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Doubly transitive lines II: Almost simple symmetries

Joseph W. Iverson & Dustin G. Mixon

Abstract We study lines through the origin of finite-dimensional complex vector spaces that enjoy a doubly transitive automorphism group. This paper classifies those lines that exhibit almost simple symmetries. We introduce a general recipe involving Schur covers to recover doubly transitive lines from their automorphism group. Combining our results with recent work on the affine case by Dempwolff and Kantor [13], we deduce a classification of all linearly dependent doubly transitive lines in real or complex space.

1. Introduction

Given a sequence $L = \{l_j\}_{j \in [n]}$ of lines through the origin of \mathbb{C}^d , consider the **automorphism group** $\text{Aut } L \leq S_n$ consisting of all permutations σ for which there exists a unitary $U \in U(d)$ satisfying $U l_j = l_{\sigma(j)}$ for every $j \in [n]$. We say L is **doubly transitive** if $\text{Aut } L$ acts doubly transitively on $[n]$, that is, for every $i, j, k, l \in [n]$ with $i \neq j$ and $k \neq l$, there exists $U \in U(d)$ such that $U l_i = l_k$ and $U l_j = l_l$. Doubly transitive lines are highly symmetric and therefore worthy of study in their own right. Case in point, G. Higman first investigated doubly transitive lines in *real* spaces (via their combinatorial *two-graph* representation) as a means of studying Conway's sporadic simple group Co_3 [49]. At the time, one might have studied two-graphs with the hope of discovering a new simple group, since the unique minimal normal subgroup of every finite doubly transitive group is either simple or elementary abelian, as detailed further below.

Today, we have another reason to study doubly transitive lines. By virtue of their high degree of symmetry, under mild conditions, doubly transitive lines are necessarily optimizers of a fundamental problem in discrete geometry, specifically, packing lines $\{l_j\}_{j \in [n]}$ through the origin of \mathbb{C}^d so as to maximize the minimum chordal distance:

$$\min_{1 \leq i < j \leq n} \overline{\|l_i - l_j\|},$$

where π_j denotes orthogonal projection onto l_j . The optimal line packings that are most frequently studied in the literature are spanned by **equiangular tight frames**

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(ETFs), which are sequences of unit vectors $\{j\}_j$ in \mathbb{C}^d with the property that there exist constants A, μ such that

$$| \langle j, j \rangle | = A, \quad | \langle i, j \rangle | = \mu \text{ for every } i, j \in [n], i \neq j.$$

See [16] for a survey. Indeed, the vectors in an ETF span lines that maximize the minimum chordal distance by achieving equality in the Welch bound [53] (also known as the simplex bound [10]). In general, optimal line packings find applications in compressed sensing [4], multiple description coding [48], digital fingerprinting [39], and quantum state tomography [43].

The proposition below makes explicit when doubly transitive lines correspond to optimal line packings. Here and throughout, the **span** of a sequence of lines is the smallest subspace containing those lines, and a sequence of lines is **linearly dependent** if one of the lines is contained in the span of the others. Observe that a sequence of n lines is linearly dependent if and only if it spans a space of dimension $d < n$.

Proposition 1.1 (Lemma 1.1 in [28]). *Given $n > d$ doubly transitive lines with span \mathbb{C}^d , select unit-norm representatives $\{j\}_j$. Then $\{j\}_j$ is an equiangular tight frame.*

This paper is the second in a series that studies doubly transitive lines. In the previous installment [28], we focused on developing the theory of *Higman pairs* and *roux*. The former is a special type of Gelfand pair (G, H) in which, as a consequence of part (b) of the Higman Pair Theorem [28] (see also Theorem 4.1 below), each of the irreducible constituents of the permutation representation of G on the cosets of H determines a sequence of linearly dependent doubly transitive lines. In fact, every sequence of linearly dependent doubly transitive lines is determined by such a Gelfand pair (by part (a) of the Higman Pair Theorem, which we prove in this paper). Meanwhile, roux provide a convenient combinatorial generalization of doubly transitive lines in complex spaces, much like how G. Higman’s two-graphs generalize doubly transitive lines in real spaces.

Both Higman pairs and roux play a pivotal role in the present installment, where we provide a general recipe involving Schur coverings to find all complex line sequences whose automorphism group contains a given doubly transitive group. We apply this machine to classify “half” of the doubly transitive lines. Specifically, the following result gives two possibilities for a doubly transitive permutation group S according to its **socle** $\text{soc } S$, which for a doubly transitive group is its unique minimal nontrivial normal subgroup.

Proposition 1.2 (Burnside [5]). *If S is a doubly transitive permutation group, then exactly one of the following holds:*

- (I) *$\text{soc } S$ is a regular elementary abelian subgroup that may be identified with F_p^m in such a way that $S = F_p^m \circ G_0 \triangleleft \text{AGL}(m, p)$, where $G_0 \triangleleft \text{GL}(m, p)$ is the stabilizer of $0 \in F_p^m$, or*
- (II) *$\text{soc } S = G$ is a nonabelian simple group, and $G \triangleleft S \triangleleft \text{Aut}(G)$.*

Groups of type (I) are said to have a **ne type**, while those of type (II) are called **almost simple**. The former are characterized by having a normal elementary abelian subgroup. (For further details, see Chapter 4 of [14].) The doubly transitive permutation groups have been classified as a consequence of the classification of finite simple groups [7]. Since the Schur covers of simple groups were already determined in order to perform this classification, our general recipe is most readily applied to the almost simple doubly transitive groups. We handle that case in this installment. The

the case was recently handled by Dempwolff and Kantor [13], and by combining our results with theirs, we are able to deduce the complete classification of linearly dependent doubly transitive lines.

In the theorem below, two line sequences $L = \{l_j\}_{j \in [n]}$ and $L' = \{l'_j\}_{j \in [n]}$ in \mathbb{C}^d are called **unitarily equivalent** if there is a unitary $U \in U(d)$ and a permutation S_n such that $U l_j = l'_{S(j)}$ for every $j \in [n]$; then we also say U **induces** a unitary equivalence $L \sim L'$. Meanwhile, a line sequence $L = \{l_j\}_{j \in [n]}$ in \mathbb{C}^d is **real** if there is a unitary $U \in U(d)$ and a choice of unit-norm representatives $\{l_j\}_{j \in [n]}$ such that $U l_j \in \mathbb{R}^d \subset \mathbb{C}^d$ for every $j \in [n]$. (This is not the definition of real lines given in [28], but the notions are equivalent. See the comments following Proposition 2.1.)

Theorem 1.3 (Classification of linearly dependent doubly transitive lines). *Let L be a sequence of $n > 2d > 2$ lines with span \mathbb{C}^d such that $S := \text{Aut } L$ is doubly transitive. Then one of the following holds:*

- (I) S has a regular normal subgroup G that is elementary abelian of order $n = p^{2m}$, where p is prime. Following bijections $L \cong G = \mathbb{F}_p^{2m}$, then S is isomorphic to a subgroup of $\text{AGL}(2m, p)$, the stabilizer $S_0 \triangleleft S$ of the line corresponding to $0 \in \mathbb{F}_p^{2m}$ can be identified with a subgroup of $\text{GL}(2m, p)$, and one of the following holds:
 - (A) The lines are real, and
 - (i) $d = 2^{m-1}(2^m - 1)$, $n = 2^{2m}$, $S_0 = \text{Sp}(2m, 2)$, $m > 1$.
 - (B) The lines are not real, and (d, n, S_0) satisfy one of the following:
 - (ii) $d = p^m(p^m - 1)/2$, $n = p^{2m}$, $S_0 = \text{Sp}(2m, p)$, $p > 2$ is prime, $m > 1$.
 - (iii) $d = 2$, $n = 4$, $|S_0| = 3$.
 - (iv) $d = 8$, $n = 64$, $S_0 = G_2(2) = \text{PSU}(3, 3)$.
- (II) S has a simple normal subgroup $G \triangleleft S \triangleleft \text{Aut } G$ such that one of the following holds:
 - (C) The lines are real, and (d, n, G) satisfy one of the following:
 - (v) $d = (q + 1)/2$, $n = q + 1$, $G = \text{PSL}(2, q)$, $q \equiv 1 \pmod{4}$ is a prime power.
 - (vi) $d = q^2 - q + 1$, $n = q^3 + 1$, $G = \text{PSU}(3, q)$, q is an odd prime power.
 - (vii) $d = q^2 - q + 1$, $n = q^3 + 1$, $G = {}^2G_2(q)$, $q = 3^{2m+1}$, $m > 0$.
 - (viii) $d = (2^{2m-1} - 3 \cdot 2^{m-1} + 1)/3$, $n = 2^{2m-1} - 2^{m-1}$, $G = \text{Sp}(2m, 2)$, $m > 3$.
 - (ix) $d = (2^{2m-1} + 3 \cdot 2^{m-1} + 1)/3$, $n = 2^{2m-1} + 2^{m-1}$, $G = \text{Sp}(2m, 2)$, $m > 3$.
 - (x) $d = 22$, $n = 176$, $G = \text{HS}$ (the Higman–Sims group).
 - (xi) $d = 23$, $n = 276$, $G = \text{Co}_3$ (the third Conway group).
 - (D) The lines are not real, and (d, n, G) satisfy one of the following:
 - (xii) $d = (q + 1)/2$, $n = q + 1$, $G = \text{PSL}(2, q)$, $q \equiv 3 \pmod{4}$ is a prime power, $q > 3$.
 - (xiii) $d = q^2 - q + 1$, $n = q^3 + 1$, $G = \text{PSU}(3, q)$, q is a prime power, $q > 2$.

Conversely, there exist lines satisfying each of (i)–(xiii). For (i)–(xii), the lines are unique up to unitary equivalence. For (xiii), the lines are unitarily equivalent to some of those constructed in Theorem 6.9(b).

Remark 1.4. The lines in case (xiii) of Theorem 1.3 are not unique, but all equivalences are detailed in Theorem 7.5. There is exactly one other equivalence among (i)–(xiii). Namely, there is just one sequence of $n = 28$ equiangular real lines in dimension $d = 7$ up to unitary equivalence, and it is repeated in (vi), (vii), and (viii).

There are no other coincidences. All the other lines can be distinguished by their realness and their automorphism groups. As described in Section 2, the real lines of (A) and (C) are equivalent to two-graphs constructed in [49], and their automorphism groups appear in [50]. They are distinct. The automorphism groups of (B) are listed above from [13], and those of (D) appear in Section 7. They are also distinct.

In Theorem 1.3, the global hypothesis $n > 2d$ is imposed only to simplify exposition. Lines spanned by the unit-norm columns of a tight frame $\mathbb{C}^{d \times n}$ naturally pair with those spanned by columns of a **Naimark complement** $\mathbb{C}^{(n-d) \times n}$ with $= \frac{n}{n-d}(I - \frac{d}{n})$. The lines spanned by Naimark complements exhibit identical automorphism groups, as an easy consequence of Proposition 2.2 below. Hence, each of the line sequences described in Theorem 1.3 with $n > 2d$ pairs with another sequence of n doubly transitive lines spanning a space of dimension $n - d$.

Real equiangular lines are equivalent to two-graphs (see Proposition 2.6), and constructions for all the examples of (A) and (C) appear in Section 6 of [49]. We provide additional descriptions of (v) and (vi) in Example 6.3 and Theorem 6.9(b), respectively. See Example 5.10 of [28] for another description of (i), and also for (ii) and (iii). A construction of (iv) appears in Example 5.11 of [28]. For (xii), see Example 6.3; another description involving Higman pairs appears in Example 5.8 of [28]. In addition to Theorem 6.9(b), the lines in (xiii) can be described using Higman pairs as in Example 5.9 of [28].

As explained in Section 2, the real cases (A) and (C) follow Taylor’s classification of doubly transitive two-graphs [50]. The cases in which $n = d^2$ (of which three exist) were classified by Zhu [58]. The almost simple case (I) is due to Dempwolff and Kantor [13]. Our contribution to the classification is the almost simple case (II).

Theorem 1.5 (Main result). *Let L be a sequence of $n > 2d > 2$ lines with span \mathbb{C}^d such that $S := \text{Aut } L$ is doubly transitive and almost simple. Then L takes the form described in Theorem 1.3(II). Conversely, there exist lines satisfying each of Theorem 1.3(v)–(xiii). For (v)–(xii), the lines are unique up to unitary equivalence. For (xiii), the lines are unitarily equivalent to some of those constructed in Theorem 6.9(b).*

In the next section, we verify that doubly transitive real lines amount to doubly transitive two-graphs. Section 3 then reviews necessary results from the previous installment [28]. We prove part (a) of the Higman Pair Theorem in Section 4, and our proof suggests a program for classifying doubly transitive lines. Section 5 builds up the machinery needed to accomplish such a classification, and then Section 6 performs this classification in the almost simple case, ultimately proving Theorems 1.3 and 1.5. Finally, Section 7 finds the automorphism groups in case (D) and uses them to sort out unitary equivalences for (xiii).

2. Doubly transitive real lines and two-graphs

Taylor [50] has completely classified doubly transitive two-graphs, which are known to be equivalent to doubly transitive lines in \mathbb{R}^d . In contrast, this series concerns doubly transitive lines in \mathbb{C}^d . Every line \mathbb{R}^d embeds into a unique line \mathbb{C}^d through the inclusion $\mathbb{R}^d \subset \mathbb{C}^d$: if $= \text{span}_{\mathbb{R}}\{ \}$, then $= \text{span}_{\mathbb{C}}\{ \}$. In this section, we show that this embedding preserves automorphism groups. Consequently, Taylor’s results apply for real lines in the complex setting.

Throughout, we abuse notation by letting $= \{ j \}_j^{[n]}$ denote both a sequence in \mathbb{C}^d and the $d \times n$ matrix whose j th column is $_j$. Its **Gram matrix** is $= [_j, _i]_{i,j}^{[n]} \in \mathbb{C}^{n \times n}$.

Proposition 2.1. *A sequence of lines in \mathbb{C}^d is real if and only if it has unit-norm representatives $\{j\}_{j \in [n]}$ whose Gram matrix G resides in $\mathbb{R}^{n \times n} \subset \mathbb{C}^{n \times n}$.*

Proof. Let $L = \{j\}_{j \in [n]}$ be a sequence of lines in \mathbb{C}^d . In the forward direction, suppose L is real, and choose unit-norm representatives $\{j\}_{j \in [n]}$ and a unitary $U \in U(d)$ such that $U_j \in \mathbb{R}^d \subset \mathbb{C}^d$ for every $j \in [n]$. Then $[G]_{ij} = \langle U_i, U_j \rangle = (U_i)^T (U_j) \in \mathbb{R}$ for every $i, j \in [n]$. In the reverse direction, suppose $\{j\}_{j \in [n]}$ is a choice of unit-norm representatives for L such that G is real-valued. Since G is positive semidefinite with rank at most d , there exists $\{j\}_{j \in [n]}$ in $\mathbb{R}^d \subset \mathbb{C}^d$ such that $G = G$. It is well known that vectors in \mathbb{C}^d are determined up to unitary equivalence by their Gram matrices, so there exists a unitary $U \in U(d)$ such that $U_j = j \in \mathbb{R}^d$ for every $j \in [n]$. Hence, L is real.

The previous installment of this series defined real lines in a third equivalent way using *signature matrices*, which play a significant role in the theory of equiangular lines. Given any sequence L of linearly dependent equiangular lines, choose unit norm representatives $\{j\}_{j \in [n]}$ satisfying $|\langle j, j \rangle| = \mu = 0$ for every $i = j \in [n]$, and consider $S := \mu^{-1}(G - I)$. We call $S \in \mathbb{C}^{n \times n}$ a **signature matrix** since it satisfies all of the following:

- (i) $S_{ii} = 0$ for every $i \in [n]$,
- (ii) $|S_{ij}| = 1$ whenever $i \neq j \in [n]$, and
- (iii) $S^T = S$.

Property (ii) says that each off-diagonal element of S is a **phase**, i.e., a member of the multiplicative group $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. (Similarly, the phase of a nonzero complex number $z \in \mathbb{C}^\times$ is defined as $z/|z| \in \mathbb{T}$.) For any choice of phases $\{\gamma_j\}_{j \in [n]}$, the matrix $D := \text{diag}(\gamma_1, \dots, \gamma_n)$ determines another signature matrix $S := D^{-1}SD$. In terms of L , S corresponds to the different choice of unit-norm representatives $\{\gamma_j j\}_{j \in [n]}$. We say that S and S are **switching equivalent**, and write $S \sim S$. The **normalized signature matrix** of L is the unique $S \sim S$ such that $S_{1j} = S_{j1} = 1$ for every $j > 1$. Conversely, any signature matrix S gives rise to a unique unitary equivalence class of linearly dependent equiangular lines. Indeed, the least eigenvalue λ_{\min} of S is strictly negative since $\text{tr } S = 0$, and so $-\lambda_{\min}^{-1}S + I > 0$ can be factored as $U^T U$ for some unit vectors U , which span linearly dependent equiangular lines having S as a signature matrix. As a consequence of Proposition 2.1, a sequence of linearly dependent equiangular lines is real if and only if its normalized signature matrix is real-valued. In [28], the latter notion was used as the definition of real lines.

The following result relates switching equivalence with unitary equivalence. It follows easily from the fact that Gram matrices determine vector sequences up to unitary equivalence.

Proposition 2.2. *Let $S, S \in \mathbb{C}^{n \times n}$ be signature matrices for equiangular line sequences L and L in \mathbb{C}^d , and let S_n be a permutation with matrix representation $P = [p_{ij}]_{i,j \in [n]} \in \mathbb{C}^{n \times n}$. Then S is switching equivalent to PSP^{-1} if and only if P induces a unitary equivalence $L \sim L$. In particular, S is switching equivalent to PSP^{-1} if and only if $P \in \text{Aut } L$.*

Definition 2.3. *Let $L = \{j\}_{j \in [n]}$ be a sequence of lines in \mathbb{R}^d , and let $L = \{j\}_{j \in [n]}$ be its embedding in \mathbb{C}^d . We define $\text{Aut}_{\mathbb{C}} L := \text{Aut } L$ and*

$\text{Aut}_{\mathbb{R}} L := \{S_n : \text{there exists } U \in O(d) \text{ such that } U_j = p_{(j)} \text{ for every } j \in [n]\}$,
 where $O(d) \subset \mathbb{R}^{d \times d}$ is the orthogonal group.

Lemma 2.4. Let $L = \{l_j\}_{j \in [n]}$ and $K = \{k_j\}_{j \in [n]}$ be sequences of lines in \mathbb{R}^d , and let $\tilde{L} = \{\tilde{l}_j\}_{j \in [n]}$ and $\tilde{K} = \{\tilde{k}_j\}_{j \in [n]}$ be their embeddings in \mathbb{C}^d . Then the following are equivalent:

- (a) there exists $U \in O(d)$ with $U l_j = k_j$ for every $j \in [n]$,
- (b) there exists $U \in U(d)$ with $U \tilde{l}_j = \tilde{k}_j$ for every $j \in [n]$.

In particular, $\text{Aut}_{\mathbb{R}} L = \text{Aut}_{\mathbb{C}} \tilde{L}$.

Proof. Choose unit-norm representatives $l_j \in \mathbb{R}^d$ and $\tilde{l}_j \in \mathbb{R}^d$ for every $j \in [n]$. Then (a) holds if and only if there exists $U \in O(d)$ and scalars $a_j \in \{\pm 1\}$ such that $U l_j = a_j \tilde{l}_j$ for every $j \in [n]$. It is well known that vector sequences in \mathbb{R}^d are determined up to orthogonal equivalence by their Gram matrices. Hence, (a) holds if and only if there exist scalars $a_j \in \{\pm 1\}$ such that

$$(1) \quad \langle l_j, l_i \rangle = a_i a_j \langle \tilde{l}_j, \tilde{l}_i \rangle \text{ for every } i, j \in [n].$$

Similarly, (b) holds if and only if there are scalars $c_j \in \mathbb{T}$ such that

$$(2) \quad \langle \tilde{l}_j, \tilde{l}_i \rangle = \overline{c_i} c_j \langle l_j, l_i \rangle \text{ for every } i, j \in [n].$$

It is now obvious that (a) implies (b). For the converse, assume (b) holds and choose scalars $c_j \in \mathbb{T}$ to satisfy (2). Define G to be the graph on the vertex set $[n]$ with an edge joining $i = j$ whenever $\langle \tilde{l}_j, \tilde{l}_i \rangle = 0$. (This is the *frame graph* of [47, 1].) Equivalently, vertices $i = j$ are adjacent when $\langle l_j, l_i \rangle = 0$.

Choose representatives $i_1, \dots, i_r \in [n]$ for each of the connected components of G , and define new scalars $\{a_j\}_{j \in [n]}$ by setting $a_j = \overline{c_{i_k}} c_j$ whenever there is a path from j to i_k . Then $\overline{a_i} a_j = \overline{c_i} c_j$ whenever $i, j \in [n]$ lie in the same connected component of G . For every $i, j \in [n]$, (2) gives $\langle \tilde{l}_j, \tilde{l}_i \rangle = \overline{a_i} a_j \langle l_j, l_i \rangle$, both sides of the equation being 0 when there is no path from i to j .

