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Simplicial resolutions of powers of square-free monomial ideals

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Abstract The Taylor resolution is almost never minimal for powers of monomial ideals, even in the square-free case. In this paper we introduce a smaller resolution for each power of any square-free monomial ideal, which depends only on the number of generators of the ideal. More precisely, for every pair of fixed integers r and q, we construct a simplicial complex that supports a free resolution of the r^{th} power of *any* square-free monomial ideal with q generators. The resulting resolution is significantly smaller than the Taylor resolution, and is minimal for special cases. Considering the relations on the generators of a fixed ideal allows us to further shrink these resolutions. We also introduce a class of ideals called "extremal ideals", and show that the Betti numbers of powers of all square-free monomial ideals are bounded by Betti numbers of powers of extremal ideals. Our results lead to upper bounds on Betti numbers of powers of any square-free monomial ideal that greatly improve the binomial bounds offered by the Taylor resolution.

1. Introduction

Important insight about the underlying structure of an ideal in a polynomial ring is gained from a careful analysis of its minimal free resolution. As such, significant effort has gone into the development of methods to compute resolutions. The approach of leveraging connections between commutative algebra and other fields, such as combinatorics and topology, has proven to be quite fruitful. Diana Taylor's thesis [19] initiated the exploration of these connections, followed by simplicial resolutions (Bayer, Peeva, and Sturmfels [1]), polytopal complexes (Nagel and Reiner [15]), and cellular complexes (Bayer and Sturmfels [2]), to name just a few. See [14, 17] for an overview of these developments.

The Taylor resolution is powerful: given *any* ideal I minimally generated by q monomials, Taylor constructed a simplicial complex $\mathrm{Taylor}(I)$ by labeling the vertices of a $(q-1)$-simplex with the monomial generators of I. She showed that this

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complex **supports** a free resolution of I, in the sense that its simplicial chain complex can be transformed, via a process called *homogenization*, to a free resolution of I, called the **Taylor resolution** of I. Unfortunately, even though every monomial ideal has a Taylor resolution, the Taylor resolution is often far from minimal. In particular, for powers of ideals it is almost never minimal due to certain syzygies that are automatically created when taking powers.

The central theme of this paper is to find an analogue for the Taylor complex for powers of square-free monomial ideals. We seek a construction that takes the automatically generated non-minimal syzygies into account and removes them from the Taylor resolution to produce a much smaller free resolution of I^r that works for any monomial ideal I. Our ultimate goal is to find a uniform combinatorial structure that depends only on the number of generators and the power of the ideal. More precisely, the question at the heart of this paper is the following:

Question 1.1. *Given positive integers r and q, is it possible to find a simplicial complex (considerably) smaller than the simplex* $\mathrm{Taylor}(I^r)$ *that supports a free resolution of I^r, where I is any ideal generated by q monomials in a polynomial ring?*

When $r = 1$, $\mathrm{Taylor}(I)$ is in fact the optimal answer to the question above, as there are ideals I for which $\mathrm{Taylor}(I)$ supports a minimal resolution. But when $r = 2$, the resolution supported on $\mathrm{Taylor}(I^2)$ is never minimal for any non-principal square-free monomial ideal I ([3]). As expected, the resolution supported on $\mathrm{Taylor}(I^r)$ becomes further from minimal as r grows.

Although Taylor's complex for the r^{th} power of a monomial ideal with q generators can be quite large - a simplex of dimension $\binom{q+r-1}{r}$ - we can improve the situation considerably by studying the general relations among the generators of I^r that must always exist for all monomial ideals regardless of the generating set for I. Similar to the idea used in Lyubeznik's resolutions [12], in the case of square-free monomial ideals, this investigation involves detecting and trimming redundant faces of the Taylor complex $\mathrm{Taylor}(I^r)$, bringing us closer to a minimal resolution.

To illustrate our underlying process, let $r = 2, q = 3$ and consider *any* ideal $I = (m_1, m_2, m_3)$ in the polynomial ring $\mathsf{k}[x_1, \dots, x_n]$ where m_1, m_2 and m_3 are minimal, square-free monomial generators and k is a field. Now $I^2 = ({m_1}^2, {m_2}^2, {m_3}^2, m_1m_2, m_1m_3, m_2m_3)$ and $\mathrm{Taylor}(I^2)$ is a 5-dimensional simplex with 6 vertices, where each vertex is labeled by a generator of I^2 and each face is labeled with the least common multiple of its vertices. A non-minimal syzygy occurs when a face and a subface have the same label (see Theorem 3.1). When considering I^2, no matter what the monomial generators of I are, when i, j, k are distinct we always have the following:

$$\mathrm{lcm}({m_i}^2, {m_j}^2) = \mathrm{lcm}({m_i}^2, m_im_j, {m_j}^2)$$
$$\mathrm{lcm}({m_i}^2, m_jm_k) = \mathrm{lcm}({m_i}^2, m_jm_k, m_im_k).$$

These equalities lead to non-minimal syzygies in the Taylor resolution of I^2, and as a result the edges

$$\{{m_1}^2, {m_2}^2\}, \{{m_1}^2, {m_3}^2\}, \{{m_2}^2, {m_3}^2\}, \{{m_1}^2, m_2m_3\}, \{{m_2}^2, m_1m_3\}, \{{m_3}^2, m_1m_2\},$$

and all faces containing these edges, can be removed from $\mathrm{Taylor}(I^2)$. The resulting 2-dimensional subcomplex L^2_3 of $\mathrm{Taylor}(I^2)$ supports a resolution of I^2 ([3]). This simple observation has a considerable impact on bounding the Betti numbers of I^2. For example, in this small case we can conclude that for every ideal I with three square-free monomial generators, the projective dimension of I^2 is at most 2 (the dimension of L^2_3), versus the bound 5 (the dimension of $\mathrm{Taylor}(I^2)$).

