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Regularity of Edge Ideals Via Suspension

Arindam Banerjee & Eran Nevo

Abstract We study the Castelnuovo–Mumford regularity of powers of edge ideals for arbitrary (finite simple) graphs. It has been repeatedly conjectured that for every graph G, $\operatorname{reg}(I(G)^s) \leqslant 2s + \operatorname{reg} I(G) - 2$ for all $s \geqslant 2$, which is the best possible upper bound for any s. We prove this conjecture for every s for all bipartite graphs, and for $s = 2$ for all graphs. The $s = 2$ case is crucial for our work and suspension plays a key role in its proof.

1. Introduction

Let M be a finitely generated graded module over a polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$, where $\mathbb{K}$ is a field. The Castelnuovo–Mumford regularity (or simply, regularity) $\operatorname{reg}(M)$ of M is defined as

$$\operatorname{reg}(M) = \max\{j - i \mid \operatorname{Tor}_i^R(M, \mathbb{K})_j \neq 0\}.$$

Regularity is an important invariant in commutative algebra and algebraic geometry that measures in some sense the complexity of ideals, modules, and sheaves. A question that has been studied by many is how the regularity behaves with respect to taking powers of homogeneous ideals. It is known that in the long-run $\operatorname{reg}(I^k)$ is linear in k, that is, there exist integers $a(I), b(I), c(I)$ such that $\operatorname{reg}(I^k) = a(I)k + b(I)$ for all $k \geqslant c(I)$ (see [9, 20]). For various classes of ideals people have studied these integers and also have looked for various upper and lower bounds for $\operatorname{reg}(I^k)$. For monomial ideals these invariants, as well as bounds on them, reflect the underlying combinatorics (see e.g. [3, 15, 16, 22, 25, 26] for various works under this theme). For monomial ideals I generated in same degree d, Kodiyalam [20] showed that $a(I) = d$.

One important class of monomial ideals is the class of edge ideals $I(G)$ of finite simple graphs, namely the ideals generated by squarefree monomials of degree two. For edge ideals, $c(I(G)) \leqslant 2$ for various cases: for example when the underlying graph is either cochordal or gap and cricket free or bipartite with $\operatorname{reg}(I(G)) \leqslant 3$ (see [1, 2, 3, 14]). As of $b(I(G))$, all known examples have

$$b(I(G)) \leqslant \operatorname{reg}(I(G)) - 2 \tag{1.1}$$

and it is conjectured by Alilooee–Banerjee–Beyarslan–Hà–Jayanthan–Selvaraja (see e.g. [3, 19]) that this inequality holds for every graph. In fact, the conjecture is slightly stronger:

Conjecture 1.1. (Alilooee–Banerjee–Beyarslan–Hà–Jayanthan–Selvaraja) *For every finite graph G, every $s \geqslant 1$ and every field $\mathbb{K}$,*

$$\operatorname{reg}(I(G)^s) \leqslant 2s + \operatorname{reg}(I(G)) - 2.$$

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Note that the values $\mathrm{reg}(I(G)^s)$ and even $b(I(G))$ may depend on the characteristic of the field, as was shown recently in [23, Exa. 1.2], and earlier for monomial ideals generated in higher degrees in [6]. For various classes of graphs (e.g. cochordal) Conjecture 1.1 is known to be true.

Our Theorem 1.2 below proves this conjecture for all graphs for second powers and for all bipartite graphs for all powers. As a result it verifies inequality (1.1) for all bipartite graphs. To demonstrate that this bound is sharp, we observe that all complete bipartite graphs G with nonempty edge set satisfy $\mathrm{reg}(I(G)^s) = 2s$ for all s, by the well known Theorem 2.7 below.

Our main theorem is the following:

Theorem 1.2. (i) *Let G be a finite simple graph. Then*

$$\mathrm{reg}(I(G)^2) \leqslant \mathrm{reg}(I(G)) + 2.$$

(ii) *Further, if G is also bipartite, then for all $s \geqslant 1$ we have*

$$\mathrm{reg}(I(G)^s) \leqslant 2s + \mathrm{reg}(I(G)) - 2.$$

Part (i) is proved topologically, via Hochster's formula and various uses of Mayer–Vietoris long exact sequence. Suspension plays a key role here. Part (ii) for $s > 2$ is proved algebraically, via various uses of short exact sequences for related ideals. Part (ii) improves the main result of [18], which proves that if G is bipartite, then for all $s \geqslant 2$ there holds $\mathrm{reg}(I(G)^s) \leqslant 2s + \mathrm{cochord(G)} - 1$, where cochord(G) is the cochordal number of G (see [18] for definition).