It remains to show that $a_j \in \{\pm 1\}$ for all $j \in [n]$. Given adjacent vertices $i, j \in [n]$, we can divide the relation $\langle \tilde{l}_j, \tilde{l}_i \rangle = \overline{a_i} a_j \langle l_j, l_i \rangle$ to deduce that $\overline{a_i} a_j \in \mathbb{R} \cap \mathbb{T} = \{\pm 1\}$. If one of a_i, a_j belongs to $\{\pm 1\}$, the other does as well. Since $a_{i_k} = \overline{c_{i_k}} c_{i_k} = 1$ for every $k \in [r]$, we quickly see that $a_j \in \{\pm 1\}$ for every $j \in [n]$. This proves (a).

Finally, we conclude that $\text{Aut}_{\mathbb{R}} L = \text{Aut}_{\mathbb{C}} \tilde{L}$ by taking any S_n and setting $K = \{\tilde{l}_j\}_{j \in [n]}$. Then $\text{Aut}_{\mathbb{R}} L$ if and only if (a) holds, while $\text{Aut}_{\mathbb{C}} \tilde{L}$ if and only if (b) holds.

Definition 2.5. A *two-graph* is a pair (V, T) where V is a finite set of vertices and T is a collection of 3-subsets of V , such that every 4-subset of V contains an even number of 3-subsets from T . The *automorphism group* of (V, T) consists of all permutations of V that preserve T .

Proposition 2.6. Given a sequence L of n real, linearly dependent, equiangular lines in \mathbb{C}^d , choose unit-norm representatives $\{l_j\}_{j \in [n]}$ and put

$$T_L := \{i, j, k\} \in [n] : i = j = k = i \text{ and } \mu^{-3} \langle l_i, l_j \rangle \langle l_j, l_k \rangle \langle l_k, l_i \rangle = -1,$$

where $\mu := |\langle l_i, l_j \rangle|$ for $i = j$. Then $([n], T_L)$ is a two-graph whose automorphism group equals $\text{Aut}(L)$. Moreover, the mapping $L \mapsto T_L$ induces a bijection between unitary equivalence classes of real, linearly dependent, equiangular lines and isomorphism equivalence classes of two-graphs. In this correspondence, doubly transitive real lines map to doubly transitive two-graphs, and vice versa.

Proof. For the first part, we may apply a global unitary to assume L is a sequence of lines in real space. Then it is well known (cf. [45]) that T_L is a two-graph, and (cf. [6]) that $\text{Aut } T_L = \text{Aut}_{\mathbb{R}} L$, which coincides with $\text{Aut}_{\mathbb{C}} \tilde{L}$ by Lemma 2.4. In particular, L is doubly transitive if and only if T_L is, too.

By Lemma 2.4, the embedding $\mathbb{R}^d \hookrightarrow \mathbb{C}^d$ induces a bijection between linearly dependent equiangular line sequences in \mathbb{R}^d up to global orthogonal transformation, and linearly dependent equiangular real line sequences in \mathbb{C}^d up to global unitarity. It is also well known (cf. [45] again) that the mapping $L \mapsto T_L$ gives a bijection between linearly independent equiangular line sequences in real space up to global orthogonal transformation, and two-graphs on vertex set $[n]$. The proof is completed by composing these bijections.

With each two-graph $(T, [n])$ we associate a normalized signature matrix $S \in \mathbb{C}^{n \times n}$, defined as follows. Set $S_{ii} = 0$ for every $i \in [n]$, and $S_{i1} = S_{1i} = 1$ for every $i \in [n]$. When $i \neq j$ are both different from 1, put $S_{ij} = -1$ if $\{1, i, j\} \in T$ and $S_{ij} = 1$ otherwise. Then S is the normalized signature matrix of a sequence L of linearly dependent, real, equiangular lines such that $T = T_L$. The two-graph $(T, [n])$ is called **regular** if S has exactly two eigenvalues $\lambda_1 > 0 > \lambda_2$. In that case, L spans a space of dimension $d = \frac{n-1}{1-\lambda_2}$, while $n-d = \frac{-n-2\lambda_2}{1-\lambda_2}$. Every doubly transitive two-graph is regular, as an easy consequence of Theorem 2.2 in [49].

In light of Proposition 2.6, doubly transitive real lines are either linearly independent or else they follow Taylor’s classification of doubly transitive two-graphs [50]. We summarize the classification of doubly transitive real lines in Proposition 2.7 below. Information about the automorphism groups is taken from [50, Theorem 1], while existence and uniqueness are provided by [50, Theorem 2]. For each of the cases (i)–(viii) below, a corresponding two-graph is described in [49, Section 6]. There one also finds eigenvalues for a normalized signature matrix. We used those eigenvalues to compute the dimensions d below. Finally, we remark that [50, Theorem 1] lists additional symmetry groups of affine type (I). However, each of those groups is contained in an affine symplectic group, and [50, Section 3] establishes that the corresponding two-graphs are instances of case (i) below.

Proposition 2.7. *If L is a sequence of $n > 2d > 2$ real lines with $\text{span } \mathbb{C}^d$ for which $S := \text{Aut } L$ is doubly transitive, then one of the following holds:*

- (I) *There is a normal elementary abelian subgroup of S that acts regularly on L . Furthermore, S has a subgroup G , where*
 - (i) $d = 2^{m-1}(2^m - 1)$, $n = 2^{2m}$, $G = \mathbb{F}_2^{2m} \circ \text{Sp}(2m, 2)$, $m \geq 1$.
- (II) *There is a nonabelian simple group G such that $G \leq S \leq \text{Aut}(G)$, where (d, n, G) is one of the following:*
 - (ii) $d = (q+1)/2$, $n = q+1$, $G = \text{PSL}(2, q)$, q is a prime power, $q \equiv 1 \pmod{4}$.
 - (iii) $d = q^2 - q + 1$, $n = q^3 + 1$, $G = \text{PSU}(3, q)$, q is an odd prime power.
 - (iv) $d = q^2 - q + 1$, $n = q^3 + 1$, $G = {}^2G_2(q)$, $q = 3^{2e+1}$.
 - (v) $d = (2^{2m-1} - 3 \cdot 2^{m-1} + 1)/3$, $n = 2^{2m-1} - 2^{m-1}$, $G = \text{Sp}(2m, 2)$, $m > 3$.
 - (vi) $d = (2^{2m-1} + 3 \cdot 2^{m-1} + 1)/3$, $n = 2^{2m-1} + 2^{m-1}$, $G = \text{Sp}(2m, 2)$, $m > 3$.
 - (vii) $d = 22$, $n = 176$, $G = \text{HS}$ (the Higman–Sims group).
 - (viii) $d = 23$, $n = 276$, $G = \text{Co}_3$ (the third Conway group).

Conversely, for each (n, G) of (i)–(viii) there exists a unique dimension d and a unique unitary equivalence class of $n > 2d > 2$ real lines spanning \mathbb{C}^d whose automorphism group contains G . In particular, $d = d$ is the dimension listed above.

3. Review of Higman pairs and roux

In \mathbb{R}^d , every sequence of real doubly transitive lines can be viewed as a two-graph. Similarly in the complex case, we will show that doubly transitive lines derive from a combinatorial object known as a **roux**. The authors introduced roux and the related notion of Higman pairs in the first paper of this series [28]. In this section, we recall

some necessary results from [28] and provide a brief summary of association schemes, roux, and Higman pairs.

3.1. **Association schemes.** Recall that a **-algebra** is a complex algebra A equipped with an involution $A \ni A \mapsto A^*$ such that $(cA + B)^* = \bar{c}A^* + B^*$ and $(AB)^* = B^*A^*$ for every $A, B \in A$ and $c \in \mathbb{C}$. An **association scheme** is a sequence $\{A_i\}_{i \in [k]}$ in $\mathbb{C}^{n \times n}$ with entries in $\{0, 1\}$ such that

- (A1) $A_1 = I$,
- (A2) $\sum_{i \in [k]} A_i = J$ (the matrix of all ones), and
- (A3) $A := \text{span}\{A_i\}_{i \in [k]}$ is a **-algebra** under matrix multiplication.

We refer to A as the **adjacency algebra** (or *Bose–Mesner algebra*) of $\{A_i\}_{i \in [k]}$. Any matrix $M \in A$ has a unique expansion $M = \sum_{i \in [k]} c_i A_i$, and M is said to **carry the scheme** when the coefficients c_i are distinct. (Then the “level sets” of M carve out the adjacency matrices $\{A_i\}_{i \in [k]}$.) An association scheme is said to be **commutative** when its adjacency algebra is commutative. In that case, the spectral theorem provides A with an alternative basis of orthogonal projections $\{E_j\}_{j \in [k]}$. These are called the **primitive idempotents** of A , since every other projection in A can be expressed as a sum of primitive idempotents. Whether or not the scheme is commutative, every orthogonal projection $G \in A$ can be factored as $G = \sum_{j \in [n]} v_j v_j^*$ where $v_j = \{v_{j,i}\}_{i \in [n]}$ is a sequence of equal-norm vectors $v_j \in \mathbb{C}^d$, $d = \text{rank } G$. By collecting the lines spanned by these vectors and eliminating duplicates as necessary, we obtain a sequence $L = \{l_j\}_{j \in [m]}$ of lines spanning \mathbb{C}^d having representatives with Gram matrix G . (Notice that each line may be represented several times in L and in G .) The resulting lines have been studied in [12, 26, 24, 25].

A large class of examples of association schemes arise from permutation groups, as follows. Let G be a finite group acting transitively on a set X (from the left). Then G also acts on $X \times X$ through the diagonal action $g \cdot (x, y) := (g \cdot x, g \cdot y)$. Collect the orbits $\{R_i\}_{i \in [k]}$ of G on $X \times X$, and let $\{A_i\}_{i \in [k]}$ be the corresponding indicator matrices in $\mathbb{C}^{X \times X}$, i.e. $[A_i]_{x,y} = 1$ if $(x, y) \in R_i$ and 0 otherwise. By reindexing if necessary, we may assume that $A_1 = I$. Then $\{A_i\}_{i \in [k]}$ is an association scheme, called a **Schurian** scheme. The corresponding adjacency algebra A consists of all G -stable matrices, that is, matrices $M \in \mathbb{C}^{X \times X}$ satisfying $M_{g \cdot x, g \cdot y} = M_{x,y}$ for every $g \in G$ and $x, y \in X$. Any vector sequence $v = \{v_x\}_{x \in X}$ with $v \in A$ is called a **homogeneous frame**, since it inherits symmetries corresponding to the action of G on X (see [25]). For an alternative description of the scheme, we may assume without loss of generality that $X = G/H$, where $H \leq G$ is the stabilizer of a point in X . Then the double cosets $\{Ha_iH\}_{i \in [k]}$ of H in G can be indexed in such a way that $[A_i]_{xH, yH} = 1$ if $y^{-1}x \in Ha_iH$ and 0 otherwise. We refer to (G, H) as a **Gelfand pair** when its Schurian scheme is commutative.

3.2. **Roux and roux lines.** Given a finite abelian group Γ of order r and a positive integer n , we denote $\mathbb{C}[\Gamma]$ for its group algebra over \mathbb{C} , and $\rho : \mathbb{C}[\Gamma]^{n \times n} \rightarrow \mathbb{C}^{rn \times rn}$ for the injective **-algebra** homomorphism that replaces each element of $\mathbb{C}[\Gamma]$ with its $r \times r$ Cayley representation, extended linearly to $\mathbb{C}[\Gamma]$.

Definition 3.1. Given an abelian group Γ and a positive integer n , an $n \times n$ **roux** for Γ is a matrix B with entries in $\mathbb{C}[\Gamma]$ such that the following hold:

- (R1) $B_{ii} = 0$ for every $i \in [n]$.
- (R2) $B_{ij} = B_{ji}^*$ for every $i, j \in [n]$, $i \neq j$.
- (R3) $B_{ji} = (B_{ij})^{-1}$ for every $i, j \in [n]$, $i \neq j$.
- (R4) The vector space $A(B) := \text{span}(\{gl : g \in \Gamma\} \cup \{gB : g \in \Gamma\}) \leq \mathbb{C}[\Gamma]^{n \times n}$ is an algebra, i.e., it is closed under matrix multiplication.

Every roux $B \in \mathbb{C}^{n \times n}$ determines an association scheme (called the **roux scheme**) with adjacency matrices $\{gI\}_g$ and $\{gB\}_g$.

Proposition 3.2 (Lemma 2.3 in [28]). *Suppose $B \in \mathbb{C}^{n \times n}$ satisfies (R1)–(R3). Then B is a roux for Γ if and only if*

$$B^2 = (n - 1)I + \sum_g c_g gB$$

for some complex numbers $\{c_g\}_g$, called the **roux parameters** for B . In this case, we necessarily have that $\{c_g\}_g$ are nonnegative integers that sum to $n - 2$, with $c_{g^{-1}} = c_g$ for every g .

Proposition 3.3 (Theorem 2.8 in [28]). *Given an $n \times n$ roux B for Γ with parameters $\{c_g\}_g$, the primitive idempotents for the corresponding roux scheme are scalar multiples of*

$$G := \sum_g (g) gI + \mu \sum_g (g) gB, \quad (\Gamma, \{+, -\}),$$

where μ is defined in terms of the Fourier transform $\hat{c} := \sum_h c_h \overline{\chi(h)}$ as follows:

$$\mu := \frac{\hat{c} + \sqrt{(\hat{c})^2 + 4(n - 1)}}{2(n - 1)}.$$

The rank of G is

$$d := \frac{n}{1 + (n - 1)(\mu)^2}.$$

Furthermore, if $\Gamma = \Gamma$, $\mu > 0$, and

$$G := \sum_g (g) gI + \mu \sum_g (g) gB$$

satisfies $G^2 = cG$ for some $c > 0$, then $\mu = \mu$ for some $\Gamma = \{+, -\}$, and a scalar multiple of $G = G$ is a primitive idempotent for the roux scheme of B .

Given a finite abelian group Γ , we write $\hat{\Gamma}$ for the Pontryagin dual group of characters $\chi: \Gamma \rightarrow \mathbb{T}$. Every $\chi \in \hat{\Gamma}$ extends by linearity to a \mathbb{C} -algebra homomorphism $\bar{\chi}: \mathbb{C}[\Gamma] \rightarrow \mathbb{C}$, which in turn extends to a \mathbb{C} -algebra homomorphism $\hat{\chi}: \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$ through entrywise application of $\bar{\chi}$.

Proposition 3.4 (Theorem 3.1 in [28]). *Suppose $B \in \mathbb{C}^{n \times n}$ satisfies (R1)–(R3). Then B is a roux if and only if for every $\chi \in \hat{\Gamma}$, $\hat{\chi}(B)$ is the signature matrix of an equiangular tight frame.*

Any sequence of lines arising from an application of Proposition 3.4 are called **roux lines**. We will see that every sequence of doubly transitive lines is roux. While it will not play a role in the sequel, we also remark that every *regular abelian distance-regular antipodal cover of the complete graph* (drackn) can be viewed as an instance of a roux, as explained in [28], and the resulting roux lines coincide with the construction of ETFs from regular abelian drackns in [19, 11].

We now have two constructions of lines from roux: one set from the primitive idempotents in Proposition 3.3, and another from Proposition 3.4. However, the resulting lines are identical. Indeed, for any choice of Γ and $\Gamma = \{1, -1\}$, it holds that $[G^{-1}]_{(i,1),(j,1)} = \mu^{-1} (B_{ij}) = (B_{ij})/\mu$ whenever $i = j$. This yields the following version of Lemma 3.2 in [28].

Proposition 3.5. For each $\hat{}$ and $\{1, -1\}$, the signature matrix $\hat{}(B)$ from Proposition 3.4 and the Gram matrix G_{-1} from Proposition 3.3 describe the same lines (each line implicated by the former is represented $| \ |$ times in the latter).

One can easily determine whether or not roux lines are real with the following result.

Proposition 3.6 (Real roux lines detector, Corollary 3.8 in [28]). Let B be a roux for $\hat{}$ and pick $\hat{}$. Then $\hat{}(B)$ is a signature matrix of real lines if and only if $\hat{}(g)$ is real for every g such that $c_g = 0$.

Finally, there are several trivial ways to deform roux, one of which is described below.

Proposition 3.7 (Lemma 2.5(a) in [28]). Let $B \in \mathbb{C}[]^{n \times n}$ be a roux with parameters $\{c_g\}_g$. Given any diagonal matrix $D \in \mathbb{C}[]^{n \times n}$ with $D_{ii} \neq 0$ for every $i \in [n]$, then DBD^{-1} is a roux with parameters $\{c_g\}_g$.

The roux B and DBD^{-1} in Proposition 3.7 are called **switching equivalent**. Given any such B and D and any $\hat{}$, it is easy to see that the signature matrices $\hat{}(B)$ and $\hat{}(DBD^{-1})$ are switching equivalent. Consequently, switching equivalent roux create the same roux lines.

3.3. Schurian roux schemes and Higman pairs.

Definition 3.8. Given a finite group G and a proper subgroup $H \subset G$, let $K := N_G(H)$ be the normalizer of H in G . We say (G, H) is a **Higman pair** if there exists a key $b \in G \setminus K$ such that

- (H1) G acts doubly transitively on G/K ,
- (H2) K/H is abelian,
- (H3) $HbH = Hb^{-1}H$,
- (H4) $aba^{-1} \in HbH$ for every $a \in K$, and
- (H5) $a \in K$ satisfies $ab \in HbH$ only if $a \in H$.

The relationship between roux and Higman pairs is given below and depicted in Figure 1.

Proposition 3.9 (Theorem 2.1 in [28]). Let G be a finite group, and pick $H \subset G$. The Schurian scheme of (G, H) is isomorphic to a roux scheme if and only if (G, H) is a Higman pair.

Proposition 3.10 (Lemma 2.6 in [28]). Given a Higman pair (G, H) , denote $K := N_G(H)$, $n := [G : K]$ and $r := [K : H]$, and select any key $b \in G \setminus K$. Then the following hold:

- (a) H has $2r$ double cosets in G : r of the form aH , and r of the form $HabH$ for some $a \in K$;
- (b) for every $a \in K$, we have $HabH = HbaH$; and
- (c) for every $a \in K$, we have $|HabH| = (n - 1)|H|$.

Proposition 3.11 (Roux from Higman pairs, Lemma 2.7 in [28]). Given a Higman pair (G, H) , denote $K := N_G(H)$ and $n := [G : K]$, and select any key $b \in G \setminus K$. Choose left coset representatives $\{x_j\}_{j \in [n]}$ for K in G , and choose coset representatives $\{a_g\}_{g \in K/H}$ for H in K . Define $B \in \mathbb{C}[K/H]^{n \times n}$ entrywise as follows: Given $i = j$, let B_{ij} be the unique $g \in K/H$ for which $x_i^{-1}x_j \in Ha_gbH$, and set $B_{ii} = 0$. Then B is a roux for K/H with roux parameters $\{c_g\}_{g \in K/H}$ given by

$$(3) \quad c_g = \frac{n - 1}{|H|} \cdot |bHb^{-1} \cap Ha_gbH|.$$

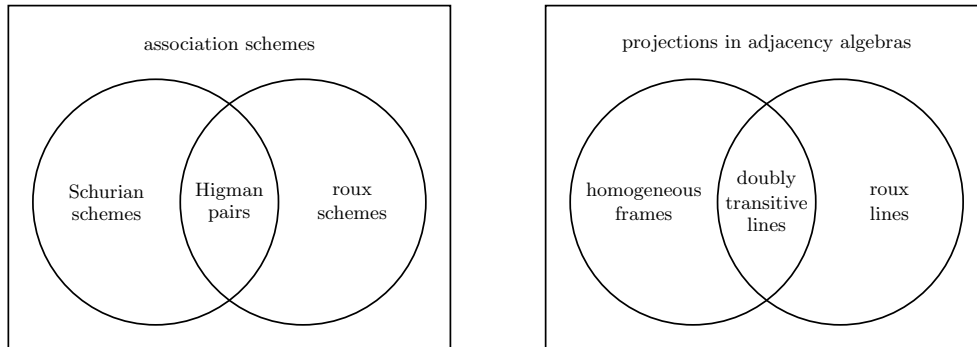


Figure 1. (left) The relationship between roux and Higman pairs, as described by Proposition 3.9. (right) The dual relationships for lines, as given by Theorem 4.1.

Furthermore, the roux scheme generated by B is isomorphic to the Schurian scheme of (G, H) .

