ALGEBRAIC COMBINATORICS

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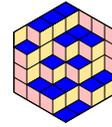
On the WL-dimension of circulant graphs of prime power order

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ABSTRACT The WL-dimension of a graph X is the smallest positive integer m such that the m-dimensional Weisfeiler–Leman algorithm correctly tests the isomorphism between X and any other graph. It is proved that the WL-dimension of any circulant graph of prime power order is at most 3, and this bound cannot be reduced. The proof is based on using theories of coherent configurations and Cayley schemes over a cyclic group.

1. INTRODUCTION

A (not necessarily undirected) graph is said to be *circulant* if it is a Cayley graph over a cyclic group. A main motivation of the present paper is the following computational problem: given a circulant graph $\mathcal{G}$ and a graph $\mathcal{G}'$ test whether or not $\mathcal{G}$ is isomorphic to $\mathcal{G}'$. A polynomial-time algorithm constructed in [7] solves this problem, but is largely based on computational group theory. It is natural to ask if the problem can be solved in a purely combinatorial way.

One of the most famous purely combinatorial methods for testing graph isomorphism goes back to paper [17], where the classical Weisfeiler–Leman algorithm was introduced. The method first uses this algorithm which, for a given graph $\mathcal{G}$, constructs a (in a sense) canonical coloring of all pairs of vertices. Then the resulting coloring is compared with similar coloring constructed for a graph $\mathcal{G}'$; the graphs $\mathcal{G}$ and $\mathcal{G}'$ are declared isomorphic if and only if the colorings have the same sets of colors (this can be checked quite easily). This method tests isomorphism of $\mathcal{G}$ and $\mathcal{G}'$ correctly in many (but not all) cases [15].

A generalization of the above method is obtained if the classical Weisfeiler–Leman algorithm is replaced by its m-dimensional ($m \geqslant 3$) analog, in which the m-tuples of vertices are canonically colored. In this case, for every graph $\mathcal{G}$ there is a minimal m with the following property: any graph $\mathcal{G}'$ such that the set of colors in the canonical coloring of the m-tuples of the vertices of $\mathcal{G}'$ is equal to that for $\mathcal{G}$, is isomorphic to $\mathcal{G}$. This minimal m is called the *Weisfeiler–Leman dimension* (the WL-*dimension*) of $\mathcal{G}$ and is denoted by $\dim_{\mathrm{WL}}(\mathcal{G})$. Now we can refine the question posed in the first paragraph as follows.

QUESTION. *Is it true that there exists $m \in \mathbb{N}$ such that $\dim_{\mathrm{WL}}(\mathcal{G}) \leqslant m$ for every circulant graph $\mathcal{G}$?*

Although we cannot answer this question in full, our main result, presented by the theorem below, says that for circulant graphs of prime power order (i.e. those with the number of vertices equal to the power of a prime), as such a constant one can take 3.

THEOREM 1.1. *The WL-dimension of every circulant graph of prime power order is at most* 3.

Every Paley graph $\mathcal{G}$ of prime order p is circulant. It is also known (see, e.g. [3, Subsec. 4.5]) that for some p, one can find a graph $\mathcal{G}'$ of order p such that $\mathcal{G}$ and $\mathcal{G}'$ are nonisomorphic but have the same sets of colors in the colorings constructed by the classical Weisfeiler–Leman algorithm. It follows that $\dim_{\mathrm{WL}}(\mathcal{G}) > 2$, which shows that the estimate in Theorem 1.1 is sharp. An infinite family of such examples is obtained if one replaces $\mathcal{G}$ and $\mathcal{G}'$ by the disjoint unions $n\mathcal{G}$ and $n\mathcal{G}'$ ($n \geqslant 2$), respectively.

Let us discuss some ideas underlying the proof of Theorem 1.1. The classical Weisfeiler–Leman algorithm can be considered as an (efficiently computable) functor which given a graph $\mathcal{G}$ defines a coherent configuration $\mathcal{X} = \mathrm{WL}(\mathcal{G})$ (the exact definitions are in Sections 2 and 3). It was proved in [10] that $\dim_{\mathrm{WL}}(\mathcal{G}) \leqslant 2$ if and only if the coherent configuration $\mathcal{X}$ is separable, i.e. every algebraic isomorphism from $\mathcal{X}$ to another coherent configuration is induced by isomorphism.

A weakening of the property of a coherent configuration to be separable is to consider not all algebraic isomorphisms, but only those that have one-point extensions. In this way, we arrive at a concept of *sesquiseparable* coherent configuration, which is introduced and studied in Section 4. In fact, if $\mathcal{G}$ is a vertex-transitive graph and the coherent configuration $\mathcal{X}$ is sesquiseparable, then $\dim_{\mathrm{WL}}(\mathcal{G}) \leqslant 3$. We show that $\mathcal{X}$ is sesquiseparable if $\mathcal{G}$ is a circulant graph of prime power order.

Now let $\mathcal{G}$ be a circulant graph. Then the coherent configuration $\mathcal{X} = \mathrm{WL}(\mathcal{G})$ is a circulant scheme (a Cayley scheme over a cyclic group; for details, see Section 6) and its structure is fairly well controlled by the Leung–Man theory (see, for example, [6]). In particular, $\mathcal{X}$ can be constructed from trivial and normal circulant schemes by using tensor and wedge products. The normal circulant schemes and the operation of wedge (or generalized wreath) product of (association) schemes have been introduced and studied in [6] and [14], respectively. It seems that a recent paper [2] opens up a way to formulate the Leung–Man theory directly in terms of graph theory, bypassing association schemes.

When $\mathcal{G}$ is of prime power order, the tensor product is irrelevant and we prove that $\mathcal{X}$ is sesquiseparable by induction on the length of a decomposition of $\mathcal{X}$ into a series of the wedge products (Lemma 6.8). The central part of the proof is focused on studying algebraic and combinatorial isomorphisms of the wedge product and establishing a sufficient condition for the wedge product to be sesquiseparable (see Section 5 and Remark 5.1).

We conclude the introduction with a brief remark about Theorem 6.4 obtained in the course of the proof and which is of independent interest. In fact, this theorem says that Theorem 1.1 holds for much wider class of circulant graphs. Among these graphs are those $\mathcal{G}$ for which the scheme $\mathcal{X}$ is the wedge product of normal circulant schemes (such schemes cover the counterexamples constructed in [5] for $p_2 = p_3$ and $n' = 1$).

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2. PRELIMINARIES

In order to make the paper as self-contained as possible, we present in this section a necessary background of the theory of coherent configurations. Our notation is

compatible with the notation in monograph [3]; the proofs of the statements below can be found there.

2.1. NOTATION. Throughout the paper, Ω denotes a finite set. For $\Delta \subseteq \Omega$, the Cartesian product $\Delta \times \Delta$ and its diagonal are denoted by $\mathbf{1}_\Delta$ and 1_Δ, respectively. If $\Delta = \{\alpha\}$, we abbreviate $1_\alpha := 1_{\{\alpha\}}$. For a relation $s \subseteq \mathbf{1}_\Omega$, we set $s^* = \{(\alpha, \beta) : (\beta, \alpha) \in s\}$, $\alpha s = \{\beta \in \Omega : (\alpha, \beta) \in s\}$ for all $\alpha \in \Omega$, and define $\langle s \rangle$ as the minimal (with respect to inclusion) equivalence relation on Ω, containing s. For any collection S of relations, we denote by $S^\cup$ the set of all unions of elements of S, and consider $S^\cup$ as a poset with respect to inclusion.

The set of classes of an equivalence relation e on Ω is denoted by Ω/e. For $\Delta \subseteq \Omega$, we set $\Delta/e = \Delta/e_\Delta$, where $e_\Delta = \mathbf{1}_\Delta \cap e$. If the classes of e_Δ are singletons, Δ/e is identified with Δ. Given a relation $s \subseteq \mathbf{1}_\Omega$, we put

$$(1) \qquad s_{\Delta/e} = \{(\Gamma, \Gamma') \in \mathbf{1}_{\Delta/e} : s_{\Gamma, \Gamma'} \neq \varnothing\},$$

where $s_{\Gamma, \Gamma'} = s \cap (\Gamma \times \Gamma')$. We also abbreviate $s_\Gamma := s_{\Gamma, \Gamma}$. Among all equivalence relations e on Ω, such that

$$(2) \qquad s = \bigcup_{(\Delta, \Delta') \in s_{\Delta/e}} \Delta \times \Delta',$$

there is the largest (with respect to inclusion) one, which is denoted by $\mathrm{Rad}(s)$ and called the *radical* of s. Obviously, $\mathrm{Rad}(s) \subseteq \langle s \rangle$.

For a set B of bijections $f : \Omega \to \Omega'$, subsets $\Delta \subseteq \Omega$ and $\Delta' \subseteq \Omega'$, equivalence relations e and e' on Ω and Ω', respectively, we put

$$B^{\Delta/e, \Delta'/e'} = \{f^{\Delta/e} : f \in B, \Delta^f = \Delta', e^f = e'\},$$

where $f^{\Delta/e}$ is the bijection from Δ/e onto Δ'/e' induced by f; we also abbreviate $B^{\Delta/e} := B^{\Delta/e, \Delta'/e}$ if Δ' is clear from context.

2.2. COHERENT CONFIGURATIONS. Let S be a partition of $\mathbf{1}_\Omega$. A pair $\mathcal{X} = (\Omega, S)$ is called a *coherent configuration* on Ω if

(C1) $1_\Omega \in S^\cup$,

(C2) $s^* \in S$ for all $s \in S$,

(C3) given $r, s, t \in S$, the number $c_{rs}^t = |\alpha r \cap \beta s^*|$ does not depend on $(\alpha, \beta) \in t$.

The number $|\Omega|$ is called the *degree* of $\mathcal{X}$. We say that $\mathcal{X}$ is *trivial* if $S = S(\mathcal{X})$ consists of 1_Ω and its complement (unless Ω is not a singleton), *homogeneous* or a *scheme* if $1_\Omega \in S$, and *commutative* if $c_{rs}^t = c_{sr}^t$ for all $r, s, t \in S$.

2.3. ISOMORPHISMS AND SCHURITY. A *combinatorial isomorphism* or, briefly, *isomorphism* from $\mathcal{X}$ to a coherent configuration $\mathcal{X}' = (\Omega', S')$ is defined to be a bijection $f : \Omega \to \Omega'$ such that $s^f = \{(\alpha^f, \beta^f) : (\alpha, \beta) \in s\}$ belongs to S' for all $s \in S$. In this case, $\mathcal{X}$ and $\mathcal{X}'$ are said to be *isomorphic*; the set of all isomorphisms from $\mathcal{X}$ to $\mathcal{X}'$ is denoted by $\mathrm{Iso}(\mathcal{X}, \mathcal{X}')$. The group of all isomorphisms of $\mathcal{X}$ to itself contains a normal subgroup

$$\mathrm{Aut}(\mathcal{X}) = \{f \in \mathrm{Sym}(\Omega) : s^f = s \text{ for all } s \in S\}$$

called the *automorphism group* of $\mathcal{X}$.

Let $G \leqslant \mathrm{Sym}(\Omega)$. Denote by S the set of all orbits $(\alpha, \beta)^G$ in the induced action of G on $\Omega \times \Omega$, where $\alpha, \beta \in \Omega$. Then the pair $\mathrm{Inv}(G) = (\Omega, S)$ is a coherent configuration. Any coherent configuration associated with a permutation group in this way is said to be *schurian*. Note that $\mathcal{X}$ is schurian if and only if $\mathcal{X} = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}))$.

2.4. Extensions. There is a natural partial order $\leqslant$ on the set of all coherent configurations on Ω. Namely, given two such coherent configurations $\mathcal{X}$ and $\mathcal{Y}$, we set

$$\mathcal{X} \leqslant \mathcal{Y} \;\Leftrightarrow\; S(\mathcal{X})^{\cup} \subseteq S(\mathcal{Y})^{\cup},$$

and say that $\mathcal{Y}$ is the *extension* of $\mathcal{X}$. The minimal and maximal elements with respect to this order are the trivial and *discrete* coherent configurations, respectively; in the last case, S consists of singletons. Note that the functor $\mathcal{X} \to \mathrm{Aut}(\mathcal{X})$ reverse the inclusion, namely,

$$(3) \qquad\qquad\qquad \mathcal{X} \leqslant \mathcal{Y} \Rightarrow \mathrm{Aut}(\mathcal{X}) \geqslant \mathrm{Aut}(\mathcal{Y}).$$

2.5. Algebraic isomorphisms and separability. A bijection $\varphi : S \to S'$ is called an *algebraic isomorphism* from $\mathcal{X}$ to $\mathcal{X}'$ if for all $r, s, t \in S$, we have

$$c_{r^{\varphi} s^{\varphi}}^{t^{\varphi}} = c_{rs}^{t}.$$

In this case, $|\Omega'| = |\Omega|$, $1_{\Omega'} = \varphi(1_{\Omega})$, and $\mathcal{X}'$ is commutative if and only if so is $\mathcal{X}$. Every $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$ induces the algebraic isomorphism $\varphi_f : \mathcal{X} \to \mathcal{X}'$, $s \mapsto s^f$; we put

$$\mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi) = \{ f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}') : \; \varphi_f = \varphi \}.$$

Note that $\mathrm{Aut}(\mathcal{X}) = \mathrm{Iso}(\mathcal{X}, \mathcal{X}, \mathrm{id})$, where id is the trivial (identical) algebraic automorphism of $\mathcal{X}$. Finally, for any $\alpha \in \Omega$ and $\alpha' \in \Omega'$, we put

$$(4) \qquad\qquad \mathrm{Iso}_{\alpha,\alpha'}(\mathcal{X}, \mathcal{X}', \varphi) = \{ f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi) : \; \alpha^f = \alpha' \}.$$

A coherent configuration $\mathcal{X}$ is said to be *separable* if every algebraic isomorphism from $\mathcal{X}$ is induced by isomorphism, equivalently, $\mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi) \neq \varnothing$ for all $\mathcal{X}'$ and φ. Any trivial coherent configuration is separable.

2.6. Relations. The elements of S and of $S^{\cup}$ are called *basis relations* and *relations* of the coherent configuration $\mathcal{X}$. The unique basis relation containing the pair (α, β) is denoted by $r(\alpha, \beta)$. The set of all relations is closed with respect to intersections and unions.

Any $\Delta \subseteq \Omega$ such that $1_{\Delta} \in S$ is called a *fiber* of $\mathcal{X}$. In view of condition (C1), the set $F = F(\mathcal{X})$ of all fibers forms a partition of Ω. Every basis relation is contained in the Cartesian product of two uniquely determined fibers. When $\mathcal{X}$ is a scheme, $F = \{\Omega\}$, whereas when $\mathcal{X}$ is schurian, F is just the set of orbits of the group $\mathrm{Aut}(\mathcal{X})$.

