, then applying Lemma 2.4 to $(Z(M), M, L)$ yields $L \cap M \nleqslant Z(M)$. Additionally, as L is non-abelian, there exists $r \in L\backslash(Z(L) \cup M)$. Applying Lemma 2.10(i) to (r, L, M)

yields $C_{L\cap M}(r) < L\cap M$, and so $Z(L)\cap M < L\cap M$. We also see, since $L\cap M \nleqslant Z(M)$, that $L\cap Z(M) < L\cap M$. Thus there exist $s \in (L\cap M)\backslash(Z(L)\cup Z(M))$ and $t\in L\backslash(C_L(s)\cup M)$. Proposition 2.5(iv) gives $d(x,s)\leqslant 2$, and as $G\backslash M$ is the neighbourhood of z in $\Xi(G)$, we obtain $s\sim t\sim z$. Hence

$$d(x,z)\leqslant d(x,s)+d(s,z)\leqslant 4.$$

(ii) Assume first that (a) and (b) hold for each maximal subgroup K of G distinct from M. As $G\backslash M$ is the set of neighbours of z in $\Xi(G)$, it suffices by (i) to show that $d(x,t)\geqslant 3$ for all $t\in G\backslash M$. Suppose for a contradiction that $d(x,t)\leqslant 2$ for some t. By (a), $\langle x,t\rangle = G$, and so $d(x,t)=2$. Thus there exists $s\in M$ such that $x\sim s\sim t$, and so $\langle s,t\rangle$ lies in a maximal subgroup R of G. However, x centralises $s\in R\cap M$ by (b), a contradiction. Thus $d(x,z)=4$.

Conversely, suppose that some maximal subgroup K of G distinct from M fails to satisfy either (a) or (b). We will prove that $d(x,z)\leqslant 3$. If K does not satisfy (b), i.e. if $x\notin C_M(K\cap M)$, then this is an immediate consequence of Lemma 4.3, with $I=K$.

Assume therefore that K does not satisfy (a), i.e. that $x\in K$. If $x\notin Z(K)$, then $C_K(x)$ and $C_K(z)=K\cap M$ are proper subgroups of K. Hence there exists $r\in K\backslash(M\cup C_K(x))$. As $G\backslash M$ is the neighbourhood of z in $\Xi(G)$, it follows that $x\sim r\sim z$ and $d(x,z)=2$.

Suppose now that $x\in Z(K)$. Then $K=C_G(x)$, and since x also lies in $M\backslash Z(G)$, applying Lemma 2.3 to (K,M) implies that $Z(M)\leqslant K\cap M$, and it is now clear that $z\in K\backslash Z(K)$. If $K\trianglelefteq G$, then we obtain $d(x,z)\leqslant 3$ by applying Lemma 2.11 to (x,M,z,K).

If instead $K\ntrianglelefteq G$, then let $g\in G\backslash K$. If $x\in K^g$, then since $x\notin Z(K^g)$, applying the second last paragraph with K^g replacing K yields $d(x,z)=2$. Otherwise, $x\notin K^g = C_G(x^g)$, and since $x^g\in K^g\cap M^g = K^g\cap M$, setting $I=K^g$ in Lemma 4.3 gives $d(x,z)\leqslant 3$.

(iii) Let $\mathcal{J}_M$ be the set of maximal subgroups of G distinct from M. Note that no $R\in\mathcal{J}_M$ is normal in G; otherwise, applying Lemma 2.10(i) to (x,M,R) would imply that $x\notin C_M(R\cap M)$, contradicting (ii)(b). Suppose for a contradiction that $K\cap M\neq\Phi(G)$ for some $K\in\mathcal{J}_M$. We will show that there exists a subset $\mathcal{I}$ of $\mathcal{J}_M$ such that $\langle I\cap M \mid I\in\mathcal{I}\rangle = M$, and it will follow from Lemma 4.3 that (ii)(b) does not hold for some $I\in\mathcal{J}_M$, a contradiction.

Suppose first that there exists $R\in\mathcal{J}_M$ with $R\cap M\ntrianglelefteq G$. Since $R\cap M$ is a maximal subgroup of R, which is not normal in G, we see that $\langle R\cap M\rangle^G\nleqslant R$. However, $\langle R\cap M\rangle^G\leqslant M$, since $M\trianglelefteq G$. If $\langle R\cap M\rangle^G\neq M$, then we can apply Lemma 2.4 to the triple $(\angle R\cap M\rangle^G, M, R)$ of distinct subgroups, and this in turn yields $R\cap M\nleqslant\langle R\cap M\rangle^G$, a contradiction. Therefore

$$M=\langle R\cap M\rangle^G=\langle R^g\cap M \mid g\in G\rangle,$$

and so we can set $\mathcal{I}=\{R^g \mid g\in G\}$.

Assume finally that $R\cap M\trianglelefteq G$ for all $R\in\mathcal{J}_M$. Since $K\cap M\neq\Phi(G)$, Lemma 3.1 implies that there exists a maximal subgroup $L\in\mathcal{J}_M$ such that $L\cap M\neq K\cap M$. Either $K\cap M\nleqslant L$ or $L\cap M\nleqslant K$, and if the former holds, then since $K\cap M\trianglelefteq G$, applying Lemma 2.4 to (K,M,L) shows that the latter holds too. Thus, in general, $L\cap M\nleqslant K$. As $L\cap M\trianglelefteq G$, we obtain $(L\cap M)(K\cap M) = ((L\cap M)K)\cap M = G\cap M = M$. We can therefore set $\mathcal{I}=\{K,L\}$.

(iv) By (i), $d(x,z)\leqslant 4$, and there exists a maximal subgroup K of G distinct from M with $K\cap M\neq Z(M)$. Suppose for a contradiction that $d(x,z)=4$. Then (ii) shows that each maximal subgroup L of G distinct from M satisfies $x\in C_M(L\cap M)\backslash L$. Additionally, (iii) implies that $\Phi(G)=L\cap M$ for each L, and so $x \in C_M(\Phi(G))\backslash\Phi(G)$. Furthermore, Lemma 3.1 implies that the finite group G contains exactly two conjugacy classes of maximal subgroups. Since M is the unique normal maximal subgroup of G (as in the proof of (iii)), and since $\Phi(G) = K \cap M$, the equivalent conditions of part (b) of Theorem 3.2(ii) hold. In addition, M is non-abelian and $\Phi(G) = K \cap M \neq Z(M)$, and so part (d) of that theorem shows that $C_M(\Phi(G)) \leqslant \Phi(G)$, contradicting $x \in C_M(\Phi(G))\backslash\Phi(G)$. Thus $d(x, z) \leqslant 3$. □

Question 4.5. *Does there exist an infinite* 2*-generated group G that contains a normal, non-abelian maximal subgroup M with $Z(G) < Z(M)$, and that satisfies the equivalent conditions of Proposition* 4.4(ii) *for some $x \in M\backslash Z(M)$, so that $d(x, z) = 4$ for $z \in Z(M)\backslash Z(G)$?*

In the following result, we write $\hat{H} := H/Z(M)$ when H is a subgroup of G containing $Z(M)$. Recall that an abstract group is primitive if it contains a core-free maximal subgroup, which is a point stabiliser for the corresponding primitive coset action.

Proposition 4.6. *Suppose that G contains a normal, non-abelian maximal subgroup M, and a maximal subgroup K with $K \cap M = Z(M)$. Then the following statements hold.*

(i) *$\hat{G}$ is primitive, and is the semidirect product of its unique minimal normal subgroup $\hat{M}$ by its point stabiliser $\hat{K}$, which has prime order.*
(ii) *$\hat{G}$ is finite if and only if it is soluble. Hence G is soluble if and only if $\hat{G}$ is finite.*
(iii) *If $\hat{G}$ is infinite, then $\hat{M}$ is an infinite simple group, and $|\hat{K}|$ is odd.*
(iv) *If G contains a maximal subgroup L with $L \neq M$ and $Z(M) < L \cap M$, then $\hat{G}$ is infinite.*

Proof. As $Z(M)$ is not a maximal subgroup of M, we deduce that K is not normal in G. However, $Z(M) = K \cap M$ is a maximal subgroup of K, and so $\mathrm{Core}_G(K) = Z(M)$. Thus $\hat{G}$ is primitive with a point stabiliser $\hat{K}$ of prime order, and $\hat{G} = \hat{M}{:}\hat{K}$. Furthermore, each nontrivial normal subgroup $\hat{N}$ of $\hat{G}$ contained in $\hat{M}$ intersects $\hat{K}$ trivially, and $\hat{N}\hat{K} = \hat{G}$. We therefore deduce that $\hat{N} = \hat{M}$, and so $\hat{M}$ is a minimal normal subgroup of $\hat{G}$.

