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Almost all wreath product character values are divisible by given primes

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Abstract For a finite group G with integer-valued character table and a prime p, we show that almost every entry in the character table of $G \wr S_N$ is divisible by p as $N \to \infty$. This result generalizes the work of Peluse and Soundararajan on the character table of S_N.

1. Introduction

Let S_N be the symmetric group on N letters. The complex irreducible characters of S_N were calculated by Frobenius in 1900; in particular, Frobenius showed that the characters are integer-valued [1]. In 2019, Alex Miller investigated the distribution of the parity of entries of the character table of S_N. He made the remarkable conjecture that for any prime p and exponent $\ell \geqslant 1$, the proportion of entries of the character table of S_N divisible by p (and later p^ℓ for $\ell \geqslant 1$) tends to 1 as $N \to \infty$ [7, 6]. This conjecture was recently proved by Peluse and Soundararajan in the case $\ell = 1$ in [8].

This leaves the question of investigating the distribution of residues modulo p for more general finite groups with integer-valued character tables. For a fixed group G with integer-valued character table, a natural infinite family of such is the wreath product $G \wr S_N$ as $N \to \infty$. When G has integer-valued character table, it is known that the characters of $G \wr S_N$ are also integer-valued [4, Corollary 4.4.11]. These families of wreath products include the Weyl group of type B_N, when $G = \mathbb{Z}/2\mathbb{Z}$, and wreath products $S_M \wr S_N$ of symmetric groups.

Our main result is a generalization of Peluse and Soundararajan's theorem:

Theorem (see Theorem 3.9 below). *Let G be a group with integer-valued character table and let $G \wr S_N$ be the wreath product of G with S_N. For all primes p, the proportion of entries in the character table of $G \wr S_N$ which are divisible by p tends to 1 as $N \to \infty$.*

The proof relies on the combinatorics of the representations of $G \wr S_N$. If G has k conjugacy classes, then conjugacy classes and representations of $G \wr S_N$ are both naturally labelled by k-multipartitions of N. One of the key inputs is characterizing when two elements of $G \wr S_N$ have columns in the character table congruent modulo p. In Lemma 3.2, we give a combinatorial characterization directly generalizing the corresponding criterion for S_N.

It is known that the character tables of all Weyl groups are integer-valued. The Weyl groups of type A are the symmetric groups, where our question was answered by Peluse and Soundararajan. The Weyl groups of type B_N and C_N are both equal to

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$\mathbb{Z}/2\mathbb{Z} \wr S_N$, handled by our main theorem. The only remaining infinite family of Weyl groups is that of type D. In Section 4, we also show that the proportion of character values of the Weyl group of type D_N divisible by a prime p tends to 1 as $N \to \infty$.

2. Preliminaries

2.1. Representation Theory of the Wreath Product. Let G be a finite group with integer-valued character table and let S_N be the symmetric group on N letters.

Definition 2.1. *The* wreath product of G with S_N, *denoted* $G \wr S_N$, *is the group of* $N \times N$ *permutation matrices with nonzero entries in* G.

We begin by recalling the representation theory of $G \wr S_N$. The representation theory of wreath products was first studied in Specht's dissertation [11], anticipated by Young's work on the case $G = \mathbb{Z}/2\mathbb{Z}$ [12]; see also [4, 13] for more modern treatments. If we take the representation theory of G as input data and let N vary, the representation theory has structural similarities to the representation theory of S_N, the case when $G = 1$. While representations and conjugacy classes of the symmetric group are labelled by partitions of N, representations and conjugacy classes of the wreath product are labelled by multipartitions:

Definition 2.2. *A* k-multipartition *of an integer* N *is* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *where* λ_i *is a partition for all* i *such that* $\sum_{i=1}^{k} |\lambda_i| = N$.

We now describe how to label conjugacy classes of $G \wr S_N$. For an element of $G \wr S_N$, assign to each cycle in its projection to S_N a conjugacy class of G, called the *cycle product*, as follows: if $(i_1 i_2 \cdots i_m)$ is a cycle of σ, then the cycle product of $(g_1, g_2, \ldots, g_N)\sigma \in G \wr S_N$ corresponding to $(i_1 i_2 \cdots i_m)$ is defined to be the conjugacy class of $g_{i_m} \cdots g_{i_2} g_{i_1}$ [4, (4.2.1)].

Proposition 2.3 ([4, Theorem 4.2.8]). *If G has k conjugacy classes, then the conjugacy classes of $G \wr S_N$ are indexed by k-multipartitions of N. Given $x \in G \wr S_N$, the multipartition λ corresponding to x is formed as follows: for each cycle in x of length ℓ, if the corresponding cycle product is the ith conjugacy class of G, then add a part of size ℓ to λ_i.*

One can check the assignment of a conjugacy class to a multipartition is well-defined by checking under conjugation by S_N and by diagonal matrices $G^N \subseteq G \wr S_N$. Conjugating an element of $G \wr S_N$ by S_N does not change the set of cycle products at all. If $(g_1, \ldots, g_N) \in G^N$ and $(12 \cdots N)$ is an N-cycle, the conjugate of $(12 \cdots N)(g_1, g_2, \ldots, g_N)$ by $(g, 1, \ldots, 1)$ is $(12 \cdots N)(g_1 g^{-1}, g_2, \ldots, g g_N)$; these two elements have conjugate cycle products $g_N \cdots g_2 g_1$ and $g(g_N \cdots g_2 g_1)g^{-1}$. The general case of conjugation by G^N reduces to the above case. We will not need to use the specific form of this bijection in this paper; it is used in the proofs of character formulas in Propositions 2.4 and 2.11, which we omit.