Outline: Preliminaries are given in Section 2, Theorem 1.2 is proved in Section 3, and concluding remarks are given in Section 4.

2. Preliminaries

In this section, we set up the basic definitions and notation needed for the main results. Let M be a finitely generated graded $R = \mathbb{K}[x_1, \dots, x_n]$-module. Write the graded minimal free resolution of M in the form:

$$0 \longrightarrow \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{p,j}(M)} \xrightarrow{\psi_p} \cdots \xrightarrow{\psi_1} \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \xrightarrow{\psi_0} M \longrightarrow 0,$$

where $p \leqslant n$, $R(-j)$ indicates the ring R with the shifted grading such that, for all $a \in \mathbb{Z}$, $R(-j)_a = R_{a-j}$. The nonnegative integers $\beta_{(i,j)}(M)$ are called i^{th}-graded Betti numbers of M in degree j.

The Castelnuovo-Mumford regularity (or regularity) of M is defined to be

$$\mathrm{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

Let I be a nonzero proper homogeneous ideal of R. Then it follows from the definition that $\mathrm{reg}(I) = \mathrm{reg}(R/I) + 1$.

Let I be any ideal of R and $a \in R$ any element, the the *colon ideal* $(I : a)$ is defined as the ideal $(I : a) \coloneqq (b \mid b \in R, ab \in I)$.

Polarization is a process that creates a squarefree monomial out of a monomial, possibly in a larger polynomial ring. If $f = x_1^{e_1} \cdots x_n^{e_n}$ is a monomial in $\mathbb{K}[x_1, \dots, x_n]$ then the polarization of f is defined as $\widetilde{f} = x_{11} \cdots x_{1e_1} x_{21} \cdots x_{2e_2} \cdots x_{n1} \cdots x_{ne_n}$ in the ring $\mathbb{K}[x_{11}, \dots x_{1e_1}, x_{21}, \dots x_{2e_2}, \dots, x_{n1}, \dots, x_{ne_n}]$. For convenience we identify the variable x_{i1} with x_i, so the new polynomial ring extends the old one. For a monomial ideal I with minimal monomial generators $\{m_1, \dots, m_k\}$, we define the *polarization* of I as $\widetilde{I} \coloneqq (\widetilde{m_1}, \cdots, \widetilde{m_k})$ in a suitable ring, see e.g [16] or [21, Sec. 1.6]. In the special case where degree of a variable $u = x_i$ is two in one or more generators then we call the unique new variable x_{i2} a *whisker variable* and denote it by u' for short. In this paper we repeatedly use an important property of polarization:

Theorem 2.1 (e.g. [21, Cor. 1.6.3(a)]). *Let I be a monomial ideal in R. Then*

$$\mathrm{reg}(I) = \mathrm{reg}(\widetilde{I}).$$

One of the main techniques that is used in this paper is that of short exact sequences. In particular we shall use the following well known result [2, Lemma. 2.11]:

Theorem 2.2. (i) *Let I be a homogeneous ideal in a polynomial ring R and m be an element of degree d in R. Then the following is a short exact sequence:*

$$0 \longrightarrow \frac{R}{(I:m)} \xrightarrow{\cdot m} \frac{R}{I} \longrightarrow \frac{R}{I+(m)} \longrightarrow 0.$$

Hence

$$\operatorname{reg}(I) \leqslant \max\{\operatorname{reg}(I:m)+d, \operatorname{reg}(I+(m))\}.$$

(ii) *In case I is squarefree and x a variable, then also* $\operatorname{reg}(I,x) \leqslant \operatorname{reg} I$.

Let G be a (finite simple) graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and the edge set $E(G)$. The *edge ideal* $I(G)$ of G is defined as the ideal in R:

$$I(G) = (x_i x_j | x_i x_j \in E(G)).$$

For example, edge ideal of a 5-cycle is $(x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$.