Now, each finite group with an abelian maximal subgroup is soluble [19]. Hence if $\hat{G}$ is finite, then it is soluble, as is G. Hence in this case $\hat{M}$ is the unique minimal normal subgroup of $\hat{G}$. If instead $\hat{G}$ is infinite, then since $\hat{K}$ is finite, [26, Theorem 1.1] shows that $\hat{M}$ is a direct product of isomorphic infinite simple groups, and is again the unique minimal normal subgroup of $\hat{G}$. Hence G is insoluble. Arguing as in the proof of [20, Theorem 4.1], we deduce that $\hat{M}$ is simple and $|\hat{K}|$ is odd. Thus we have proved (i)–(iii).

Finally, suppose that G has a maximal subgroup L as in (iv). As $\hat{M}$ is maximal and the unique minimal normal subgroup of $\hat{G}$, the maximal subgroup $\hat{L}$ is core-free. Additionally, $\hat{K} \cap \hat{M} = 1 \neq \hat{L} \cap \hat{M}$, and thus the core-free maximal subgroup $\hat{K}$ is not conjugate to $\hat{L}$. Theorem 2.2 therefore implies that $\hat{G}$ is either infinite or insoluble, and (iv) follows from (ii). □

Lemma 4.7. *Suppose that G contains a normal, non-abelian maximal subgroup M. In addition, let $x \in M\backslash Z(M)$ and $y \in G\backslash M$. Finally, suppose that R is a maximal subgroup of G containing y, with $C_{R\cap M}(y) < R \cap M \neq Z(M)$. Then $d(x, y) \leqslant 3$.*

Proof. Either $Z(M) \nleqslant R$ or $R \cap M \nleqslant Z(M)$, and if the former holds, then applying Lemma 2.4 to $(Z(M), M, R)$ shows that the latter also holds. Thus, in general,

$R \cap M \nleqslant Z(M)$, and so $R \cap Z(M) < R \cap M$. Since $C_{R\cap M}(y) < R \cap M$, there exists $h \in R \cap M$ with $h \notin C_{R\cap M}(y) \cup Z(M)$. We see that $h \sim y$, while $d(x,h) \leqslant 2$ by Proposition 2.5(iv). Therefore, $d(x,y) \leqslant 3$. ☐

LEMMA 4.8. *Suppose that G contains a normal, non-abelian maximal subgroup M, with $Z(G) < Z(M)$. In addition, suppose that G contains maximal subgroups K and L, with $K \cap M = Z(M) \nleqslant L$ and $L \ntrianglelefteq G$. Then the following statements hold.*

(i) *$U := L \cap M$ is a normal subgroup of G.*
(ii) *Let $s \in L \cap (K \backslash Z(M))$. Then $S := \{[s,r] \mid r \in Z(M)\} \trianglelefteq G$, and $SU = M$.*
(iii) *Each element of $K \backslash Z(M)$ lies in some G-conjugate of L.*

Proof.

(i) As $U \trianglelefteq L$ and $U \leqslant M = C_M(Z(M))$, we obtain $U \trianglelefteq Z(M)L = G$.

(ii) Let $x, y \in Z(M)$. As $Z(M) \trianglelefteq G$, it follows that $[s,y] \in Z(M)$. We calculate $[s,x][s,y] = [s,yx]$, and it follows that $S \leqslant Z(M)$, and hence $S \trianglelefteq M$. Additionally, $x^s \in Z(M)$, and so $[s,x]^s = [s,x^s] \in S$. Thus $S^s = S$. As $s \in K \backslash Z(M)$, we see that $s \notin M$ and $S \trianglelefteq \langle M, s\rangle = G$.

Now, by (i), $U \trianglelefteq G$. For a subgroup T of G, let $\overline{T} := TU/U$, and for an element $g \in G$, let $\overline{g} := Ug$. We observe that U is maximal in L, and since $L \ntrianglelefteq G$, it follows that $\overline{G}$ is primitive with point stabiliser $\overline{L}$. Since $|\overline{G}|$ is not prime, it follows that $Z(\overline{G}) = 1$.

Let $r \in Z(M) \backslash L$. Then $C_{\overline{G}}(\overline{r}) = \overline{M}$. As $s \notin M$, we deduce that $\overline{[s,r]} = [\overline{s}, \overline{r}] \neq 1$, and thus $[s,r] \notin U = L \cap M$. Since $S \leqslant M$, it follows that $S \nleqslant L$. Thus $SL = G$, and we conclude that $SU = S(L \cap M) = SL \cap M = G \cap M = M$.

(iii) Let $k \in K \backslash Z(M)$. As $G = Z(M)L$, it follows that $k = zf$ for some $z \in Z(M)$ and some $f \in L \backslash Z(M)$. In fact, since $Z(M) \leqslant K$, we see that $f = z^{-1}k \in K \backslash Z(M)$. Hence $f^{-1} \in L \cap (K \backslash Z(M))$.

Finally, let $S := \{[f^{-1}, r] \mid r \in Z(M)\}$. As $U \trianglelefteq G$ by (i), we deduce that $Z(M)U/U \leqslant M/U$, which is equal to SU/U by (ii). Thus there exists $r \in Z(M)$ such that $Uz^{-1} = U[f^{-1}, r]$, and hence $[f^{-1}, r]z = z[f^{-1}, r] \in U$. As $U = U^r \leqslant L^r$, it follows that $k = zf = z[f^{-1}, r]f^r \in L^r$. ☐

We now bound distances in $\Xi(G)$ between elements of $M \backslash Z(M)$ and elements of $G \backslash M$.

PROPOSITION 4.9. *Suppose that G is 2-generated and contains a normal, non-abelian maximal subgroup M, with $Z(G) < Z(M)$. In addition, let $x \in M \backslash Z(M)$ and $y \in G \backslash M$.*

(i) *x and y lie in distinct connected components of $\Xi(G)$ if and only if $K \cap M = Z(M)$ for every maximal subgroup K of G distinct from M.*
(ii) *If x and y lie in the same connected component of $\Xi(G)$, then $d(x,y) \leqslant 4$.*
(iii) *If $d(x,y) = 4$, then $\Phi(G) = Z(M)$, and $G/Z(M)$ is primitive with unique minimal normal subgroup $M/Z(M)$, which is infinite and simple. Moreover, each maximal subgroup K of G containing y satisfies $K \cap M = Z(M)$, and $K/Z(M)$ is a point stabiliser of $G/Z(M) = (M/Z(M)) : (K/Z(M))$ of odd prime order. Additionally, G contains a maximal subgroup L such that $Z(M) < L \cap M$.*

Proof. First, Proposition 4.1 shows that G contains no abelian maximal subgroups. Let $z \in Z(M) \backslash Z(G)$. Then $y \sim z$ by Lemma 4.2, and so x and y lie in the same connected component of $\Xi(G)$ if and only if x and z lie in the same component. Thus (i) follows from Proposition 4.4(i).

Assume now that x and y lie in the same connected component of $\Xi(G)$.

Case (a): $\Phi(G) = Z(M)$. Let K be a maximal subgroup of G containing y, so that $K \neq M$, and suppose that $d(x,y) > 3$. Since $K \cap M$ contains $\Phi(G) = Z(M) > Z(G)$, we deduce from Lemma 2.3, applied to (M,K), that $Z(K) \leqslant K \cap M$. As $y \notin M$, it follows that $y \notin Z(K)$. Thus applying Lemma 2.10(i) to (y,K,M) yields $C_{K\cap M}(y) < K \cap M$. Since $d(x,y) > 3$, Lemma 4.7 shows that $K \cap M = Z(M)$. On the other hand, as x and y lie in the same component of $\Xi(G)$, we see from (i) that there exists a maximal subgroup L of G with $L \neq M$ and $L \cap M \neq Z(M)$. This means that $\Phi(G) = Z(M) < L \cap M$, and so Proposition 4.4 gives $d(x,z) \leqslant 3$. As $y \sim z$, it follows that $d(x,y) = 4$. Furthermore, as $K \cap M = Z(M) < L \cap M$, Proposition 4.6(iv) shows that $G/Z(M)$ is infinite, and hence the claims about $G/Z(M)$, $M/Z(M)$ and $K/Z(M)$ in (iii) follow from Proposition 4.6(i)–(iii).

Case (b): $\Phi(G) \neq Z(M)$. To complete the proof of (ii) and (iii), it suffices to show that $d(x,y) \leqslant 3$. Since $y \sim z$, some maximal subgroup K of G contains y and z. Note that $z \in (K \cap M)\backslash C_G(y)$, and so $C_{K\cap M}(y) < K \cap M$. By Lemma 4.7, we may assume that $K \cap M = Z(M)$, and so $\Phi(G) < Z(M)$. Hence some maximal subgroup L satisfies $Z(M) \nleqslant L$.