To find the complex irreducible representations of $G \wr S_N$, we need the complex irreducible representations of G as input; call the irreducible G-representations $V_1, \ldots, V_k$.

Proposition 2.4 ([4, Theorem 4.4.3]). *If G has k conjugacy classes, then the irreducible representations of $G \wr S_N$ are in bijection with k-multipartitions of N. For $\lambda = (\lambda_1, \ldots, \lambda_k)$ a k-multipartition of N, let $a_i = |\lambda_i|$ and $G_a = G \wr S_a$. Then the irreducible representation of $G \wr S_N$ corresponding to λ is*

$$V^\lambda = \operatorname{Ind}_{G_{a_1} \times \cdots \times G_{a_k}}^{G_N} \left(\boxtimes_{i=1}^{k} \left(S^{\lambda_i} \otimes V_i^{\otimes a_i} \right) \right)$$

where S^{λ_i} is the Specht module for S_N corresponding to λ_i.

Character values of wreath products can be calculated using a modified version of the Murnaghan–Nakayama rule for the symmetric group. Let χ^λ be the character of V^λ and χ^λ_μ be the value of χ^λ on the conjugacy class corresponding to μ. Then χ^λ_μ is calculated by decomposing the Young diagrams of the partitions λ_i for all i using rimhooks:

Definition 2.5. *A* rimhook *of a k-multipartition $\lambda = (\lambda_1, \dots, \lambda_k)$ consists of k adjacent boxes in the Young diagram of a single λ_i such that no other boxes are remaining south or east after the rimhook has been removed and no box in the rimhook has a southeast neighbor.*

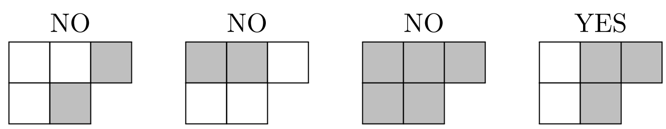


Figure 1. Examples of three invalid and one valid rimhooks in $\lambda = ((3^1 2^1))$.

Definition 2.6. *For k-multipartitions λ and μ, a* rimhook decomposition *of λ by μ is obtained by repeatedly removing rimhooks in λ according to a fixed ordering of the parts of μ. Further, we define $RHD(\lambda, \mu)$ to consist of all rimhook decompositions of λ by μ.*

As in the Murnaghan–Nakayama rule for the usual symmetric group, formulas for irreducible characters involve the height of a rimhook decomposition:

Definition 2.7. *The* height *of a rimhook is one less than the number of rows included in that rimhook. The* height *of a rimhook decomposition ρ, denoted $ht(\rho)$, is the sum of the heights of all rimhooks in the decomposition.*

The Murnaghan–Nakayama rule can be modified for wreath products as follows:

Proposition 2.8 ([4, Theorem 4.4.10]). *Let λ and μ be k-multipartitions of N. Let $\chi^1, \chi^2, \dots, \chi^k$ be the irreducible characters of G and $c_1, c_2 \dots, c_k$ the conjugacy classes of G. For $\rho \in RHD(\lambda, \mu)$, let $\psi(\rho)$ be defined by*

$$\psi(\rho) = \prod_{i=1}^{k} \prod_{j=1}^{k} \left(\chi^i(c_j)^{\#\{rimhooks\ h \in \rho\ from\ \mu_j\ in\ \lambda_i\}} \right).$$

Then

$$\chi^\lambda_\mu = \sum_{\rho \in RHD(\lambda,\mu)} (-1)^{ht(\rho)} \psi(\rho).$$

The permutation module characters of wreath products form another basis for the space of class functions of $G \wr S_N$ that is easier to work with.

Definition 2.9. *Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a k-multipartition of N and let $a_i = |\lambda_i|$. For each λ_i, let S_{λ_i} be the Young subgroup of S_{a_i} corresponding to λ_i and let $G_{\lambda_i} = G \wr S_{\lambda_i}$. Then the* permutation module *M^λ for $G \wr S_N$ is defined by*

$$M^\lambda = \mathrm{Ind}_{G_{\lambda_1} \times \cdots \times G_{\lambda_k}}^{G \wr S_N} \left(\boxtimes_{i=1}^{k} V_i^{\otimes a_i} \right).$$

There is a character formula for M^λ using row decompositions instead of rimhook decompositions. It is as follows:

Definition 2.10. *Let λ and μ be k-multipartitions of N. A* row decomposition *of λ by μ is a function $\rho : \{\text{rows of } \mu\} \to \{\text{rows of } \lambda\}$ such that if r is a row of λ, then the rows in $\rho^{-1}(r)$ have the same total length as r. The set of all row decompositions of λ by μ is denoted $RD(\lambda, \mu)$.*

We will think of row decompositions of λ by μ as a tiling of the Young diagrams of λ by rows, where rows of μ are placed in a fixed ordering.