The next couple of theorems allow for induction when increasing the power of an edge ideal.

Theorem 2.3 ([2, Thm. 5.2]). *For any graph G and any $s \geqslant 1$, let the set of minimal monomial generators of $I(G)^s$ be $\{m_1, \ldots, m_k\}$. Then*

$$\operatorname{reg}(I(G)^{s+1}) \leqslant \max\{\operatorname{reg}(I(G)^s), \operatorname{reg}((I(G)^{s+1}) : m_l) + 2s, 1 \leqslant l \leqslant k\}.$$

By identifying the variables with the vertices of G, interpreting edges as squarefree quadratic monomials, defining neighborhood for any vertex c, $N(c) \coloneqq \{z \in V(G) : cz \in E(G)\}$ and using [2, Thm. 5.2] we get the following corollary:

Corollary 2.4. (i) *The ideal $(I(G)^{s+1} : m_l)$ is a quadratic monomial ideal.*

(ii) *For the special case where $s = 1$ and $m = ab$ is an edge we have*

$$(I(G)^2 : ab) = I(G) + (xy | x \in N(a), y \in N(b)).$$

For bipartite graphs we further have:

Theorem 2.5 ([1, Lem. 3.2, Prop. 3.4]). *Let G be a bipartite graph and $s \geqslant 1$ an integer. Then $(I(G)^{s+1} : e_1 \cdots e_s)$ is a quadratic squarefree monomial ideal where for all i we have $e_i \in E(G)$. Moreover, the graph G' associated to $(I(G)^{s+1} : e_1 \cdots e_s)$ is bipartite on the same vertex set and same bipartition as G.*

For any bipartite graph G we have

$$(I(G)^{s+1} : e_1 \cdots e_s) = ((I(G)^2 : e_i)^s : \Pi_{j \neq i} e_j)$$

where for all $l \in \{1, \ldots, s\}$ we have $e_l \in E(G)$.

Remark 2.6. From Theorem 2.5 one can show a slightly more general equality involving colons, under the same assumptions: for all k and any $s \geqslant k+1$,

$$(I(G)^{s+1} : e_1 \cdots e_s) = ((I(G)^{k+1} : e_1 \cdots e_k)^{s-k+1} : e_{k+1} \cdots e_s).$$

This follows from the following sequence of arguments:

(a) From Theorem 2.5 for $i = 1$ it follows that:

$$(I(G)^{s+1} : e_1 \cdots e_s) = ((I(G)^2 : e_1)^s : e_2 \cdots e_s).$$

(b) The first part of the Theorem 2.5 tells us that $(I(G)^2 : e_1)$ is a bipartite edge ideal.

(c) By Corollary 2.4 above we get that each e_i is an edge of the bipartite graph associated to $(I(G)^2 : e_1)$.

(d) Applying the Theorem 2.5 again on the edge ideal $(I(G)^2 : e_1)$ we get:

$$((I(G)^2 : e_1)^s : e_2 \cdots e_s) = (((I(G)^2 : e_1)^2 : e_2)^{s-1} : e_3 \cdots e_s).$$

(e) By a similar application of the last result with $s = 2$ we have

$$(I(G)^3 : e_1 e_2) = ((I(G)^2 : e_1)^2 : e_2).$$

(f) Combining these we get
$$(I(G)^{s+1} : e_1 \cdots e_s) = ((I(G)^3 : e_1e_2)^{s-1} : e_3 \cdots e_s).$$
(g) Inductively, assuming that for a fixed k one has for any $s \geqslant k+1$
$$(I(G)^{s+1} : e_1 \cdots e_s) = ((I(G)^{k+1} : e_1e_2 \cdots e_k)^{s-k+1} : e_{k+1} \cdots e_s).$$
(h) Again applying Theorem 2.5 on the bipartite edge ideal $(I(G)^{k+1} : e_1 \cdots e_k)$ we get
$$((I(G)^{k+1} : e_1 \cdots e_k)^{s-k+1} : e_{k+1} \cdots e_s) = (((I(G)^{k+1} : e_1 \cdots e_k)^2 : e_{k+1})^{s-k} : e_{k+2} \cdots e_s).$$
(i) But by induction
$$(I(G)^{k+2} : e_1 \cdots e_{k+1}) = ((I(G)^{k+1} : e_1 \cdots e_k)^2 : e_{k+1}).$$
So we get
$$((I(G)^{k+1} : e_1 \cdots e_k)^{s-k+1} : e_{k+1} \cdots e_s) = ((I(G)^{k+2} : e_1 \cdots e_{k+1})^{s-k} : e_{k+2} \cdots e_s),$$
as claimed.