Every algebraic isomorphism $\varphi : \mathcal{X} \to \mathcal{X}'$ is extended in a natural way to a poset isomorphism $S^{\cup} \to (S')^{\cup}$, denoted also by φ. Then φ induces a poset isomorphism $F^{\cup} \to (F')^{\cup}$, $\Delta \mapsto \Delta^{\varphi}$, where the set Δ^{φ} is defined by the equality $\varphi(1_{\Delta}) = 1_{\Delta^{\varphi}}$.

Let $\mathcal{X} \leqslant \mathcal{Y}$ and $\mathcal{X}' \leqslant \mathcal{Y}'$. We say that the algebraic isomorphism φ is extended to an algebraic isomorphism $\psi : \mathcal{Y} \to \mathcal{Y}'$ if $\psi(s) = \varphi(s)$ for all $s \in S$.

2.7. Parabolics and quotients. A relation of $\mathcal{X}$ that is an equivalence relation on Ω is called a *parabolic* of $\mathcal{X}$; the set of all parabolics is denoted by $E = E(\mathcal{X})$. It contains the equivalence relations $\langle s \rangle$ and $\mathrm{Rad}(s)$ for $s \in S^{\cup}$. Given a parabolic $e \in E$, we define $S_{\Omega/e} = \{ s_{\Omega/e} : \; s \in S \}$, and $S_{\Delta} = \{ s_{\Delta} : \; s \in S, \; s_{\Delta} \neq \varnothing \}$ for any $\Delta \in \Omega/e$. Then

$$\mathcal{X}_{\Omega/e} = (\Omega/e, S_{\Omega/e}) \quad \text{and} \quad \mathcal{X}_{\Delta} = (\Delta, S_{\Delta})$$

are coherent configurations, called a *quotient* of $\mathcal{X}$ modulo e and a *restriction* of $\mathcal{X}$ to Δ, respectively. Their fibers are obtained from fibers $\Gamma \in F$ as follows: $\Gamma_{\Omega/e}$ in the first case, and $\Gamma \cap \Delta$ in the second. In particular, $\mathcal{X}_{\Omega/e}$ and $\mathcal{X}_{\Delta}$ are schemes if so is $\mathcal{X}$.

Let $\varphi : \mathcal{X} \to \mathcal{X}'$ be an algebraic isomorphism and $e \in E$. Then $\varphi(e)$ is a parabolic of $\mathcal{X}'$ with the same number of classes. Moreover,

$$(5) \qquad\qquad \varphi(\langle s \rangle) = \langle \varphi(s) \rangle \quad \text{and} \quad \varphi(\mathrm{Rad}(s)) = \mathrm{Rad}(\varphi(s)).$$

The algebraic isomorphism φ induces the algebraic isomorphism

$$\varphi_{\Omega/e} : \mathcal{X}_{\Omega/e} \to \mathcal{X}'_{\Omega'/e'}, \ s_{\Omega/e} \mapsto s'_{\Omega'/e'},$$

where $e' = \varphi(e)$ and $s' = \varphi(s)$. Now let $\Delta \in \Omega/e$ and $\Delta' \in \Omega/e'$. Assume that there is $\Gamma \in F$ such that $\Delta \cap \Gamma \neq \varnothing \neq \Delta' \cap \Gamma'$, where $\Gamma' = \Gamma^\varphi$. Then φ induces an algebraic isomorphism

(6) $$\varphi_{\Delta,\Delta'} : \mathcal{X}_\Delta \to \mathcal{X}'_{\Delta'}, \ s_\Delta \mapsto s'_{\Delta'}.$$

This algebraic isomorphisms always exists for all Δ and Δ' if $\mathcal{X}$ (and hence $\mathcal{X}'$) is a scheme.

2.8. SECTIONS. Let $\mathcal{X}$ be a coherent configuration, $e \in E$, and Δ a class of a parabolic containing e. The quotient set $\mathfrak{S} = \Delta/e$ is called a *section* of $\mathcal{X}$. Any element of $\mathfrak{S}$ is of the form $\alpha_\mathfrak{S} = \alpha e$ for some $\alpha \in \Delta$; when Δ is implicit, we write $\alpha_\mathfrak{S} = \varnothing$ for all $\alpha \notin \Delta$. For any $s \in S$, we define the relation $s_\mathfrak{S}$ on $\mathfrak{S}$ by formula (1). The set of all sections of $\mathcal{X}$ is denoted by $\mathfrak{S}(\mathcal{X})$. This set is partially ordered: $\Delta/e \leqslant \Delta'/e'$, whenever $\Delta \subseteq \Delta'$ and $e' \subseteq e$.

LEMMA 2.1. *Let $\mathcal{X}$ be a commutative scheme, $\mathfrak{S} \in \mathfrak{S}(\mathcal{X})$, and $s \in S$. Then $\alpha_\mathfrak{S} s_\mathfrak{S} = (\alpha s)_\mathfrak{S}$ for all $\alpha \in \Omega$ such that $\alpha_\mathfrak{S} \neq \varnothing$.*

Proof. Obviously, $\alpha_\mathfrak{S} s_\mathfrak{S} \supseteq (\alpha s)_\mathfrak{S}$. Conversely, without loss of generality, we assume that $s_\mathfrak{S} \neq \varnothing$. Let $\mathfrak{S} = \Delta/e$ and $\alpha \in \Delta$. Assume that $\beta_\mathfrak{S} \in \alpha_\mathfrak{S} s_\mathfrak{S}$ for some $\beta \in \Delta$. Then there are $\alpha' \in \alpha e$ and $\beta' \in \beta e$ such that $(\alpha', \beta') \in s$. Then $c_{rs}^t \neq 0$, where $r = r(\alpha, \alpha')$ and $t = r(\alpha, \beta')$. By the commutativity, $c_{sr}^t \neq 0$. It follows that there exists $\beta'' \in \beta e$ such that $(\alpha, \beta'') \in s$. Thus, $\beta_\mathfrak{S} = \beta e = \beta'' e \subseteq (\alpha s)_\mathfrak{S}$. $\qquad\square$

For a section $\mathfrak{S} = \Delta/e \in \mathfrak{S}(\mathcal{X})$, we put $\mathcal{X}_\mathfrak{S} = (\mathcal{X}_\Delta)_{\Delta/e}$. Then $\mathcal{X}_\mathfrak{S}$ is schurian if so is $\mathcal{X}$. Now let $\varphi : \mathcal{X} \to \mathcal{X}'$ be an algebraic isomorphism, $e' = \varphi(e)$, and Δ' a class of the φ-image of the parabolic of $\mathcal{X}$, containing the class Δ. The algebraic isomorphism (6) (if it is defined) induces the algebraic isomorphism

$$\varphi_{\mathfrak{S},\mathfrak{S}'} = (\varphi_{\Delta,\Delta'})_{\Delta/e,\Delta'/e'}$$

between the coherent configurations $\mathcal{X}_\mathfrak{S}$ and $\mathcal{X}'_{\mathfrak{S}'}$.

2.9. POINT EXTENSIONS. The *point extension* $\mathcal{X}_{\alpha,\beta,\dots}$ of the coherent configuration $\mathcal{X}$ with respect to the points $\alpha, \beta, \dots \in \Omega$ is defined to be the smallest coherent configuration $\mathcal{Y} = (\Omega, T)$ such that $\mathcal{Y} \geqslant \mathcal{X}$ and $1_\alpha, 1_\beta, \dots \in T$. When the points are irrelevant, we use the term "point extension" and "one-point extension" if $\alpha = \beta = \dots$.

Let $\varphi : \mathcal{X} \to \mathcal{X}'$ be an algebraic isomorphism, and let $\alpha \in \Omega^m$, $\alpha' \in \Omega'^m$. We say that algebraic isomorphism $\psi : \mathcal{X}_\alpha \to \mathcal{X}'_{\alpha'}$ extending φ is an (α, α')-*extension* of φ if

$$\psi(1_{\alpha_i}) = 1_{\alpha'_i}, \quad i = 1, \dots, m.$$

Note that the (α, α')-extension is unique if it exists.

When $m = 1$, the (α, α')-extension ψ exists only if $\psi(r(\alpha, \alpha)) = r(\alpha', \alpha')$, or, equivalently, $\Delta^\varphi = \Delta'$, where Δ (respectively, Δ') is the fiber of $\mathcal{X}$ (respectively, $\mathcal{X}'$), containing the point α (respectively, α').

2.10. PARTLY REGULAR COHERENT CONFIGURATIONS. A coherent configuration $\mathcal{X}$ is said to be *partly regular* if there exists a point $\alpha \in \Omega$ such that $|\alpha s| \leqslant 1$ for all $s \in S$; the point α is said to be *regular*. In this case, the set $\alpha^{\mathrm{Aut}(\mathcal{X})}$ is a faithful regular orbit of $\mathrm{Aut}(\mathcal{X})$. Note that every extension of partly regular coherent configuration is partly regular.

LEMMA 2.2 ([3, Theorem 3.3.19]). *Every partly regular coherent configuration is schurian and separable.*

Corollary 2.3. *Let $\mathcal{X}$ be a partly regular coherent configuration and $\mathcal{X}' \geqslant \mathcal{X}$. Then $\mathcal{X}' = \mathcal{X}$ if and only if $F(\mathcal{X}') = F(\mathcal{X})$.*

Proof. It suffices to verify the "if" part only. Assume that $F(\mathcal{X}') = F(\mathcal{X})$. We take $\Delta \in F(\mathcal{X})$ that contains a regular point of $\mathcal{X}$. Since $\mathcal{X}$ is schurian (Lemma 2.2), Δ is a faithful regular orbit of the group $G = \mathrm{Aut}(\mathcal{X})$ and hence $|G| = |\Delta|$. The same argument applied to partly regular coherent configuration $\mathcal{X}' \geqslant \mathcal{X}$ and $\Delta \in F(\mathcal{X}')$, shows that $|G'| = |\Delta|$, where $G' = \mathrm{Aut}(\mathcal{X}')$. Thus,

$$|G| = |\Delta| = |G'|.$$

Since $G' \leqslant G$ (see (3)), this yields $G = G'$, and $\mathcal{X} = \mathrm{Inv}(G) = \mathrm{Inv}(G') = \mathcal{X}'$. $\qquad\square$

3. Multidimensional coherent configurations and WL-dimension

The concept of the WL-dimension of a graph was introduced in [12] in terms of the multidimensional Weisfeiler–Leman algorithm. A goal of this section is to analyze this definition in terms of coherent configurations. Our approach is based on the multidimensional coherent configurations defined in [1]; all the necessary information and results about them are taken from [13].

Throughout this section, we fix an integer $m \geqslant 2$, and put $M = \{1, \dots, m\}$. The monoid of all mappings $\sigma : M \to M$ is denoted by $\mathrm{Mon}(M)$. Elements of the Cartesian m-power Ω^m are m-tuples $x = (x_1, \dots, x_m)$ with $x_i \in \Omega$ for all $i \in M$. For a tuple $x \in \Omega^m$, denote by $\rho(x)$ the equivalence relation on M such that $(i, j) \in \rho(x)$ if and only if $x_i = x_j$. For $\alpha \in \Omega$, we put

$$(7) \qquad x_{i \leftarrow \alpha} = (x_1, \dots, x_{i-1}, \alpha, x_{i+1}, \dots, x_m).$$

For a set $X \subseteq \Omega^m$ and a mapping $\sigma \in \mathrm{Mon}(M)$, we put $X^\sigma = \{x^\sigma : x \in X\}$, where $x^\sigma = (x_{1^\sigma}, \dots, x_{m^\sigma})$.

3.1. The multidimensional Weisfeiler–Leman algorithm.
The m-dimensional Weisfeiler–Leman algorithm constructs for a given graph $\mathcal{G} = (\Omega, D)$ a certain coloring $c(m, \mathcal{G})$ of the set Ω^m; a *coloring* is meant as a function from Ω^m to a linearly ordered set the elements of which are called *colors*.

At the first stage, an initial coloring $c_0 = c_0(m, \mathcal{G})$ of Ω^m is constructed from a coloring c' defined by the following condition: given $x, y \in \Omega^m$, we have $c'(x) = c'(y)$ if and only if $\rho(x) = \rho(y)$ and for all $i, j \in M$,

$$(x_i, x_j) \in D \Leftrightarrow (y_i, y_j) \in D.$$

Namely, the color $c_0(x)$ of every x is set to be the tuple $(c'(x^\sigma))_{\sigma \in \mathrm{Mon}(M)}$.

Starting from the second stage, the initial coloring is refined step by step. Namely, if c_i is the coloring constructed at the ith step ($i \geqslant 0$), then the color of an m-tuple x in the coloring c_{i+1} is defined to be

$$c_{i+1}(x) = (c_i(x), \{\!\{ (c_i(x_{1 \leftarrow \alpha}), \dots, c_i(x_{m \leftarrow \alpha})) : \alpha \in \Omega \}\!\}),$$

where $\{\!\{ \cdot \}\!\}$ denotes a multiset. The algorithm stops when $|\mathrm{im}(c_i)| = |\mathrm{im}(c_{i+1})|$ and the final coloring $c(m, \mathcal{G})$ is set to be c_i.

The coloring $c_\mathcal{G} = c(m, \mathcal{G})$ defines a partition $\mathrm{WL}_m(\mathcal{G})$ of Ω^m into the color classes $X = c_\mathcal{G}^{-1}(i)$, where i runs over the colors of $c_\mathcal{G}$. In particular, the color $c_\mathcal{G}(X) = c_\mathcal{G}(x)$ does not depend on $x \in X$. Since the final coloring is a refinement of the initial one, we have the following statement.

Lemma 3.1. *Let $X \in \mathrm{WL}_m(\mathcal{G})$ and $i, j \in M$. Then $(x_i, x_j) \in D$ either for all or for no $x \in X$.*

Another important property of the coloring $c_{\mathcal{G}}$ is that it defines a monoid homomorphism $\tau : \mathrm{Mon}(M) \to \mathrm{Mon}(\mathrm{im}(c_{\mathcal{G}}))$ such that for all $X \in \mathrm{WL}_m(\mathcal{G})$ and all $\sigma \in \mathrm{Mon}(M)$, we have

$$\tag{8} c_{\mathcal{G}}(X)^{\tau(\sigma)} = c_{\mathcal{G}}(X^{\sigma}).$$

Note that this equality holds for $c_{\mathcal{G}}$ replaced with c_i for $i = 0, 1 \ldots$ (this is obvious for $i = 0$ and then one can use induction on the number of iterations). According with interpretation given in [13], equality (8) means that the color of X "knows" the color of X^{σ}.