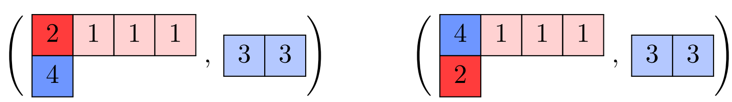


Figure 2. All valid row decompositions of (⊞□□, □□) by (31, 21). The numbers in the boxes indicate the order in which parts of μ are placed into rows of λ, with fixed right-to-left placement.

Proposition 2.11. *Let λ and μ be k-multipartitions of N. Let $\chi^1, \chi^2, \dots, \chi^k$ be the irreducible characters of G and $c_1, \dots, c_k$ the conjugacy classes of G. For $\rho \in RD(\lambda, \mu)$, let $\alpha(\rho)$ be defined by*

$$\alpha(\rho) = \prod_{i=1}^{k} \prod_{j=1}^{k} \left(\chi^i(c_j)^{\#\{\text{rows from } \mu_j \text{ placed into } \lambda_i \text{ by } \rho\}} \right).$$

Then the character for permutation module M^λ at μ is

$$M^\lambda_\mu = \sum_{\rho \in RD(\lambda,\mu)} \alpha(\rho).$$

The proof follows from the character formula for induced representations.

We now describe the change-of-basis between irreducible and permutation characters.

Definition 2.12. *The* dominance order *on partitions is defined by $\lambda \succcurlyeq \eta$ if the row lengths $\lambda^1 \geqslant \lambda^2 \geqslant \dots$ and $\eta^1 \geqslant \eta^2 \geqslant \dots$ of λ and η satisfy $\sum_{i=1}^{j} \lambda^i \geqslant \sum_{i=1}^{j} \eta^i$ for all $j \geqslant 1$. The dominance order on k-multipartitions is defined by $\lambda \succcurlyeq \eta$ if and only if $\lambda_i \succcurlyeq \eta_i$ for all i.*

Lemma 2.13. *The matrix of multiplicities $[M^\lambda : V^\eta]$ of the irreducible representations of $G \wr S_N$ in permutation modules is unimodular and lower-triangular with respect to dominance order.*

Proof. Recall that the Kostka numbers $K^{\beta,\gamma}$ for β, γ partitions of N are defined by

$$M^\beta = \operatorname{Ind}_{S_\beta}^{S_N} 1 = \bigoplus_\gamma (V^\gamma)^{\oplus K^{\gamma,\beta}},$$

where S_β is the Young subgroup corresponding to β and V^γ is the Specht module corresponding to γ. Note that our notation for M^β and V^γ agrees with that of wreath products $G \wr S_N$ when $G = 1$. The Kostka numbers satisfy $K^{\beta,\beta} = 1$ and $K^{\gamma,\beta} > 0$ if and only if $\gamma \succcurlyeq \beta$ in dominance order [5, I, (6.5)].

We claim that

$$M^\lambda = \bigoplus_\eta (V^\eta)^{\oplus c(\lambda,\eta)}, \qquad c(\lambda, \eta) = \left(\prod_{i=1}^{k} K^{\eta_i, \lambda_i} \right). \tag{1}$$

By Definition 2.9, if $a_i = |\lambda_i|$ for all i and $H = G_{a_1} \times G_{a_2} \times \dots \times G_{a_k}$, then

$$M^\lambda = \operatorname{Ind}_{G_\lambda}^{G_N} \left(\boxtimes_{i=1}^{k} V_i^{\otimes a_i} \right) = \operatorname{Ind}_{H}^{G_N} \left(\boxtimes_{i=1}^{k} M^{\lambda_i} \otimes V_i^{\otimes a_i} \right),$$

where S_{a_i} acts diagonally on the tensor product $M^{\lambda_i} \otimes V_i^{\otimes a_i}$ and G^{a_i} acts naturally on $V_i^{\otimes a_i}$. Then (1) follows from multilinearity of the tensor product and linearity of induction.

Now since the matrix of Kostka numbers is unimodular and upper-triangular with respect to dominance order, the matrix $\{c(\lambda, \mu)\}_{\lambda,\mu}$ is unimodular and lower-triangular with respect to dominance order. ☐

2.2. Asymptotics of Partitions. We recall a form of the Hardy–Ramanujan asymptotic for the number of partitions of N, denoted $p(N)$.

Proposition 2.14 ([3, (1.36)]). *If $\delta > 0$, then*

$$\left(\frac{2\pi}{\sqrt{6}} - \delta\right)\sqrt{N} \leqslant \log p(N) \leqslant \left(\frac{2\pi}{\sqrt{6}} + \delta\right)\sqrt{N}$$

for sufficiently large N.

