ALGEBRAIC COMBINATORICS

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Volume 6, issue 6 (2023), p. 1697-1734.
https://doi.org/10.5802/alco.309

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Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

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Cyclic independence: Boolean and monotone

Octavio Arizmendi, Takahiro Hasebe & Franz Lehner

Abstract The present paper introduces a modified version of cyclic-monotone independence which originally arose in the context of random matrices, and also introduces its natural analogy called cyclic-Boolean independence. We investigate formulas for convolutions, limit theorems for sums of independent random variables, and also classify infinitely divisible distributions with respect to cyclic-Boolean convolution. Finally, we provide applications to the eigenvalues of the adjacency matrices of iterated star products of graphs and also iterated comb products of graphs.

1. Introduction

The present paper takes its origin in the concept of *cyclic-monotone independence* which appeared in the study of random matrices [5, 20] and which deserves separate treatment; see [2] for further work. The term "cyclic-monotone independence" was coined in [5] because of its apparent similarity with monotone independence except that it involves two linear functionals: a state and a tracial linear functional. It abstracts an asymptotic formula for the mixed moments, with respect to the non-normalized trace, of a random rotation of two sets $\mathbf{A}^N$ and $\mathbf{B}^N$ consisting of $N \times N$ deterministic matrices such that all mixed moments of $\mathbf{A}^N$ have finite limits with respect to the *non-normalized trace* as N tends to infinity, and all mixed moments of $\mathbf{B}^N$ have finite limits with respect to the *normalized trace*. More precisely, suppose that $\{A_i^N : 1 \leqslant i \leqslant k\}$ and $\{B_i^N : 1 \leqslant i \leqslant k\}, N = 1, 2, 3, \ldots$ are families of $N \times N$ deterministic matrices that satisfy the following conditions: for any $*$-polynomial $P(x_1, x_2, \ldots, x_k)$ in non-commuting variables $x_1, x_2, \ldots, x_k$ over the field $\mathbb{C}$ without a constant term (e.g. $P(x_1, x_2) = x_1^2 x_2 x_1^*$), the limits

$$\lim_{N\to\infty} \mathrm{Tr}_N[P(A_1^N, A_2^N, \ldots, A_k^N)] \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}_N[P(B_1^N, B_2^N, \ldots, B_k^N)]$$

Manuscript received 30th August 2022, revised 27th March 2023 and 6th April 2023, accepted 6th April 2023.

Keywords. Boolean independence, monotone independence, star product, comb product.

Acknowledgements. O.A. gratefully acknowledges financial support by the grants Conacyt A1-S-9764 and SFB TRR 195.
T.H. is supported by JSPS Grant-in-Aid for Early-Career Scientists 19K14546 and for Scientific Research (B) 18H0111. This work was supported by JSPS Open Partnership Joint Research Projects grant no. JPJSBP120209921.
Co-funded by the European Union F.L. was supported by the H2020-MSCA-RISE project 734922 - CONNECT.

exist in $\mathbb{C}$. According to [5, Theorem 4.3], for an $N \times N$ Haar unitary matrix U^N, $N = 1, 2, 3, \dots$, we have the almost sure convergence

$$\begin{aligned} &\lim_{N\to\infty} \mathrm{Tr}_N[A_1^N (U^N)^* B_1^N U^N A_2^N (U^N)^* B_2^N U^N \cdots A_k^N (U^N)^* B_k^N U^N] \\ &\quad = \lim_{N\to\infty} \mathrm{Tr}_N[A_1^N A_2^N \cdots A_k^N] \prod_{j=1}^{k} \left[\lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}_N[B_j^N] \right]. \end{aligned} \tag{1.1}$$

The formula (1.1) shows some similarity with monotone independence, but they are not the same because the formula involves both normalized trace and non-normalized trace.

The present paper offers a simple operator model for cyclic-monotone independence realized on the tensor product of Hilbert spaces. This construction also uncovers the associativity of cyclic-monotone independence with respect to a state and a trace. In order to ensure associativity, we modify the definition of cyclic-monotone independence. The new definition consists of two conditions: one is basically the condition in [5, Definition 3.2] referring to both the state and the tracial linear functional, and the other is monotone independence with respect to the state (see Definition 7.2). The modified definition of cyclic-monotone independence shares the same spirit with c-monotone independence [10] (and c-freeness [3]) because they are all associative notions of independence referring to two linear functionals.

However, the relationship between the random matrix model and the operator model is not perfectly understood. Curiously monotone independence does not appear in the random matrix model above, although it appears very naturally in the operator model. This is related to the fact that the random matrix model above is limited to two families of random matrices $\mathbf{A}^N$ and $\mathbf{B}^N$ and hence the question of associativity is not relevant.

Our operator realization of (modified) cyclic-monotone independence also indicates that a similar construction works for Boolean independence, which therefore leads to a notion of *cyclic-Boolean independence*. We develop a general theory of these two independences: computing generating functions for the sum of independent random variables, limit theorems, cyclic-Boolean cumulants which are governed by cyclic-interval partitions and infinitely divisible distributions with respect to cyclic-Boolean convolution. We do not know how to define cyclic-monotone cumulants and therefore this question is not addressed in the present paper.

Moreover, the operator models for cyclic-Boolean and cyclic-monotone independences are directly connected to the star product of (rooted) graphs (see Section 2.6) and the comb product of (rooted) graphs (see Section 2.7). Specifically, the eigenvalues of their adjacency matrices can be analyzed by means of cyclic-Boolean independence and cyclic-monotone independence, respectively.

The techniques are motivated by the relations between the adjacency matrix, the spectrum, the characteristic polynomial and walk generating functions of a graph. These form the core subject of algebraic graph theory, which deals with various matrices, polynomials and generating functions and other invariants carrying information about graphs.

It was shown by Schwenk [19] (later generalized by Godsil and McKay [9]) that the characteristic polynomial of the star product (or coalescence) and the comb product (or rooted product) of graphs only depends on the characteristic polynomials of the factors and the walk generating function at the roots of the factors and he gave an explicit formula. Similar simple formulas for the generating function of closed walks starting at the root hold. While Schwenk's proofs are combinatorial, we will give algebraic proofs based on the Schur complement which can be generalized to

arbitrary matrices and operators. Accardi, Ben Ghorbal and Obata [1] and Obata [15] initiated the application of monotone independence and Boolean independence to the asymptotic spectral analysis of adjacency matrices of iterated comb products and star products, respectively. The operator models for cyclic-monotone and cyclic-Boolean independences extend their work in the sense that the new framework also enables one to analyze refined properties of eigenvalues of the adjacency matrices. These generalizations are the subject of the present paper and illustrate the emergence of new notions of noncommutative independence, i.e., cyclic-monotone and cyclic-Boolean ones.

To summarize, the main contributions of the present paper are:

(1) the new notion of cyclic-Boolean independence (Sections 3) and a modification of the definition of cyclic-monotone independence given in [5] (Section 7);
(2) operator models for cyclic-Boolean independence (Sections 3) and for cyclic-monotone independence (Section 7);
(3) convolution formulas for the sum of independent random variables (Sections 4, 7) and their relationships to algebraic graph theory (Section 2);
(4) limit theorems for sums of independent random variables (Sections 3, 7);
(5) cyclic-Boolean cumulants and the relevant partition structure with cyclic-interval partitions (Section 5);
(6) classification of infinitely divisible distributions for cyclic-Boolean convolution (Section 6);
(7) analysis of the asymptotics of the eigenvalues of the adjacency matrices of iterated star product of graphs and iterated comb product of graphs (Sections 4, 7).

Recently Collins, Leid and Sakuma found a different matrix model for monotone independence and cyclic monotone independence [6]. So far a connection between their model and ours is not clear.

2. Preliminaries

2.1. Adjacency matrix. Let $\mathbf{\Gamma} = (V, E)$ be a graph on a vertex set $V = \{v_1, v_2, \dots, v_d\}$ with edge set E. We always consider finite undirected graphs without loops or multiple edges. An edge between two vertices u and v is denoted by uv. The *adjacency matrix* of $\mathbf{\Gamma}$ is the matrix $A_{\mathbf{\Gamma}} = [a_{ij}]_{i,j=1}^d$ with entries

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Since the graph has no loops, the diagonal entries of $A_{\mathbf{\Gamma}}$ are all zero. The *spectrum of the graph* $\mathbf{\Gamma}$ is the spectrum of its adjacency matrix. It consists of the eigenvalues λ_i of $A_{\mathbf{\Gamma}}$ which are the roots of the *characteristic polynomial*

$$\phi_{\mathbf{\Gamma}}(x) = \det(xI - A_{\mathbf{\Gamma}}) = \prod_{i=1}^{d} (x - \lambda_i).$$

Alternatively, the eigenvalues of $A_{\mathbf{\Gamma}}$ are the poles of the (tracial) *Cauchy transform*

$$\mathfrak{g}_{\mathbf{\Gamma}}(z) = \mathrm{Tr}((zI - A_{\mathbf{\Gamma}})^{-1}). \tag{2.1}$$

The Cauchy transform and the characteristic polynomial are mutually related by the logarithmic derivative

$$\mathfrak{g}_{\mathbf{\Gamma}}(z) = \sum_{i=1}^{d} \frac{1}{z - \lambda_i} = \frac{d}{dz} \log \phi_{\mathbf{\Gamma}}(z) = \frac{\phi'_{\mathbf{\Gamma}}(z)}{\phi_{\mathbf{\Gamma}}(z)}. \tag{2.2}$$

For the generalization of this identity to trace class operators it will be convenient to remove the moment of order zero and work with the "renormalized" Cauchy transform

$$\tilde{\mathfrak{g}}_{\mathbf{\Gamma}}(z) = \mathfrak{g}_{\mathbf{\Gamma}}(z) - \frac{d}{z} = \mathrm{Tr}((zI - A_{\mathbf{\Gamma}})^{-1} - z^{-1}I) \tag{2.3}$$

instead.

2.2. Walk generating functions. Let $(\mathbf{\Gamma}, o)$ be a finite rooted graph, i.e., a graph on vertices $v_1, v_2, \dots, v_d$ where we single out the vertex $o = v_1$ as the root of the graph. The number m_n of closed walks of length n starting at the root o is equal to $\langle A_{\mathbf{\Gamma}}^n e_1, e_1\rangle$ where $A_{\mathbf{\Gamma}}$ is the adjacency matrix of $\mathbf{\Gamma}$ and e_1 is the vector $(1, 0, 0, \dots, 0) \in \mathbb{C}^d$. Denote by

$$M_{\mathbf{\Gamma}}(z) = \sum_{n=0}^{\infty} m_n z^n = \langle (I - zA_{\mathbf{\Gamma}})^{-1} e_1, e_1\rangle \tag{2.4}$$

the *walk generating function*. One caution is in place here. To keep notation simple here and below we do not explicitly write the root in subscripts, although the generating functions depend on the choice of the root.

It will be more convenient to rather work with the resolvent of $A_{\mathbf{\Gamma}}$ and with the *Green function* (evaluated at the root)

$$G_{\mathbf{\Gamma}}(z) = \langle (zI - A_{\mathbf{\Gamma}})^{-1} e_1, e_1\rangle = \frac{1}{z} M_{\mathbf{\Gamma}}\left(\frac{1}{z}\right) \tag{2.5}$$

and its reciprocal

$$F_{\mathbf{\Gamma}}(z) = \frac{1}{G_{\mathbf{\Gamma}}(z)}. \tag{2.6}$$

We can obtain a relation between the Green function (2.5) and the Cauchy transform (2.1) from the *Schur complement*.

2.3. Schur complement. Let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

be a block matrix and assume D is invertible. Then the *Schur complement* [24] is defined as

$$M/D = A - BD^{-1}C. \tag{2.7}$$

It appears in *Aitken's factorization*

$$M = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}, \tag{2.8}$$

which is obtained by Gaussian elimination on the original matrix M. From this factorization we infer the following assertions:

(1) M is invertible if and only if M/D is invertible. If this is the case, then the *Banachiewicz inversion formula*

$$M^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix} \tag{2.9}$$

holds.

(2) The determinant factorizes and *Jacobi's identity*

$$\det M = \det(M/D) \det D \tag{2.10}$$

holds.

2.4. Relation between the Green function and the characteristic polynomial. Let $(\mathbf{\Gamma}, o)$ be a rooted graph on vertices $v_1, v_2, \dots, v_d$. If we decompose its adjacency matrix

$$A_{\mathbf{\Gamma}} = \begin{bmatrix} 0 & b^* \\ b & D \end{bmatrix}$$

into block form with $D = A_{\mathbf{\Gamma} \smallsetminus o}$. Then the resolvent of the matrix

$$M = M(z) = zI - A_{\mathbf{\Gamma}} = \begin{bmatrix} z & -b^* \\ -b & zI - D \end{bmatrix}$$

can be determined from the Banachiewicz formula (2.9). The Schur complement is

$$S = z - b^*(zI - D)^{-1} b$$

and the Green function

$$G_{\mathbf{\Gamma}}(z) = \langle (zI - A_{\mathbf{\Gamma}})^{-1} e_1, e_1 \rangle = S^{-1}$$

i.e., the Schur complement is $S = G_{\mathbf{\Gamma}}(z)^{-1} = F_{\mathbf{\Gamma}}(z)$. Being a matrix of dimension 1 it equals its determinant and Jacobi's identity (2.10) yields

$$\phi_{\mathbf{\Gamma}}(x) = \det S \det(zI - D) = F_{\mathbf{\Gamma}}(x)\, \phi_{\mathbf{\Gamma} \smallsetminus o}(x), \qquad x \in \mathbb{C} \smallsetminus \mathbb{R}. \tag{2.11}$$

2.5. Relation between the Green function and the Cauchy transform of a general matrix. Let A be a $d \times d$ matrix. We want to understand the relation between the functions

$$G_A(z) = \langle (z - A)^{-1} e_1, e_1 \rangle \quad \text{and} \quad \mathfrak{g}_A(z) = \operatorname{Tr}((z - A)^{-1}).$$

To this end, we partition the matrix into blocks of dimension 1 and $d - 1$

$$A = \begin{bmatrix} \alpha & a_1^* \\ a_2 & \mathring{A} \end{bmatrix}. \tag{2.12}$$

Now the corresponding Schur complement (2.7) of $z - A$ is a scalar

$$S = z - \alpha - a_1^*(z - \mathring{A})^{-1} a_2$$

and we conclude from the Banachiewicz inversion formula (2.9) that

$$G_A(z) = S^{-1} = \frac{1}{z - \alpha - a_1^*(z - \mathring{A})^{-1} a_2} \tag{2.13}$$

and

$$\begin{aligned} \mathfrak{g}_A(z) &= \operatorname{Tr} \begin{bmatrix} S^{-1} & * \\ * & (z - \mathring{A})^{-1} + (z - \mathring{A})^{-1} a_2 S^{-1} a_1^* (z - \mathring{A})^{-1} \end{bmatrix} \\ &= G_A(z) + \mathfrak{g}_{\mathring{A}}(z) + G_A(z) \operatorname{Tr}((z - \mathring{A})^{-1} a_2 a_1^* (z - \mathring{A})^{-1}) \\ &= G_A(z)(1 + \operatorname{Tr}(a_1^*(z - \mathring{A})^{-2} a_2)) + \mathfrak{g}_{\mathring{A}}(z) \\ &= \frac{F_A'(z)}{F_A(z)} + \mathfrak{g}_{\mathring{A}}(z). \end{aligned}$$

After subtracting the unit matrix according to (2.3) we obtain the identity

$$\tilde{\mathfrak{g}}_{\mathring{A}}(z) = \tilde{\mathfrak{g}}_A(z) + \frac{d}{dz} \log(z G_A(z)) = \tilde{\mathfrak{g}}_A(z) + \frac{1}{z} + \frac{G_A'(z)}{G_A(z)}, \tag{2.14}$$

which can be extended to trace class operators.

2.6. The star product and its adjacency matrix. Let $(\mathbf{\Gamma}_1, o_1) = (V_1, E_1, o_1)$ and $(\mathbf{\Gamma}_2, o_2) = (V_2, E_2, o_2)$ be rooted graphs. The star product, denoted by $(\mathbf{\Gamma}_1, o_1) \circledast (\mathbf{\Gamma}_2, o_2)$, is defined by conglutinating the graphs $(\mathbf{\Gamma}_1, o_1), (\mathbf{\Gamma}_2, o_2)$ at their roots; see Fig. 1. Formally the vertex set of $\mathbf{\Gamma}$ can be realized as a subset of $V_1 \times V_2$,

$$V = \{(x_1, o_2) : x_1 \in V_1\} \cup \{(o_1, x_2) : x_2 \in V_2\}.$$

Two vertices (x_1, x_2) and (y_1, y_2) of V are connected by an edge if either $x_1 y_1 \in E_1$ and $x_2 = y_2 = o_2$, or $x_1 = y_1 = o_1$ and $x_2 y_2 \in E_2$. The cartesian product of the vertices corresponds to the tensor product of the vector spaces and the adjacency matrix has entries

$$a_{(x_1,x_2),(y_1,y_2)} = a^{(1)}_{x_1,y_1} \delta_{x_2,o_2} \delta_{y_2,o_2} + \delta_{x_1,o_1} \delta_{y_1,o_1} a^{(2)}_{x_2,y_2}$$

i.e.,

$$A = A_1 \otimes P_2 + P_1 \otimes A_2$$

where P_i is the orthogonal projection of $\ell^2(V_i)$ onto the one-dimensional subspace spanned by the delta function δ_{o_i}; see [12, Proposition 8.50].

The star product is associative and hence one may define by iteration the star product $(\mathbf{\Gamma}, o) = (V, E, o)$ of rooted graphs $(\mathbf{\Gamma}_i, o_i) = (V_i, E_i, o_i), i = 1, 2, \ldots, N$. Suppose further that those graphs are finite and simple. Then the vertex set V of $\mathbf{\Gamma}$ can be regarded as a subset of $V_1 \times \cdots \times V_N$ and hence the adjacency matrix $A_{\mathbf{\Gamma}}$ can be regarded as an operator on $\ell^2(V_1) \otimes \cdots \otimes \ell^2(V_N)$. Under this identification one has

$$A_{\mathbf{\Gamma}} = \sum_{i=1}^{N} P_1 \otimes P_2 \otimes \cdots \otimes P_{i-1} \otimes A_{\mathbf{\Gamma}_i} \otimes P_{i+1} \otimes \cdots \otimes P_N. \tag{2.15}$$

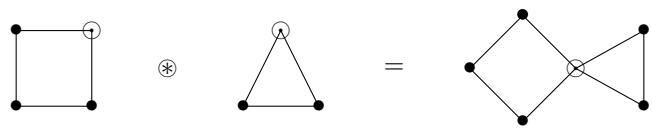

Figure 1. Star product of rooted graphs

2.7. The comb product and its adjacency matrix. Given a graph $\mathbf{\Gamma}_1 = (V_1, E_1)$ and a rooted graph $(\mathbf{\Gamma}_2, o_2) = (V_2, E_2, o_2)$, a new graph $\mathbf{\Gamma} = \mathbf{\Gamma}_1 \triangleright (\mathbf{\Gamma}_2, o_2)$ is defined by gluing a copy of $\mathbf{\Gamma}_2$ to every vertex of $\mathbf{\Gamma}_1$ at the root o_2: The vertex set V of $\mathbf{\Gamma}$ is $V_1 \times V_2$ and two vertices (x_1, x_2) and (y_1, y_2) are connected by an edge if and only if either $x_1 y_1 \in E_1$ and $x_2 = y_2 = o_2$, or $x_1 = y_1$ and $x_2 y_2 \in E_2$. If we further specify a root of $\mathbf{\Gamma}_1$, then the natural root for the comb product is (o_1, o_2), which makes the comb product associative (but non-commutative) in the category of rooted graphs; see Fig. 2. The adjacency matrix now can be written as

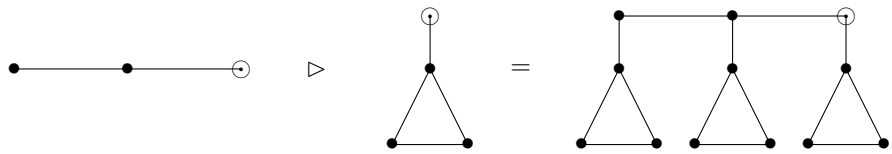

Figure 2. Comb product of rooted graphs

$$A_1 \otimes P_2 + I_1 \otimes A_2$$

and by iteration one may define the comb product $(\mathbf{\Gamma}, o)$ of a sequence of rooted graphs $(\mathbf{\Gamma}_i, o_i) = (V_i, E_i, o_i)$, $i = 1, 2, \dots, N$. The adjacency matrix $A_{\mathbf{\Gamma}}$ can then be regarded as an operator on $\ell^2(V_1) \otimes \cdots \otimes \ell^2(V_N)$ and it has the form

$$A_{\mathbf{\Gamma}} = \sum_{i=1}^{N} I_1 \otimes I_2 \otimes \cdots \otimes I_{i-1} \otimes A_{\mathbf{\Gamma}_i} \otimes P_{i+1} \otimes \cdots \otimes P_N; \tag{2.16}$$

see [12, Proposition 8.38].