Two graphs $\mathcal{G}$ and $\mathcal{G}'$ are said to be WL_m-*equivalent*[1] if $\mathrm{im}(c_{\mathcal{G}}) = \mathrm{im}(c_{\mathcal{G}'})$. The *Weisfeiler–Leman dimension* $\dim_{\mathrm{WL}}(\mathcal{G})$ of a graph $\mathcal{G}$ is defined to be the smallest natural m such that every graph WL_m-equivalent to $\mathcal{G}$ is isomorphic to $\mathcal{G}$. It should be noted that these definitions can be extended to the case $m = 1$.

For $m = 2$, the partition $\mathrm{WL}(\mathcal{G}) = \mathrm{WL}_2(\mathcal{G})$ coincides with the set $S(\mathcal{X})$ for some coherent configuration on Ω, called the coherent configuration of the graph $\mathcal{G}$; in fact, $\mathcal{X}$ is the smallest coherent configuration for which $D \in S^{\cup}$. The graphs $\mathcal{G}$ and $\mathcal{G}'$ are WL_2-equivalent (briefly, WL-equivalent) if and only if the mapping $X \to c_{\mathcal{G}'}^{-1}(c_{\mathcal{G}}(X))$ is an algebraic isomorphism from $\mathrm{WL}(\mathcal{G})$ to $\mathrm{WL}(\mathcal{G}')$ (see [10]), where $c_{\mathcal{G}} = c(2, \mathcal{G})$ and $c_{\mathcal{G}'} = c(2, \mathcal{G}')$. A goal of this section is to prove an analog of the "only if" part of this statement for $m \geqslant 3$.

3.2. MULTIDIMENSIONAL COHERENT CONFIGURATIONS. For any $X_1, \ldots, X_m \subseteq \Omega^m$, denote by $n(x; X_1, \ldots, X_m)$ the number of $\alpha \in \Omega$ such that $x_{i \leftarrow \alpha} \in X_i$ for all $i \in M$.

DEFINITION 3.2. *A partition $\mathfrak{X}$ of Ω^m is called an m-ary coherent configuration on Ω if the following conditions are satisfied for all $X \in \mathfrak{X}$:*

- (C1') $\rho(X) := \rho(x)$ *does not depend on $x \in X$,*
- (C2') $X^{\sigma} \in \mathfrak{X}$ *for all $\sigma \in \mathrm{Mon}(M)$,*
- (C3') *for any $X_0, X_1, \ldots, X_m \in \mathfrak{X}$, the number $n_{X_1, \ldots, X_m}^{X_0} = n(x_0; X_1, \ldots, X_m)$ does not depend on $x_0 \in X_0$.*

For $m = 2$, conditions (C1'), (C2'), and (C3') imply conditions (C1), (C2), and (C3), respectively. In fact, the coherent configurations are just the 2-ary coherent configurations. An example of m-ary coherent configuration is a partition of Ω^m into the orbits of a permutation group on Ω, acting on Ω^m coordinatewise.

The set of all m-ary configurations on Ω is partially ordered. Namely, $\mathfrak{X} \leqslant \mathfrak{Y}$ if every class of $\mathfrak{X}$ is a union of some classes of $\mathfrak{Y}$, or equivalently, if $\mathfrak{X}^{\cup} \subseteq \mathfrak{Y}^{\cup}$, where $\mathfrak{X}^{\cup}$ (respectively, $\mathfrak{Y}^{\cup}$) is the set of all unions of classes of $\mathfrak{X}$ (respectively, $\mathfrak{Y}$). The largest m-ary coherent configuration is the discrete one in which every class is a singleton; the smallest m-ary coherent configuration consists of the orbits of the symmetric group $\mathrm{Sym}(\Omega)$ in its componentwise action on Ω^m (this easily follows from the fact that the orbits are in one-to-one correspondence with the equivalence relations on M).

An *algebraic isomorphism* of m-ary coherent configurations $\mathfrak{X}$ and $\mathfrak{X}'$ is a bijection $\varphi : \mathfrak{X} \to \mathfrak{X}'$ such that for all $X, X_0, \ldots, X_m \in \mathfrak{X}$ and $\sigma \in \mathrm{Mon}(M)$,

$$\tag{9} \varphi(X^{\sigma}) = \varphi(X)^{\sigma} \quad \text{and} \quad n_{X_1, \ldots, X_m}^{X_0} = n_{\varphi(X_1), \ldots, \varphi(X_m)}^{\varphi(X_0)}.$$

For $m = 2$, one can easily verify that our definition is agreed with that in Subsection 2.5.

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[1]In terms of [12], this means that the m-dimensional Weisfeiler–Leman algorithm does not distinguish $\mathcal{G}$ and $\mathcal{G}'$.

Note that if $X \in \mathfrak{X}$ and $i, j \in M$, then $(i, j) \in \rho(X)$ if and only if $X^\sigma = X$, where $\sigma \in \mathrm{Mon}(M)$ is identical on $M \smallsetminus \{i\}$ and takes i to j. It follows that

$$\rho(\varphi(X)) = \rho(X). \tag{10}$$

We extend φ to a bijection $\mathfrak{X}^\cup \to (\mathfrak{X}')^\cup$ in a natural way; then $X \subseteq Y$ implies $\varphi(X) \subseteq \varphi(Y)$ for all $X, Y \in \mathfrak{X}^\cup$.

An example of an algebraic isomorphism of m-ary coherent configurations is obtained for any two WL_m-equivalent graphs $\mathcal{G}$ and $\mathcal{G}'$. In this case, we have $\mathrm{im}(c_\mathcal{G}) = \mathrm{im}(c_{\mathcal{G}'})$ and the mapping

$$\varphi_{\mathcal{G}, \mathcal{G}'} : \mathrm{WL}_m(\mathcal{G}) \to \mathrm{WL}_m(\mathcal{G}'), \ X \mapsto c_{\mathcal{G}'}^{-1}(c_\mathcal{G}(X)), \tag{11}$$

is a bijection. In view of equality (8), it satisfies the first relation in (9), whereas the second relation follows from the description of the m-dimensional Weisfeiler–Leman algorithm. Thus, $\varphi_{\mathcal{G}, \mathcal{G}'}$ is an algebraic isomorphism from $\mathrm{WL}_m(\mathcal{G})$ to $\mathrm{WL}_m(\mathcal{G}')$.

3.3. PROJECTIONS. Let $k \in M$. The *k-projection* of Ω^m is defined to be the mapping

$$\mathrm{pr}_k : \Omega^m \to \Omega^k, \ (\alpha_1, \ldots, \alpha_m) \mapsto (\alpha_1, \ldots, \alpha_k).$$

The statement below follows from [13, Exercises 2.7, 2.11] (and their proofs). Below, for every $X \subseteq \Omega^k$, we put

$$\widehat{X} = \{(x_1, \ldots, x_k, x_k, \ldots, x_k) \in \Omega^m : \ (x_1, \ldots, x_k) \in X\}. \tag{12}$$

LEMMA 3.3. *Let $\mathfrak{X}$ be an m-ary coherent configuration. Then*

$$\mathrm{pr}_k(\mathfrak{X}) = \{\mathrm{pr}_k(X) : \ X \in \mathfrak{X}\}$$

is a k-ary coherent configuration.[2] *Moreover,*

$$n_{X_1, \ldots, X_k}^{X_0} = \sum_{Y_1, \ldots, Y_{m-k} \in \mathfrak{X}} n_{\widehat{X}_1, \ldots, \widehat{X}_k, Y_1, \ldots, Y_{m-k}}^{\widehat{X}_0} \tag{13}$$

for all $X_0, X_1, \ldots, X_k \in \mathrm{pr}_k(\mathfrak{X})$.

Let $\mathcal{G} = (\Omega, D)$ be a graph, $\mathfrak{X} = \mathrm{WL}_m(\mathcal{G})$, and $k = 2$. From Lemma 3.1, it follows that $\widehat{D} \in \mathfrak{X}^\cup$. Therefore, $D = \mathrm{pr}_2(\widehat{D})$ is a relation of the coherent configuration $\mathrm{pr}_2(\mathfrak{X})$. Thus,

$$\mathrm{pr}_2(\mathrm{WL}_m(\mathcal{G})) \geqslant \mathrm{WL}(\mathcal{G}). \tag{14}$$

Let $\varphi : \mathfrak{X} \to \mathfrak{X}'$ be an algebraic isomorphism of m-ary coherent configurations. We define a mapping $\varphi_k : \mathrm{pr}_k(\mathfrak{X}) \to \mathrm{pr}_k(\mathfrak{X}')$ by setting

$$\varphi_k(\mathrm{pr}_k(X)) = \mathrm{pr}_k(\varphi(X))$$

for all $X \in \mathfrak{X}$. Note that if $\mathrm{pr}_k(X) = \mathrm{pr}_k(Y)$ for some $Y \in \mathfrak{X}$, then $X^\sigma = Y^\sigma$, where $\sigma \in \mathrm{Mon}(M)$ takes i to $\min\{i, k\}$. For this σ, we have

$$\mathrm{pr}_k(\varphi(X)) = \mathrm{pr}_k(\varphi(X)^\sigma) \quad \text{and} \quad \mathrm{pr}_k(\varphi(Y)) = \mathrm{pr}_k(\varphi(Y)^\sigma).$$

Since also $\varphi(X)^\sigma = \varphi(X^\sigma) = \varphi(Y^\sigma) = \varphi(Y)^\sigma$, we conclude that $\mathrm{pr}_k(\varphi(X)) = \mathrm{pr}_k(\varphi(Y))$. Thus the mapping φ_k is well-defined. Reversing the above argument, one can see that it is injective. Finally, φ_k is surjective, because it is a composition of the surjections pr_k and φ.

LEMMA 3.4. *Let $\varphi : \mathfrak{X} \to \mathfrak{X}'$ be an algebraic isomorphism of m-ary coherent configurations. Then the bijection $\varphi_k : \mathrm{pr}_k(\mathfrak{X}) \to \mathrm{pr}_k(\mathfrak{X}')$ is an algebraic isomorphism.*

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[2]In [1, 13], the partition $\mathrm{pr}_k(\mathfrak{X})$ is called the *k-skeleton* of $\mathfrak{X}$.

Proof. Let $X \in \mathrm{pr}_k(\mathfrak{X})$ and $\sigma \in \mathrm{Mon}(K)$ with $K = \{1, \ldots, k\}$. Then, obviously,

$$X^\sigma = \mathrm{pr}_k(\widehat{X})^\sigma = \mathrm{pr}_k(\widehat{X^{\widehat{\sigma}}}),$$

where $\widehat{\sigma} \in \mathrm{Mon}(M)$ is the mapping identical on $M \smallsetminus K$ and coinciding with σ on K. It follows that

$$\varphi_k(X^\sigma) = \varphi_k(\mathrm{pr}_k(\widehat{X^{\widehat{\sigma}}})) = \mathrm{pr}_k(\varphi(\widehat{X^{\widehat{\sigma}}})) = \mathrm{pr}_k(\varphi(\widehat{X})^{\widehat{\sigma}}) = (\mathrm{pr}_k(\varphi(\widehat{X})))^\sigma = \varphi_k(X)^\sigma,$$

which proves the first part of (9). To prove the second relation, let $X_i \in \mathrm{pr}_k(\mathfrak{X})$, where $i = 0$ or $i \in K$. In view of formula (10), we have $\rho(\varphi(\widehat{X}_i)) = \rho(\widehat{X}_i)$. It follows that $\varphi(\widehat{X}_i) = \widehat{X'}$ for some $X' \in \mathrm{pr}_k(\mathfrak{X}')$. Moreover,

$$X' = \mathrm{pr}_k(\widehat{X'}) = \mathrm{pr}_k(\varphi(\widehat{X}_i)) = \varphi_k(\mathrm{pr}_k(\widehat{X}_i)) = X'_i,$$

where $X'_i = \varphi_k(X_i)$. Consequently, $\varphi(\widehat{X}_i) = \widehat{X'_i}$ for all i. Now, the required statement follows from formula (13). $\square$

For arbitrary $K \subseteq M$ and $X \subseteq \Omega^m$, one can define $\mathrm{pr}_K(X) = (\mathrm{pr}_k(X^\sigma))^{\sigma^{-1}}$, where $\sigma \in \mathrm{Sym}(M)$ is such that $K^\sigma = \{1, \ldots, k\}$ with $k = |K|$. Using condition (C2'), one can easily prove Lemmas 3.3 and 3.4 with pr_k and φ_k replaced by pr_K and φ_K, respectively.

3.4. RESIDUES. Let again $k \in M$ and $y \in \Omega^{m-k}$. The *residue* of $X \subseteq \Omega^m$ with respect to y is defined to be the set

$$X_y = \{\overline{x} \in \mathrm{pr}_k(X) : (\overline{x}_1, \ldots, \overline{x}_k, y_1, \ldots, y_{m-k}) \in X\}.$$

For any $\overline{X} \in \mathrm{pr}_k(\mathfrak{X})$, there is a unique $X \in \mathfrak{X}$ such that $\overline{X} = X_y$; we denote X by Xy. The statement below follows from [13, Exercise 2.13] (and its proof).

LEMMA 3.5. *Let $\mathfrak{X}$ be an m-ary coherent configuration. Then for any $y \in \Omega^{m-k}$,*

$$\mathfrak{X}_y = \{X_y : X \in \mathfrak{X}, \ X_y \neq \varnothing\}$$

is a k-ary coherent configuration. Moreover, $\mathfrak{X}_y \geqslant \mathrm{pr}_k(\mathfrak{X})$ and

$$(15) \qquad n_{X_1, \ldots, X_k}^{X_0} = \sum_{Y_1, \ldots, Y_{m-k} \in \mathfrak{X}} n_{X_1 y, \ldots, X_k y, Y_1, \ldots, Y_{m-k}}^{X_0 y}$$

for all $X_0, X_1, \ldots, X_k \in \mathfrak{X}_y$.