We will show that $Z(L) \leqslant Z(G)$. Observe that

$$G/Z(M) = Z(M)L/Z(M) \cong L/(L \cap Z(M)).$$

Since $G/Z(M)$ is primitive by Proposition 4.6 (and $|G/Z(M)|$ is not prime), $L/(L\cap Z(M))$ has trivial centre. Thus $Z(L) \leqslant L \cap Z(M) \leqslant M$. As $Z(M) \nleqslant L \cap M$, the contrapositive of Lemma 2.3, applied to (L,M), shows that $Z(L) = Z(L) \cap M \leqslant Z(G)$, as claimed.

We divide the remainder of Case (b) into three (not all mutually exclusive) sub-cases.

Case (b)(α): $y \in L^g$ for some $g \in G$. Since $Z(L^g) \leqslant Z(G)$, applying Lemma 2.10(i) to (y, L^g, M) yields $C_{L^g\cap M}(y) < L^g \cap M \neq Z(M)$. Thus $d(x,y) \leqslant 3$ by Lemma 4.7.

Case (b)(β): $L \ntrianglelefteq G$. Since $y \in K\backslash Z(M)$, it follows from Lemma 4.8(iii) that $y \in L^g$ for some $g \in G$. Thus by the previous sub-case, $d(x,y) \leqslant 3$.

Case (b)(γ): $L \trianglelefteq G$ and $y \notin L$. Applying Lemma 2.10(i) to (y,K,L) shows that $r \sim y$ for some $r \in K \cap L$, and that $C_{K\cap L}(y) < K \cap L$. If $x \in L$, then (since $Z(L) \leqslant Z(G)$) Proposition 2.5(iv) yields $d(x,r) \leqslant 2$, and so $d(x,y) \leqslant 3$.

If instead $x \notin L$, then since $K \cap M = Z(M) \nleqslant L$, applying Lemma 2.4 to (K,M,L) shows that $L \cap M \nleqslant K$. Therefore, applying the same proposition to (M,L,K) yields $K \cap L \nleqslant M$. Thus there exists $t \in (K \cap L)\backslash M$. In particular, $t \notin Z(G)$, and hence $t \notin Z(L)$. It follows from Lemma 2.10(ii), applied to the triple (x,M,L) and the element t, that $x \sim s \sim t$ for some $s \in L \cap M$. As $s \sim t \in K \cap L$, we see that $C_{K\cap L}(s) < K \cap L$. Additionally, $C_{K\cap L}(y) < K \cap L$ by the previous paragraph. Hence there exists an element $f \in K \cap L$ that centralises neither s nor y. Since $y \in K$, we see that $x \sim s \sim f \sim y$ and $d(x,y) \leqslant 3$. □

Question 4.10. *Does there exist an infinite 2-generated group G that contains a normal, non-abelian maximal subgroup M with $Z(G) < Z(M)$, that satisfies all necessary conditions given in Proposition 4.9(iii)? If yes, is $d(x,y) = 4$ possible for $x \in M\backslash Z(M)$ and $y \in G\backslash M$?*

Remark 4.11. Let $x \in M\backslash Z(M)$, $z \in Z(M)\backslash Z(G)$ and $y \in G\backslash M$. Propositions 4.4 and 4.9 show that if $d(x,z) = 4$, then $K \cap M = \Phi(G) \neq Z(M)$ for each maximal subgroup K of G distinct from M, while if $d(x,y)=4$, then there exists a maximal subgroup $L \neq M$ such that $Z(M)=\Phi(G) < L \cap M$. Hence $d(x,z)$ and $d(x,y)$ cannot both be equal to 4.

Our next result specifies exactly when $\Xi(G)$ is connected, assuming that G contains a maximal subgroup M as above. In the next section, we will consider in more detail the diameters of the connected components of this graph, and discuss several concrete examples.

Lemma 4.12. *Suppose that G is 2-generated and contains a normal, non-abelian maximal subgroup M, with $Z(G) < Z(M)$. Then $\Xi(G)$ is not connected if and only if $K \cap M = Z(M)$ for every maximal subgroup K of G distinct from M, in which case Proposition 4.6 applies to G for any choice of K. In particular, if G is finite, then $\Xi(G)$ is not connected if and only if G satisfies Assumption 3.3.*

Proof. Suppose first that $K \cap M = Z(M)$ for every maximal subgroup K of G distinct from M. Then Propositions 4.4 and 4.9 show that there is no path in $\Xi(G)$ between any element of $M \backslash Z(M)$ and any element of $(G \backslash M) \cup (Z(M) \backslash Z(G))$. Hence $\Xi(G)$ is not connected. In addition, Proposition 4.6 applies to G for any choice of K, and so $\mathrm{Core}_G(K) = Z(M)$. Thus M is the unique normal maximal subgroup of G. It follows from Lemma 3.1 that if G is finite, then it is soluble and contains exactly two conjugacy classes of maximal subgroups, and $Z(M)=\Phi(G)$ is the intersection of each pair of distinct maximal subgroups. Hence in this case G satisfies all conditions of Theorem 3.2(i). In particular, G has exactly two conjugacy classes of maximal subgroups, and a unique non-cyclic Sylow subgroup P. Furthermore, Theorem 3.2(ii) shows that G has a nontrivial cyclic Sylow subgroup whose maximal subgroup is normal in G, and that $\Phi(P)=Z(P)$. As $Z(G) < Z(M) = \Phi(G)$, we also observe from this theorem that $\Phi(P) \nleqslant Z(G)$. Thus G satisfies Assumption 3.3.

If instead G has a maximal subgroup $L \neq M$ with $L \cap M \neq Z(M)$, then Propositions 4.4 and 4.9 imply that $\Xi(G)$ is connected. Suppose that G is finite in this case. To show that Assumption 3.3 does not hold for G, we may assume that G has exactly two conjugacy classes of maximal subgroups, and a nontrivial cyclic Sylow subgroup whose maximal subgroup is normal in G. Then Theorem 3.2(ii)(c)–(d) implies that $\Phi(G) = L \cap M \neq Z(M)$, and hence that the unique non-cyclic Sylow subgroup P of G does not satisfy $\Phi(P)=Z(P)$. As $\Phi(S) < S = Z(S)$ for each nontrivial Sylow subgroup $S \neq P$, Assumption 3.3 is not satisfied by G. ☐

We note that Proposition 4.6 and Lemma 4.12 show that if G is infinite and $\Xi(G)$ is not connected, then $G/Z(M)$ is primitive with a unique minimal normal subgroup, which is infinite and simple, and each point stabiliser of $G/Z(M)$ has odd prime order.

5. Non-central by non-cyclic groups

In this section, we will determine upper bounds (or exact values in some cases) for the diameters of the connected components of $\Xi(G)$ whenever G satisfies the following assumption.

Assumption 5.1. *Assume that G contains a normal subgroup N, such that G/N is not cyclic and $N \nleqslant Z(G)$. Additionally, let $C := C_G(N)$.*

Note that this assumption holds whenever $G/Z(G)$ has a proper non-cyclic quotient. Additionally, C and $Z(C)$ are normal subgroups of G, and $Z(G) \leqslant C < G$.

Throughout this section, we will implicitly use Proposition 2.5(i), which states that each nontrivial connected component of $\Xi(G)$ has diameter at least 2.

Lemma 5.2. *Let G, N and C be as in Assumption 5.1.*

(i) *Let $h, h' \in G\backslash C$. Then there exists $n \in N\backslash Z(G)$ such that $h \sim n \sim h'$.*
(ii) *Let $c \in C\backslash Z(G)$ and $g \in G\backslash Z(G)$. If $d(c,g) > 2$, then either $G/\langle c\rangle^G$ and $G/\langle g\rangle^G$ are both cyclic, or one of these quotients is cyclic and $[c,g] = 1$.*

Proof. To prove (i), note that since $h, h' \notin C$, each of $C_N(h)$ and $C_N(h')$ is a proper subgroup of N. Thus there exists $n \in N\backslash(C_N(h) \cup C_N(h'))$, and Lemma 2.8 yields $h \sim n \sim h'$.

Next, we prove the contrapositive of (ii). If $[c,g] \neq 1$ and either $G/\langle c\rangle^G$ or $G/\langle g\rangle^G$ is not cyclic, then Lemma 2.8 yields $d(c,g) = 1$. Suppose therefore that $[c,g] = 1$, with $G/\langle c\rangle^G$ and $G/\langle g\rangle^G$ both non-cyclic. Since the non-commuting graph of G has diameter 2 by Proposition 2.7, there exists $k \in G\backslash Z(G)$ such that (c,k,g) is a path in that graph. By Lemma 2.8, this is also a path in $\Xi(G)$, and hence $d(c,g) \leqslant 2$. ☐

We now split the investigation of the structure of $\Xi(G)$ into three cases: G/C non-cyclic; G/C cyclic and C abelian; and G/C cyclic and C non-abelian. In the second and third cases, we will see that more can be said if we know whether or not C is a maximal subgroup of G.