We emphasize that a Higman pair (G, H) does not, in general, carry a unique key, and distinct keys may create distinct roux through Proposition 3.11. However, for any choice of key the resulting roux scheme is isomorphic to the Schurian scheme of (G, H) . Since roux lines correspond with primitive idempotents in the roux scheme, their ranks do not depend on the choice of key. In particular, the ranks of the primitive idempotents for the Schurian scheme of (G, H) can be computed using (3) and Proposition 3.3.

4. Radicalization and the Higman Pair Theorem

As mentioned in the introduction, every sequence of linearly dependent doubly transitive lines is determined by a Higman pair. This fact will enable us to classify doubly transitive lines that exhibit almost simple symmetries. The following result makes this pivotal correspondence explicit.

Theorem 4.1 (Higman Pair Theorem, Theorem 1.3 in [28]).

- (a) Assume $n > 3$. Given $n > d$ doubly transitive lines that span \mathbb{C}^d , there exists r such that one may select r equal-norm representatives from each of the n lines that together carry the association scheme of a Higman pair (G, H) with $r = [N_G(H) : H]$ and $n = [G : N_G(H)]$. Moreover, their Gram matrix is a primitive idempotent for this scheme.
- (b) Every Higman pair (G, H) is a Gelfand pair. Every primitive idempotent of its association scheme is the Gram matrix of $r := [N_G(H) : H]$ equal-norm representatives from each of $n := [G : N_G(H)]$ doubly transitive lines that span \mathbb{C}^d with $d < n$, and the phase of each entry is an r th root of unity. Moreover, the automorphism group of the lines contains the doubly transitive action of G on $G/N_G(H)$.

The proof of Theorem 4.1(b) appeared in [28], but the proof of Theorem 4.1(a) was saved for the current installment. Our proof technique relies on the following notions. A **projective unitary representation** of a finite group G is a mapping $\rho : G \rightarrow \text{U}(d)$ such that $\rho(1) = I$ and such that there exists $f : G \times G \rightarrow \mathbb{T}$ satisfying $\rho(x)\rho(y) = f(x, y)\rho(xy)$ for every $x, y \in G$. If $\rho : G \rightarrow \text{U}(d)$ is a homomorphism (i.e.,

	Group	Stabilizer	Character	Kernel
Radicalization	$\tilde{G} = G \times C_r$	$\tilde{G}_1 = G_1 \times C_r$	$\tilde{\alpha}: \tilde{G}_1 \rightarrow C_r$ extends	$H = \ker \tilde{\alpha}$
Schur cover	\check{G}	$\check{G}_1 = \pi^{-1}(G_1)$	$\alpha: \check{G}_1 \rightarrow C_{r'}$	
Permutation group	G	G_1		

Figure 2. Overview of notation in Example 4.5, the proof of Theorem 4.7, and Section 5. Here, $\check{\cdot} : G \rightarrow G$ is a Schur covering, $r = 2r'$, and (\tilde{G}, H) is the radicalization of (G, G_1, \cdot) from Definition 4.6.

if $f = 1$), then we call ρ an (honest) **unitary representation**. The function f may be formalized as follows.

Definition 4.2. Let G be a finite group. A **2-cocycle** is a function $f: G \times G \rightarrow \mathbb{C}^\times$ such that $f(x, y)f(xy, z) = f(y, z)f(x, yz)$ for all $x, y, z \in G$. A **2-coboundary** is a function $f: G \times G \rightarrow \mathbb{C}^\times$ for which there exists $t: G \rightarrow \mathbb{C}^\times$ such that $f(x, y) = t(x)t(y)t(xy)^{-1}$ for all $x, y \in G$. The **Schur multiplier** of G is the abelian group $M(G)$ of 2-cocycles under pointwise multiplication, modulo the subgroup of 2-coboundaries. A **Schur cover** of G is a group \check{G} with a subgroup A such that:

- (i) the center of G and the commutator subgroup of G both contain A ,
- (ii) $A = M(G)$, and
- (iii) there is an epimorphism $\check{\cdot} : G \rightarrow G$ (a **Schur covering**) with kernel A .

The Schur multiplier of a finite group is finite, and every finite group has a (necessarily finite) Schur cover. We refer to [30, 31] for background. The following result explains how Schur covers help convert projective unitary representations into honest unitary representations.

Proposition 4.3. Let G be a finite group with a projective unitary representation $\rho : G \rightarrow \text{U}(d)$. Given any Schur covering $\check{\cdot} : G \rightarrow G$, there exists a unitary representation $\tilde{\rho} : G \rightarrow \text{U}(d)$ and phases $\{ \chi_x \}_{x \in G}$ such that $\tilde{\rho}(x) = \chi_x \rho(\check{x})$ for every $x \in G$.

Proof. A theorem of Schur [30, Lemma 3.1] provides the existence of a representation $\rho : G \rightarrow \text{GL}(d, \mathbb{C})$ and an ensemble of complex scalars $\{ \chi_x \}_{x \in G}$ satisfying $\tilde{\rho}(x) = \chi_x \rho(\check{x})$ for every $x \in G$. In order to verify that $\tilde{\rho}$ takes its image in $\text{U}(d) \subset \text{GL}(d, \mathbb{C})$, we need only check that each scalar χ_x is unimodular. This can be seen by considering the eigenvalues of the diagonalizable operators $\rho(\check{x})$ and $\tilde{\rho}(x)$. The first has unimodular eigenvalues because it is unitary; the second has unimodular eigenvalues since its order is finite. As $\tilde{\rho}(x) = \chi_x \rho(\check{x})$, we conclude that χ_x is a phase.

Lemma 4.4. Let G be a doubly transitive group of automorphisms of a sequence $L = \{ \ell_i \}_{i \in [n]}$ of $n > d$ lines spanning \mathbb{C}^d . For each $g \in G$, choose a unitary $U(g) \in \text{U}(d)$ such that $U(1) = I$ and $U(g)_i = U(g)_{(i)}$ for every $i \in [n]$. Then $\rho : G \rightarrow \text{U}(d)$ is a projective unitary representation.

Proof. Choose unit-norm representatives $\ell_i = \{ \ell_i \}_{i \in [n]}$ for L . Then ℓ_i is an equiangular tight frame, by Proposition 1.1. Since $n > d$, the common value of $|\langle \ell_i, \ell_j \rangle|$ for $i \neq j$ is not zero. The desired result now follows immediately from [9, Lemma 6.5].

The following example demonstrates the fundamental connection between doubly transitive lines and Higman pairs.

Example 4.5. We produce a Higman pair (\tilde{G}, H) from four doubly transitive lines in \mathbb{C}^2 . Put $\alpha := e^{i/4}$, $a := \sqrt{2 - \sqrt{3}}$, $b := \sqrt{1 + a^2} = \sqrt{3 - \sqrt{3}}$, and consider the lines $L = \{l_1, l_2, l_3, l_4\}$ spanned by columns of

$$L := \frac{1}{b} \begin{bmatrix} a & a \\ a - a & - \end{bmatrix}.$$

It is easy to check that L is an equiangular tight frame, and it is well known that $\text{Aut } L = A_4 =: G$. Indeed, the permutation $\pi_1 := (12)(34)$ is performed by the unitary $U(\pi_1) := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and the permutation $\pi_2 := (234)$ is performed by the unitary $U(\pi_2) := \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$. Consequently, $\text{Aut } L \cong \langle \pi_1, \pi_2 \rangle = A_4$. Equality holds since there do not exist $n = 4$ lines in $d = 2$ dimensions that have triply transitive automorphism group S_4 ; see Lemma 6.2. By Lemma 4.4, we may extend L to a projective unitary representation $\rho : G \rightarrow \text{U}(2)$ with the property that $(\rho(g))_{ii} = \rho(g)_{ii}$ for each $g \in G$ and $i \in [4]$; in fact, the extension is unique up to a choice of unimodular constants. We desire an honest unitary representation, and so we consider the Schur cover $\tilde{G} := \text{SL}(2, 3)$. Denoting $x_1 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $x_2 := \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$, we have that $\tilde{G} = \langle x_1, x_2 \rangle$, and a Schur covering $\rho : \tilde{G} \rightarrow G$ is given by $\rho(x_1) = \pi_1$ and $\rho(x_2) = \pi_2$. As in Proposition 4.3, the data $(\rho(x_1)) := i \cdot U(\pi_1)$ and $(\rho(x_2)) := -e^{i/12} U(\pi_2)$ then determine a unitary representation $\rho : \tilde{G} \rightarrow \text{U}(2)$ with the property that $(\rho(x))_{ii} = \rho(x)_{ii}$ for each $x \in \tilde{G}$ and $i \in [4]$.

Next, we consider how a subgroup of G holds a line invariant via L . The stabilizer $G_1 \triangleleft A_4$ of $l_1 \in [4]$ is generated by π_2 , and $G_1 := \rho^{-1}(G_1)$ is the preimage of G_1 in \tilde{G} . As such, $G_1 = \{x \in \tilde{G} : \rho(x)_{11} = 1\}$ is the stabilizer of l_1 . Since l_1 is spanned by $\frac{1}{b} \begin{bmatrix} a \\ a \end{bmatrix}$, the action of G_1 on l_1 determines a homomorphism $\chi : G_1 \rightarrow \mathbb{T}$ such that $\chi(x) = \rho(x)_{11}$ for every $x \in G_1$. Explicitly, $\chi(x_2) = -1$, since $\begin{bmatrix} a \\ a \end{bmatrix}$ is an eigenvector of $\rho(x_2)$ with eigenvalue -1 . It follows that the G -orbit of l_1 consists of the columns of z for each $z \in \{\pm 1\} = C_r$, where $r := 2$. Meanwhile, the off-diagonal entries of the signature matrix of L are all members of C_r with $r := 2r = 4$.

It will be convenient to extend G and L so that the resulting orbit of l_1 consists of the columns of z for every $z \in C_r$, as suggested by the signature matrix. To accomplish this, we put $\tilde{G} := G \times C_r$ and define $\tilde{\rho} : \tilde{G} \rightarrow \text{U}(2)$ by $\tilde{\rho}(x, z) := z \cdot \rho(x)$. Then the stabilizer of l_1 in \tilde{G} is $\tilde{G}_1 := G_1 \times C_r$, and the resulting character is $\tilde{\chi} : \tilde{G}_1 \rightarrow \mathbb{T}$ given by $\tilde{\chi}(x, z) := z \cdot \chi(x)$. Taking $H := \{\tilde{x} \in \tilde{G} : \tilde{\rho}(\tilde{x}) = 1\} = \ker \tilde{\chi}$, then it turns out that (\tilde{G}, H) is a Higman pair. Furthermore:

We can recover the doubly transitive lines from the Higman pair.

Indeed, the \tilde{G} -orbit of l_1 consists of $rn = 16$ vectors, and their Gram matrix is a scalar multiple of an (easy-to-find) primitive idempotent in the adjacency algebra of (\tilde{G}, H) .

Following the previous example, we note that when $r = 2r$, each element of C_r can be obtained by squaring an appropriate element of C_r . In this sense, C_r consists of radicals of C_r . This fact motivates the following terminology, which will play a crucial role in our proofs of the Higman Pair Theorem and the main result.

Definition 4.6. Given a group G , a subgroup $G_1 \triangleleft G$, and a linear character $\chi : G_1 \rightarrow \mathbb{T}$, put $r = |\text{im } \chi|$, $r = 2r$, and define $\rho : G_1 \times C_r \rightarrow C_r$ by $\rho(x, z) = \chi(x)z$. (Here, $C_r \triangleleft \mathbb{C}^\times$ is the group of r th roots of unity under multiplication.) We say $(G \times C_r, \ker \rho)$ is the **radicalization** of (G, G_1, χ) .

Theorem 4.7 (Lines from permutations). Let $G \triangleleft S_n$ be a doubly transitive permutation group with $n > 3$, and let $G_1 \triangleleft G$ be the stabilizer of $1 \in [n]$. Choose any Schur covering $\rho : G \rightarrow G$, and put $G_1 := \rho^{-1}(G_1)$. If G is a group of automorphisms of a

sequence L of $n > d$ lines spanning \mathbb{C}^d , then there is a linear character $\chi : G_1 \rightarrow \mathbb{T}$ such that:

- (a) the radicalization (G, H) of (G, G_1, χ) is a Higman pair, and
- (b) the Schurian scheme of (G, H) has a primitive idempotent that is the Gram matrix of $r := [N_G(H) : H]$ equal-norm representatives from each of the $n := [G : N_G(H)]$ lines in L , and furthermore, this Gram matrix carries the scheme.

Our proof of Theorem 4.7 uses the following well-known characterizations of doubly transitive actions. For background on permutation groups, we refer the reader to [14].

Proposition 4.8. *Let G be a finite group acting transitively on a set X , and let $G_1 \triangleleft G$ be the stabilizer of some point $x_1 \in X$. Then the following are equivalent:*

- (a) G acts doubly transitively on X .
- (b) G_1 acts transitively on $X \setminus \{x_1\}$.
- (c) For any $g \in G \setminus G_1$, $G = G_1 \cup gG_1$.
- (d) For any epimorphism $\chi : G \rightarrow G/G_1$, G acts doubly transitively on G/G_1 , where $G_1 := \chi^{-1}(1)$.

When these conditions hold, $G_1 \triangleleft G$ is a maximal subgroup.

Proof of Theorem 4.7. Denote $L = \{l_j\}_{j \in [n]}$. Since G consists of automorphisms of L , we can choose unitaries $\{U(i)\}_{i \in [n]}$ such that $U(1) = I$ and $U(i) = U(j)$ for every $i, j \in [n]$. Then $\rho : G \rightarrow U(d)$ is a projective unitary representation, by Lemma 4.4. Applying Proposition 4.3, we lift ρ to an honest unitary representation $\tilde{\rho} : G \rightarrow U(d)$ of the Schur cover, where $\tilde{\rho}(x)_i = \rho(x)_{(i)}$ for every $x \in G$ and $i \in [n]$.

Fix a unit vector χ_1 . Since G_1 holds χ_1 invariant, every $g \in G_1$ produces a unimodular constant $\chi(g)$ such that $\tilde{\rho}(g)\chi_1 = \chi(g)\chi_1$. Moreover, $\chi : G_1 \rightarrow \mathbb{T}$ is a linear character since ρ is a homomorphism. Put $r := \chi_1$ and $\chi_r := 2r$, so that χ_r maps G_1 onto $C_r \triangleleft C_r \triangleleft \mathbb{T}$. We would like to extend χ_r in such a way that its image includes all of $\{zI : z \in C_r\}$. To that end, we define extension groups $G := G \times C_r$ and $G_1 := G_1 \times C_r$, and let $\tilde{\rho} : G \rightarrow U(d)$ be the unitary representation given by $\tilde{\rho}(x, z) = z \rho(x)$. Then G permutes L according to the rule $\tilde{\rho}(x, z)_i = \rho(x)_{(i)}$, and in particular

$$(4) \quad G_1 = \bar{\chi} : G \rightarrow \mathbb{T} \quad \tilde{\rho}(\bar{\chi})_1 = \chi_1.$$

As before, this implies that $\tilde{\rho}(x, z) = \chi_r(z) \rho(x)$ for every $(x, z) \in G_1$, where $\chi_r : G_1 \rightarrow C_r$ is the character $\chi_r(z) = z \chi(z)$. Let $H \triangleleft G_1$ be the group that stabilizes not only χ_1 , but also χ_r :

$$H = \bar{\chi} : G \rightarrow \mathbb{T} \quad \tilde{\rho}(\bar{\chi}) = \ker \chi_r = \{z \in C_r : \chi_r(z) = 1\} = \{z \in C_r : z^2 = 1\} = G_1.$$

Then (G, H) is the radicalization of (G, G_1, χ) . The notation established thus far is partially summarized in Figure 2.

To prove (G, H) is a Higman pair, first observe that $H = \ker \chi_r \in G_1$, and so $N_G(H) > G_1$. By Proposition 4.8 and (4), G acts doubly transitively on G/G_1 , and $G_1 \triangleleft G$ is a maximal subgroup. We claim that H is not normal in G , and therefore $G_1 = N_G(H)$. Notice that G_1 is not normal in G since it has $n > 3$ left cosets but only two double cosets. Choose $g \in G_1$ and $x \in G$ such that $xg \notin G_1$. Then $(x, 1) \in H$, but $(x, 1)(g, 1)^{-1}(x, 1)^{-1} = (xg^{-1}, 1)^{-1} \notin H$. This proves the claim, and (H1) follows. We have (H2) since $G_1/\ker \chi_r = C_r$.

We now obtain a candidate key for (G, H) . For each $i \in [n]$, choose $x_i \in G$ such that $\rho(x_i)_1 = i$. Then $\{x_i\}_{i \in [n]}$ is a transversal for G/G_1 , and $\chi := \{\chi(x_i)\}_{i \in [n]}$

is an ETF by Proposition 1.1. Every $x \in G$ can be written uniquely in the form $x = x_i$ for some $i \in [n]$ and $x \in G_1$, and so

$$(5) \quad \langle x, x \rangle = \begin{cases} 1 & \text{if } x \in G_1, \\ \mu & \text{if } x \notin G_1, \end{cases}$$

where $\mu = \frac{n-d}{d(n-1)} \in \{0, 1\}$ by the Welch bound [53]. We claim that $\mu^{-1} \langle x, x \rangle \in C_r$ whenever $x \notin G_1$. Choose such an x . Then $\langle x, x \rangle = 1$. Since G acts doubly transitively, there exists $y \in G$ such that $\langle y, (x)(1) \rangle = 1$ and $\langle y, (1) \rangle = \langle x, (1) \rangle$. We have $yx, x^{-1}y \in G_1$, and so

$$(6) \quad \begin{aligned} \langle x, x \rangle &= \langle yx, yx \rangle = \langle xx^{-1}y, xx^{-1}y \rangle \\ &= \langle yx, x \rangle \langle x, x^{-1}y \rangle \\ &= \langle yxy^{-1}x, x \rangle = \mu. \end{aligned}$$

Write $\langle x, x \rangle = z\mu$. Then (6) says that $z\mu = \langle yxy^{-1}x, z^{-1}\mu \rangle$, so that $z^2 = \langle yxy^{-1}x, z^{-1}\mu \rangle \in C_r$. Therefore, $z \in C_r$, as desired. In particular, there exists $b = x, z^{-1} \in G \cap G_1$ such that $\langle b, b \rangle = \mu$. Fix such b for the remainder of the proof.

Next, we demonstrate the crucial identity

$$(7) \quad \text{for any } \bar{x}, \bar{y} \in G, \quad H\bar{x}H = H\bar{y}H \implies \langle \bar{x}, \bar{x} \rangle = \langle \bar{y}, \bar{y} \rangle.$$

If $\bar{x} = a\bar{y}a$ for $a, a \in H$, then

$$\langle \bar{x}, \bar{x} \rangle = \langle \bar{y}, \bar{y} \rangle \langle a, a \rangle \langle a^{-1}, a^{-1} \rangle = \langle \bar{y}, \bar{y} \rangle.$$

Conversely, suppose $\bar{x} = (x, z)$ and $\bar{y} = (y, w)$ satisfy $\langle \bar{x}, \bar{x} \rangle = \langle \bar{y}, \bar{y} \rangle$, that is,

$$z \langle x, x \rangle = w \langle y, y \rangle.$$

By (5), either $x, y \in G_1$ or $x, y \notin G_1$. In the former case, we have $z \langle x, x \rangle = w \langle y, y \rangle$, so that $\langle y^{-1}x, w^{-1}z \rangle = 1$ and $\langle x, z \rangle = \langle y, w \rangle \langle y^{-1}x, w^{-1}z \rangle = H(y, w)H$. On the other hand, if $x, y \notin G_1$ then there exist $x', y' \in G_1$ such that $x = y'$, by Proposition 4.8. In that case,

$$w \langle y, y \rangle = z \langle x, x \rangle = z \langle y', y' \rangle, \quad (z')^{-1} = z^{-1} \langle y' \rangle^{-1} \langle y \rangle,$$

and $z = (z')^{-1}w \langle y' \rangle^{-1}$, so that $\langle x, z \rangle = \langle y', (z')^{-1} \langle y, w \rangle \rangle = (z')^{-1} H(y, w)H$. This proves (7).

The remaining Higman pair axioms follow quickly from (7). Since $\langle b, b \rangle = \mu \in \mathbb{R}$, we have $\langle b^{-1}, b^{-1} \rangle = \mu$, and so $b^{-1} \in HbH$. This is (H3). For every $a \in G_1$,

$$\begin{aligned} \langle aba^{-1}, aba^{-1} \rangle &= \langle b, b \rangle \langle a^{-1}, a^{-1} \rangle \langle a, a \rangle \\ &= \langle b, b \rangle \langle a^{-1}, a^{-1} \rangle \\ &= \langle b, b \rangle, \end{aligned}$$

so $aba^{-1} \in HbH$. This is (H4). Finally,

$$\langle ab, ab \rangle = \langle b, b \rangle \langle a^{-1}, a^{-1} \rangle = \langle a \rangle \langle b \rangle,$$

so $ab \in HbH$ only if $a \in \ker = H$. This is (H5), and the proof of (a) is complete.