The k-ary coherent configuration $\mathfrak{X}_y$ contains the singleton $\{(y_i, \ldots, y_i)\}$ for $i = 1, \ldots, m-k$. For $k = 2$, it is just 1_{y_i}, and the coherent configuration $\mathrm{WL}_m(\mathcal{G})_y$ is larger than or equal to the smallest coherent configuration on Ω, for which D and the 1_{y_i} are relations. The latter coherent configuration is the point extension of $\mathrm{WL}(\mathcal{G})$ with respect to $y_1, \ldots, y_{m-2}$. Thus,

$$(16) \qquad (\mathrm{WL}_m(\mathcal{G}))_y \geqslant \mathrm{WL}(\mathcal{G})_{y_1, \ldots, y_{m-2}}.$$

Let $\varphi : \mathfrak{X} \to \mathfrak{X}'$ be an algebraic isomorphism of m-ary coherent configurations and $K' = \{k+1, \ldots, m\}$. Assume that $Y \in \mathrm{pr}_{K'}(\mathfrak{X})$ and $Y' \in \mathrm{pr}_{K'}(\mathfrak{X}')$ be such that $\varphi_{K'}(Y) = Y'$; in this case any two tuples one from Y and another from Y' will be called *φ-similar*. By Lemma 3.4,

$$\mathrm{pr}_{K'}(Z) = Y \quad \Leftrightarrow \quad \mathrm{pr}_{K'}(\varphi(Z)) = Y'$$

for all $Z \in \mathfrak{X}$. It follows that if $y \in Y$ and $y' \in Y'$ (i.e. y and y' are φ-similar), then the sets $\{Xy : X \in \mathfrak{X}_y\}$ and $\{X'y' : X' \in \mathfrak{X}'_{y'}\}$ are in a natural one-to-one correspondence. This enables us to define a mapping

$$\varphi_{y, y'} : \mathfrak{X}_y \to \mathfrak{X}'_{y'}, \ X \mapsto \varphi(Xy)_{y'},$$

which is a bijection, because the mappings $\mathfrak{X}_y \to \mathfrak{X}$, $X \mapsto Xy$, and $\mathfrak{X}'_{y'} \to \mathfrak{X}'$, $X' \mapsto X'y'$, are injective.

LEMMA 3.6. *Let $\varphi : \mathfrak{X} \to \mathfrak{X}'$ be an algebraic isomorphism of m-ary coherent configurations, and let $y \in \Omega^{m-k}$ and $y' \in \Omega'^{m-k}$ be φ-similar tuples. Then the bijection $\varphi_{y,y'} : \mathfrak{X}_y \to \mathfrak{X}'_{y'}$ is an algebraic isomorphism. Moreover, it extends the algebraic isomorphism φ_k.*

Proof. Let $X \in \mathfrak{X}_y$ and $\sigma \in \operatorname{Mon}(K)$ with $K = \{1, \ldots, k\}$. Then, obviously, $X^\sigma y = (Xy)^{\widehat{\sigma}}$, where $\widehat{\sigma} \in \operatorname{Mon}(M)$ is the mapping identical on $M \setminus K$ and coinciding with σ on K. It follows that

$$\varphi_{y,y'}(X^\sigma) = \varphi(X^\sigma y)_{y'} = \varphi((Xy)^{\widehat{\sigma}})_{y'} = (\varphi(Xy)^{\widehat{\sigma}})_{y'} = (\varphi(X_y)_{y'})^\sigma = \varphi_{y,y'}(X)^\sigma,$$

which proves the first part of (9). To prove the second relation, it suffices to note that $\varphi(Xy) = \varphi_{y,y'}(X)_{y'}$ and make use of formula (15). $\square$

3.5. REDUCTION TO COHERENT CONFIGURATIONS. The main result of this subsection (Theorem 3.7 below) establishes a necessary condition for two graphs to be WL_m-equivalent in terms of their coherent configurations.

THEOREM 3.7. *Let $\mathcal{G} = (\Omega, D)$ and $\mathcal{G}' = (\Omega', D')$ be WL_m-equivalent graphs, $m \geqslant 2$. Then there is an algebraic isomorphism $\varphi : \mathrm{WL}(\mathcal{G}) \to \mathrm{WL}(\mathcal{G}')$ such that $\varphi(D) = D'$. Moreover, if $m \geqslant 3$, and $x \in \Omega^m$ and $x' \in \Omega'^m$ are such that $c_\mathcal{G}(x) = c_{\mathcal{G}'}(x')$, then φ has the (y, y')-extension, where $y = \operatorname{pr}_K(x)$, $y' = \operatorname{pr}_K(x')$, and $K = \{3, \ldots, m\}$.*

Proof. Denote by ψ the algebraic isomorphism defined by (11). By Lemma 3.4 for $k = 2$, there is an algebraic isomorphism $\psi_2 : \operatorname{pr}_2(\mathrm{WL}_m(\mathcal{G})) \to \operatorname{pr}_2(\mathrm{WL}_m(\mathcal{G}'))$. Note that D is a relation of the coherent configuration $\mathcal{X} = \mathrm{WL}(\mathcal{G})$ (see formula (14)), and also

$$\psi_2(D) = \operatorname{pr}_2(\psi(\widehat{D})) = \operatorname{pr}_2(\widehat{D'}) = D',$$

where the second equality holds by formula (8) and the fact that ψ respects the initial colorings $c_0(m, \mathcal{G})$ and $c_0(m, \mathcal{G}')$. Since $\mathcal{X}$ is the smallest coherent configuration containing D as a relation, $\psi_2(\mathcal{X})$ is the smallest coherent configuration containing D' as a relation, i.e. $\psi_2(\mathcal{X})$ is equal to $\mathcal{X}' = \mathrm{WL}(\mathcal{G}')$. Thus as the required algebraic isomorphism φ one can take the restriction of ψ_2 to $\mathcal{X}$.

Now, let $x \in \Omega^m$ and $x' \in \Omega'^m$ be such that $c_\mathcal{G}(x) = c_{\mathcal{G}'}(x')$. Then the algebraic isomorphism ψ takes the class $X \ni x$ to the class $X' \ni x'$. It follows that

$$\psi_K(\operatorname{pr}_K(X)) = \operatorname{pr}_K(X').$$

Since $y \in \operatorname{pr}_K(X)$ and $y' \in \operatorname{pr}_K(X')$, we can apply Lemma 3.6 to find an algebraic isomorphism

$$\psi_{y,y'} : \mathrm{WL}_m(\mathcal{G})_y \to \mathrm{WL}_m(\mathcal{G}')_{y'}$$

which extends ψ_2 and hence extends φ. On the other hand, $y \in \Omega^{m-2}$, $y' \in \Omega'^{m-2}$, and the inclusion (16) yields

$$\mathrm{WL}_m(\mathcal{G})_y \geqslant \mathcal{X}_{y_1, \ldots, y_{m-2}} \quad \text{and} \quad \mathrm{WL}_m(\mathcal{G}')_{y'} \geqslant \mathcal{X}'_{y'_1, \ldots, y'_{m-2}}.$$

Furthermore, the pairs $(1, 2), (1, i+2)$ belong to the equivalence relation $\rho(1_{y_i} y)$. By formula (10), this implies that these pairs also belong to $\rho(\psi(1_{y_i} y))$. It follows that $\psi_{y,y'}(1_{y_i}) = \psi(1_{y_i} y)_{y'} = 1_{y'_i}$ for $i = 1, \ldots, m-2$. Thus,

$$\psi_{y,y'}(\mathcal{X}_{y_1, \ldots, y_{m-2}}) = \mathcal{X}'_{y'_1, \ldots, y'_{m-2}},$$

and the restriction of $\psi_{y,y'}$ to $\mathcal{X}_{y_1, \ldots, y_{m-2}}$ is the (y, y')-extension of φ. $\square$

4. Sesquiclosed coherent configurations and algebraic isomorphisms

4.1. Sesquiclosed algebraic isomorphisms. In this subsection, we introduce a notion of a sesquiclosed algebraic isomorphism, which weakens the notion of an (algebraic) 2-isomorphism studied in [3, Section 3.5.2].

DEFINITION 4.1. *An algebraic isomorphism $\varphi : \mathcal{X} \to \mathcal{X}'$ is said to be sesquiclosed if φ has the (α, α')-extension for all $\alpha \in \Delta \in F(\mathcal{X})$ and $\alpha' \in \Delta' \in F(\mathcal{X}')$ such that $\Delta^\varphi = \Delta'$.*

Examples of sesquiclosed algebraic isomorphisms arise naturally in the context of WL_3-equivalent graphs, namely the following statement is an almost immediate consequence of Theorem 3.7.

LEMMA 4.2. *Let $\mathcal{G}$ and $\mathcal{G}'$ be WL_3-equivalent graphs. Assume that $\mathcal{G}$ is vertex-transitive. Then there is a sesquiclosed algebraic isomorphism $\varphi : \mathrm{WL}(\mathcal{G}) \to \mathrm{WL}(\mathcal{G}')$ such that $\varphi(D) = D'$, where D and D' are the arc sets of $\mathcal{G}$ and $\mathcal{G}'$, respectively.*

Proof. By the first part of Theorem 3.7 (for $m = 3$), there is an algebraic isomorphism $\varphi : \mathrm{WL}(\mathcal{G}) \to \mathrm{WL}(\mathcal{G}')$ such that $\varphi(D) = D'$. It suffices to verify that φ has the (α, α')-extension for all $\alpha \in \Omega$ and $\alpha' \in \Omega'$. Because $\mathcal{G}$ is vertex-transitive, $X = \mathrm{Diag}(\Omega^3)$ is a class of $\mathrm{WL}_3(\mathcal{G})$. Since $\mathcal{G}$ and $\mathcal{G}'$ be WL_3-equivalent, $X' = \mathrm{Diag}(\Omega'^3)$ is a class of $\mathrm{WL}_3(\mathcal{G}')$, and $c_\mathcal{G}(X) = c_{\mathcal{G}'}(X')$. It follows that $c_\mathcal{G}(x) = c_{\mathcal{G}'}(x')$, where $x = (\alpha, \alpha, \alpha)$ and $x' = (\alpha', \alpha', \alpha')$. By the second part of Theorem 3.7, this implies that φ has (α, α')-extension. $\square$

Not every algebraic isomorphism is sesquiclosed, e.g. a straightforward computation shows that a unique nontrivial algebraic automorphism of the antisymmetric scheme of rank 3 and degree 15 is not sesquiclosed. On the other hand, [3, Lemma 3.5.25] shows that every algebraic 2-isomorphism is sesquiclosed.

LEMMA 4.3. *Let $\mathcal{X}$ be a coherent configuration, $\varphi : \mathcal{X} \to \mathcal{X}'$ a sesquiclosed algebraic isomorphism, $\mathfrak{S} \in \mathfrak{S}(\mathcal{X})$, and $\mathfrak{S}' \in \mathfrak{S}(\mathcal{X}')$. Assume that φ induces an algebraic isomorphism $\psi : \mathcal{X}_\mathfrak{S} \to \mathcal{X}'_{\mathfrak{S}'}$. Then ψ is sesquiclosed.*

Proof. Let $\bar{\alpha} \in \bar{\Delta} \in F(\mathcal{X}_\mathfrak{S})$ and $\bar{\alpha}' \in \bar{\Delta}' \in F(\mathcal{X}'_{\mathfrak{S}'})$ be such that $\bar{\Delta}^\psi = \bar{\Delta}'$. Then there exist $\Delta \in F(\mathcal{X})$ and $\Delta' \in F(\mathcal{X}')$ for which

$$\Delta_\mathfrak{S} = \bar{\Delta}, \quad \Delta'_{\mathfrak{S}'} = \bar{\Delta}', \quad \Delta^\varphi = \Delta'.$$

Let $\alpha \in \Delta$ and $\alpha' \in \Delta'$ be such that $\alpha_\mathfrak{S} = \overline{\alpha}$ and $\alpha'_{\mathfrak{S}'} = \alpha'$. The algebraic isomorphism φ being sesquiclosed has (α, α')-extension $\varphi_{\alpha,\alpha'}$. Since φ induces ψ, one can define the algebraic isomorphism

$$\psi_{\overline{\alpha},\overline{\alpha}'} = (\varphi_{\alpha,\alpha'})_{\mathfrak{S},\mathfrak{S}'}$$

from $(\mathcal{X}_\alpha)_\mathfrak{S}$ to $(\mathcal{X}'_{\alpha'})_{\mathfrak{S}'}$. For any $\overline{s} \in S(\mathcal{X}_\mathfrak{S})$, we have

$$\psi_{\overline{\alpha},\overline{\alpha}'}(\overline{s}) = (\varphi_{\alpha,\alpha'})_{\mathfrak{S},\mathfrak{S}'}(s_\mathfrak{S}) = (\varphi_{\alpha,\alpha'}(s))_{\mathfrak{S}'} = \varphi(s)_{\mathfrak{S}'} = \psi(s_\mathfrak{S}),$$

where $s \in S$ is such that $s_\mathfrak{S} = \overline{s}$, and

$$\psi_{\alpha,\alpha'}(1_{\bar{\alpha}}) = (\varphi_{\alpha,\alpha'})_\mathfrak{S}((1_\alpha)_\mathfrak{S}) = (\varphi_{\alpha,\alpha'}(1_\alpha))_\mathfrak{S} = (1_{\alpha'})_\mathfrak{S} = 1_{\bar{\alpha}'},$$

Thus the algebraic isomorphism $\psi_{\bar{\alpha},\bar{\alpha}'}$ is the $(\bar{\alpha}, \bar{\alpha}')$-extension of the algebraic isomorphism ψ. $\square$

A coherent configuration $\mathcal{X}$ is said to be *sesquiseparable* if every sesquiclosed algebraic isomorphism is induced by isomorphism. Clearly, every separable coherent configuration is sesquiseparable; the converse is not true, see Subsection 6.2.

Lemma 4.4. *A coherent configuration $\mathcal{X}$ is sesquiseparable if every one-point extension of $\mathcal{X}$ is separable.*

Proof. Let $\varphi : \mathcal{X} \to \mathcal{X}'$ be a sesquiclosed algebraic isomorphism. Then it has an (α, α')-extension $\varphi_{\alpha,\alpha'}$ for all appropriate α and α'. Since the coherent configuration $\mathcal{X}_\alpha$ is separable, $\varphi_{\alpha,\alpha'}$ is induced by a certain isomorphism f. Then $s^f = \varphi_{\alpha,\alpha'}(s) = \varphi(s)$ for all $s \in S$. Hence, $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ and $\mathcal{X}$ is sesquiseparable. $\square$

4.2. Sesquiclosed coherent configurations. In parallel with sesquiclosed algebraic isomorphisms, we introduce a notion which weakens the notion of 2-closed coherent configuration studied in [3, Section 3.5.3]. Namely, in Lemma 3.5.25 there, it was proved that a coherent configuration $\mathcal{X}$ is 2-closed only if for every $\alpha \in \Omega$ the following conditions are satisfied:

(S1) $F(\mathcal{X}_\alpha) = \{\alpha s : \ s \in S, \ \alpha s \neq \varnothing\}$,
(S2) the trivial algebraic automorphism of $\mathcal{X}$ is sesquiclosed.