In this paper we show that a similar argument can be made more generally. If I is minimally generated by square-free monomials $m_1, \ldots, m_q$, then we can identify faces of the simplex Taylor(I^r) which lead to non-minimal syzygies for I^r. Eliminating these faces results in a much smaller subcomplex of Taylor(I^r), which we call L_q^r (Definition 4.2).

Our main result, Theorem 5.9, shows that L_q^r supports a resolution of I^r. This echoes the power of Taylor's complex in that we have a topological structure depending solely on r and q that supports a resolution of I^r for *any* square-free monomial ideal I with q minimal generators. By further deleting redundancies specific to the generators of I, we obtain a subcomplex $\mathsf{L}^r(I)$ of L_q^r, so that we have

$$\mathsf{L}^r(I) \subseteq \mathsf{L}_q^r \subseteq \mathrm{Taylor}(I^r).$$

Theorem 5.9 also shows that $\mathsf{L}^r(I)$ supports a free resolution of I^r. Our approach to proving this involves showing that $\mathsf{L}^r(I)$ and L_q^r are both *quasi-trees* (see Definition 2.2), meaning (roughly) that are built from a special ordering on their facets. In Theorem 3.6, we show that for a quasi-tree to support a free resolution one only needs to check that certain induced subcomplexes are connected. This extends the main result of [9].

Our complex $\mathsf{L}^r(I)$ gives natural and useful bounds on homological information of I^r. Indeed, the Betti numbers of I^r are bounded in terms of the number and size of faces of $\mathsf{L}^r(I)$, yielding bounds that are significantly smaller than those given by the Taylor resolution. In the later sections of this paper we examine just how much smaller these bounds on the Betti numbers are and when the resolutions obtained are minimal. We define a class of square-free monomial ideals, which we call *extremal ideals*, whose Betti numbers of powers bound the Betti numbers of powers of any square-free monomial ideal with the same number of generators. As a result, we reduce the question of minimality of L_q^r to the study of when L_q^r supports a minimal free resolution of the r^{th} power of an extremal ideal.

To put this work in context of the broader literature, studying powers of ideals and bounding their invariants has received much attention in recent years. Powers play an important role in Rees algebras and associated graded rings among other uses, making understanding their behavior desirable but difficult. In another direction, there has been considerable interest in describing minimal topological resolutions for all monomial ideals using a variety of methods, such as using chain maps from multiple simplicial complexes (see [16, 20]). There are situations where the structure of a minimal topological resolution leads to minimal topological resolutions for powers (see, for example [4, 5, 10]), but in general, this is a challenging task. In this paper we combine the two interests and seek, for powers of monomial ideals, resolutions that are supported on a single topological structure which is practical to determine based on the generators of the original ideal I.

The paper is organized as follows. Section 2 contains basics of simplicial resolutions. In Section 3 we use simplicial collapsing and the Bayer–Peeva–Sturmfels criterion to prove the above-mentioned criterion for quasi-trees (Theorem 3.6). In Section 4 we introduce the definition of the simplicial complex L_q^r, and we prove in Proposition 4.6 that L_q^r is a quasi-tree. In Section 5 we define $\mathsf{L}^r(I)$, discuss some examples, and then prove the main result, Theorem 5.9. Section 6 investigates the bounds on the Betti numbers of I^r that follow from the main result. Finally, Section 7 introduces extremal ideals, which have maximal Betti numbers among powers of square-free monomial ideals. In particular, Proposition 7.11 provides a full characterization of the conditions on r and q that guarantee L_q^r supports a minimal free resolution of I^r for some ideal I.

This paper is an extension of the work in [3] where the focus is on the second power of the ideal I. The collaboration was initiated at the 2019 workshop "Women in Commutative Algebra" hosted by the Banff International Research Station.

2. Simplicial Resolutions

Fix $S = \mathsf{k}[x_1, \ldots, x_n]$ to be a polynomial ring over a field k. We begin by reviewing necessary background for simplicial complexes and simplicial resolutions and then demonstrate the potential relationship to resolutions of ideals.

A **simplicial complex** Δ on a **vertex set** V is a set of subsets of V such that if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$. We use the following terminology for simplicial complexes:

Definition 2.1. *Let Δ be a simplicial complex.*

- *An element of Δ is called a* **face***.*
- *The* **facets** *of Δ are the maximal faces under inclusion.*
- *The* **dimension** *of a face $F \in \Delta$ is $\dim(F) = |F| - 1$.*
- *The* **dimension** *of Δ is the maximum of the dimensions of its faces.*
- *Δ is called a* **simplex** *if it has one facet.*
- *The* **f-vector** $\mathbf{f}(\Delta) = (f_0, \ldots, f_d)$ *of a d-dimensional simplicial complex Δ has $f_i =$ the number of i-dimensional faces of Δ.*

Note that a simplicial complex can be uniquely determined by its facets. One writes

$$\Delta = \langle F_1, \ldots, F_q \rangle$$

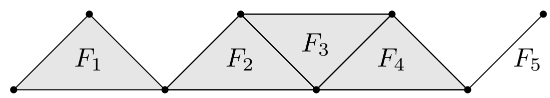
to denote a simplicial complex Δ with facets $F_1, \ldots, F_q$.