Now we recall some basic definitions about graphs and simplicial complexes that will be useful.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A subgraph $H \subseteq G$ is called *induced* if $\{u, v\}$ is an edge of H if and only if u and v are vertices of H and $\{u, v\}$ is an edge of G. A *clique* in a graph is an induced subgraph that is a *complete graph*.

For $u \in V(G)$, let $N_G(u) = \{v \in V(G) \mid \{u, v\} \in E(G)\}$ and $N_G[u] = N_G(u) \cup \{u\}$. For $U \subseteq V(G)$, denote by $G \smallsetminus U$ the induced subgraph of G on the vertex set $V(G) \smallsetminus U$.

Let G be a graph. We denote the graph consisting of two disjoint edges by $2K_2$. A graph without induced copy of $2K_2$ is called $2K_2$-free or *gap-free* graph. The *complement* of a graph G, denoted by G^c, is the graph on the same vertex set in which $\{u, v\}$ is an edge of G^c if and only if it is not an edge of G. Then G is gap-free if and only if G^c contains no induced 4-cycle.

A graph G is chordal (also called triangulated) if every induced cycle in G has length 3, and is co-chordal if the complement graph G^c is chordal. The following important theorem(s) characterizes the edge ideals with regularity 2, and the regularity of their powers.

Theorem 2.7. (1) (*Fröberg* [14, Thm.1]) *For any graph G, we have G^c is chordal if and only if* $\operatorname{reg}(I(G)) = 2$.
(2) (*Herzog–Hibi–Zheng* [17, Thm.1.2]) *Further, in this case* $\operatorname{reg}(I(G)^s) = 2s$ *for any s.*

A *simplicial complex* Δ on a vertex set $\{1, \ldots, n\}$ is a collection of subsets of $\{1, \ldots, n\}$ such that if $\tau \in \Delta, \sigma \subseteq \tau$ then we have $\sigma \in \Delta$ and all singletons belong to that collection; if τ is such a subset belonging to Δ then τ is called a *face* of Δ. The *induced subcomplex* $\Delta[A]$ of Δ on vertex set $A \subset \{1, \ldots, n\}$ is the collection of faces τ' of Δ such that $\tau' \subset A$. Clearly the induced subcomplex is a simplicial complex itself. We denote by $\mathrm{V}(\Delta)$ the set of vertices of Δ.

For sets A and B we sometimes write $A \cup_C B$ for the set $A \cup B$ as a shorthand for denoting their intersection by $C = A \cap B$. Likewise for simplicial complexes.

The *link* of a vertex v in Δ is
$$\operatorname{link}_v(\Delta) = \{\tau | \tau \cup \{v\} \in \Delta, \{v\} \cap \tau = \varnothing\}.$$

The (open) *star* of a vertex v in a simplicial complex Δ is the set of all faces that contain v, namely $\operatorname{st}_v(\Delta) = \{\tau | \tau \in \Delta, v \in \tau\}$. The *closed star* $\bar{\operatorname{st}}_v(\Delta)$ of v is defined by the smallest subcomplex that contains $\operatorname{st}_v(\Delta)$. The *antistar* of vertex v is defined as the subcomplex $\operatorname{ast}_v(\Delta) = \{\tau \in \Delta | \tau \cap \{v\} = \varnothing\}$.

The *join* of two simplicial complexes Δ_1 and Δ_2 is defined by $\Delta_1 * \Delta_2 = \{\sigma \cup \tau | \sigma \in \Delta_1, \tau \in \Delta_2\}$. The *suspension* of a simplicial complex Δ, w.r.t. two points a and b not vertices of Δ, is the join defined by $\Sigma_{a,b}\Delta = \Delta * \{\{a\}, \{b\}, \varnothing\}$; its geometric realization is homeomorphic to the topological suspension of the space Δ.