Definition 4.5. *A coherent configuration $\mathcal{X}$ satisfying* (S1) *and* (S2) *for all α is said to be sesquiclosed.*

Though conditions (S1) and (S2) are satisfied in the schurian case, not every sesquiclosed coherent configuration is schurian, for example, those coherent configurations constructed in [3, Theorem 4.2.4]. The same example shows that the class of sesquiclosed coherent configurations is not invariant with respect to algebraic isomorphisms.

Lemma 4.6. *Let $\mathcal{X}$ be a sesquiclosed coherent configuration and $\mathfrak{S} \in \mathfrak{S}(\mathcal{X})$. Then the coherent configuration $\mathcal{X}_\mathfrak{S}$ is sesquiclosed.*

Proof. Condition (S2) is satisfied for the coherent configuration $\mathcal{X}_\mathfrak{S}$ by Lemma 4.3 with $\mathcal{X} = \mathcal{X}'$ and $\varphi = \mathrm{id}$. Assume that condition (S1) is not satisfied for $\mathcal{X}_\mathfrak{S}$. Then there are $\alpha \in \Omega$ and $s \in S$ such that

$$(17) \qquad \varnothing \neq \alpha_\mathfrak{S} s_\mathfrak{S} \notin F((\mathcal{X}_\mathfrak{S})_{\alpha_\mathfrak{S}}).$$

Without loss of generality, we may assume that $\alpha s \neq \varnothing$. By condition (S1), we have $\alpha s \in F(\mathcal{X}_\alpha)$ and hence $(\alpha s)_\mathfrak{S} \in F((\mathcal{X}_\alpha)_\mathfrak{S})$. However, $(\mathcal{X}_\alpha)_\mathfrak{S} \geqslant (\mathcal{X}_\mathfrak{S})_{\alpha_\mathfrak{S}}$. Therefore, $(\alpha s)_\mathfrak{S}$ is contained in some $\Delta \in F((\mathcal{X}_\mathfrak{S})_{\alpha_\mathfrak{S}})$. Since $(\alpha s)_\mathfrak{S} \subseteq \alpha_\mathfrak{S} s_\mathfrak{S}$, we have

$$(18) \qquad (\alpha s)_\mathfrak{S} \subseteq \Delta \subseteq \alpha_\mathfrak{S} s_\mathfrak{S}.$$

In view of (17), there is $\Delta' \in F((\mathcal{X}_\mathfrak{S})_{\alpha_\mathfrak{S}})$ such that $\Delta \neq \Delta' \subseteq \alpha_\mathfrak{S} s_\mathfrak{S}$. Take any point α' for which $(\alpha' s)_\mathfrak{S}$ intersects Δ'. Then as above

$$(19) \qquad (\alpha' s)_\mathfrak{S} \subseteq \Delta' \subseteq \alpha_\mathfrak{S} s_\mathfrak{S}.$$

Note that α and α' lie in the same fiber of $\mathcal{X}$, because $\alpha s \neq \varnothing \neq \alpha' s$ and $s \in S$. By condition (S2), the trivial algebraic automorphism of $\mathcal{X}$ has the (α, α')-extension; denote it by ψ. Then there is an algebraic isomorphism $\psi_\mathfrak{S} : (\mathcal{X}_\alpha)_\mathfrak{S} \to (\mathcal{X}_{\alpha'})_\mathfrak{S}$ that induces the trivial algebraic automorphism of $\mathcal{X}_\mathfrak{S}$. Using formulas (18) and (19), and taking into account that αs is a union of fibers of $\mathcal{X}_\alpha$, we obtain

$$\Delta^{\psi_\mathfrak{S}} \supseteq ((\alpha s)_\mathfrak{S})^{\psi_\mathfrak{S}} = ((\alpha s)^\psi)_\mathfrak{S} = (\alpha' s)_\mathfrak{S} \subseteq \Delta',$$

whence $\Delta = \Delta'$, a contradiction. $\square$

Lemma 4.7. *Let $\mathcal{X}$ be a sesquiclosed coherent configuration and $\varphi : \mathcal{X} \to \mathcal{X}'$ a sesquiclosed algebraic isomorphism. Then the coherent configuration $\mathcal{X}'$ is sesquiclosed.*

Proof. Let $\alpha' \in \Omega'$ and $s' \in S'$. Since the algebraic isomorphism $\varphi : \mathcal{X} \to \mathcal{X}'$ is sesquiclosed, it has the (α, α')-extension ψ for some $\alpha \in \Omega$ such that $\alpha s \neq \varnothing$, where $s = \varphi^{-1}(s')$. Since the coherent configuration $\mathcal{X}$ is sesquiclosed, $\alpha s \in F(\mathcal{X}_\alpha)$. Consequently, the set $\alpha' s' = \psi(\alpha s)$ belongs to $F(\mathcal{X}'_{\alpha'})$. This shows that $\mathcal{X}'$ satisfies condition (S1).

Let $\alpha', \alpha'' \in \Omega'$ lie in the same fiber of $\mathcal{X}'$. Since the algebraic isomorphism φ is sesquiclosed, it has an (α, α')-extension ψ and (α, α'')-extension ψ' for some $\alpha \in \Omega$. It follows that $\psi^{-1}\psi'$ is the (α', α'')-extension of the trivial algebraic automorphism of $\mathcal{X}'$. This shows that $\mathcal{X}'$ satisfies condition (S2). $\qquad\square$

When a coherent configuration $\mathcal{X}$ is not sesquiclosed, one can construct a uniquely determined sesquiclosed extension $\mathcal{X}'$ such that $\mathrm{Aut}(\mathcal{X}') = \mathrm{Aut}(\mathcal{X})$. However, the explicit definition of $\mathcal{X}'$ is outside the scope of the present paper.

4.3. PARTLY REGULAR SECTIONS. In this subsection, we prove two auxiliary lemmas for the sections of sesquiclosed schemes, the restriction to which is partly regular. They will be used in Subsection 5.4.

LEMMA 4.8. *Let $\mathcal{X}$ be a commutative sesquiclosed scheme, $\mathfrak{S}_0, \mathfrak{S} \in \mathfrak{S}(\mathcal{X})$, and α a point such that $\alpha_{\mathfrak{S}_0} \neq \varnothing$. Assume that $\mathfrak{S}_0 \leqslant \mathfrak{S}$ and $((\mathcal{X}_\mathfrak{S})_{\alpha_\mathfrak{S}})_{\mathfrak{S}_0}$ is partly regular. Then*

$$(20) \qquad ((\mathcal{X}_\mathfrak{S})_{\alpha_\mathfrak{S}})_{\mathfrak{S}_0} = ((\mathcal{X}_\alpha)_\mathfrak{S})_{\mathfrak{S}_0}.$$

Proof. The obvious inclusion $(\mathcal{X}_\mathfrak{S})_{\alpha_\mathfrak{S}} \leqslant (\mathcal{X}_\alpha)_\mathfrak{S}$ implies $((\mathcal{X}_\mathfrak{S})_{\alpha_\mathfrak{S}})_{\mathfrak{S}_0} \leqslant ((\mathcal{X}_\alpha)_\mathfrak{S})_{\mathfrak{S}_0}$. The coherent configuration on the left-hand side is partly regular by the hypothesis. By Corollary 2.3, it suffices to verify that every fiber Δ of $((\mathcal{X}_\alpha)_\mathfrak{S})_{\mathfrak{S}_0}$ is also a fiber of $((\mathcal{X}_\mathfrak{S})_{\alpha_\mathfrak{S}})_{\mathfrak{S}_0}$.

By condition (S1), one can find $s \in S$ such that $\Delta = (\alpha s)_{\mathfrak{S}_0}$. Since $\mathcal{X}$ is commutative, Lemma 2.1 yields

$$\Delta = (\alpha s)_{\mathfrak{S}_0} = ((\alpha s)_\mathfrak{S})_{\mathfrak{S}_0} = (\alpha_\mathfrak{S} s_\mathfrak{S})_{\mathfrak{S}_0}.$$

On the other hand, the scheme $\mathcal{X}_\mathfrak{S}$ is sesquiclosed by Lemma 4.6. Therefore, the set $\alpha_\mathfrak{S} s_\mathfrak{S}$ is a fiber of the coherent configuration $(\mathcal{X}_\mathfrak{S})_{\alpha_\mathfrak{S}}$. It follows that $(\alpha_\mathfrak{S} s_\mathfrak{S})_{\mathfrak{S}_0}$ and hence Δ is a fiber of the coherent configuration $((\mathcal{X}_\mathfrak{S})_{\alpha_\mathfrak{S}})_{\mathfrak{S}_0}$. $\qquad\square$

LEMMA 4.9. *Let $\mathcal{X}$ and $\mathcal{X}'$ be schemes, $\varphi : \mathcal{X} \to \mathcal{X}'$ an algebraic isomorphism having an (α, α')-extension $\varphi_{\alpha,\alpha'}$ for some α and α', and $f \in \mathrm{Iso}_{\alpha,\alpha'}(\mathcal{X}, \mathcal{X}', \varphi)$. Assume that*

(i) *$\mathcal{X}$ is schurian,*
(ii) *$(\mathcal{X}_\alpha)_\mathfrak{S}$ is partly regular for some $\mathfrak{S} \in \mathfrak{S}(\mathcal{X})$ such that $\alpha_\mathfrak{S} \neq \varnothing$.*

Then

$$(21) \qquad \mathrm{Iso}(\mathcal{X}_\alpha, \mathcal{X}'_{\alpha'}, \varphi_{\alpha,\alpha'})^\mathfrak{S} = \mathrm{Iso}((\mathcal{X}_\alpha)_\mathfrak{S}, (\mathcal{X}'_{\alpha'})_{\mathfrak{S}'}, (\varphi_{\alpha,\alpha'})_{\mathfrak{S},\mathfrak{S}'}),$$

where $\mathfrak{S}' = \mathfrak{S}^f$.

Proof. By formula (4), the bijection $f^\mathfrak{S}$ belongs to the left-hand side of (21). Furthermore, since $\alpha^f = \alpha'$ and $\mathfrak{S}' = \mathfrak{S}^f$, this bijection takes $(\mathcal{X}_\alpha)_\mathfrak{S}$ to $(\mathcal{X}'_{\alpha'})_{\mathfrak{S}'}$. It follows that $f^\mathfrak{S}$ induces $(\varphi_{\alpha,\alpha'})_{\mathfrak{S},\mathfrak{S}'}$ and hence belongs to the right-hand side of (21). Thus the sets on the left- and right-hand sides are not empty. Consequently,

$$|\mathrm{Iso}(\mathcal{X}_\alpha, \mathcal{X}'_{\alpha'}, \varphi_{\alpha,\alpha'})^\mathfrak{S}| = |\mathrm{Aut}(\mathcal{X}_\alpha)^\mathfrak{S}|,$$

and

$$|\mathrm{Iso}((\mathcal{X}_\alpha)_\mathfrak{S}, (\mathcal{X}'_{\alpha'})_{\mathfrak{S}'}, (\varphi_{\alpha,\alpha'})_{\mathfrak{S},\mathfrak{S}'})| = |\mathrm{Aut}((\mathcal{X}_\alpha)_\mathfrak{S})|$$

and it suffices to verify that

$$(22) \qquad \mathrm{Aut}((\mathcal{X}_\alpha)_\mathfrak{S}) = \mathrm{Aut}(\mathcal{X}_\alpha)^\mathfrak{S}.$$

Obviously, $\mathrm{Aut}(\mathcal{X}_\alpha)^\mathfrak{S} \leqslant \mathrm{Aut}((\mathcal{X}_\alpha)_\mathfrak{S})$. Conversely, by condition (ii), the group $\mathrm{Aut}((\mathcal{X}_\alpha)_\mathfrak{S})$ has a faithful regular orbit Δ. Since the coherent configuration $(\mathcal{X}_\alpha)_\mathfrak{S}$ is schurian (Lemma 2.2), $\Delta = \Lambda_\mathfrak{S}$ for some fiber $\Lambda \in F(\mathcal{X}_\alpha)$. By the first part of condition (i), Λ is an orbit of the group $\mathrm{Aut}(\mathcal{X}_\alpha)$. It follows that $\Delta = \Lambda_\mathfrak{S}$ is an orbit of the group $\mathrm{Aut}(\mathcal{X}_\alpha)^\mathfrak{S} \leqslant \mathrm{Aut}((\mathcal{X}_\alpha)_\mathfrak{S})$. This proves (22), because Δ is a faithful regular orbit of the last group. $\qquad\square$

5. e_1/e_0-condition

5.1. DEFINITION. Let $\mathcal{X} = (\Omega, S)$ be a scheme, and let e_0 and e_1 be parabolics of $\mathcal{X}$ such that $e_0 \subseteq e_1$. Following [7], we say that $\mathcal{X}$ satisfies the e_1/e_0-*condition* if for all $s \in S$,

$$s \cap e_1 = \varnothing \quad \Rightarrow \quad e_0 \subseteq \mathrm{Rad}(s).$$

This is always true if $e_0 = 1_\Omega$ or $e_1 = \mathbf{1}_\Omega$; in the other cases, we say that the e_1/e_0-condition is satisfied *nontrivially*. When $e_0 = e_1$, the condition exactly means that the scheme $\mathcal{X}$ is isomorphic to the wreath product $\mathcal{X}_\Delta \wr \mathcal{X}_{\Omega/e_0}$ for any $\Delta \in \Omega/e_0$. Finally, we note that if $e_0' \subseteq e_0$ and $e_1' \supseteq e_1$ are parabolics, then $\mathcal{X}$ satisfies also the e_1'/e_0'-condition.

REMARK 5.1. The e_1/e_0-condition provides internal definition of the operation wedge product of association schemes, introduced and studied in [14]. Namely, $\mathcal{X}$ satisfies the e_1/e_0-condition if and only if $\mathcal{X}$ is isomorphic to the wedge product of the family of schemes $\mathcal{X}_\Delta$, $\Delta \in \Omega/e_1$, and the scheme $\mathcal{X}_{\Omega/e_0}$.

In the sequel, we simplify the notation as follows. First, we put $\Omega_0 = \Omega/e_0$, $\mathcal{X}_0 = \mathcal{X}_{\Omega/e_0}$, and $\Omega_1 = \Omega/e_1$. Second, for every $\Delta \in \Omega_1$, we put $\Delta_0 = \Delta/e_0$. Finally, for an algebraic isomorphism $\varphi : \mathcal{X} \to \mathcal{X}'$, we put $\varphi_0 = \varphi_{\Omega_0}$.