Let $p_k(N)$ denote the number of k-multipartitions of N.

Claim 2.15. *If $\delta > 0$, then*

$$\left(\frac{2\pi}{\sqrt{6}} - \delta\right)\sqrt{kN} \leqslant \log p_k(N) \leqslant \left(\frac{2\pi}{\sqrt{6}} + \delta\right)\sqrt{kN}$$

for sufficiently large N.

This formula also appears in [9]. We provide an elementary inductive proof.

Proof. We proceed by induction on k. The base case $k = 1$ is Proposition 2.14.

For $\delta > 0$, let $\delta' = \frac{4}{5}\delta$. By inductive hypothesis, there exists a constant B such that if $C \geqslant B$, then

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right)\left(\sqrt{(k-1)C}\right)\right) \leqslant p_{k-1}(C) \leqslant \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\left(\sqrt{(k-1)C}\right)\right)$$

and

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right)\left(\sqrt{C}\right)\right) \leqslant p(C) \leqslant \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\left(\sqrt{C}\right)\right).$$

By considering the size of the first partition in a k-multipartition, it follows that

$$p_k(N) = \sum_{a=0}^{N} p(a)p_{k-1}(N-a).$$

Now assume that $N \geqslant 2B - 2$; we can then break up the sum for $p_k(N)$ into the following distinct parts: let

$$D_1 = \sum_{a=0}^{B-1} p(a)p_{k-1}(N-a),$$

$$D_2 = \sum_{a=B}^{N-B} p(a)p_{k-1}(N-a),$$

$$D_3 = \sum_{a=N-B+1}^{N} p(a)p_{k-1}(N-a).$$

In D_2, for $B \leqslant a \leqslant N - B$, we have

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right)\left(\sqrt{a} + \sqrt{(k-1)(N-a)}\right)\right) \leqslant p(a)p_{k-1}(N-a),$$

and the right hand side in turn satisfies

$$p(a)p_{k-1}(N-a) \leqslant \exp\left(\left(\frac{2\pi}{\sqrt{6}}+\delta'\right)\left(\sqrt{a}+\sqrt{(k-1)(N-a)}\right)\right).$$

Note that $\sqrt{a}+\sqrt{(k-1)(N-a)} \leqslant \sqrt{kN}$, with equality achieved at $a=\frac{N}{k}$. Summing over $a \in [B, N-B]$, we get

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}}-\delta\right)\sqrt{kN}\right) \leqslant D_2 \leqslant (N-2B+1)\exp\left(\left(\frac{2\pi}{\sqrt{6}}+\delta'\right)\sqrt{kN}\right). \tag{2}$$

We now consider D_1 and D_3. Note that for $a \in [0, B)$, we have $p(a)p_{k-1}(N-a) \leqslant p(B)p_{k-1}(N)$, and for $a \in (N-B, N]$, we have $p(a)p_{k-1}(N-a) \leqslant p(N)p_{k-1}(B)$. Hence

$$0 \leqslant D_1 \leqslant Bp(B)p_{k-1}(N) \leqslant B\exp\left(\left(\frac{2\pi}{\sqrt{6}}+\delta'\right)\sqrt{kN}\right) \tag{3}$$

for sufficiently large N. Likewise,

$$0 \leqslant D_3 \leqslant Bp(N)p_{k-1}(B) \leqslant B\exp\left(\left(\frac{2\pi}{\sqrt{6}}+\delta'\right)\sqrt{kN}\right). \tag{4}$$

Combining (2), (3), and (4), we have that

$$\begin{aligned}\exp\left(\left(\frac{2\pi}{\sqrt{6}}-\delta\right)\sqrt{kN}\right) \leqslant p_k(N) &\leqslant (N+1)\exp\left(\left(\frac{2\pi}{\sqrt{6}}+\delta'\right)\sqrt{kN}\right)\\ &\leqslant \exp\left(\left(\frac{2\pi}{\sqrt{6}}+\delta\right)\sqrt{kN}\right)\end{aligned}$$

for sufficiently large N. □

The above estimate implies k-multipartitions concentrate around having close to equal-size parts:

Corollary 2.16. *For all $\delta > 0$, the proportion of k-multipartitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash N$ such that*

$$\frac{N}{k}(1-\delta) < |\lambda_i| < \frac{N}{k}(1+\delta)$$

for all $1 \leqslant i \leqslant k$ goes to 1 as $N \to \infty$.