For H a graph let $\operatorname{cl}(H)$ denote its clique complex, i.e. the simplicial complex whose faces are the cliques of H.

The following formulation of regularity follows from the so called Hochster's formula (see [22] for further details):

Theorem 2.8 (Hochster's formula). *For every graph G whose edge set is nonempty and $\Delta = \mathrm{cl}(G^c)$ we have:*

$$\mathrm{reg}(G) \coloneqq \mathrm{reg}(I(G)) = \max\{l+2 : \exists W \subseteq \mathrm{V}(\Delta), \widetilde{H}_l(\Delta[W];k) \neq 0\}.$$

(Sometimes Hochster's formula is phrased with the extra condition on W that $|W| = l+2+i$ for some $i \geqslant 0$. However, this condition is superfluous: if $|W| \leqslant l+1$ then either $\dim(\Delta[W]) < l$ or $\Delta[W]$ is the simplex on $l+1$ vertices. In both cases $\widetilde{H}_l(\Delta[W]) = 0$.)

3. Main Results

We first prove that $\mathrm{reg}(I(G)^2 : e) \leqslant \mathrm{reg}(I(G))$ for every edge e of a graph G. This will lead us to our main result via a series of short exact sequence arguments. For that we first prove the following:

Theorem 3.1. *Let G be a graph, $ab \in E(G)$ and $G' = G \cup \{xy : x \neq y, ax, by \in E(G)\}$. Then*

$$\mathrm{reg}(I(G')) \leqslant \mathrm{reg}(I(G)).$$

Proof. Denote $\Delta = \mathrm{cl}(G^c)$ and $\Delta' = \mathrm{cl}(G'^c)$. Let $A = \mathrm{V}(\bar{\mathrm{st}}_a(\Delta)), B = \mathrm{V}(\bar{\mathrm{st}}_b(\Delta)), C = A \cap B$ and $D = \mathrm{V}(\Delta) - (A \cup B)$. With this notation we observe the following crucial relation between Δ and Δ':

$$\Delta' = (\Delta[A] \cup_{\Delta[C]} \Delta[B]) \cup_{\Delta[C]} (\Delta[C] \cup_{d \in D} (\{d\} * \Delta[C \cap \mathrm{V}(\mathrm{st}_d(\Delta))])). \tag{3.1}$$

We obtain this equality simply from the definition of G', where every neighbour of a is connected to every neighbour of b. The above decomposition of Δ' shows us how to prove the theorem using Hochster's formula and Mayer–Vietoris sequences, detailed next.

Let $\mathrm{reg}(I(G')) = l+2$ and W be a subset of the vertices of minimal size such that $\widetilde{H}_l(\Delta'[W]) \neq 0$. Decompose $W = W_1 \cup_{W_C} W_2 \subseteq V$ where $W_1 = W \cap (A \cup_C B)$ (recall $C = A \cap B$), $W_2 = W \cap (C \cup D)$ and $W_C = W_1 \cap W_2 = C \cap W$.

If $W \cap D = \varnothing$ then Eq.(3.1) implies $\Delta'[W] = \Delta[W]$ and thus $\mathrm{reg}(I(G)) \geqslant l+2$ as claimed. Thus, we may assume $W \cap D \neq \varnothing$, so $|W_1| < |W|$ and thus, by minimality of W, $\widetilde{H}_l(\Delta'[W_1]) = 0$.

If $|W_2| < |W|$ then minimality of $|W|$ also imply $\widetilde{H}_l(\Delta'[W_2]) = 0$. By (3.1) the decomposition $\Delta'[W] = \Delta'[W_1] \cup_{\Delta'[W_C]} \Delta'[W_2]$ holds. Consider the following Mayer–Vietoris exact sequence:

$$\widetilde{H}_l\Delta'[W_1] \bigoplus \widetilde{H}_l\Delta'[W_2] \longrightarrow \widetilde{H}_l\Delta'[W] \longrightarrow \widetilde{H}_{l-1}(\Delta'[W_C]).$$

As the two summands on the left term vanish, exactness implies $\widetilde{H}_{l-1}(\Delta'[W_C]) \neq 0$. Now by (3.1) we have $\Delta'[W_C] = \Delta[W_C]$, hence suspension gives $0 \neq \widetilde{H}_l(\Delta[W_C] * \{\{a\},\{b\},\varnothing\}) = \widetilde{H}_l(\Delta[\{a,b\} \cup W_C])$. We conclude $\mathrm{reg}(\Delta) \geqslant l+2$ as claimed.

