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Abstract We establish an isomorphism between the center $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ of the Heisenberg category defined by Khovanov in [13] and the algebra Λ^* of shifted symmetric functions defined by Okounkov–Olshanski in [18]. We give a graphical description of the shifted power and Schur bases of Λ^* as elements of $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$, and describe the curl generators of $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ in the language of shifted symmetric functions. This latter description makes use of the transition and co-transition measures of Kerov [10] and the noncommutative probability spaces of Biane [2]

1. Introduction

In [13], Khovanov introduces a graphical calculus of oriented planar diagrams and uses it to define a linear monoidal category $\mathcal{H}'$, which he proposes as a categorification of the Heisenberg algebra. We denote by $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ the endomorphism algebra of the monoidal unit in $\mathcal{H}'$. The commutative algebra $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ is, by definition, the algebra of closed oriented planar diagrams modulo the relations of the Khovanov graphical calculus. In his study of morphism spaces of $\mathcal{H}'$, Khovanov introduces two sets of generators for $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$: the clockwise curls $\{c_k\}_{k\geqslant 0}$ and the counterclockwise curls $\{\tilde{c}_k\}_{k\geqslant 2}$. He then establishes algebra isomorphisms

$$\mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \cong \mathbb{C}[c_0, c_1, c_2, \dots] \cong \mathbb{C}[\tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \dots],$$

and describes a recursion for expressing the clockwise and counterclockwise curls in terms of each other. He then relates $\mathcal{H}'$ to representation theory by defining a sequence of functors $f_k^{\mathcal{H}'}$ from $\mathcal{H}'$ to bimodule categories for symmetric groups. A consequence of the existence of these functors is the existence of surjective algebra homomorphisms,

$$f_n^{\mathcal{H}'} : \mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow Z(\mathbb{C}[S_n]),$$

from $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ to the center of the group algebra of each symmetric group. Based in part on this, Khovanov suggests that there should be a close connection between $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ and the asymptotic representation theory of symmetric groups. Furthermore, one might hope that $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ in fact gives a diagrammatic description of some algebra of pre-existing combinatorial interest.

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The main goal of the current paper is to make precise the connection between $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ and both the asymptotic representation theory of symmetric groups and algebraic combinatorics. We do this by establishing an isomorphism between

$$\varphi : \mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow \Lambda^*,$$

where Λ^* is the *shifted symmetric functions* of Okounkov–Olshanski [18]. (See Theorem 5.3.) The algebra of shifted symmetric functions Λ^* is a deformation of the algebra of symmetric functions. As is the case for $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$, there are surjective algebra homomorphisms

$$f_n^{\Lambda^*} : \Lambda^* \longrightarrow Z(\mathbb{C}[S_n]),$$

to the center of the group algebra of each symmetric group. The isomorphism $\varphi : \mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow \Lambda^*$ is canonical, in that it intertwines the homomorphisms $f_n^{\mathcal{H}'}$ and $f_n^{\Lambda^*}$.

The isomorphism $\varphi : \mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow \Lambda^*$ allows us to give a graphical description of several important bases of Λ^*. For example, the shifted power sum denoted $p^{\#}_\lambda$ in [18] appears in $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ as the closure of a permutation of cycle type λ. The shifted Schur function s^*_λ appears as the closure of a Young symmetrizer of type λ. (See Theorem 5.4).

In the other direction, it is also reasonable to ask for a description of the image of Khovanov's curl generators c_k and $\tilde{c}_k$ as elements of Λ^*. It turns out that the right language for such a description is that of noncommutative probability theory. In [10], Kerov introduces, for each partition λ, a pair of finitely supported probability measures on $\mathbb{R}$; these probability measures are known as the *transition* and *co-transition* measures, or sometimes as growth and decay. In work of Biane [2], these probability measures appear as the compactly-supported measures associated to self-adjoint operators on a noncommutative probability space, and as a result they are basic objects of interest at the intersection of representation theory and noncommutative probability theory. In particular, the *moments* and *Boolean cumulants* of the transition and co-transition measures may be regarded as elements of Λ^*. In Theorem 5.5, we show that the isomorphism φ takes Khovanov's curl generators c_k and $\tilde{c}_k$ to scalar multiples of the kth moments of Kerov's transition and co-transition measures. In fact, the close relationship between the transition and co-transition measures themselves yields two independent descriptions of the image of the curl generator c_k: it is equal to a scalar multiple of both the kth moment of the co-transition measure and the $(k+2)$th Boolean cumulant of the transition measure. The observation that the Boolean cumulants of the transition measure are equal to the moments of the co-transition measure is closely connected to the adjointness of induction and restriction functors between representation categories of symmetric groups. A dictionary between several of the bases of $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ and Λ^* is given in Table 1 below.

The existence of a relationship between $\mathcal{H}'$ and free probability, and indeed, much of this paper, was anticipated by Khovanov in [13]. The relationship between generators of $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ and the noncommutative probability spaces of [2] may be seen as a further manifestation of the "planar structure" of free probability; the many connections between noncommutative probability and other mathematical subjects with planar structure are emphasized in the work of Guionnet, Jones and Shlyakhtenko [6].

In addition to the center of $\mathcal{H}'$, another algebra of interest in the study of $\mathcal{H}'$ is its trace (or zeroth Hochschild homology). The trace of $\mathcal{H}'$ is an infinite-dimensional noncommutative algebra, which may be defined diagrammatically as the algebra of diagrams on an annulus; the trace acts naturally on $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ by gluing annular diagrams around planar ones. In [4], the trace of $\mathcal{H}'$ is shown to be isomorphic to the $W_{1+\infty}$ algebra of conformal field theory. An action of $W_{1+\infty}$ on Λ^* appears to be well known in the vertex algebra community, and such an action is constructed explicitly

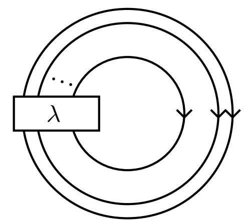
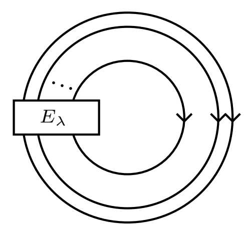
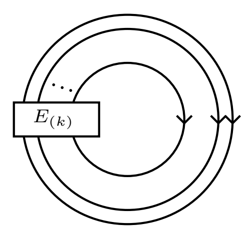
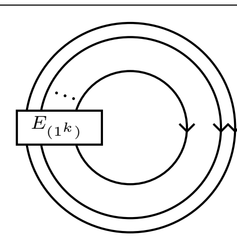
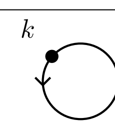
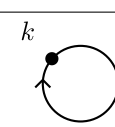
Λ^*	diagram in $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$
$p^{\#}_{\lambda}$	
s^*_{λ}	$\frac{1}{\dim L^{\lambda}}$ 
h^*_k	
e^*_k	
$\widehat{m}_k$	
$\widehat{b}_{k+2} = p^{\#}_1 \check{m}_k$	

TABLE 1. A dictionary between Λ^* and diagrams in $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$.

in the work of Lascoux–Thibon [14]. Thus the isomorphism $\varphi : \mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow \Lambda^*$ of Theorem 5.3, together with the main result of [4], gives a purely planar realization, via Khovanov's graphical calculus, of Lascoux–Thibon's construction.

2. The symmetric group and its normalized character theory

We begin by establishing notation related to partitions and Young diagrams. Let $\mathcal{P}_n$ be the set of partitions of n and

$$\mathcal{P} := \bigcup_{n \geqslant 0} \mathcal{P}_n.$$

For this section let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathcal{P}_n$ and $\mu = (\mu_1, \dots, \mu_t) \in \mathcal{P}_k$ with $n \geqslant k$. We assume that $\lambda_1 \geqslant \dots \geqslant \lambda_r > 0$ and $\mu_1 \geqslant \dots \geqslant \mu_t > 0$. When $i > r$ (respectively $i > t$) we understand that $\lambda_i = 0$ (resp. $\mu_i = 0$). We use the following notation throughout:

- $n = \lambda_1 + \lambda_2 + \dots + \lambda_r =: |\lambda|$.
- $\lambda \cup \mu$ is the partition formed from the union of the parts of λ and μ.
- $\mu \subseteq \lambda$ if $\mu_i \leqslant \lambda_i$ for all $i \geqslant 1$. When this is the case, we write λ/μ for the associated skew diagram.
- $\phi_{k,n} : \mathcal{P}_k \hookrightarrow \mathcal{P}_n$ is the function defined by $\phi_{k,n}(\mu) = \mu \cup 1^{n-k} \in \mathcal{P}_n$.

Example 2.1. If $\mu = (3,2,1,1,1) \in \mathcal{P}_8$ then $\phi_{8,10}(\mu) = (3,2,1,1,1,1,1) \in \mathcal{P}_{10}$.

We freely identify $\mu \in \mathcal{P}$ with its corresponding Young diagram, which we draw using Russian notation (see Example 2.2). If $\square$ is a cell in the ith row and jth column of μ then the *content* of $\square$ is defined to be the integer

$$\mathrm{cont}(\square) := j - i.$$

We say that a cell $\square \notin \mu$ is i-addable with respect to μ if it has content i and adding it to μ gives a Young diagram. We say that a cell $\square \in \mu$ is i-removable with respect to μ if it has content i and removing it from μ gives a Young diagram. We call two sequences $a_1, \dots, a_d$ and $b_1, \dots, b_{d-1}$ *interlacing* when

$$a_1 < b_1 < a_2 < \dots < a_{d-1} < b_{d-1} < a_d.$$

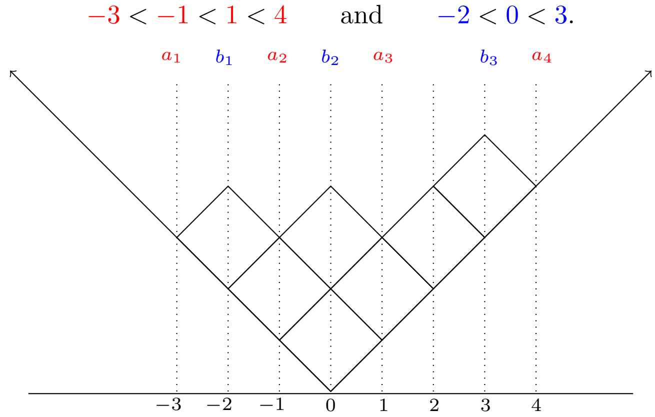
The *center* of this pair of sequences is defined as the quantity $(a_1 + \dots + a_d) - (b_1 + \dots + b_{d-1})$. Each Young diagram μ uniquely defines two integer valued interlacing sequences $a_1, \dots, a_d$ and $b_1, \dots, b_{d-1}$ where:

- $a_1, \dots, a_d$ is the ordered list of all a_j such that there exists an a_j-addable cell with respect to μ.
- $b_1, \dots, b_{d-1}$ is the ordered list of all b_j such that there exists a b_j-removable cell with respect to μ.