For comb products of identical rooted graphs, Accardi, Ben Ghorbal and Obata used monotone independence satisfied by the summands in (2.16) in order to study the asymptotics of the Green function of $\mathbf{\Gamma}$ as $N \to \infty$; see the original article [1, Theorem 5.1] or the book [12, Theorem 8.40]. On the other hand, for the star product, the summands in (2.15) are Boolean independent, which provides another type of asymptotics of Green function; see the original article of Obata [15, Theorem 3.7] or the book [12, Theorem 8.53].

In the present paper we study the **asymptotic behavior of eigenvalues or empirical eigenvalue distributions** of $A_{\mathbf{\Gamma}}$ for large N using the asymptotics of the characteristic polynomial $\phi_{\mathbf{\Gamma}}(z)$ or the Cauchy transform $\mathfrak{g}_{\mathbf{\Gamma}}(z)$.

2.8. Identities for the star product. For the sake of notational convenience we denote by $\mathbf{\Gamma}_1 \circledast \mathbf{\Gamma}_2$ the star product of two rooted graphs $(\mathbf{\Gamma}_1, o_1)$ and $(\mathbf{\Gamma}_2, o_2)$. The Green function of the star product satisfies the following relation, which follows from the decomposition (2.15) of the adjacency matrix into Boolean independent operators and the linearization formula for Boolean convolution in [23, Section 2]; see also [17] for another proof of the latter.

Proposition 2.1. *For rooted graphs $(\mathbf{\Gamma}_1, o_1)$ and $(\mathbf{\Gamma}_2, o_2)$ the following formula holds.*

$$F_{\mathbf{\Gamma}_1 \circledast \mathbf{\Gamma}_2}(z) = F_{\mathbf{\Gamma}_1}(z) + F_{\mathbf{\Gamma}_2}(z) - z. \tag{2.17}$$

The Cauchy transform of the star product is computed by the formula below.

Proposition 2.2. *For rooted graphs $(\mathbf{\Gamma}_1, o_1)$ and $(\mathbf{\Gamma}_2, o_2)$ the following formula holds.*

$$\mathfrak{g}_{\mathbf{\Gamma}_1 \circledast \mathbf{\Gamma}_2}(z) + \frac{G'_{\mathbf{\Gamma}_1 \circledast \mathbf{\Gamma}_2}(z)}{G_{\mathbf{\Gamma}_1 \circledast \mathbf{\Gamma}_2}(z)} = \mathfrak{g}_{\mathbf{\Gamma}_1}(z) + \mathfrak{g}_{\mathbf{\Gamma}_2}(z) + \frac{G'_{\mathbf{\Gamma}_1}(z)}{G_{\mathbf{\Gamma}_1}(z)} + \frac{G'_{\mathbf{\Gamma}_2}(z)}{G_{\mathbf{\Gamma}_2}(z)}. \tag{2.18}$$

Remark 2.3. Later we will give two alternative proofs in a more general setting; see Theorem 4.2.

Proof. The key is the simple identity

$$\phi_{(\mathbf{\Gamma}_1 \circledast \mathbf{\Gamma}_2) \smallsetminus o}(x) = \phi_{\mathbf{\Gamma}_1 \smallsetminus o_1}(x)\, \phi_{\mathbf{\Gamma}_2 \smallsetminus o_2}(x), \tag{2.19}$$

for the star product, which follows from the fact that the removal of the root splits the graph into two disjoint connected components

$$(\mathbf{\Gamma}_1 \circledast \mathbf{\Gamma}_2) \smallsetminus o = (\mathbf{\Gamma}_1 \smallsetminus o_1) \cup (\mathbf{\Gamma}_2 \smallsetminus o_2).$$

Using the Schur identity (2.11) we can rewrite (2.19) as

$$\phi_{\mathbf{\Gamma}_1 \circledast \mathbf{\Gamma}_2}(z)\, G_{\mathbf{\Gamma}_1 \circledast \mathbf{\Gamma}_2}(z) = \phi_{\mathbf{\Gamma}_1}(z)\, G_{\mathbf{\Gamma}_1}(z)\, \phi_{\mathbf{\Gamma}_2}(z)\, G_{\mathbf{\Gamma}_2}(z) \tag{2.20}$$

and taking the logarithmic derivative of (2.20) together with (2.2) yields (2.18). □

Finally, the characteristic polynomial of the star product satisfies the following identity proved by Schwenk, for which we give an alternative proof.

Theorem 2.4 ([19, Corollary 2b], [8, Lemma 9.1]). *For rooted graphs $(\mathbf{\Gamma}_1, o_1)$ and $(\mathbf{\Gamma}_2, o_2)$ the following formula holds.*

$$\phi_{\mathbf{\Gamma}_1 \circledast \mathbf{\Gamma}_2}(x) = \phi_{\mathbf{\Gamma}_1}(x)\, \phi_{\mathbf{\Gamma}_2 \smallsetminus o_2}(x) + \phi_{\mathbf{\Gamma}_1 \smallsetminus o_1}(x)\, \phi_{\mathbf{\Gamma}_2}(x) - x \phi_{\mathbf{\Gamma}_1 \smallsetminus o_1}(x)\, \phi_{\mathbf{\Gamma}_2 \smallsetminus o_2}(x). \tag{2.21}$$

Proof. Equations (2.20) and (2.17) give rise to

$$\begin{aligned}\phi_{\mathbf{\Gamma}_1\circledast\mathbf{\Gamma}_2}(x) &= \phi_{\mathbf{\Gamma}_1}(x)\phi_{\mathbf{\Gamma}_2}(x)G_{\mathbf{\Gamma}_1}(x)G_{\mathbf{\Gamma}_2}(x)(F_{\mathbf{\Gamma}_1}(x)+F_{\mathbf{\Gamma}_2}(x)-x)\\ &= \phi_{\mathbf{\Gamma}_1}(x)\phi_{\mathbf{\Gamma}_2}(x)(G_{\mathbf{\Gamma}_1}(x)+G_{\mathbf{\Gamma}_2}(x)-xG_{\mathbf{\Gamma}_1}(x)G_{\mathbf{\Gamma}_2}(x)).\end{aligned}$$

Substituting the formula (2.11), $G_{\mathbf{\Gamma}_i}(x)=\phi_{\mathbf{\Gamma}_i\setminus o}(x)/\phi_{\mathbf{\Gamma}_i}(x)$, into the above yields the desired formula. □

2.9. Identities for the comb product. In this section, the comb product is denoted simply by $\mathbf{\Gamma}_1\rhd\mathbf{\Gamma}_2$, the root being omitted for simplicity, for a graph $\mathbf{\Gamma}_1$ and a rooted graph $(\mathbf{\Gamma}_2,o_2)$. The relation between the Green functions is simple and follows from the decomposition (2.16) of the adjacency matrix together with Muraki's formula [14, Theorem 3.1]; see also [16, Theorem 3.2] for another proof of Muraki's formula.

Proposition 2.5. *For rooted graphs $(\mathbf{\Gamma}_1,o_1)$ and $(\mathbf{\Gamma}_2,o_2)$ the following formula holds:*

$$F_{\mathbf{\Gamma}_1\rhd\mathbf{\Gamma}_2}(z)=F_{\mathbf{\Gamma}_1}(F_{\mathbf{\Gamma}_2}(z)). \tag{2.22}$$

For the characteristic polynomial Schwenk proved the following relation by combinatorial arguments; we give an algebraic proof based on the simpler relation (2.22).

Theorem 2.6 ([19, Theorem 5]). *Let $\mathbf{\Gamma}$ be a graph on d vertices and $(\mathbf{H},o)$ a rooted graph. Then*

$$\phi_{\mathbf{\Gamma}\rhd\mathbf{H}}(x)=\phi_{\mathbf{H}\setminus o}(x)^d\phi_{\mathbf{\Gamma}}(\phi_{\mathbf{H}}(x)/\phi_{\mathbf{H}\setminus o}(x))=\phi_{\mathbf{H}\setminus o}(x)^d\phi_{\mathbf{\Gamma}}(F_{\mathbf{H}}(x)). \tag{2.23}$$

Proof. We proceed by induction. Fix an arbitrary vertex o' of $\mathbf{\Gamma}$ as a root and denote by o'' the root of $\mathbf{\Gamma}\rhd\mathbf{H}$. Removing it from $\mathbf{\Gamma}\rhd\mathbf{H}$ splits off an extra copy of $\mathbf{H}\setminus o$ (cf. Fig. 2)

$$(\mathbf{\Gamma}\rhd\mathbf{H})\setminus o''=((\mathbf{\Gamma}\setminus o')\rhd\mathbf{H})\cup(\mathbf{H}\setminus o)$$

and therefore

$$\phi_{(\mathbf{\Gamma}\rhd\mathbf{H})\setminus o''}(x)=\phi_{(\mathbf{\Gamma}\setminus o')\rhd\mathbf{H}}(x)\,\phi_{\mathbf{H}\setminus o}(x).$$

We proceed with identities (2.11) and (2.22) to conclude by induction

$$\begin{aligned}\phi_{\mathbf{\Gamma}\rhd\mathbf{H}}(x) &= \phi_{(\mathbf{\Gamma}\rhd\mathbf{H})\setminus o'}(x)F_{\mathbf{\Gamma}\rhd\mathbf{H}}(x)\\ &= \phi_{(\mathbf{\Gamma}\setminus o')\rhd\mathbf{H}}(x)\,\phi_{\mathbf{H}\setminus o}(x)\,F_{\mathbf{\Gamma}\rhd\mathbf{H}}(x)\\ &= \phi_{\mathbf{H}\setminus o}(x)^{d-1}\,\phi_{\mathbf{\Gamma}\setminus o'}(F_{\mathbf{H}}(x))\,\phi_{\mathbf{H}\setminus o}(x)\,F_{\mathbf{\Gamma}}(F_{\mathbf{H}}(x))\\ &= \phi_{\mathbf{H}\setminus o}(x)^{d}\,\phi_{\mathbf{\Gamma}}(F_{\mathbf{H}}(x)).\end{aligned}$$

□

Finally, formula (2.23) gives rise to an equivalent formula for the renormalized Cauchy transform.

Proposition 2.7. *In the setting of Theorem 2.6, one has*

$$\tilde{\mathfrak{g}}_{\mathbf{\Gamma}\rhd\mathbf{H}}(z)=d\,\tilde{\mathfrak{g}}_{\mathbf{\Gamma}}(z)+F'_{\mathbf{H}}(z)\;\tilde{\mathfrak{g}}_{\mathbf{\Gamma}}(F_{\mathbf{H}}(z)).$$

Remark 2.8. Later we will give two more proofs in a more general setting; see Theorem 7.7.

Proof. Combining (2.2), (2.23) and (2.11) yields

$$\begin{aligned}\mathfrak{g}_{\mathbf{\Gamma}\rhd\mathbf{H}}(z) &= d\frac{d}{dz}\log\frac{\phi_{\mathbf{H}}(z)}{F_{\mathbf{H}}(z)}+F'_{\mathbf{H}}(z)\;\mathfrak{g}_{\mathbf{\Gamma}}(F_{\mathbf{H}}(z))\\ &= d\left(\mathfrak{g}_{\mathbf{H}}(z)-\frac{F'_{\mathbf{H}}(z)}{F_{\mathbf{H}}(z)}\right)+F'_{\mathbf{H}}(z)\;\mathfrak{g}_{\mathbf{\Gamma}}(F_{\mathbf{H}}(z)),\end{aligned}$$

which can be rewritten as

$$\tilde{\mathfrak{g}}_{\mathbf{\Gamma}\triangleright\mathbf{H}}(z)+\frac{dd'}{z}=d\left(\tilde{\mathfrak{g}}_{\mathbf{H}}(z)+\frac{d'}{z}-\frac{F'_{\mathbf{H}}(z)}{F_{\mathbf{H}}(z)}\right)+F'_{\mathbf{H}}(z)\left(\tilde{\mathfrak{g}}_{\mathbf{\Gamma}}(F_{\mathbf{H}}(z))+\frac{d}{F_{\mathbf{H}}(z)}\right),$$

where d' is the number of vertices of $\mathbf{H}$. □

3. Cyclic-Boolean independence

In order to study the eigenvalues of the adjacency matrix of star product graphs, we will compute traces of powers of the adjacency matrix. These computations can be abstracted and formulated as a new notion of independence, which we call cyclic-Boolean independence.

3.1. Definition and example.

The definition of cyclic-Boolean independence is motivated by the star product from Section 2.6, which can be extended to the general setting of Hilbert spaces as follows.

Example 3.1. Let $H_i, i\in\mathbb{N}$, be Hilbert spaces with distinguished unit vectors $\xi_i\in H_i$, $P_i\colon H_i\to H_i$ the orthogonal projection onto $\mathbb{C}\xi_i$, $T(H_i)$ the $*$-algebra of trace-class operators on H_i and φ_i the vector state on $B(H_i)$ defined by $\varphi_i(A)=\langle A\xi_i,\xi_i\rangle$.

Let $H=H_1\otimes\cdots\otimes H_N$, $\xi=\xi_1\otimes\cdots\otimes\xi_N$ and φ be the vacuum state on $B(H)$ defined by ξ. Let $\pi_i\colon B(H_i)\to B(H)$ be the $*$-homomorphism defined by

$$\pi_i(A)=P_1\otimes\cdots\otimes P_{i-1}\otimes A\otimes P_{i+1}\otimes\cdots\otimes P_N. \tag{3.1}$$

The family of $*$-subalgebras $\{\pi_i(B(H_i))\}_{i=1}^N$ is Boolean independent in $(B(H),\varphi)$; e.g. see [12, Theorem 8.8]. Furthermore, we compute the mixed moments with respect to the trace. A key formula is

$$P_iAP_i=\varphi_i(A)P_i,\qquad A\in B(H_i). \tag{3.2}$$

For any cyclically alternating tuple $(k_1,\dots,k_n)\in\mathbb{N}^n$, namely those satisfying $k_1\neq k_2\neq\cdots\neq k_n\neq k_1$, and for any $A_i\in T(H_{k_i})$ a direct computation using formula (3.2) yields

$$\mathrm{Tr}_H(\pi_{k_1}(A_1)\cdots\pi_{k_n}(A_n))=\begin{cases}\mathrm{Tr}_{H_{k_1}}(A_1) & \text{if } n=1,\\ \varphi_{k_1}(A_1)\,\varphi_{k_2}(A_2)\cdots\varphi_{k_n}(A_n) & \text{if } n\geqslant 2.\end{cases}$$

Let us raise this identity to an abstract concept.

Definition 3.2. *Let $\mathcal{A}$ be a $*$-algebra over $\mathbb{C}$, φ a positive linear functional on $\mathcal{A}$ and ω a positive tracial linear functional on $\mathcal{A}$. The triplet $(\mathcal{A},\varphi,\omega)$ is called a* cyclic non-commutative probability space *(cncps). Here the term "cyclic" reflects the invariance of the trace ω under cyclic permutations of the argument. The* distribution *of a self-adjoint element $a\in\mathcal{A}$ is the data $\{(\varphi(a^n),\omega(a^n)):n\geqslant 1\}$.*

Definition 3.3. *Let $(\mathcal{A},\varphi,\omega)$ be a cncps. A family of $*$-subalgebras $\{\mathcal{A}_k\}_{k\in K}$ is said to be* cyclic-Boolean independent *if*

(i) *it is Boolean independent with respect to φ, that is, for any $n\geqslant 2$, alternating tuple $(k_1,\dots,k_n)\in K^n$ (namely, with $k_1\neq k_2\neq\cdots\neq k_n$) and $a_i\in\mathcal{A}_{k_i}, i=1,2,\dots,n$, one has the factorization*

$$\varphi(a_1\cdots a_n)=\varphi(a_1)\varphi(a_2)\cdots\varphi(a_n);$$

(ii) *for any $n\geqslant 1$, any cyclically alternating tuple $(k_1,\dots,k_n)\in K^n$ and any choice of $a_i\in\mathcal{A}_{k_i}, i=1,2,\dots,n$, one has*

$$\omega(a_1a_2\cdots a_n)=\begin{cases}\omega(a_1) & \text{if } n=1,\\ \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n) & \text{if } n\geqslant 2.\end{cases}$$

A family of elements $\{a_k\}_{k\in K}$ of $\mathcal{A}$ is said to be cyclic-Boolean independent if this is the case for $\{\mathcal{A}_k\}_{k\in K}$, where $\mathcal{A}_k$ is the $*$-subalgebra generated by a_k without unit.

Example 3.4. Suppose that $\{a, b, c\}$ is cyclic-Boolean independent in $(\mathcal{A}, \varphi, \omega)$. Then

$$\varphi(ba^2bc^2b) = \varphi(b)\varphi(a^2)\varphi(b)\varphi(c^2)\varphi(b)$$

and

$$\omega(ba^2bc^2b) = \varphi(b^2)\varphi(a^2)\varphi(b)\varphi(c^2).$$

Another operator model occurs on star products of Hilbert spaces.

Example 3.5. Let $\mathcal{H}_i$ be separable Hilbert spaces with distinguished unit vectors ξ_i as above and $\mathring{\mathcal{H}}_i = (\mathbb{C}\xi_i)^\perp$. The *star product* of the Hilbert spaces $\mathcal{H}_i$ is the direct sum

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_i \mathring{\mathcal{H}}_i.$$

Then each $\mathcal{H}_i$ can be identified with the subspace $\mathbb{C}\xi \oplus \mathring{\mathcal{H}}_i \subseteq \mathcal{H}$ and there is a canonical representation of $B(\mathcal{H}_i)$ on $\mathcal{H}$ which acts by simply annihilating the complement of H_i. More precisely we decompose $\mathcal{H}$ as a direct sum $\mathcal{H} \simeq \mathcal{H}_i \oplus \mathcal{H}_i^\perp$ where $\mathcal{H}_i^\perp = \bigoplus_{j\neq i} \mathring{\mathcal{H}}_j$ and define the representation $\pi_i(X) = X \oplus 0$. Then the algebras $\mathcal{A}_i = \pi_i(B(\mathcal{H}_i))$ are Boolean independent with respect to the vacuum expectation $\varphi = \langle .\, \xi, \xi\rangle$ and moreover, the algebras $\mathcal{A}_i$ are cyclic-Boolean independent with respect to the trace. Indeed, let $P_0 \in B(\mathcal{H})$ be the projection onto $\mathbb{C}\xi$ and P_i the projections onto $\mathring{\mathcal{H}}_i$; then $P_0, P_1, P_2, \dots$ form a partition of unity and by definition we have $X = (P_0 + P_i)X(P_0 + P_i)$ for all $X \in \mathcal{A}_i$. Let $X_1X_2 \dots X_n$ be a cyclically alternating product of trace class operators, i.e., $X_k \in \mathcal{A}_{i_k}$ with $i_k \neq i_{k+1}$ for all k and $i_n \neq i_1$, then

$$\begin{aligned}\operatorname{Tr}(X_1X_2\cdots X_n) &= \operatorname{Tr}((P_0 + P_{i_1})X_1(P_0 + P_{i_1})(P_0 + P_{i_2})X_2(P_0 + P_{i_2}) \\ &\qquad \cdots (P_0 + P_{i_n})X_n(P_0 + P_{i_n})) \\ &= \operatorname{Tr}(P_0X_1P_0X_2P_0\cdots P_0X_nP_0) \\ &= \varphi(X_1)\,\varphi(X_2)\cdots\varphi(X_n)\end{aligned}$$

where we used the identity $(P_0 + P_i)(P_0 + P_j) = P_0$, valid for $i \neq j$.