Thus we may assume additionally that $W_2 = W$. Denote $X = \Delta'[W] = \Delta'[W_2]$ and let $d \in W \cap D$. By (3.1) $\mathrm{link}_d X$ is an induced subcomplex of $\Delta[C]$, and further, the suspension of $\mathrm{link}_d X$ by the vertices a and b is an induced subcomplex of Δ.

Thus, if $\widetilde{H}_{l-1}(\mathrm{link}_d X) \neq 0$ then $0 \neq \widetilde{H}_l(\{\{a\},\{b\},\varnothing\} * \mathrm{link}_d X) = \widetilde{H}_l(\Delta[\{a,b\} \cup V(\mathrm{link}_d X)])$, and thus $\mathrm{reg}(I(G)) \geqslant l+2$ as claimed. So we may assume $\widetilde{H}_{l-1}(\mathrm{link}_d X) = 0$.

Consider the Mayer–Vietoris long exact sequence corresponding to the union $X = \mathrm{ast}_d X \cup_{\mathrm{link}_d X} \bar{\mathrm{St}}_d X$:

$$\widetilde{H}_l(\mathrm{ast}_d X) \bigoplus \widetilde{H}_l(\overline{\mathrm{St}}_d X) \longrightarrow \widetilde{H}_l(X) \longrightarrow \widetilde{H}_{l-1}(\mathrm{link}_d X).$$

By assumption, the right term is zero while the middle term is nonzero, and closed stars have vanishing homology, hence $\widetilde{H}_l(\mathrm{ast}_d X) = \widetilde{H}_l(\Delta'[W \smallsetminus \{d\}]) \neq 0$. This contradicts the minimality of W, so in fact this last case cannot occur. $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. (i) By Theorem 2.3 we just need to prove

$$\operatorname{reg}(I(G)^2 : ab) \leqslant \operatorname{reg}(I(G)) \quad \text{for all} \quad ab \in E(G). \tag{3.2}$$

Let $J = I(G) + (uv | u \neq v, u \in N(a), v \in N(b))$. By Corollary 2.4 we have:

$$(I(G)^2 : ab) = J + (u^2 | u \in N(a) \cap N(b)).$$

By Theorem 2.1, $\operatorname{reg}(I(G)^2 : ab) = \operatorname{reg}((\widetilde{I(G)^2 : ab}))$, the polarization. Here $L := (\widetilde{I(G)^2 : ab}) = J + (uu' | u \in N(a) \cap N(b))$ for new whisker variables u' in a larger polynomial ring (defined in Section 2). So, it is enough to prove that $\operatorname{reg}(L) \leqslant \operatorname{reg}(I(G))$.

If $N(a) \cap N(b) = \varnothing$ then the assertion follows by Theorem 3.1. Now let $N(a) \cap N(b) = \{u_1, \dots, u_k\}$. Consider the following short exact sequences:

$$0 \longrightarrow \frac{R}{(L : u_1)}(-1) \to \frac{R}{L} \to \frac{R}{(L, u_1)} \to 0$$

$$0 \longrightarrow \frac{R}{((L, u_1) : u_2)}(-1) \to \frac{R}{(L, u_1)} \to \frac{R}{(L, u_1, u_2)} \to 0$$

$$\vdots$$

$$0 \longrightarrow \frac{R}{((L, u_1, \dots, u_{k-1}) : u_k)}(-1) \to \frac{R}{(L, u_1, \dots, u_{k-1})} \to \frac{R}{(L, u_1, \dots, u_k)} \to 0.$$

Now observe that $(L, u_1, \dots, u_k) = J + (\text{variables})$ and for every i, $((L, u_1, \dots u_{i-1}) : u_i) = (L : u_i) + (\text{variables})$. By repeated use of Theorem 2.2 (both parts) we have that $\operatorname{reg}(L) \leqslant \max\{\operatorname{reg}(J), \operatorname{reg}(L : u) + 1, u \in N(a) \cap N(b)\}$. Now by Theorem 3.1 $\operatorname{reg}(J) \leqslant \operatorname{reg}(I(G))$. It is enough to show that $\operatorname{reg}(L : u) + 1 \leqslant \operatorname{reg}(I(G))$ for all $u \in N(a) \cap N(b)$.