To prove (b), put $\mathcal{L} = \{ \langle x_i, z \rangle \}_{i \in [n], z \in C_r} = \{ z \cdot \langle x_i \rangle \}_{i \in [n], z \in C_r}$. As $\text{span}\{ \langle x_i \rangle \} = \langle x_i \rangle_{i=1}^n = \mathbb{R}^n$, \mathcal{L} consists of r representatives from each line in L . Since $\mathcal{L} = \{ \langle x_i \rangle \}_{i \in [n]}$ is an ETF, Proposition 3.20 in [25] implies that \mathcal{L} is a tight frame. In particular, its Gram matrix $G := \langle \langle x_i, z \rangle, \langle x_j, z \rangle \rangle$ is a scalar multiple of an

orthogonal projection. Notice that $\{(x_i, z)\}_{i \in [n], z \in C_r}$ is a transversal for G/H , since $|G/H| = |G/G_1| \cdot |G_1/H| = nr$ and

$$(x_i^{-1}x_j, z^{-1}w) \in H \iff x_i^{-1}x_j \in G_1 \text{ and } (x_i^{-1}x_j) = w^{-1}z \iff i = j \text{ and } z = w.$$

It follows from Theorem 3.2 in [25] that G lies in the adjacency algebra of (G, H) . Moreover, (7) implies that two entries

$$G_{(i,z),(j,w)} = \sum_{x \in G_1} (x_i^{-1}x_j, z^{-1}w) \cdot x, \quad \text{and} \quad G_{(i,z),(j,w)} = \sum_{x \in G_1} (x_i^{-1}x_j, (z^{-1}w)) \cdot x,$$

of G are equal if and only if $(x_i, z)^{-1}(x_j, w) \in H(x_i, z)^{-1}(x_j, w)H$. Therefore, G carries the Schurian scheme of (G, H) .

It remains only to prove that a scalar multiple of G is a *primitive* idempotent for (G, H) . To that end, let $B \in \mathbb{C}[C_r]^{n \times n}$ be the roux given by Proposition 3.11 for the Higman pair (G, H) . Specifically, choose $\{\tilde{x}_i\}_{i \in [n]} := \{(x_i, 1)\}_{i \in [n]}$ as a transversal for G/G_1 , and $\{\tilde{a}_z\}_{z \in C_r} := \{(1, z)\}_{z \in C_r}$ as a transversal for \tilde{G}_1/H . For $i = j$, (7) implies that $\tilde{x}_i^{-1}\tilde{x}_j \in H\tilde{a}_z bH$ if and only if $(x_i^{-1}x_j) = z\mu$. By Proposition 3.11,

$$B_{ij} = \begin{cases} \mu^{-1} (x_i^{-1}x_j) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

If \hat{C}_r is given by $(z) = z^{-1}$, then comparison of the entries shows that

$$G = \sum_{z \in C_r} (z) zI + \mu \sum_{z \in C_r} (z) zB.$$

Then Proposition 3.3 implies that a scalar multiple of G is a primitive idempotent for (G, H) .

Proof of Theorem 4.1(a). Immediate from Theorem 4.7.

The language of the following corollary intentionally mirrors that of Theorem 3.3 in [28], which gave a parallel characterization of roux lines.

Corollary 4.9. *Let L be a sequence of linearly dependent complex lines. Then L is doubly transitive if and only if all of the following occur simultaneously:*

- (a) L is equiangular,
- (b) there exist unit-norm representatives $\{z_j\}_{j \in [n]}$ of L whose signature matrix is comprised of r th roots of unity for some r , and
- (c) the Gram matrix G of $\{z_j\}_{j \in [n], z \in C_r}$ carries a Schurian association scheme.

In this case, G carries the Schurian association scheme of a Higman pair, and a scalar multiple of G is a primitive idempotent for that scheme.

Proof. If L is doubly transitive, then (a)–(c) were verified in the proof of Theorem 4.7. Conversely, if (a)–(c) are satisfied, then Theorem 3.3 in [28] gives that there is a roux having L as roux lines and the Schurian association scheme carried by G as its roux scheme, with a scalar multiple of G as a primitive idempotent. By Proposition 3.9, G carries the Schurian association scheme of a Higman pair. Then Theorem 4.1 implies that G is the Gram matrix of r unit-norm representatives from a sequence of doubly transitive lines, namely L .

5. Roux from radicalization

The previous section leveraged Theorem 4.7 to prove part (a) of the Higman Pair Theorem, and in the next section, we will again apply Theorem 4.7 to classify sequences of doubly transitive lines that exhibit almost simple symmetries. In particular, for every almost simple doubly transitive permutation group, we will determine whether

that group is a subgroup of the automorphism group of some sequence of lines. By Theorem 4.7, it suffices to find a linear character for which a certain radicalization is a Higman pair. In the event that such a character exists, we will need to dig further to describe the lines, find the dimension of their span, and determine whether the lines are real. The purpose of this section is to build up the machinery needed to answer such questions.

Remark 5.1. Throughout this section, the results we present will assume the following setting without mention. Let G be a doubly transitive permutation group on $n > 3$ points, and let $G_1 \triangleleft G$ be the stabilizer of some point. Given a Schur covering $\pi : G \twoheadrightarrow G_1$, put $G_1 = \pi^{-1}(G_1)$. Fix a linear character $\chi : G_1 \rightarrow C_r$ and denote $r := 2r = 2/\text{im } \chi$, $G := G \times C_r$, and $G_1 := G_1 \times C_r$. Let $\pi : G_1 \twoheadrightarrow C_r$ be the epimorphism given by $\pi(x, z) = z \pi(x)$, and put

$$(8) \quad H := \ker \pi = \{ (x, z) \in G_1 \mid \pi(x) = 1 \},$$

so that (G, H) is the radicalization of (G, G_1, π) . This notation is summarized in Figure 2.

The following lemma is proved as in the proof of Theorem 4.7.

Lemma 5.2. G acts doubly transitively on G/G_1 . Furthermore, $N_G(H) = G_1$.

Lemma 5.3 (Key finder). Assume that (G, H) is a Higman pair. Given $x \in G \setminus G_1$, there exist $\pi, \sigma \in G_1$ such that $x^{-1} = \pi x \sigma$. Pick either of the $z \in C_r$ such that $z^2 = \pi(\sigma)$. Then (x, z) is a key for (G, H) .

Proof. First observe that $x \notin G_1$, so $(x, z) \notin G_1 = N_G(H)$. To prove (H3) for b , notice that $z^{-1} = z \pi(\sigma)^{-1}$, so that

$$(9) \quad (x, z)^{-1} = \pi x \sigma, z \pi(\sigma)^{-1} = \pi, \pi^{-1}(x, z) \in \pi^{-1} H(x, z)H.$$

For (H4), take any $(\pi, w) \in G_1$ and compute

$$(10) \quad (\pi, w)(x, z)(\pi, w)^{-1} = \pi x^{-1} z = \pi, \pi^{-1}(x, z) \pi^{-1}, \pi^{-1} H(x, z)H.$$

It remains to prove (H5) for b . To that end, assume $(\pi, w) \in G_1$ satisfies $(\pi, w)(x, z) \in H(x, z)H$. Then there exist $\sigma, \tau \in G_1$ such that

$$(\pi, w)(x, z) = \sigma, \pi^{-1}(x, z) \tau, \pi^{-1}.$$

In other words, $x = \pi x$ and $w = \pi(\sigma)^{-1}$. Since (G, H) is a Higman pair by assumption, we may select a key (x_0, z_0) . As $x, x_0 \notin G_1$, Lemma 5.2 and Proposition 4.8 provide $\pi, \sigma \in G_1$ such that $x = \pi x_0$. Substituting this relation into $x = \pi x$, we obtain $x_0 = \pi x_0$. Recalling that $w = \pi(\sigma)^{-1}$, we have

$$\begin{aligned} \pi, w \pi^{-1}(x_0, z_0) &= \pi x_0 \pi^{-1}, \pi^{-1} \pi^{-1} \pi^{-1} z_0 \\ &= \pi, \pi^{-1}(x_0, z_0) \pi^{-1}, \pi^{-1} H(x_0, z_0)H. \end{aligned}$$

Since (x_0, z_0) is a key, (H5) implies that $\pi, w \pi^{-1} \in H$. In other words, $\pi^{-1} = w \pi^{-1}$, so that $(\pi, w) = \pi, \pi^{-1} \in H$. Therefore (H5) holds for b .

Theorem 5.4 (Higman pair detector). The following are equivalent for any choice of $x \in G \setminus G_1$:

- (a) The radicalization of (G, G_1, π) is a Higman pair.
- (b) Every $\pi \in G_1$ for which $x \pi x^{-1} \in G_1$ satisfies $(x \pi x^{-1}) = \pi$.

We emphasize that the equivalence of (a) and (b) holds for *any* choice of $x \in G \cap G_1$, despite the fact that x appears in (b) but not in (a). As we will see, that flexibility makes this tool powerful.

Theorem 5.4 implies that the radicalization of (G, G_1, \sim) is a Higman pair only if $\ker \sim$ is **strongly closed** in G_1 with respect to G . That is, for any $x \in G$, $x(\ker \sim)x^{-1} \cap G_1 \subseteq \ker \sim$. Related conditions for doubly transitive two-graphs appear in Theorem 6.1 of [49], Theorem 6.1 of [6], and Remark 7.6 of [44].

Proof of Theorem 5.4. Assume (a) holds. By Lemma 5.3, there exists $z \in C_r$ such that $b := (x, z)$ is a key for the Higman pair (G, H) . Choose $\gamma \in G_1$ such that $x\gamma x^{-1} = \gamma$, as in (b). Then

$$\gamma^{-1} \gamma (x, z) = \gamma^{-1} x^{-1} x \gamma z = \gamma^{-1} (\gamma)^{-1} (x, z) \gamma^{-1} (\gamma) = H(x, z)H.$$

By (H5), $\gamma^{-1} \gamma \in H$. Consequently, $\gamma^{-1} = 1$ and $x\gamma x^{-1} = (\gamma) = (\gamma)$.

Conversely, suppose (b) holds. By Lemma 5.2 and Proposition 4.8, there exist $\gamma \in G_1$ such that $x\gamma x^{-1} = \gamma$. Let $z \in C_r$ be such that $z^2 = (\gamma)$. We claim that (G, H) is a Higman pair with key $b := (x, z)$. Lemma 5.2 supplies (H1), while (H2) holds by the first isomorphism theorem $G_1/H = C_r$. Calculations similar to those in (9) and (10) verify (H3) and (H4).

To prove (H5), suppose $(\gamma, w) \in G_1$ satisfies $(\gamma, w)(x, z) = H(x, z)H$. Then there exist $\gamma', \gamma'' \in G_1$ such that

$$(\gamma', w) = \gamma', (\gamma')^{-1} (x, z) \gamma'', (\gamma'')^{-1}.$$

That is, $x = x$ and $w = (\gamma'')^{-1}$. We can rewrite the former relation as $x\gamma' x^{-1} = (\gamma')^{-1} \in G_1$. Then (b) gives $(\gamma') = (\gamma')^{-1}$. Therefore $(\gamma')^{-1} = (\gamma')^{-1} = w$, and $(\gamma', w) \in H$.

Lemma 5.5 (Roux from radicalization). *Suppose the radicalization (\tilde{G}, H) of (G, G_1, \sim) is a Higman pair with key (x, z) . Then the roux $B \in \mathbb{C}[C_r]^{n \times n}$ constructed in Proposition 3.11 can be found as follows: Choose left coset representatives $\{x_j\}_{j \in [n]}$ for G_1 in G . Given $i = j$, there exist $\gamma_i \in G_1$ for which $x_i^{-1}x_j = \gamma_i$. For any such choice of γ_i and γ_j , define $B_{ij} = (\gamma_i)z^{-1}$, and set $B_{ij} = 0$. Then B is a roux for C_r , and the roux scheme generated by B is isomorphic to the Schurian scheme of (\tilde{G}, H) . In particular, if γ_i is real valued and $z \in \{\pm 1\}$, then all lines arising from B (in the sense of Proposition 3.4) are real.*

Proof. We apply Proposition 3.11 with $b := (x, z)$. Lemma 5.2 gives that $K := N_{\tilde{G}}(H) = G_1 \times C_r$. We may therefore select $\{\tilde{x}_j\}_{j \in [n]} := \{(x_j, 1)\}_{j \in [n]}$ as left coset representatives of K in $\tilde{G} = G \times C_r$. Furthermore, we may select $\{a_w\}_{w \in C_r} := \{(1, w)\}_{w \in C_r}$ as coset representatives of H in K ; indeed, \sim maps K onto C_r with kernel H , and $\sim(1, w) = w$ for each $w \in C_r$. Following Proposition 3.11, we seek $w \in C_r$ for which $\tilde{x}_i^{-1}\tilde{x}_j = Ha_w bH$. Recalling the definition of H in (8), then for any choice of $w \in C_r$ and $\gamma_i \in G_1$, we have

$$(\gamma_i, (\gamma_i)^{-1})a_w b (\gamma_j, (\gamma_j)^{-1}) = (x, (\gamma_j)^{-1}wz).$$

This equals $\tilde{x}_i^{-1}\tilde{x}_j$ if and only if $x_i^{-1}x_j = x$ and $w = (\gamma_j)z^{-1}$. There exist $\gamma_j \in G_1$ for which this occurs by Proposition 3.10, and the desired result follows immediately from Proposition 3.11. For the “in particular” part of the result, assume γ_i is real valued and $z \in \{\pm 1\}$. Then every off-diagonal entry of B has order 2, and evaluating at any character results in a real signature matrix, which in turn determines real lines, by Proposition 2.1.

Lemma 5.6 (Roux parameters for radicalization). *Assume (G, H) is a Higman pair, and let (x, z) be any key. Then the roux for $C_r = G_1/H$ constructed in Proposition 3.11 has parameters $\{c_w\}_w$ given by*

$$c_w = \frac{n-1}{|G_1|} \cdot \quad G_1: \quad , \quad G_1 \text{ s.t. } x x^{-1} = x \text{ and } \quad^{-1} z^{-1} = w \quad .$$

Proof. It follows from (3). Specifically, we choose $\{a_w\}_w := \{(1, w)\}_w$ as a transversal for C_r in G_1 . Then

$$\begin{aligned} Ha_w bH &= \quad , \quad ()^{-1} (1, w)(x, z) \quad , \quad ()^{-1} : \quad , \quad G_1 \\ &= \quad x \quad , \quad ()^{-1} w z : \quad , \quad G_1 \quad , \end{aligned}$$

while

$$bHb^{-1} = \quad x x^{-1} \quad , \quad ()^{-1} : \quad G_1 \quad .$$

Consequently,

$$\begin{aligned} |bHb^{-1} \quad Ha_w bH| &= \quad G_1 : \quad , \quad G_1 \text{ s.t.} \\ &\quad x x^{-1} = x \quad \text{and} \quad ()^{-1} w z = ()^{-1} \quad . \end{aligned}$$

The formula for c_w now follows from (3) since $|H| = |G_1|$.

Lemma 5.7. *If $\quad = 1$, then all roux lines coming from the Higman pair (G, H) span a space of dimension 1 or $n - 1$.*

Proof. We have $r = 1$ and $r = 2$. Notice that (G, H) is a Higman pair by Theorem 5.4. Given any $x \in G \cap G_1$, Lemma 5.3 implies that $b = (x, 1)$ is a key. There are just two roux parameters $\{c_1, c_{-1}\}$, and $c_{-1} = 0$ by Lemma 5.6. The sum of the roux parameters is therefore $c_1 = n - 2$, by Proposition 3.2. In particular, the Fourier transform of $\{c_w\}_w$ is given by $\hat{c} \quad n - 2, \quad C_2$. In the notation of Proposition 3.3, we have $\mu \quad \frac{n-2+\quad}{2n-2}$. That is, $\mu^+ \quad 1$ and $\mu^- \quad -1/(n - 1)$. Therefore, $d^+ \quad 1$ and $d^- \quad n - 1$. By Proposition 3.3, any sequence of n roux lines coming from the Higman pair (G, H) spans a space of dimension 1 or $n - 1$.

Lemma 5.8. *Assume \quad is real valued and (G, H) is a Higman pair. If $b = (x, z)$ is any key for (G, H) and $z \in \{\pm 1\}$, then all roux lines coming from (G, H) are real.*

Proof. We apply Proposition 3.6, where $\quad = C_r$. Since \quad is real valued, we either have $r = 1$ or $r = 2$. If $r = 1$, then $r = 2$, and the desired result is immediate from Proposition 3.6. Now assume $r = 2$, so that $r = 4$. Since $z \in \{\pm 1\}$, the roux parameters $\{c_w\}_w$ satisfy $c_i = c_{-i} = 0$ by Lemma 5.6. For any $\quad C_4$, $(1) = 1$ and $(-1) \in \{\pm 1\}$. In particular, both are real. By Proposition 3.6, the roux lines corresponding to any $\quad C_4$ are real.

Combining Lemmas 5.3 and 5.8, we immediately obtain the following.

Lemma 5.9. *Assume \quad is real-valued and (G, H) is a Higman pair. If there exists $x \in G \cap G_1$ with $x = x^{-1}$, then all roux lines coming from (G, H) are real.*

6. Partial classification of doubly transitive lines

Having developed the necessary machinery, we now put our tools to use in order to partially classify doubly transitive lines, as summarized in Theorem 1.5. In this section, we determine all sequences of linearly dependent doubly transitive lines whose automorphism groups are almost simple.

6.1. Overview of classification strategy. Doubly transitive permutation groups have been classified as a consequence of the classification of finite simple groups. The following list is taken from [29]. Most of the groups below are not literally permutation groups, but each has a well-known permutation representation whose image is a doubly transitive subgroup of S_n . Some of the groups have two different doubly transitive permutation representations, but when that happens they are exchanged by an outer automorphism of the source of the representation, and in particular, they produce the same range in S_n . In that sense, each item below corresponds to a unique permutation group up to permutation equivalence.

Proposition 6.1 ([29]). *If S is a doubly transitive permutation group of a set of size n , then one of the following cases occurs:*

- (I) S is of affine type.
- (II) S has a normal subgroup $G \in S \triangleleft \text{Aut}(G)$, where G and n are
 - (1) Alternating.
 - (i) A_n , $n > 5$.
 - (2) Lie type.
 - (ii) $\text{PSL}(m, q)$, $m > 2$, $n = (q^m - 1)/(q - 1)$, $(m, q) = (2, 2), (2, 3)$.
 - (iii) $\text{PSU}(3, q)$, $n = q^3 + 1$, $q > 2$.
 - (iv) $\text{Sz}(q)$, $n = q^2 + 1$, $q = 2^{2m+1} > 2$.
 - (v) ${}^2G_2(q)$, $n = q^3 + 1$, $q = 3^{2m+1}$.
 - (3) Other.
 - (vi) $\text{Sp}(2m, 2)$, $m > 3$, $n = 2^{2m-1} \pm 2^{m-1}$.
 - (vii) $\text{PSL}(2, 11)$, $n = 11$.
 - (viii) A_7 , $n = 15$.
 - (ix) M_n , $n \in \{11, 12, 22, 23, 24\}$.
 - (x) M_{11} , $n = 12$.
 - (xi) HS (the Higman–Sims group), $n = 176$.
 - (xii) Co_3 (the third Conway group), $n = 276$.

To prove our main result, we will consider each permutation group G above in turn, and find all instances of doubly transitive lines whose automorphism group contains G .

We can quickly cross some groups off our list with the help of the following lemma, which extends Theorem 11.6 in [33]. Recall that a permutation group $G \triangleleft S_n$ is **triplly transitive** if for any ordered triple of distinct indices (i, j, k) in $[n]$ and any other such ordered triple (i', j', k') there exists $g \in G$ such that $g(i) = i'$ for every $i \in \{i, j, k\}$.