Next we show that any Boolean independent family can be represented on a star product space.

3.2. Construction of a cyclic-Boolean trace. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, where $\mathcal{A}$ is a $*$-algebra and $\mathcal{A}_i$ are Boolean independent subalgebras. In the following assume that $\mathcal{A}$ is faithfully represented on a Hilbert space $\mathcal{H}$ and that the state φ is realized as a vector state $\varphi(X) = \langle X\xi, \xi\rangle$. One way to achieve this under certain conditions is the *GNS-construction*.

Recall that the GNS-representation consists of the Hilbert space $\mathcal{H}_\varphi$ obtained by completing the quotient space $\mathcal{A}/N_\varphi$, where $N_\varphi = \{x \in \mathcal{A} \mid \varphi(x^*x) = 0\}$, with respect to the scalar product

$$\langle [x]_\varphi, [y]_\varphi\rangle = \varphi(y^*x).$$

The action of the GNS representation is $\pi_\varphi(x)[y]_\varphi = [xy]_\varphi$.

Lemma 3.6. *The GNS representation is faithful if and only if the state φ is nondegenerate in the sense that if $\varphi(axb) = 0$ for all $a, b \in \mathcal{A}$, then $x = 0$.*

Proof. Let $x \in \mathcal{A}$, then $\pi_\varphi(x) = 0 \iff \pi_\varphi(x)[y]_\varphi = 0$ for all $y \in \mathcal{A} \iff \langle [xy]_\varphi, [z]_\varphi\rangle_\varphi = 0$ for all $y, z \in \mathcal{A} \iff \varphi(z^*xy) = 0$ for all $y, z \in \mathcal{A}$. □

If $\mathcal{A}$ is unital, then the state vector $\xi = [1]_\varphi$ comes for free, otherwise the state must satisfy the Cauchy-Schwarz condition

$$|\varphi(x)|^2 \leqslant C\varphi(x^*x)$$

for some fixed constant C in order to allow a positive extension to the unitization of $\mathcal{A}$, see [18, Theorem 4.5.11].

Assuming that $\mathcal{A}$ and the state φ are faithfully represented on some Hilbert space $\mathcal{H}$ we identify $\mathcal{A}$ with a subalgebra of $B(\mathcal{H})$ and we are now going to reconstruct the star product space from this data. Let $\mathcal{H}_0 = [\xi] = \mathbb{C}\xi$ be the subspace spanned by ξ and P_0 the orthogonal projection onto it. Adjoining this projection to the algebra $\mathcal{A}$ and to each subalgebra $\mathcal{A}_i$, Boolean independence is preserved because $\varphi(P_0) = 1$ and $P_0XP_0 = \varphi(X)P_0$ for any $X \in B(\mathcal{H})$; therefore without loss of generality we may assume that $P_0 \in \mathcal{A}_i$ for every i. Let now $\mathring{\mathcal{A}}_i = \ker\varphi \cap \mathcal{A}_i$, then we can construct the components of the star product space as follows.

Lemma 3.7. *Let $\mathcal{H}_0 = [\xi]$ and $\mathring{\mathcal{H}}_i = [\mathring{\mathcal{A}}_i\xi]$ be the closed invariant subspace generated by ξ. Then*

(i) *$\mathcal{H}_0 \perp \mathring{\mathcal{H}}_i$ for all i.*
(ii) *$\mathring{\mathcal{H}}_i \perp \mathring{\mathcal{H}}_j$ for all $i \neq j$.*

Proof. It suffices to verify orthogonality on the dense subspaces $\mathring{\mathcal{A}}_i\xi$.

(i) Let $X \in \mathring{\mathcal{A}}_i$, then
$$\langle X\xi, \xi\rangle = \varphi(X) = 0.$$
(ii) Let $X \in \mathring{\mathcal{A}}_i$ and $Y \in \mathring{\mathcal{A}}_j$ with $i \neq j$, then
$$\langle X\xi, Y\xi\rangle = \varphi(Y^*X) = \varphi(Y^*)\,\varphi(X) = 0.$$

□

We now construct the decomposition. Under the assumption that $P_0 \in \mathcal{A}_i$ we have $\mathcal{H}_i := \mathcal{H}_0 \oplus \mathring{\mathcal{H}}_i = [\mathcal{A}_i\xi]$. Let P_i be the projection onto $\mathring{\mathcal{H}}_i$, $\hat{\mathcal{A}}$ the $*$-subalgebra of $\mathcal{A}$ generated by $(\mathcal{A}_i)_{i\in I}$ and $\hat{\mathcal{H}} = [\hat{\mathcal{A}}\xi] \subseteq \mathcal{H}$ the closed invariant subspace generated by ξ. Let further $\hat{P}$ and $\hat{P}^\perp$ be the respective projections onto the space $\hat{\mathcal{H}}$ and its orthogonal complement $\hat{\mathcal{H}}^\perp$.

Proposition 3.8.

(i) *For each i the space $\mathcal{H}_i$ is invariant under $\mathcal{A}_i$, i.e., for $X \in \mathcal{A}_i$*
$$X(P_0 + P_i) = (P_0 + P_i)X(P_0 + P_i). \tag{3.3}$$
(ii) *For $i \neq j$ the subspace $\mathring{\mathcal{H}}_j$ is annihilated by $\mathcal{A}_i$, i.e., for $X \in \mathcal{A}_i$*
$$XP_j = P_jX = 0. \tag{3.4}$$
(iii) *The space $\hat{\mathcal{H}}$ is the closed linear span of the subspaces $\mathcal{A}_i\xi$, i.e.,*
$$\hat{\mathcal{H}} = \mathcal{H}_0 \oplus \bigoplus_i \mathring{\mathcal{H}}_i. \tag{3.5}$$

Proof.

(i) This is an immediate consequence of the definition.
(ii) It suffices to show that $XP_j = 0$, i.e., X vanishes on $\mathring{\mathcal{H}}_j$. We verify this on the dense subspace $\mathring{\mathcal{A}}_j\xi$. Indeed, let $Y \in \mathring{\mathcal{A}}_j$, then
$$\|XY\xi\|^2 = \langle XY\xi, XY\xi\rangle = \varphi(Y^*X^*XY) = \varphi(Y^*)\,\varphi(X^*X)\,\varphi(Y) = 0;$$
finally $P_jX = (X^*P_j)^* = 0$ because $\mathcal{A}_i$ is a $*$-algebra.

(iii) The space $\hat{\mathcal{H}}$ is the closure of the span of the alternating words $X_1X_2\dots X_n\xi$ with $X_j \in \mathcal{A}_{i_j}$ and $i_j \neq j_{j+1}$. We claim that such a word satisfies $X_1X_2\cdots X_n\xi \in \mathcal{H}_{i_1}$. We proceed by induction. The claim is obviously true for $n=1$. Let now $\eta = X_1X_2\cdots X_n\xi$ be a given word. Then $\eta = X_1\eta'$ with $\eta' = X_2\cdots X_n\xi$ and by induction hypothesis $\eta' \in H_{i_2}$, say $\eta' = \alpha\xi + \eta''$ with $\eta'' \in \mathring{\mathcal{H}}_{i_2}$. But then from item ((ii)) we infer that $X_1\eta' = \alpha X_1\xi + 0 \in \mathcal{H}_{i_1}$. □

Corollary 3.9. *Every $X \in \hat{\mathcal{A}}$ has block decomposition*

$$X = \hat{P}X\hat{P} + \hat{P}^{\perp}X\hat{P}^{\perp} \tag{3.6}$$

and more precisely every $X \in \mathcal{A}_i$ has the block decomposition

$$X = (P_0 + P_i)X(P_0 + P_i) + \hat{P}^{\perp}X\hat{P}^{\perp}. \tag{3.7}$$

Theorem 3.10. *The functional $\omega(X) = \mathrm{Tr}(\hat{P}X\hat{P})$ is a semifinite trace on the algebra $\hat{\mathcal{A}}$ and the subalgebras $\mathcal{A}_i \cap T(\mathcal{H})$ are cyclic-Boolean independent with respect to ω.*

Proof. ω is a trace on $\hat{\mathcal{A}}$ because $\hat{P}$ is in the commutant of $\hat{\mathcal{A}}$. Now let $X_1X_2\cdots X_n$ be a cyclically alternating product with $X_j \in \mathcal{A}_{i_j}$ for $j = 1, 2, \dots, n$, then we have

$$\begin{aligned}
\omega(X_1X_2\cdots X_n) = {} & \mathrm{Tr}\big(\hat{P}((P_0 + P_{i_1})X_1(P_0 + P_{i_1}) + \hat{P}^{\perp}X_1\hat{P}^{\perp}) \\
& \times ((P_0 + P_{i_2})X_2(P_0 + P_{i_2}) + \hat{P}^{\perp}X_2\hat{P}^{\perp}) \\
& \cdots ((P_0 + P_{i_n})X_n(P_0 + P_{i_n}) + \hat{P}^{\perp}X_n\hat{P}^{\perp})\hat{P}\big) \\
= {} & \mathrm{Tr}\big((P_0 + P_{i_1})X_1(P_0 + P_{i_1})((P_0 + P_{i_2})X_2(P_0 + P_{i_2}) \\
& \cdots ((P_0 + P_{i_n})X_n(P_0 + P_{i_n}))\big) \\
= {} & \mathrm{Tr}\big((P_0 + P_{i_1})X_1P_0X_2P_0\cdots P_0X_n(P_0 + P_{i_n})\big) \\
= {} & \mathrm{Tr}(P_0X_1P_0X_2P_0\cdots P_0X_nP_0)
\end{aligned}$$

and it follows that $\hat{P}X_1X_2\cdots X_n$ is trace class. □

4. Convolution and central limit theorem

4.1. Cyclic-Boolean convolution. Let $(\mathcal{A}, \varphi, \omega)$ be a cncps. If $a, b \in \mathcal{A}$ are cyclic-Boolean independent and self-adjoint, then the distribution of the sum $a + b$, i.e., the collection of moments $\{(\varphi((a+b)^n), \omega((a+b)^n)) : n \geqslant 1\}$, is uniquely determined by the marginal distributions, i.e., the individual moments $\{(\varphi(a^n), \omega(a^n))\}_{n\geqslant 1}$ and $\{(\varphi(b^n), \omega(b^n))\}_{n\geqslant 1}$, of a and b, and is called the *cyclic-Boolean convolution* of (the distributions of) a and b. The description of convolutions in noncommutative probability is very much facilitated with the help of generating functions and we will see that cyclic-Boolean convolution makes no exception.

We start by generalizing the Cauchy transform (2.1): the renormalized (tracial) Cauchy transform is the formal Laurent series

$$\tilde{\mathfrak{g}}_a(z) = \sum_{n=1}^{\infty} \frac{\omega(a^n)}{z^{n+1}}.$$

By slight abuse of terminology, we call G_a the Green function (evaluated at the state φ) of a. It has formal Laurent expansion

$$G_a(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\varphi(a^n)}{z^{n+1}},$$

and we denote the reciprocal Green function by $F_a(z) = 1/G_a(z)$.

If a is a trace class operator on a Hilbert space and ω is the trace then $|\operatorname{Tr}(a^n)| \leqslant \||a|^{n-1}\| \operatorname{Tr}(|a|) \leqslant \|a\|^{n-1} \operatorname{Tr}(|a|)$, and hence $\tilde{\mathfrak{g}}_a(z)$ is absolutely convergent in $\{z \in \mathbb{C} : |z| > \|a\|\}$. Moreover, if a is selfadjoint then $\tilde{\mathfrak{g}}_a$ has analytic extension to $\mathbb{C} \smallsetminus \operatorname{spec}(a)$ by Lidskii's theorem

$$\tilde{\mathfrak{g}}_a(z) = \operatorname{Tr}((z-a)^{-1} - z^{-1}) = \sum_{i=1}^{\infty} \frac{\lambda_i}{z(z-\lambda_i)}, \tag{4.1}$$

where $\{\lambda_i\}_{i \geqslant 1}$ is the multiset of eigenvalues of a. In particular, the non-zero eigenvalues of a can be detected from $\tilde{\mathfrak{g}}_a$ as poles. If the Hilbert space is finite-dimensional, then we also have the formula

$$\tilde{\mathfrak{g}}_a(z) = \sum_{i=1}^{\dim(H)} \frac{1}{z-\lambda_i} - \frac{\dim(H)}{z}, \tag{4.2}$$

and hence

$$\lim_{z \to 0} z\, \tilde{\mathfrak{g}}_a(z) = \text{the multiplicity of the eigenvalue zero} - \dim(H).$$

Remark 4.1. By [5, Corollary 2.2], the tracial moments $\operatorname{Tr}(a^n)$ for all but finitely many natural numbers n determine the eigenvalues of a. So, for any $p \in \mathbb{N}$, we can generalize the above setting to the Schatten class S_p by using the truncated generating function

$$\tilde{\mathfrak{g}}_p(z) = \sum_{n=p}^{\infty} \frac{\omega(a^n)}{z^{n+1}}.$$

Let a and b be cyclic-Boolean independent in $(\mathcal{A}, \varphi, \omega)$. It is known [23] (and will be shown in Remark 4.4 below) that the Green function of $a+b$ can be computed via the formula

$$\frac{1}{G_a(z)} + \frac{1}{G_b(z)} - z = \frac{1}{G_{a+b}(z)}; \tag{4.3}$$

i.e.,

$$B_{a+b}(z) = B_a(z) + B_b(z),$$

where

$$B_a(z) = \frac{z}{G_a(1/z)} - 1 \tag{4.4}$$

is the *Boolean cumulant transform*. The next theorem generalizes this identity to an analogous formula for the generating function $\tilde{\mathfrak{g}}_{a+b}(z)$ which gives information on the eigenvalues of $a+b$.

Theorem 4.2. *Let a and b be cyclic-Boolean independent elements. Then the renormalized Cauchy transform of their sum is*

$$\tilde{\mathfrak{g}}_{a+b}(z) = \tilde{\mathfrak{g}}_a(z) + \tilde{\mathfrak{g}}_b(z) + \frac{G_a'(z)}{G_a(z)} + \frac{G_b'(z)}{G_b(z)} - \frac{G_{a+b}'(z)}{G_{a+b}(z)} + \frac{1}{z};$$

i.e., if we define (cf. (2.14))

$$\mathfrak{h}_a(z) = \tilde{\mathfrak{g}}_a(z) + \frac{d}{dz} \log z G_a(z) \tag{4.5}$$

then

$$\mathfrak{h}_{a+b}(z) = \mathfrak{h}_a(z) + \mathfrak{h}_b(z). \tag{4.6}$$

Remark 4.3. While $\mathfrak{h}$ linearizes independent sums and is useful for analyzing convolutions, we will later introduce a modification which deserves to be called the cyclic-Boolean cumulant transform; see Section 5.

Algebraic proof. We expand the power $(a+b)^n$ and regroup the resulting monomials into those ending in a and those ending in b:

$$(a+b)^n = a^n + \sum_{\substack{k\geqslant 1\\ p_0\geqslant 0,\, p_1,q_1,\dots,p_k,q_k\geqslant 1\\ p_0+p_1+q_1+\cdots+p_k+q_k=n}} a^{p_0}b^{q_1}a^{p_1}\cdots b^{q_k}a^{p_k} + b^n + \sum_{\substack{k\geqslant 1\\ q_0\geqslant 0,\, p_1,q_1,\dots,p_k,q_k\geqslant 1\\ q_0+q_1+p_1+\cdots+p_k+q_k=n}} b^{q_0}a^{p_1}b^{q_1}\cdots a^{p_k}b^{q_k},$$

and applying ω yields

$$\begin{aligned}\omega((a+b)^n) &= \omega(a^n)+\omega(b^n)+\sum_{\substack{k\geqslant 1\\ p_0\geqslant 0,\, p_1,q_1,\dots,p_k,q_k\geqslant 1\\ p_0+p_1+q_1+\cdots+p_k+q_k=n}} \omega(a^{p_0+p_k}b^{q_1}a^{p_1}\cdots b^{q_k})\\ &\qquad + \sum_{\substack{k\geqslant 1\\ q_0\geqslant 0,\, p_1,q_1,\dots,p_k,q_k\geqslant 1\\ q_0+p_1+q_1+\cdots+p_k+q_k=n}} \omega(b^{q_0+q_k}a^{p_1}b^{q_1}\cdots a^{p_k})\\ &= \omega(a^n)+\omega(b^n)+\sum_{\substack{k\geqslant 1\\ p_0\geqslant 0,\, p_1,q_1,\dots,p_k,q_k\geqslant 1\\ p_0+p_1+q_1+\cdots+p_k+q_k=n}} \varphi(a^{p_0+p_k})\,\varphi(b^{q_1})\,\varphi(a^{p_1})\cdots\varphi(b^{q_k})\\ &\qquad + \sum_{\substack{k\geqslant 1\\ q_0\geqslant 0,\, p_1,q_1,\dots,p_k,q_k\geqslant 1\\ q_0+p_1+q_1+\cdots+p_k+q_k=n}} \varphi(b^{q_0+q_k})\,\varphi(a^{p_1})\,\varphi(b^{q_1})\cdots\varphi(a^{p_k}).\end{aligned}$$

Multiplying the above identity by z^{-n-1} and taking the summation over n yields

$$\begin{aligned}\tilde{\mathfrak{g}}_{a+b}(z) &= \sum_{n\geqslant 0}\frac{\omega((a+b)^n)}{z^{n+1}}\\ &= \sum_{n\geqslant 1}\frac{\omega(a^n)}{z^{n+1}}+\sum_{n\geqslant 1}\frac{\omega(b^n)}{z^{n+1}}+\sum_{\substack{k\geqslant 1\\ p_0\geqslant 0,\, p_1,q_1,\dots,p_k,q_k\geqslant 1}}\frac{\varphi(a^{p_0+p_k})}{z^{p_0+p_k+1}}\,\frac{\varphi(b^{q_1})}{z^{q_1}}\,\frac{\varphi(a^{p_1})}{z^{p_1}}\cdots\frac{\varphi(b^{q_k})}{z^{q_k}}\\ &\quad + \sum_{\substack{k\geqslant 1\\ q_0\geqslant 0,\, p_1,q_1,\dots,p_k,q_k\geqslant 1}}\frac{\varphi(b^{q_0+q_k})}{z^{q_0+q_k+1}}\,\frac{\varphi(a^{p_1})}{z^{p_1}}\,\frac{\varphi(b^{q_1})}{z^{q_1}}\cdots\frac{\varphi(a^{p_k})}{z^{p_k}}.\end{aligned}$$