Lemma 5.3. *Let G, N and C be as in Assumption* 5.1*, and suppose that G/C is not cyclic. Then $\Xi(G)$ is connected with diameter* 2 *or* 3*. Moreover, if $d(x,y) = 3$ for $x, y \in G\backslash Z(G)$, then one of these elements lies in C, the other lies in $G\backslash(N \cup C)$, and $[x,y] = 1$. Hence* $\mathrm{diam}(\Xi(G)) = 2$ *if $C_G(x) \subseteq N \cup C$ for all $x \in C\backslash Z(G)$, and in particular if $C = Z(G)$.*

Proof. By Lemma 5.2(i), any two elements of $G\backslash C$ are joined in $\Xi(G)$ by a path of length at most two. Thus it suffices to consider distances in $\Xi(G)$ involving elements of $C\backslash Z(G)$.

Suppose that $x \in C\backslash Z(G)$ and $y \in G\backslash Z(G)$ satisfy $d(x,y) > 2$. As G/N and G/C are not cyclic, neither is $G/\langle r\rangle^G$ for any $r \in N \cup C$. In particular, $G/\langle x\rangle^G$ is not cyclic. Therefore, Lemma 5.2(ii) implies that $y \in G\backslash(N \cup C)$ and $[x,y] = 1$.

Now, Lemma 5.2(i) shows that $n \sim y$ for some $n \in N\backslash Z(G)$. By the previous paragraph, $d(x,n) \leqslant 2$, and so $d(x,y) \leqslant d(x,n) + d(n,y) \leqslant 3$. ☐

Using Magma, we see that the groups S_4 and $C_2^2 \times S_3$ satisfy the hypotheses of Lemma 5.3, and have non-commuting, non-generating graphs of diameter 3 and 2, respectively. In fact, in the latter case, $C = Z(G)$. On the other hand, if $G = S_3 \times S_3$, then G satisfies the hypotheses of Lemma 5.3 and $\mathrm{diam}(\Xi(G)) = 2$, even though $C_G(x) \nsubseteq N \cup C$ for some $x \in C\backslash Z(G)$.

Example 5.4. Consider the infinite, 2-generated Thompson's group F. The derived subgroup F' of F is infinite and simple, $F/F' \cong \mathbb{Z}^2$, and every proper quotient of F is abelian [4, §1.4]. Hence F' is the unique minimal normal subgroup of F, and it follows that $C_F(F') = 1$. As F/F' is not cyclic, we can apply Lemma 5.3 with $G = F$ and $N = F'$ to deduce that $\mathrm{diam}(\Xi(F)) = 2$.

Next, we prove useful properties of subgroups of G containing C, when G/C is cyclic.

Lemma 5.5. *Let G, N and C be as in Assumption* 5.1*, and suppose that G/C is cyclic. Additionally, let H be a subgroup of G properly containing C. Then H is non-abelian, $Z(H) < Z(C)$, and $H \trianglelefteq G$. In particular, $Z(G) < Z(C)$.*

Proof. Since G/C is cyclic, so is its subgroup $NC/C \cong N/(N \cap C) = N/Z(N)$. Thus N is abelian, and it follows that $N \leqslant Z(C)$. In particular, H contains N. Hence each of N and C is centralised by $Z(H)$, and so $Z(H) \leqslant Z(C)$. However, H does not centralise N. Thus $N \nleqslant Z(H)$, and it follows that H is non-abelian and

$Z(H) < Z(C)$. Additionally, H/C is a normal subgroup of the cyclic group G/C, and thus $H \trianglelefteq G$. $\square$

Our next proposition explores the case where G/C is cyclic and C is abelian. Recall that $\Xi^+(G)$ is the subgraph of $\Xi(G)$ induced by its non-isolated vertices. As in much of the previous section, we assume that G is 2-generated; otherwise, $\mathrm{diam}(\Xi(G)) = 2$ by Proposition 2.5(ii).

Lemma 5.6. *Let G, N and C be as in Assumption* 5.1*. Suppose also that G is 2-generated, G/C is cyclic, and C is abelian, so that G is soluble. Then the following statements hold.*

(i) *Each isolated vertex of $\Xi(G)$ lies in $C\backslash N$.*
(ii) *Suppose that C is maximal in G. Then $\Xi^+(G)$ is connected with diameter* 2*.*
(iii) *Suppose that C is not maximal in G, and let M be a maximal subgroup of G containing C. Then Table* 1 *lists upper bounds for distances between vertices of $\Xi(G)$, depending on the subsets of $G\backslash Z(G)$ that contain them. In particular,* $\mathrm{diam}(\Xi(G)) \leqslant 3$.

Table 1. Upper bounds for distances between vertices $x \in A$ and $y \in B$ of $\Xi(G)$, with $A, B \subseteq G\backslash Z(G)$, and C and M as in Lemma 5.6(iii).

B A	$Z(M)\backslash Z(G)$	$C\backslash Z(M)$	$M\backslash C$	$G\backslash M$
$G\backslash M$	1	3 2, if $[x,y] \neq 1$	2	2
$M\backslash C$	3	2	2	
$C\backslash Z(M)$	3	2		
$Z(M)\backslash Z(G)$	2			

Proof.

(i) By Lemma 5.2(i), any two elements of $G\backslash C$ have distance at most two in $\Xi(G)$. Additionally, as G/N is not cyclic, and as the non-commuting graph of G is connected by Proposition 2.7, it follows from Lemma 2.8 that each isolated vertex lies in $C\backslash N$.

(ii) By Lemma 5.2(i), it suffices to show that $d(x,y) \leqslant 2$ whenever $x \in C\backslash Z(G)$ and $y \in G\backslash Z(G)$ are distinct non-isolated vertices. If $G/\langle x\rangle^G$ is not cyclic, then Lemma 2.8 shows that $G\backslash C_G(x) = G\backslash C$ is the neighbourhood of x in $\Xi(G)$. In particular, if $y \in G\backslash C$, then $d(x,y) = 1$. If instead the non-isolated vertex y lies in the abelian group C, then $k \sim y$ for some $k \in G\backslash C$. Hence $x \sim k \sim y$ and $d(x,y) = 2$.

Suppose now that $G/\langle x\rangle^G$ is cyclic. By Proposition 2.5(iii), there exist maximal subgroups L and K of G with $x \in L\backslash Z(L)$ and $y \in K\backslash Z(K)$. Then $x \in C \cap L$, and applying Lemma 2.3 to (C, L) gives $Z(L) \leqslant Z(G)$. Note also that $G = CL = C_G(x)L$, and so $\langle x\rangle^G = \langle x\rangle^L \leqslant L$. Thus $L/\langle x\rangle^G \trianglelefteq G/\langle x\rangle^G$, and hence $L \trianglelefteq G$. This implies that $C_L(x) = C \cap L \trianglelefteq G$. Additionally, $y \notin Z(L) \leqslant Z(G)$, and x is centralised by C, and hence not by K. We therefore obtain $d(x,y) \leqslant 2$ by applying Lemma 2.11 to (x, L, y, K).

(iii) Since $Z(G) \leqslant C < M$, it follows that $Z(G) \leqslant Z(M)$. Additionally, Lemma 5.5 shows that M is non-abelian and normal in G, with $Z(M) < Z(C) = C$. We observe from Proposition 2.5(iv) and Lemma 5.2(i) that any two vertices of $\Xi(G)$ in $M\backslash Z(M) = (M\backslash C) \cup (C\backslash Z(M))$ have distance at most two, as do any two vertices in $G\backslash C = (G\backslash M) \cup (M\backslash C)$. This yields the $(1,3)$, $(1,4)$, $(2,2)$, $(2,3)$ and $(3,2)$ entries of Table 1.

Now, suppose that $Z(G) < Z(M)$, and let $z \in Z(M)\backslash Z(G)$. Since M is a non-abelian, normal subgroup of G and $\langle z\rangle^G \leqslant Z(M)$, the quotient $G/\langle z\rangle^G$ is not cyclic. Additionally, $C_G(z) = M$, and so Lemma 2.8 gives $z \sim r$ for each $r \in G\backslash M$, hence the $(1,1)$ entry of Table 1. Thus if $z' \in Z(M)\backslash Z(G)$ is not equal to z, then $z \sim r \sim z'$ and $d(z,z') = 2$, yielding the $(4,1)$ entry of the table. Moreover, if $m \in M\backslash C$, then $d(r,m) \leqslant 2$ by the (1,3) entry of the table, and so $d(z,m) \leqslant d(z,r) + d(r,m) \leqslant 3$. This gives the $(2,1)$ entry of the table.