Lemma 6.2. *Let L be a sequence of $n > d$ lines with $\text{span } C^d$. If $\text{Aut } L$ is triply transitive, then $d \in \{1, n - 1\}$, L is real, and $\text{Aut } L = S_n$.*

Proof. If $n = 2$, then $d = 1$ and the result is trivial. Hence we can assume $n > 3$. Then $\text{Aut } L$ is doubly transitive, and L is equiangular by Proposition 1.1. Choose unit-norm representatives $\{e_j\}_{j \in [n]}$ for $L = \{e_j\}_{j \in [n]}$, and put $\mu := |e_i \cdot e_j|$ for $i \neq j$. By choosing new representatives for e_2, \dots, e_n if necessary, we may assume that $e_1 \cdot e_j = \mu$ whenever $2 \leq j \leq n$. Given any pair of 3-subsets $\{i, j, k\}, \{i', j', k'\} \subseteq [n]$ there exists a unitary $U \in \text{U}(d)$ and constants $\alpha_i \in \mathbb{T}$ such that $U e_i = \alpha_i e_{i'}$ for $i \in \{i, j, k\}$. Consequently,

$$(11) \quad e_i \cdot e_{j'} = \alpha_j \mu, \quad e_j \cdot e_{k'} = \alpha_k \mu, \quad e_i \cdot e_{k'} = \alpha_k \mu, \quad e_{i'} \cdot e_{j'} = \alpha_j \mu, \quad e_{j'} \cdot e_{k'} = \alpha_k \mu, \quad e_{i'} \cdot e_{k'} = \alpha_k \mu.$$

Taking $k = 1$ shows that $e_i \cdot e_{j'} = C/\mu^2$ whenever $2 \leq i = j \leq n$, and $C = \overline{C}$ since we can switch i and j . Consequently, L is real. In the notation of Proposition 2.6, (11) implies that we either have $T_L = ?$ or $T_L = \{\text{all 3-subsets of } [n]\}$. In either case, the two-graph $([n], T_L)$ has automorphism group S_n . By Proposition 2.6, $\text{Aut } L = S_n$,

and some choice of unit-norm representatives for L has signature matrix $J - I$ or $I - J$. In the former case $d = 1$, and in the latter case $d = n - 1$.

For the remaining groups on our list, we apply the following broad strategy. Let $G \leq S_n$ be a doubly transitive group from Proposition 6.1. Our first step is to obtain a Schur covering $\pi : \tilde{G} \rightarrow G$. (This is the most difficult part in practice. Thankfully, the multipliers of all the finite simple groups were computed as part of their classification, and a covering group can be found.) Let $G_1 \leq \tilde{G}$ be the stabilizer of a point, and put $\tilde{G}_1 := \pi^{-1}(G_1)$. If there exists a sequence L of $n > d$ lines spanning \mathbb{C}^d with $G \leq \text{Aut } L$, then Theorem 4.7 explains how to recover L from the radicalization (G, H) of (G, G_1, π) , for some choice of linear character $\chi : G_1 \rightarrow \mathbb{T}$. In particular, (G, H) must be a Higman pair. Our plan is to iterate through all linear characters $\chi : G_1 \rightarrow \mathbb{T}$, find those for which (G, H) is a Higman pair, and describe the resulting lines.

In particular, for any choice of linear character $\chi : G_1 \rightarrow \mathbb{T}$, Theorem 5.4 provides a simple test to determine whether or not the radicalization (G, H) of (G, G_1, π) is a Higman pair. When this test is affirmative, a sequence L of doubly transitive lines satisfying $G \leq \text{Aut } L$ exists by Theorem 4.1. For every such L , it then remains to describe L , determine whether or not L is real, and find the dimension of its span.

In many cases, Lemma 5.9 provides a fast judgment that all lines coming from (G, H) are real. In that case, our job is simple. All real doubly transitive lines derive from Taylor's classification of doubly transitive two-graphs, hence they appear in Proposition 2.7.

For the remaining cases, we dig deeper. First, we apply Lemma 5.3 to find a key for (G, H) . Then, we find the resulting roux with Lemma 5.5. Next, we obtain roux parameters using Lemma 5.6, and compute ranks of the primitive idempotents in the roux scheme with Proposition 3.3. Finally, with the roux parameters in hand, Propositions 3.5 and 3.6 team up to give a final determination of which primitive idempotents correspond to real lines, and which do not.

6.2. Linear groups. We first consider $\text{PSL}(m, q)$ and its doubly transitive action on the 1-dimensional subspaces of \mathbb{F}_q^m .

Example 6.3. Let q be an odd prime power, take $X := \mathbb{F}_q \setminus \{0\}$, and consider the vectors $\{t_i\}_{i \in X}$ in \mathbb{F}_q^2 defined by $t_a := [a, 1]$ for $a \in \mathbb{F}_q$ and $t_0 := [1, 0]$. We will construct doubly transitive lines in $\mathbb{C}^{(q+1)/2}$ that are indexed by X and have $\text{PSL}(2, q)$ in their automorphism group. The signature matrix can be described by composing a multiplicative character with a symplectic form. Let $Q \leq \mathbb{F}_q^\times$ denote the multiplicative subgroup of index 2, let $\chi : \mathbb{F}_q \rightarrow \mathbb{R}$ denote the quadratic character defined by

$$\chi(a) := \begin{cases} 1 & \text{if } a \in Q, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a \in Q \setminus \{0\}, \end{cases}$$

and consider the symplectic form $[\cdot, \cdot] : \mathbb{F}_q^2 \times \mathbb{F}_q^2 \rightarrow \mathbb{F}_q$ defined by

$$[a, b], [c, d] := ad - bc = \det \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Finally, we define $A \in \mathbb{R}^{X \times X}$ by $A_{ij} = ([t_i, t_j])$, and put $S := zA \in \mathbb{C}^{X \times X}$, where

$$z := \begin{cases} 1 & \text{if } q \equiv 1 \pmod{4}, \\ i & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

Then S is the signature matrix of an ETF of $q + 1$ vectors in $\mathbb{C}^{(q+1)/2}$; see [42] or [57, p. 67], for example.

To see that $\text{PSL}(2, q)$ is in the automorphism group of these lines, first observe that each member of $\text{PSL}(2, q)$ permutes the lines in \mathbb{F}_q^2 spanned by $\{t_i\}_i$, which in turn determines a permutation of X . Select any such permutation σ , and let C denote a corresponding member of $\text{SL}(2, q)$. Then there exist scalars $\{c_i\}_i$ in \mathbb{F}_q^\times such that $t_{\sigma(i)} = c_i C t_i$. Since $[\cdot, \cdot]$ is invariant under the action of $\text{SL}(2, q)$, it follows that

$$A_{(\sigma(i), \sigma(j))} = ([t_{\sigma(i)}, t_{\sigma(j)}]) = ([c_i C t_i, c_j C t_j]) = (c_i c_j [C t_i, C t_j]) = (c_i c_j) A_{ij} (c_i c_j).$$

Letting $P \in \mathbb{R}^{X \times X}$ denote the matrix representation of σ and $D \in \mathbb{R}^{X \times X}$ denote the diagonal matrix whose i th diagonal entry equals $(c_i c_j)$, this implies $PAP^{-1} = DAD$. Multiplying both sides by z then gives that S is switching equivalent to PSP^{-1} . Now Proposition 2.2 shows that $\text{Aut } L = \text{Aut } L$. (See Theorem 7.1 for the full automorphism group of L .)

Theorem 6.4. *Let L be a sequence of $n > 2d > 2$ lines with span \mathbb{C}^d and doubly transitive automorphism group containing $\text{PSL}(m, q)$ as in Proposition 6.1(ii). Then $m = 2$, $d = (q + 1)/2$, $n = q + 1$, and L is unitarily equivalent to the lines of Example 6.3, where either*

- (a) L is real and $q \equiv 1 \pmod{4}$, or
- (b) L is not real and $3 < q \equiv 3 \pmod{4}$.

In general, the Schur cover of $\text{PSL}(m, q)$ is $\text{SL}(m, q)$, with a few exceptions, which we address with the following lemmas.

Lemma 6.5 (Computer-assisted result). *Let L be a sequence of $n = 10 > 2d > 2$ lines with span \mathbb{C}^d and $\text{Aut } L > \text{PSL}(2, 9)$ as in Proposition 6.1(ii). Then L is real and unitarily equivalent to the lines of Example 6.3 with $q = 9$ and $d = 5$.*

Computer-assisted proof. We used GAP [17] to find a Schur covering $\rho : G \rightarrow G$ of $G = \text{PSL}(2, 9)$. Denote $G_1 \leq G$ for the stabilizer of a point, and $G_1 = \rho^{-1}(G_1)$. For each nontrivial linear character $\chi : G_1 \rightarrow \mathbb{T}$ we applied Theorem 5.4 to determine if the radicalization of (G, G_1, χ) was a Higman pair (the trivial character being handled by Lemma 5.7). Exactly one choice of χ led to a Higman pair in this way. For that choice of χ , we found a key using Lemma 5.3 and then computed the roux parameters using Proposition 3.11. An application of Proposition 3.6 showed that the resulting lines were real. Proposition 2.7 then implied that these lines were unique up to equivalence, from which it followed that they were constructed in Example 6.3. (Our code is available online [27].) By Theorem 4.7, this procedure captures every sequence L of $n = 10 > d$ lines spanning \mathbb{C}^d for which $\text{Aut } L$ contains $\text{PSL}(2, 9)$ in its doubly transitive action on 1-dimensional subspaces of \mathbb{F}_9^2 .

Lemma 6.6 (Computer-assisted result). *For each of*

$$(m, q) \in \{(2, 4), (3, 2), (4, 2), (3, 3), (3, 4)\},$$

there does not exist a sequence L of $n = \frac{q^m - 1}{q - 1} > 2d > 2$ doubly transitive lines with span \mathbb{C}^d and $\text{Aut } L > \text{PSL}(m, q)$ as in Proposition 6.1(ii).

Computer-assisted proof. The action of $\text{PSL}(2, 4) = A_5$ on $n = 5$ points is triply transitive, so this case was eliminated by Lemma 6.2. For every other choice of (m, q) above, we proceeded as in the proof of Lemma 6.5. In every such case, there were no nontrivial characters $\chi : G_1 \rightarrow \mathbb{T}$ for which the radicalization of (G, G_1, χ) was a Higman pair. (Our code is available online [27].)

While the following result may be known, we were not able to locate a reference. Hence, we supply our own proof.

Proposition 6.7. Consider the action of $SL(m, q)$ on the one-dimensional subspaces of F_q^m , where matrices act on column vectors from the left. Then the stabilizer of the line spanned by $[1, 0, \dots, 0] \in F_q^m$ is

$$G_1 = \left\{ \begin{pmatrix} (\det M)^{-1} & u \\ 0 & M \end{pmatrix} : u \in F_q^{m-1}, M \in GL(m-1, q) \right\}.$$

Furthermore, if $(m, q) \notin \{(2, 2), (2, 3), (3, 2)\}$ then the linear characters $\chi : G_1 \rightarrow F_q^\times$ are in one-to-one correspondence with the characters $\psi : F_q^\times \rightarrow F_q^\times$ through the relation

$$\chi \left(\begin{pmatrix} a & u \\ 0 & M \end{pmatrix} \right) = \psi(a).$$

Proof. It is easy to see that G_1 is the stabilizer of the line spanned by $[1, 0, \dots, 0] \in F_q^m$. For the other statement, consider the epimorphism $\pi : G_1 \rightarrow F_q^\times$ given by

$$\pi \left(\begin{pmatrix} a & u \\ 0 & M \end{pmatrix} \right) = a = (\det M)^{-1}, \text{ with}$$

$$\ker \pi = \left\{ \begin{pmatrix} 1 & u \\ 0 & M \end{pmatrix} : u \in F_q^{m-1}, M \in SL(m-1, q) \right\}.$$

Every $\psi \in F_q^\times$ produces a character $\chi := \psi \circ \pi$ of G_1 , and the mapping is injective since π is surjective. To complete the proof, it suffices to show that the commutator subgroup $[G_1, G_1]$ contains (hence equals) $\ker \pi$; in that case the kernel of every character $\chi : G_1 \rightarrow F_q^\times$ contains $[G_1, G_1] \supseteq \ker \pi$, and so χ factors through π .

We may assume that $m > 2$. To begin, take any $M, N \in SL(m-1, q)$ and observe that $\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}$ have commutator $\begin{pmatrix} 1 & 0 \\ 0 & MNMN^{-1} \end{pmatrix}$. Since $(m, q) \neq (3, 2)$, the group $SL(m-1, q)$ is perfect (cf. Ch. XIII of [35]). Consequently, $[G_1, G_1]$ contains every matrix $\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$ with $M \in SL(m-1, q)$.

Since $\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & I \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & M \end{pmatrix}$, it remains only to show that $[G_1, G_1]$ contains every matrix $\begin{pmatrix} 1 & u \\ 0 & I \end{pmatrix}$ with $u \in F_q^{m-1}$. To that end, we claim there exists $M \in GL(m-1, q)$ such that $\det M = a^{-1}$ is not an eigenvalue of M^{-1} . Assume this is true for the moment. Then $M^{-1} - a^{-1}I$ is invertible. Given $u \in F_q^{m-1}$, we can find $v \in F_q^{m-1}$ such that

$$u = av (M^{-1} - a^{-1}I) = av M^{-1} - v.$$

Then $\begin{pmatrix} a & 0 \\ 0 & M \end{pmatrix}$ and $\begin{pmatrix} 1 & v \\ 0 & I \end{pmatrix}$ have commutator

$$\begin{pmatrix} 1 & -v + av M^{-1} \\ 0 & I \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & I \end{pmatrix},$$

as desired.

To prove the claim, we find an invertible matrix $N = M^{-1}$ not having $(\det N)^{-1} = \det M$ as an eigenvalue. If $m-1 = 1$, then $q > 3$ by assumption, and so we can choose $N = b \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \in \{ \pm 1 \}$. If $m-1 = 2$, we take $N = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$. Finally, if $m-1 > 3$, we take N to have entries $N_{ij} = 1$ if $j > i$ or $(i, j) = (m-1, 1)$, and $N_{ij} = 0$ otherwise. In other words, N has all 1's on and above the diagonal, and all 0's below

the diagonal, except for a 1 in the bottom-left corner. For example, $N = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ when

$m - 1 = 3$. A cofactor expansion along the bottom row shows that $\det N = 1$ (when the first column and bottom row are deleted, the remaining matrix has a repeated column). Meanwhile, the columns of $N - I$ are linearly independent, and so 1 is not an eigenvalue of N .

Lemma 6.8. *For each $m > 2$ and prime power q , there does not exist a sequence L of $n = \frac{q^m - 1}{q - 1} > 2d > 2$ doubly transitive lines with span \mathbb{C}^d and $\text{Aut } L \cong \text{PSL}(m, q)$ as in Proposition 6.1(ii).*

Proof. We may assume that (m, q) is not among the cases handled by Lemma 6.6. Then the quotient map $\pi : \text{SL}(m, q) \rightarrow \text{PSL}(m, q)$ is a Schur covering, by Theorem 7.1.1 of [31]. As in Proposition 6.7, the pre-image of a point stabilizer $G_1 \in \text{PSL}(m, q)$ is

$$G_1 := \pi^{-1}(G_1) = \begin{pmatrix} (\det M)^{-1} u & \\ & M \end{pmatrix} : u \in \mathbb{F}_q^{m-1}, M \in \text{GL}(m-1, q).$$

Choose any nontrivial linear character $\chi : G_1 \rightarrow \mathbb{T}$. We will apply Theorem 5.4 to show that the radicalization of $(\text{SL}(m, q), G_1, \chi)$ is not a Higman pair.

By Proposition 6.7 there is a nontrivial character $\chi : \mathbb{F}_q^\times \rightarrow \mathbb{T}$ such that $\begin{pmatrix} a u & \\ & 0 M \end{pmatrix} =$

(a) for every $\begin{pmatrix} a u & \\ & 0 M \end{pmatrix} \in G_1$. Choose any $a \in \mathbb{F}_q^\times$ for which $\chi(a) = 1$, and any $N \in \text{GL}(m-2, q)$ such that $\det N = a^{-1}$. Put

$$M := \begin{pmatrix} a 0 & \\ & 0 N \end{pmatrix} \in \text{SL}(m-1, q), \quad \gamma := \begin{pmatrix} 1 & (1-a) & \\ & M & \\ & & 1 \end{pmatrix} \in G_1,$$

$$x := \begin{pmatrix} 1 & 0 & \\ & 1 & \\ & & I \end{pmatrix} \in \text{SL}(m, q) \cap G_1,$$

where $\gamma^{-1} := [1, 0, \dots, 0] \in \mathbb{F}_q^{m-1}$. Then a straightforward calculation shows that

$$x \gamma x^{-1} = \begin{pmatrix} 1 & 0 & & \\ & 1 & (1-a) & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & (1-a) & & \\ & M & & \\ & & -1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} a & (1-a) & & \\ & 0 & (1-a) & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in G_1,$$

and yet $\chi(x \gamma x^{-1}) = \chi(\gamma) = 1 = \chi(\gamma)$. By Theorem 5.4, the radicalization of $(\text{SL}(m, q), G_1, \chi)$ is not a Higman pair. The desired result now follows from Theorem 4.7 and Lemma 5.7.

Proof of Theorem 6.4. We may assume that $m = 2$ and $q \notin \{4, 9\}$, since all other cases were handled above. Then the quotient map $\pi : \text{SL}(2, q) \rightarrow \text{PSL}(2, q)$ is a Schur covering, by Theorem 7.1.1 of [31]. According to Proposition 6.7, the pre-image of a point stabilizer $G_1 \in \text{PSL}(2, q)$ is

$$G_1 := \pi^{-1}(G_1) = \begin{pmatrix} a & b \\ & 0 a^{-1} \end{pmatrix} : a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q,$$

and the linear characters $\chi : G_1 \rightarrow \mathbb{T}$ are in one-to-one correspondence with the characters $\chi : \mathbb{F}_q^\times \rightarrow \mathbb{T}$ through the relation $\begin{pmatrix} a & b \\ & 0 a^{-1} \end{pmatrix} = \chi(a)$.

Choose any $\chi : \mathbb{F}_q^\times \rightarrow \mathbb{T}$, and let $\chi : G_1 \rightarrow \mathbb{T}$ be the corresponding character of G_1 . To begin, we apply the Higman pair detector (Theorem 5.4) to show that the radicalization of $(\text{SL}(2, q), G_1, \chi)$ is a Higman pair if and only if χ is real valued. Take

Let $x := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{SL}(2, q) \cap G_1$. To identify which choices of $\chi := \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in G_1$ satisfy the condition $x \chi x^{-1} \in G_1$, we compute $x \chi x^{-1} = \begin{pmatrix} a^{-1} & 0 \\ -b & a \end{pmatrix}$. Thus, $x \chi x^{-1} \in G_1$ if and only if $b = 0$, and for such χ we have $(x \chi x^{-1}) = (a^{-1}) = \overline{(a)} = \overline{\chi}$. Comparing $(x \chi x^{-1})$ and (χ) , we find that condition (b) of Theorem 5.4 holds if and only if (or equivalently, χ) is real valued. The claim now follows from Theorem 5.4.

If q is even, the only real-valued character of F_q^\times is trivial. By Lemma 5.7 and Theorem 4.7, we conclude that no sequence of $n = q + 1 > 2d > 2$ lines spanning C^d has automorphism group containing $\text{PSL}(2, q)$ when q is even.

For the remainder of the proof, we assume that q is odd. Let $\chi \in F_q^\times$ be the unique nontrivial real-valued character. Explicitly, $\chi(a) = 1$ if a is a quadratic residue and -1 otherwise. Let $\tau : G_1 \rightarrow \mathbb{T}$ be the linear character corresponding to χ . We begin by finding a key for the radicalization of $(\text{SL}(2, q), G_1, \tau)$ via Lemma 5.3.

Let $x := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, as above. Then $\chi := I$ and $\tau := -I$ satisfy $x \chi = -\chi = x^{-1} \chi$ and

$$\chi(-1) = \begin{cases} 1 & \text{if } q \equiv 1 \pmod{4}, \\ -1 & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

Put $z := 1$ when $q \equiv 1 \pmod{4}$, and $z := i$ when $q \equiv 3 \pmod{4}$. Then (x, z) is a key for the radicalization of $(\text{SL}(2, q), G_1, \tau)$, by Lemma 5.3.

When $q \equiv 1 \pmod{4}$, the resulting lines are real by Lemma 5.5. By Proposition 2.7, these lines are unique, and so they are constructed in Example 6.3. For the case $q \equiv 3 \pmod{4}$, the resulting lines are not real by Proposition 2.7. We proceed by constructing the roux $B \in C[C_4]^{n \times n}$ using Lemma 5.5. Following the notation in Example 6.3, let $[\cdot, \cdot] : F_q^2 \times F_q^2 \rightarrow F_q$ be the symplectic form given by $[a, b], [c, d] = ad - bc$, and take $t_a := [a, 1]$ for each $a \in F_q$ and $t := [1, 0]$. We denote $X := F_q \setminus \{0\}$, so that $\{t_i\}_i \subset X$ is a full set of representatives for lines through the origin of F_q^2 . Here, $G_1 \subset G$ is the stabilizer of $\text{span}\{t\}$. For each $i \in X$, we select a representative x_i of the coset of G_1 that sends $\text{span}\{t\}$ to $\text{span}\{t_i\}$. To this end, it suffices to take $x_a := \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ for each $a \in F_q$, and $x := I$. Then $\{x_i\}_i \subset X$ is a full set of left coset representatives for G_1 in G . Choose any $i, j \in X$ with $i \neq j$. We apply Lemma 5.5 in cases to show $B_{ij} = z \cdot ([t_i, t_j])$. In the case where $i = a$, then $\chi := \begin{pmatrix} -1 & -j \\ 0 & -1 \end{pmatrix}$ and $\tau := I$ satisfy $x \chi = x_j = x_i^{-1} x_j$. Then $\chi(-1) = -1$, and so Lemma 5.5 gives $B_{ij} = \chi(-1) z^{-1} = z = z \cdot ([t_i, t_j])$. Next, if $j = a$, then $B_{ij} = B_{ji}^{-1} = z^{-1} = z \cdot ([t_i, t_j])$. Otherwise, we have $i, j \in F_q$, in which case $\chi := \begin{pmatrix} (j-i)^{-1} & -1 \\ 0 & j-i \end{pmatrix}$ and $\tau := \begin{pmatrix} 1 & (i-j)^{-1} \\ 0 & 1 \end{pmatrix}$ satisfy $x \chi = \begin{pmatrix} 1 & 0 \\ j-i & 1 \end{pmatrix} = x_i^{-1} x_j$. Then $\chi(-1) = -(i-j)$, and so Lemma 5.5 gives $B_{ij} = \chi(-1) z^{-1} = z \cdot (i-j) = z \cdot ([t_i, t_j])$. This proves the claim.