Proof. Pick $1 \leqslant i \leqslant k$. By partitioning k-multipartitions according to the size of λ_i, the number of k-multipartitions of N where $|\lambda_i| \notin \left(\frac{N}{k}(1-\epsilon), \frac{N}{k}(1+\epsilon)\right)$ is

$$\sum_{a \notin \left(\frac{N}{k}(1-\varepsilon), \frac{N}{k}(1+\varepsilon)\right)} p(a)p_{k-1}(N-a). \tag{5}$$

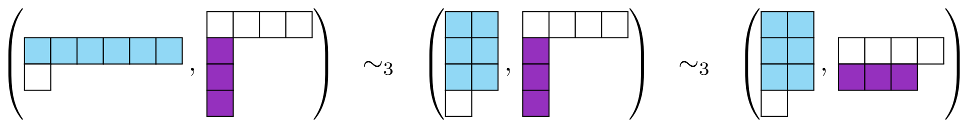
Note that $\sqrt{a}+\sqrt{(k-1)(N-a)}$ increases for $a < \frac{N}{k}$ and decreases for $a > \frac{N}{k}$. Then by Claim 2.15, the rate of growth (5) is significantly slower than the rate of growth of $p_k(N)$. □

3. Main Results

3.1. Character Table Column Congruences. Corollary 3.3 below, which we call "the mashing rule," gives a criterion for mod p congruence of two columns of the character table of $G \wr S_N$ in terms of k-multipartitions.

In this section, we must assume that G has integer-valued character table. By [10, §13.1], the group G has integer-valued character table if and only if $\sigma \in G$ is conjugate to σ^j whenever j is prime to the order of σ.

Definition 3.1. *Let $\sim_p$ be the equivalence relation on k-multipartitions generated by the following: $\mu \sim_p \nu$ if there is j such that $\mu_i = \nu_i$ for $i \neq j$, and ν_j is formed by replacing one part of size mp in μ_j with p parts of size m in ν_j.*



$(61, 41^3)$ $\sim_3$ $(2^3 1, 41^3)$ $\sim_3$ $(2^3 1, 43)$

Figure 3. Example of three conjugacy classes which are congruent mod 3 in $\mathbb{Z}/2\mathbb{Z} \wr S_N$ (note that $m = 2$ in the first cycle type and $m = 1$ in the second).

Lemma 3.2. *Let p be a prime and G be a group with integer-valued character table. Let $\mu = (\mu_1, \ldots, \mu_k)$ and $\nu = (\nu_1, \ldots, \nu_k)$ be k-multipartitions of N, indexing conjugacy classes of $G \wr S_N$. If $\mu \sim_p \nu$, then $M^\lambda_\mu \equiv M^\lambda_\nu \pmod p$ for all k-multipartitions λ of N.*

Proof. It suffices to show $M^\lambda_\mu \equiv M^\lambda_\nu \pmod p$ if there exists j such that $\mu_i = \nu_i$ for all $i \neq j$, $\mu_j = (\xi, mp)$ for some ξ, and $\nu_j = (\xi, m^p)$. We break $RD(\lambda, \nu)$ into two cases. In case one, we consider the row decompositions of ν where the p rows not in ξ are tiled into the same row of λ. In case two we consider the row decompositions where the p rows not in ξ are not tiled in the same row. Recalling our formula for characters of permutation modules in Proposition 2.11, let

$$\beta = \sum_{\substack{\rho \in RD(\lambda,\nu) \text{ s.t. } m^p \text{ is tiled} \\ \text{in the same row}}} \alpha(\rho) \tag{6}$$

and

$$\gamma = \sum_{\substack{\rho \in RD(\lambda,\nu) \text{ s.t. } m^p \text{ is not tiled} \\ \text{in the same row}}} \alpha(\rho), \tag{7}$$

so that $M^\lambda_\nu = \beta + \gamma$. In case one, we will show $\beta \equiv M^\lambda_\mu \pmod p$ and in case two that $\gamma \equiv 0 \pmod p$. Together, these two congruences imply $M^\lambda_\mu \equiv M^\lambda_\nu \pmod p$.

In both cases, we break into subcases based on the ways to tile μ_i for $i \neq j$ and ξ. In case one, we have compatible tilings for μ and ν, and in case two, we have additional tilings for ν.

In case one, assume we have tiled all rows of μ_i for all $i \neq j$ and we have tiled ξ. We now have one row remaining. There is only one way to tile the last row for both μ and ν: put the remaining pieces into the remaining row. Let these row decompositions be denoted ρ_μ and ρ_ν respectively.

For ρ_μ, say that we place the final row r of size mp in the partition λ_q. The associated cycle product is c_j because mp comes from μ_j. Then mp contributes $\chi_q(c_j)$ to the product $\alpha(\rho_\mu)$. Then for ρ_ν, the p rows of size m are placed into λ_q. The conjugacy class of G associated with the p rows of size m is again c_j, so m^p contributes a factor of $\chi_q(c_j)^p$ to $\alpha(\rho_\nu)$.

By assumption, the character values of G are integral, so by Fermat's little theorem, $\chi_q(c_j) \equiv \chi_q(c_j)^p \pmod p$. All other factors in $\alpha(\rho_\mu)$ contributed by μ_i for $i \neq j$ and ξ are identical to the corresponding factors in $\alpha(\rho_\nu)$. Hence, $\alpha(\rho_\mu) \equiv \alpha(\rho_\nu) \pmod p$.

Summing over all the tilings in case one, we find $M^\lambda_\mu \equiv \beta \pmod p$.