Now

$$\sum_{\substack{p_0\geqslant 0\\ p\geqslant 1}}\frac{\varphi(a^{p_0+p_1})}{z^{p_0+p+1}} = \sum_{m\geqslant 1}\sum_{\substack{p_0\geqslant 0\\ p\geqslant 1\\ p_0+p=m}}\frac{\varphi(a^m)}{z^{m+1}} = \sum_{m\geqslant 1} m\frac{\varphi(a^m)}{z^{m+1}} = -(zG_a(z))'$$

and thus

$$\begin{aligned}\tilde{\mathfrak{g}}_{a+b}(z) &= \tilde{\mathfrak{g}}_a(z)+\tilde{\mathfrak{g}}_b(z)-\sum_{k\geqslant 1}(zG_a(z))'(zG_a(z)-1)^{k-1}(zG_b(z)-1)^k\\ &\qquad -\sum_{k\geqslant 1}(zG_b(z))'(zG_a(z)-1)^k(zG_b(z)-1)^{k-1}\\ &= \tilde{\mathfrak{g}}_a(z)+\tilde{\mathfrak{g}}_b(z)-\frac{(zG_a(z))'(zG_b(z)-1)+(zG_a(z)-1)(zG_b(z))'}{1-(zG_a(z)-1)(zG_b(z)-1)}\\ &= \tilde{\mathfrak{g}}_a(z)+\tilde{\mathfrak{g}}_b(z)-\frac{(zG_a(z))'}{G_a(z)}\frac{z-F_b(z)}{F_a(z)F_b(z)-(z-F_a(z))(z-F_b(z))}\\ &\qquad -\frac{(zG_b(z))'}{G_b(z)}\frac{z-F_a(z)}{F_a(z)F_b(z)-(z-F_a(z))(z-F_b(z))}\end{aligned}$$

$$
\begin{aligned}
&= \tilde{\mathfrak{g}}_a(z) + \tilde{\mathfrak{g}}_b(z) - \left(\frac{1}{z} + \frac{G_a'}{G_a}\right) \frac{z - F_b}{F_a + F_b - z} - \left(\frac{1}{z} + \frac{G_b'}{G_b}\right) \frac{z - F_a}{F_a + F_b - z} \\
&= \tilde{\mathfrak{g}}_a(z) + \tilde{\mathfrak{g}}_b(z) + \left(\frac{F_a'}{F_a}(z - F_b) + \frac{F_b'}{F_b}(z - F_a) - 1\right) \frac{1}{F_a + F_b - z} + \frac{1}{z} \\
&= \tilde{\mathfrak{g}}_a(z) + \tilde{\mathfrak{g}}_b(z) + \frac{1}{z} \\
&\qquad + \left(\frac{F_a'}{F_a}(z - F_a - F_b) + \frac{F_b'}{F_b}(z - F_a - F_b) - 1 + F_a' + F_b'\right) \frac{1}{F_a + F_b - z} \\
&= \tilde{\mathfrak{g}}_a(z) + \tilde{\mathfrak{g}}_b(z) + \frac{1}{z} - \frac{F_a'}{F_a} - \frac{F_b'}{F_b} + \frac{F_{a+b}'}{F_{a+b}} \\
&= \tilde{\mathfrak{g}}_a(z) + \tilde{\mathfrak{g}}_b(z) + \frac{1}{z} + \frac{G_a'}{G_a} + \frac{G_b'}{G_b} - \frac{G_{a+b}'}{G_{a+b}}.
\end{aligned}
$$

□

Analytic proof in the setting of Example 3.5. Under the assumption that our $*$-algebras are represented as trace class operators on the star product Hilbert space

$$\mathcal{H} = \mathbb{C}\xi \oplus \mathring{\mathcal{H}}_1 \oplus \mathring{\mathcal{H}}_2$$

equipped with a vacuum state $\varphi = \langle .\xi, \xi \rangle$ and the trace $\omega = \mathrm{Tr}$ we can use this decomposition and represent the involved operators as block operator matrices

$$
A = \begin{bmatrix} \alpha & a' & 0 \\ a & \mathring{A} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \beta & 0 & b' \\ 0 & 0 & 0 \\ b & 0 & \mathring{B} \end{bmatrix}, \qquad A + B = \begin{bmatrix} \alpha + \beta & a' & b' \\ a & \mathring{A} & 0 \\ b & 0 & \mathring{B} \end{bmatrix}. \tag{4.7}
$$

In other words, $(A + B)^\circ = \mathring{A} \oplus \mathring{B}$ is a direct sum and therefore

$$\tilde{\mathfrak{g}}_{(A+B)^\circ}(z) = \tilde{\mathfrak{g}}_{\mathring{A}}(z) + \tilde{\mathfrak{g}}_{\mathring{B}}(z)$$

and we conclude with the identity (2.14). □

Remark 4.4. The idea of the above algebraic/analytic proofs can also be used to verify the known formula (4.3). For example, the Banachiewicz formula (2.9) applied to the decomposition (4.7) yields the Green function in the form

$$\frac{1}{G_{A+B}(z)} = z - (\alpha + \beta) - a'(z - \mathring{A})^{-1} a - b'(z - \mathring{B})^{-1} b.$$

Combining this with the formulas

$$\frac{1}{G_A(z)} = z - \alpha - a'(z - \mathring{A})^{-1} a \qquad \text{and} \qquad \frac{1}{G_B(z)} = z - \beta - b'(z - \mathring{B})^{-1} b$$

we obtain (4.3).

4.2. Examples from star product graphs. For a rooted graph $(\mathbf{\Gamma}, o)$, its N-fold star product $(\mathbf{\Gamma}_N, o_N) = (\mathbf{\Gamma}, o) \circledast (\mathbf{\Gamma}, o) \circledast \cdots \circledast (\mathbf{\Gamma}, o)$ has the adjacency matrix that is the sum of cyclic-Boolean independent copies of the adjacency matrix of $\mathbf{\Gamma}$; see (2.15). Therefore, Theorem 4.2 and (2.17) imply that

$$\tilde{\mathfrak{g}}_{\mathbf{\Gamma}_N}(z) = N\, \tilde{\mathfrak{g}}_a(z) + N \frac{G_{\mathbf{\Gamma}}'(z)}{G_{\mathbf{\Gamma}}(z)} - \frac{G_{\mathbf{\Gamma}_N}'(z)}{G_{\mathbf{\Gamma}_N}(z)} + \frac{N-1}{z} \tag{4.8}$$

where

$$\frac{1}{G_{\mathbf{\Gamma}_N}(z)} = \frac{N}{G_{\mathbf{\Gamma}}(z)} - (N-1)z. \tag{4.9}$$

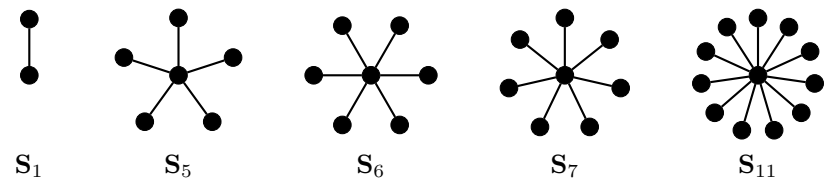

Figure 3. Star graphs

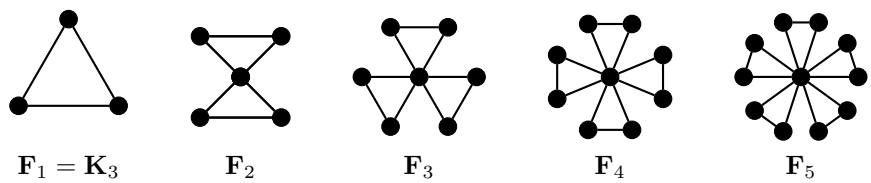

Figure 4. Friendship graphs

Example 4.5 (Star graph). The star graph $(\mathbf{S}_N, o_N)$ on $N+1$ vertices $\{0, 1, \dots, N\}$ has edges $\{0, i\}, i = 1, 2, \dots, N$. It is the N-fold star product of the complete graph $(\mathbf{K}_2, o)$ (Figure 3). The eigenvalues of the adjacency matrix of $\mathbf{K}_2$ are ± 1, and hence

$$\tilde{\mathfrak{g}}_{\mathbf{K}_2}(z) = \frac{1}{z-1} + \frac{1}{z+1} - \frac{2}{z} \qquad \text{and} \qquad G_{\mathbf{K}_2}(z) = \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z-1}\right),$$

where the latter formula can be computed via (2.11). Using (4.9) entails

$$G_{\mathbf{S}_N}(z) = \frac{1}{2}\left(\frac{1}{z-\sqrt{N}} - \frac{1}{z+\sqrt{N}}\right) \qquad \text{and} \qquad \frac{G'_{\mathbf{S}_N}(z)}{G_{\mathbf{S}_N}(z)} = \frac{1}{z} - \frac{1}{z-\sqrt{N}} - \frac{1}{z+\sqrt{N}}.$$

The renormalized Cauchy transform of $(\mathbf{S}_N, o_N)$ may be calculated from (4.8) as follows:

$$\begin{aligned}\tilde{\mathfrak{g}}_{\mathbf{S}_N}(z) &= N\left(\frac{1}{z-1} + \frac{1}{z+1} - \frac{2}{z}\right) + N\left(\frac{1}{z} - \frac{1}{z-1} - \frac{1}{z+1}\right) \\ &\quad + \left(\frac{1}{z-\sqrt{N}} + \frac{1}{z+\sqrt{N}} - \frac{1}{z}\right) + \frac{N-1}{z} \\ &= \frac{1}{z-\sqrt{N}} + \frac{1}{z+\sqrt{N}} - \frac{2}{z}.\end{aligned}$$

The Cauchy transform is given by

$$\mathfrak{g}_{\mathbf{S}_N}(z) = \frac{1}{z-\sqrt{N}} + \frac{1}{z+\sqrt{N}} + \frac{N-1}{z}.$$

This recovers the fact that the multiset of eigenvalues of the adjacency matrix of $\mathbf{S}_N$ is given by $\{[\sqrt{N}]^1, [-\sqrt{N}]^1, [0]^{N-1}\}$.

Example 4.6 (Friendship graph). The friendship graph $\mathbf{F}_N$ is the graph with $2N+1$ vertices $\{0, \dots, 2N\}$ in which 0 is connected to every other vertex and the only other edges are $\{2i-1, 2i\}$ for $1 \leqslant i \leqslant N$. The friendship graph is the N-fold star product of the complete graph $(\mathbf{K}_3, o)$ with itself; see Figure 4.

In this case

$$\tilde{\mathfrak{g}}_{\mathbf{K}_3}(z) = \frac{2}{z+1} + \frac{1}{z-2} - \frac{3}{z} \qquad \text{and} \qquad G_{\mathbf{K}_3}(z) = \frac{1}{3}\left(\frac{2}{z+1} + \frac{1}{z-2}\right),$$

from which

$$\frac{G'_{\mathbf{K}_3}(z)}{G_{\mathbf{K}_3}(z)} = \frac{-z^2+2z-3}{z^3-2z^2-z+2} = \frac{1}{z-1} - \frac{1}{z+1} - \frac{1}{z-2}.$$

On the other hand

$$\frac{1}{G_{\mathbf{F}_N}(z)} = \frac{N}{G_{\mathbf{K}_3}(z)} - (N-1)z = \frac{z(z-1)-2N}{z-1},$$

and then

$$\begin{aligned}\frac{G'_{\mathbf{F}_N}(z)}{G_{\mathbf{F}_N}(z)} &= \frac{1}{z-1} + \frac{1-2z}{z^2-z-2N} \\ &= \frac{1}{z-1} - \frac{1}{z-(1+\sqrt{1+8N})/2} - \frac{1}{z-(1-\sqrt{1+8N})/2}.\end{aligned}$$

Thus the renormalized Cauchy transform may be calculated as follows.

$$\begin{aligned}&\tilde{\mathfrak{g}}_{\mathbf{F}_N}(z) \\ &= N\left(\frac{2}{z+1} + \frac{1}{z-2} - \frac{3}{z}\right) + N\left(\frac{1}{z-1} - \frac{1}{z+1} + \frac{1}{2-z}\right) \\ &\quad - \left(\frac{1}{z-1} - \frac{1}{z-(1+\sqrt{1+8N})/2} - \frac{1}{z-(1-\sqrt{1+8N})/2}\right) + \frac{N-1}{z} \\ &= \frac{N-1}{z-1} + \frac{N}{z+1} + \frac{1}{z-(1+\sqrt{1+8N})/2} + \frac{1}{z-(1-\sqrt{1+8N})/2} - \frac{2N+1}{z}.\end{aligned}$$

Then

$$\begin{aligned}\mathfrak{g}_{\mathbf{F}_N}(z) &= \tilde{\mathfrak{g}}_{\mathbf{F}_N}(z) + \frac{2N+1}{z} \\ &= \frac{N-1}{z-1} + \frac{N}{z+1} + \frac{1}{z-(1+\sqrt{1+8N})/2} + \frac{1}{z-(1-\sqrt{1+8N})/2}.\end{aligned}$$

This recovers the fact that the multiset of eigenvalues of the adjacency matrix of $\mathbf{F}_N$ is given by

$$\left\{\left[\frac{1}{2} - \frac{1}{2}\sqrt{1+8N}\right]^1, [-1]^N, [1]^{N-1}, \left[\frac{1}{2} + \frac{1}{2}\sqrt{1+8N}\right]^1\right\}.$$

4.3. Cyclic-Boolean central limit theorem. Since we have an appropriate linearization (4.6) for cyclic-Boolean convolution, we are able to determine the central limit law.

Theorem 4.7. *For each $N \in \mathbb{N}$, let $\{a_i^{(N)}\}_{i=1}^N$ be self-adjoint cyclic-Boolean independent random variables in a cncps $(\mathcal{A}_N, \varphi_N, \omega_N)$. Assume that, for each fixed $k \in \mathbb{N}$, the moments $\varphi_N((a_i^{(N)})^k)$ and $\omega_N((a_i^{(N)})^k)$ do not depend on i or N, and also $\omega_N(a_i^{(N)}) = \varphi_N(a_i^{(N)}) = 0$, $\varphi_N((a_i^{(N)})^2) = 1$ for all i and N. Then, for the normalized sum*

$$s_N = \frac{a_1^{(N)} + a_2^{(N)} + \cdots + a_N^{(N)}}{\sqrt{N}},$$

it holds that

$$\lim_{N\to\infty} \varphi_N(s_N^k) = \begin{cases} 1 & \text{if } k \in 2\mathbb{N}, \\ 0 & \text{if } k \in 2\mathbb{N}-1, \end{cases} \quad \text{and} \quad \lim_{N\to\infty} \omega_N(s_N^k) = \begin{cases} 2 & \text{if } k \in 2\mathbb{N}+2, \\ 0 & \text{if } k \in 2\mathbb{N}+1. \end{cases}$$

Proof. The Boolean central limit theorem [23] asserts that $G_{s_N}(z) \to z/(z^2-1)$. Let $\alpha = \omega_N((a_i^{(N)})^2)$, then Theorem 4.2 yields

$$\mathfrak{h}_{s_N}(z) = N^{3/2}\,\mathfrak{h}_{a_1^{(N)}}(\sqrt{N}z) = N^{3/2}\left(\frac{\alpha-2}{(\sqrt{N}z)^3} + O(N^{-2})\right) \to \frac{\alpha-2}{z^3}.$$

Therefore

$$\tilde{\mathfrak{g}}_{s_N}(z) \to \frac{2z}{z^2-1} - \frac{2}{z} + \frac{\alpha-2}{z^3} = \sum_{n\geqslant 1} \frac{\tilde{\mathfrak{g}}_n}{z^{n+1}},$$

where

$$\tilde{\mathfrak{g}}_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even and } n \geqslant 4, \\ \alpha & \text{if } n = 2. \end{cases}$$

□

The limit law exhibits a large spectral gap:

Corollary 4.8. *In addition to the setting of Theorem 4.7, suppose that $\mathcal{A}_N$ is the $*$-algebra $T(H_N)$ of trace class operators on some Hilbert space H_N and $\omega_N = \mathrm{Tr}_{H_N}$. Let λ_N and μ_N be the largest and smallest eigenvalues of s_N, respectively. The following assertions hold:*

(i) *the multiplicities of λ_N and μ_N are both one for sufficiently large N;*
(ii) *λ_N converges to 1 and μ_N converges to -1 as $N \to \infty$;*
(iii) *the remaining eigenvalues accumulate around 0:*

$$\lim_{N\to\infty} \mathrm{dist}(\mathrm{spec}(s_N) \smallsetminus \{\lambda_N, \mu_N\}, 0) = 0$$

Proof. Let s be a self-adjoint operator of rank two on a Hilbert space K having eigenvalues $-1, 1, 0$. Then $\mathrm{Tr}_K(s^n) = 2$ for even $n \geqslant 2$ and $\mathrm{Tr}_K(s^n) = 0$ for odd $n \geqslant 1$. The convergence of $\mathrm{Tr}_H(s_N^k)$ in Theorem 4.7 and [5, Proposition 2.8] imply that $s_N \to s$ in eigenvalues and this concludes the argument. □

Remark 4.9. As shown in the proof of Theorem 4.7, $\mathrm{Tr}_{H_N}(s_N^2)$ converges (actually is equal) to α which might not equal $2 = \mathrm{Tr}_K(s^2)$. This difference of Hilbert-Schmidt norms is due to a large number of small eigenvalues of s_N and does not contradict the convergence of eigenvalues; see [5, Proposition 2.8, Proposition 2.10 and Remark 2.11].

Now we come back to the original model, the adjacency matrix of the star product of rooted graphs.

Corollary 4.10. *Suppose that $(\mathbf{\Gamma}, o)$ is a rooted graph with $\deg(o) \geqslant 1$. Let A_N be the adjacency matrix of the N-fold star product graph $(\mathbf{\Gamma}, o) \circledast (\mathbf{\Gamma}, o) \circledast \ldots \circledast (\mathbf{\Gamma}, o)$ and denote by λ_N and μ_N the largest and smallest eigenvalues of $(\deg(o)N)^{-\frac{1}{2}} A_N$, respectively. The following assertions hold:*

(i) *the multiplicities of λ_N and μ_N are both one for sufficiently large N;*
(ii) *λ_N converges to 1 and μ_N converges to -1 as $N \to \infty$;*
(iii) $\lim_{N\to\infty} \mathrm{dist}(\mathrm{spec}((\deg(o)N)^{-\frac{1}{2}} A_N) \smallsetminus \{\lambda_N, \mu_N\}, 0) = 0$

Proof. This is a combination of Corollary 4.8, formula (2.15) and Example 3.1. The factor $(\deg(o)N)^{-\frac{1}{2}}$ appears because of the variance $\langle A_{\mathbf{\Gamma}}^2 \delta_o, \delta_o\rangle_{\ell^2(V)} = \deg(o)$, where V is the vertex set of $\mathbf{\Gamma}$. □

Remark 4.11. In the setting of Corollary 4.10 it is already known that, according to the Boolean central limit theorem, the distribution of $(\deg(o)N)^{-\frac{1}{2}}A_N$ regarding the vector state $\varphi_N = \langle \cdot\delta_o, \delta_o\rangle$ converges weakly to $\frac{1}{2}(\delta_{-1}+\delta_1)$. This fact entails an intuitive consequence of Corollary 4.10: the vector δ_o in the tensor product Hilbert space $\ell^2(V)^{\otimes N}$ is almost orthogonal to the subspace spanned by eigenvectors corresponding to small eigenvalues, or equivalently, δ_o is almost contained in the two-dimensional subspace spanned by the eigenvectors corresponding to the eigenvalues near ± 1.

Example 4.12. Corollary 4.10 can be directly confirmed in the following examples.

(i) For the star graph on $N+1$ vertices, its adjacency matrix divided by $\sqrt{N}$ has eigenvalues $\{[1]^1, [-1]^1, [0]^{N-1}\}$; see Example 4.5. Eigenvectors corresponding to the eigenvalues 1 and -1 are $f_1 = (\sqrt{N}, 1, 1, \dots, 1)$ and $f_2 = (-\sqrt{N}, 1, 1, \dots, 1)$, respectively, and hence, the function δ_o, which corresponds to the vector $(1, 0, 0, \dots, 0)$, is exactly contained in the subspace spanned by f_1 and f_2.

(ii) For the friendship graph on $2N+1$ vertices, its adjacency matrix divided by $\sqrt{2N}$ has eigenvalues

$$\left\{\left[-\sqrt{1+\frac{1}{8N}}-\frac{1}{2\sqrt{2N}}\right]^1, \left[-\frac{1}{\sqrt{2N}}\right]^N, \left[\frac{1}{\sqrt{2N}}\right]^{N-1}, \left[\sqrt{1+\frac{1}{8N}}+\frac{1}{2\sqrt{2N}}\right]^1\right\};$$

see Example 4.6.