In subsequent sections, we will use the tool of trimming simplicial complexes via certain rules. Essentially, we will delete vertices in a specified fashion. Vertex deletions naturally lead to the consideration of subcomplexes of simplicial complexes, which are defined below, together with additional structures and notions that will be used throughout the paper.

Definition 2.2. *Let Δ be a simplicial complex on a vertex set V.*

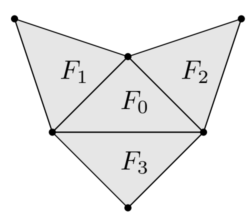
- *If v is a vertex of Δ, then the* **deletion** *of v from Δ is the simplicial complex*
$$\Delta \smallsetminus \{v\} = \{\sigma \in \Delta \mid v \notin \sigma\}.$$
- *A* **subcomplex** *of Δ is a subset of Δ which is also a simplicial complex.*
- *Given $W \subseteq V$, the* **induced subcomplex** *of Δ on W is the subcomplex*
$$\Delta_W = \{\sigma \in \Delta \mid \sigma \subseteq W\}.$$
- *A* **leaf** [8] *is a facet F of Δ such that F is the only facet of Δ, or there is a facet G of Δ with $F \neq G$ such that*
$$F \cap H \subseteq G$$
for all facets $H \neq F$. The facet G is called a **joint** *of F. (Note that the joint of a leaf need not be unique (see* [8]*).*
- *Δ is called a* **quasi-forest** [22] *if the facets of Δ can be ordered as $F_1, \ldots, F_q$ such that for $i = 1, \ldots, q$, the facet F_i is a leaf of the simplicial complex $\langle F_1, \ldots, F_i \rangle$.*
- *Δ is a* **quasi-tree** *if it is a connected quasi-forest.*

Example 2.3. The simplicial complex below is a quasi-tree. The leaf order is $F_1, \ldots, F_5$, meaning that each F_i is a leaf of $\langle F_1, \ldots, F_i \rangle$. In this example the joint of F_i is F_{i-1} for all $i > 1$.

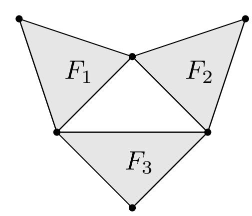


Example 2.4. The star-shaped complex drawn below on the left is a quasi-tree, with leaf order F_0, F_1, F_2, F_3. In particular, the center facet F_0 is the joint of F_i for every $i > 1$. This complex is a standard example of a quasi-tree which is not a simplicial tree in the sense of [8]. This particular quasi-tree is shown in [3] to support a free resolution of the second power of any ideal with three square-free monomial generators.

If one removes F_0 from the center, the remaining complex is shown in the picture on the right. This simplicial complex is not a quasi-tree, since no facet is a leaf.



quasi-tree



not a quasi-tree

A **free resolution** of I is an exact sequence of the form

$$0 \to S^{\beta_t} \to S^{\beta_{t-1}} \to \cdots \to S^{\beta_1} \to S^{\beta_0} \to I \to 0,$$

where S^{β_j} is a free S-module of rank β_j and $t \in \mathsf{N}$. When β_j is the smallest possible rank of a free module in the j^{th} spot of any free resolution of I for each j, the resolution is **minimal**. In this case, the numbers β_j are invariants of I and are called the **Betti numbers** of I.

In the 1960s, Diana Taylor demonstrated a striking connection between a $(q-1)$-simplex and a resolution of a monomial ideal I in S. If I is a monomial ideal in S minimally generated by monomials $m_1, \ldots, m_q$, then Taylor(I) denotes the simplex with q vertices indexed by the set $[q] = \{1, \ldots, q\}$, where each vertex i is labeled with one of the monomials m_i, and each face σ is labeled with the monomial

$$M_\sigma = \mathrm{lcm}(m_i \colon i \in \sigma).$$

In her Ph.D thesis [19], Taylor proved that the simplicial chain complex of Taylor(I) gives rise to a multigraded free resolution of I. In particular, the i^{th} Betti number of I is bounded above by the number of i-dimensional faces of Taylor(I), which is $\binom{q}{i+1}$. This method has been generalized so that if Δ is a simplicial or cellular complex whose vertices are labeled with the monomial generators $m_1, \ldots, m_q$ of an ideal I and whose faces are labeled with the least common multiple of the vertex labels as above, then we say that Δ **supports a free resolution** of I if the homogenization of the simplicial (or cellular) chain complex of Δ is a multi-graded free resolution of I, denoted by F_Δ, see [1, 2]. The multi-graded complex F_Δ is described as follows. For each $t > 0$, the free module $(\mathsf{F}_\Delta)_t$ has basis elements denoted by e_σ, where σ ranges over all faces of Δ with $|\sigma| = t+1$, and e_σ is considered to have multi-degree M_σ. The differential is described by

$$\partial(e_\sigma) = \sum_{j=0}^{t} (-1)^j \frac{M_\sigma}{M_{\sigma \smallsetminus \{v_{i_j}\}}} e_{\sigma \smallsetminus \{v_{i_j}\}}\,, \tag{1}$$

where $\sigma = \{v_{i_0}, \ldots, v_{i_t}\}$ with $i_0 < i_1 < \cdots < i_t$.