From this description it is clear that $a_1, \dots, a_d$ and $b_1, \dots, b_{d-1}$ are interlacing.

Example 2.2. Let $\mu = (4,2,1)$. Then μ yields the interlacing sequences

$$-3 < -1 < 1 < 4 \qquad \text{and} \qquad -2 < 0 < 3.$$



Proposition 2.3 ([11]). *If $a_1, \dots, a_d$ and $b_1, \dots, b_{d-1}$ are the pair of interlacing sequences associated to a Young diagram then their center is 0. Conversely, any pair of integer valued interlacing sequences with center 0 is associated to a Young diagram.*

When $\mu \subseteq \lambda$ and $\lambda/\mu = \square$, then we write $\mu \nearrow \lambda$. In other words, $\mu \nearrow \lambda$ whenever we can obtain λ from μ by adding a single cell. If $a_1, \dots, a_d$ and $b_1, \dots, b_{d-1}$ are the interlacing sequences associated to μ, then we denote by $\mu^{(i)}$ the Young diagram that we get by adding a cell of content a_i, so that

$$\mathrm{cont}(\mu^{(i)}/\mu) = a_i.$$

Similarly, we denote by $\mu_{(i)}$ the Young diagram that we get by removing a cell of content b_i from μ, so that

$$\mathrm{cont}(\mu/\mu_{(i)}) = b_i.$$

Note that $\mu_{(i)} \nearrow \mu$, while $\mu \nearrow \mu^{(i)}$.

Example 2.4. If $\mu = (4, 2, 1)$ as in Example 2.2, we have

$$\begin{array}{lll} \mu^{(1)} = (4,2,1,1) & & \\ \mu^{(2)} = (4,2,2) & & \mu_{(1)} = (4,2) \\ \mu^{(3)} = (4,3,1) & \text{and} & \mu_{(2)} = (4,1,1) \\ \mu^{(4)} = (5,2,1) & & \mu_{(3)} = (3,2,1). \end{array} \tag{1}$$

Let S_n be the symmetric group. S_n is generated by Coxeter generators $s_1, \dots, s_{n-1}$ where s_i is the adjacent transposition $(i, i+1)$. We identify $\mathbb{C}[S_0] \cong \mathbb{C}$. If $g \in S_n$ has cycle type $\lambda \vdash n$, then we write $\mathrm{sh}(g) := \lambda$. For $k \leqslant n$, there is an embedding $S_k \hookrightarrow S_n$ called the *standard embedding* which sends S_k to the subgroup generated by $s_1, \dots, s_{k-1}$, which stabilizes $\{k+1, \dots, n\}$ pointwise. We extend this embedding by linearity to get an embedding of group algebras which we denote by $\iota_{k,n} : \mathbb{C}[S_k] \hookrightarrow \mathbb{C}[S_n]$. We write 1_k for the identity element in $\mathbb{C}[S_k]$ so that $\iota_{k,n}(1_k) = 1_n$. We write $w_{0,n}$ for the longest element of S_n by Coxeter length. $w_{0,n}$ is the product of disjoint transpositions $(1, n), (2, n-1), (3, n-2), \dots$ and in terms of Coxeter generators can be written

$$w_{0,n} = (s_1 s_2 \dots s_{n-1})(s_1 s_2 \dots s_{n-2}) \dots (s_1 s_2)(s_1).$$

For $\lambda \vdash n$, let L^λ be the simple $\mathbb{C}[S_n]$-module (the irreducible S_n representation) corresponding to λ, $E_\lambda \in Z(\mathbb{C}[S_n])$ its associated Young idempotent (or central idempotent), and $\chi^\lambda : \mathbb{C}[S_n] \to \mathbb{C}$ its associated character. Abusing notation, we write $\chi^\lambda(\mu)$ for $\chi^\lambda(g)$ when $\mathrm{sh}(g) = \mu$ (this notation is well-defined since χ^λ is a class function). The *normalized character* $\widetilde{\chi}^\lambda : \bigoplus_{k \leqslant n} \mathbb{C}[S_k] \to \mathbb{C}$ associated to λ is defined so that for $x \in \mathbb{C}[S_k]$,

$$\widetilde{\chi}^\lambda(x) := \frac{\chi^\lambda(\iota_{k,n}(x))}{\dim L^\lambda} = \frac{\chi^\lambda(\iota_{k,n}(x))}{\chi^\lambda(1_n)}. \tag{2}$$

Let $\mu = (\mu_1, \dots, \mu_t) \vdash k \leqslant n$ and set $\pi_\mu = 1_k$ if $\mu = (1^k)$ and otherwise

$$\begin{aligned} \pi_\mu &= \left(s_{k-1} \dots s_{k-\mu_t+1}\right) \dots \left(s_{\mu_1+\mu_2-1} \dots s_{\mu_1+1}\right) \cdot \left(s_{\mu_1-1} \dots s_2 s_1\right) \\ &= (k, k-1, \dots, k-\mu_t+1) \dots (\mu_1+\mu_2, \dots, \mu_1+1) \cdot (\mu_1, \dots, 2, 1) \in S_k. \end{aligned}$$

We define

$$\sigma_{\mu,n} := w_{0,n}^{-1}(\iota_{k,n}(\pi_\mu)) w_{0,n} \in S_n.$$

Observe that $\sigma_{\mu,n}$ has cycle type $\phi_{k,n}(\mu)$ and fixes $1, 2, \dots, n-k$ pointwise.

Example 2.5. Let $\mu = (3,2) \vdash 5$, then

$$\pi_\mu = (s_4)(s_2 s_1) = (5,4)(3,2,1)$$

and we see that $\mathrm{sh}(\pi_\mu) = \mu$. For $n = 8$,

$$\sigma_{\mu,8} = s_4 s_6 s_7 = (4,5)(6,7,8),$$

while for $n = 10$,

$$\sigma_{\mu,10} = (6,7)(8,9,10).$$

The elements

$$\{\sigma_{(1),n}, \sigma_{(2),n}, \sigma_{(3),n}, \dots, \sigma_{(n),n}\} = \{1_n,\ s_{n-1},\ s_{n-2}s_{n-1},\ \dots,\ s_1 s_2 \dots s_{n-1}\}$$

are the minimal length left coset representatives of S_{n-1} in S_n. We extend this observation in the following lemma.

Lemma 2.6. *For $k < n$, the elements of the set*

$$\{\sigma_{(i_n),n}\sigma_{(i_{n-1}),n-1} \dots \sigma_{(i_{k+1}),k+1} \mid 1 \leqslant i_j \leqslant j\ \}$$

are the minimal length left coset representatives of S_k in S_n. We denote this set by $\mathcal{LC}^n_k$.

Proof. Let $G_{n-2} \subset G_{n-1} \subset G_n$ be a sequence of nested groups and $\{g_1^{(n-1)}, \dots, g_{t_{n-1}}^{(n-1)}\}$ and $\{g_1^{(n)}, \dots, g_{t_n}^{(n)}\}$ be a collection of left coset representatives of G_{n-2} in G_{n-1} and G_{n-1} in G_n respectively. Then $\{g_{i_2}^{(n)} g_{i_1}^{(n-1)} \mid 1 \leqslant i_1 \leqslant t_{n-1}$ and $1 \leqslant i_2 \leqslant t_n\}$ is a collection of left coset representatives of G_{n-2} in G_n. This fact can be extended inductively to calculate the left coset representatives of G_k in G_n for any nested sequence $G_k \subset G_{k+1} \subset \dots \subset G_n$ when a collection of left coset representatives of G_{i-1} in G_i is known for $k+1 \leqslant i \leqslant n$. The lemma then follows from the observation that

$$\begin{aligned}\{\sigma_{(i_j),j} \mid 1 \leqslant i_j \leqslant j\} &= \{\sigma_{(1),j}, \sigma_{(2),j}, \dots, \sigma_{(j),j}\} \\ &= \{1_j, s_{j-1}, s_{j-2}s_{j-1}, \dots, s_1 s_2 \dots s_{j-1}\}\end{aligned}$$

is a collection of left coset representatives of S_{j-1} in S_j. □

We note that $|\mathcal{LC}^n_k| = (n \downharpoonleft n-k)$, where the *falling factorial power* is defined as

$$(x \downharpoonleft j) = \begin{cases} x(x-1)\dots(x-j+1), & \text{if } j = 1,2,\dots \\ 1, & \text{if } j = 0. \end{cases} \tag{3}$$

Example 2.7. We have

$$\begin{aligned}\mathcal{LC}^4_3 &= \{1_4,\ s_3,\ s_2 s_3,\ s_1 s_2 s_3\}, \\ \mathcal{LC}^3_2 &= \{1_3,\ s_2,\ s_1 s_2\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{LC}^4_2 = \{1_4,\ & s_3,\ s_2 s_3,\ s_1 s_2 s_3, \\ & s_2,\ s_3 s_2,\ s_2 s_3 s_2,\ s_1 s_2 s_3 s_2, \\ & s_1 s_2,\ s_3 s_1 s_2,\ s_2 s_3 s_1 s_2,\ s_1 s_2 s_3 s_1 s_2\}.\end{aligned}$$

2.1. The center of $\mathbb{C}[S_n]$. For $\mu \vdash k \leqslant n$, set

$$C_{\mu,n} := \sum_{\substack{g \in S_n, \\ \mathrm{sh}(g) = \phi_{k,n}(\mu)}} g.$$

The elements $\{C_{\mu,n}\}_{\mu \vdash n}$ are a basis for the center of the symmetric group algebra, $Z(\mathbb{C}[S_n])$. We write $z_{\mu,n}$ for the size of the centralizer of an element in S_n with cycle type $\phi_{k,n}(\mu)$. Note that when $\mu \vdash n$, then $z_{\mu,n} = z_\mu$.

Definition 2.8. *For $\mu = (\mu_1, \ldots, \mu_t) \vdash k \leqslant n$, set*

$$A_{\mu,n} := \sum_{g \in \mathcal{LC}^n_{n-k}} g\sigma_{\mu,n} g^{-1}. \tag{4}$$

$A_{\mu,n}$ is called the normalized conjugacy class sum *associated to μ in $\mathbb{C}[S_n]$.*

Alternatively, $A_{\mu,n}$ may be written as

$$A_{\mu,n} = \sum (i_1, \ldots, i_{\mu_1}) \ldots (i_{k-\mu_t+1}, \ldots, i_k) \tag{5}$$

where this sum is taken over all distinct k-tuples $(i_1, \ldots, i_k)$ of elements from $\{1, 2, \ldots, n\}$. From (5) a counting argument shows that

$$A_{\mu,n} = \frac{z_{\mu,n}}{(n-k)!} C_{\mu,n}. \tag{6}$$

It follows from (6) that $A_{\mu,n} \in Z(\mathbb{C}[S_n])$.