It remains to determine upper bounds for $d(c,g)$, where $c \in C\backslash Z(M)$ and $g \in G\backslash M$, and for $d(c,z)$ when the element z exists. As g does not lie in C, it is a non-isolated vertex by (i). It follows from Proposition 2.5(iii) that $g \in K\backslash Z(K)$ for some maximal subgroup K of G. Additionally, the abelian group C lies in $C_G(c)$, and the cyclic group G/C normalises $C_G(c)/C$. Thus $C_G(c) \trianglelefteq G$. Since $M \trianglelefteq G$, it follows that $C_M(c) = C_G(c) \cap M \trianglelefteq G$. Therefore, applying Lemma 2.11 to (c, M, g, K) gives $d(c,g) \leqslant 3$. Moreover, since $g \notin Z(M)$, that lemma shows that if $d(c,g) = 3$, then $c \in Z(K)$, and in particular, $[c,g] = 1$. Thus we obtain the $(1,2)$ entry of Table 1.

Finally, since $C_G(c) < G$ and $M < G$, there exists $h \in G\backslash(M \cup C_G(c))$. The $(1,1)$ and $(1,2)$ entries of Table 1 yield $h \sim z$ and $d(c,h) \leqslant 2$. Hence $d(c,z) \leqslant d(c,h) + d(h,z) \leqslant 3$. This gives the $(3,1)$ entry of the table, completing the proof. □

Using Magma, we observe that if G is equal to $C_3{:}S_3$ or the dihedral group D_{12} of order 12, then Lemma 5.6(ii) applies, and $\Xi(G)$ is connected only in the former case (in both cases, $\Xi^+(G)$ is connected with diameter 2). If instead G is equal to $C_2 \times \mathrm{AGL}(1,5)$ or $C_3{:}\mathrm{AGL}(1,5)$, then Lemma 5.6(iii) applies, and $\Xi(G)$ has diameter 2 or 3, respectively.

The following result is a more detailed version of Lemma 4.12, with a weaker hypothesis.

Lemma 5.7. *Let G and C be as in Assumption* 5.1. *Suppose also that G is* 2*-generated, G/C is cyclic, and C is non-abelian. Then the following statements hold.*

(i) *$\Xi(G)$ is not connected if and only if C is maximal in G and $K \cap C = Z(C)$ for every maximal subgroup K of G distinct from C. In this case, $\Xi(G)$ is the union of two connected components of diameter* 2*, and one component consists of the elements of $C\backslash Z(C)$. In particular, if G is finite, then $\Xi(G)$ is not connected if and only if G satisfies Assumption* 3.3.

(ii) *Suppose that C is not maximal in G, and let M be a maximal subgroup of G containing C. Then Table* 2 *lists upper bounds for distances between vertices of $\Xi(G)$, depending on the subsets of $G\backslash Z(G)$ that contain them. In particular,* $\mathrm{diam}(\Xi(G)) \leqslant 4$*, and* $\mathrm{diam}(\Xi(G)) \leqslant 3$ *if G is finite or if $Z(M) = Z(G)$.*

(iii) *Suppose that C is maximal in G, and that $\Xi(G)$ is connected. Then Table* 3 *lists upper bounds for distances between vertices of $\Xi(G)$, depending on the subsets of $G\backslash Z(G)$ that contain them. In particular,* $\mathrm{diam}(\Xi(G)) \leqslant 4$*, and* $\mathrm{diam}(\Xi(G)) \leqslant 3$ *if G is finite.*

Proof. Proposition 2.5(iv) and Lemma 5.2(i) show that any two vertices in $C\backslash Z(C)$ are joined by a path of length at most two, as are any two vertices in $G\backslash C$, and any two vertices in $M\backslash Z(M)$ when M is as in (ii). We therefore obtain the $(1,4)$, $(1,5)$, $(2,2)$,

TABLE 2. Upper bounds for distances between vertices $x \in A$ and $y \in B$ of $\Xi(G)$, with $A, B \subseteq G \backslash Z(G)$, and C and M as in Lemma 5.7(ii). Additionally, $\mathcal{M}$ denotes the family of groups for which any two distinct maximal subgroups intersect in $\Phi(G)$.

A \ B	$Z(M)\backslash Z(G)$	$Z(C)\backslash Z(M)$	$C\backslash Z(C)$	$M\backslash C$	$G\backslash M$
$G\backslash M$	1	3	3	2	2
$M\backslash C$	3	2	2	2	
$C\backslash Z(C)$	4 3, if $\lvert G\rvert < \infty$ 3, if $G \notin \mathcal{M}$	2	2		
$Z(C)\backslash Z(M)$	2	2			
$Z(M)\backslash Z(G)$	2				

TABLE 3. Upper bounds for distances between vertices $x \in A$ and $y \in B$ of $\Xi(G)$, with $A, B \subseteq G \backslash Z(G)$, and C as in Lemma 5.7(iii). Additionally, $\mathcal{M}$ denotes the family of groups for which any two distinct maximal subgroups intersect in $\Phi(G)$.

A \ B	$Z(C)\backslash Z(G)$	$C\backslash Z(C)$	$G\backslash C$
$G\backslash C$	1	4 3, if $\lvert G/Z(C)\rvert < \infty$ 3, if $G \in \mathcal{M}$	2
$C\backslash Z(C)$	4 3, if $\lvert G\rvert < \infty$ 3, if $G \notin \mathcal{M}$	2	
$Z(C)\backslash Z(G)$	2		

$(2,3)$, $(2,4)$, $(3,2)$, $(3,3)$ and $(4,2)$ entries of Table 2, and the $(1,3)$ and $(2,2)$ entries of Table 3. Additionally, Lemma 5.5 implies that $Z(G) < Z(C)$. Let $z \in Z(C)\backslash Z(G)$. Since C is non-abelian and normal in G and $\langle z\rangle^G \leqslant Z(C)$, the quotient $G/\langle z\rangle^G$ is not cyclic. We split the remainder of the proof into two cases, depending on whether C is maximal in G.

Case (a): C is maximal in G. Here, $C = C_G(z)$. Since $G/\langle z\rangle^G$ is not cyclic, it follows from Lemma 2.8 that $z \sim k$ for all $k \in G\backslash C$. Thus we obtain the $(1,1)$ entry of Table 3. If z' is another element of $Z(C)\backslash Z(G)$, then $z \sim k \sim z'$, yielding the $(3,1)$ entry of Table 3. Furthermore, since $Z(G) < Z(C)$, Lemma 4.12 shows that if G is finite, then $\Xi(G)$ is not connected if and only G satisfies Assumption 3.3, and in general, $\Xi(G)$ is not connected if and only if $K \cap C = Z(C)$ for every maximal subgroup $K \neq C$ of G.

Suppose first that $\Xi(G)$ is connected, and let $x \in C\backslash Z(C)$ and $y \in G\backslash C$. Then Propositions 4.4 shows that $d(x,z) \leqslant 4$, and that if $d(x,z) = 4$, then $|G| = \infty$ and $K \cap C = \Phi(G)$ for each maximal subgroup K of G distinct from C. It follows from Lemma 3.1 that if $d(x,z) = 4$, then any two distinct maximal subgroups of G intersect in $\Phi(G)$, and we obtain the $(2,1)$ entry of Table 3. Additionally, Proposition 4.9 shows that $d(x,y) \leqslant 4$, and that if $d(x,y) = 4$, then $|G/Z(C)| = \infty$ and G contains maximal subgroups K and

$L \neq C$ such that $K \cap C = Z(C) \neq L \cap C$. This yields the $(1,2)$ entry of Table 3. We have therefore proved (iii).

Next, suppose that $\Xi(G)$ is not connected. We have shown that if $g, h \in C \backslash Z(C)$ or $g, h \in (G \backslash C) \cup (Z(C) \backslash Z(G))$, then $d(g,h) \leqslant 2$. Hence the components of $\Xi(G)$ and their diameters are as in (i). To complete the proof of (i), it remains to show that if C is not maximal in G, then $\Xi(G)$ is connected, and if G is also finite, then it does not satisfy Assumption 3.3.