Example 2.5. Let $I = (xy, yz, zu)$ in $R = \mathsf{k}[x, y, z, u]$. The labeled simplicial chain complex

$$\begin{tikzcd} xy \arrow[r, dash, "xyz"] & yz \arrow[r, dash, "yzu"] & zu \end{tikzcd}$$

supports a free resolution of I. The chain complex of Δ is

$$0 \longrightarrow \mathsf{k}^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}} \mathsf{k}^3 \to \mathsf{k}$$

and homogenization results in the free resolution

$$0 \longrightarrow R(xyz) \oplus R(yzu) \xrightarrow{\begin{bmatrix} z & 0 \\ -x & u \\ 0 & -y \end{bmatrix}} R(xy) \oplus R(yz) \oplus R(zu) \longrightarrow I \longrightarrow 0\,,$$

where the notation $R(x^a y^b z^c u^d)$ refers to the R-free module with one generator in multi-degree (a, b, c, d).

Remark 2.6. Simplicial complexes that support a free resolution of a monomial ideal are usually constructed such that the vertices correspond to and are labeled by a *minimal* set of generators of the ideal. However, one can also work with non-minimal generators, at the expense of producing a larger complex. In particular, one can mimic the construction of $\mathrm{Taylor}(I)$, but use instead any set of monomial generators of I. The same considerations show that this complex supports a free resolution of I.

There are various combinatorial ways to build subcomplexes Δ of $\mathrm{Taylor}(I)$ that support a free resolution of I. One such well known complex is the *Lyubeznik complex*, which supports the *Lyubeznik resolution* of I ([12]). The Lyubeznik resolution is the main inspiration for the complexes L^r_q and $\mathsf{L}^r(I)$ which appear later in this paper. We will not define this resolution since it is not used in this paper, but refer the reader to [12, 13] for additional information.

3. Resolutions supported on quasi-trees

Taylor's resolution is usually far from minimal. However, for a given monomial ideal I, a criterion of Bayer, Peeva and Sturmfels (see Theorem 3.1) allows one to check if a subcomplex of $\mathrm{Taylor}(I)$ supports a free resolution of I. In this section, using the above criterion and by observing that quasi-trees are collapsible, we show that a quasi-tree Δ supports a free resolution of a given monomial ideal if and only if certain subcomplexes of Δ are connected.

For a subcomplex Δ of $\mathrm{Taylor}(I)$ and a monomial M in S, let Δ_M be the subcomplex of Δ induced on the vertices of Δ whose labels divide M.

The following is the criterion of Bayer, Peeva and Sturmfels [1, Lemma 2.2]; see also [6, Theorem 2.2] for the statement on minimality.

Theorem 3.1 (**Criterion for a simplicial complex to support a free resolution**). *Let Δ be a simplicial complex whose vertices are labeled with a monomial generating set of a monomial ideal I in a polynomial ring S over a field. Then Δ supports a free resolution of I over S if and only if for every monomial M, the induced subcomplex Δ_M of Δ on the vertices whose labels divide M is empty or acyclic.*

Furthermore, a free resolution supported on Δ is minimal if and only if $M_\sigma \neq M_{\sigma'}$ for every proper subface σ' of a face σ of Δ.

Remark 3.2. The results of Theorem 3.1 are usually stated with the assumption that one uses the *minimal* monomial generating set of I for the labels. However, the proof of [1, Lemma 2.2] does not make use of this assumption, hence we formulated the result above to reflect this observation.

In particular, Theorem 3.1 implies that the **f**-vector of a complex Δ supporting a resolution of a monomial ideal I is an upper bound for the vector of Betti numbers of I. In other words, for each $i \leqslant d = \dim(\Delta)$,

$$\beta_i(I) \leqslant f_i \quad \text{where} \quad \mathbf{f}(\Delta) = (f_0, \dots, f_d).$$

In particular, if Δ supports a minimal free resolution of I, then equality holds above.

Using Theorem 3.1 it is straightforward to see that to determine whether Δ supports a free resolution of I, it suffices to check that Δ_M is empty or acyclic only for monomials M in the lcm lattice of I; that is, for monomials M that are least common multiples of sets of vertex labels.

If the complex Δ under consideration in Theorem 3.1 is a simplicial tree, then it suffices to show that Δ_M is connected, instead of acyclic, see [9]. More precisely, it is established in [9] that every induced subcomplex of a simplicial tree is a simplicial forest, and then it is shown that simplicial trees are acyclic, and hence an induced subcomplex of Δ is acyclic if and only if it is empty or connected (see [9, Theorems 2.5, 2.9, 3.2]).

We now generalize the work in [9] by showing that the criterion in Theorem 3.1 can be extended to the class of quasi-trees. To do so we need to argue that quasi-trees, and their connected induced subcomplexes, are acyclic. We do so using the following series of results.