5. Cyclic-Boolean cumulants

5.1. Univariate cumulants. Let $(\mathcal{A}, \varphi, \omega)$ be a cncps and $a \in \mathcal{A}$. The generating function $\mathfrak{h}_a$ defined in (4.5) has the series expansion

$$\mathfrak{h}_a(z) = \sum_{n=1}^{\infty} \frac{h_n(a)}{z^{n+1}},$$

where the first two coefficients are $h_1(a) = \omega(a) - \varphi(a)$ and $h_2(a) = \omega(a^2) + \varphi(a)^2 - 2\varphi(a^2)$. In general, $n \geqslant 2$ there is a universal polynomial $Q_n(x_1, x_2, \dots, x_{n-1})$ such that

$$h_n(a) = \omega(a^n) - n\varphi(a^n) + Q_n(\varphi(a), \varphi(a^2), \dots, \varphi(a^{n-1})). \tag{5.1}$$

We can modify $\mathfrak{h}_a(z)$ by adding the Boolean cumulants to delete $-n\varphi(a^n)$ from $h_n(a)$. We switch from $\tilde{\mathfrak{g}}_a$ and G_a to the moment generating functions

$$\mathfrak{m}_a(z) = \frac{1}{z}\,\tilde{\mathfrak{g}}_a\left(\frac{1}{z}\right) = \sum_{n\geqslant 1}\omega(a^n)z^n, \qquad M_a(z) = \frac{1}{z}\left(G_a\left(\frac{1}{z}\right) - z\right) = \sum_{n\geqslant 1}\varphi(a^n)z^n.$$

The Boolean cumulant transform (4.4) is then expressed by

$$B_a(z) = \frac{M_a(z)}{1+M_a(z)} = \sum_{n\geqslant 1} b_n(a)z^n.$$

We introduce the new generating function

$$\begin{aligned}\mathfrak{c}_a(z) &= \frac{1}{z}\,\mathfrak{h}_a\left(\frac{1}{z}\right) + zB_a'(z) && (5.2)\\ &= \mathfrak{m}_a(z) - \frac{zM_a(z)M_a'(z)}{(1+M_a(z))^2}\\ &= \mathfrak{m}_a(z) - zM_a(z)B_a'(z), && (5.3)\end{aligned}$$

which linearizes the convolution

$$\mathfrak{c}_{a+b}(z) = \mathfrak{c}_a(z) + \mathfrak{c}_b(z).$$

The function $\mathfrak{c}_a$ will be called the *cyclic-Boolean cumulant transform of* a and the coefficients $c_n(a)$ appearing as

$$\mathfrak{c}_a(z) = \sum_{n\geqslant 1} c_n(a) z^n \tag{5.4}$$

are called the (univariate) *cyclic-boolean cumulants* of a. The first two cumulants are

$$c_1(a) = \omega(a) \qquad \text{and} \qquad c_2(a) = \omega(a^2) - \varphi(a)^2.$$

For general $n \geqslant 2$, we combine identities (5.1), (5.2) and $B_a'(z) = M_a'(z)/(1+M_a(z))^2$ to conclude that there exists a universal polynomial $P_n(x_1, \dots, x_{n-1})$ depending only on n such that

$$c_n(a) = \omega(a^n) + P_n(\varphi(a), \dots, \varphi(a^{n-1})).$$

5.2. Cyclic-interval partitions. Cyclic-Boolean independence gives rise to an exchangeability system and we can define and compute the (multivariate) cyclic-Boolean cumulants using the methods of [13, 11]. The relevant partition structure turns out to be cyclic-interval partitions, which were already discussed in [7] in their search for notions of independence, similar to Boolean and monotone ones, but such that the algebra of scalars, $\mathbb{C}$, is independent from any other algebra.

Before embarking on we recall some basic concepts on set partitions.

Definition 5.1. *Let $k \in \mathbb{N}$. We often use the notation $[k] = \{1, 2, \dots, k\}$.*

(i) *A* set partition *of $[k]$ is a set $\pi = \{B_1, B_2, \dots, B_p\}$ of nonempty and disjoint subsets $B_1, \dots, B_p$ of $[k]$, called* blocks*, such that their union is $[k]$. The length $|\pi|$ of a partition π is the number of blocks. The set of partitions of the set $[k]$ is denoted by $\mathcal{P}(k)$. Set partitions are in one-to-one correspondence with equivalence relations: Any set partition $\pi \in \mathcal{P}(k)$ determines an equivalence relation $i \sim_\pi j$ on $[k]$ by requiring that i, j belong to the same block of π; conversely, for an equivalence relation $\sim$ on $[k]$ its equivalence classes determine disjoint subsets of $[k]$ and hence a set partition.*

(ii) *A subset of $[k]$ of form $\{i, i+1, \dots, j\}$ is called an* interval *and a set partition of $[k]$ is called an* interval partition *if all its blocks are intervals. The set of interval partitions is denoted by $\mathcal{I}(k)$.*

(iii) *For set partitions $\sigma, \pi \in \mathcal{P}(k)$ we write $\sigma \leqslant \pi$ if every block of σ is a subset of a block of π. This makes $\mathcal{P}(k)$ a poset. The trivial set partition $\{[k]\}$ is the maximum of $\mathcal{P}(k)$, which is denoted by $\hat{1}_k$.*

(iv) *A tuple $(i_1, \dots, i_k) \in \mathbb{N}^k$ induces a unique equivalence relation $\sim$ on $[k]$ by the requirement that $p \sim q$ holds if and only if $i_p = i_q$. The corresponding set partition is called the* kernel set partition*, denoted by $\kappa(i_1, \dots, i_k)$.*

Example 5.2. Some kernel set partitions are

$$\kappa(6, 3, 2, 3, 6) = \{\{3\}, \{2, 4\}, \{1, 5\}\},$$
$$\kappa(2, 7, 4, 7, 4, 2, 4) = \{\{1, 6\}, \{3, 5, 7\}, \{2, 4\}\}.$$

We will see that cyclic-Boolean cumulants c_π (defined in the next section) vanish identically unless π is a *cyclic-interval partition.*

Definition 5.3. *A partition $\pi \in \mathcal{P}(n)$ is called a* cyclic-interval partition *if every block is an interval or the complement of an interval. In other words, there is a cyclic permutation $\sigma \in \mathfrak{S}_n$ such that $\sigma \cdot \pi$ is an interval partition. We denote by $\mathcal{CI}(n)$ the set of cyclic-interval partitions of $[n]$.*

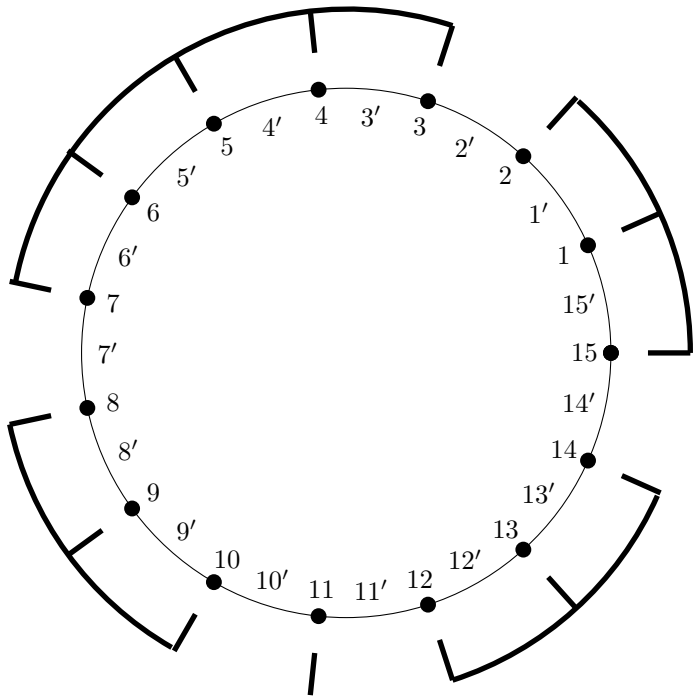

FIGURE 5. The cyclic interval partition $\{1, 2, 15/3, 4, 5, 6, 7/8/9/10/11/12/13/14\}$ has separating set $\{2', 7', 10', 11', 14'\}$

As already noticed in [7, Corollary 1], it is not difficult to see that the number of cyclic-interval partitions is $|\mathcal{CI}(n)| = 2^n - n$. To see this, the most convenient picture of cyclic-interval partitions is obtained by actually drawing them on a circle as shown in Fig. 5. Then it is clear that a cyclic-interval partition is uniquely determined by the set of separators of the blocks. For the maximal partition $\hat{1}_n$ this set is empty, while for all other cyclic-interval partitions there must be at least two separators.

5.3. MULTIVARIATE CUMULANTS. In order to avoid the discussion of positivity (see Remark 5.4 below) we notice that one can easily extend the definition of independence to a purely algebraic setting without positivity. Thus in this section we will focus on an algebraic cyclic probability space $(\mathcal{A}, \varphi, \omega)$ without positivity structure, that is, $\mathcal{A}$ is an algebra over $\mathbb{C}$, φ is a linear functional and ω is a tracial linear functional.

Take copies $\mathcal{A}_k$ of $\mathcal{A}$ and define the nonunital algebraic free product

$$\mathcal{U} = \mathop{*}_{k\in\mathbb{N}} \mathcal{A}_k = \bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{(k_1,\dots,k_n)\in\mathbb{N}^n \\ k_1\neq\cdots\neq k_n}} \mathcal{A}_{k_1} \otimes \cdots \otimes \mathcal{A}_{k_n}.$$

Let $\pi^{(i)} \colon a \mapsto a^{(i)}$ denote the embedding of $\mathcal{A}$ into $\mathcal{U}$ as the i-th copy $\mathcal{A}_i$. By the universality of tensor products we can define $(\tilde{\varphi}, \tilde{\omega})$ on $\mathcal{U}$ as follows: for $n \geqslant 1$, an alternating tuple $(k_1, \dots, k_n) \in \mathbb{N}^n$ and $a_i \in \mathcal{A}_{k_i}, i = 1, 2, \dots, n$, set

$$\tilde{\varphi}(a_1 a_2 \cdots a_n) := \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n),$$

$$\tilde{\omega}(a_1 a_2 \cdots a_n) := \begin{cases} \omega(a_1) & \text{if } n = 1, \\ \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n) & \text{if } n \geqslant 2, k_1 \neq k_n, \\ \varphi(a_n a_1)\varphi(a_2)\cdots\varphi(a_{n-1}) & \text{if } n \geqslant 2, k_1 = k_n. \end{cases}$$

One can check that $\tilde{\omega}$ is a trace on $\mathcal{U}$, $\varphi = \tilde{\varphi} \circ \pi^{(i)}$, $\omega = \tilde{\omega} \circ \pi^{(i)}$ and the family of subalgebras $\{\pi^{(i)}(\mathcal{A})\}_{i=1}^{\infty}$ is cyclic-Boolean independent in $(\mathcal{U}, \tilde{\varphi}, \tilde{\omega})$.

Remark 5.4 (Positivity). It is not clear under what conditions the product trace $\tilde{\omega}$ preserves positivity. First observe that the trivial example $\omega = 0$ shows that some conditions are necessary. Indeed, if $\omega = 0$ and $a, b \in \mathcal{A}$ are self-adjoint then $\tilde{\omega}((a^{(1)} + b^{(2)})^2) = 2\varphi(a)\varphi(b)$, which can be negative, although both φ and ω are positive.

This example suggests that in order to expect positivity of $\tilde{\omega}$ one should at least require $\omega(x^*x) \geqslant |\varphi(x)|^2$ or $\omega(x^*x) \geqslant \varphi(x^*x)$. Both conditions however are not promoted to the cyclic free product. Although the proof of [3, Theorem 2.2] adapts well to show $\tilde{\omega}(x^*x) \geqslant |\tilde{\varphi}(x)|^2$ for $x \in \mathcal{U}_{11}$ or $x \in \mathcal{U}_{22}$, where

$$\mathcal{U}_{11} = \bigoplus_{n \geqslant 3} \bigoplus_{\substack{(k_1,\dots,k_n) \in \mathbb{N}^n \\ k_1 \neq \cdots \neq k_n \\ k_1 = k_n = 1}} \mathcal{A}_{k_1} \otimes \cdots \otimes \mathcal{A}_{k_n}$$

etc., the following example shows that positivity fails on elements mixing these subspaces. Choose $x, y, z \in \mathcal{A}_1$ and $w \in \mathcal{U}_{22}$ and put $a = x + ywz$. Note that ywz is alternating and we compute

$$\begin{aligned}\tilde{\omega}(a^*a) &= \omega(x^*x) + \tilde{\omega}(x^*ywz) + \tilde{\omega}(z^*w^*y^*x) + \tilde{\omega}(z^*w^*y^*ywz) \\ &= \omega(x^*x) + \varphi(zx^*y)\,\varphi(w) + \varphi(w^*)\,\varphi(y^*xz^*) + \varphi(zz^*)\,\varphi(w^*)\,\varphi(y^*y)\,\varphi(w)\end{aligned}$$

and

$$\begin{aligned}\tilde{\varphi}(a^*a) &= \varphi(x^*x) + \tilde{\varphi}(x^*ywz) + \tilde{\varphi}(z^*w^*y^*x) + \tilde{\varphi}(z^*w^*y^*ywz) \\ &= \varphi(x^*x) + \varphi(x^*y)\,\varphi(w)\,\varphi(z) + \varphi(z^*)\,\varphi(w^*)\,\varphi(y^*x) \\ &\qquad + \varphi(z^*)\,\varphi(w^*)\,\varphi(y^*y)\,\varphi(w)\,\varphi(z).\end{aligned}$$

Now choose x, y, z, w such that

$$\omega(x^*x) = \varphi(x^*x), \qquad \varphi(z) = 0, \qquad \varphi(zx^*y), \varphi(w) \in \mathbb{R} \setminus \{0\}.$$

Such a choice is possible, e.g., when $\omega = \varphi$ is a tracial state: first choose a unitary u such that $\varphi(u) = 0$ and set $z = u$, $y = u^*$, then for selfadjoint w and x we obtain

$$\tilde{\omega}(a^*a) - \tilde{\varphi}(a^*a) = 2\varphi(x)\varphi(w) + |\varphi(w)|^2$$

which can be made negative by an appropriate rescaling of x.

The positivity condition $\omega(x^*x) \geqslant |\varphi(x)|^2$ proves to be inappropriate as well. In the last specification, we further choose x so that $|\varphi(x)|^2 < \varphi(x^2) < 2|\varphi(x)|^2$. Then

$$\tilde{\omega}(a^*a) - |\tilde{\varphi}(a)|^2 = \varphi(x^2) - |\varphi(x)|^2 + 2\varphi(x)\varphi(w) + |\varphi(w)|^2 .$$

Replacing x with $\lambda x, \lambda \in \mathbb{R}$, will change this value into

$$\left[\varphi(x^2) - |\varphi(x)|^2\right] \left[\lambda + \frac{\varphi(x)\varphi(w)}{\varphi(x^2) - |\varphi(x)|^2}\right]^2 + \frac{|\varphi(w)|^2[\varphi(x^2) - 2|\varphi(x)|^2]}{\varphi(x^2) - |\varphi(x)|^2}.$$

Taking $\lambda = -\frac{\varphi(x)\varphi(w)}{\varphi(x^2) - |\varphi(x)|^2}$ will make this value negative.

Both pairs $(\mathcal{U}, \tilde{\varphi})$ and $(\mathcal{U}, \tilde{\omega})$ are exchangeability systems in the sense of [13, Definition 1.8] except that we are not assuming unitality, which however is not essential for the theory of cumulants. For the first pair we will get Boolean cumulants B_π which are well known; therefore we will focus on $(\mathcal{U}, \tilde{\omega})$ from now on. The exchangeability of $(\mathcal{U}, \tilde{\omega})$ means that, for any $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathcal{A}$, the value of the function

$$\mathbb{N}^n \ni (i_1, \dots, i_n) \mapsto \tilde{\omega}(a_1^{(i_1)} a_2^{(i_2)} \cdots a_n^{(i_n)}) \in \mathbb{C}$$

is determined by the kernel set partition $\pi = \kappa(i_1, \dots, i_n)$. This value is denoted by $\omega_\pi(a_1, \dots, a_n)$, which then gives an n-linear functional $\omega_\pi \colon \mathcal{A}^n \to \mathbb{C}$. For each $\pi \in \mathcal{P}(n)$ a partitioned cumulant $\mathrm{c}_\pi \colon \mathcal{A}^n \to \mathbb{C}$ is then defined by

$$\mathrm{c}_\pi(a_1, a_2, \dots, a_n) = \sum_{\substack{\rho \in \mathcal{P}(n) \\ \rho \leqslant \pi}} \omega_\rho(a_1, a_2, \dots, a_n)\, \mu(\rho, \pi), \tag{5.5}$$

where μ is the Möbius function for the poset $\mathcal{P}(n)$. By [11, Lemma 4.18 (ii)] or imitating the proof of [13, Proposition 4.11] we can prove that $\mathrm{c}_\pi = 0$ if $\pi \in \mathcal{P}(n) \smallsetminus \mathcal{CI}(n)$. This vanishing property and the Möbius inversion of (5.5) imply that

$$\omega(a_1 a_2 \cdots a_n) = \sum_{\pi \in \mathcal{CI}(n)} \mathrm{c}_\pi(a_1, a_2, \dots, a_n). \tag{5.6}$$

To compute the non-vanishing cumulants we distinguish three cases.

(i) Let $\pi \in \mathcal{CI}(n)$ and first assume that $\pi \in \mathcal{I}(n)$. Now if $\pi < \hat{1}_n$ then $1 \not\sim_\pi n$ and moreover $1 \not\sim_\rho n$ for any ρ satisfying $\rho \leqslant \pi$. This allows us to replace $\tilde{\omega}$ by $\tilde{\varphi}$ to obtain

$$\mathrm{c}_\pi(a_1, a_2, \dots, a_n) = \sum_{\rho \leqslant \pi} \varphi_\rho(a_1, a_2, \dots, a_n)\, \mu(\rho, \pi) = B_\pi(a_1, a_2, \dots, a_n);$$

see [13, Proposition 4.11] for the last equality.

(ii) Let us assume next that $\pi \in \mathcal{CI}(n) \smallsetminus \mathcal{I}(n)$. This means that $\pi < \hat{1}_n$ and that $1 \sim_\pi n$. In this case we cannot immediately replace ω by φ, but first must use the traciality of ω and rotate the partition π into an element of $\mathcal{I}(n)$. Indeed fix a cyclic permutation $\sigma \in \mathfrak{S}_n$ such that $\sigma \cdot \pi \in \mathcal{I}(n)$. Then $1 \not\sim_\pi n$ and also $1 \not\sim_\rho n$ for any $\rho \leqslant \pi$ and we have

$$\begin{aligned} \mathrm{c}_\pi(a_1, a_2, \dots, a_n) &= \mathrm{c}_{\sigma\cdot\pi}(a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \dots, a_{\sigma^{-1}(n)}) \\ &= B_{\sigma\cdot\pi}(a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \dots, a_{\sigma^{-1}(n)}); \end{aligned}$$

notice that the rotation cannot be reversed now because the Boolean cumulants are nontracial.

(iii) Finally if $\pi = \hat{1}_n$ there is no direct formula but we infer from the moment-cumulant formula (5.6) that

$$\mathrm{c}_n(a_1, a_2, \dots, a_n) = \omega(a_1, a_2, \dots, a_n) - \sum_{\pi \in \mathcal{CI}(n), \pi < \hat{1}_n} B_{\sigma\cdot\pi}(a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \dots, a_{\sigma^{-1}(n)}),$$

where for each π an appropriate cyclic permutation σ is chosen.

Remark 5.5. Note that for univariate cumulants the rotation does not change the value of the cumulant and we can write

$$\omega(a^n) = c_n(a) + \sum_{\pi \in \mathcal{CI}(n), \pi < \hat{1}_n} b_\pi(a), \tag{5.7}$$

where $b_\pi(a) = B_\pi(a, a, \dots, a)$ and $c_n(a) = \mathrm{c}_{\hat{1}_n}(a, a, \dots, a)$.