In case two, assume we have tiled all rows of μ_i for $i \neq j$ and ξ, after which there are $t > 1$ remaining unfilled rows of the Young diagrams of λ. If $T \subseteq RD(\lambda, \nu)$ is the set of row decompositions extending our given tiling by μ_i for $i \neq j$ and ξ, then we will show

$$\sum_{\rho \in T} \alpha(\rho) \equiv 0 \pmod{p}.$$

Then γ is the sum over all such T of $\sum_{\rho \in T} \alpha(\rho)$, from which it will follow $\gamma \equiv 0$ mod p.

Call the lengths of the remaining rows $(m\ell_1, m\ell_2, \ldots, m\ell_t)$. Since the elements of T are in bijection with choices of placements of p cycles of length m into these rows,

$$|T| = \binom{p}{\ell_1, \ell_2, \ldots, \ell_t}. \tag{8}$$

Let $\rho \in T$. Note that all pieces of m^p come from μ_j, and thus have cycle product c_j, while all other cycles in μ are in the same place in T. Thus $\alpha(\rho) = \alpha(\rho')$ for all $\rho, \rho' \in T$. Hence $\sum_{\rho \in T} \alpha(\rho)$ is a sum of $|T|$ identical terms. Then $\sum_{\rho \in T} \alpha(\rho) \equiv 0 \pmod{p}$ because $|T|$ is divisible by p.

Case one has shown that $M_\mu^\lambda \equiv \beta \pmod{p}$, and case two has shown that $\gamma \equiv 0 \pmod{p}$. Since $M_\nu^\lambda = \beta + \gamma$, we conclude $M_\mu^\lambda \equiv M_\nu^\lambda \pmod{p}$. □

Corollary 3.3 (The mashing rule). *Let G have integer-valued character table and k conjugacy classes. Let μ and ν be k-multipartitions of N. If $\mu \sim_p \nu$, then $\chi_\mu^\lambda \equiv \chi_\nu^\lambda \pmod{p}$ for all irreducible characters χ^λ of $G \wr S_N$.*

Proof. The set of irreducible characters and the set of characters of permutation modules form bases for the space of class functions on $G \wr S_N$. Since the change of basis matrix between these two bases is unimodular and lower-triangular, as stated in Lemma 2.13, χ^λ can be expressed as an integral linear combination of M^η for all k-multipartitions λ. It follows from Lemma 3.2 that $\mu \sim_p \nu$ implies $\chi_\mu^\lambda \equiv \chi_\nu^\lambda \pmod{p}$. □

Remark 3.4. Corollary 3.3 may also be proved using the following criterion from modular representation theory: if G has integer-valued character table, $g \in G$, and g' is the p-prime part of g, then $\chi(g) \equiv \chi(g') \mod p$ for all characters χ of G [10, 18.1(v)]. In this situation, it may be seen that the mashing preserves the conjugacy class of the p-prime part of a class in $G \wr S_N$. Nonetheless, we have chosen to give a combinatorial proof.

3.2. Proof of Main Theorem. Using Corollary 3.3, the existence of one zero in the character table implies many more entries are divisible by p. We proceed, following Peluse and Soundararajan in [8], by using Proposition 2.8 to show sufficiently many entries of the character table are zero.

Definition 3.5. *A partition is called a* t-core *if none of the hook lengths of its Young diagram are divisible by t where $t \in \mathbb{Z}$. For example, from Figure 4 one can see that $(4, 2, 1)$ is a 5-core.*

6	4	2	1
3	1		
1			

Figure 4. Hook-lengths for $\lambda_i = (4, 2, 1)$

In the course of proving [8, Proposition 1], Peluse and Soundararajan proved the following estimate of the number of t-cores when t is slightly larger than the typical longest cycle in a random conjugacy class:

Proposition 3.6. *Let L be a positive integer, and let A be a real number with $1 \leqslant A \leqslant \log L/\log\log L$. Additionally suppose that t is a positive integer with*

$$t \geqslant \frac{\sqrt{6}}{2\pi}\sqrt{L}(\log L)\left(1+\frac{1}{A}\right). \tag{9}$$

Then the number of partitions λ of L which are not t-cores is at most

$$O\left(p(L)\frac{\log L}{L^{\frac{1}{2A}}}\right),$$

independent of t satisfying (9)*.*

Complementing the estimate in Proposition 3.6, Peluse and Soundararajan also estimated how many columns of the character table are congruent to a column corresponding to a partition with a large first part:

Proposition 3.7 ([8, Proposition 2]). *Let $p \leqslant \frac{(\log L)}{(\log\log L)^2}$ be a prime. Starting with a partition μ of L, we repeatedly replace every occurrence of p parts of the same size m by one part of size mp until we arrive at a partition $\tilde{\mu}$ where no part appears more than $p-1$ times. Then the largest part of $\tilde{\mu}$ exceeds*

$$\frac{\sqrt{6}}{2\pi}\sqrt{L}\,(\log L)\left(1+\frac{1}{5p}\right),$$

except for at most

$$O\left(p(L)\exp\left(-L^{\frac{1}{15p}}\right)\right)$$

partitions μ.

We now extend Peluse and Soundarajan's estimate in Proposition 3.7 to k-multipartitions.