Example 2.9. Let $k \leqslant n$. When $\mu = (k) \vdash k$, then $z_{(k),n} = k(n-k)!$ so that

$$A_{(k),n} = kC_{(k),n}.$$

The elements $A_{\mu,n}$ are important in the study of the asymptotic character theory of symmetric groups [12]. They also appear in connection with the algebra of partial permutations [8]. If $\mu \vdash k \leqslant n$ and $\lambda \vdash n$ then

$$\widetilde{\chi}^\lambda(A_{\mu,n}) = (n \downharpoonright k) \frac{\chi^\lambda(\phi_{k,n}(\mu))}{\dim L^\lambda}. \tag{7}$$

The following is well-known.

Proposition 2.10. *When restricted to $Z(\mathbb{C}[S_n])$, the normalized character $\widetilde{\chi}^\lambda$ is an algebra homomorphism from $Z(\mathbb{C}[S_n])$ to $\mathbb{C}$.*

$Z(\mathbb{C}[S_n])$ is also generated by symmetric polynomials in the Jucys–Murphy elements $\{J_i\}_{1 \leqslant i \leqslant n} \subseteq \mathbb{C}[S_n]$, where

$$J_1 = 0, \qquad \text{and} \qquad J_k = (1,k) + (2,k) + \cdots + (k-1,k), \qquad 2 \leqslant k \leqslant n.$$

Written in terms of the Coxeter generators

$$J_k = \sum_{i=1}^{k-1} s_i \ldots s_{k-2} s_{k-1} s_{k-2} \ldots s_i. \tag{8}$$

2.2. The transition measure and co-transition measure. In this section we recall the notion of transition and co-transition measures, also known as growth and decay, respectively. Assume that $\lambda \vdash n$ and let $a_1, \ldots, a_d$ and $b_1, \ldots, b_{d-1}$ be the interlacing sequences associated to λ. Recall that $\lambda^{(1)}, \ldots, \lambda^{(d)}$ are the partitions of $n+1$ such that $\mathrm{cont}(\lambda^{(i)}/\lambda) = a_i$, while $\lambda_{(1)}, \ldots, \lambda_{(d-1)}$ are the partitions of $n-1$ such that $\mathrm{cont}(\lambda/\lambda_{(i)}) = b_i$.

For $1 \leqslant i \leqslant d$, the *transition probabilities* for λ are defined as

$$\widehat{q}_\lambda(\lambda^{(i)}) := \frac{\dim(L^{\lambda^{(i)}})}{(n+1)\dim(L^\lambda)}.$$

The *transition measure* $\widehat{\omega}_\lambda$ is then the probability measure on $\mathbb{R}$ defined by

$$\widehat{\omega}_\lambda := \sum_{i=1}^{d} \widehat{q}_\lambda(\lambda^{(i)})\delta_{a_i} \tag{9}$$

where δ_{a_i} is the Dirac delta measure with support on $a_i \in \mathbb{R}$. Dually, for $1 \leqslant i \leqslant d-1$ the *co-transition probabilities* of λ are

$$\check{q}_\lambda(\lambda_{(i)}) := \frac{\dim(L^{\lambda_{(i)}})}{\dim(L^\lambda)}$$

and the *co-transition measure* $\check{\omega}_\lambda$ is

$$\check{\omega}_\lambda := \sum_{i=1}^{d-1} \check{q}_\lambda(\lambda_{(i)})\delta_{b_i}. \tag{10}$$

These probability measures were first investigated by Kerov ([10, 11]). They are fundamental tools in the study of the asymptotic representation theory of symmetric groups. For example, in [11] Kerov shows that

$$\sum_{i=1}^{d} \frac{\widehat{q}_\lambda(\lambda^{(i)})}{z-a_i} = \frac{(z-b_1)\dots(z-b_{d-1})}{(z-a_1)\dots(z-a_{d-1})(z-a_d)} \tag{11}$$

and

$$z - |\lambda| \sum_{i=1}^{d-1} \frac{\check{q}_\lambda(\lambda_{(i)})}{z-b_i} = \frac{(z-a_1)\dots(z-a_{d-1})(z-a_d)}{(z-b_1)\dots(z-b_{d-1})}. \tag{12}$$

The kth moment associated to the transition measure $\widehat{\omega}_\lambda$ is given by

$$\widehat{m}_k(\lambda) = \sum_{i=1}^{d} a_i^k \widehat{q}_\lambda(\lambda^{(i)})$$

while the kth moment associated to the co-transition measure $\check{\omega}_\lambda$ is given by

$$\check{m}_k(\lambda) = \sum_{i=1}^{d-1} b_i^k \check{q}_\lambda(\lambda_{(i)}).$$

Consider the series

$$\widehat{\mathcal{M}}_\lambda(z) := \sum_{k=0}^{\infty} \widehat{m}_k(\lambda) z^{-k-1} \quad \text{and} \quad \check{\mathcal{M}}_\lambda(z) := z - \sum_{k=0}^{\infty} |\lambda| \check{m}_k(\lambda) z^{-k-1}.$$

Lemma 2.11. *For $\lambda \in \mathcal{P}$*

$$\widehat{\mathcal{M}}_\lambda(z) = (\check{\mathcal{M}}_\lambda(z))^{-1}. \tag{13}$$

Proof. This follows directly from equations (11), (12) and Lemma 5.1 in [11]. ☐

Boolean cumulants linearize convolution of probability measures under the notion of Boolean independence [19]. The most convenient way to define the boolean cumulants $\{\widehat{b}_k(\lambda)\}_{k \geqslant 1}$ associated to $\widehat{\omega}_\lambda$ in our case is as the coefficients on the multiplicative inverse of $\widehat{\mathcal{M}}_\lambda(z)$. We write

$$\widehat{\mathcal{B}}_\lambda(z) = z - \sum_{k=-1}^{\infty} \widehat{b}_{k+2}(\lambda) z^{-k-1} = (\widehat{\mathcal{M}}_\lambda(z))^{-1} \tag{14}$$

(for the equivalence of this definition and other analytic definitions see Section 2 of [19]). Given Lemma 2.11 this definition immediately implies the following proposition.

Proposition 2.12. *Let* $\lambda \in \mathcal{P}$ *and* $k \geqslant 0$*, then* $\widehat{b}_1(\lambda) = 0$ *and*

$$\widehat{b}_{k+2}(\lambda) = |\lambda| \check{m}_k(\lambda). \tag{15}$$

Remark 2.13. The equality (14) can be rewritten as

$$\sum_{i=1}^{k} \widehat{m}_{k-i}(\lambda) \widehat{b}_i(\lambda) = \widehat{m}_k(\lambda). \tag{16}$$

For general information about the relationship between moments, Boolean cumulants, and other families of cumulants see [1].

There is a more algebraic approach to the transition measure due to Biane [2]. Let

$$\mathrm{pr}_{n-1} : \mathbb{C}[S_n] \to \mathbb{C}[S_{n-1}] \subset \mathbb{C}[S_n]$$

be the projection map defined on S_n by

$$\mathrm{pr}_{n-1}(g) = \begin{cases} g & \text{if } g \in \mathbb{C}[S_{n-1}] \\ 0 & \text{otherwise.} \end{cases}$$

In the context of probability theory, pr_{n-1} is sometimes known as the *conditional expectation.*

Proposition 2.14. *For* $\lambda \vdash n$*,*

$$\widehat{m}_k(\lambda) = \widetilde{\chi}^\lambda[\mathrm{pr}_n(J_{n+1}^k)] \tag{17}$$

and

$$\widehat{b}_{k+2}(\lambda) = |\lambda| \check{m}_k(\lambda) = \widetilde{\chi}^\lambda \left(\sum_{i=1}^{n} s_i \dots s_{n-1} J_n^k s_{n-1} \dots s_i \right). \tag{18}$$

Proof. The statement of (17) appears in [3, Section 4]. A detailed proof is given in Theorem 9.23 of [7]. To get (18) note that since characters are class functions,

$$\widetilde{\chi}^\lambda \left(\sum_{i}^{n} s_i \dots s_{n-1} J_n^k s_{n-1} \dots s_i \right) = |\lambda| \widetilde{\chi}^\lambda(J_n^k).$$

As J_n eigenspaces, L^λ decomposes as

$$L^\lambda \cong \bigoplus_{i=1}^{d-1} L^{\lambda_{(i)}}$$

with $L^{\lambda_{(i)}}$ corresponding to eigenvalue b_i [21]. Hence,

$$|\lambda| \widetilde{\chi}^\lambda(J_n^k) = |\lambda| \sum_{i=1}^{d-1} \frac{\dim(\lambda_{(i)}) b_i^k}{\dim(\lambda)} = |\lambda| \check{m}_k(\lambda) = \widehat{b}_{k+2}(\lambda). \qquad \square$$

Proposition 2.14 is related to the fact that we are working in a noncommutative probability space (that is, a von Neumann algebra equipped with a normal faithful trace). In our case the algebra is $\mathrm{End}(L^\lambda) \otimes M_{n+1}(\mathbb{C})$ and $\widehat{\omega}_\lambda$ then arises from the distribution of a self-adjoint element in this algebra (see Proposition 3.3 in [2]).

3. Symmetric functions and shifted symmetric functions

In order to define the algebra of shifted symmetric functions, we first recall the classical symmetric functions. Let Λ_n be the algebra of symmetric polynomials over $\mathbb{C}$ in $x_1, \dots, x_n$. This algebra is graded by polynomial degree. Recall that for $n \geqslant 0$ there is a homomorphism

$$\Lambda_{n+1} \to \Lambda_n \tag{19}$$

given by setting $x_{n+1} = 0$ in Λ_{n+1}. One can define the algebra of symmetric functions as the projective limit $\Lambda = \varprojlim \Lambda_n$ taken in the category of graded algebras. We recall three collections of algebraically independent generators of Λ:

- elementary symmetric functions $e_1, e_2, e_3, \dots,$
- complete homogeneous symmetric functions $h_1, h_2, h_3, \dots,$
- power sum symmetric functions $p_1, p_2, p_3, \dots$

For $\{f_k\}_{k \geqslant 1}$ equal to any of these three sets of generators and $\lambda = (\lambda_1, \dots, \lambda_r)$ we write $f_\lambda := f_{\lambda_1} \dots f_{\lambda_r}$. We denote the basis of Schur functions by $\{s_\lambda\}_{\lambda \in \mathcal{P}}$. We refer the reader to [16] and [20] for background on Λ.

Let Λ_n^* be the algebra of polynomials over $\mathbb{C}$ in $x_1, \dots, x_n$, which become symmetric in the new variables $x_i' = x_i - i$. This algebra is filtered by polynomial degree. In analogy to Λ_{n+1}, setting $x_{n+1} = 0$ in Λ_{n+1}^* gives a homomorphism

$$\Lambda_{n+1}^* \to \Lambda_n^* \tag{20}$$

which respects the filtration. Using (20), set

$$\Lambda^* := \varprojlim \Lambda_n^*,$$

where this limit is taken in the category of filtered algebras. Λ^* is called the *algebra of shifted symmetric functions*.

Because Λ^* is filtered, we can consider the associated graded algebra $\mathrm{gr}(\Lambda^*)$.