We can see that the definition of $c_n(a)$ in Remark 5.5 coincides with that in (5.4). This can be confirmed from the definition and uniqueness of cumulants, but here we directly prove the formula (5.3) for $c_n(a) = \mathrm{c}_{\hat{1}_n}(a, a, \dots, a)$ using the recurrence relation (5.7). Decomposing $\mathcal{CI}(n)$ into $\mathcal{I}(n)$ and $\mathcal{CI}(n) \smallsetminus \mathcal{I}(n)$ we obtain

$$\sum_{\pi \in \mathcal{CI}(n), \pi \neq \hat{1}_n} b_\pi(a) = \sum_{\pi \in \mathcal{I}(n), \pi \neq \hat{1}_n} b_\pi(a) + \sum_{\substack{k, \ell \geqslant 1 \\ k+\ell \leqslant n-1}} b_{k+\ell}(a) \sum_{\sigma \in \mathcal{I}(n-k-\ell)} b_\sigma(a)$$

$$= \varphi(a^n) - b_n(a) + \sum_{\substack{k,\ell \geqslant 1 \\ k+\ell \leqslant n-1}} b_{k+\ell}(a)\, \varphi(a^{n-k-\ell}).$$

Multiplying the above by z^n and taking the sum over n in (5.7) yields

$$\begin{aligned}\mathfrak{m}_a(z) &= \mathfrak{c}_a(z) + M_a(z) - B_a(z) + \sum_{k,\ell \geqslant 1} b_{k+\ell}(a) z^{k+\ell} \sum_{n \geqslant k+\ell+1} \varphi(a^{n-k-\ell}) z^{n-k-\ell} \\ &= \mathfrak{c}_a(z) + M_a(z) - B_a(z) + (zB_a'(z) - B_a(z))M_a(z) = \mathfrak{c}_a(z) + zM_a(z)B_a'(z).\end{aligned}$$

6. Cyclic-Boolean infinite divisibility

This section is devoted to the definition and classification of infinite divisibility. Due to the lack of a precise notion of positivity (see Remark 5.4), we are not able to treat general $*$-algebras with a state and a tracial linear functional. Hence, we give the definition of infinitely divisible distributions in the special setting of operators on Hilbert spaces where the linear functional ω is chosen to be the trace.

Definition 6.1. *Let H be a Hilbert space and φ a state on $B(H)$. An element $a \in T(H)_{\mathrm{sa}}$ is said to be* cyclic-Boolean infinitely divisible *if for any $n \in \mathbb{N}$ there exist a Hilbert space H_n and a state φ_n on $B(H_n)$ and cyclic-Boolean i.i.d. elements $a_1, \dots, a_n \in T(H_n)_{\mathrm{sa}}$ such that a with respect to (φ, Tr_H) has the same distribution as $a_1 + \cdots + a_n$ with respect to $(\varphi_n, \mathrm{Tr}_{H_n})$.*

Suppose that a is a trace class selfadjoint operator and cyclic-Boolean ID. For each $n \geqslant 2$, a equals the sum of certain cyclic-Boolean iid random variables $a_{n,1}, \dots, a_{n,n}$ in distribution, and let $t = 1/n$ and denote $\tilde{\mathfrak{g}}_t = \tilde{\mathfrak{g}}_{a_{n,i}}$ and $G_t = G_{a_{n,i}}$. Let $\{\lambda_i\}_{i \in I}$ be the set of mutually distinct eigenvalues of a and m_i the multiplicity of λ_i. Setting $I_0 = \{i \in I : \lambda_i \neq 0\}$ we have

$$\tilde{\mathfrak{g}}_a(z) = \sum_{i \in I_0} \frac{m_i \lambda_i}{z(z - \lambda_i)}.$$

Moreover, let E_a be the spectral decomposition of a and $p_i = \varphi(E_a(\{\lambda_i\})) \geqslant 0$; then we have

$$G_a(z) = \sum_{i \in I'} \frac{p_i}{z - \lambda_i}, \qquad I' = \{i \in I : p_i > 0\}.$$

For later use we also set

$$I_0' = I' \cap I_0.$$

By calculus, we see that G_a has a unique zero in each interval between neighboring poles and has no other zeros off the real line. Hence the set $\{\mu_j\}_{j \in J}$ of zeros of G_a is contained in $\mathbb{R}$ and is interlacing with $\{\lambda_i\}_{i \in I'}$.

Lemma 6.2. *The factorization*

$$G_a(z) = \frac{1}{z} \prod_{j \in J} \left(1 - \frac{\mu_j}{z}\right) \prod_{i \in I'} \left(1 - \frac{\lambda_i}{z}\right)^{-1}$$

holds for every $z \in \mathbb{C} \setminus (\{0\} \cup \{\lambda_i\}_{i \in I'})$.

Proof. When the set I' is finite, the conclusion is easily proved since G_a is a rational function. We may then assume that I' and hence I_0' is an infinite set. We decompose the set $\{\lambda_i\}_{i \in I_0'}$ into the positive part $\{\lambda_k^+\}_{k=1}^{n_+}$ and the negative part $\{\lambda_k^-\}_{k=1}^{n_-}$ arranged in the way

$$\lambda_1^- < \lambda_2^- < \cdots < 0 < \cdots < \lambda_2^+ < \lambda_1^+,$$

where $n_\pm \in \mathbb{N} \cup \{0, \infty\}$. By our assumption, n_- or n_+ is infinity. We rewrite the function G_a into the form

$$G_a(z) = \frac{p}{z} + \sum_{k=1}^{n_+} \frac{p_k^+}{z - \lambda_k^+} + \sum_{k=1}^{n_-} \frac{p_k^-}{z - \lambda_k^-}, \qquad p = \sum_{i \in I' : \lambda_i = 0} p_i \in [0,1],$$

where $p_k^\pm$ is the weight of $\lambda_k^\pm$ with respect to φ. Now introduce $n_\pm(\epsilon) = \max\{k \geqslant 1 : |\lambda_k^\pm| > \epsilon\} < \infty$ and then the truncated function

$$G_a^\epsilon(z) = \frac{p(\epsilon)}{z} + \sum_{k=1}^{n_+(\epsilon)} \frac{p_k^+}{z - \lambda_k^+} + \sum_{k=1}^{n_-(\epsilon)} \frac{p_k^-}{z - \lambda_k^-}, \qquad p(\epsilon) = p + \sum_{k : |\lambda_k^+| \leqslant \epsilon} p_k^+ + \sum_{k : |\lambda_k^-| \leqslant \epsilon} p_k^-.$$

The fact that n_- or n_+ is infinity implies that $p(\epsilon) > 0$ for every $\epsilon > 0$. Since G_a^ϵ is a rational function, we have

$$G_a^\epsilon(z) = \frac{1}{z} \prod_{k=1}^{n_-(\epsilon)} \left(\frac{z - \mu_k^-(\epsilon)}{z - \lambda_k^-} \right) \prod_{k=1}^{n_+(\epsilon)} \left(\frac{z - \mu_k^+(\epsilon)}{z - \lambda_k^+} \right), \tag{6.1}$$

where $\mu_k^\pm(\epsilon)$ is the unique zero of G_a^ϵ on the interval between $\lambda_k^\pm$ and $\lambda_{k+1}^\pm$ for $k = 1, 2, \ldots, n_\pm(\epsilon) - 1$ and $\mu_{n_\pm(\epsilon)}^\pm(\epsilon)$ is the unique zero of G_a on the interval between 0 and $\lambda_{n_\pm(\epsilon)}^\pm$.

In order to pass to the limit in (6.1), first note that $n_\pm(\epsilon) \to n_\pm$ as $\epsilon \to 0$. Since G_a^ϵ converges locally uniformly to G_a on $\mathbb{C} \setminus \{0, \lambda_k^+, \lambda_k^- : k \geqslant 1\}$, we conclude that $\mu_k^\pm(\epsilon)$ converges to $\mu_k^\pm$ as $\epsilon \to 0$ for each k. From (the product version of) Weierstrass' M-test (recall that $|\mu_k^\pm(\epsilon)| \leqslant |\lambda_k^\pm|$) we obtain

$$G_a(z) = \frac{1}{z} \prod_{k=1}^{n_-} \left(\frac{z - \mu_k^-}{z - \lambda_k^-} \right) \prod_{k=1}^{n_+} \left(\frac{z - \mu_k^+}{z - \lambda_k^+} \right),$$

the desired formula. □

Now we are able to characterize infinitely divisible measures with respect to cyclic-Boolean independence. First, notice that taking the logarithmic derivative in Lemma 6.2 yields that

$$\frac{G_a'(z)}{G_a(z)} = -\frac{1}{z} - \sum_{i \in I_0'} \frac{\lambda_i}{z(z - \lambda_i)} + \sum_{j \in J} \frac{\mu_j}{z(z - \mu_j)}.$$

As before, let $t = 1/n$ and denote $\tilde{\mathfrak{g}}_t = \tilde{\mathfrak{g}}_{a_{n,i}}$ and $G_t = G_{a_{n,i}}$. Let $\{\lambda_i(t)\}_{i \in I_0'(t)}$ be the set of (mutually distinct) non-zero poles of G_t. Since

$$G_t(z) = \frac{G_a(z)}{(1-t)zG_a(z) + t},$$

the set $\{\lambda_i(t)\}_{i \in I_0'(t)}$ is exactly the set of zeros of $(1-t)zG_a(z) + t$. On the other hand, the set of zeros of G_t is exactly the set of zeros of G_a, and hence

$$\frac{G_t'(z)}{G_t(z)} = -\frac{1}{z} - \sum_{i \in I_0'(t)} \frac{\lambda_i(t)}{z(z - \lambda_i(t))} + \sum_{j \in J} \frac{\mu_j}{z(z - \mu_j)}.$$

By Theorem 4.2 we have

$$\tilde{\mathfrak{g}}_a + \frac{G_a'}{G_a} + \frac{1}{z} = n \left(\tilde{\mathfrak{g}}_t + \frac{G_t'}{G_t} + \frac{1}{z} \right)$$

and hence

$$\tilde{\mathfrak{g}}_t = t\, \tilde{\mathfrak{g}}_a + t \frac{G_a'}{G_a} - \frac{G_t'}{G_t} + \frac{t-1}{z}$$

$$
\begin{aligned}
&= t\sum_{i\in I_0}\frac{m_i\lambda_i}{z(z-\lambda_i)} + t\left(-\frac{1}{z} - \sum_{i\in I_0'}\frac{\lambda_i}{z(z-\lambda_i)} + \sum_{j\in J}\frac{\mu_j}{z(z-\mu_j)}\right)\\
&\qquad - \left(-\frac{1}{z} - \sum_{i\in I_0'(t)}\frac{\lambda_i(t)}{z(z-\lambda_i(t))} + \sum_{j\in J}\frac{\mu_j}{z(z-\mu_j)}\right) + \frac{t-1}{z}\\
&= t\sum_{i\in I_0'}\frac{(m_i-1)\lambda_i}{z(z-\lambda_i)} + t\sum_{i\in I_0\smallsetminus I_0'}\frac{m_i\lambda_i}{z(z-\lambda_i)}\\
&\qquad - (1-t)\sum_{j\in J}\frac{\mu_j}{z(z-\mu_j)} + \sum_{i\in I_0'(t)}\frac{\lambda_i(t)}{z(z-\lambda_i(t))}.
\end{aligned}
$$

Now, for each non-zero real α the number $\lim_{z\to\alpha}(z-\alpha)\,\tilde{\mathfrak{g}}_t(z)$ is non-negative, as it is a positive integer if α is a non-zero eigenvalue of $a_{n,1}$ and zero otherwise.

Therefore, to cancel the negative coefficient $-(1-t)$ above, the only possibility is that each non-zero μ_j must be a member of $\{\lambda_i\}_{i\in I_0}\cup\{\lambda_i(t)\}_{i\in I_0'(t)}$ and because of interlacing of the zeros and poles of G_a, one sees that μ_j can not be included in $\{\lambda_i\}_{i\in I_0'}\cup\{\lambda_i(t)\}_{i\in I_0'(t)}$ as it is a zero of G_a.

It is possible that $\mu_j=\lambda_i$ for some $i\in I_0\smallsetminus I_0'$. In this case, however, for $t>0$ sufficiently small (namely, n sufficiently large) the coefficient $tm_i-(1-t)$ is negative; therefore, we conclude that $\mu_j=0$ for all $j\in J$ or $J=\varnothing$, and hence $\#J=0$ or 1. This happens only if $\#I'=0,1$ or 2.

On the other hand, for $i\in I_0'$ we have $\lim_{z\to\lambda_i}(z-\lambda_i)\,\tilde{\mathfrak{g}}_t(z)=t(m_i-1)$ which must be a non-negative integer for any $t=1/n$. Therefore, we conclude that $m_i=1$ for all $i\in I_0'$. For $i\in I_0\smallsetminus I_0'$ we have $\lim_{z\to\lambda_i}(z-\lambda_i)\,\tilde{\mathfrak{g}}_t(z)=tm_i$, which cannot be an integer for sufficiently small t, and hence $I_0'=I_0$.

Now it remains to study the possible cases for $I_0'=I_0=0,1$ or 2.

Case 1: $\#I_0'=\#I_0=0$ or 1. Then $G_a(z)=1/(z-\alpha)$ for some $\alpha\in\mathbb{R}$ and $J=\varnothing$, and consequently $\tilde{\mathfrak{g}}_a(z)=\frac{\alpha}{z(z-\alpha)}$ and $\tilde{\mathfrak{g}}_t(z)=t\alpha/(z-t\alpha)$.

Case 2: $\#I_0'=\#I_0=2$. Note that G_a cannot have a pole at 0 because it would create a non-zero μ_j. Hence $G_a(z)=p/(z-\alpha)+(1-p)/(z-\beta)$ for some $\alpha,\beta\neq 0,\alpha<\beta,0<p<1$, and $G_a(0)=-p/\alpha-(1-p)/\beta=0$. The last condition yields the restriction that $\alpha<0<\beta$ and $G_a(z)=z/[(z-\alpha)(z-\beta)]$. Solving the equation $(1-t)zG_a(z)+t=0$ we obtain two solutions

$$\lambda_\pm(t)=\frac{t(\alpha+\beta)\pm\sqrt{t^2(\alpha+\beta)^2-4t\alpha\beta}}{2}.$$

The eigenvalues can be retrieved from the formula

$$\tilde{\mathfrak{g}}_t(z)=\frac{\lambda_+(t)}{z(z-\lambda_+(t))}+\frac{\lambda_-(t)}{z(z-\lambda_-(t))}.$$

It is easy to see that the above cases are actually cyclic Boolean ID. Thus we arrive to the following.

Theorem 6.3. *Let H be a Hilbert space and φ be a state on $T(H)$. An element $a\in T(H)_{\mathrm{sa}}$ is cyclic-Boolean ID with respect to (φ,Tr_H) if and only if a has either*

(i) *only zero eigenvalues (that is, $a=0$),*
(ii) *only one non-zero eigenvalue and its multiplicity is one, or*
(iii) *exactly two non-zero eigenvalues α,β, their multiplicities are one, $\alpha\beta<0$ and the distribution of a with respect to φ is*

$$\frac{-\alpha}{\beta-\alpha}\delta_\alpha+\frac{\beta}{\beta-\alpha}\delta_\beta.$$

In the last case, for every $n \geqslant 2$ an n-th root of a has two non-zero eigenvalues α_n, β_n given as the solutions to the equation

$$x^2 - \frac{\alpha+\beta}{n}x + \frac{\alpha\beta}{n} = 0,$$

and the distribution with respect to the state is

$$\frac{-\alpha_n}{\beta_n - \alpha_n}\delta_{\alpha_n} + \frac{\beta_n}{\beta_n - \alpha_n}\delta_{\beta_n}.$$

Example 6.4. The matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has eigenvalues $\{[1]^1, [-1]^1\}$ and its distribution with respect to the unit vector $e_1 = {}^t(1,0)$ is

$$\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1.$$

By Theorem 6.3, the matrix A is cyclic-Boolean infinitely divisible with respect to $(\langle \cdot\, e_1, e_1\rangle_{\mathbb{C}^2}, \mathrm{Tr}_{\mathbb{C}^2})$.

7. Cyclic-monotone independence

7.1. Definition and example. We perform an investigation for monotone independence in a spirit similar to cyclic-Boolean independence. To this end, we start from a specific operator model inspired by the comb product of rooted graphs in Section 2.7.

Example 7.1. Let $N \in \mathbb{N}$ be a fixed number and assume that for each $i \in \{1, 2, \dots, N\}$ a *finite-dimensional* Hilbert space H_i with distinguished unit vector $\xi_i \in H_i$ be given. Denote by $P_i \colon H_i \to H_i$ the orthogonal projection onto $\mathbb{C}\xi_i$ and by φ_i the vector state on $B(H_i)$ induced by ξ_i. Let $H = H_1 \otimes \cdots \otimes H_N$, $\xi = \xi_1 \otimes \cdots \otimes \xi_N$ and φ the vacuum state on $B(H)$ defined by ξ. This is the same setting as in Example 3.1 with the additional requirement of finite dimensionality. Analogously to the embedding (3.1) we introduce another embedding of $B(H_i)$ into $B(H)$:

$$\sigma_i(A) = I_{H_1} \otimes \cdots \otimes I_{H_{i-1}} \otimes A \otimes P_{i+1} \otimes \cdots \otimes P_N. \tag{7.1}$$

Note that this embedding does not preserve trace class and therefore the construction is restricted to finite dimensional spaces. It is known that the family $\{\sigma_i(B(H_i))\}_{i=1}^N$ is monotonically independent with respect to φ; see [12, Theorem 8.9].

In addition, we can compute moments with respect to the trace. Again formula (3.2) is crucial: for $n \geqslant 2$, a cyclically alternating tuple $(i_1, \dots, i_n) \in [N]^n$ and $A_k \in B(H_{i_k})$, if $p \in [n]$ is such that $i_{p-1} < i_p > i_{p+1}$ (with the conventions $i_0 = i_n$ and $i_{n+1} = i_1$) then direct computations entail

$$\begin{aligned}\mathrm{Tr}_H(\sigma_{i_1}(A_1)\cdots\sigma_{i_n}(A_n)) \\ = \varphi_{i_p}(A_p)\,\mathrm{Tr}_H\big[\sigma_{i_1}(A_1)\cdots\sigma_{i_{p-1}}(A_{p-1})\sigma_{i_{p+1}}(A_{p+1})\cdots\sigma_{i_n}(A_n)\big].\end{aligned}$$

Observe that if $2 \leqslant p \leqslant n$ then the assumption that $(i_1, \dots, i_n) \in I^n$ is alternating is sufficient. However, if $p = 1$ or n then the assumption that $(i_1, \dots, i_n) \in I^n$ is *cyclically alternating* is crucial. Indeed, consider for instance the alternating tuple $(3, 2, 3)$; then for $A, C \in B(H_3)$ and $B \in B(H_2)$

$$\mathrm{Tr}_H(\sigma_3(A)\sigma_2(B)\sigma_3(C)) = \mathrm{Tr}_H(\sigma_3(CA)\sigma_2(B)) = \varphi_3(CA)\,\mathrm{Tr}_H(\sigma_2(B)),$$

which is different from $\varphi_3(A)\,\mathrm{Tr}_H\,[\sigma_2(B)\sigma_3(C)]$ in general.

This example can be abstracted in the following way.