Proposition 3.8. *Let $p \ll N$ be a prime. Given a k-multipartition $\mu = (\mu_1, \dots \mu_k)$ of N, for all μ_i with $1 \leqslant i \leqslant k$, we repeatedly replace every occurrence of p parts of the same size m by one part of size mp until we arrive at a k-multipartition $\tilde{\mu}$ where no part in any $\tilde{\mu}_i$ appears more than $p-1$ times.*

Then the largest part of $\tilde{\mu}$ is of size at least

$$\frac{\sqrt{6}}{2\pi}\sqrt{\frac{N}{k}}\left(\log\frac{N}{k}\right)\left(1+\frac{1}{5p}\right) \tag{10}$$

except for a number of multipartitions μ which is at most

$$O\left(\exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right)p_k(N)\right).$$

Proof. For a k-multipartition $\mu = (\mu_1, \mu_2, \dots \mu_k)$ of N, let $\tilde{\mu}$ be as above. We will bound above the number of k-multipartitions μ such that $\tilde{\mu}$ has largest part less than (10).

For any μ, we know that for some $1 \leqslant i \leqslant k$, $|\mu_i| \geqslant \frac{N}{k}$. Fix i such that μ_i has size $|\mu_i| = a \geqslant \frac{N}{k}$. Then Proposition 3.7 tells us that the largest part of $\tilde{\mu}_i$ exceeds

$$\frac{\sqrt{6}}{2\pi}\sqrt{a}\,(\log a)\left(1+\frac{1}{5p}\right) \geqslant \frac{\sqrt{6}}{2\pi}\sqrt{\frac{N}{k}}\left(\log\frac{N}{k}\right)\left(1+\frac{1}{5p}\right)$$

except for at most
$$O\left(p(a)\exp\left(-a^{\frac{1}{15p}}\right)\right)$$
partitions μ_i of size a and therefore at most
$$O\left(p(a)\exp\left(-a^{\frac{1}{15p}}\right)p_{k-1}(N-a)\right)$$
total k-multipartitions μ with $|\mu_i| = a$. Furthermore, since $a \geqslant \frac{N}{k}$,
$$\exp\left(-a^{\frac{1}{15p}}\right) \leqslant \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right),$$
and therefore summing over all $a \geqslant \frac{N}{k}$ we have that the number of multipartitions μ such that $|\mu_i| \geqslant \frac{N}{k}$ with no part in $\tilde{\mu}_i$ exceeding (10) is at most an absolute constant times

$$\begin{aligned}\sum_{a=\frac{N}{k}}^{N} \exp\left(-a^{\frac{1}{15p}}\right) p(a)p_{k-1}(N-a) &\ll \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) \sum_{a=\frac{N}{k}}^{N} p(a)p_{k-1}(N-a)\\ &\ll \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) \sum_{a=0}^{N} p(a)p_{k-1}(N-a)\\ &= \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) p_k(N).\end{aligned}$$

Since this bound is identical for each i, the number of k-multipartitions μ such that $\tilde{\mu}$ does not have a part of size greater than (10) is at most a factor of k greater than the bound above, and therefore also at most
$$O\left(\exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) p_k(N)\right). \qquad \square$$

Theorem 3.9. *Let G be a group with integer-valued character table, and let $G \wr S_N$ be the wreath product of G with the symmetric group S_N. For all primes p, the proportion of entries in the character table of $G \wr S_N$ divisible by p tends to 1 as $N \to \infty$.*

Proof. Let k be the number of conjugacy classes of G. Given a k-multipartition μ, let $\tilde{\mu}$ be the multipartition obtained by repeatedly replacing p parts of μ_i of size m with one part of size mp until no μ_i has a part appearing more than $p-1$ times. For $A = 5p$, Proposition 3.8 implies that the largest part of $\tilde{\mu}$ has size
$$t \geqslant \frac{\sqrt{6}}{2\pi}\sqrt{\frac{N}{k}}\left(\log\frac{N}{k}\right)\left(1+\frac{1}{A}\right) \tag{11}$$
for a proportion of μ tending to 1 as $N \to \infty$. Now pick $A' \geqslant 1$ and $\delta > 0$ such that
$$\left(\log\frac{N}{k}\right)\left(1+\frac{1}{A}\right) \geqslant \sqrt{1+\delta}\left(\log\left(\frac{N}{k}(1+\delta)\right)\right)\left(1+\frac{1}{A'}\right).$$
By Corollary 2.16, the proportion of k-multipartitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash N$ such that $|\lambda_i| \in \left(\frac{N}{k}(1-\delta), \frac{N}{k}(1+\delta)\right)$ for all i tends to 1 as $N \to \infty$. Thus, consider only (λ, μ) satisfying the above conditions.