A signature matrix of our lines is given by $\hat{\chi}(B)$ for some character $\chi : C_4 \rightarrow \mathbb{T}$. Furthermore, χ is not real valued since the lines are not real. When χ is the identity character, $\hat{\chi}(B)$ is exactly the signature matrix S given in Example 6.3. Otherwise, χ is defined by $\chi(w) = \overline{w}$, in which case $\hat{\chi}(B) = \overline{S}$. In what follows, we show that \overline{S} is equivalent to S up to switching and permutation; then Proposition 2.2 implies the underlying line sets are equivalent, too. Let $\sigma : X \rightarrow X$ be the permutation given by $\sigma(a) = -a$ and $\sigma(a) = a$ for $a \in F_q$, define $\chi := -I$ and $\tau_a := I$ for $a \in F_q$, and

take $F := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then for any $i, j \in X$ with $i \neq j$, we have

$$\begin{aligned} \langle t_i, t_j \rangle &= \langle Ft_i, Ft_j \rangle \\ &= z \det(F) \langle [t_i, t_j] \rangle = \overline{\langle [t_i, t_j] \rangle} = \overline{S_{ij}}. \end{aligned}$$

This completes the proof.

6.3. Unitary groups. In this subsection, we fix a prime power $q > 2$, which may be even or odd. Throughout, matrices act on column vectors from the left. Take $\text{GU}(3, q) \leq \text{GL}(3, q^2)$ to stabilize the Hermitian form $(u, v) = u_1 v_3^q + u_2 v_2^q + u_3 v_1^q$ on $\mathbb{F}_{q^2}^3$, let $\text{SU}(3, q) \leq \text{GU}(3, q)$ be the subgroup of matrices with determinant 1, and let $\text{PSU}(3, q)$ be the quotient of $\text{SU}(3, q)$ by its scalar subgroup. We next consider the doubly transitive action of $\text{PSU}(3, q)$ on isotropic lines in $\mathbb{F}_{q^2}^3$, that is, on the lines spanned by $u \in \mathbb{F}_{q^2}^3 \setminus \{0\}$ with $(u, u) = 0$. There are $n = q^3 + 1$ such lines. Explicitly, take $X := T \setminus \{0\}$, where

$$T := \{[a, b] \in \mathbb{F}_{q^2}^2 : a^{q+1} + b + b^q = 0\},$$

and define the vectors $t_i := [1, 0, 0]$ and $t_{a,b} := [b, a, 1]$ for $[a, b] \in T$. Then $\{t_i\}_{i \in X}$ gives a full set of representatives for the $q^3 + 1$ isotropic lines in $\mathbb{F}_{q^2}^3$. The theorem below constructs doubly transitive lines in $\mathbb{C}^{q^2 - q + 1}$ that are indexed by X and have $\text{PSU}(3, q)$ in their automorphism group. These lines can be described in terms of a roux over the unique subgroup $\mathbb{T}_q \leq \mathbb{F}_{q^2}^\times$ of order $q + 1$.

Theorem 6.9. *With notation as above, define $B \in \mathbb{C}[\mathbb{T}_q]^{X \times X}$ by*

$$B_{ij} = \begin{cases} -(t_i, t_j)^{q-1} & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

Then the following hold:

- (a) B is a roux with parameters $c_1 = q - 1$ and $c_a = q^2 - 1$ for $a \in \mathbb{T}_q \setminus \{1\}$.
- (b) For each nontrivial character $\chi : \mathbb{T}_q \rightarrow \mathbb{C}$, it holds that $\widehat{\chi}(B) \in \mathbb{C}^{X \times X}$ is the signature matrix of a $d \times n$ ETF, where $d = q^2 - q + 1$ and $n = q^3 + 1$. Furthermore, the lines L spanned by these ETF vectors have $\text{Aut } L \cong \text{PSU}(3, q)$, and exactly one of the following holds:
 - (i) The lines are real and χ is real valued.
 - (ii) The lines are not real and χ takes non-real values.
- (c) For any dimension d , any sequence L of $n > 2d > 2$ lines in \mathbb{C}^d for which $\text{Aut } L \cong \text{PSU}(3, q)$ arises as in (b). In particular, $d = d$.

We prove Theorem 6.9 at the end of this subsection. Different choices of the character χ in Theorem 6.9(b) may or may not give equivalent lines, as detailed in Theorem 7.5 below.

Remark 6.10. As explained in [28], every roux B yields a **roux graph** with adjacency matrix B . In Theorem 6.9, the roux graph of B is a (cyclic) distance regular antipodal $(q + 1)$ -fold cover of $K_{q^3 + 1}$ with the property that vertices at distance two have exactly $q^2 - 1$ neighbors in common. This follows from Theorem 4.2 in [28]; in the terminology of that paper, the graph is a $(q^3 + 1, q + 1, q^2 - 1)$ -drackn. Furthermore, one can show that this graph admits an arc-transitive group of automorphisms isomorphic to $\text{SU}(3, q)$. By a recent theorem of Tsiovkina [51, Theorem 5.11], there is only one such graph up to isomorphism. It was constructed by Godsil [18] and rediscovered by Fickus, et. al, in the context of equiangular lines [15]. When q is odd, the construction of this graph in terms of Theorem 6.9 is new, as far as the authors

know. When q is even, Cameron gave an equivalent construction in Proposition 5.1 of [8]. For any choice of $m \in [q]$, the roux graph of B^m is also distance regular, and when q or m is even, our construction also amounts to that of Cameron [8]. (Here, \circ indicates a Hadamard power.) When q and m are both odd, the difference between our construction and Cameron's amounts to the leading minus sign in the expression $-(t_i, t_j)^{q-1}$.

The following is surely known. We include a proof for the sake of completeness.

Proposition 6.11. *The quotient mapping $\pi : \text{SU}(3, q) \rightarrow \text{PSU}(3, q)$ is a Schur covering.*

Proof. The Schur multiplier of $\text{PSU}(3, q) = \text{SU}(3, q)/Z(\text{SU}(3, q))$ is isomorphic to $Z(\text{SU}(3, q))$, by Theorem 3 in [20]. Since $q > 2$, $\text{SU}(3, q)$ is perfect (cf. [22, Theorem 11.22]), and in particular $Z(\text{SU}(3, q)) \trianglelefteq [\text{SU}(3, q), \text{SU}(3, q)]$.

Our proof of Theorem 6.9 requires detailed knowledge about $G_1 := \pi^{-1}(G_1)$, where $G_1 \trianglelefteq \text{PSU}(3, q)$ is the stabilizer of $\text{span}\{t\}$. In particular, we must understand all of its linear characters, as well as their interactions with the double coset structure of G_1 in $G := \text{SU}(3, q)$. This information is provided below.

Proposition 6.12. *Consider the action of $\text{SU}(3, q)$ on isotropic lines in $\mathbb{F}_{q^2}^3$. The stabilizer of the line spanned by $t \in \mathbb{F}_{q^2}^3$ is $G_1 = K \cap N$, where*

$$K := \{ e : e \in \mathbb{F}_{q^2}^\times \}, \quad N := \{ a, b : a, b \in \mathbb{F}_{q^2}, a^{q+1} + b + b^q = 0 \},$$

with e and a, b given by

$$e := \begin{pmatrix} e & 0 & 0 \\ 0 & e^{q-1} & 0 \\ 0 & 0 & e^{-q} \end{pmatrix}, \quad a, b := \begin{pmatrix} 1 & a & b \\ 0 & 1 & -a^q \\ 0 & 0 & 1 \end{pmatrix} \quad (e \in \mathbb{F}_{q^2}^\times, a^{q+1} + b + b^q = 0).$$

Furthermore, the linear characters $\chi : G_1 \rightarrow \mathbb{T}$ are in one-to-one correspondence with the characters $\chi : \mathbb{F}_{q^2}^\times$ through the relation $(\chi|_{e, a, b}) = \chi(e)$.

We were not able to locate a reference for Proposition 6.12, so we supply our own proof.

Proof. Notice that $Z(\text{SU}(3, q)) = \{ e : e \in \mathbb{F}_{q^2}, e^{q+1} = e^3 = 1 \}$. O'Nan [40] establishes that $G_1 \trianglelefteq \text{SU}(3, q)$ is the stabilizer of the line spanned by t , and that $G_1 = K \cap N$. The latter are subgroups, and the group operations for K , N , and G_1 follow the basic identities

$$(12) \quad \begin{aligned} e h &= eh, & a, b f, g &= a + f, b + g - af^q, \\ \frac{-1}{a, b} &= -a, b^q, & e a, b e^{-1} &= e^{2-q} a, e^{q+1} b. \end{aligned}$$

Identifying $G_1/N = K$ with $\mathbb{F}_{q^2}^\times$, we obtain the epimorphism $\pi : G_1 \rightarrow \mathbb{F}_{q^2}^\times$, $(\chi|_{e, a, b}) = \chi(e)$. Consequently, every $\chi : \mathbb{F}_{q^2}^\times \rightarrow \mathbb{T}$ defines a character $\chi : G_1 \rightarrow \mathbb{T}$ of G_1 , and the mapping π is injective. It remains to show that every linear character of G_1 factors through π , or equivalently, that $[G_1, G_1]$ contains $\ker \pi = N$.

To begin, we consider the \mathbb{F}_q -linear subspace $V \trianglelefteq \mathbb{F}_{q^2}$ of all b for which $b + b^q = 0$. The \mathbb{F}_q -linear transformation $\pi : \mathbb{F}_{q^2} \rightarrow \mathbb{F}_{q^2}$ given by $\pi(a) = a - a^q$ maps into V , with one-dimensional kernel $\mathbb{F}_q \trianglelefteq \mathbb{F}_{q^2}$. Consequently, π maps \mathbb{F}_{q^2} onto V . Similarly, the mapping $a \mapsto a + a^q$ is a surjection of \mathbb{F}_{q^2} onto \mathbb{F}_q .

Given $b \in V$, choose $a \in \mathbb{F}_{q^2}$ such that $a - a^q = b$. Then find $f \in \mathbb{F}_{q^2}$ such that $f + f^q = -a^{q+1}$, and $g \in \mathbb{F}_{q^2}$ satisfying $g + g^q = -1$. A simple computation shows that $[\pi(a, f, -1, g)] = \pi(0, a - a^q) = \pi(0, b)$. Therefore, $[G_1, G_1]$ contains the group $\{ \pi(0, b) : b \in V \}$.

Finally, let $\varphi_{a,b} \in \mathcal{N}$ be arbitrary. Notice that $a^{2-q} - 1 = 0$ since $q^2 - 1 > 2q - 1$ by the assumption $q > 2$. Put $f := a(a^{2-q} - 1)^{-1}$, and find $g \in \mathbb{F}_{q^2}^\times$ such that $g + g^q = -f^{q+1}$. Then one computes $[\varphi_{f,g}, \varphi_{a,h}] = \varphi_{f(a^{2-q}-1),h} = \varphi_{a,h}$ for some appropriate choice of $h \in \mathbb{F}_{q^2}^\times$, which necessarily satisfies

$$a^{q+1} + h + h^q = 0 = a^{q+1} + b + b^q.$$

Hence $b - h \in V$, and $[G_1, G_1]$ contains $\varphi_{a,h,0,b-h} = \varphi_{a,b}$.

Lemma 6.13. *Adopt the notation of Proposition 6.12, and put $x := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Let*

G_1 be arbitrary. If $\varphi \in K$, then $x x^{-1} \varphi \in G_1$. If $\varphi \notin K$, then $x x^{-1} \varphi \in G_1 x G_1$. Furthermore, suppose $\mathbb{F}_{q^2}^\times$ satisfies $\text{im } \varphi \subset C_{q+1}$, and let φ be the corresponding linear character of G_1 . If $\varphi = \varphi_{e,a,b} \notin K$, then every decomposition $x x^{-1} = x$ with $\varphi \in G_1$ satisfies $(\varphi^{-1}) = (b)$.

Proof. First, we compute

$$(13) \quad x e x^{-1} = e^{-q}, \quad x \varphi_{a,b} x^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ a^q & 1 & 0 \\ b & -a & 1 \end{pmatrix} \quad (e = 0, a^{q+1} + b + b^q = 0).$$

Therefore $x K x^{-1} = K$, and $x(\varphi_{e,a,b})x^{-1} = (x e x^{-1})(x \varphi_{a,b} x^{-1}) \in G_1$ if and only if $a = b = 0$. Since $G := \text{SU}(3, q)$ acts doubly transitively on G / G_1 , there are just two double cosets of G_1 in G . When $\varphi \notin K$ we have $x x^{-1} \varphi \notin G_1$, and so $x x^{-1} \varphi \in G_1 x G_1$.

Now let $\varphi_{k,l,m} \in \mathbb{F}_{q^2}^\times$ be such that $\text{im } \varphi \subset C_{q+1}$. Suppose $\varphi = \varphi_{e,a,b} \notin K$, and $x x^{-1} = (\varphi_{h,f,g})x(\varphi_{k,l,m})$ for some $h, f, g, k, l, m \in G_1$. Using (12) and (13), we rewrite

$$\varphi_{h^{-q}k^{-1}} = x^{-1} \varphi_{k^{2q-1}f, k^{q+1}g} x \varphi_{l,m}.$$

The left side of this equation is

$$\varphi_{h^q k^{-1}} = \varphi_{h^q k^{-1} e, a, b} = \begin{pmatrix} h^q k^{-1} e & h^q k^{-1} a & h^q k^{-1} eb \\ 0 & h^{1-q} k^{1-q} e^{q-1} & -h^{1-q} k^{1-q} e^{q-1} a^q \\ 0 & 0 & h^{-1} k^q e^{-q} \end{pmatrix},$$

while the right side takes the form

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \dots,$$

where \dots marks (possibly different) entries we have not computed. Comparing the top-right entries, we see that $b = h^{-q} k e^{-1}$. We have $(\varphi^{-1}) = (h)$ since $\text{im } \varphi \subset C_{q+1}$, and therefore

$$(\varphi_{h,f,g} \varphi_{k,l,m}^{-1}) = (h k e^{-1}) = (h^{-q} k e^{-1}) = (b).$$

Proof of Theorem 6.9. To begin, we find all characters $\varphi : G_1 \rightarrow \mathbb{T}$ for which the radicalization of (G, G_1, φ) is a Higman pair. Fix a linear character $\varphi \in \mathbb{F}_{q^2}^\times$, and

let ψ be the corresponding character of G_1 . Put $x := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Given $\varphi \in G_1$, we have $x x^{-1} \varphi \in G_1$ if and only if $\varphi = \varphi_e$ for $e = 0$, in which case $x e x^{-1} = e^{-q}$. By Theorem 5.4, the radicalization (G, H) of (G, G_1, φ) is a Higman pair if and only if $(\varphi_e) = (\varphi_e)^{-q}$ for every $e = 0$, if and only if $\text{im } \varphi \subset C_{q+1}$.

Now fix a choice of χ for which $\text{im } \chi \in C_{q+1}$, and put $r := |\text{im } \chi|$, $r := 2r$. Then the radicalization of (G, G_1, χ) is a Higman pair. To find a key for this Higman pair, first observe that $F_{q^2}^\times \in \ker \chi$. Consequently, there is a unique character $\psi: \mathbb{T}_q \rightarrow C_r$ for which $\psi(a) = (a^{q-1})$ for each $a \in F_{q^2}^\times$. Put $z := (-1)^{\chi(\pm 1)}$. Then since $x = x^{-1}$, Lemma 5.3 implies that (x, z) is a key for the radicalization of (G, G_1, χ) . With this key, Proposition 3.11 produces a roux $\tilde{B} \in C[C_r]^{X \times X}$.

We now construct the entries of \tilde{B} using Lemma 5.5. To begin, we fix a choice of left coset representatives for G_1 in G . Since G acts doubly transitively on isotropic lines in $F_{q^2}^3$, where G_1 stabilizes $\text{span}\{t\}$, it suffices to find $x_i \in G$ for which

$$x_i t = t_i \text{ for each } i \in X. \text{ We take } x := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and for } [a, b] \in T \text{ we define}$$

$$x_{a,b} := \begin{pmatrix} b - a^q & 1 & 0 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ where } x_{a,b}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -a \\ 1 & a^q & b^q \end{pmatrix}. \text{ Then } \{x_i\}_{i \in X} \text{ is a full set of left coset}$$

representatives for G_1 in G . For any choice of $i, j \in X$ with $i \neq j$, a straightforward calculation gives that the $(3, 1)$ entry of the matrix $x_i^{-1} x_j$ equals $(t_j, t_i) = (t_i, t_j)^q$. For any $[a, b] \in G_1$ such that $x = x_i^{-1} x_j$, one may examine the $(3, 1)$ entry to conclude that $\tilde{B}([a, b]) = ((t_i, t_j)^q) = \overline{((t_i, t_j))}$. By Lemma 5.5, it follows that $\tilde{B}_{ij} = z \overline{((t_i, t_j))} = -\overline{(t_i, t_j)^{q-1}}$. At this point, we observe that the entries of \tilde{B} reside in the subring $C[C_r] \in C[C_r]$, and so we may view \tilde{B} as a roux for either C_r or C_r . Furthermore, the lines given by \tilde{B} do not change when we view it as a roux over C_r instead of C_r , in light of Proposition 3.5 and the fact that every character of C_r extends to a character of C_r .

To prove (a) and (b), we focus on a special choice of χ . Namely, we fix a character $\psi: F_{q^2}^\times \rightarrow C_{q+1}$ for which $\psi: \mathbb{T}_q \rightarrow C_{q+1}$ is an isomorphism. Take $\chi = \psi$ above, and let $\tilde{B} \in C[C_{2(q+1)}]^{X \times X}$ be the resulting roux with entries $\tilde{B}_{ii} = 0$ and $\tilde{B}_{ij} = \overline{\psi(- (t_i, t_j))}$ for $i \neq j$ in X . For the moment, we view \tilde{B} as a roux over $C_{2(q+1)}$, and we compute its roux parameters $\{c_w\}_{w \in C_{2(q+1)}}$. For any $w \in C_{2(q+1)}$, Lemmas 5.6 and 6.13 combine to show that

$$(14) \quad c_w = \frac{q^3}{|G_1|} \cdot |\{e_{a,b} \in G_1 : b = 0 \text{ and } \psi(b) = zw\}| \\ = \frac{q^3(q^2 - 1)}{|K||N|} \cdot N_b = N_b, \\ \begin{matrix} b \in F_{q^2}^\times, \\ (b) = zw \end{matrix} \quad \begin{matrix} b \in F_{q^2}^\times, \\ (b) = zw \end{matrix}$$

where

$$N_b = |\{a \in F_{q^2} : a^{q+1} + b + b^q = 0\}| \quad (b \in F_{q^2}^\times).$$

Since $a \mapsto a^{q+1}$ is a surjective homomorphism of $F_{q^2}^\times$ onto F_q^\times , and since $-(b + b^q) \in F_q$ for every $b \in F_{q^2}$, we deduce that

$$N_b = \begin{cases} 1 & \text{if } b + b^q = 0 \\ q + 1 & \text{else} \end{cases} = \begin{cases} 1 & \text{if } b^{q-1} = -1 \\ q + 1 & \text{else} \end{cases} \quad (b \in F_{q^2}^\times).$$

Therefore,

$$c_w = \begin{cases} b \in F_{q^2}^\times : b^{q-1} = -1 \text{ and } \psi(b) = zw \\ + (q + 1) \cdot \{b \in F_{q^2}^\times : b^{q-1} = -1 \text{ and } \psi(b) = zw\} \end{cases}$$

Our choice of $z = \alpha(-1)$ ensures that $\alpha(b) = \alpha(b^{q-1}) = z$ whenever $b^{q-1} = -1$. There are exactly $q - 1$ choices of b for which $b^{q-1} = -1$, and so

$$c_w = (q - 1) \alpha_{w,1} + (q + 1) \cdot \sum_{b \in \mathbb{F}_{q^2}^\times : b^{q-1} = -1} \alpha(b) = zw.$$

For $w \in C_{q+1} = \text{im } \alpha$, we have $w = 1$, and there does not exist $b \in \mathbb{F}_{q^2}^\times$ for which $w = z \alpha(b)$ since $z \notin \text{im } \alpha$. Hence, $c_w = 0$ whenever $w \in C_{q+1}$. Given $w \in C_{q+1}$, we have $zw \in \text{im } \alpha$, and there are exactly $|\ker \alpha| = q - 1$ choices of $b \in \mathbb{F}_{q^2}^\times$ for which $\alpha(b) = zw$. For $w = 1$, any such b satisfies $b^{q-1} = -1$ by our choice of z , and so $c_1 = q - 1$. For $w \in C_{q+1} \setminus \{1\}$, any such b satisfies $b^{q-1} = -1$, and so $c_w = q^2 - 1$.