Case (b): C is not maximal in G. Let M be a maximal subgroup of G containing C. Then M is non-abelian, and Lemma 5.5 gives $M \trianglelefteq G$ and $Z(M) < Z(C)$. As $Z(G) < C$, it also follows that $Z(G) \leqslant Z(M)$. Let $z' \in Z(C) \backslash (Z(G) \cup \{z\})$. Since $G/\langle z \rangle^G$ and $G/\langle z' \rangle^G$ are not cyclic, we can apply Lemma 5.2(ii) (and the fact that $[z, z'] = 1$) to obtain $d(z, z') = 2$. This gives the $(4,1)$ and $(5,1)$ entries of Table 2. Note also that as $z \in Z(C) \backslash Z(G)$, there exists $h \in G \backslash C$ with $[z, h] \neq 1$, and Lemma 2.8 gives $z \sim h$. Letting $g \in G \backslash C$, the known entries of Table 2 show that $d(h, g) \leqslant 2$, and so $d(z, g) \leqslant 3$. This yields the $(1,2)$ and $(2,1)$ entries of the table.

Now, let $x \in C \backslash Z(C)$. Then $x \notin Z(M)$, and so $C_M(x) < M$. Thus there exists $k \in M \backslash (C \cup C_M(x))$, and we observe that $x \sim k$. The $(1,4)$ entry of Table 2 gives $d(k, g) \leqslant 2$ for each $g \in G \backslash M$, and hence $d(x, g) \leqslant 3$, yielding the $(1,3)$ entry of the table.

Next, we will consider the remaining entries in the first column of Table 2, which apply only when $Z(G) < Z(M)$. Assume that $z \in Z(M) \backslash Z(G)$. Then the $(2,1)$ entry of Table 2 shows that $d(z, m) < \infty$ for each $m \in M \backslash C$. The $(3,1)$ entry of the table therefore follows from Proposition 4.4 and Lemma 3.1. In addition, $M = C_G(z)$, and so Lemma 2.8 gives the $(1,1)$ entry of the table. This completes the proof of (ii).

We have shown that $\Xi(G)$ is connected, which partially proves (i). To complete the proof, suppose for a contradiction that G is finite and satisfies Assumption 3.3. Then the unique non-cyclic Sylow subgroup G of P satisfies $Z(P) \nleqslant Z(G)$; the unique maximal subgroup of each nontrivial cyclic Sylow subgroup of G is normal in G; and M is precisely the maximal subgroup M specified in Theorem 3.2(ii) (by part (a) of that theorem). Part (c) of that theorem therefore implies that $Z(M)$ contains $Z(P) \nleqslant Z(G)$, and so $Z(G) < Z(M)$. Hence Lemma 4.12 applies, and shows that G does not in fact satisfy Assumption 3.3. Thus (i) holds. □

Notice from Propositions 4.4 and 4.9 that the conditions $G \notin \mathcal{M}$ and $G \in \mathcal{M}$ in the $(2,1)$ and $(1,2)$ entries of Table 3, respectively, are stronger than those necessary to ensure that the specified distances cannot be equal to 4 (and similarly for the $(3,1)$ entry of Table 2). However, the chosen conditions highlight the fact that there is no group for which these two entries of Table 3 are simultaneously equal to 4, as discussed in Remark 4.11.

We compute via Magma that $S_3 \times \mathrm{AGL}(1,5)$ and the group numbered $(192, 30)$ in the Small Groups Library [5] satisfy the hypotheses of Lemma 5.7(ii), and have non-commuting, non-generating graphs of diameter 2 and 3, respectively. In addition, $S_3 \times S_3$ and SmallGroup$(48, 15)$ satisfy the hypotheses of Lemma 5.7(iii), and their graphs have diameter 2 and 3, respectively.

Before presenting examples of groups that satisfy Lemma 5.7(i), we further clarify how Assumption 3.3 relates to this lemma (and to Assumption 5.1).

Proposition 5.8. *Suppose that G satisfies Assumption 3.3. Then G is 2-generated and has a normal subgroup $N \nleqslant Z(G)$, with G/N non-cyclic, $G/C_G(N)$ cyclic, and*

$C_G(N)$ non-abelian. Thus G satisfies the hypotheses of Lemma 5.7*, and so $\Xi(G)$ is the union of two connected components of diameter* 2*.*

Proof. Let P and Q be as in Assumption 3.3, and let R be the unique maximal subgroup of Q. Theorem 3.2 shows that G contains a normal maximal subgroup $M := P \times R$, and that $G = \langle x, y \rangle$ for each $x \in P \backslash \Phi(P)$ and generator y for Q. Additionally, $N := Z(M) = Z(P) \times R$ is not a subgroup of $Z(G)$ by assumption. Since $Z(P) = \Phi(P) < P$, both P and M are non-abelian. Thus $G/N = G/Z(M)$ is not cyclic, and so G and N are as in Assumption 5.1. Moreover, $C_G(N) = M$, and hence $G/C_G(N)$ is cyclic. Therefore, G satisfies the hypotheses of Lemma 5.7, and the final part of the result follows from Lemma 5.7(i). ☐

We will call a group G a $[2,2]$-*group* if $\Xi(G)$ is the union of two connected components of diameter 2. The following example describes an infinite family of such groups.

Example 5.9. Let G be the finite simple Suzuki group $\mathrm{Sz}(q)$, where $q := 2^i$ with i an odd integer at least 3. Additionally, let P be a Sylow 2-subgroup of G, and $N := N_G(P)$. Then $|P| = q^2$, and N is a maximal subgroup of G isomorphic to the Frobenius group $P{:}C_{q-1}$ [27, §4, p. 133, & Theorem 9]. Given a primitive prime divisor r of $2^i - 1$, let $N_r := P{:}C_r \leqslant N$. Then N_r is also Frobenius, hence $Z(N_r) = 1$. Moreover, each cyclic subgroup of N_r of order r acts irreducibly on $P/\Phi(P)$ [18, Theorem 3.5]. Thus N_r satisfies all conditions of Theorem 3.2(i).

We claim that N_r is a $[2,2]$-group. By Proposition 5.8, it suffices to show that G satisfies Assumption 3.3. As the unique maximal subgroup of C_r is the trivial subgroup, it remains only to prove that $\Phi(P) = Z(P) \nleqslant Z(N_r) = 1$. Let θ be the automorphism $\alpha \mapsto \alpha^{\sqrt{2q}}$ of $\mathbb{F}_q$. Then [27, pp. 111-112 & Theorem 7] shows that P is isomorphic to the group $\{(\alpha,\beta) \mid \alpha,\beta \in \mathbb{F}_q\}$, where $(\alpha,\beta)(\gamma,\delta) := (\alpha+\gamma, \alpha\gamma^\theta + \beta + \delta)$ for all $(\alpha,\beta),(\gamma,\delta) \in P$.

Now, $Z(P) = \{(0,\beta) \mid \beta \in \mathbb{F}_q\} \nleqslant 1 = Z(N_r)$. Using the fact that $\alpha \mapsto \alpha\alpha^\theta$ is an automorphism of $\mathbb{F}_q^\times$, we calculate that $Z(P)$ contains P' and is equal to the subgroup K generated by all squares in P. Since $\Phi(P) = KP'$, it follows that $\Phi(P) = Z(P)$. Hence N_r is a $[2,2]$-group.

We observe using Magma that SmallGroup$(96,3)$ is the unique smallest finite $[2,2]$-group. Additionally, there exist $[2,2]$-groups with no Sylow subgroup of prime order, e.g. SmallGroup$(288,3)$, and with odd order, e.g. SmallGroup$(9477,4035)$.

The following theorem summarises this section's main results.

Theorem 5.10. *Suppose that G contains a normal subgroup N, such that G/N is not cyclic and $N \nleqslant Z(G)$, and let $C := C_G(N)$. Then one of the following holds.*

(i) *$\Xi(G)$ has an isolated vertex, and $\Xi^+(G)$ is connected with diameter* 2*. Additionally, G is soluble, C is abelian and maximal in G, and each isolated vertex lies in $C \backslash N$.*
(ii) *$\Xi(G)$ is connected with diameter* 2*,* 3 *or* 4*. If* $\mathrm{diam}(\Xi(G)) = 4$*, then G is infinite, G/C is cyclic, and C is non-abelian.*
(iii) *$\Xi(G)$ is the union of two connected components of diameter* 2*, with one component consisting of the elements of $C \backslash Z(C)$. Moreover, C is non-abelian and maximal in G.*

Furthermore, if G is finite, then (iii) holds if and only if G satisfies Assumption 3.3*.*

Proof. We may assume that G is 2-generated; otherwise, $\mathrm{diam}(\Xi(G)) = 2$ by Proposition 2.5(ii). If G/C is not cyclic, then Lemma 5.3 applies, and (ii) holds. Otherwise, either Lemma 5.6 or Lemma 5.7 applies, depending on whether C is abelian. Specifically, if C is abelian, then (i) or (ii) holds, and otherwise, (ii) or (iii) holds. Thus we observe from Lemma 5.7 and Proposition 5.8 that if G is finite, then (iii) holds if and only if G satisfies Assumption 3.3. □

The above theorem implies Theorems 1.2 and 1.3 in the case where $G/Z(G)$ has a proper non-cyclic quotient. Additionally, if $\Xi(G)$ has two nontrivial components, then Lemma 5.7(i) shows that $K \cap C = Z(C)$ for each maximal subgroup K of G distinct from the normal, non-abelian maximal subgroup C. As $Z(G) < Z(C)$ by Lemma 5.5, we can use Lemma 4.12 and Proposition 4.6 to deduce further information about infinite groups in this case. It is also easy to show using Theorem 5.10 that if a group G contains non-central normal subgroups N_1 and N_2, with G/N_1 and G/N_2 non-cyclic and $C_G(N_1) \neq C_G(N_2)$, then $\Xi(G)$ is connected.