Proposition 3.3 (**Induced subcomplexes of quasi-forests are quasi-forests**). *If a simplicial complex Δ is a quasi-forest, then every induced subcomplex of Δ is a quasi-forest.*

Proof. By [10, Proposition 6], a simplicial complex Δ with vertex set V is a quasi-forest if and only if for every subset $W \subseteq V$, the induced subcomplex Δ_W has a leaf. If $W \subseteq V$, consider the induced subcomplex Δ_W of Δ. If $U \subseteq W$, then $\Delta_U = (\Delta_W)_U$ has a leaf, and hence, Δ_W is a quasi-forest.

A face σ of a simplicial complex Δ is called a **free face** if it is properly contained in a unique facet F of Δ. A **collapse** of Δ along the free face σ is the simplicial complex obtained by removing the faces τ such that $\sigma \subseteq \tau \subseteq F$ from Δ. If additionally $\dim(\sigma) = \dim(F) - 1$, then the collapse is called an **elementary collapse**. The simplicial complex Δ is called **collapsible** if it can be reduced to a point via a series of (elementary) collapses.

Example 3.4. Consider the quasi-tree in Example 2.4. To illustrate that this complex is collapsible to a point, we label in Figure 1 the vertices and then demonstrate one collapsing sequence. At each step, the two faces that play the roles of σ and F in the exposition above are indicated. Note that the first two steps are elementary collapses for demonstration purposes. Alternate collapsing sequences exist.

In general, by starting with a leaf and the face that contains it, any quasi-tree can be collapsed to a point.

Proposition 3.5. *Every quasi-tree is collapsible.*

Proof. Proceed by induction on q, the number of facets of Δ. When $q = 1$, Δ is a simplex, and known to be collapsible (e.g. [9, Proposition 2.7]). Suppose the statement holds for all quasi-trees with fewer than q facets.

Let Δ be a quasi-tree with facet ordering $F_1, \dots, F_q$ so that, for $i = 1, \dots, q$, each F_i is a leaf of $\langle F_1, \dots, F_i \rangle$. Since F_q is a leaf of Δ, it intersects $\Delta' = \langle F_1, \dots, F_{q-1} \rangle$ in

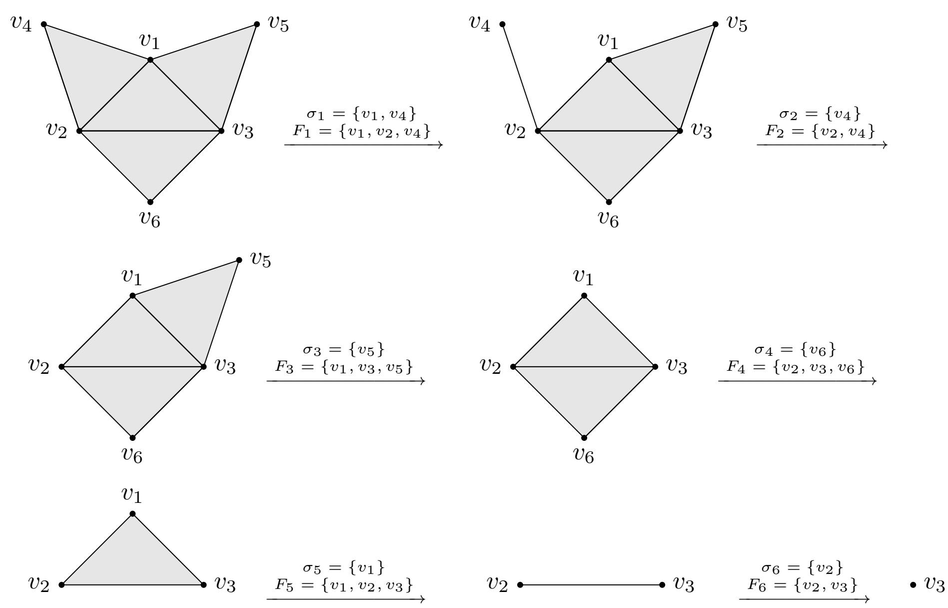


Figure 1. An example of collapsing

a face F' of Δ'. By [9, Proposition 2.7], F_q collapses down to the face F' by removing faces outside Δ'. As a result, this series of collapses brings Δ to the quasi-tree Δ', which, by the induction hypothesis, is collapsible.

We now use Proposition 3.5 and Proposition 3.3 to show that the Bayer-Peeva-Sturmfels criterion in Theorem 3.1 can be extended to the class of quasi-trees.

Theorem 3.6 (**Criterion for a quasi-tree to support a free resolution**). *Let Δ be a quasi-tree whose vertices are labeled with a monomial generating set of a monomial ideal I in a polynomial ring S over a field. Then Δ supports a free resolution of I over S if and only if for every monomial M, the induced subcomplex Δ_M of Δ on the vertices whose labels divide M is empty or connected.*

Proof. By Theorem 3.1, Δ supports a resolution of I if and only if Δ_M is empty or acyclic for every monomial $M \in S$. If $M \in S$ is a monomial, then by Proposition 3.3 above, Δ_M is a quasi-forest. Moreover, by Proposition 3.5, every connected component of Δ_M is contractible, i.e. homotopy equivalent to a point, and hence acyclic. The only possible homology that Δ_M could have would be that which comes from it being disconnected. As a result, Δ supports a resolution of I if and only if Δ_M is empty or connected for every monomial $M \in S$.

4. The quasi-tree L_q^r

Recall that our goal is to find a simplicial complex smaller than Taylor's complex that supports a free resolution of I^r when I has q square-free monomial generators. In this section we construct a subcomplex L_q^r of the Taylor simplex, which depends only on the integers r and q, and contains a subcomplex supporting a free resolution of I^r.