To prove this we use three facts:

A. Going mod any set of variables makes the regularity stay same or go down (follows from Theorem 2.2 (ii)).

B. Adjoining a disjoint set of variables keeps the regularity same (follows directly from Hochster's formula (Theorem 2.7)).

C. Adding a disjoint edge makes the regularity go up by one. (This follows from Hochster's formula (Theorem 2.7) applied for the suspension of the original clique complex; e.g. by Künneth formula (see e.g. [5, 9.12]) or [30, Lemma.8]).

Now, $(L : u) = L + (u') + (s : u \neq s \in N_G(u) \cup N_G(a) \cup N_G(b)) = I(G'[U]) + (s : s \in N_{G'}(u))$ where G' is the polarization graph with $L = I(G')$ and U is the set of non-neighbours of u in G'. By Fact B, $\operatorname{reg}((L : u)) = \operatorname{reg}(I(G'[U]))$.

Note that $a, b \notin U$, and that $G'[U] = G[U]$, as $u \in N_G(a) \cap N_G(b)$ and no edge involving a whisker vertex has its other vertex in U. By Fact C, $\operatorname{reg}(I(G[U]) + (ab)) = 1 + \operatorname{reg}(I(G[U]))$.

By Fact A, modding $I(G)$ by the variables in $(N(a) \cup N(b)) \smallsetminus \{a, b\}$, we obtain the inequality $\operatorname{reg}(I(G[U]) + (ab)) \leqslant \operatorname{reg}(I(G))$.

Combined, we have:

$$\operatorname{reg}(L : u) + 1 = \operatorname{reg}(I(G[U])) + 1 \leqslant \operatorname{reg}(I(G)),$$

as desired. This proves (3.2) and hence proves part (i) of the theorem.

(ii) Here the underlying graph is bipartite. In part (i) we have already proved that $\operatorname{reg}(I(G)^2 : e) \leqslant \operatorname{reg}(I(G))$ for any graph G with $e \in E(G)$. We may assume $s \geqslant 3$ by part (i).

By the second part of Theorem 2.5 we get

$$\begin{aligned}(I(G)^s : e_1 \dots e_{s-1}) &= ((I(G)^2 : e_1)^{s-1} : e_2 \cdots e_{s-1}) = (((I(G)^2 : e_1)^2 : e_2)^{s-2} : e_3 \cdots e_{s-1}) \\ &= \dots = (\dots((I(G)^2 : e_1)^2 : e_2)^2 \dots : e_{s-2})^2 : e_{s-1}).\end{aligned}$$

Hence using the first part of Theorem 2.5 and (3.2) that $\operatorname{reg}(I(G)^2 : e) \leqslant \operatorname{reg}(I(G))$ for all graph G and for all $e \in E(G)$ we get that:

$$\operatorname{reg}(I(G)^s : e_1 \dots e_{s-1}) = \dots = \operatorname{reg}(\dots((I(G)^2 : e_1)^2 : e_2)^2 \dots : e_{s-2})^2 : e_{s-1})$$

$$\leqslant \operatorname{reg}(\ldots((I(G)^2 : e_1)^2 : e_2)^2 \ldots : e_{s-1})$$
$$\leqslant \operatorname{reg}(\ldots((I(G)^2 : e_1)^2 : e_2)^2 \ldots : e_{s-2}) \leqslant \ldots \leqslant \operatorname{reg}(I(G)^2 : e_1) \leqslant \operatorname{reg}((I(G)).$$

This gives us the result by Theorem 2.3. □

4. Further Research

In this section we discuss some questions for further research.

In the introduction we mentioned the constants $a(I), b(I), c(I)$ related to asymptotic stability of the regularity of powers of a homogenous ideal I. So far we focused on $b(I)$ for edge ideals, now we turn to discuss $c(I)$.