Proposition 3.1 ([18, Proposition 1.5]). $\mathrm{gr}(\Lambda^*)$ *is canonically isomorphic to* Λ.

Remark 3.2. It is noted in Remark 1.7 of [18] that we may also view Λ^* as a deformation of Λ. Let $\Lambda_n^*(\theta)$ be the algebra of polynomials in $x_1, \dots, x_n$ which are symmetric in the new variables $x_i' = x_i + c - i\theta$ for $1 \leqslant i \leqslant n$ and where $c \in \mathbb{C}$. Define $\Lambda^*(\theta) = \varprojlim \Lambda_n^*(\theta)$. Then $\Lambda^*(0) = \Lambda$ and $\Lambda^*(1) = \Lambda^*$. In fact for all $\theta \neq 0$, $\Lambda^*(\theta) \cong \Lambda^*$.

3.1. Bases of Λ^*.

In [18] Okounkov and Olshanski introduced a remarkable basis for Λ^* called the shifted Schur functions. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition with $\lambda_1 \geqslant \dots \geqslant \lambda_n \geqslant 0$ (note that here we allow components of a partition to be zero). The *shifted Schur polynomial in n variables, indexed by λ* is the ratio of two $n \times n$ determinants,

$$s_\lambda^*(x_1, \dots, x_n) = \frac{\det[(x_i + n - i \downharpoonright \lambda_j + n - j)]}{\det[(x_i + n - i \downharpoonright n - j)]}, \tag{21}$$

where $1 \leqslant i, j \leqslant n$ and $(x \downharpoonright k)$ is defined in (3). This polynomial belongs to Λ_n^*. It is shown in [18] that

$$s_\lambda^*(x_1, \dots, x_n, 0) = s_\lambda^*(x_1, \dots, x_n). \tag{22}$$

This implies that for fixed λ, letting $n \to \infty$ gives a well-defined element s_λ^* of Λ^*. The elements $\{s_\lambda^*\}_{\lambda \in \mathcal{P}} \in \Lambda^*$ are called the *shifted Schur functions* and form a basis for Λ^*. There is a linear map $\Lambda^* \to \mathrm{gr}(\Lambda^*) \cong \Lambda$ which sends $f \in \Lambda^*$ to its top homogeneous component which is an element of Λ. Under this map

$$s_\lambda^* \mapsto s_\lambda$$

or alternatively,

$$s^*_\lambda = s_\lambda + \text{l.o.t.} \tag{23}$$

where l.o.t. means lower order terms in polynomial degree.

In analogy to the classical case, the *elementary shifted functions* can be defined as $e^*_k := s^*_{(1^k)}$, while the *complete shifted functions* can be defined as $h^*_k := s^*_{(k)}$. More explicitly:

$$e^*_k(x_1, x_2, \dots) = \sum_{1 \leqslant i_1 < \dots < i_k < \infty} (x_{i_1} + k - 1)(x_{i_2} + k - 2) \dots x_{i_k}$$

and

$$h^*_k(x_1, x_2, \dots) = \sum_{1 \leqslant i_1 \leqslant \dots \leqslant i_k < \infty} (x_{i_1} - k + 1)(x_{i_2} - k + 2) \dots x_{i_k}.$$

Let F be the linear isomorphism $F : \Lambda \to \Lambda^*$ which sends $s_\lambda \mapsto s^*_\lambda$. Define the element $p^\#_\lambda \in \Lambda^*$ to then be

$$p^\#_\lambda := F(p_\lambda), \tag{24}$$

where p_λ is the power sum symmetric function. The elements $p^\#_\lambda$ are one of several shifted analogues of the power sums. For $\lambda \vdash n$, the transition coefficients between the power-sum and Schur bases are given by the character tables of the symmetric group (see [20]):

$$p_\lambda = \sum_{\mu \vdash n} \chi^\mu(\lambda) s_\mu.$$

It follows directly from definition (24) that

$$p^\#_\lambda = \sum_{\mu \vdash n} \chi^\mu(\lambda) s^*_\mu. \tag{25}$$

Note also that by (23) and (25),

$$p^\#_\lambda = p_\lambda + \text{l.o.t.} \tag{26}$$

Since the power symmetric functions $p_1, p_2, \dots$ are algebraically independent and generate Λ, it follows from Proposition 3.1 and (26) that $p^\#_1, p^\#_2, \dots$ are algebraically independent and generate Λ^*. Similarly, since $\{p_\lambda\}_{\lambda \in \mathcal{P}}$ is a basis for Λ, $\{p^\#_\lambda\}_{\lambda \in \mathcal{P}}$ is a basis for Λ^*. For more properties of the basis $\{p^\#_\lambda\}$ see [9].

Remark 3.3. Let $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$. While it is true that in Λ, $p_{\lambda_1} \dots p_{\lambda_r} = p_\lambda$, in general

$$p^\#_{\lambda_1} \dots p^\#_{\lambda_r} \neq p^\#_\lambda.$$

However, by (26)

$$p^\#_{\lambda_1} \dots p^\#_{\lambda_r} = p^\#_\lambda + \text{l.o.t.}$$

3.2. Λ^* as functions on $\mathcal{P}$. Let $\mathrm{Fun}(\mathcal{P}, \mathbb{C})$ be the algebra of functions from $\mathcal{P}$ to $\mathbb{C}$ with pointwise multiplication. Viewing $\mu = (\mu_1, \dots, \mu_t) \vdash k$ as the sequence $(\mu_1, \dots, \mu_t, 0, 0, \dots)$, we can evaluate $f \in \Lambda^*$ on μ by setting

$$f(\mu) = f(\mu_1, \dots, \mu_t, 0, 0, \dots). \tag{27}$$

Since $(\mu_1, \dots, \mu_t, 0, 0, \dots)$ has only a finite number of nonzero values, it is clear that (27) is well-defined. In fact f is uniquely defined by its values on $\mathcal{P}$. Thus Λ^* may be realized as a subalgebra of $\mathrm{Fun}(\mathcal{P}, \mathbb{C})$. This fact is used repeatedly en route to establishing many of the fundamental results about shifted symmetric functions in [12] and [18].

For $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ and α a cell in the Young diagram corresponding to λ with coordinates (i, j), the hook length of α is defined as $h(\alpha) := \lambda_i - j + \lambda'_j - i + 1$, where $\lambda' = (\lambda'_1, \dots, \lambda'_{\lambda_1})$ is the partition conjugate to λ. Set $H(\lambda)$ as the product of all hooklengths in λ,

$$H(\lambda) := \prod_{\alpha \in \lambda} h(\alpha).$$

The following is known as the "Characterization Theorem" of [17].

Theorem 3.4. *For $\mu \vdash k$, s^*_μ is the unique element of Λ^* such that* $\deg(s^*_\mu) \leqslant k$ *and*

$$s^*_\mu(\lambda) = \delta_{\mu\lambda} H(\mu)$$

for all $\lambda \in \mathcal{P}$ such that $|\lambda| \leqslant |\mu|$.

This theorem along with (25) then give the following proposition.

Proposition 3.5 ([18]). *For $\mu \vdash k$, $\lambda \vdash n$,*

$$p^\#_\mu(\lambda) = \begin{cases} \frac{(n\downarrow k)}{\dim L^\lambda} \chi^\lambda(\phi_{k,n}(\mu)) & k \leqslant n \\ 0 & \textit{otherwise.} \end{cases} \tag{28}$$

Remark 3.6. We will later use the fact that $p^\#_1 = x_1 + x_2 + \cdots = p_1$, so that $p^\#_1(\lambda) = |\lambda|$ for all $\lambda \in \mathcal{P}$.

In Section 2.2 we introduced the moments $\{\widehat{m}_k(\lambda)\}$ (resp. $\{\check{m}_k(\lambda)\}$) of the transition measure (resp. co-transition measure) associated to a partition λ and the corresponding Boolean cumulants $\{\widehat{b}_k(\lambda)\}$. We can interpret all of these as elements of $\mathrm{Fun}(\mathcal{P}, \mathbb{C})$ via

$$\lambda \xmapsto{\widehat{m}_k} \widehat{m}_k(\lambda), \qquad \lambda \xmapsto{\check{m}_k} \check{m}_k(\lambda), \quad \text{and} \quad \lambda \xmapsto{\widehat{b}_k} \widehat{b}_k(\lambda).$$

Henceforth we omit the partition argument from $\widehat{m}_k$, $\check{m}_k$, and $\widehat{b}_k$ in cases where we want to emphasize that we are considering these as elements of $\mathrm{Fun}(\mathcal{P}, \mathbb{C})$.

Proposition 3.7 ([15, Theorem 6.4]). *As elements of* $\mathrm{Fun}(\mathcal{P}, \mathbb{C})$*, $\widehat{m}_k$ and $\widehat{b}_k$ belong to Λ^*.*

Remark 3.8. In [15] Section 5, Lassalle shows that with the appropriate alphabet A_λ (which is specific to each partition λ),

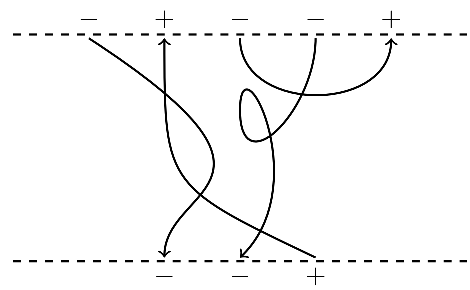
$$\widehat{m}_k(\lambda) = h_k(A_\lambda) \qquad \text{and} \qquad \widehat{b}_k(\lambda) = (-1)^{k-1} e_k(A_\lambda). \tag{29}$$

4. The Heisenberg category $\mathcal{H}'$

In [13], Khovanov defines an additive $\mathbb{C}$-linear monoidal category $\mathcal{H}'$ which we will call the *Heisenberg category*. The objects in $\mathcal{H}'$ are generated by two objects Q_+ and Q_-. Following the notation of [13], we denote $Q_{\epsilon_1} \otimes \cdots \otimes Q_{\epsilon_m}$ by Q_ϵ where $\epsilon = \epsilon_1 \dots \epsilon_m$ is a finite sequence of pluses and minuses. The unit object, $\mathbb{1}$, corresponds to the empty sequence $Q_\varnothing$.

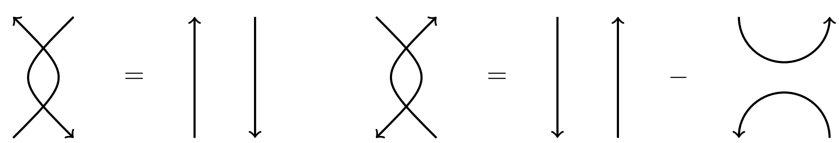
The collection of morphisms $\mathrm{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'})$, for two sequences ϵ and ϵ' is the $\mathbb{C}$-vector space spanned by planar diagrams modulo some local relations. The diagrams are oriented compact 1-manifolds immersed in the strip $\mathbb{R} \times [0, 1]$, modulo rel boundary isotopies. The endpoints of the 1-manifolds are located at $\{1, \dots, m\} \times \{0\}$ and $\{1, \dots, n\} \times \{1\}$, where m and n are the lengths of ϵ and ϵ', respectively. Further, the orientation of the 1-manifolds at the endpoints must match the signs in the sequences ϵ and ϵ'. Triple intersections are not allowed.