Note that Theorem 5.10 applies whenever G is a free product $\langle a \rangle * \langle b \rangle$ of nontrivial cyclic groups, e.g. with $N = \langle (ab)^k \rangle$ for some $k \geqslant 2$ and $C = \langle ab \rangle$. By appealing to the nature of G as a free product, we prove in [16, §5.10] that $\operatorname{diam}(\Xi(G)) = 2$ unless $|a| = |b| = 2$, in which case G is the infinite dihedral group, $\Xi(G) \backslash \Xi^+(G) = \{ab, ba\}$, and $\operatorname{diam}(\Xi^+(G)) = 2$.

6. Groups with each proper quotient cyclic

In order to prove Theorems 1.2 and 1.3, it remains to consider the case where every proper quotient of $G/Z(G)$ is cyclic. As in the previous section, we will implicitly use Proposition 2.5(i), which states that each nontrivial component of $\Xi(G)$ has diameter at least 2.

The following lemma generalises the classification given in [23, §3] of *finite* groups whose proper quotients are all cyclic. Here, by a *central extension* of G, we mean a group H such that $H/Z(H) \cong G$. As above, an abstract group is primitive if it has a core-free maximal subgroup, which is a point stabiliser for the corresponding primitive action.

Lemma 6.1. *Suppose that each proper quotient of G is cyclic. Then one of the following holds:*

(i) *for each central extension H of G (including G itself), every maximal subgroup of H is normal in H, and hence G is not primitive;*

(ii) *G is soluble and primitive with a (unique) minimal normal subgroup and a cyclic point stabiliser; or*

(iii) *G is insoluble and primitive, and $C_G(N) = 1$ for each normal subgroup $N \neq 1$ of G.*

Proof. We split the proof into three cases, which together account for all possibilities.

Case (a): G is not primitive. Suppose that G contains a maximal subgroup M, and let $J := \operatorname{Core}_G(M)$. Then $J \neq 1$, and so G/J is cyclic. Thus $M/J \trianglelefteq G/J$, and so $M \trianglelefteq G$. As each maximal subgroup of H not containing $Z(H)$ is normal, (i) follows.

Case (b): G is primitive, and $C_G(J) \neq 1$ for some nontrivial normal subgroup J of G. Then $N := C_G(J)$ is a minimal normal subgroup of G. If G contains a distinct minimal normal subgroup K, then K is non-abelian and equal to $K/(K \cap N) \cong NK/N$. In particular, NK/N is not cyclic, and so neither is G/N, a contradiction. Hence N is the unique minimal normal subgroup of G. Moreover, since $C_G(N)$ contains the nontrivial subgroup J (which is now clearly equal to N), it follows that N is abelian. As G/N is cyclic, G is soluble, and (ii) holds.

Case (c): G is primitive, and $C_G(N) = 1$ for each nontrivial normal subgroup N of G. If G is soluble, then the penultimate subgroup R in the derived series of G is nontrivial and abelian, and hence $C_G(R) \neq 1$. However, $R \trianglelefteq G$, a contradiction. Thus (iii) holds. $\square$

If case (i) of Lemma 6.1 holds, then Theorem 1.1 applies. Hence it remains to consider cases (ii) and (iii). We will write $\overline{H} := H/Z(G)$ when H is a subgroup of G containing $Z(G)$.

PROPOSITION 6.2. *Suppose that $\overline{G}$ is soluble and primitive with every proper quotient cyclic, and let L be a non-abelian maximal subgroup of G. Then $L \trianglelefteq G$ and $Z(L) \leqslant Z(G)$.*

Proof. It follows from Lemma 6.1 that $\overline{G}$ contains a unique minimal normal subgroup $\overline{N}$, and a cyclic point stabiliser $\overline{M}$ such that $\overline{G} = \overline{N}{:}\overline{M}$. Suppose first that $Z(G) \nleqslant L$. Then $L \trianglelefteq G$. Additionally, there is no $x \in G$ satisfying $L = C_G(x)$, and thus $Z(L) < Z(G)$.

Assume from now on that $Z(G) \leqslant L$, and note that $\overline{L}$ is a non-cyclic maximal subgroup of $\overline{G}$. Thus $\overline{L}$ is not a complement of $\overline{N}$ in $\overline{G}$, and so $\overline{L}$ is not core-free in $\overline{G}$. Hence $\overline{N} \leqslant \overline{L}$. Moreover, as $\overline{G}/\overline{N}$ is cyclic, we see that $\overline{L}/\overline{N} \trianglelefteq \overline{G}/\overline{N}$, and it follows that $L \trianglelefteq G$.

Next, $\overline{G}/\overline{Z(L)}$ is isomorphic to $G/Z(L)$, which is not cyclic, and so $\overline{N} \nleqslant \overline{Z(L)}$. As $\overline{N}$ lies in each nontrivial normal subgroup of $\overline{G}$, we conclude that $\overline{Z(L)} = 1$, and so $Z(L) = Z(G)$. $\square$

LEMMA 6.3. *Suppose that $\overline{G}$ is soluble and primitive with every proper quotient cyclic, and that $\Xi(G)$ has an edge. Then $\Xi(\overline{G})$ has isolated vertices, and* $\operatorname{diam}(\Xi^+(G)) = 2$.

Proof. Since the primitive group $\overline{G}$ has a minimal normal subgroup by Lemma 6.1, it is clear that each point stabiliser $\overline{K}$ of $\overline{G}$ is cyclic. Additionally, $Z(\overline{G}) = 1$, and since $\overline{G} = \langle k, g \rangle$ whenever k is a generator for $\overline{K}$ and $g \in \overline{G} \backslash \overline{K}$, every such k is an isolated vertex of $\Xi(\overline{G})$.

Now, let $x, y \in \Xi^+(G)$, and assume that G is 2-generated; else $\operatorname{diam}(\Xi(G)) = 2$ by Proposition 2.5(ii). By Propositions 2.5(iii) and 6.2, there exist normal maximal subgroups L and M of G with $x \in L \backslash Z(L)$, $y \in M \backslash Z(M)$ and $Z(L), Z(M) \leqslant Z(G)$. By Lemma 2.11, $d(x, y) \leqslant 2$. $\square$

As we mentioned in §1, the generating graph of a finite group is connected precisely when all proper quotients of that group are cyclic [9, Theorem 1 & Corollary 2]. However, this is not the case for the non-commuting, non-generating graph. Indeed, $G := \mathrm{AGL}(1,5) = \overline{G}$ satisfies the hypotheses of Lemma 6.3, and so $\Xi^+(G) \neq \Xi(G)$. On the other hand, Magma computations show that $H := D_{10}{:}C_8$ is a non-split extension of $Z(H)$ by G, and $\operatorname{diam}(\Xi(H)) = 2$.

In the following theorem, we assume that G itself is primitive, so that $Z(G) = 1$.

LEMMA 6.4. *Suppose that G is insoluble, non-simple and primitive with every proper quotient cyclic. Then $\Xi^+(G)$ is connected with diameter 2 or 3. Moreover, if $\Xi(G)$ has an isolated vertex r, then G is infinite and each proper subgroup of G containing r is core-free. Finally, if G is 2-generated, then it contains a normal maximal subgroup with trivial centre.*

Proof. We may again assume that G is 2-generated, else $\operatorname{diam}(\Xi(G)) = 2$ by Proposition 2.5(ii). Let N be a nontrivial proper normal subgroup of G, and M a maximal subgroup containing N. Then for each overgroup H of N in G, the cyclic group G/N normalises H/N, and hence $H \trianglelefteq G$. Thus each non-normal subgroup of G is core-free.

In particular, $M \trianglelefteq G$, and $Z(M) = 1$ by Lemma 6.1. Since no maximal subgroup of a finite insoluble group is abelian [19], the statement about an isolated vertex follows from Propositions 2.5(iii) and 2.6.