Due to the asymptotic stability we have that for a homogeneous ideal I generated in degree d there exists a minimal integer $c(I)$ such that $\operatorname{reg}(I^{s+1}) - \operatorname{reg}(I^s) = d$ for all $s \geqslant c(I)$. We have proved that for all bipartite graphs G we have $\operatorname{reg}(I(G)^s) - \operatorname{reg}(I(G)) \leqslant 2s - 2$. However the behaviour of the sequence $\{\operatorname{reg}(I^s)\}$ can be irregular for smaller s values even for edge ideals. In fact there are examples of bipartite graphs where $\operatorname{reg}(I(G)^2) = \operatorname{reg}(I(G)) + 1$ (for example one can check that this is the case for the bipartite edge ideal of the 8-cycle $(x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_1y_2, x_2y_4, x_3y_1, x_4y_3)$).

Can $c(I)$ be bounded by some simple invariants of I, for homogenous ideals? Conca [7] showed that for any given integer $d > 1$ there exists an ideal J generated by $d+5$ monomials of degree $d+1$ in 4 variables such that $\operatorname{reg}(J^k) = k(d+1)$ for every $k < d$ and $\operatorname{reg}(J^d) \geqslant d(d+1) + d - 1$. In particular, $c(I)$ cannot be bounded above in terms of the number of variables only, not even for monomial ideals in general. Further, a result of Raicu [29] gives binomial ideals I_n on n^2 variables, generated in degree 2, with $c(I_n) = n - 1$. All these fit into the framework of the following question that served as broad aim for various researchers since the works of Cutkosky, Herzog and Trung, and Kodiyalam ([3, 7, 8, 9, 10, 15, 17, 19, 20, 26, 27]):

Question 4.1. *For homogeneous ideals I on n variables, generated in degree d, is $c(I)$ bounded above by a function of d and n?*

It has been conjectured by Banerjee and Mukundan [4] that for all bipartite graphs G, we have $c(I(G)) \leqslant 2$. It is known for cochordal, gap free plus cricket/diamond/4-cycle free graphs [2, 17, 13, 12]. Apart from edge ideals, it was shown by Conca and Herzog [8] that polymatroidal ideals have linear resolutions and powers of polymatroidal ideals are polymatroidal ideals. So for the class of all polymatroidal ideals $c(I) = 1$.

Finally we conclude by a discussion on a related conjecture by [27]:

Conjecture 4.2 ([27]). *If $\operatorname{reg}(I(G)) \leqslant 3$ and G^c has no induced 4-cycle then for all $s \geqslant 2$ we have $\operatorname{reg}(I(G)^s) = 2s$.*

Very recently this conjecture was verified for $s = 2$ and $s = 3$ by Minh and Vu [24]. Theorem 2.3 was proved by Banerjee in his thesis to study this conjecture and related other problems, based on Theorem 2.2. We now explain why this inductive approach via colon ideals that is used in this paper (and also in [1, 2, 3, 26, 27, 18, 19]) can *not* be used directly to settle Conjecture 4.2.

Every 2-dimensional simplicial complex Δ can be subdivided so that the resulted complex is flag-no-square, see [11] or [28, Lem. 2.3], i.e. $\Delta = \operatorname{cl}(H)$ where H is a graph with no *induced* 4-cycles. In particular, we choose such H so that Δ triangulates the dunce hat, a contractible 2-dimensional complex. Thus, all subcomplexes of Δ have vanishing homology in dimension $\geqslant 2$. Further, the link of every vertex $a \in \Delta$ is an induced subcomplex (as Δ is a clique complex) with nonzero first homology. For an edge $ab \in G \coloneqq H^c$, the construction of $\Delta' = \operatorname{cl}(G'^c)$ from Theorem 3.1 satisfies $\operatorname{link}_a \Delta' = \operatorname{link}_a \Delta$ is an induced subcomplex of Δ'.

We conclude that $\operatorname{reg}(I(G)) = 3$ and by Corollary 2.4(ii) for every edge $ab \in G$ also $\operatorname{reg}((I(G)^2 : ab) = 3$. Thus, if $\operatorname{reg}(I(G)^2) = 4$ as Conjecture 4.2 suggests, then Theorem 2.2 cannot be directly applied to prove it.

On the other hand, if $\operatorname{reg}(I(G)^2) > 4$ then this will be a counterexample. Unfortunately we could not verify the value of $\operatorname{reg}(I(G)^2)$ due to computational limitations. It will be great if this can be verified in the future.

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