To prove (a), view \tilde{B} as a roux over C_{q+1} , and observe that its roux parameters $\{c_w\}_{w \in C_{q+1}}$ are as given above. Indeed, $c_w = 0$ when $w \in C_{2(q+1)} \setminus C_{q+1}$, and so $\tilde{B}^2 = (n - 1)I + \sum_{w \in C_{2(q+1)}} c_w w \tilde{B} = (n - 1)I + \sum_{w \in C_{q+1}} c_w w \tilde{B}$. Next, define $\tilde{\cdot} : C_{q+1} \rightarrow \mathbb{T}_q$ to be the inverse mapping of α , so that $\tilde{\alpha(a)} = a$ for every $a \in \mathbb{T}_q$. Then $\tilde{\cdot}$ extends to an isomorphism $\tilde{\cdot} : \mathbb{C}[C_{q+1}]^{\times \times \times} \rightarrow \mathbb{C}[\mathbb{T}_q]^{\times \times \times}$, and $B := \tilde{\cdot}(\tilde{B}) \in \mathbb{C}[\mathbb{T}_q]^{\times \times \times}$ is a roux with entries $B_{ii} = 0$ and $B_{ij} = \tilde{\cdot}(\tilde{B}_{ij}) = -(t_i, t_j)^{q-1}$ for every $i \neq j$. Considering the parameters for \tilde{B} we found above, then B has roux parameters $\{c_a\}_{a \in \mathbb{T}_q}$ with $c_1 = q - 1$ and $c_a = q^2 - 1$ for $a \in \mathbb{T}_q \setminus \{1\}$. This proves (a).

For (b), we apply Proposition 3.3 to compute the ranks $\{d\}_{\mathbb{T}_q \setminus \{1, -1\}}$ of the primitive idempotents in the roux scheme given by B . Given a character $\chi : \mathbb{T}_q \rightarrow \mathbb{T}$, we compute

$$\hat{c}_\chi = \sum_{a \in \mathbb{T}_q} c_a \overline{\chi(a)} = c_1 \overline{\chi(1)} + \sum_{a \in \mathbb{T}_q \setminus \{1\}} c_a \overline{\chi(a)} = (q - 1) + (q^2 - 1) \cdot \overline{\sum_{a \in \mathbb{T}_q \setminus \{1\}} \chi(a)}.$$

When $\chi = 1$ is the trivial character, this yields $\hat{c}_1 = (q - 1) + (q^2 - 1)q = q^3 - 1 = n - 2$. In Proposition 3.3, we obtain $\mu_1^+ = 1$ and $\mu_1^- = (1 - n)^{-1}$, so that $d_1^+ = 1$ and $d_1^- = n - 1$. Now assume χ is nontrivial. Then $\hat{c}_\chi = (q - 1) - (q^2 - 1) = q - q^2$. In Proposition 3.3, we obtain $\mu^+ = q^{-2}$ and $\mu^- = -q^{-1}$, so that $d^+ = q(q^2 - q + 1)$ and $d^- = q^2 - q + 1$. In particular, Proposition 3.5 implies that $\hat{\cdot}(B)$ is the signature matrix of a $d \times n$ ETF with $d = d^- = q^2 - q + 1$ and $n = |X| = q^3 + 1$. Furthermore, the lines L spanned by the vectors in this ETF are real if and only if $\hat{\cdot}$ takes real values, by Proposition 3.6. To see that $\text{Aut } L \cong \text{PSU}(3, q)$, let $\sigma : X \rightarrow X$ be a permutation of isotropic line indices that is implemented by a unitary $U \in \text{SU}(3, q)$. Then for each $i \in X$ there exists $a_i \in \mathbb{F}_{q^2}^\times$ such that $U a_i t_i = t_{\sigma(i)}$. Given $i, j \in X$ with $i \neq j$, it holds that

$$\begin{aligned} \hat{\cdot}(B)_{\sigma(i), \sigma(j)} &= -(t_{\sigma(i)}, t_{\sigma(j)})^{q-1} \\ &= -(U a_i t_i, U a_j t_j)^{q-1} \\ &= -a_i^{q-1} a_j^{q(q-1)} (t_i, t_j)^{q-1} = a_i^{q-1} \overline{a_j^{q-1}} \hat{\cdot}(B)_{ij}. \end{aligned}$$

Overall, the result of permuting the entries of $\hat{\cdot}(B)$ with σ is the same as that of applying a switching equivalence to $\hat{\cdot}(B)$. Then $\text{Aut } L \cong \text{PSU}(3, q)$, by Proposition 2.2. This proves (b).

It remains to prove (c). Let L be a sequence of $n > 2d > 2$ lines for C^d for which $\text{Aut } L \cong \text{PSU}(3, q)$. Then we can take X as an indexing set for L , and $n = |X| = q^3 + 1$. By Theorem 4.7, there is a character $\chi : G_1 \rightarrow \mathbb{T}$ for which the radicalization of (G, G_1, χ) is a Higman pair whose Schurian scheme has a primitive idempotent that represents L . By the above, $(e_{a,b}) = (e^{q-1})$ for some character $\chi : \mathbb{T}_q \rightarrow \mathbb{T}$. L can be obtained from a primitive idempotent in the roux scheme of $\tilde{B} \in \mathbb{C}[C_r]^{\times \times \times}$ with entries $\tilde{B}_{ii} = 0$ and $\tilde{B}_{ij} = \overline{-(t_i, t_j)^{q-1}}$ for $i \neq j$, where

$r := \lfloor \text{im } \chi \rfloor$. Considering Proposition 3.5, there exists a character $\chi : C_r \rightarrow \mathbb{T}$ and $\epsilon \in \{1, -1\}$ such that $\hat{\chi}(\tilde{B})$ is the signature matrix of an ETF whose vectors span L . To see this has the form of (b), take $\chi : \mathbb{T}_q \rightarrow \mathbb{T}$ to be the character given by $\chi(a) = \overline{\chi(\bar{a})}$ for every $a \in \mathbb{T}_q$. Then $\hat{\chi}(\tilde{B}) = \hat{\chi}(B)$. By Proposition 3.5, $d = d_{-1}$ is as computed above. Then the condition $n > 2d > 2$ ensures that χ is nontrivial and $\epsilon = -1$. This completes the proof.

6.4. Suzuki groups. We now consider the doubly transitive action of a Suzuki group $\text{Sz}(q)$, where $q = 2^{2m+1} > 2$ as in Proposition 6.1(iv). Our primary reference is §XI.3 in [23]. Fix the following notation for the duration of this subsection. Set $t = 2^{m+1}$. Given $a, b \in \mathbb{F}_q$ and $e \in \mathbb{F}_q^\times$, put

$$a, b := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & a^t & 1 & 0 \\ a^{2+t} + ab + b^t a^{t+1} + b a & 1 & 0 & 1 \end{pmatrix}, \quad e := \begin{pmatrix} e^{1+2^m} & 0 & 0 & 0 \\ 0 & e^{2^m} & 0 & 0 \\ 0 & 0 & e^{-2^m} & 0 \\ 0 & 0 & 0 & e^{-1-2^m} \end{pmatrix}.$$

Also define $x \in \text{GL}(4, q)$ to be the anti-diagonal matrix with entries $x_{ij} = \delta_{i,5-j}$. By definition,

$$\text{Sz}(q) := \langle a, b, e, x : a, b \in \mathbb{F}_q, e \in \mathbb{F}_q^\times \rangle \leq \text{GL}(4, q).$$

It has a doubly transitive permutation representation with point stabilizer

$$G_1 := \langle a, b, e : a, b \in \mathbb{F}_q, e \in \mathbb{F}_q^\times \rangle.$$

Theorem 6.14. *There does not exist a sequence L of $n = q^2 + 1 > 2d > 2$ doubly transitive lines with span \mathbb{C}^d and $\text{Aut } L \cong \text{Sz}(q)$ as in Proposition 6.1(iv).*

By [2], the Schur multiplier of $\text{Sz}(q)$ is trivial with a single exception, addressed below.

Lemma 6.15 (Computer-assisted result). *There does not exist a sequence L of $n > 2d > 2$ doubly transitive lines with span \mathbb{C}^d and $\text{Aut } L \cong \text{Sz}(8)$ as in Proposition 6.1(iv).*

Computer-assisted proof. The Schur multiplier of $G := \text{Sz}(8)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, by Theorem 2 in [2]. We used the AtlasRep [56] package in GAP [17] to obtain a Schur covering $\tilde{G} : \tilde{G} \rightarrow G$. Then we proceeded as in the proof of Lemma 6.5. No nontrivial character $\chi : G_1 \rightarrow \mathbb{T}$ produced a Higman pair through radicalization. Consequently, any sequence of lines whose automorphism group contains $\text{Sz}(8)$ spans a space of dimension $d \in \{1, n - 1\}$. (Our code is available online [27].)

We supply our own proof for the following proposition, which we could not locate elsewhere.

Proposition 6.16. *The point stabilizer $G_1 \leq \text{Sz}(q)$ is a semidirect product $G_1 = N \circ K$, where*

$$N := \{ a, b : a, b \in \mathbb{F}_q \}, \quad K := \{ e : e \in \mathbb{F}_q^\times \}.$$

Furthermore, the linear characters $\chi : G_1 \rightarrow \mathbb{T}$ are in one-to-one correspondence with the characters $\chi \in \mathbb{F}_q^\times$ through the relation $\chi(a, b, e) = \chi(e)$.

Proof. In [23, Lemma XI.3.1] it is shown that $G_1 = N \circ K$. Moreover, the group operation for G_1 satisfies the basic identities

$$\begin{aligned} a, b, f, g &= a + f, b + g + a^t f \\ e, h &= eh & (a, b, f, g \in \mathbb{F}_q, e, h \in \mathbb{F}_q^\times). \\ e^{-1} a, b, e &= ea, e^{t+1} b \end{aligned}$$

Consequently, there is a well-defined epimorphism $\pi : G_1 \rightarrow F_q^\times$ given by $\pi(a, b, e) = e$. Each $\chi \in F_q^\times$ determines a linear character $\chi : G_1 \rightarrow \mathbb{T}$, and the mapping $\chi \mapsto \ker \chi$ is injective. It remains to prove that $[G_1, G_1]$ contains $\ker \pi = N$, so that every linear character $\chi : G_1 \rightarrow \mathbb{T}$ factors through π .

Since $m > 0$ we have $2^{m+1} + 1 < 2^{2m+1} - 1 = q - 1$, and so there exists $e \in F_q^\times$ such that $e^{t+1} = 1$. Fix such an e . Given any $a, b \in F_q$, define

$$f := (e - 1)^{-1}a, \quad g := (e^{t+1} - 1)^{-1}[b - (1 - e^t)f^{t+1}].$$

Then

$$e^{-1} f, g e^{-1} = e f, e^{t+1} g - f, -g + f^{t+1} = (e-1)^{-1} f, (e^{t+1}-1)g + (1-e^t)f^{t+1} = a, b,$$

so that $(a, b) \in [G_1, G_1]$.

Proof of Theorem 6.14. We can assume $q > 8$. Then the Schur multiplier is trivial by [2, Theorem 1], and so $G = G$ is its own Schur cover. Let $\chi : G_1 \rightarrow \mathbb{T}$ be any nontrivial linear character of G_1 , and let $\chi \in F_q^\times$ be the corresponding character in Proposition 6.16. Choose any $e \in F_q^\times$ such that $\chi(e) = 1$. Since $F_q^\times = C_{q-1}$ and q is even, there exists $h \in F_q^\times$ with $h^2 = e$. Then $x = hx^{-1} \in G_1$, and yet $\chi(h) = \chi(h^{-1})$. By Theorem 5.4, the radicalization of (G, G_1, χ) is not a Higman pair. From Theorem 4.7 and Lemma 5.7 we conclude that any sequence of lines admitting $Sz(q)$ as a group of automorphisms spans a space of dimension $d \in \{1, n - 1\}$.

6.5. Ree groups. Next, we find all line sequences whose automorphism group contains the Ree group ${}^2G_2(q)$, where $q = 3^{2m+1}$. Our primary reference is [3, Section 3], which compiles information about Ree groups taken from [23, 32, 36, 52, 55]. We use the following notation in this subsection. Set $t = 3^m$. Given $a, b, c \in F_q$, put $(a, b, c) := (a, 0, 0, 0, b, 0, 0, 0, c)$, where

$$(a, 0, 0) := \begin{pmatrix} 1 & a^t & 0 & 0 & -a^{3t+1} & -a^{3t+2} & a^{4t+2} \\ 0 & 1 & a & a^{t+1} & -a^{2t+1} & 0 & -a^{3t+2} \\ 0 & 0 & 1 & a^t & -a^{2t} & 0 & a^{3t+1} \\ 0 & 0 & 0 & 1 & a^t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -a & a^{t+1} \\ 0 & 0 & 0 & 0 & 0 & 1 & -a^t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(0, b, 0) := \begin{pmatrix} 1 & 0 & -b^t & 0 & -b & 0 & -b^{t+1} \\ 0 & 1 & 0 & b^t & 0 & -b^{2t} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 1 & 0 & b^t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(0, 0, c) := \begin{pmatrix} 1 & 0 & 0 & -c^t & 0 & -c & -c^{2t} \\ 0 & 1 & 0 & 0 & -c^t & 0 & c \\ 0 & 0 & 1 & 0 & 0 & c^t & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -c^t \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $e \in F_q^\times$, put

$$e := \text{diag}(e^t, e^{1-t}, e^{2t-1}, 1, e^{1-2t}, e^{t-1}, e^{-t}).$$

Finally, define $x \in \text{GL}(7, q)$ to be the anti-diagonal matrix with entries $x_{ij} = -\delta_{i,8-j}$. By definition,

$${}^2G_2(q) := (a, b, c, e, x : a, b, c \in F_q, e \in F_q^\times) \leq \text{GL}(7, q).$$

It acts doubly transitively on $n := q^3 + 1$ points with point stabilizer

$$G_1 := (a, b, c, e : a, b, c \in F_q, e \in F_q^\times) = N \circ K,$$

where

$$N := \{_{a,b,c} : a, b, c \in \mathbb{F}_q\}, \quad K := \{e : e \in \mathbb{F}_q^\times\}.$$

Multiplication in G_1 follows the basic identities

$$\begin{aligned} _{a,b,c} f, g, h &= a + f, b + g - af^{3t}, c + h - fb + af^{3t+1} - a^2 f^{3t}, & _{a,b,c}^{-1} &= -a, -b - a^{3t+1}, -c - ab + a^{3t+2}, \\ _e^{-1} _{a,b,c} e &= e^{3t-2} a, e^{1-3t} b, e^{-1} c, & e k &= ek. \end{aligned}$$

Theorem 6.17. *Let L be a sequence of $n > d$ doubly transitive lines with span \mathbb{C}^d and $\text{Aut } L \cong {}^2G_2(q)$ as in Proposition 6.1(v). Then L is real, hence described by Proposition 2.7(iv).*

Our technique for proving Theorem 6.17 requires $q > 3$. We dispose of the case $q = 3$ by computer, as below.

Lemma 6.18 (Computer-assisted result). *Assume $q = 3$. Let L be a sequence of $n > d$ doubly transitive lines with span \mathbb{C}^d and $\text{Aut } L \cong {}^2G_2(3)$ as in Proposition 6.1(v). Then L is real.*

Computer-assisted proof. We proceeded as in the proof of Lemma 6.5, using GAP to compute a Schur cover G . For every linear character $\chi : G_1 \rightarrow \mathbb{T}$, we checked to see if the radicalization of (G, G_1, χ) was a Higman pair. If it was, we computed roux parameters. Then we applied the real roux lines detector of Propositions 3.5 and 3.6. In every case, we found that the resulting lines were real. (Our code is available online [27].)

Proposition 6.19. *Assume $q > 3$. Then the linear characters $\chi : G_1 \rightarrow \mathbb{F}_q$ of the point stabilizer $G_1 \triangleleft {}^2G_2(q)$ are in one-to-one correspondence with the characters $\chi \in \mathbb{F}_q^\times$ through the relation $(_{a,b,c} \chi) = (\chi)$.*

We provide a proof since we could not locate a reference.

Proof. The quotient map of G onto $G/N = K = \mathbb{F}_q^\times$ gives an epimorphism $\chi : G_1 \rightarrow \mathbb{F}_q^\times$, $(_{a,b,c} \chi) = \chi$. Therefore every $\chi \in \mathbb{F}_q^\times$ produces a linear character $\chi : G_1 \rightarrow \mathbb{T}$, and the mapping $\chi \mapsto (_{a,b,c} \chi)$ is injective. It suffices to prove that $[G_1, G_1]$ contains $\ker \chi = N$, so that every linear character $\chi : G_1 \rightarrow \mathbb{T}$ factors through χ .

Since $q = 3^{2m+1} > 3$, a simple counting argument shows there exists $e \in \mathbb{F}_q^\times$ such that $1 \notin \{e, e^{3t-2}, e^{3t-1}\}$. Fix such an e . Given $a, b, c \in \mathbb{F}_q$, put

$$\begin{aligned} f &:= (e^{3t-2} - 1)^{-1} a, & g &:= (e^{1-3t} - 1)^{-1} [b - (e^{3t-2} - 1) f^{3t+1}], \\ h &:= (e^{-1} - 1)^{-1} [c - (e^{1-3t} - 1) fg - (1 - e^{3t-2} + e^{6t-4}) f^{3t+2}]. \end{aligned}$$

Then a direct calculation shows that

$$\begin{aligned} _e^{-1} f, g, h e^{-1} f, g, h &= (e^{3t-2} - 1) f, (e^{1-3t} - 1) g + (e^{3t-2} - 1) f^{3t+1}, (e^{-1} - 1) h + (e^{1-3t} - 1) fg + (1 - e^{3t-2} + e^{6t-4}) f^{3t+2}, \end{aligned}$$

which equals $(_{a,b,c} \chi)$ by construction.

Proof of Theorem 6.17. We may assume that $q > 3$. The Schur multiplier of $G = {}^2G_2(q)$ is trivial, by [2, Theorem 1]. Hence $G = G$ is its own Schur cover. Choose any linear character $\chi : G_1 \rightarrow \mathbb{T}$, and let $\chi \in \mathbb{F}_q$ be the corresponding character from Proposition 6.19. Given any $e \in \mathbb{F}_q^\times$, we have $x \mapsto ex^{-1} = e^{-1}$ on G_1 . By Theorem 5.4, the radicalization (G, H) of (G, G_1, χ) is a Higman pair only if $(\chi) = (\chi)^{-1}$ for every $e \in \mathbb{F}_q$, only if χ is real valued. In that case, Lemma 5.9 shows that the resulting doubly transitive lines are real. We conclude from Theorem 4.7 that every sequence of linearly dependent lines L satisfying ${}^2G_2(q) \triangleleft \text{Aut } L$ is real.

6.6. **Symplectic groups.** Next we consider the doubly transitive actions of symplectic groups $\text{Sp}(2m, 2)$, summarized below. For details, see [14, Section 7.7], [22, Chapters 12–14], [34, Chapter 2]. Fix $m > 3$. We consider \mathbb{F}_2^{2m} to consist of column vectors $u = \begin{pmatrix} u_{(1)} \\ u_{(2)} \end{pmatrix}$ with $u_{(i)} \in \mathbb{F}_2^m$, $i \in \{1, 2\}$. Then $\text{Sp}(2m, 2) \leq \text{GL}(2m, 2)$ is the group of all invertible operators that stabilize the symplectic form $[\cdot, \cdot] : \mathbb{F}_2^{2m} \times \mathbb{F}_2^{2m} \rightarrow \mathbb{F}_2$ given by

$$[u, v] = u_{(1)}v_{(2)} + u_{(2)}v_{(1)}.$$

Put $e_1 = [1, 0, \dots, 0] \in \mathbb{F}_2^{2m}$. For each $\epsilon \in \{\pm 1\}$, we define a quadratic form $Q : \mathbb{F}_2^{2m} \rightarrow \mathbb{F}_2$ given by

$$Q^+(u) = u_{(1)}u_{(2)}, \quad Q^-(u) = u_{(1)}u_{(2)} + (u_{(1)} + u_{(2)}) \cdot e_1.$$

Thus $Q(u+v) + Q(u) + Q(v) = [u, v]$ for every $u, v \in \mathbb{F}_2^{2m}$. We take the orthogonal group $\text{O}(2m, 2) \leq \text{Sp}(2m, 2)$ to be the group of all linear operators stabilizing Q . Then $\text{Sp}(2m, 2)$ acts doubly transitively on $n = 2^{m-1}(2^m + \epsilon)$ points, and $\text{O}(2m, 2)$ is the stabilizer of a point.