Definition 7.2. *Let $(\mathcal{A}, \varphi, \omega)$ be a cncps, I a totally ordered set, and $\hat{I} := \{-\infty\} \cup I$ an enlargement of I, where $-\infty$ is the minimum of $\hat{I}$. An ordered family of $*$-subalgebras $\{\mathcal{A}_i\}_{i \in I}$ of $\mathcal{A}$ is said to be cyclic-monotone independent if*

(i) *it is monotonically independent with respect to φ, that is, for any $n \geqslant 2$, any alternating tuple $(i_1, \dots, i_n) \in I^n$ (namely $i_1 \neq \cdots \neq i_n$) and $a_k \in \mathcal{A}_{i_k}, k = 1, 2, \dots, n$, if $p \in [n]$ is such that $i_{p-1} < i_p > i_{p+1}$ (with the convention that $i_0 = i_{n+1} = -\infty$) then*

$$\varphi(a_1 \cdots a_n) = \varphi(a_p)\varphi(a_1 \cdots a_{p-1} a_{p+1} \cdots a_n);$$

(ii) *for any $n \geqslant 2$, any cyclically alternating tuple $(i_1, \dots, i_n) \in I^n$ (namely $i_1 \neq \cdots \neq i_n \neq i_1$) and $a_k \in \mathcal{A}_{i_k}, k = 1, 2, \dots, n$, if $p \in [n]$ is such that $i_{p-1} < i_p > i_{p+1}$ (with the different convention that $i_0 = i_n$ and $i_{n+1} = i_1$) then*

$$\omega(a_1 \cdots a_n) = \varphi(a_p)\omega(a_1 \cdots a_{p-1} a_{p+1} \cdots a_n).$$

Remark 7.3. In the case $n = 2$ condition ((ii)) of the preceding definition boils down to multiplicativity of ω in the sense that

$$\omega(a_1 a_2) = \omega(a_1)\varphi(a_2)$$

whenever $a_1 \in \mathcal{A}_{i_1}$ and $a_2 \in \mathcal{A}_{i_2}$ with $i_1 < i_2$.

For $n \geqslant 3$ it is important to insist on cyclically alternating tuples, see the observation at the end of Example 7.1.

Definition 7.4. *Let $(\mathcal{A}, \varphi, \omega)$ be a cncps and I a totally ordered set. An ordered family of elements $\{a_i\}_{i \in I}$ of $\mathcal{A}$ is said to be cyclic-monotone independent if so is $\{\mathcal{A}_i\}_{i \in I}$, where $\mathcal{A}_i$ is the $*$-algebra generated by a_i without unit.*

Example 7.5. Suppose that (a, b, c) is cyclic-monotone independent in $(\mathcal{A}, \varphi, \omega)$. Then

$$\varphi(ba^2bac^2b) = \varphi(c^2)\varphi(b)^3\varphi(a^3)$$

and

$$\omega(ba^2bac^2b) = \varphi(c^2)\varphi(b)\varphi(b^2)\omega(a^3).$$

Remark 7.6. Cyclic-monotone independence already appeared in the random matrix model in [5] (see also [20, 2] and Section 1), where independence was defined for a pair of $*$-subalgebras and only for ω. For a random matrix model for monotone independence see [4].

In [5] the trace functional ω is unbounded, because it can diverge in the large dimensional limit, and therefore a domain for ω was specified. To avoid this problem in the present paper we focus on finite dimensional Hilbert spaces and $\omega = \mathrm{Tr}$.

It should be noticed that Example 7.1 does not provide an i.i.d. operator model even when $H_i = K$ does not depend on i; for $A \in B(K)$ the operators $\{\sigma_i(A)\}_{i=1}^N$ are identically distributed with respect to φ, but not with respect to ω, because

$$\omega(\sigma_i(A)) = d^{i-1}\,\mathrm{Tr}_K(A),$$

where $d = \dim(K)$. In fact, we do not know of any non-trivial operator model for cyclic monotone i.i.d. random variables and for this reason we do not see any meaningful notions of cumulants and of infinitely divisible distributions.

7.2. Cyclic-monotone convolution. Similar to the cyclic-Boolean convolution, for a, b in a cncps which are cyclic-monotone independent, we call the distribution of the sum $a + b$ the *cyclic-monotone convolution* of (the distributions of) a and b. The convolution formula can be verified in several ways.

THEOREM 7.7. *Let $(\mathcal{A},\varphi,\omega)$ be a cncps and $a,b\in\mathcal{A}$. Suppose that (a,b) is cyclic-monotone independent. Then we have*

$$\tilde{\mathfrak{g}}_{a+b}(z)=\tilde{\mathfrak{g}}_b(z)+F_b'(z)\,\tilde{\mathfrak{g}}_a(F_b(z)).$$

Algebraic proof. Expand $(a+b)^n$ into

$$(a+b)^n=b^n+\sum_{\substack{k\geqslant 1\\ q_1,q_2,\dots,q_{k+1}\geqslant 0\\ q_1+\cdots+q_{k+1}+k=n}} b^{q_1}ab^{q_2}a\cdots ab^{q_{k+1}},$$

and applying ω yields

$$\omega((a+b)^n)=\omega(b^n)+\sum_{\substack{k\geqslant 1\\ q_1,q_2,\dots,q_{k+1}\geqslant 0\\ q_1+\cdots+q_{k+1}+k=n}} \varphi(b^{q_1+q_{k+1}})\varphi(b^{q_2})\cdots\varphi(b^{q_k})\omega(a^k).$$

Multiplying the above identity by z^{-n-1} and taking the summation over n yields

$$\begin{aligned}\tilde{\mathfrak{g}}_{a+b}(z)&=\tilde{\mathfrak{g}}_b(z)+\sum_{k\geqslant 1}\sum_{q_1,\dots,q_{k+1}\geqslant 0}\frac{\varphi(b^{q_1+q_{k+1}})}{z^{q_1+q_{k+1}}}\frac{\varphi(b^{q_2})}{z^{q_2}}\cdots\frac{\varphi(b^{q_k})}{z^{q_k}}\frac{\omega(a^k)}{z^{k+1}}\\
&=\tilde{\mathfrak{g}}_b(z)-z^2G_b'(z)\sum_{k\geqslant 1}[zG_b(z)]^{k-1}\frac{\omega(a^k)}{z^{k+1}}\\
&=\tilde{\mathfrak{g}}_b(z)-\frac{G_b'(z)}{G_b(z)^2}\sum_{k\geqslant 1}G_b(z)^{k+1}\omega(a^k)\\
&=\tilde{\mathfrak{g}}_b(z)+F_b'(z)\,\tilde{\mathfrak{g}}_a(F_b(z)).\end{aligned}$$

Note here that the identity

$$\sum_{m\geqslant 0,n\geqslant 0}^{\infty}\frac{\varphi(b^{m+n})}{z^{m+n}}=-z^2G_b'(z)$$

is used above. □

Analytic proof in the setting of Example 7.1: Schur complement approach. Let $A\in B(\mathcal{H}_1)$ and $B\in B(\mathcal{H}_2)$ be operators with block decompositions

$$A=\begin{bmatrix}\alpha & a'\\ a & \mathring{A}\end{bmatrix}\qquad\text{and}\qquad B=\begin{bmatrix}\beta & b'\\ b & \mathring{B}\end{bmatrix}$$

according to the decomposition $\mathcal{H}_i=\mathbb{C}\xi_i\oplus\mathring{\mathcal{H}}_i$ as in (2.12) and let $\sigma_1(A)=A\otimes P_2$ and $\sigma_2(B)=I_1\otimes B$ act on $\mathcal{H}_1\otimes\mathcal{H}_2\simeq\mathbb{C}\xi\oplus\mathring{\mathcal{H}}_1\oplus(\mathcal{H}_1\otimes\mathring{\mathcal{H}}_2)$ according to Example 7.1, i.e., if we denote by $\eta_1:\mathring{\mathcal{H}}_1\to\mathcal{H}_1$ the embedding and $\eta_1^*:\mathcal{H}_1\to\mathring{\mathcal{H}}_1$ the projection, then

$$\begin{aligned}\sigma_1(A)\xi&=\alpha\xi\oplus a\oplus 0 & \sigma_2(B)\xi&=\beta\xi\oplus 0\oplus(\xi_1\otimes b)\\
\sigma_1(A)\mathring{h}_1&=(a'\mathring{h}_1)\xi\oplus\mathring{A}\mathring{h}_1\oplus 0 & \sigma_2(B)\mathring{h}_1&=0\oplus\beta\mathring{h}_1\oplus(\eta_1(\mathring{h}_1)\otimes b)\\
\sigma_1(A)(h_1\otimes\mathring{h}_2)&=0\end{aligned}$$

and

$$\sigma_2(B)(h_1\otimes\mathring{h}_2)=(b'\mathring{h}_2)(\xi_1^*h_1)\xi\oplus(b'\mathring{h}_2)\eta_1^*(h_1)\oplus(h_1\otimes\mathring{B}\mathring{h}_2);$$

thus we obtain the block decompositions

$$\sigma_1(A)=\begin{bmatrix}\alpha & a' & 0\\ a & \mathring{A} & 0\\ 0 & 0 & 0\end{bmatrix},\qquad \sigma_2(B)=\begin{bmatrix}\beta & 0 & \xi_1^*\otimes b'\\ 0 & \beta\mathring{I}_1 & \eta_1^*\otimes b'\\ \xi_1\otimes b & \eta_1\otimes b & I_1\otimes\mathring{B}\end{bmatrix}$$

and together

$$\sigma_1(A)+\sigma_2(B)=\begin{bmatrix}\alpha+\beta & a' & \xi_1^*\otimes b'\\ a & \mathring{A}+\beta\mathring{I}_1 & \eta_1^*\otimes b'\\ \xi_1\otimes b & \eta_1\otimes b & I_1\otimes\mathring{B}\end{bmatrix}.$$

We compute the resolvent

$$(z-\sigma_1(A)-\sigma_2(B))^{-1}=\left[\begin{array}{c|cc} z-\alpha-\beta & -a' & -\xi_1^*\otimes b'\\ \hline -a & (z-\beta)\mathring{I}_1-\mathring{A} & -\eta_1^*\otimes b'\\ -\xi_1\otimes b & -\eta_1\otimes b & I_1\otimes(z\mathring{I}_2-\mathring{B})\end{array}\right]^{-1} \tag{7.2}$$

via the Schur complement. To this end, we first compute the lower resolvent

$$L^{-1}=\begin{bmatrix}(z-\beta)\mathring{I}_1-\mathring{A} & -\eta_1^*\otimes b'\\ -\eta_1\otimes b & I_1\otimes(z\mathring{I}_2-\mathring{B})\end{bmatrix}^{-1} \tag{7.3}$$

on $\mathring{\mathcal{H}}_1\oplus(\mathcal{H}_1\otimes\mathring{\mathcal{H}}_2)$. The corresponding Schur complement of L is

$$\begin{aligned}S_L&=(z-\beta)\mathring{I}_1-\mathring{A}-(\eta_1^*\otimes b')(I_1\otimes(z\mathring{I}_2-\mathring{B})^{-1})(\eta_1\otimes b)\\ &=(z-\beta)\mathring{I}_1-\mathring{A}-\eta_1^*I_1\eta_1\otimes b'(z\mathring{I}_2-\mathring{B})^{-1}b\\ &=F_B(z)\mathring{I}_1-\mathring{A}.\end{aligned}$$

If we denote by $\mathfrak{R}_{\mathring{A}}(F_B(z))=(F_B(z)\mathring{I}_1-\mathring{A})^{-1}$ and $\mathfrak{R}_{\mathring{B}}(z)=(z\mathring{I}_2-\mathring{B})^{-1}$ the resolvents of $\mathring{A}$ and $\mathring{B}$, respectively, then with the help of Banachiewicz' formula (2.9) the resolvent (7.3) can be written as

$$\begin{aligned}L^{-1}&=\begin{bmatrix}S_L^{-1} & S_L^{-1}(\eta_1^*\otimes b')(I_1\otimes\mathfrak{R}_{\mathring{B}}(z))\\ I_1\otimes\mathfrak{R}_{\mathring{B}}(z)(\eta_1\otimes b)S_L^{-1} & (I_1\otimes\mathfrak{R}_{\mathring{B}}(z))\big(1+(\eta_1\otimes b)S_L^{-1}(\eta_1^*\otimes b')(I_1\otimes\mathfrak{R}_{\mathring{B}}(z))\big)\end{bmatrix}\\ &=\begin{bmatrix}\mathfrak{R}_{\mathring{A}}(F_B(z)) & \mathfrak{R}_{\mathring{A}}(F_B(z))\eta_1^*\otimes b'\,\mathfrak{R}_{\mathring{B}}(z)\\ \eta_1\,\mathfrak{R}_{\mathring{A}}(F_B(z))\otimes\mathfrak{R}_{\mathring{B}}(z)b & I_1\otimes\mathfrak{R}_{\mathring{B}}(z)+\eta_1\,\mathfrak{R}_{\mathring{A}}(F_B(z))\eta_1^*\otimes\mathfrak{R}_{\mathring{B}}(z)bb'\,\mathfrak{R}_{\mathring{B}}(z)\end{bmatrix}.\end{aligned}$$

Now we plug L^{-1} into Banachiewicz' formula (2.9) for (7.2):

$$(z-\sigma_1(A)-\sigma_2(B))^{-1}=\left[\begin{array}{c|c} S^{-1} & S^{-1}\left[a'\ \xi_1^*\otimes b'\right]L^{-1}\\ \hline L^{-1}\begin{bmatrix}a\\ \xi_1\otimes b\end{bmatrix}S^{-1} & L^{-1}+L^{-1}\begin{bmatrix}a\\ \xi_1\otimes b\end{bmatrix}S^{-1}\left[a'\ \xi_1^*\otimes b'\right]L^{-1}\end{array}\right].$$

After some cancellations the Schur complement evaluates to

$$\begin{aligned}S&=F_{\sigma_1(A)+\sigma_2(B)}(z)\\ &=z-\alpha-\beta-\left[-a'\ -\xi_1^*\otimes b'\right]L^{-1}\begin{bmatrix}-a\\ -\xi_1\otimes b\end{bmatrix}\\ &=z-\alpha-\beta-a'(F_B(z)\mathring{I}_1-\mathring{A})^{-1}a-\xi_1^*\xi_1b'(z\mathring{I}_2-\mathring{B})^{-1}b\\ &=F_B(z)-\alpha-a'(F_B(z)\mathring{I}_1-\mathring{A})^{-1}a\\ &=F_A(F_B(z))\end{aligned}$$

where we used the Schur complement representation (2.13). Finally the resolvent is

$$(z-\sigma_1(A)-\sigma_2(B))^{-1}=G_A(F_B(z))\left[\begin{array}{c|cc} 1 & a'\,\mathfrak{R}_{\mathring{A}}(F_B(z)) & u'\\ \hline \mathfrak{R}_{\mathring{A}}(F_B(z))a & F_A(F_B(z))L^{-1}+L_2 & \\ u & & \end{array}\right]$$

where

$$\begin{aligned}u'&=[a'\,\mathfrak{R}_{\mathring{A}}(F_B(z))\eta_1^*+\xi_1^*]\otimes b'\,\mathfrak{R}_{\mathring{B}}(z)\\ u&=[\eta_1\,\mathfrak{R}_{\mathring{A}}(F_B(z))a+\xi_1]\otimes\mathfrak{R}_{\mathring{B}}(z)b\end{aligned}$$

$$L_2 = \begin{bmatrix} T & T\eta_1^* + \mathfrak{R}_{\mathring{A}}(F_B(z))a\xi_1^* \otimes b'\,\mathfrak{R}_{\mathring{B}}(z) \\ \eta_1 T + \xi_1 a'\,\mathfrak{R}_{\mathring{A}}(F_B(z)) \otimes \mathfrak{R}_{\mathring{B}}(z)b & L_{22} \end{bmatrix}$$

with

$$\begin{aligned} T &= (F_B(z) - \mathring{A})^{-1} a a' (F_B(z) - \mathring{A})^{-1} \\ L_{22} &= \big(\eta_1 T \eta_1^* + \eta_1\,\mathfrak{R}_{\mathring{A}}(F_B(z))a\xi_1^* + \xi_1 a'\,\mathfrak{R}_{\mathring{A}}(F_B(z))\eta_1^* + \xi_1\xi_1^*\big) \otimes \mathfrak{R}_{\mathring{B}}(z). \end{aligned}$$

Finally the trace of the resolvent evaluates to

$$\begin{aligned} \mathfrak{g}_{\sigma_1(A)+\sigma_2(B)}(z) &= G_A(F_B(z)) + \operatorname{Tr}(L^{-1}) + G_A(F_B(z))\operatorname{Tr}(L_2) \\ &= G_A(F_B(z)) + \mathfrak{g}_{\mathring{A}}(F_B(z)) + \operatorname{Tr}[(z-\mathring{B})^{-1} b b' (z-\mathring{B})^{-1}] \\ &\qquad + \operatorname{Tr}(I_1)\,\mathfrak{g}_{\mathring{B}}(z) + \operatorname{Tr}[\eta_1 (F_B(z) - \mathring{A})^{-1}\eta_1^*] \\ &\qquad + G_A(F_B(z))\operatorname{Tr}[T(F_B(z)-\mathring{A})^{-1} a a' (F_B(z)-\mathring{A})^{-1}] \\ &\qquad + G_A(F_B(z))\operatorname{Tr}(\eta_1 T\eta_1^* + \xi_1\xi_1^*)\operatorname{Tr}[(z-\mathring{B})^{-1} b b' (z-\mathring{B})^{-1}] \\ &= G_A(F_B(z)) + \mathfrak{g}_{\mathring{A}}(F_B(z)) + d_1\,\mathfrak{g}_{\mathring{B}}(z) + \mathfrak{g}_{\mathring{A}}(F_B(z))(F_B'(z)-1) \\ &\qquad + G_A(F_B(z))[F_A'(F_B(z)) - 1 + F_A'(F_B(z))(F_B'(z)-1)] \\ &= d_1\,\mathfrak{g}_{\mathring{B}}(z) + \mathfrak{g}_A(F_B(z))F_B'(z). \end{aligned}$$

□

We note here that another equivalent convolution formula can be given in terms of

$$\mathfrak{l}_a(z) = -\sum_{n\geqslant 1} \frac{\omega(a^n)}{n z^n}.$$

The convolution formula in Theorem 7.7 then reads $\mathfrak{l}_{a+b}' = \mathfrak{l}_b' + (\mathfrak{l}_a \circ F_b)'$ and hence

$$\mathfrak{l}_{a+b}(z) = \mathfrak{l}_b(z) + \mathfrak{l}_a(F_b(z)).$$

7.3. Limit theorem. In the setting of Example 7.1, let H_i be the same Hilbert space K with $d = \dim(K) \in \{2,3,4,\dots\}$ and with a distinguished unit vector ξ and let $a^{(i)} = \sigma_i(a)$ for some $a \in B(K)_{\mathrm{sa}}$. Let ω be the trace on H, Tr the trace on K and ψ the vector state on K determined by ξ.

Our main object in this section is the sum

$$b_N = a^{(1)} + \cdots + a^{(N)}. \tag{7.4}$$

In order to see the convergence of trace moments of b_N we start from some examples. Since $\operatorname{Tr}(I_K) = d$, we obtain

$$\omega(b_N) = \operatorname{Tr}(a) + d\operatorname{Tr}(a) + \cdots + d^{N-1}\operatorname{Tr}(a) = [N]_d \operatorname{Tr}(a),$$

where $[N]_d = 1 + d + d^2 + \cdots + d^{N-1}$, and

$$\begin{aligned} \omega(b_N^2) &= \sum_{i<j} \omega(a^{(i)}a^{(j)}) + \sum_{i>j} \omega(a^{(i)}a^{(j)}) + \sum_i \omega((a^{(i)})^2) \\ &= \sum_{i<j} \omega(a^{(i)})\varphi(a^{(j)}) + \sum_{i>j} \varphi(a^{(i)})\omega(a^{(j)}) + [N]_d \operatorname{Tr}(a^2) \\ &= \sum_{j=2}^{N} \frac{d^{j-1}-1}{d-1}\operatorname{Tr}(a)\psi(a) + \sum_{i=2}^{N} \frac{d^{i-1}-1}{d-1}\operatorname{Tr}(a)\psi(a) + [N]_d\operatorname{Tr}(a^2) \\ &= \frac{2}{d-1}\left[\frac{d(d^{N-1}-1)}{d-1} - (N-1)\right]\operatorname{Tr}(a)\psi(a) + [N]_d\operatorname{Tr}(a^2) \\ &= \frac{2}{d-1}([N]_d - N)\operatorname{Tr}(a)\psi(a) + [N]_d\operatorname{Tr}(a^2). \end{aligned}$$

A similar computation yields that

$$\omega(b_N^3) = 6\left[\frac{[N]_d - N}{(d-1)^2} - \frac{N(N-1)}{2(d-1)}\right]\operatorname{Tr}(a)\psi(a)^2 + \frac{3}{d-1}([N]_d - N)\operatorname{Tr}(a^2)\psi(a) + \frac{3}{d-1}([N]_d - N)\operatorname{Tr}(a)\psi(a^2) + [N]_d \operatorname{Tr}(a^3).$$

Therefore, the normalized traces converge as $N \to \infty$ without rescaling of b_N:

$$\begin{aligned} d^{-N}\omega(b_N) &\to \frac{\operatorname{Tr}(a)}{d-1}, \\ d^{-N}\omega(b_N^2) &\to \frac{2}{(d-1)^2}\operatorname{Tr}(a)\psi(a) + \frac{1}{d-1}\operatorname{Tr}(a^2), \\ d^{-N}\omega(b_N^3) &\to \frac{6}{(d-1)^3}\operatorname{Tr}(a)\psi(a)^2 + \frac{3}{(d-1)^2}\operatorname{Tr}(a^2)\psi(a) \\ &\qquad + \frac{3}{(d-1)^2}\operatorname{Tr}(a)\psi(a^2) + \frac{1}{d-1}\operatorname{Tr}(a^3). \end{aligned}$$

In order to describe the general situation, we need some concepts on ordered set partitions.