The base case for our construction is the case $r = 1$. In this case L_q^1 is the well-known Taylor complex [19]: a simplex with q vertices that supports a free resolution of any monomial ideal with q generators. The case $r = 2$ was investigated in the earlier work of the authors [3]. For example, it was shown in [3] that L_3^2 is the quasi-tree in Example 2.4.

We now collect and formalize the notation needed for the extension to the case $r \geqslant 3$.

Notation 4.1. Let r and q be two positive integers.

- Let $\mathbf{e}_1, \dots, \mathbf{e}_q$ denote the standard basis vectors for R^q; i.e. for each $i \in [q]$, $\mathbf{e}_i$ is the q-tuple with 1 in the i^{th} coordinate and 0 elsewhere.

- Define $\mathcal{N}_q^r$ to be the set of points in $\mathsf{Z}^q_{\geqslant 0}$ whose coordinates add up to r:

$$\begin{aligned}\mathcal{N}_q^r =& \{(a_1, \dots, a_q) \in \mathsf{Z}^q_{\geqslant 0} \colon a_1 + \cdots + a_q = r\}\\ =& \{a_1\mathbf{e}_1 + \cdots + a_q\mathbf{e}_q \colon a_i \in \mathsf{Z}_{\geqslant 0} \text{ and } a_1 + \cdots + a_q = r\}.\end{aligned}$$

- Set $s = \lceil \frac{r}{2} \rceil$; that is, when r is odd, $r = 2s-1$, and when r is even, $r = 2s$.

Definition 4.2 (**The simplicial complex L_q^r - see also Proposition 4.3**). *Let $r, q \geqslant 1$ be two integers. Following Notation 4.1 we define L_q^r to be the simplicial complex with vertex set $\mathcal{N}_q^r$ whose faces are all subsets of the (not necessarily distinct) sets $F_1^r, \dots, F_q^r, G_1^r, \dots, G_q^r$ defined as*

$$F_i^r = \{(a_1, \dots, a_q) \in \mathcal{N}_q^r \colon a_i \leqslant \max\{r-1, s\} \textit{ and } a_j \leqslant s \textit{ for } j \neq i\}$$

$$G_i^r = \{(a_1, \dots, a_q) \in \mathcal{N}_q^r \colon a_i \geqslant r-1\} = \{(r-1)\mathbf{e}_i + \mathbf{e}_j \colon j \in [q]\}$$

for each $i \in [q]$. We refer to the set $\{F_1^r, \dots, F_q^r\}$ as the **first layer** *of L_q^r and the set $\{G_1^r, \dots, G_q^r\}$ as the* **second layer** *of L_q^r. We define the* **base** *of L_q^r to be the face*

$$B^r = \left\{(a_1, \dots, a_q) \in \mathcal{N}_q^r \colon a_i \leqslant s \textit{ for all } i \in [q]\right\},$$

so that

$$F_i^r = B^r \cup \{(a_1, \dots, a_q) \in \mathcal{N}_q^r \colon s+1 \leqslant a_i \leqslant r-1\}.$$

In general, $F_1^r, \dots F_q^r, G_1^r, \dots, G_q^r$ are the facets of L_q^r (Proposition 4.3); however, for small values of r and q, these sets need not be distinct. We summarize these facts in Proposition 4.3 to give a more precise description of the facets of L_q^r.

Proposition 4.3 (**Equivalent definition of L_q^r**). *The simplicial complex L_q^r in Definition 4.2 can be described in terms of its distinct facets:*

$$\mathsf{L}_q^r = \begin{cases} \langle F_1^r, F_2^r, \dots, F_q^r, G_1^r, \dots, G_q^r \rangle & \textit{if } r > 3 \textit{ and } q \geqslant 2\\ \langle B^r, G_1^r, \dots, G_q^r \rangle & \textit{if } r = 3 \textit{ and } q \geqslant 2 \textit{ or } r = 2 \textit{ and } q > 2\\ \langle G_1^2, G_2^2 \rangle & \textit{if } r = 2 \textit{ and } q = 2\\ \langle \mathcal{N}_q^r \rangle & \textit{if } r = 1 \textit{ or } q = 1. \end{cases}$$

Proof. When $r = 1$, then $s = 1$ and $\mathcal{N}_q^1 = \{\mathbf{e}_1, \dots, \mathbf{e}_q\}$. It follows that

$$B^1 = F_i^1 = G_i^1 = \mathcal{N}_q^1 \quad \text{for all} \quad i \in [q].$$

In this case L_q^1 is a simplex with q vertices (the Taylor simplex).

If $q = 1$ and $r > 1$, then

$$B^r = F_1^r = \emptyset \quad \text{and} \quad G_1^r = \mathcal{N}_1^r = \{(r)\}.$$

If $q = r = 2$, then

$$G_1^2 = \{(2,0), (1,1)\} \quad \text{and} \quad G_2^2 = \{(0,2), (1,1)\}$$

and since $s = 1 = r - 1$,

$$F_1^2 = F_2^2 = B^2 = \{(1,1)\} \subseteq G_1^2 \cap G_2^2.$$

When $q \geqslant 2$ and $r = 3$, or $q > 2$ and $r = 2$, then the G_i^r are distinct as each G_i^r is the unique facet containing the vertex $r\mathbf{e}_i$. Furthermore, since $s + 1 = r$,

$$F_1^r = \cdots = F_q^r = B^r$$

contains all the vertices $(r-1)\mathbf{e}_i + \mathbf{e}_j$ where $i \neq j$, and so B^r is not embedded in any of the G_i^r.