5.2. ADMISSIBLE PAIRS. Let $\varphi : \mathcal{X} \to \mathcal{X}'$, $s \mapsto s'$ be an algebraic isomorphism, where $\mathcal{X}' = (\Omega', S')$. In view of formula (5), $\mathcal{X}'$ satisfies the e_1'/e_0'-condition, where $e_0' = \varphi(e_0)$ and $e_1' = \varphi(e_1)$. As above, we define the sets Ω_0', Ω_1', and Δ_0' for all $\Delta' \in \Omega_1'$. In what follows, we are interested in determining the set $\mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$; for $\varphi = \mathrm{id}$, this was done in [7].

Suppose we are given a bijection $f_0 : \Omega_0 \to \Omega_0'$ taking $(e_1)_{\Omega_0}$ to $(e_1')_{\Omega_0'}$, and bijections $f_\Delta : \Delta \to \Delta'$ for each $\Delta \in \Omega_1$, taking $(e_0)_{\Delta_0}$ to $(e_0')_{\Delta_0'}$, where Δ' is a uniquely determined class of e_1', for which $(\Delta_0)^{f_0} = \Delta_0'$. The pair $P = (\{f_\Delta\}_{\Delta \in \Omega_1}, f_0)$ is said to be e_1/e_0-*admissible* if

$$(23) \qquad\qquad (f_\Delta)^{\Delta_0} = (f_0)^{\Delta_0}$$

for all $\Delta \in \Omega_1$. In this case there exists a uniquely determined bijection $f : \Omega \to \Omega'$ for which $f^{\Omega_0} = f_0$ and $f^\Delta = f_\Delta$ for all $\Delta \in \Omega_1$. We say that f is induced by the pair P.

THEOREM 5.2. *Let $\mathcal{X}$ be a scheme satisfying the e_1/e_0-condition, $\mathcal{X}'$ a scheme on Ω', and $\varphi : \mathcal{X} \to \mathcal{X}'$ an algebraic isomorphism. Then $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ if and only if f is induced by an e_1/e_0-admissible pair $(\{f_\Delta\}, f_0)$ such that*

$$(24) \qquad f_0 \in \mathrm{Iso}(\mathcal{X}_0, \mathcal{X}_0', \varphi_0) \quad and \quad f_\Delta \in \mathrm{Iso}(\mathcal{X}_\Delta, \mathcal{X}_{\Delta'}', \varphi_{\Delta, \Delta'}) \ for\ all\ \Delta \in \Omega_1.$$

Proof. The "only if" part is clear, because every $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ is obviously induced by the pair $(\{f^\Delta\}, f^{\Omega_0})$ which satisfies the conditions (24). Conversely, suppose that f is induced by an e_1/e_0-admissible pair $(\{f_\Delta\}, f_0)$ satisfying conditions (24). We need to verify that $s^f = \varphi(s)$ for all $s \in S$.

Let $s \subseteq e_1$. Then s is equal to the union of s_Δ, $\Delta \in \Omega_1$. By the second part of (24), this implies that

$$s^f = \bigcup_{\Delta \in \Omega_1} (s_\Delta)^{f_\Delta} = \bigcup_{\Delta \in \Omega_1} \varphi_{\Delta, \Delta'}(s_\Delta) = \bigcup_{\Delta' \in \Omega_1'} \varphi(s)_{\Delta'} = \varphi(s).$$

Now, we may assume that $s \cap e_1 = \varnothing$. Then $e_0 \subseteq \mathrm{Rad}(s)$, because $\mathcal{X}$ satisfies the e_1/e_0-condition. It follows that s is equal to the union of all $\Gamma \times \Lambda$ for which (Γ, Λ) lies in $s_0 = s_{\Omega/e_0}$. Because $f^{\Omega/e_0} = f_0$, we obtain

$$s^f = \big(\bigcup_{(\Gamma, \Lambda) \in s_0} \Gamma \times \Lambda \big)^f = \bigcup_{(\Gamma^f, \Lambda^f) \in s_0^f} \Gamma^f \times \Lambda^f = \bigcup_{(\Gamma', \Lambda') \in s_0'} \Gamma' \times \Lambda',$$

where s_0', Γ', and Λ' are the f_0-images of s_0, Γ, and Λ, respectively. As was noted above, $\mathcal{X}'$ satisfies the e_1'/e_0'-condition with $e_0' = \varphi(e_0)$, and $e_1' = \varphi(e_1)$. Since also f_0 induces φ_0, the relation $\varphi(s)$ equals the union of all $\Gamma' \times \Lambda'$ with $(\Gamma', \Lambda') \in s_0'$. Thus,

$$s^f = \bigcup_{(\Gamma', \Lambda') \in s_0'} \Gamma' \times \Lambda' = \varphi(s),$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

5.3. GENERAL SUFFICIENT CONDITION. A quite natural sufficient condition for a bijection, induced by an admissible e_1/e_0-pair, to satisfy condition (24), could be given in the notation of Subsection 5.1 by the equality

$$\mathrm{Iso}(\mathcal{X}_0, \mathcal{X}_0', \varphi_0)^{\Delta_0} = \mathrm{Iso}(\mathcal{X}_\Delta, \mathcal{X}_{\Delta'}', \varphi_{\Delta, \Delta'})^{\Delta_0}$$

for all $\Delta \in \Omega_1$ (cf. (23)). In this subsection, we weaken this equality by considering smaller sets of isomorphisms, while assuming that the algebraic isomorphism φ has a one-point extension.

LEMMA 5.3. *Let $\mathcal{X}$ be a scheme on Ω, satisfying the e_1/e_0-condition, $\mathcal{X}'$ a scheme on Ω', and $\varphi : \mathcal{X} \to \mathcal{X}'$ an algebraic isomorphism. Assume that for all $\alpha \in \Omega$, $\alpha' \in \Omega'$,*

$$(25) \qquad \mathrm{Iso}_{\alpha_0, \alpha_0'}(\mathcal{X}_0, \mathcal{X}_0', \varphi_0)^{\Delta_0} = \mathrm{Iso}_{\alpha, \alpha'}(\mathcal{X}_1, \mathcal{X}_1', \varphi_1)^{\Delta_0},$$

where $\alpha_0 = \alpha e_0$, $\alpha_0' = \alpha' e_0'$, $\mathcal{X}_1 = \mathcal{X}_\Delta$ with $\Delta \in \Omega_1$ containing α, $\mathcal{X}_1' = \mathcal{X}_{\Delta'}'$ with $\Delta' \in \Omega_1'$ containing α', and $\varphi_0 = \varphi_{\Omega_0}$, $\varphi_1 = \varphi_{\Delta, \Delta'}$. Then for all $\alpha \in \Omega$ and $\alpha' \in \Omega'$,

(i) $\mathrm{Iso}_{\alpha, \alpha'}(\mathcal{X}, \mathcal{X}', \varphi)^{\Omega_0} = \mathrm{Iso}_{\alpha_0, \alpha_0'}(\mathcal{X}_0, \mathcal{X}_0', \varphi_0),$
(ii) $\mathrm{Iso}_{\alpha, \alpha'}(\mathcal{X}, \mathcal{X}', \varphi)^{\Delta} = \mathrm{Iso}_{\alpha, \alpha'}(\mathcal{X}_1, \mathcal{X}_1', \varphi_1).$

Proof. In both cases, it suffices to verify only the inclusion $\supseteq$ under the assumption that the set on the right-hand side is nonempty. In the case (i), we take arbitrary $f_0 \in \mathrm{Iso}_{\alpha_0, \alpha_0'}(\mathcal{X}_0, \mathcal{X}_0', \varphi_0)$. For every $\Delta \in \Omega_1$, we define a certain bijection $f_\Delta : \Delta \to \Delta'$, where Δ' is a unique class of e_1', for which $(\Delta_0)^{f_0} = \Delta_0'$. First, assume that $\alpha \in \Delta$. Then $\alpha' \in \Delta'$ and

$$(f_0)^{\Delta_0} \in \mathrm{Iso}_{\alpha_0, \alpha_0'}(\mathcal{X}_0, \mathcal{X}_0', \varphi_0)^{\Delta_0}.$$

In view of (25), one can find an isomorphism $f_\Delta \in \mathrm{Iso}_{\alpha, \alpha'}(\mathcal{X}_1, \mathcal{X}_1', \varphi_1)$ for which $(f_\Delta)^{\Delta_0} = (f_0)^{\Delta_0}$; in particular, $\alpha^{f_\Delta} = \alpha'$. Now if $\alpha \notin \Delta$, then we define f_Δ as above, but instead of α and α' we take arbitrary points $\delta \in \Delta$ and $\delta' \in (\delta_0)^{f_0}$, respectively.

The constructed family $\{f_\Delta : \Delta \in \Omega_1\}$ together with the isomorphism f_0 forms an e_1/e_0-admissible pair. Let $f : \Omega \to \Omega'$ be the bijection induced by this pair. By definition, f satisfies conditions (24) for all Δ. Hence, $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ by Theorem 5.2. Since also $\alpha^f = \alpha'$ and $f^{\Omega_0} = f_0$, we are done.

In the case (ii), let $f_1 \in \mathrm{Iso}_{\alpha, \alpha}(\mathcal{X}_1, \mathcal{X}_1', \varphi_1)$, where $\alpha \in \Delta$ and $\alpha' \in \Delta'$. In view of (25), one can find

$$f_0 \in \mathrm{Iso}_{\alpha_0, \alpha_0'}(\mathcal{X}_0, \mathcal{X}_0', \varphi_0)$$

such that $(f_1)^{\Delta_0} = f_0$. Let $\Gamma \in \Omega_1$ be the class of e_1, other that Δ. Denote by Γ' a unique class of e_1', for which $(\Gamma_0)^{f_0} = \Gamma_0'$, and take arbitrary $\gamma \in \Gamma$. Then $(\gamma_0)^{f_0} = \gamma_0'$ for some $\gamma_0' \in \Gamma'$. Using (25) again, we find $f_\Gamma \in \mathrm{Iso}_{\gamma,\gamma'}(\mathcal{X}_\Gamma, \mathcal{X}_{\Gamma'}', \varphi_{\Gamma,\Gamma'})$ such that $(f_\Gamma)^{\Gamma_0} = (f_0)^{\Gamma_0}$.

The constructed family $\{f_\Lambda : \Lambda \in \Omega_1\}$ together with the isomorphism f_0 forms an e_1/e_0-admissible pair. Let $f : \Omega \to \Omega'$ is the bijection induced by this pair. By definition, f satisfies conditions (24). Hence, $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ by Theorem 5.2. Since also $\alpha' = \alpha^{f_1} = \alpha^f$ and $f^\Delta = f_1$, we are done. $\qquad\square$

COROLLARY 5.4. *In the hypothesis and notation of Lemma* 5.3*, the following statements are equivalent for all $\alpha \in \Omega$ and $\alpha' \in \Omega'$:*

(1) $\mathrm{Iso}_{\alpha,\alpha'}(\mathcal{X}, \mathcal{X}', \varphi) \neq \varnothing$,
(2) $\mathrm{Iso}_{\alpha_0,\alpha_0'}(\mathcal{X}_0, \mathcal{X}_0', \varphi_0) \neq \varnothing$,
(3) $\mathrm{Iso}_{\alpha,\alpha'}(\mathcal{X}_1, \mathcal{X}_1', \varphi_1) \neq \varnothing$.

COROLLARY 5.5. *In the hypothesis and notations of Lemma* 5.3*, assume that $\mathcal{X} = \mathcal{X}'$ and $\varphi = \mathrm{id}$. Then $\mathcal{X}$ is schurian if and only if $\mathcal{X}_0$ is schurian and $\mathcal{X}_1$ is schurian for some α.*

Proof. It suffices to prove the "if" part only. Assume that $\mathcal{X}_0$ is schurian and $\mathcal{X}_1$ is schurian for some $\alpha \in \Omega$. Since $\varphi = \mathrm{id}$, we have $\mathcal{X}_0' = \mathcal{X}_0$ (but not necessarily $\mathcal{X}_1' = \mathcal{X}_1$, because, in general, $\alpha \neq \alpha'$). Let $\alpha' \in \Omega$. By the schurity of $\mathcal{X}_0$, there exists $f \in \mathrm{Aut}(\mathcal{X}_0)$ such that $(\alpha_0)^f = \alpha_0'$, or, equivalently,

$$\mathrm{Iso}_{\alpha_0,\alpha_0'}(\mathcal{X}_0, \mathcal{X}_0, \mathrm{id}) \neq \varnothing.$$

By Lemma 5.3(i), this yields $\mathrm{Iso}_{\alpha,\alpha'}(\mathcal{X}, \mathcal{X}, \mathrm{id}) \neq \varnothing$. Since α' is arbitrary, this means that the group $\mathrm{Aut}(\mathcal{X}) = \mathrm{Iso}(\mathcal{X}, \mathcal{X}, \mathrm{id})$ is transitive. Moreover, by statements (i) and (ii) of Lemma 5.3 for $\alpha = \alpha'$, we have, respectively,

$$(\mathrm{Aut}(\mathcal{X})_\alpha)^{\Omega_0} = \mathrm{Aut}(\mathcal{X}_0)_{\alpha_0} \quad \text{and} \quad (\mathrm{Aut}(\mathcal{X})_\alpha)^\Delta = \mathrm{Aut}(\mathcal{X}_1)_\alpha.$$

By the transitivity of $\mathrm{Aut}(\mathcal{X})$ and the schurity of $\mathcal{X}_0$ and $\mathcal{X}_1$, this implies that $\mathcal{X}$ is schurian. $\qquad\square$

5.4. CONCRETE SUFFICIENT CONDITION. The key point in our general sufficiency condition established in Lemma 5.3 is equality (25). In general, checking this equality is not an easy task. The lemma below gives a more or less simple criterion to be used in Section 6.

LEMMA 5.6. *Let $\mathcal{X}$ be a sesquiclosed commutative scheme on Ω, satisfying the e_1/e_0-condition, $\mathcal{X}'$ a scheme on Ω', and $\varphi : \mathcal{X} \to \mathcal{X}'$ a sesquiclosed algebraic isomorphism. In the notation of Lemma* 5.3*, assume that for $\alpha \in \Omega$ and $\alpha' \in \Omega'$,*

(i) *$\mathcal{X}_0$ and $\mathcal{X}_1$ are schurian, $\mathrm{Iso}(\mathcal{X}_0, \mathcal{X}_0', \varphi_0) \neq \varnothing \neq \mathrm{Iso}(\mathcal{X}_1, \mathcal{X}_1', \varphi_1)$,*
(ii) *$((\mathcal{X}_0)_{\alpha_0})_{\Delta_0}$ and $((\mathcal{X}_1)_\alpha)_{\Delta_0}$ are partly regular.*

Then equality (25) holds.