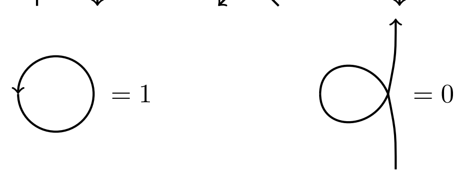
EXAMPLE 4.1. The diagram

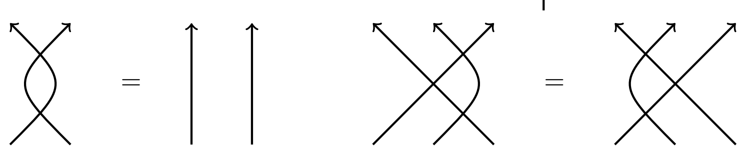


is a morphism from Q_{--+} to Q_{-+--+}.

The composition of two morphisms is achieved by stacking diagrams. The local relations for diagrams are:

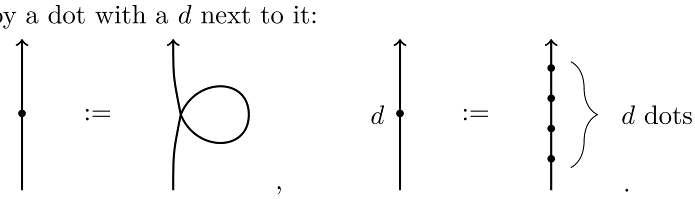
(30) 

(31) 

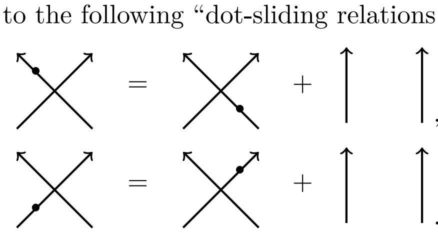
(32) 

The relations (30) and (31) are motivated by the Heisenberg relation $pq = qp + 1$, while the relations (32) are motivated by the symmetric group relations.

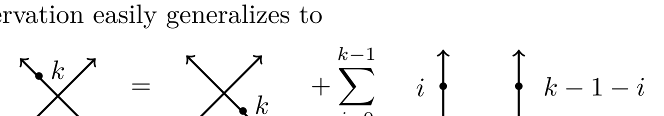
It is convenient to denote a right curl by a dot on a strand, and a sequence of d right curls by a dot with a d next to it:

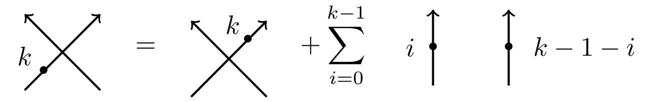


Using relations (30)–(32) it can be shown that a dot can be moved across intersection points, according to the following "dot-sliding relations" [13]:

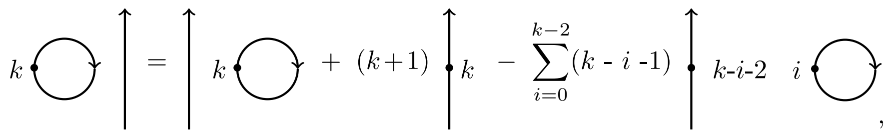


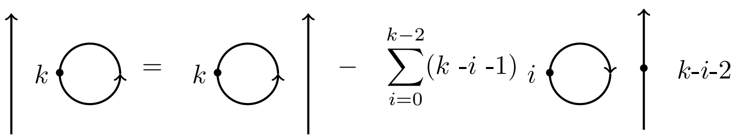
This observation easily generalizes to

(33) 

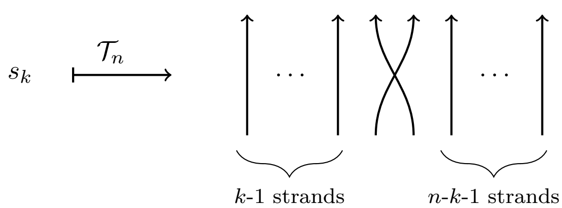
(34) 

Another consequence of relations (30)–(32) are the "bubble moves" [13]:

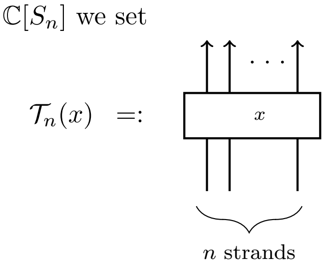
(35) 

(36) 

Note that relations (32) imply that there is a homomorphism $\mathcal{T}_n : \mathbb{C}[S_n] \to \mathrm{End}_{\mathcal{H}'}(Q_{+^n})$ which sends



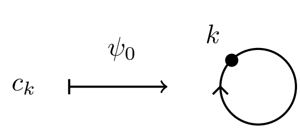
Diagrammatically, for $x \in \mathbb{C}[S_n]$ we set



The appearance of the group algebra $\mathbb{C}[S_n]$ as endomorphisms in $\mathcal{H}'$ is responsible for the connection between $\mathcal{H}'$ and the representation theory of symmetric groups.

4.1. The endomorphism algebra $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$. The center of $\mathcal{H}'$ is $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$, that is, the algebra of endomorphisms of the monoidal unit object $\mathbb{1}$. Diagrammatically, the algebra $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ is the commutative $\mathbb{C}$-algebra spanned by all closed diagrams, with multiplication given by juxtaposition of diagrams. The algebra structure of $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ was determined by Khovanov in [13]. Let $\mathbb{C}[c_0, c_1, c_2, \dots]$ be the polynomial algebra in countably many indeterminants $\{c_i\}_{i \geqslant 0}$.

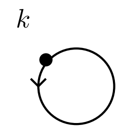
Theorem 4.2 ([13, Proposition 3]). *The map* $\psi_0 : \mathbb{C}[c_0, c_1, \dots] \to \mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ *which sends*

(37) 

is an algebra isomorphism.

Henceforth we will freely identify c_k with its image in $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$. Another natural set of diagrams to consider are the counterclockwise-oriented circles with k right-twist curls on them. Set

$\tilde{c}_k \quad :=$

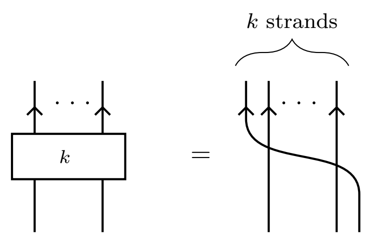


It follows from the relations in (31) that $\tilde{c}_0 = 1$ and $\tilde{c}_1 = 0$.

Lemma 4.3 ([13, Proposition 2]). *For* $k > 0$*,*

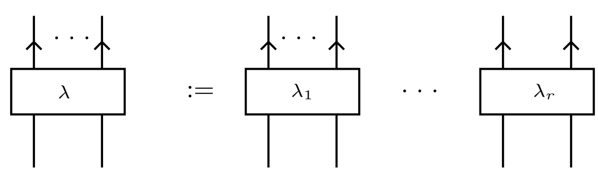
(38)
$$\tilde{c}_{k+1} = \sum_{i=0}^{k-1} c_{k-1-i}\tilde{c}_i.$$

Another class of elements in $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ we consider are those arising from the closure of permutations (that is, closures of morphisms in the image of $\mathcal{T}_n$). We define



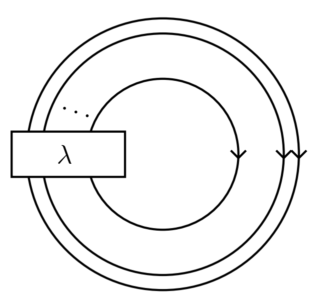
For $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$, let

(39)



then we define

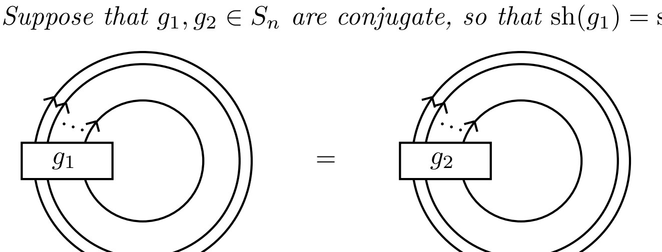
$\alpha_\lambda \quad :=$



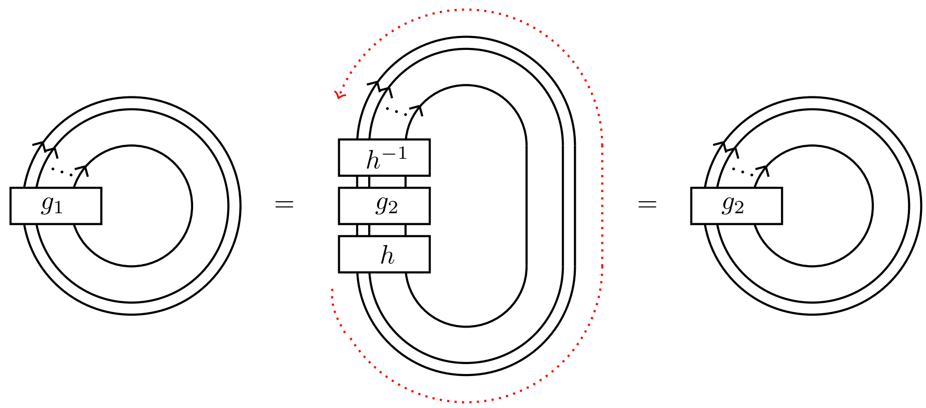
with $\alpha_k := \alpha_{(k)}$.

Lemma 4.4 below shows that we could replace the permutation in (39) by the image under $\mathcal{T}_n$ of any $g \in S_n$ such that $\mathrm{sh}(g) = \lambda$. We choose (39) because it will be convenient for later calculations.

Lemma 4.4. *Suppose that* $g_1, g_2 \in S_n$ *are conjugate, so that* $\mathrm{sh}(g_1) = \mathrm{sh}(g_2)$*. Then*



Proof. We use the fact that $g_1 = h^{-1}g_2h$ for some $h \in S_n$. Replacing g_1 by $h^{-1}g_2h$, we slide h around the closed diagram to cancel it with h^{-1},



□

Remark 4.5. In [13] Khovanov also studies $\mathcal{H}$, the Karoubi envelope (or idempotent completion) of $\mathcal{H}'$. It is $\mathcal{H}$ which conjecturally categorifies the Heisenberg algebra. Since taking the Karoubi envelope of a category does not change its center, all the results we prove about $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ also hold for $\mathrm{End}_{\mathcal{H}}(\mathbb{1})$.

4.2. Diagrams as bimodule homomorphisms. In order to establish an isomorphism between $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ and Λ^*, we will make use of a family of representations of the monoidal category $\mathcal{H}'$ constructed in [13].