Finally, when $q \geqslant 2$ and $r > 3$, then $s < r - 1$. Therefore, the G_i^r are distinct, since each G_i^r is the unique facet containing the vertex $r\mathbf{e}_i$.

For $i, j \in [q]$, $i \neq j$, F_j^r is not contained in F_i^r since

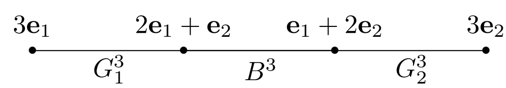
$$(s+1)\mathbf{e}_j + (r-s-1)\mathbf{e}_i \in F_j^r \smallsetminus F_i^r.$$

Moreover, $r\mathbf{e}_j \notin F_i^r$, showing that

$$G_j^r \nsubseteq F_i^r.$$

Lastly, no F_i^r can be contained in G_j^r since $B^r \subseteq F_i^r$, but $B^r \cap G_j^r = \varnothing$ when $s < r - 1$.

Example 4.4. The geometric realization of the simplicial complex L_2^3 is a path of length 3, as can be seen in the figure below.



Since

$$\mathcal{N}_2^3 = \{(3,0),(2,1),(1,2),(0,3)\} = \{3\mathbf{e}_1, 2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + 2\mathbf{e}_2, 3\mathbf{e}_2\},$$

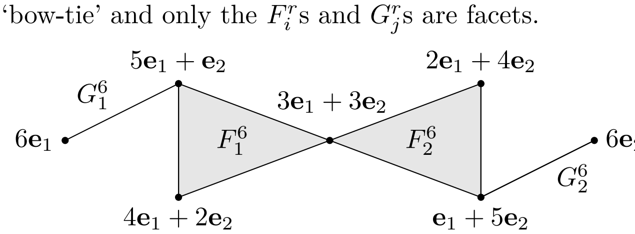
according to Proposition 4.3, the facets of L_2^3 are

$$\begin{aligned} B^3 &= \{(2,1),(1,2)\} = \{2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + 2\mathbf{e}_2\} \\ G_1^3 &= \{(3,0),(2,1)\} = \{3\mathbf{e}_1, 2\mathbf{e}_1 + \mathbf{e}_2\} \\ G_2^3 &= \{(0,3),(1,2)\} = \{3\mathbf{e}_2, \mathbf{e}_1 + 2\mathbf{e}_2\}. \end{aligned}$$

By contrast, if $q = 2$ and $r \geqslant 4$ is even, then B^r is a single vertex

$$B^r = \{(r/2)\mathbf{e}_1 + (r/2)\mathbf{e}_2\} \subsetneq F_i^r \quad \text{for all} \quad i \in [q].$$

For instance, if $r = 6$, then L_2^6 is pictured below, and B^6 is the single point at the middle of the 'bow-tie' and only the F_i^rs and G_j^rs are facets.



Note that the points in $\mathcal{N}_q^r$ can be viewed as lattice points in R^q. Indeed, they are precisely the integer lattice points in a hyperplane section of the first octant, cut out by the hyperplane whose equation is

$$x_1 + \cdots + x_q = r.$$

While for small values of q this gives a natural way to depict the points, it does not illustrate the simplicial structure well. For instance, in Example 4.4, the 6 points would be co-linear, while our depiction emphasizes the existence of the two triangles. Using the combinatorial approach rather than the embedding in R^q also allows for a