Theorem 6.20. *Let L be a sequence of $n > d$ doubly transitive lines with span \mathbb{C}^d and $\text{Aut } L \cong \text{Sp}(2m, 2)$ as in Proposition 6.1(vi). Then L is real, hence described by Proposition 2.7(v,vi).*

The Schur multiplier of $\text{Sp}(2m, 2)$ is trivial for $m > 4$. We dispose of the case $m = 3$ by computer.

Lemma 6.21 (Computer-assisted result). *Assume $m = 3$. Let L be a sequence of $n > d$ doubly transitive lines with span \mathbb{C}^d and $\text{Aut } L \cong \text{Sp}(6, 2)$ as in Proposition 6.1(vi). Then L is real.*

Computer-assisted proof. We used GAP [17] to verify that the Schur multiplier of $G = \text{Sp}(6, 2)$ is \mathbb{Z}_2 . (Note that in some places it is falsely reported as being trivial.) Using the AtlasRep package [56], we obtained a double cover $\tilde{G} : G \rightarrow G$, which we checked was a Schur covering. Then we proceeded as in the proof of Lemma 6.5, computing the cases $\epsilon = +$ and $\epsilon = -$ separately. For each linear character $\chi : G_1 \rightarrow \mathbb{T}$, we checked to see if the radicalization (G, H) of (\tilde{G}, G_1, χ) was a Higman pair. When it was, we computed roux parameters. In every such case, the real roux lines detector of Propositions 3.5 and 3.6 verified that all lines coming from (G, H) were real. (Our code is available online [27].)

Proof of Theorem 6.20. We may assume that $q > 4$. Then the Schur multiplier of $G = \text{Sp}(2m, 2)$ is trivial [46], so that $\tilde{G} = G$ is its own Schur cover. Fix $\epsilon \in \{\pm 1\}$, and let $G_1 = \text{O}(2m, 2)$. By [22, Proposition 14.23], $[G_1, G_1] \leq G_1$ is a subgroup of index 2. Hence G_1 has only real-valued linear characters.

Put $w := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ when $\epsilon = +$, and $w := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ when $\epsilon = -$. Define the transvection $\tau : \mathbb{F}_2^{2m} \rightarrow \mathbb{F}_2^{2m}$ by $\tau(u) = u + [u, w]w$. Then $\tau \in \text{Sp}(2m, 2)$ satisfies $\tau^2 = 1$, but one easily checks that $\tau \notin G_1$. If $\chi : G_1 \rightarrow \{\pm 1\}$ is any linear character for which the radicalization (G, H) of (\tilde{G}, G_1, χ) is a Higman pair, then Lemma 5.9 implies that all lines coming from (G, H) are real. By Theorem 4.7, every sequence of linearly dependent lines L such that $\text{Sp}(2m, 2) \leq \text{Aut } L$ is real.

6.7. **Proof of the main result.** So far, we have applied our technology to all but four of the almost simple groups in Proposition 6.1. The remaining groups can be ticked off one by one. We found it convenient to use a computer for this task in the following lemma, but all the relevant computations could also be done by hand.

Lemma 6.22 (Computer-assisted result). *Suppose L is a sequence of $n > d$ lines spanning \mathbb{C}^d , and assume that $\text{Aut } L$ contains one of the groups from cases (vii), (viii), (xi), (xii) of Proposition 6.1. Then L is real, hence described by Proposition 2.7.*

Computer-assisted proof. For each group G with point stabilizer $G_1 \triangleleft G$, we found a Schur covering $\pi : \tilde{G} \twoheadrightarrow G$ and put $\tilde{G}_1 := \pi^{-1}(G_1)$. Then we used the Higman pair detector (Theorem 5.4) to find every linear character $\chi : \tilde{G}_1 \rightarrow \mathbb{T}$ for which the radicalization of $(\tilde{G}, \tilde{G}_1, \chi)$ was a Higman pair. In every case, we were able to apply Lemma 5.7 or Lemma 5.9 to deduce that the corresponding lines were real. It follows by Theorem 4.7 that L is real whenever $G \triangleleft \text{Aut } L$.

Specifically, for case (vii) the quotient mapping $\pi : \text{SL}(2, 11) \twoheadrightarrow \text{PSL}(2, 11)$ is a Schur covering by [31, Theorem 7.1.1]. Here $G_1 = \text{SL}(2, 5)$ has no nontrivial linear characters at all.

For (viii), GAP [17] provided a Schur covering of A_7 . No nontrivial linear character produced a Higman pair through radicalization.

In case (xi), the Schur multiplier of the Higman–Sims group G is Z_2 [38]. We obtained a double cover \tilde{G} of G through the AtlasRep package [56] of GAP, and verified that it was a Schur cover of G . Only real-valued characters of \tilde{G}_1 produced Higman pairs through radicalization, and there exists an involution $x \in \tilde{G} \setminus \tilde{G}_1$. By Lemma 5.9, these Higman pairs produce only real lines.

Finally, for (xii) the Schur multiplier of $G = \text{Co}_3$ is trivial [21], and so $\tilde{G} = G$ is its own Schur cover. Here, G_1 has only real-valued characters to begin with, and there is an involution $x \in G \setminus G_1$ because G is generated by involutions [37]. By Lemma 5.9, any Higman pair that arises from radicalization can only create real lines.

For all cases, our code is available online [27].

Proof of Theorem 1.5. Let L be a sequence of $n > 2d > 2$ lines that span \mathbb{C}^d , and assume that $S = \text{Aut } L$ is doubly transitive and almost simple. Then S contains a subgroup G such that $G \in S \triangleleft \text{Aut}(G)$ and G satisfies one of the cases (i)–(xii) of Proposition 6.1. In Subsections 6.2–6.6 and Lemma 6.22, we settled what happens in each instance except (i), (ix), and (x). In each of those cases, G is triply transitive: for (i), see [54, Theorem 9.7]; for (ix), see Theorems XII.1.3 and XII.1.4 in [23]; and for (x) consult [23, Theorem XII.1.9]. By Lemma 6.2, each remaining case violates the hypothesis $n > 2d > 2$.

Proof of Theorem 1.3. By Proposition 1.2, S is either a line (as in case (I)) or almost simple (as in case (II)). The line case was settled by Theorem 1.1 and Remark 1.2 of Dempwolff and Kantor [13], and the almost simple case is settled by Theorem 1.5.

7. Equivalence and automorphisms

In this section, we compute the automorphism groups of complex doubly transitive lines with almost simple symmetry groups, as in Theorem 1.3(D). As an application, we sort out equivalences in the unitary case (xiii). The remaining automorphism groups of doubly transitive lines are known. Type (B) appears in [13], while the two-graphs behind types (A) and (C) are treated in [50].

7.1. Linear symmetry. We begin with the lines of Example 6.3, as in Theorem 1.3(xii). Let $q \equiv 3 \pmod{4}$ be a prime power with prime divisor p . We write $\text{L}(2, q)$ for the group of all permutations $f : \mathbb{F}_q^2 \rightarrow \mathbb{F}_q^2$ of the form $f(u) = (Mu)^{p^k}$ for $M \in \text{GL}(2, q)$ and $k > 0$, while $\text{L}(2, q) \triangleleft \text{L}(2, q)$ is the subgroup of all those f with $M \in \text{SL}(2, q)$. (Here, p indicates a Hadamard power which implements a field automorphism to the entries of Mu .) Both $\text{L}(2, q)$ and $\text{L}(2, q)$ permute the one-dimensional subspaces of \mathbb{F}_q^2 , and in each case the kernel of the action

coincides with the center consisting of scalar functions. This produces embeddings of $P\ L(2, q) := L(2, q)/Z(L(2, q))$ and $P\ L(2, q) := L(2, q)/Z(L(2, q))$ in S_X , where $X = \{ \} F_q$ indexes the one-dimensional subspaces of F_q^2 as in Example 6.3.

Theorem 7.1. *Let $q \equiv 3 \pmod 4$ be a prime power, and let L be the sequence of lines constructed in Example 6.3. Then $\text{Aut } L = P\ L(2, q)$.*

Our proof uses the following detail.

Proposition 7.2. *Let $S \triangleleft S_n$ be primitive and almost simple with socle G . If $H \triangleleft S$ and $H = G$, then $H = G$.*

Proof. Suppose instead that $H = G$. Then $H \triangleleft (G \triangleleft H) = \{1\}$ since H is simple and normalizes G . Hence, $S > G, H = G \circ H$, and in particular, $|G|^2 \triangleleft |S|$. Meanwhile, $C_S(G) = \{1\}$ by Theorem 4.2A of [14], and so conjugation provides an embedding $S \triangleleft \text{Aut } G$. This is impossible since every finite simple group satisfies $|\text{Aut } G| \triangleleft \frac{1}{30}|G|^2$, as detailed in Lemma 2.2 of [41].

Proof of Theorem 7.1. We follow the notation of Example 6.3, and we also denote $S := \text{Aut } L > \text{PSL}(2, q)$. The doubly transitive group S must appear in Proposition 6.1. It also acts as automorphisms of $n = q + 1$ complex lines in dimension $d = n/2$. If S is a line, then Theorem 1.1 of [13] reports that (d, n) are listed somewhere in Theorem 1.3(I). The only parameters with $d = n/2$ are in case (ii) with $d = 2, n = 4$, and $q = 3$. These lines are unique, and they indeed have $S = A_4 = P\ L(2, 3)$. Now assume S has simple socle G . If $G = \text{PSU}(3, q_0)$ for some prime power $q_0 > 2$, then $q + 1 = n = q_0^3 + 1$, and Theorem 6.9(c) shows $d = q_0^2 - q_0 + 1$. This is impossible since $d = n/2$. We ruled out all other possibilities from Proposition 6.1(II), and so $G = \text{PSL}(2, q)$ and $q > 3$. By Proposition 7.2, $G = \text{PSL}(2, q) \triangleleft S_X$ (in its usual embedding). In particular,

$$S \triangleleft N_{S_X} \text{PSL}(2, q) = \text{Aut } \text{PSL}(2, q) = P\ L(2, q),$$

where the equalities follow from [14, Theorem 4.2A] and [55, §3.3.4].

Now it suffices to prove $P\ L(2, q) \triangleleft S = P\ L(2, q)$. Select $M \in \text{GL}(2, q)$ and $k > 0$, and let $P\ L(2, q) \triangleleft S_X$ be the corresponding permutation given by $\text{span}\{t_{(i)}\} = \text{span}\{(Mt_i)^{p^k}\}$ for $i \in X$. We use Proposition 2.2 to show $S \triangleleft S$ if and only if $P\ L(2, q)$. For each $i \in X$, select $t_i \in F_q^\times$ with $t_{(i)} = (Mt_i)^{p^k}$. Then for $i, j \in X$,

$$[t_{(i)}, t_{(j)}] = (i) (j) (\det M)^{p^k} [t_i, t_j]^{p^k} = (i) (j) (\det M) [t_i, t_j],$$

since the Frobenius automorphism preserves the set Q of quadratic residues. Thus,

$$(15) \quad S_{(i), (j)} = (i) (j) (\det M) S_{i,j} \quad \text{for every } i, j \in X.$$

When $M = cM$ for some $c \in F_q^\times$ and $M \in \text{SL}(2, q)$, we have $\det M = c^2$ and $(\det M) = 1$, and then Proposition 2.2 implies $S \triangleleft S$. That is, $P\ L(2, q) \triangleleft P\ L(2, q) \triangleleft S$.

To prove the reverse inclusion, assume for the sake of contradiction that $S \triangleleft S$ and yet $M = cM$ for any $c \in F_q^\times$ and $M \in \text{SL}(2, q)$. Then it is easy to show $(\det M) = -1$. By Proposition 2.2 and (15), there are unimodular constants $\{c_i\}_i \subset F_q^\times$ such that $c_i \bar{c}_j S_{i,j} = S_{(i), (j)} = - (i) (j) S_{i,j}$ for every $i, j \in X$. Equivalently,

$$(16) \quad c_i \bar{c}_j [t_i, t_j] = - (i) (j) [t_i, t_j] \quad \text{for every } i = j \text{ in } X.$$

Taking $i = a$ and $j = a \in F_q \setminus X$ in (16), we find $c \bar{c}_a = - () (a)$, so that

$$(17) \quad c_a (a) = -c ()$$

is constant for all $a \in \mathbb{F}_q$. Taking $i = 0$ and $j = -1$ in (16), we find $c_0 \overline{c_{-1}} = - (c_0) (c_{-1})$, so that $c_0 (c_0) = -c_{-1} (c_{-1})$. This contradicts (17).

7.2. Unitary symmetry. Now we handle the lines of Theorem 6.9, as in Theorem 1.3(xiii). Let $q > 2$ be a prime power with prime divisor p , and let notation be as in the first paragraph of Subsection 6.3. We write $\text{U}(3, q)$ for the group of all functions $f: \mathbb{F}_{q^2}^3 \rightarrow \mathbb{F}_{q^2}^3$ with $f(v) = (Uv)^{\rho^k}$ for $U \in \text{GU}(3, q)$ and $k > 0$. Then $\text{U}(3, q)$ permutes the isotropic lines in $\mathbb{F}_{q^2}^3$, and the kernel of this action consists of scalar functions, which form the center of $\text{U}(3, q)$. This provides an embedding of $\text{P U}(3, q) := \text{U}(3, q)/Z \text{U}(3, q)$ in S_X .

Lemma 7.3. *Let $\text{P U}(3, q) \hookrightarrow S_X$, where there is a unitary $U \in \text{GU}(3, q)$ and $k > 0$ such that $\text{span}\{t_{(i)}\} = \text{span}\{(Ut_i)^{\rho^k}\}$ for all $i \in X$. For any choice of nontrivial characters $\chi, \tau: \mathbb{T}_q \rightarrow \mathbb{T}$, it holds that $\chi = \rho^k \tau$ if and only if χ induces a unitary equivalence $L \cong L$ between the corresponding lines of Theorem 6.9(b).*

Proof. We follow the notation of Theorem 6.9, and we also write

$$S = - (t_i, t_j)^{q-1} \quad i, j \in X$$

for the signature matrix that produces L . Select constants $\{c_i\}_{i \in X}$ in $\mathbb{F}_{q^2}^\times$ such that $c_i t_{(i)} = (Ut_i)^{\rho^k}$ for each $i \in X$, and put $c_i := \rho^k (c_i^{q-1})$. For $i = j$ in X , we have $(t_{(i)}, t_{(j)}) = c_i^{\rho^k} c_j^{q\rho^k} (t_i, t_j)^{\rho^k}$, so that

$$(18) \quad [S]_{(i), (j)} = - (t_{(i)}, t_{(j)}) \\ = - c_i^{\rho^k} c_j^{q\rho^k} (t_i, t_j)^{\rho^k} = - (t_i, t_j)^{q-1} = c_i \overline{c_j} [S_{\rho^k}]_{i,j}.$$

By Proposition 2.2, χ induces a unitary equivalence $L \cong L_{\rho^k}$.

Conversely, suppose χ induces a unitary equivalence $L \cong L$. By Proposition 2.2 and (18), there are unimodular constants $\{c_i\}_{i \in X}$ such that $c_i \overline{c_j} [S]_{i,j} = [S]_{(i), (j)} = c_i \overline{c_j} [S_{\rho^k}]_{i,j}$ for every $i, j \in X$. In other words,

$$(19) \quad c_i \overline{c_j} - (t_i, t_j)^{q-1} = c_i \overline{c_j} \rho^k - (t_i, t_j)^{q-1} \quad \text{for every } i = j \text{ in } X.$$

Take $i = T$ and $j = -1$ in (19) to obtain $c_i \overline{c_{-1}} (-1) = c_i \overline{c_{-1}} (-1)$, so that

$$(20) \quad c_i \overline{c_i} = c \overline{c} (-1) (-1) =: C \in \mathbb{T}$$

is constant for $i \in T$. Select any $e \in \mathbb{T}_q$, and find $b \in \mathbb{F}_{q^2}^\times$ with $b^{q-1} = -e$. Then find $a \in \mathbb{F}_{q^2}$ with $a^{q+1} = -b - b^q$, so that $i := [a, b] \in T$. For $j := [0, 0] \in T$, we have $(t_i, t_j)^{q-1} = b^{q-1} = -e$. Then (19) produces $c_i \overline{c_j} (e) = c_i \overline{c_j} \rho^k (e)$, and $\rho^k (e) = c_i \overline{c_i} c_j (e) = C \overline{C} (e) = (e)$ by (20).

Theorem 7.4. *Let $\chi: \mathbb{T}_q \rightarrow \mathbb{T}$ be a nontrivial character that takes non-real values, and let L be the lines of Theorem 6.9(b). Then $\text{PSU}(3, q) \in \text{Aut } L \hookrightarrow \text{P U}(3, q)$. Specifically, $\text{Aut } L$ consists of all $\rho^k \chi$ for which there exist $U \in \text{GU}(3, q)$ and $k > 0$ such that $\text{span}\{t_{(i)}\} = \text{span}\{(Ut_i)^{\rho^k}\}$ for all $i \in X$ and $\rho^k \chi = \chi$.*

The case where χ is real is treated in [50]: $\text{Aut } L = \text{P U}(3, q)$ unless $q = 3$, in which case $\text{Aut } L = \text{Sp}(6, 2)$.

Proof of Theorem 7.4. The condition that χ takes non-real values ensures L are not real lines. Put $S := \text{Aut } L$. We first show S is almost simple with socle $\text{PSU}(3, q)$. As in the proof of Theorem 7.1, S is doubly transitive and must appear in Proposition 6.1. If S is a line, then (d, n) appear somewhere in Theorem 1.3(B). We have $d = q^2 - q + 1$

and $n = q^3 + 1$, which excludes (iii) and (iv). The only other possibility is (ii), with $\frac{n}{d} = \frac{2\sqrt{n}}{n-1}$ and n a square prime power. Meanwhile, the true parameters satisfy

$$\frac{n}{d} = \frac{q^3 + 1}{q^2 - q + 1} > \frac{q^3}{q^2 - q + q} = q = \sqrt[3]{n-1}.$$

For $n > 18$, we have $\frac{n}{d} > \sqrt[3]{n-1} > \frac{2\sqrt{n}}{n-1}$, and after checking all possibilities for $n \leq 18$ we find that case (ii) does not occur. Overall, S is not a 2 -design, and so it is almost simple. Theorem 6.4 shows its socle cannot be $\text{PSL}(2, q)$ for any prime power q , since $\frac{n}{d} > q > 2$. We have ruled out all other possibilities, and so the socle S is isomorphic to $\text{PSU}(3, q)$. By Proposition 7.2, the socle equals $\text{PSU}(3, q) \leq S_X$ (in its usual embedding). In particular,

$$(21) \quad S \leq N_{S_X} \text{PSU}(3, q) = \text{Aut PSU}(3, q) = \text{P}\Omega^{\epsilon}(3, q),$$

where the equalities follow from [14, Theorem 4.2A] and [55, §3.6.3]. The desired result now follows immediately from Lemma 7.3.

Theorem 7.5. *Let $\chi, \psi : \Gamma_q \rightarrow \mathbb{C}^*$ be nontrivial characters, both of which take non-real values. Then the lines L_χ and L_ψ constructed in Theorem 6.9(b) are unitarily equivalent if and only if $\chi = \psi^k$ for some $k > 0$.*

Proof. Given $k > 0$, let U be the identity operator on $\mathbb{F}_{q^2}^3$, and let $\rho = \text{P}\Omega^{\epsilon}(3, q) \leq S_X$ be given by $\text{span}\{t_{(i)}\} = \text{span}\{(Ut_i)^{\rho^k}\}$ for every $i \in X$. Then Lemma 7.3 establishes that ρ induces a unitary equivalence $L_\chi = L_{\chi \circ \rho^k}$. Conversely, suppose S_X induces a unitary equivalence $L_\chi = L_\psi$. Then $(\text{Aut } L_\chi)^{-1} = \text{Aut } L_\psi$, and in particular, ρ conjugates the socle of $\text{Aut } L_\chi$ to that of $\text{Aut } L_\psi$. Both automorphism groups have socle $\text{PSU}(3, q)$ by Theorem 7.4, so $N_{S_X} \text{PSU}(3, q) = \text{P}\Omega^{\epsilon}(3, q)$ as in (21). Now Lemma 7.3 shows that $\chi = \psi^k$ for some $k > 0$.

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