Proof. Let $f_0 \in \mathrm{Iso}(\mathcal{X}_0, \mathcal{X}_0', \varphi_0)$. Because $\mathcal{X}_0$ is a schurian scheme, we may assume that $(\alpha_0)^{f_0} = \alpha_0'$. Then the algebraic isomorphism ψ_0 induced by f_0 is the (α_0, α_0')-extension of the algebraic isomorphism φ_0. It easily follows that

(26) $$\mathrm{Iso}_{\alpha_0,\alpha_0'}(\mathcal{X}_0, \mathcal{X}_0', \varphi_0)^{\Delta_0} = \mathrm{Iso}((\mathcal{X}_0)_{\alpha_0}, (\mathcal{X}_0')_{\alpha_0'}, \psi_0)^{\Delta_0}.$$

Next, by the first parts of conditions (i) and (ii), the scheme $\mathcal{X}_0$ and algebraic isomorphism φ_0 satisfy the hypothesis of Lemma 4.9 for $\mathfrak{S} = \Delta_0$, $\mathfrak{S}' = \Delta_0'$, and $\alpha = \alpha_0$. Therefore,

(27) $$\mathrm{Iso}((\mathcal{X}_0)_{\alpha_0}, (\mathcal{X}_0')_{\alpha_0'}, \psi_0)^{\Delta_0} = \mathrm{Iso}(((\mathcal{X}_0)_{\alpha_0})_{\Delta_0}, ((\mathcal{X}_0')_{\alpha_0'})_{\Delta_0'}, (\psi_0)_{\Delta_0,\Delta_0'}).$$

Furthermore, the scheme $\mathcal{X}$, sections $\mathfrak{S} = \Omega_0$, $\mathfrak{S}_0 = \Delta_0$, and point α satisfy the hypothesis of Lemma 4.8. Therefore,

$$(28) \qquad ((\mathcal{X}_0)_{\alpha_0})_{\Delta_0} = ((\mathcal{X}_\alpha)_{\Omega_0})_{\Delta_0} = (\mathcal{X}_\alpha)_{\Delta_0}.$$

By Lemma 4.7, the scheme $\mathcal{X}'$ is sesquiclosed. Furthermore, the isomorphism f_0 induces an isomorphism from $((\mathcal{X}_0)_{\alpha_0})_{\Delta_0}$ to the coherent configuration $((\mathcal{X}'_0)_{\alpha'_0})_{\Delta'_0}$, which is therefore partly regular. Thus, the scheme $\mathcal{X}'$, sections $\mathfrak{S} = \Omega'_0$, $\mathfrak{S}_0 = \Delta'_0$, and point α' satisfy the hypothesis of Lemma 4.8, whence

$$(29) \qquad ((\mathcal{X}'_0)_{\alpha'_0})_{\Delta_0} = ((\mathcal{X}'_{\alpha'})_{\Omega'_0})_{\Delta'_0.} = (\mathcal{X}'_{\alpha'})_{\Delta'_0}.$$

Since the algebraic isomorphism φ is sesquiclosed, it has the (α, α')-extension $\varphi_{\alpha,\alpha'}$ taking α_0 to α'_0, and hence extending φ_0 and taking Δ_0 to Δ'_0. It follows that the algebraic isomorphism $\psi_{\Delta_0,\Delta'_0}$ defined as the restriction of $\varphi_{\alpha,\alpha'}$ to $((\mathcal{X}_0)_{\alpha_0})_\Delta$ coincides with $(\psi_0)_{\Delta_0,\Delta'_0}$. With taking equalities (28) and (29) into account, we can continue formula (27) as follows:

$$\mathrm{Iso}(((\mathcal{X}_0)_{\alpha_0})_{\Delta_0}, ((\mathcal{X}'_0)_{\alpha'_0})_{\Delta'_0}, (\psi_0)_{\Delta_0,\Delta'_0}) = \mathrm{Iso}((\mathcal{X}_\alpha)_{\Delta_0}, (\mathcal{X}'_{\alpha'})_{\Delta'_0}, \psi_{\Delta_0,\Delta'_0}).$$

This together with (26), yields

$$\mathrm{Iso}_{\alpha_0,\alpha'_0}(\mathcal{X}_0, \mathcal{X}'_0, \varphi_0)^{\Delta_0} = \mathrm{Iso}((\mathcal{X}_\alpha)_{\Delta_0}, (\mathcal{X}'_{\alpha'})_{\Delta'_0}, \psi_{\Delta_0,\Delta'_0}).$$

Arguing in a similar way with $\mathcal{X}_0$ and φ_0 replaced by $\mathcal{X}_1$ and φ_1, we obtain

$$\mathrm{Iso}_{\alpha,\alpha'}(\mathcal{X}_1, \mathcal{X}'_1, \varphi_1)^{\Delta_0} = \mathrm{Iso}((\mathcal{X}_\alpha)_{\Delta_0}, (\mathcal{X}'_{\alpha'})_{\Delta'_0}, \psi_{\Delta_0,\Delta'_0}).$$

Comparing the expressions on the right-hand sides of the last two equalities, we get (25). $\qquad\square$

6. Circulant sesquiseparable schemes

6.1. Circulant schemes. In this subsection, we follow [3, Sections 2.4 and 4.4]. Let G be a cyclic group. A coherent configuration $\mathcal{X}$ on the elements of G is said to be *circulant* if for every $g \in G$, the permutation

$$\rho_g : x \mapsto xg, \ x \in G,$$

is an automorphism of $\mathcal{X}$; in particular, $\mathcal{X}$ is a commutative scheme. There is a one-to-one correspondence between circulant schemes and the Schur rings over cyclic groups (see, e.g. [6]). The reader should not be embarrassed that most of the results cited in the present paper were formulated and proved in the language of Schur rings. In the rest of this subsection, we will change our terminology slightly to match that used for circulant schemes (and Schur rings). In the sequel, $\mathcal{X}$ is a circulant scheme on G, $S = S(\mathcal{X})$, $E = E(\mathcal{X})$, and α is the identity element of G.

Every group $H \leqslant G$ defines an equivalence relation $e = e(H)$ on G, the classes of which are the cosets of H in G. In particular, $e(\alpha) = \mathrm{Diag}(G \times G)$ and $e(G) = G \times G$. Note that $H = \alpha e$ and $H \leqslant H'$ if and only if $e(H) \subseteq e(H')$.

We say that H is an $\mathcal{X}$-*group* if $e(H) \in E$. The mapping $H \mapsto e(H)$ defines a one-to-one correspondence between the $\mathcal{X}$-groups and parabolics of $\mathcal{X}$. The parabolics $\langle s \rangle$ and $\mathrm{Rad}(s)$, where $s \in S$, correspond to $e(H)$ with $H = \langle \alpha s \rangle$ and $H = \{h \in G : (\alpha s)h = \alpha s\}$, respectively. Note that the parabolic $\mathrm{Rad}(s) = e(H)$, and hence the group H, does not depend on the relation $s = r(\alpha, g)$, where g is a generator of G. This group is called the *radical* of $\mathcal{X}$ and is denoted by $\mathrm{Rad}(\mathcal{X})$.

In the context of circulant schemes, we are interested in those sections $\mathfrak{S} = \Delta/e$ for which $\alpha \in \Delta$. In this case, $\Delta = U$ and $e = e(L)$ for some $\mathcal{X}$-groups U and L such that $L \leqslant U$; in particular, $\mathfrak{S} = U/L$. The set of all such sections is denoted by $\mathfrak{S}_0(\mathcal{X})$. It is important to note that every $\mathfrak{S} \in \mathfrak{S}_0(\mathcal{X})$ is treated as a cyclic group

and the scheme $\mathcal{X}_{\mathfrak{S}}$ is treated as a circulant scheme on $\mathfrak{S}$. We say that a section $\mathfrak{S}$ is *trivial* (respectively, *primitive*) if the scheme $\mathcal{X}_{\mathfrak{S}}$ is trivial (respectively, primitive, i.e. contains no $\mathcal{X}_{\mathfrak{S}}$-group different from the trivial one and $\mathfrak{S}$). It is known that every primitive section of composite order is trivial.

6.2. Normal circulant schemes. The scheme $\mathcal{X}$ is said to be *normal* if the group $\langle \rho_g : \ g \in G \rangle$ is normal in $\operatorname{Aut}(\mathcal{X})$, or equivalently, $\operatorname{Aut}(\mathcal{X})_\alpha \leqslant \operatorname{Aut}(G)$. This condition is always satisfied when G is of prime order and $\mathcal{X}$ is not trivial.

Lemma 6.1 ([6, Theorem 6.1 and Corollary 6.2]). *Every normal circulant scheme $\mathcal{X}$ is schurian. Moreover, any one-point extension of $\mathcal{X}$ is partly regular.*

There are normal circulant schemes that are not separable, see, e.g. [3, Section 4.5]. However, from Lemmas 6.1 and 2.2, it follows that every normal circulant scheme is schurian and any one-point extension of it is separable. By Lemma 4.4, this proves the following statement.

Corollary 6.2. *Every normal circulant scheme is sesquiseparable.*

The following auxiliary statement is crucial for the proof of Theorem 6.4 below, because it enables us to verify condition (ii) in Lemma 5.6.

Lemma 6.3. *Let $\mathcal{X}$ be a normal circulant scheme and α the identity element of G. Then the coherent configuration $(\mathcal{X}_\alpha)_{\mathfrak{S}}$ is partly regular for every section $\mathfrak{S} \in \mathfrak{S}_0(\mathcal{X})$.*

Proof. Let $\mathfrak{S} = U/L$, where U and L are $\mathcal{X}$-subgroups, and let u be a generator of U. It suffices to verify that $\bar{u} = uL$ is a regular point of the coherent configuration $(\mathcal{X}_\alpha)_{\mathfrak{S}}$. Assume on the contrary that there exist $x, x' \in U$ such that

$$(30) \qquad\qquad \bar{x} \neq \bar{x}' \quad \text{and} \quad \bar{r}(\bar{u}, \bar{x}) = \bar{r}(\bar{u}, \bar{x}'),$$

where $\bar{x} = xL$, $\bar{x}' = x'L$, and for all $\bar{y}, \bar{z} \in \mathfrak{S}$ we denote by $\bar{r}(\bar{y}, \bar{z})$ the basis relation of the coherent configuration $(\mathcal{X}_\alpha)_{\mathfrak{S}}$, containing the pair $(\bar{y}, \bar{z})$. By the second equality in (30), there is $u' \in \bar{u}$ such that

$$r(u, x) = r(u', x').$$

The normality of $\mathcal{X}$ implies that every element of $\operatorname{Aut}(\mathcal{X}_\alpha) = \operatorname{Aut}(\mathcal{X})_\alpha$ is induced by raising to a power coprime to the order n of the underlying cyclic group G. Furthermore, the coherent configuration $\mathcal{X}_\alpha$ is partly regular by Lemma 6.1 and hence is schurian by Lemma 2.2. Thus there is an integer m coprime to n and such that

$$x^m = x' \quad \text{and} \quad u^m = u'.$$

In particular, $\bar{x}^m = \bar{x}'$ and $\bar{u}^m = \bar{u}' = \bar{u}$. On the other hand, since u is a generator of $U \geqslant L$, there is an integer a for which $x = u^a$. Furthermore, $(uL)^m = \bar{u}^m = \bar{u} = uL$ and hence $u^m = yu$ for some $y \in L$. Thus for every $\ell \in L$, we have

$$(x\ell)^m = (u^a \ell)^m = (u^a)^m \ell^m = (u^m)^a \ell^m = (yu)^a \ell^m = u^a(y^a \ell^m) = x\ell',$$

where the element $\ell' = y^a \ell^m$ belongs to L. Consequently, $\bar{x}^m = \bar{x}$, implying $\bar{x}' = \bar{x}^m = \bar{x}$ in contrast to the first inequality in (30). $\qquad\square$

Let $\mathcal{X}$ be an arbitrary circulant scheme. A section $\mathfrak{S} \in \mathfrak{S}_0(\mathcal{X})$ is said to be *normal* if the (circulant) scheme $\mathcal{X}_{\mathfrak{S}}$ is normal, and *subnormal* if there exists a normal section $\mathfrak{S}' \in \mathfrak{S}_0(\mathcal{X})$ such that $\mathfrak{S} \leqslant \mathfrak{S}'$. Note that every normal section is subnormal and any primitive section of composite order greater than 4 is not subnormal.

6.3. THE $\mathfrak{S}$-WREATH PRODUCT OF CIRCULANT SCHEMES. Let $\mathcal{X}$ be a circulant scheme and $\mathfrak{S} \in \mathfrak{S}_0(\mathcal{X})$. Assume that $\mathcal{X}$ satisfies the $e(U)/e(L)$-condition, where U and L are $\mathcal{X}$-groups such that $\mathfrak{S} = U/L$. In this case, we say that $\mathcal{X}$ is the $\mathfrak{S}$-wreath product of schemes $\mathcal{X}_1 = \mathcal{X}_U$ and $\mathcal{X}_0 = \mathcal{X}_{G/L}$. The $\mathfrak{S}$-wreath product is nontrivial if $1 < L \leqslant U < G$, i.e. if $\mathcal{X}$ satisfies the $e(U)/e(L)$-condition nontrivially. When $U = L$, we say that $\mathcal{X}$ is a wreath product.

The main result of this subsection is to establish a simple criterion for the $\mathfrak{S}$-wreath product to be schurian and sesquiseparable.

THEOREM 6.4. *Let $\mathcal{X}$ be a circulant sesquiclosed scheme on G. Assume that $\mathcal{X}$ is the $\mathfrak{S}$-wreath product of (circulant) schemes $\mathcal{X}_0$ and $\mathcal{X}_1$ for some $\mathfrak{S} \in \mathfrak{S}_0$, and also*

(1) *the section $\mathfrak{S}$ is subnormal in both $\mathcal{X}_0$ and $\mathcal{X}_1$,*
(2) *the schemes $\mathcal{X}_0$ and $\mathcal{X}_1$ are schurian and sesquiseparable.*

Then $\mathcal{X}$ is schurian and sesquiseparable.