Recall that for algebras A and B, M is an (A, B)-bimodule if it is a left A-module and a right B-module, and the actions of A and B are compatible. To describe the representations $\mathcal{H}'$, we start by setting some notation for $(\mathbb{C}[S_{k_1}], \mathbb{C}[S_{k_2}])$-bimodules. All inclusions are assumed to be the standard ones $\iota_{k,n} : \mathbb{C}[S_k] \to \mathbb{C}[S_n]$ introduced in Section 2. Suppose that $k_1, k_2 \leqslant n$. We write:

- (n) for $\mathbb{C}[S_n]$ considered as a $(\mathbb{C}[S_n], \mathbb{C}[S_n])$-bimodule.
- $(n)_{k_2}$ for $\mathbb{C}[S_n]$ considered as a $(\mathbb{C}[S_n], \mathbb{C}[S_{k_2}])$-bimodule.
- ${}_{k_1}(n)$ for $\mathbb{C}[S_n]$ considered as a $(\mathbb{C}[S_{k_1}], \mathbb{C}[S_n])$-bimodule.
- ${}_{k_1}(n)_{k_2}$ for $\mathbb{C}[S_n]$ considered as a $(\mathbb{C}[S_{k_1}], \mathbb{C}[S_{k_2}])$-bimodule.

Let $\mathcal{S}'$ be the category whose objects are compositions of induction and restriction functors of symmetric groups. We write

$$\mathrm{Ind}_n^{n+1} := \mathrm{Ind}_{S_n}^{S_{n+1}} \qquad \text{and} \qquad \mathrm{Res}_n^{n+1} := \mathrm{Res}_{S_n}^{S_{n+1}}\,.$$

Since induction from S_n to S_{n+1} is given by tensoring on the left by $(n+1)_n$ and restriction from S_{n+1} to S_n is given by tensoring on the left by ${}_n(n+1)$, the objects in $\mathcal{S}'$ can be reinterpreted as $(\mathbb{C}[S_{k_1}], \mathbb{C}[S_{k_2}])$-bimodules for $k_1, k_2 \geqslant 0$. Most of our calculations will use this interpretation.

Example 4.6. One object in $\mathcal{S}'$ is the composition

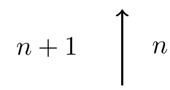
$$\mathrm{Res}_4^5 \circ \mathrm{Ind}_4^5 \circ \mathrm{Ind}_3^4 \circ \mathrm{Res}_3^4\,. \tag{40}$$

In the language of bimodules, this is the $(\mathbb{C}[S_4], \mathbb{C}[S_4])$-bimodule

$${}_4(5)_4(4)_3(4).$$

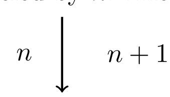
The morphisms in $\mathcal{S}'$ are certain natural transformations of these compositions (or, equivalently, certain bimodule homomorphisms). Like $\mathcal{H}'$, morphisms in $\mathcal{S}'$ can be presented diagrammatically as oriented compact 1-manifolds immersed in $\mathbb{R} \times [0, 1]$.

Unlike $\mathcal{H}'$, in $\mathcal{S}'$ we label the regions of the strip $\mathbb{R}\times[0,1]$ by non-negative integers, so that if there is an upwards oriented line separating two regions and the right region is labeled by n, then the left region must be labeled by $n+1$. The diagram



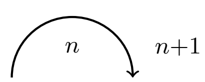
denotes the identity endomorphism of the induction functor Ind_n^{n+1} or alternatively the identity endomorphism of the bimodule $(n+1)_n$.

If there is a downward oriented line separating two regions and the right is labeled by $n+1$ then the left must be labeled by n. The diagram

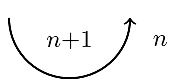


denotes the identity endomorphism of the restriction functor Res_n^{n+1} or alternatively the identity endomorphism of the bimodule ${}_n(n+1)$.

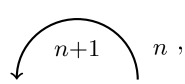
The bimodule maps associated to the four cups and caps are:

(41) 

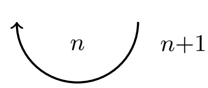
, $\quad (n+1)_n(n+1)\to(n+1),\quad g\otimes h\mapsto gh,\quad g,h\in S_{n+1}$,

(42) 

, $\quad (n)\to {}_n(n+1)_n,\quad g\mapsto g,\quad g\in S_n$,

(43) 

, $\quad {}_n(n+1)_n\to(n),\quad g\mapsto \mathrm{pr}_n(g)=\begin{cases} g & g(n+1)=n+1\\ 0 & \text{otherwise},\end{cases}$

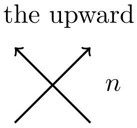
(44) 

, $\quad (n+1)\to(n+1)_n(n+1)$,

where (44) is determined by the condition that

$$1_{n+1}\mapsto\sum_{i=1}^{n+1}s_is_{i+1}\cdots s_n\otimes s_n\cdots s_{i+1}s_i=\sum_{g\in\mathcal{LC}_n^{n+1}}g\otimes g^{-1}.$$

Finally, the upward crossing is the bimodule map

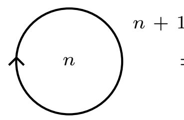
(45) 

, $\quad (n+2)_n\to(n+2)_n,\quad g\mapsto gs_{n+1},\quad g\in S_{n+2}$.

Any diagram that has a region labeled with a negative number is set to **0**. It is shown in [13] that all diagrams are compatible with isotopy.

Remark 4.7. Closed diagrams in $\mathcal{S}'$ with outside region labeled by n correspond to $(\mathbb{C}[S_n],\mathbb{C}[S_n])$-bimodule endomorphisms of (n). The algebra of such bimodule endomorphisms is isomorphic to $Z(\mathbb{C}[S_n])$ via the map which sends $f\in\mathrm{End}_{(\mathbb{C}[S_n],\mathbb{C}[S_n])}(\mathbb{C}[S_n])$ to $f(1_n)$. In this way closed diagrams in $\mathcal{S}'$ may be regarded as elements of the center of the appropriate group algebra.

Khovanov shows that the diagrams in $\mathcal{S}'$ satisfy the defining relations for morphisms in $\mathcal{H}'$. As a result, given an endomorphism of $\mathcal{H}'$, after labeling the far right region by a non-negative integer, one obtains a well-defined bimodule homomorphism in $\mathcal{S}'$. For example, using the bimodule definitions of cups and caps (41)–(44) we can compute that the clockwise-oriented curl generator c_0 maps to multiplication by $n+1$ when we label the outside region with $n+1$:

(46) 

$= \; n+1.$

In other words, the endomorphism $c_0 \in \mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ becomes the scalar $n+1$ in $Z(\mathbb{C}[S_{n+1}])$.

$\mathcal{S}'$ is the direct sum of categories

$$\mathcal{S}' = \bigoplus_{k=0}^{\infty} \mathcal{S}'_k,$$

where $\mathcal{S}'_k$ contains all objects such that induction or restriction starts at S_k (i.e. the rightmost region of the diagram is labeled by k). There are functors $f_k^{\mathcal{H}'} : \mathcal{H}' \to \mathcal{S}'_k$ such that the object $\epsilon_1\epsilon_2 \dots \epsilon_n$ is taken to a composition of induction and restriction functors with $+$ sent to Ind_i^{i+1} and $-$ sent to Res_{i-1}^{i} where i in each case is determined by the requirement that induction/restriction begin from S_k. $f_k^{\mathcal{H}'}$ takes a diagram from $\mathcal{H}'$ to $\mathcal{S}'_k$ by labeling regions so that the rightmost region is labeled with a k and then interpreting the diagram as an element of $\mathcal{S}'_k$.

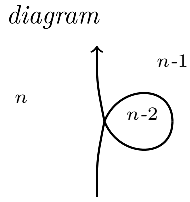
Example 4.8. $f_5^{\mathcal{H}'} : \mathcal{H}' \to \mathcal{S}'_5$ takes

$$(++-+-) \xmapsto{f_5^{\mathcal{H}'}} \mathrm{Ind}_5^6 \circ \mathrm{Ind}_4^5 \circ \mathrm{Res}_4^5 \circ \mathrm{Ind}_4^5 \circ \mathrm{Res}_4^5,$$

$$(-++) \xmapsto{f_5^{\mathcal{H}'}} \mathrm{Res}_6^7 \, \mathrm{Ind}_6^7 \, \mathrm{Ind}_5^6 \,.$$

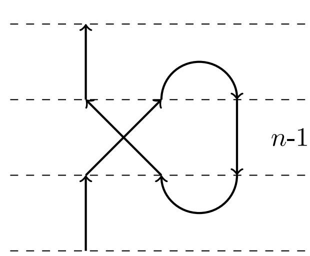
In the remainder of this section we calculate the image of a number of important diagrams in $\mathcal{H}'$ under the functors $f_k^{\mathcal{H}'}$.

Lemma 4.9 ([13, Section 4]). *The diagram*



is the endomorphism of $(n)_{n-1}$ which is right multiplication by J_n.

Proof. The right twist curl can be written as the composition of a cup, a crossing, and a cap.



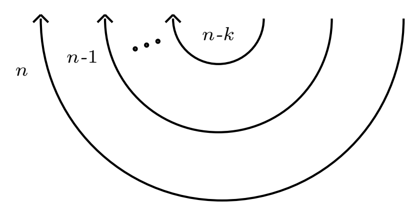
Applying the endomorphism to 1_n gives

$$1_n \mapsto \sum_{i=1}^{n-1} s_i \cdots s_{n-2} \otimes s_{n-2} \cdots s_i \mapsto \sum_{i=1}^{n-1} s_i \cdots s_{n-2} s_{n-1} \otimes s_{n-2} \cdots s_i$$
$$\mapsto \sum_{i=1}^{n-1} s_i \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_i = J_n$$

where the equality holds by (8). □

Lemma 4.10. *Let $k \leqslant n$:*

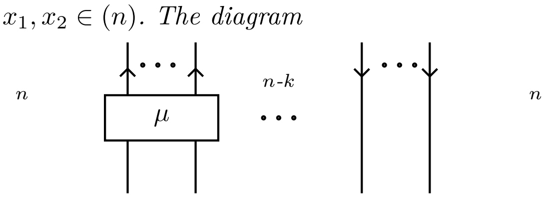
(1) *The diagram*



corresponds to the bimodule homomorphism $(n) \to (n)_{n-k}(n)$ which sends

$$1_n \mapsto \sum_{g \in \mathcal{LC}^n_{n-k}} g \otimes g^{-1}.$$

(2) *Let $\mu \vdash k$ and $x_1, x_2 \in (n)$. The diagram*

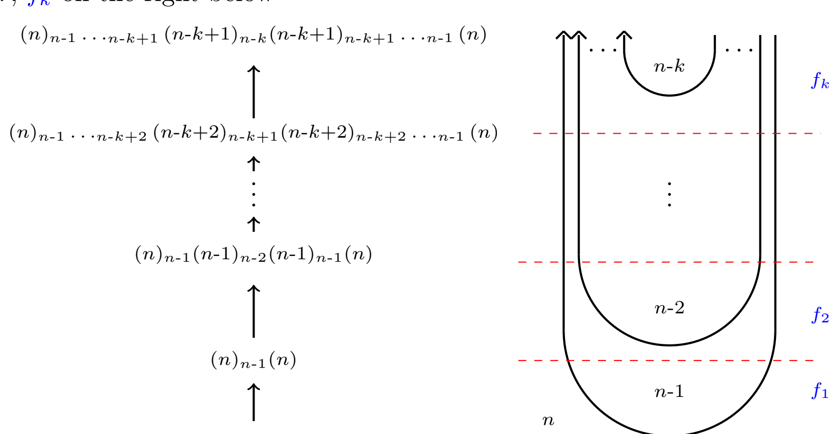


corresponds to the bimodule homomorphism $(n)_{n-k}(n) \to (n)_{n-k}(n)$ which sends

$$x_1 \otimes x_2 \mapsto x_1 \sigma_{\mu,n} \otimes x_2.$$

Proof.