Our choice of δ and A' imply that
$$\frac{\sqrt{6}}{2\pi}\sqrt{\frac{N}{k}}\left(\log\frac{N}{k}\right)\left(1+\frac{1}{A}\right) \geqslant \frac{\sqrt{6}}{2\pi}\sqrt{\frac{N(1+\delta)}{k}}\left(\log\left(\frac{N}{k}(1+\delta)\right)\right)\left(1+\frac{1}{A'}\right).$$

If $(N_1,\dots,N_k)$ is a partition of sufficiently large N with $N_i \in \left(\frac{N}{k}(1-\delta), \frac{N}{k}(1+\delta)\right)$, we can assume $A' \leqslant \frac{\log N_i}{\log\log N_i}$ for each i, as $N_i \geqslant \frac{N}{k(1-\delta)} \to \infty$. Then by Proposition 3.6, the proportion of k-multipartitions λ with $|\lambda_i| = N_i$ such that some λ_i is not a t-core is

$$\sum_{i=1}^{k} O\left(\frac{\log N_i}{N_i^{\frac{1}{2A'}}}\right)$$

for all t satisfying (11), independent of t. Hence, over all k-multipartitions λ satisfying $|\lambda_i| \in \left(\frac{N}{k}(1-\delta), \frac{N}{k}(1+\delta)\right)$, the proportion of λ such that some λ_i is not a t-core is

$$O\left(\frac{\log\left(\frac{N}{k}(1+\delta)\right)}{\left(\frac{N}{k}(1-\delta)\right)^{\frac{1}{2A'}}}\right).$$

It follows that most (λ,μ) satisfy that λ_i is a t-core for t the largest part of $\tilde{\mu}$. Thus, for a proportion of (λ,μ) tending to 1 as $N\to\infty$, we have $\chi^\lambda_{\tilde{\mu}} = 0$ by Proposition 2.8 and therefore $\chi^\lambda_\mu \equiv 0 \mod p$ by Corollary 3.3. □

4. Weyl groups of type D

Definition 4.1. *The* Weyl group of type D_N *is the group of $N\times N$ signed permutation matrices with an even number of entries equal to -1.*

We will denote this group by D_N also (note that it is distinct from the dihedral group). The study of representations of D_N was taken up by Young in [12, §7-10]; see [2] for a more modern treatment.

Let $B_N = \{\pm 1\} \wr S_N$; then D_N is a subgroup of B_N of index two. Recall that conjugacy classes of B_N are labelled by 2-multipartitions (η,ν) of N, where we take ν to be those cycles with nontrivial cycle product.

Proposition 4.2 ([2, Proposition 3.4.12]). *The conjugacy classes B_N which meet D_N correspond to 2-multipartitions (η,ν) of N where ν has an even number of parts. The conjugacy classes of B_N which split in D_N are exactly those (η,ν) where $\nu=\varnothing$ and η has only even parts.*

Since D_N is a subgroup of B_N of index two, Clifford theory determines its representations:

Proposition 4.3 ([2, §5.6.1]). *The irreducible representations of the Weyl group of type D_N are as follows:*

(1) *if (λ,μ) is a 2-multipartition of N such that $\lambda\neq\mu$, then*

$$\operatorname{Res}^{B_N}_{D_N} V^{\lambda,\mu} = \operatorname{Res}^{B_N}_{D_N} V^{\mu,\lambda}$$

is an irreducible representation of D_N;

(2) *if (λ,λ) is a 2-multipartition of N with equal parts, then*

$$Res^{B_N}_{D_N} V^{\lambda,\lambda}$$

is the sum of two irreducible representations of D_N.

(3) *Each irreducible representation of D_N appears exactly once in (1) or (2).*

Corollary 4.4. *For all primes p, the proportion of entries in the character table of D_N which are divisible by p tends to 1 as $N\to\infty$.*

Proof. The number of irreducible representations of D_N of the form $\operatorname{Res}^{B_N}_{D_N} V^{\lambda,\mu}$ for $\lambda\neq\mu$ equals $\frac{1}{2}\left(p_2(N) - p(N/2)\right)$ when N is even, and $\frac{1}{2}p_2(N)$ when N is odd. The number of irreducible representations appearing as a summand of $\operatorname{Res}^{B_N}_{D_N} V^{\lambda,\lambda}$

is $2p(N/2)$ when N is even and 0 when N is odd. By Claim 2.15, we have $p_2(N) \gg p(N/2)$ for large enough N, so the proportion of irreducibles of the form $\mathrm{Res}^{B_N}_{D_N} V^{\lambda,\mu}$ goes to 1 as $N \to \infty$.

We must also analyze the splitting of conjugacy classes from B_N in D_N. By Proposition 4.2, the number of conjugacy classes of B_N which split in D_N is $p(N/2)$ when N is even and 0 when N is odd. We conclude just as above that the proportion of non-split conjugacy classes tends to 1 as $N \to \infty$.

We have shown that almost every entry of the character table of D_N is of the form $V^{\lambda,\mu}_{\eta,\nu}$ where $\lambda \neq \mu$ and (η, ν) is a non-split conjugacy class in D_N. Such entries occupy at least a constant fraction of the character table of B_N as $N \to \infty$. By Theorem 3.9, almost all of such entries are divisible by p as $N \to \infty$. Hence, the proportion of entries of the character table of D_N which are divisible by p goes to 1 as $N \to \infty$. ☐

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