Now, let x and y be non-isolated vertices of $\Xi(G)$. Then Proposition 2.5(iii) shows that $x \in K \backslash Z(K)$ and $y \in L \backslash Z(L)$ for some maximal subgroups K and L of G. We may assume that $K \neq L$, as otherwise $d(x, y) \leqslant 2$ by Proposition 2.5(iv). Observe also that $K \cap M$ and $L \cap M$ are nontrivial maximal subgroups of K and L, respectively.

Suppose first that $K \trianglelefteq G$, so that $Z(K) = 1$ by this proof's first paragraph. We may assume that $y \notin K$, as otherwise we could set $L = K$. Applying Lemma 2.10(i) to (y, L, K) yields $y \sim h$ for some $h \in K \cap L \backslash C_{K \cap L}(y)$. As $d(x, h) \leqslant 2$ by Proposition 2.5(iv), we obtain $d(x, y) \leqslant 3$.

By symmetry, we may assume from now on that neither K nor L is normal in G, i.e. that both are core-free in G, and that $x, y \notin M$. Then the nontrivial subgroups $K \cap M$ and $L \cap M$ are not normal in G. Since $K \cap M \trianglelefteq K$ and $\langle K, L \rangle = G$, it follows that L does not centralise $K \cap M$. Additionally, applying Lemma 2.10(i) to (x, K, M) gives $C_{K \cap M}(x) < K \cap M$. Thus if $K \cap M \leqslant L$, then there exists an element $a \in K \cap M$ that centralises neither x nor L. Hence $x \sim a$, and Proposition 2.5(iv) yields $d(a, y) \leqslant 2$. Therefore, $d(x, y) \leqslant 3$.

If instead $K \cap M \nleqslant L$, then $\langle K \cap M, L \rangle = G$. As $L \cap M$ is normalised by L but not G, we deduce that $K \cap M \nleqslant C_G(L \cap M)$. Since $C_{K \cap M}(x) < K \cap M$, and similarly $C_{L \cap M}(y) < L \cap M$, it follows that there exists an element $b \in K \cap M$ that centralises neither $L \cap M$ nor x, and an element $c \in L \cap M$ that centralises neither b nor y. Thus $x \sim b \sim c \sim y$ and $d(x, y) \leqslant 3$. □

Using Magma, we see that $G_1 \coloneqq A_5 \wr C_2$ and $G_2 \coloneqq (A_5 \times A_5){:}C_4$ (with a point stabiliser of index 25) satisfy the hypotheses of Lemma 6.4, with $\operatorname{diam}(\Xi(G_1)) = 2$ and $\operatorname{diam}(\Xi(G_2)) = 3$.

Lemma 6.5. *Suppose that $\overline{G}$ is insoluble, non-simple and primitive with every proper quotient cyclic.*

(i) *The subgraph X of $\Xi(G)$ induced by the vertices in the set*

$$\{g \in G \backslash Z(G) \mid Z(G)g \in \Xi^+(\overline{G})\}$$

has diameter at most $k \coloneqq \operatorname{diam}(\Xi^+(\overline{G})) \in \{2, 3\}$. Hence if $\Xi(\overline{G})$ has no isolated vertices, and in particular if $\overline{G}$ is finite, then $\operatorname{diam}(\Xi(G)) \leqslant k$.

(ii) *If $X \neq \Xi^+(G)$, then $\Xi^+(G)$ is connected with diameter at most 4.*

Proof. As $Z(\overline{G}) = 1$, (i) follows from Lemma 6.4, and Lemma 2.9(ii) with $N = Z(G)$.

To prove (ii), we may assume that $X \neq \Xi^+(G)$ and (by Proposition 2.5(ii)) that G is 2-generated. Then $\overline{G}$ is also 2-generated, and Lemma 6.4 implies that G contains a normal maximal subgroup M with $Z(M) = Z(G)$, and that each element of $M \backslash Z(G)$ lies in X.

Let $y, y' \in \Xi^+(G) \backslash X$, so that $y, y' \notin M$. Then Proposition 2.5(iii) shows that $y \in K \backslash Z(K)$ for some maximal subgroup K of G, and applying Lemma 2.10(i) to (y, K, M) yields $y \sim m$ for some $m \in K \cap M$. Similarly, $y' \sim m'$ for some $m' \in M$. Proposition 2.5(iv) gives $d(m, m') \leqslant 2$, and so $d(y, y') \leqslant d(y, m) + d(m, m') + d(m', y) \leqslant 4$.

By (i), it remains to consider $d(y, x)$, with $x \in X$. We also observe from (i) that $d(m, x) \leqslant 3$. Hence $d(y, x) \leqslant d(y, m) + d(m, x) \leqslant 4$, and we conclude that $\operatorname{diam}(\Xi^+(G)) \leqslant 4$. □

Recall from above that if $G = \overline{G} = (A_5 \times A_5){:}C_4$, then $\operatorname{diam}(\Xi(G)) = 3$. We can use Magma to show that the non-commuting, non-generating graphs of the central extensions $G \times C_2$ and $G \times C_3$ of G are connected with diameter 2 and 3, respectively (cf. [11, Proposition 16]).

QUESTION 6.6. *Does there exist an infinite group G such that $\overline{G}$ satisfies the hypotheses of Lemma* 6.5 *and either* $\operatorname{diam}(\Xi^+(G)) = 4$*, or* $\operatorname{diam}(\Xi^+(G)) = 3$ *and* $\Xi^+(G) \neq \Xi(G)$*?*

In the following example, we prove that $\Xi(G_k)$ is connected (and hence has no isolated vertices) for a certain infinite family of infinite groups G_k that satisfy the hypotheses of Lemma 6.4.

EXAMPLE 6.7. For each positive integer k, let G_k be the group of permutations of $\mathbb{Z}$ generated by the simple alternating group $\operatorname{Alt}(\mathbb{Z})$ and the translation t_k that maps x to $x + k$ for all $x \in \mathbb{Z}$. Then $\operatorname{Alt}(\mathbb{Z})$ is the unique minimal normal subgroup of the insoluble group G_k [13, Proposition 2.5]. Moreover, G_k is 2-generated, and every proper quotient of G_k is cyclic [14, Theorem 4.1]. Observe that any maximal subgroup of G_k containing t_k is core-free, and so G_k is primitive.

Assume now that $k \geqslant 3$, and let $g \in G_k \backslash \{1\}$. Since $\operatorname{Alt}(\mathbb{Z})$ is generated by its 3-cycles and $C_{G_k}(\operatorname{Alt}(\mathbb{Z})) = 1$, it follows that there exists a 3-cycle $\alpha \in \operatorname{Alt}(\mathbb{Z})$ with $[g, \alpha] \neq 1$. Furthermore, the proofs of [14, Lemmas 4.2–4.3] show that no 3-cycle in $\operatorname{Alt}(\mathbb{Z})$ lies in a generating pair for G_k. Hence $g \sim \alpha$. In particular, $\Xi(G_k) = \Xi^+(G_k)$. Using Lemmas 6.4 and 6.5(i), we conclude that $\Xi(G_k)$ is connected with diameter 2 or 3, as is $\Xi(H)$ for each central extension H of G_k.

It would be interesting to determine $\operatorname{diam}(\Xi(G_k))$ precisely for each $k \geqslant 3$, and to investigate $\Xi(G_k)$ when $k \in \{1, 2\}$, where each $g \in G_k \backslash \{1\}$ lies in a generating pair [14, Theorem 6.1].

We now prove this paper's main theorems.

Proof of Theorems 1.2 and 1.3. If $\overline{G} = G/Z(G)$ has a proper non-cyclic quotient, then G contains a normal subgroup N such that $Z(G) < N$ and G/N is not cyclic. Thus in this case Theorem 5.10 applies, and case (i), (iii), (iv) or (v) of Theorem 1.2 holds. Otherwise, one of the three cases in Lemma 6.1 applies, with $\overline{G}$ in place of G. If every maximal subgroup of G is normal, or if $\overline{G}$ is soluble and primitive, then we can use Theorem 1.1 or Lemma 6.3, respectively, to show that case (i) or (iii) of Theorem 1.2 holds. If instead $\overline{G}$ is insoluble and primitive, then Lemma 6.5 shows that case (ii) or (iii) holds. This completes the proof of Theorem 1.2.

Assume now that G is finite. By the previous paragraph, if $\Xi(G)$ is the union of two connected components of diameter 2, then Theorem 5.10 applies. That theorem and Proposition 5.8 imply that $\Xi(G)$ is such a union if and only if G satisfies Assumption 3.3, and Theorem 1.3 follows. □

We see from the above proof that if $\Xi(G)$ has two nontrivial connected components, then Theorem 5.10 applies. Hence, as discussed below the proof of that theorem, Lemma 4.12 and Proposition 4.6 yield further information about the structures of infinite groups in this case.

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