(1) We can factor the diagram in part (1) of the lemma into a vertically stacked composition of k (S_n, S_n)-bimodule homomorphisms which we label f_1, f_2, …, f_k on the right below



The homomorphism corresponding to f_1 sends

$$1_n \overset{f_1}{\longmapsto} \sum_{g \in \mathcal{LC}^n_{n-1}} g \otimes g^{-1}$$

and for $2 \leqslant i \leqslant k$, and $x^{(j)}, y^{(j)} \in \mathbb{C}[S_j]$, the homomorphism corresponding to f_i sends

$$\begin{aligned}&x^{(n)} \otimes \cdots \otimes x^{(n-i+2)} \otimes y^{(n-i+2)} \otimes \cdots \otimes y^{(n)} \\ &\quad \mapsto \sum_{g \in \mathcal{LC}^{n\text{-}i+1}_{n-i}} x^{(n)} \otimes \cdots \otimes x^{(n-i+2)} \otimes g \otimes g^{-1} \otimes y^{(n-i+2)} \otimes \cdots \otimes y^{(n)}.\end{aligned}$$

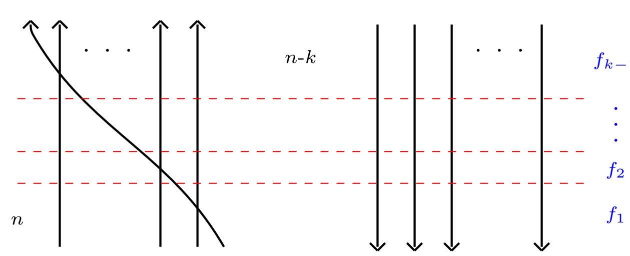
Noting that the final (S_n, S_n)-bimodule in the composition above is isomorphic to $(n)_{n-k}(n)$ by the map which sends

$$\begin{aligned}&x^{(n)} \otimes \cdots \otimes x^{(n\text{-}k+1)} \otimes y^{(n\text{-}k+1)} \otimes \cdots \otimes y^{(n)} \\ &\qquad\qquad \mapsto x^{(n)} \dots x^{(n\text{-}k+1)} \otimes y^{(n\text{-}k+1)} \dots y^{(n)}\end{aligned}$$

we get that the composition of $f_1, \dots, f_k$ sends

$$\begin{aligned}1_n \mapsto \sum_{g^{(n)} \in \mathcal{LC}^n_{n-1}} \cdots \sum_{g^{(n\text{-}k+1)} \in \mathcal{LC}^{n\text{-}k+1}_{n-k}} & g^{(n)} \dots g^{(n\text{-}k+1)} \otimes (g^{(n\text{-}k+1)})^{\text{-}1} \dots (g^{(n)})^{\text{-}1} \\ &= \sum_{g \in \mathcal{LC}^n_{n-k}} g \otimes g^{-1}.\end{aligned}$$

(2) For simplicity we let $\mu = (k)$. The calculation for arbitrary μ easily generalizes from this case. As in part (1) of the proof, we factor the diagram in part (2) into a vertically stacked composition of diagrams



with corresponding (S_n, S_n)-bimodule endomorphisms of $(n)_{n-k}(n)$ labeled by $f_1, f_2, \dots, f_{k-1}$. In particular, reading left to right the diagram for f_i consists of k upward pointing strands on the left with the $(k-i)$th strand and $(k-i+1)$th strand crossed, and k downward pointing strands on the right. The region to the right of the crossing is labeled by $n-k+i-1$, the regions above and below are labeled by $n-k+i$, and the region to the left is labeled by $n-k+i+1$. In order to use (45) to apply f_i to element $x'_1 \otimes x'_2 \in (n)_{n-k}(n)$, we use the isomorphism

$$\begin{aligned}(n)_{n\text{-}k}(n) &\overset{\sim}{\to} (n)_{n\text{-}k+i+1}(n-k+i+1)_{n\text{-}k+i\text{-}1}(n-k+i-1)_{n\text{-}k}(n) \\ x'_1 \otimes x'_2 &\mapsto x'_1 \otimes 1_{n\text{-}k+i+1} \otimes 1_{n\text{-}k+i\text{-}1} \otimes x'_2.\end{aligned} \tag{47}$$

Applying f_i via (45) gives

$$x'_1 \otimes 1_{n\text{-}k+i+1} \otimes 1_{n\text{-}k+i\text{-}1} \otimes x'_2 \overset{f_i}{\longmapsto} x'_1 \otimes s_{n\text{-}k+i} \otimes 1_{n\text{-}k+i\text{-}1} \otimes x'_2$$

and using the inverse of (47) we obtain $x_1' s_{n\text{-}k+i} \otimes x_2' \in (n)_{n-k}(n)$. Starting with $x_1 \otimes x_2 \in (n)_{n-k}(n)$ and using this procedure we calculate

$$\begin{aligned} f_{k-1} \circ \cdots \circ f_2 \circ f_1(x_1 \otimes x_2) &= x_1 s_{n\text{-}k+1} s_{n\text{-}k+2} \dots s_{n\text{-}1} \otimes x_2 \\ &= x_1 \sigma_{(k),n} \otimes x_2. \end{aligned}$$ □

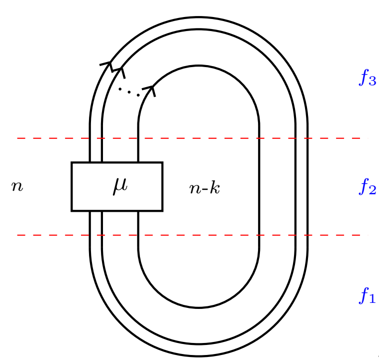
Lemma 4.11. *As elements of* $Z(\mathbb{C}[S_n])$*:*

(1) $f_n^{\mathcal{H}'}(c_k) = \sum_{i=1}^{n} s_i \cdots s_{n-1} J_n^k s_{n-1} \cdots s_i$,

(2) $f_n^{\mathcal{H}'}(\tilde{c}_k) = \mathrm{pr}_n(J_{n+1}^k)$.

(3) $f_n^{\mathcal{H}'}(\alpha_\mu) = \begin{cases} A_{\mu,n} & \text{if } |\mu| \leqslant n \\ 0 & \text{otherwise.} \end{cases}$

Proof. (1)–(2) are found in [13, Section 4] and can be computed from the definitions of cups and caps and Lemma 4.9. To compute (3), we can decompose the diagram for α_μ into three vertically stacked diagrams



.

Using Lemma 4.10 we see that the homomorphisms f_1 and f_2 map

$$1_n \overset{f_1}{\longmapsto} \sum_{g \in \mathcal{LC}^n_{n-k}} g \otimes g^{-1} \overset{f_2}{\longmapsto} \sum_{g \in \mathcal{LC}^n_{n-k}} g\sigma_{\mu,n} \otimes g^{-1}.$$

Then (41) implies that f_3 sends

$$\sum_{g \in \mathcal{LC}^n_{n-k}} g\sigma_{\mu,n} \otimes g^{-1} \overset{f_3}{\longmapsto} \sum_{g \in \mathcal{LC}^n_{n-k}} g\sigma_{\mu,n} g^{-1} = A_{\mu,n}.$$ □

5. The isomorphism $\varphi : \mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow \Lambda^*$

In this section we establish the algebra isomorphism $\mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \cong \Lambda^*$. The proof is somewhat analogous to Ivanov and Kerov's proof of a related isomorphism connecting shifted symmetric functions to the representation theory of symmetric groups (see [8, Theorem 9.1]).

In [13, Section 4], Khovanov defines a grading on $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ by setting

$$\deg(c_0) := 0, \quad \text{and} \quad \deg(c_k) := k+1, \quad \text{for } k \geqslant 1. \tag{48}$$

We will consider the increasing filtration induced by this grading. A relationship between the elements $\{c_k\}_{k\geqslant 0}$ and $\{\alpha_k\}_{k\geqslant 1}$ is then given in terms of this filtration as follows.

Proposition 5.1. *For any* $k \geqslant 1$*,*

$$\alpha_k = c_{k-1} + l.o.t.$$

Proof. This follows from repeated application of the dot sliding moves (33)–(34) and bubble sliding move (35). Notice that with each application of these moves, we get a single term from the same filtered part plus additional terms of lower degree. □

Since the elements $c_0, c_1, \dots$ are algebraically independent generators of $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$, we immediately obtain the following.

Corollary 5.2. *The elements $\alpha_1, \alpha_2, \dots$ are algebraically independent generators of $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$.*

For any $\lambda \vdash n$, composing $f_n^{\mathcal{H}'}$ with the normalized character $\widetilde{\chi}^\lambda$ gives a map

$$(\widetilde{\chi}^\lambda \circ f_n^{\mathcal{H}'}) : \mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \to \mathbb{C}$$

and allows us to define a homomorphism $\varphi : \mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \to \mathrm{Fun}(\mathcal{P}, \mathbb{C})$. Specifically, for $x \in \mathrm{End}_{\mathcal{H}'}(\mathbb{1})$, we write

$$[\varphi(x)](\lambda) := (\widetilde{\chi}^\lambda \circ f_n^{\mathcal{H}'})(x).$$

Combining Lemma 4.11.3 with (7) implies that for $\mu \vdash k$

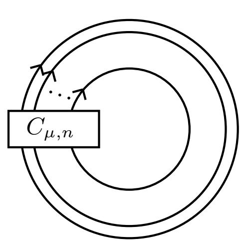
(49) $$[\varphi(\alpha_\mu)](\lambda) = \begin{cases} \frac{(n\downarrow k)}{\dim L^\lambda} \chi^\lambda(\phi_{k,n}(\mu)) & \text{if } k \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

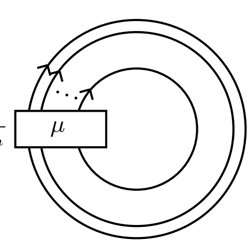
Theorem 5.3. *The map φ induces an algebra isomorphism $\mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \to \Lambda^* \subseteq \mathrm{Fun}(\mathcal{P}, \mathbb{C})$ with*

$$\alpha_\mu \overset{\varphi}{\longmapsto} p_\mu^\#.$$

Proof. Let $\lambda \vdash n$. φ is an algebra homomorphism because $f_n^{\mathcal{H}'}$ is a homomorphism from $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ to $Z(\mathbb{C}[S_n])$ and $\widetilde{\chi}^\lambda$ is a homomorphism when restricted to $Z(\mathbb{C}[S_n])$. By Proposition 3.5 and (49), α_μ maps to $p_\mu^\#$. Since the $\{p_k^\#\}_{k\geqslant 1}$ (respectively $\{\alpha_k\}_{k\geqslant 1}$) are algebraically independent generators of Λ^* (resp. $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$), φ must be an isomorphism. □

Note that Theorem 5.3 along with Lemma 4.4 imply that when $\mu \vdash n$,

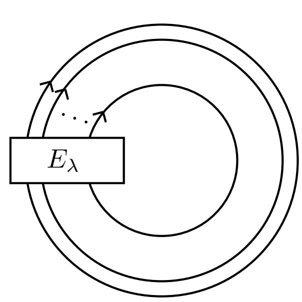
(50) 

$= \frac{n!}{z_{\mu,n}}$ 

$\overset{\varphi}{\longmapsto} \frac{n!}{z_{\mu,n}} p_\mu^\#.$

For $\lambda \vdash n$ recall that E_λ is the Young idempotent associated to λ.