Definition 7.8. *Let $k \in \mathbb{N}$.*

(i) *An ordered set partition of $[k]$ is a tuple $\pi = (B_1, B_2, \dots, B_p)$ of subsets of $[k]$ such that $\{B_1, \dots, B_p\}$ is a set partition of $[k]$; that is, $B_1, \dots, B_p$ are non-empty and mutually disjoint subsets of $[k]$, and their union is $[k]$. The length p of π is denoted by $|\pi|$. The set of ordered set partitions of $[k]$ is denoted by $\mathcal{OP}(k)$.*

(ii) *For a tuple $\mathbf{i} = (i_1, \dots, i_k) \in \mathbb{N}^k$, the ordered kernel set partition $\ker(\mathbf{i}) \in \mathcal{OP}(k)$ is defined as follows: first, pick the smallest value p_1 among $i_1, \dots, i_k$ and then define the subset $B_1 = \{j \in [k] : i_j = p_1\}$; secondly, pick the second smallest value p_2 among $i_1, \dots, i_k$ and define the subset $B_2 = \{j \in [k] : i_j = p_2\}$; continuing this procedure until the end we arrive at an ordered set partition $(B_1, B_2, \dots)$, which is denoted by $\ker(\mathbf{i})$.*

Example 7.9.

$$\ker(6,3,2,3,6) = (\{3\}, \{2,4\}, \{1,5\}),$$
$$\ker(2,7,4,7,4,2,4) = (\{1,6\}, \{3,5,7\}, \{2,4\}).$$

For further information on ordered (kernel) set partitions the reader is referred to [11].

For an ordered set partition π of $[k]$ there exists a unique *packed word*, i.e., a tuple $\mathbf{i}(\pi) = (i_1(\pi), \dots, i_k(\pi)) \in [|\pi|]^k$ such that $\pi = \ker(\mathbf{i}(\pi))$. Using this tuple we define $\omega(\pi)$ to be $\omega(a^{(i_1(\pi))} \cdots a^{(i_k(\pi))})$.

Example 7.10. If $\pi = (\{1,3\}, \{2\})$ then $\mathbf{i}(\pi) = (1,2,1)$ and

$$\omega(\pi) = \omega(a^{(1)}a^{(2)}a^{(1)}) = \varphi(a)\operatorname{Tr}(a^2).$$

If $\pi = (\{3\}, \{2,4,6\}, \{1,5\})$ then $\mathbf{i}(\pi) = (3,2,1,2,3,2)$ and

$$\omega(\pi) = \omega(a^{(3)}a^{(2)}a^{(1)}a^{(2)}a^{(3)}a^{(2)}) = \varphi(a)\varphi(a)\varphi(a^3)\operatorname{Tr}(a).$$

With those notions, we have

$$\omega(b_N^k) = \sum_{\pi \in \mathcal{OP}(k)} \sum_{\substack{\mathbf{i} = (i_1 \dots, i_k) \in [N]^k \\ \ker(\mathbf{i}) = \pi}} \omega(a^{(i_1)} \cdots a^{(i_k)})$$

$$= \sum_{\pi \in \mathcal{OP}(k)} \sum_{\substack{\mathbf{i}=(i_1 \ldots, i_k) \in [N]^k \\ \ker(\mathbf{i})=\pi}} d^{\min\{i_1,\ldots,i_k\}-1} \omega(\pi)$$
$$= \sum_{\pi \in \mathcal{OP}(k)} \alpha_{|\pi|}(d,N)\omega(\pi),$$

where

$$\alpha_p(d,N) := \sum_{\substack{(j_1,\ldots,j_p)\in[N]^p \\ j_1<\cdots<j_p}} d^{j_1-1}, \quad N \geqslant p; \qquad \alpha_p(d,N) := 0, \quad 0 \leqslant N < p.$$

In order to investigate the asymptotics of $\omega(b_N^k)$ it suffices to understand the function $\alpha_k(d,N)$.

Lemma 7.11. *For each $k \in \mathbb{N}$ there exists a polynomial P_k in two variables such that*

$$\alpha_k(d,N) = \frac{d^N}{(d-1)^k} + P_k((d-1)^{-1}, N), \qquad d \geqslant 2, N \geqslant 0.$$

Proof. The proof goes by induction on k. For $k=1$, $\alpha_1(d,N) = (d^N-1)/(d-1)$, and hence $P_1(x,y) = -x$. For general $k \geqslant 2$ we proceed as

$$\alpha_k(d,N) = \sum_{j=1}^{N} \sum_{\substack{(j_1,\ldots,j_{k-1})\in[j-1]^{k-1} \\ j_1<\cdots<j_{k-1}}} d^{j_1-1} = \sum_{j=1}^{N} \alpha_{k-1}(d,j-1)$$
$$= \sum_{j=1}^{N} \left[\frac{d^{j-1}}{(d-1)^{k-1}} + P_{k-1}((d-1)^{-1}, j-1) \right]$$
$$= \frac{d^N}{(d-1)^k} - \frac{1}{(d-1)^k} + \sum_{j=1}^{N} P_{k-1}((d-1)^{-1}, j-1), \qquad N \geqslant 1.$$

By Faulhaber's formula and induction hypothesis, there is a polynomial $Q_k(x,y)$ such that $Q_k(x,0)=0$ and

$$Q_k(x,N) = \sum_{j=1}^{N} P_{k-1}(x, j-1), \qquad N \geqslant 1,$$

which implies the desired formula for $N \geqslant 1$ by taking $P_k(x,y) = -x^k + Q_k(x,y)$. Since $Q_k(x,0)=0$ the formula holds for $N=0$ as well. □

By Lemma 7.11 we obtain the limit

$$\lim_{N\to\infty} d^{-N}\alpha_k(d,N) = \frac{1}{(d-1)^k}$$

and conclude the following.

Theorem 7.12. *In the setting above we have*

$$\lim_{N\to\infty} d^{-N}\omega(b_N^k) = \sum_{\pi\in\mathcal{OP}(k)} \frac{\omega(\pi)}{(d-1)^{|\pi|}}. \tag{7.5}$$

Thus the empirical eigenvalue distributions of b_N converge (in the sense of moments) to a probability measure whose k-th moment is the above limit. Of course the empirical eigenvalue distributions of the rescaled sum $N^{-1/2}b_N$ converge weakly to δ_0, which means that the number of eigenvalues of $N^{-1/2}b_N$ outside a fixed neighborhood of 0 is of the order $o(d^N)$. Combining this with the monotone CLT, which asserts that

vacuum spectral distribution of $N^{-1/2}b_N$ weakly converges an arcsine distribution, it turns out that the vacuum vector captures a relatively small number of eigenvalues of $N^{-1/2}b_N$ that lie outside the neighborhood of 0.

The limit moments (7.5) depend on a lot of information about trace and vacuum moments of the original matrix a. This is in sharp contrast with the fact that if $\psi(a) = 0$ and $\psi(a^2) = 1$ then the distribution of the rescaled sum $N^{-1/2}b_N$ with respect to the vacuum state φ converges weakly to the same arcsine law.

We come back to the original model of comb product graphs in Section 2.7 (cf: Example 7.1), and compute the limit empirical eigenvalue distribution of the adjacency matrix of the iterated comb product of the complete graph $\mathbf{K}_2$. Even for this simplest graph, the limit moments (7.5) are not explicit; they only satisfy a recurrence relation. Fortunately, we can describe the limit distribution with the help of work of Smyth [21], who defined a distribution function $L_+\colon [0,\infty) \to [0,1)$ (denoted as F therein) characterized by the property that L_+ is strictly increasing, $L_+(0) = 0$ and

$$|2L_+(x) - 1| = L_+(|x - x^{-1}|), \qquad x > 0.$$

Let λ_+ be the distribution associated with L_+ and λ the symmetrization of λ_+. It is known that L_+ is continuous and hence λ has no atoms.

Theorem 7.13. *Let A_N be the adjacency matrix of the N-fold comb product of $(\mathbf{K}_2, o)$ with itself. Then the empirical eigenvalue distribution of A_N converges weakly to λ as $N \to \infty$.*

Proof. In the notation of this section, we are dealing with

$$K = \mathbb{C}^2, \qquad a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

As already verified, the limiting p-th moment of b_N is described by

$$\sum_{\pi \in \mathcal{OP}(p)} \omega(\pi), \tag{7.6}$$

where $\omega(\pi)$ in (7.5) is determined by the moment sequence $\{\omega(a^p)\}_{p\geqslant 0} = (2, 0, 2, 0, \dots) = \{2\psi(a^p)\}_{p\geqslant 0}$. It is easy to see that $\omega(\pi) = 0$ for all $\pi \in \mathcal{OP}(p)$ if p is odd, and hence all odd moments vanish. We will compute the numbers

$$\gamma_{n,k} := \sum_{\pi \in \mathcal{OP}(2n), |\pi| = k} \omega(\pi).$$

The very definition of $\omega(\pi)$ shows that $\gamma_{n,1} = 2$ and $\omega(\pi)$ is either 0 or 2. Below we identify $[p]$ with $\mathbb{Z}_p$ regarded as points on a circle. Let $p \geqslant 2$, then a *maximal arc* in a subset $B \subset \mathbb{Z}_p$ is a maximal cyclic interval $I \subseteq \mathbb{Z}_p$ contained in B.

Any subset $B \subset \mathbb{Z}_p$ is a union of maximal arcs of B in $\mathbb{Z}_p$ (see Fig. 6). This notion is important since for $\pi = (B_1, \dots, B_k) \in \mathcal{OP}(2n)$ the factorization

$$\omega(\pi) = \omega(\pi|_{[2n]\smallsetminus B_k}) \prod_{I:\text{maximal arc of } B_k \text{ in } \mathbb{Z}_{2n}} \psi(a^{\#I}) \tag{7.7}$$

holds. Observe from the repeated use of (7.7) that $\omega(\pi) = 2$ if and only if

(a) each maximal arc of B_k in $[2n] \simeq \mathbb{Z}_{2n}$ has even size,
(b) each maximal arc of B_i in $[2n] \smallsetminus (B_{i+1} \cup \dots \cup B_k) \simeq \mathbb{Z}_{2n - \sum_{j=i+1}^{k} \#B_j}$ has even size for all $i = 1, 2, \dots, k-1$.

Note that these conditions imply that all B_i have even size. Moreover, since B_i are not empty, we must have $2 \leqslant \#B_i \leqslant 2n - 2k + 2$ for all $i \in [k]$.

To find a recursive formula for $\gamma_{n,k}$, we count the number $\xi_{n,m}$ of all subsets $B_k \subset [2n]$ satisfying ((a)) and with $\#B_k = 2m$ for each $1 \leqslant k \leqslant n$ and $1 \leqslant m \leqslant n - k + 1$.

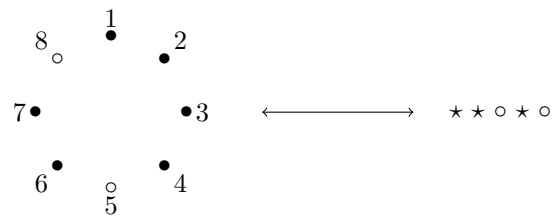

FIGURE 6. $B_k = \{1,2,3,4,6,7\} \subset [8]$ and its unfolded line, cut between 1 and 8. The maximal arcs of B_k in $[8]$ are $\{1,2,3,4\}$ and $\{6,7\}$.

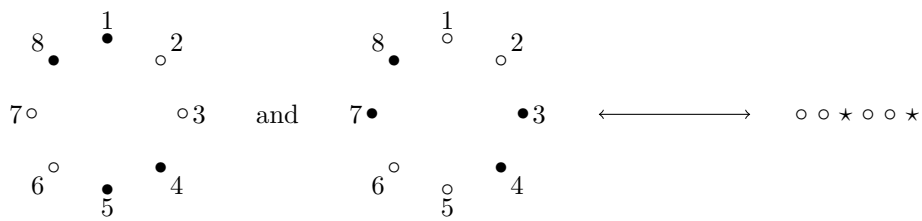

FIGURE 7. $\{1,4,5,8\} \subset [8]$ and its counterclockwise rotation $\{3,4,7,8\}$

As in Fig. 6, two neighboring elements $\bullet$ of B_k can be joined to a single element $\star$ and the elements on the circle can be cut between 1 and $2n$ and be opened to a line, so the problem comes to counting the number of arranging m elements $\star$ and $2n-2m$ elements $\circ$ on one line; however, subsets like $B_k = \{1,4,5,8\}$ do not correspond to such a line arrangement, so we adjust such a case by rotating the circle counterclockwise as in Fig. 7. Therefore, a line arrangement of m elements $\star$ and $2n-2m$ elements $\circ$ corresponds to a single B_k if it ends with $\circ$, while it corresponds to two B_k's if it ends with $\star$. Altogether, we arrive at

$$\xi_{n,m} = \binom{m+2n-2m-1}{m} + 2\binom{m+2n-2m-1}{m-1} = \binom{2n-m}{m}\frac{2n}{2n-m}.$$

From this counting, (7.7) gives the recursive formulas

$$\begin{aligned}\gamma_{n,k} &= \sum_{\substack{B_k\subset[2n]\\ 2\leqslant \#B_k\leqslant 2n-2k+1\\ B_k \text{ satisfies ((a))}}} \gamma_{n-\frac{1}{2}\#B_k,k-1} = \sum_{m=1}^{n-k+1}\xi_{n,m}\gamma_{n-m,k-1}\\ &= \sum_{m=1}^{n-k+1}\binom{2n-m}{m}\frac{2n}{2n-m}\gamma_{n-m,k-1} = \sum_{\ell=k-1}^{n-1}\binom{n+\ell}{n-\ell}\frac{2n}{n+\ell}\gamma_{\ell,k-1}, \quad n\geqslant k\geqslant 2,\end{aligned} \tag{7.8}$$

with $\gamma_{n,1} = 2$ for $n \geqslant 1$. Let

$$\beta_n := \sum_{k=1}^{n}\gamma_{n,k}, \qquad n\geqslant 1; \qquad \beta_0 := 1.$$

Note that β_n is the limiting moment (7.6) of order $p = 2n$. The recursive formula (7.8) implies

$$\beta_n = \sum_{\ell=0}^{n-1} \binom{n+\ell}{n-\ell} \frac{2n}{n+\ell} \beta_\ell, \qquad n \geqslant 1, \tag{7.9}$$

and thus β_n satisfies the recurrence in [22, Theorem 6] and hence coincides with the $2n$-th moment of λ. Some examples are: $\{\beta_n\}_{n=1}^7 = \{2, 10, 80, 874, 12092, 202384, 3973580\}$.

It then suffices to show the determinacy of the moment problem to conclude the weak convergence. According to Lemma 7.14 below, one easily sees Carleman's condition

$$\sum_{n \geqslant 1} \beta_n^{-\frac{1}{2n}} = \infty,$$

which shows that the moment problem for (7.6) is determinate. □

Lemma 7.14. *Let $\{\beta_n\}_{n\geqslant 0}$ be the sequence defined by* (7.9) *with $\beta_0 = 1$. There exists $C > 0$ such that the inequality $\beta_n \leqslant (Cn)^{2n}$ holds for all $n \geqslant 1$.*

Remark 7.15. The proof below shows that $C = 11$ suffices. Moreover, according to OEIS A048286, a more precise asymptotics $\beta_n \sim c(2/(e\log 2))^n n^{n+\frac{1}{2}-\frac{\log 2}{4}}$ holds, where $c = 1.6463....$

Proof. We proceed by induction using the recursion formula (7.9) and Stirling's formula

$$\sqrt{2\pi} n^{n+1/2} e^{-n} \leqslant n! \leqslant e n^{n+1/2} e^{-n}, \qquad n \geqslant 1.$$

Let $n \geqslant 2$. We adopt the notation $0^0 = 1$. Assuming the desired inequality holds until $n-1$ for some constant $C > 1$, one has

$$\begin{aligned}\beta_n &= 2 + \sum_{\ell=1}^{n-1} \binom{n+\ell}{n-\ell} \frac{2n}{n+\ell} \beta_\ell \leqslant 2 + 2\sum_{\ell=1}^{n-1} \binom{n+\ell}{n-\ell} (C\ell)^{2\ell} \\ &\leqslant 2 + \frac{e}{\pi\sqrt{2}} \sum_{\ell=1}^{n-1} \frac{(n+\ell)^{n+\ell+1/2}}{\sqrt{\ell} 2^{2\ell} (n-\ell)^{n-\ell+1/2}} C^{2\ell} \leqslant 2 + \frac{e}{\pi\sqrt{2}} \sum_{\ell=1}^{n-1} \frac{(n+\ell)^{n+\ell+1/2}}{2^{2\ell}(n-\ell)^{n-\ell+1/2}} C^{2\ell}.\end{aligned}$$

We split the sum into the two parts $1 \leqslant \ell \leqslant n-4$ and $n-3 \leqslant \ell \leqslant n-1$ (the arguments below are valid even for $2 \leqslant n \leqslant 4$ by setting the irrelevant terms to be 0). The first part is estimated as

$$\begin{aligned}\sum_{\ell=1}^{n-4} \frac{(n+\ell)^{n+\ell+1/2}}{2^{2\ell}(n-\ell)^{n-\ell+1/2}} C^{2\ell} &\leqslant \sum_{\ell=1}^{n-4} \frac{(2n)^{n+n-4+1/2}}{2^{2\ell} 4^{n-\ell}} C^{2\ell} \leqslant \frac{1}{n} \sum_{\ell=1}^{n-4} n^{2n} C^{2\ell} \\ &\leqslant \frac{n-4}{nC^8} n^{2n} C^{2n} \leqslant \frac{1}{C^2} n^{2n} C^{2n},\end{aligned}$$

and the second part is estimated as

$$\sum_{\ell=n-3}^{n-1} \frac{(n+\ell)^{n+\ell+1/2}}{2^{2\ell}(n-\ell)^{n-\ell+1/2}} C^{2\ell} \leqslant \frac{3}{C^2} \frac{(2n)^{2n} C^{2n}}{2^{2n-6}} = \frac{3 \cdot 2^6}{C^2} n^{2n} C^{2n}.$$

Obviously, $2 \leqslant \frac{1}{C^2} \frac{e}{\pi\sqrt{2}} n^{2n} C^{2n}$. Putting everything together, we obtain

$$\beta_n \leqslant \frac{194e}{C^2 \pi\sqrt{2}} (Cn)^{2n}.$$

By taking $C > 1$ such that $C^2 \geqslant 194 \frac{e}{\pi\sqrt{2}}$ ($C = 11$ suffices) we obtain $\beta_n \leqslant (Cn)^{2n}$. □

Acknowledgements. We thank the anonymous referees for careful readings and extensive comments which helped to improve the paper.

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