Proof. By condition (1), the section $\mathfrak{S}$ is contained in certain sections $\mathfrak{S}_0 \in \mathfrak{S}(\mathcal{X}_0)$ and $\mathfrak{S}_1 \in \mathfrak{S}(\mathcal{X}_1)$ such that the schemes

$$\mathcal{X}_{\mathfrak{S}_0} = (\mathcal{X}_0)_{\mathfrak{S}_0} \quad \text{and} \quad \mathcal{X}_{\mathfrak{S}_1} = (\mathcal{X}_1)_{\mathfrak{S}_1}$$

are normal circulant. By Lemma 6.1, the one-point extensions $(\mathcal{X}_{\mathfrak{S}_0})_\alpha$ and $(\mathcal{X}_{\mathfrak{S}_1})_\beta$ are partly regular, where $\alpha = 1_{\mathfrak{S}_0}$ and $\beta = 1_{\mathfrak{S}_1}$ treated as points of the schemes $\mathcal{X}_{\mathfrak{S}_0}$ and $\mathcal{X}_{\mathfrak{S}_1}$, respectively. By Lemma 6.3, the coherent configurations

$$((\mathcal{X}_{\mathfrak{S}_0})_\alpha)_{\mathfrak{S}} \leqslant ((\mathcal{X}_0)_\alpha)_{\mathfrak{S}} \quad \text{and} \quad ((\mathcal{X}_{\mathfrak{S}_1})_\beta)_{\mathfrak{S}} \leqslant ((\mathcal{X}_1)_\beta)_{\mathfrak{S}}$$

are partly regular. It follows that so are the coherent configurations $((\mathcal{X}_0)_\alpha)_{\mathfrak{S}}$ and $((\mathcal{X}_{\mathfrak{S}_1})_\beta)_{\mathfrak{S}}$. Consequently, condition (ii) of Lemma 5.6 is satisfied for $e_0 = e(L)$, $e_1 = e(U)$, and $\alpha = 1_G$. Since $\mathrm{Aut}(\mathcal{X})$ is transitive, this condition is true for all points of $\mathcal{X}$.

The schemes $\mathcal{X}_0$ and $\mathcal{X}_1$ are schurian by the first part of condition (2). Thus condition (i) of Lemma 5.6 is satisfied for $\varphi = \mathrm{id}$ (the condition $\mathrm{Iso}(\mathcal{X}_1, \mathcal{X}_1', \varphi_1) \neq \varnothing$ is true, because the group $\mathrm{Aut}(\mathcal{X})$ is transitive). By that lemma, equality (25) holds for all α and α', and hence the scheme $\mathcal{X}$ satisfies the hypothesis of Corollary 5.5. Thus, $\mathcal{X}$ is schurian.

Let $\varphi : \mathcal{X} \to \mathcal{X}'$ be a sesquiclosed algebraic isomorphism. Then the algebraic isomorphisms φ_0 and φ_1 are also sesquiclosed by Lemma 4.3. By the second part of condition (2), this implies that $\mathrm{Iso}(\mathcal{X}_0, \mathcal{X}_0', \varphi_0) \neq \varnothing$, and $\mathrm{Iso}(\mathcal{X}_1, \mathcal{X}_1', \varphi_1) \neq \varnothing$. Thus condition (i) of Lemma 5.6 is satisfied. By that lemma, equality (25) holds for all $\alpha \in \Omega$ and $\alpha' \in \Omega$. Thus, $\mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi) \neq \varnothing$ by Corollary 5.4, and hence $\mathcal{X}$ is sesquiseparable. $\square$

The conditions of Theorem 6.4 are always true when the circulant schemes $\mathcal{X}_0$ and $\mathcal{X}_1$ are normal. Thus this theorem implies the following statement.

COROLLARY 6.5. *Let $\mathcal{X}$ be a circulant sesquiclosed scheme on G. Assume that $\mathcal{X}$ is the $\mathfrak{S}$-wreath product for some $\mathfrak{S} \in \mathfrak{S}_0(\mathcal{X})$ such that the both operands are normal. Then $\mathcal{X}$ is schurian and sesquiseparable.*

6.4. CIRCULANT SCHEMES OF PRIME POWER DEGREE. In this subsection, $\mathcal{X}$ is a circulant scheme on a cyclic p-group G. The structure of $\mathcal{X}$ has been completely described in [16] and [11] for $p \geqslant 3$ and $p = 2$, respectively. In particular, $\mathcal{X}$ is schurian. Another important consequence of the description is the radical monotonicity for $p \geqslant 3$, namely,

$$(31) \qquad \qquad \langle s \rangle \leqslant \langle t \rangle \quad \Rightarrow \quad \mathrm{Rad}(s) \leqslant \mathrm{Rad}(t).$$

for all $s, t \in S$. Using this fact, we show in the lemma below that the wreath product can be lifted from any section of $\mathcal{X}$.

LEMMA 6.6. *Let $p \geqslant 3$ and $\mathfrak{S} \in \mathfrak{S}_0(\mathcal{X})$. If $\mathcal{X}_\mathfrak{S}$ is a nontrivial wreath product, then so is $\mathcal{X}$.*

Proof. Let $\mathfrak{S} = U/L$ and $\mathcal{X}_\mathfrak{S}$ a nontrivial wreath product. Then there is an $\mathcal{X}$-subgroup H such that $L < H < U$ and for every $s \in S$,

$$s \subseteq e(U) \smallsetminus e(H) \quad \Rightarrow \quad e(H/L) \subseteq \mathrm{Rad}(s_\mathfrak{S}),$$

Without loss of generality, we may assume that H is the only $\mathcal{X}$-subgroup strictly between L and U (here we use the fact that G is a cyclic p-group and hence the $\mathcal{X}$-groups are linearly ordered with respect to inclusion). It suffices to verify that

$$(32) \qquad\qquad s \subseteq e(U) \smallsetminus e(H) \quad \Rightarrow \quad \mathrm{Rad}(s) \geqslant e(H).$$

Indeed, then by formula (31), we have $\mathrm{Rad}(s) \geqslant e(H)$ for all $s \in S$ not contained in e_H. But this means that $\mathcal{X}$ satisfies the $e(H)/e(H)$-condition, i.e. $\mathcal{X}$ is a nontrivial wreath product.

Suppose on the contrary that (32) is not true for some s. Then the $\mathcal{X}$-subgroup L', for which $\mathrm{Rad}(s) = e(L')$, is strictly contained in H. By the assumption on H, this implies that $L' < L$, and hence $\mathfrak{S}$ is a subsection of $\mathfrak{S}' = U/L'$. Note that

$$\mathrm{Rad}(\mathcal{X}_\mathfrak{S}) \neq 1 \quad \text{and} \quad \mathrm{Rad}(\mathcal{X}_{\mathfrak{S}'}) = 1.$$

However, this contradicts the fact (see [4, Corollary 7.4]) that if a circulant scheme (in our case $\mathcal{X}_{\mathfrak{S}'}$) of odd prime power degree is of trivial radical, then every its section (in our case $\mathfrak{S}$) is also of trivial radical. $\qquad\square$

In the following lemma, we collect some known properties of circulant schemes of prime power degree, that will be used in the proof of Theorem 6.8. Below, we say that a circulant scheme $\mathcal{X}$ is *dense* if $\mathfrak{S}_0(\mathcal{X})$ contains no trivial section of composite order.

LEMMA 6.7. *Let $\mathcal{X}$ be a circulant scheme of prime power degree. Then*

  (1) *if $\mathrm{Rad}(\mathcal{X}) = 1$, then $\mathcal{X}$ is trivial or normal,*
  (2) *if $\mathrm{Rad}(\mathcal{X}) \neq 1$ and $\mathcal{X}$ is dense, then $\mathcal{X}$ is a nontrivial $\mathfrak{S}$-wreath product, and also $\mathrm{Rad}(\mathcal{X}_\mathfrak{S}) = 1$ or $|\mathfrak{S}| = 4$,*
  (3) *if $\mathfrak{S}_0(\mathcal{X})$ contains a proper primitive non-subnormal section, then $\mathcal{X}$ is a nontrivial wreath product.*

Proof. Statements (1) and (3) are consequences of [6, Corollary 6.4], and the statement cited in [9, p.29] (and proved in [8, Theorem 4.6]), specified for schemes of prime power degree. Statement (2) is exactly [9, Lemma 6.1]. $\qquad\square$

Now we are ready to prove the main result of this subsection.

THEOREM 6.8. *Every circulant scheme of prime power degree is sesquiseparable.*

Proof. Let $\mathcal{X}$ be a circulant scheme of degree p^n, where p is a prime and $n \geqslant 1$. Assume first that $\mathcal{X}$ is trivial or normal. In the former case, $\mathcal{X}$ is separable and hence sesquiseparable. In the latter case, $\mathcal{X}$ is sesquiseparable by Corollary 6.2. Thus, the statement is true if $n = 1$. Assume by induction that it is true for all circulant scheme of degree p^m, where $m < n$.

Claim. *Let $\mathcal{X}$ be neither trivial nor normal. Then $\mathcal{X}$ is a nontrivial $\mathfrak{S}$-wreath product with $\mathfrak{S}$ subnormal in both operands.*

Proof. By Lemma 6.7(3), we may assume that $\mathfrak{S}_0(\mathcal{X})$ contains no proper primitive non-subnormal section (otherwise we have a section $\mathfrak{S}$ for which $|\mathfrak{S}| = 1$). Then $\mathfrak{S}_0(\mathcal{X})$ contains no trivial section of composite order, i.e. $\mathcal{X}$ is dense. By statements (1) and (2) of Lemma 6.7, the scheme $\mathcal{X}$ is a nontrivial $\mathfrak{S}$-wreath product, and also $\operatorname{Rad}(\mathcal{X}_\mathfrak{S}) = 1$ or $|\mathfrak{S}| = 4$. In the last case, $\mathfrak{S}$ is normal (because all circulant schemes of degree at most 4 are normal) and hence is subnormal in both operands.

Let $\operatorname{Rad}(\mathcal{X}_\mathfrak{S}) = 1$. Then by Lemma 6.7(1), the section $\mathfrak{S}$ is either normal or trivial. In the first case, we are done as above. In the second case, $|\mathfrak{S}| = p \geqslant 5$ (otherwise $\mathfrak{S}$ is normal). It remains to verify that $\mathfrak{S}$ is subnormal in both operands. But if this is not true, then one of the operands is a nontrivial wreath product by Lemma 6.7(3). Then so is $\mathcal{X}$ by Lemma 6.6, and we are done as above. The claim is proved. $\qquad\square$

To complete the proof of the theorem, we may assume by the claim that $\mathcal{X}$ is a nontrivial $\mathfrak{S}$-wreath product of circulant schemes $\mathcal{X}_0$ and $\mathcal{X}_1$ such that the section $\mathfrak{S}$ is subnormal both in $\mathcal{X}_0$ and $\mathcal{X}_1$. Note that $\mathcal{X}_0$ and $\mathcal{X}_1$ are circulant schemes of prime power degrees, which are less than p^n. By induction, this implies that they are sesquiseparable. Because they are also schurian, the scheme $\mathcal{X}$ is sesquiseparable by Theorem 6.4. $\qquad\square$

6.5. PROOF OF THEOREM 1.1. Let $\mathcal{G}$ be a circulant graph of prime power order; in particular, $\mathcal{G}$ is vertex-transitive and $\operatorname{WL}(\mathcal{G})$ is a schurian circulant scheme. Let $\mathcal{G}'$ be a graph WL_3-equivalent to $\mathcal{G}$. By Lemma 4.2, there is a sesquiclosed algebraic isomorphism $\varphi : \operatorname{WL}(\mathcal{G}) \to \operatorname{WL}(\mathcal{G}')$ such that $\varphi(D) = D'$, where D and D' are the arc sets of $\mathcal{G}$ and $\mathcal{G}'$, respectively. On the other hand, the scheme $\operatorname{WL}(\mathcal{G})$ is sesquiseparable by Theorem 6.8. Thus the algebraic isomorphism φ is induced by an isomorphism f. It follows that $D^f = \varphi(D) = D'$. Hence, $f \in \operatorname{Iso}(\mathcal{G}, \mathcal{G}')$, i.e. the graphs $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic. Consequently, $\dim_{\operatorname{WL}}(\mathcal{G}) \leqslant 3$.

REFERENCES

[1] L. Babai, *Group, graphs, algorithms: the graph isomorphism problem*, in Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. IV. Invited lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 3319–3336.

[2] J. Bagherian and H. Memarzadeh, *The generalized X-join of Cayley graphs and their automorphisms*, Ars Math. Contemp. **24** (2024), article no. 6 (23 pages).

[3] G. Chen and I. Ponomarenko, *Coherent configurations*, Central China Normal University Press, 2019, the updated version is available at `http://www.pdmi.ras.ru/~inp/ccNOTES.pdf`.

[4] S. Evdokimov, I. Kovács, and I. Ponomarenko, *Characterization of cyclic Schur groups*, St. Petersburg Math. J. **25** (2014), 755–773.

[5] S. Evdokimov and I. Ponomarenko, *On a family of Schur rings over a finite cyclic group*, St. Petersburg Math. J. **13** (2002), no. 3, 441–451.

[6] _____, *Characterization of cyclotomic schemes and normal Schur rings over a cyclic group*, St. Petersburg Math. J. **14** (2003), no. 2, 189–221.

[7] _____, *Recognizing and isomorphism testing circulant graphs in polynomial time*, St. Petersburg Math. J. **15** (2004), no. 6, 813–835.

[8] _____, *Schurity of S-rings over a cyclic group and generalized wreath product of permutation groups*, St. Petersburg Math. J. **24** (2013), no. 3, 431–460.

[9] _____, *On the separability problem for circulant S-rings*, St. Petersburg Math. J. **28** (2017), no. 1, 21–35.

[10] F. Fuhlbrück, J. Köbler, and O. Verbitsky, *Identifiability of graphs with small color classes by the Weisfeiler–Leman algorithm*, SIAM J. Discrete Math. **35** (2021), no. 3, 1792–1853.

[11] Y. Y. Gol'fand, M. H. Klin, and N. L. Naimark, *The structure of S-rings over Z_{2^m}*, Sixteenth All Union Algebraic Conference. Part 2, Leningrad, 1981, pp. 195–196.

[12] M. Grohe, *Descriptive complexity, canonisation, and definable graph structure theory*, Cambridge University Press, 2017.

[13] H. Helfgott, J. Bajpai, and D. Dona, *Graph isomorphisms in quasi-polynomial time*, 2017, `https://arxiv.org/abs/1710.04574`.

[14] M. Muzychuk, *A wedge product of association schemes*, European J. Combin. **30** (2009), no. 3, 705–715.

[15] O. Pikhurko and O. Verbitsky, *Logical complexity of graphs: a survey*, in Model theoretic methods in finite combinatorics, Amer. Math. Soc., Providence, RI, 2011, pp. 129–179.

[16] R. Pöschel, *Untersuchungen von S-Ringen, insbesondere im Gruppenring von p-Gruppen*, Math. Nachr. **60** (1974), 1–27.

[17] B. Yu. Weisfeiler and A. A. Leman, *The reduction of a graph to canonical form and the algebra which appears therein*, NTI, Ser. 2 **9** (1968), 12–16, English translation is available at `https://www.iti.zcu.cz/wl2018/pdf/wl_paper_translation.pdf`.

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