Theorem 5.4. *The isomorphism φ sends*

$\frac{1}{\dim L^\lambda}$ 

$\overset{\varphi}{\longmapsto} s_\lambda^*.$

Proof. Recall that

$$\left(\frac{1}{\dim L^\lambda}\right) E_\lambda = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{n!} C_{\mu,n}$$

(see for example [16]), while

$$s^*_\lambda = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{z_{\mu,n}} p^\#_\mu.$$

The result then follows from (50). ☐

The previous theorems gave graphical realizations of some important bases of Λ^*. Now we go the other way, and describe Khovanov's curl generators $\tilde{c}_k$ and c_k as elements of Λ^*. It is this description that makes an explicit connection between $\mathcal{H}'$ and the transition and co-transition measures of Kerov.

Theorem 5.5. *The isomorphism φ sends:*

(1) $\tilde{c}_k \mapsto \widehat{m}_k \in \Lambda^*$,
(2) $c_k \mapsto p^\#_1 \check{m}_k = \widehat{b}_{k+2} \in \Lambda^*$.

Proof. Let $\lambda \vdash n$, then from Lemma 4.11 and Proposition 2.14 we have

$$[\varphi(\tilde{c}_k)](\lambda) = \widetilde{\chi}^\lambda(\mathrm{pr}_n(J^k_{n+1})) = \widehat{m}_k(\lambda)$$

and

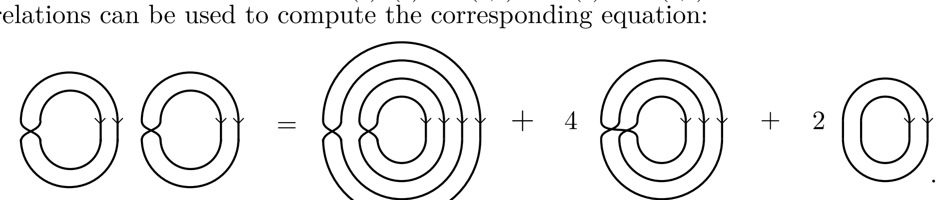
$$[\varphi(c_k)](\lambda) = \widetilde{\chi}^\lambda\left(\sum_{i=1}^n s_i \cdots s_{n-1} J^k_n s_{n-1} \cdots s_i\right) = p^\#_1(\lambda)\check{m}_k(\lambda) = \widehat{b}_{k+2}(\lambda).$$ ☐

Remark 5.6. In [5], Farahat and Higman used the inductive structure of symmetric groups to construct a $\mathbb{C}$-algebra known as the Farahat–Higman algebra $\mathcal{K}_\mathbb{C}$ (see also [16, Example 24, Section I.7]). It follows from, for example [8], that there is an algebra isomorphism $\mathcal{K}_\mathbb{C} \cong \Lambda^*$, and the functors $f^{\mathcal{H}'}_n$ can also be used to give a direct isomorphism between $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ and $\mathcal{K}_\mathbb{C}$. So in principle all of the appearances of shifted symmetric functions in the previous sections could be rephrased in the language of the Farahat–Higman algebra.

Remark 5.7. Theorem 5.5 and Remark 3.8 together imply that the recursive relationships for $\{\widehat{m}_k\}$ and $\{\widehat{b}_k\}$ in Remark 2.13 and $\{c_k\}$ and $\{\tilde{c}_k\}$ in Lemma 4.3 are both consequences of the well-known relationship between the elementary and homogeneous symmetric functions:

$$\sum_{i=0}^k (-1)^i e_i h_{n-i} = 0.$$

Example 5.8. In Λ^* we have $p^\#_{(2)}p^\#_{(2)} = p^\#_{(2,2)} + 4p^\#_{(3)} + 2p^\#_{(1,1)}$. In $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ the local relations can be used to compute the corresponding equation:



5.1. Involutions on $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$. In [13], Khovanov introduced three involutive autoequivalences on $\mathcal{H}'$. Only one of these, which we denote as ξ, acts non-trivially on $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ where it gives an involutive algebra automorphism. For $D \in \mathrm{Hom}_{\mathcal{H}'}(Q_{\epsilon_1}, Q_{\epsilon_2})$, ξ is defined so that

$$\xi(D) := (-1)^{c(D)} D$$

where $c(D)$ is the total number of dots and crossings in the diagram. Thus, in $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$:

$$c_k \overset{\xi}{\longmapsto} (-1)^k c_k, \tag{51}$$

$$\tilde{c}_k \overset{\xi}{\longmapsto} (-1)^k \tilde{c}_k, \tag{52}$$

$$\alpha_k \overset{\xi}{\longmapsto} (-1)^{k-1} \alpha_k. \tag{53}$$

In Section 4 of [18], Okounkov and Olshanski identified an involutive algebra automorphism $I : \Lambda^* \to \Lambda^*$ which acts on $f \in \Lambda^*$ such that for $\lambda \in \mathcal{P}$,

$$[I(f)](\lambda) = f(\lambda'),$$

where λ' is the conjugate partition to λ. In particular

$$I(s^*_\lambda) = s^*_{\lambda'}, \tag{54}$$

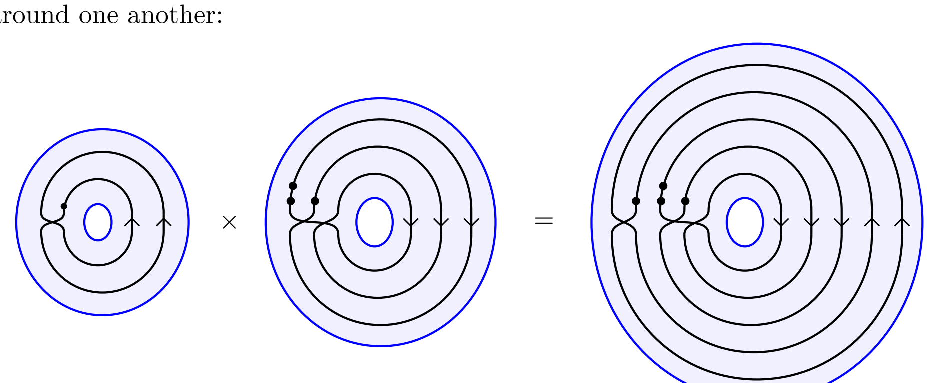
$$I(e^*_k) = h^*_k, \tag{55}$$

$$I(p^\#_k) = (-1)^{k-1} p^\#_k. \tag{56}$$

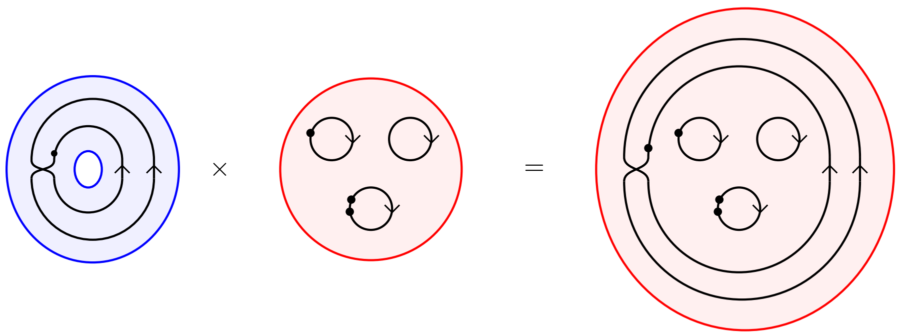
Proposition 5.9. *The involution ξ on $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ coincides with the involution I on Λ^*.*

Proof. This follows from the fact that $\{\alpha_k\}_{k \geqslant 1}$, (respectively $\{p^\#_k\}_{k\geqslant 1}$) generate $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ (resp. Λ^*), $\varphi(\alpha_k) = p^\#_k$, and a comparison of (53) and (56). □

5.2. A graphical construction of the action of $W_{1+\infty}$ on Λ^*. In [4], the trace $\mathrm{Tr}(\mathcal{H}')$ (or zeroth Hochschild homology) of $\mathcal{H}'$ is shown to be isomorphic as an algebra to a quotient of the W-algebra $W_{1+\infty}$. Like the center $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$, which is the algebra of closed planar diagrams, the trace $\mathrm{Tr}(\mathcal{H}')$ has a purely graphical description, as the space of annular diagrams modulo Khovanov's local diagrammatic relations. More precisely, the underlying vector space of $\mathrm{Tr}(\mathcal{H}')$ is isomorphic to the span of annular diagrams, where an annular diagram $\tilde{f}$ is by definition a diagram obtained by taking an endomorphism $f \in \mathrm{End}_{\mathcal{H}'}(X)$ for some object $X \in \mathcal{H}'$, and closing it up to the right in an annulus. The multiplication in $\mathrm{Tr}(\mathcal{H}')$ is given by gluing annuli around one another:



The action of $\mathrm{Tr}(\mathcal{H}')$ on $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ then acquires a graphical description: given an annular diagram $\tilde{f} \in \mathrm{Tr}(\mathcal{H}')$ and a closed planar diagram $g \in \mathrm{End}_{\mathcal{H}'}(\mathbb{1})$, the closed planar diagram $\tilde{f}g \in \mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ is given by inserting a planar neighborhood of the closed diagram g into the middle of the annulus:



Thus, via the isomorphisms

$$\mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \cong \Lambda^*, \qquad \mathrm{Tr}(\mathcal{H}') \cong W_{1+\infty}$$

of Theorem 5.3 and [4], respectively, we obtain a purely graphical construction of the action of $W_{1+\infty}$ on Λ^*. Such an action was first considered by Lascoux–Thibon in [14].

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