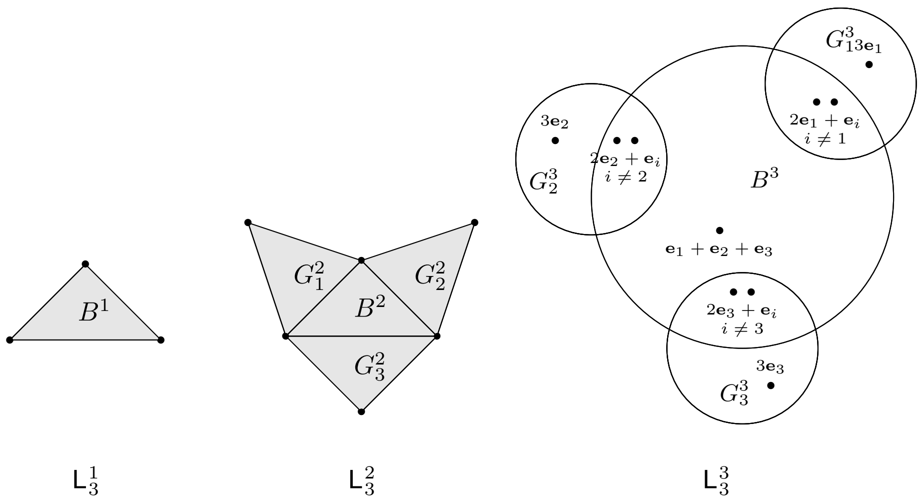


FIGURE 2. A picture of L_3^1, L_3^2 and L_3^3

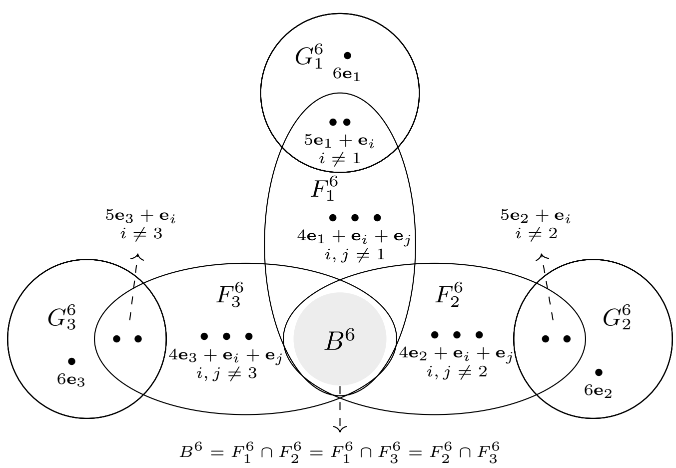


FIGURE 3. A picture of L_3^6

generalized depiction, seen in the following example, that can easily be extended to higher q and r.

EXAMPLE 4.5. We examine the case $q=3$ in the same manner as above. The simplicial complexes L_3^1, L_3^2 and L_3^3 appear in Figure 2, and L_3^6 appears in Figure 3.

The following statement generalizes [3, Proposition 3.3], which deals with the special case $r=2$.

PROPOSITION 4.6 (**L_q^r is a quasi-tree**). *The simplicial complex L_q^r is a quasi-tree.*

Proof. Following Proposition 4.3 for a description of the facets of L_q^r, we consider each case separately.

When $r=1$ or $q=1$, L_1^1 is a simplex, and hence by definition a quasi-tree.

When $r=q=2$, L_2^2 has only two facets, and it is trivially a quasi-tree.

When $r=2$ and $q>2$, then order the facets of L_q^2 by

$$B^2, G_1^2, \dots, G_q^2.$$

In this case, if $i \neq j$ with $i, j \in [q]$, then

$$G_i^2 \cap B^2 \subseteq B^2 \quad \text{and} \quad G_i^2 \cap G_j^2 = \{\mathbf{e}_i + \mathbf{e}_j\} \subseteq B^2.$$

Thus each G_i^2 is a leaf of $\langle B^2, G_1^2, \ldots, G_i^2 \rangle$ with joint B^2.

When $r = 3$ and $q > 2$, then as in the previous case,

$$B^3, G_1^3, \ldots, G_q^3$$

is a leaf order. Since $G_i^3 \cap G_j^3 = ?$ when $i \neq j$, for each $i \in [q]$ the facet G_i^3 is a leaf of

$$\langle B^3, G_1^3, \ldots, G_i^3 \rangle$$

with joint B^3.

When $r > 3$ and $q > 2$, we claim

$$F_1^r, F_2^r, \ldots, F_q^r, G_1^r, \ldots, G_q^r \tag{2}$$

is a leaf order for L_q^r, making it a quasi-tree. To see this note that if $i \neq j$, $i, j \in [q]$, then

$$\begin{aligned} &F_i^r \cap G_j^r = ?, \quad F_i^r \cap F_j^r = B^r, \quad G_i^r \cap G_j^r = ?, \\ &F_i^r \cap G_i^r = \{(r-1)\mathbf{e}_i + \mathbf{e}_h \colon h \neq i\}. \end{aligned}$$

These observations show that for each $j \in [q]$, G_j^r is a leaf of

$$\langle F_1^r, \ldots, F_q^r, G_1^r, \ldots, G_j^r \rangle$$

with joint F_j^r, and for each $j \in \{2, \ldots, q\}$ F_j^r is a leaf of

$$\langle F_1^r, \ldots, F_j^r \rangle$$

with joint, say, F_1^r. Hence (2) is a leaf order and L_q^r is a quasi-tree.

5. The quasi-tree $\mathsf{L}^r(I)$ supporting a free resolution of I^r

Given an ideal I with q square-free monomial generators, we now define an induced subcomplex of L_q^r, denoted $\mathsf{L}^r(I)$, which is obtained by deleting vertices representing redundant generators of I^r. We show in Theorem 5.9 that both L_q^r and $\mathsf{L}^r(I)$ support a free resolution of I^r.

Definition 5.1 (**The simplicial complex $\mathsf{L}^r(I)$**). *Let I be an ideal minimally generated by monomials $m_1, \ldots, m_q$ in the polynomial ring S. For $\mathbf{a} = (a_1, \ldots, a_q) \in \mathcal{N}_q^r$ we set*

$$\mathbf{m}^{\mathbf{a}} = {m_1}^{a_1} \cdots {m_q}^{a_q}.$$

Define a partition of $\mathcal{N}_q^r$ into equivalence classes $V_1, \ldots, V_t$ by

$$\mathbf{a} \sim \mathbf{b} \iff \mathbf{m}^{\mathbf{a}} = \mathbf{m}^{\mathbf{b}}.$$

We use these equivalence classes to build an induced subcomplex $\mathsf{L}^r(I)$, of L_q^r using the following steps:

Step 1. *From each equivalence class V_i pick a unique representative $\mathbf{c}_i$ in the following way: if $V_i \cap B^r \neq ?$, then $\mathbf{c}_i \in B^r$. Otherwise, choose any $\mathbf{c}_i \in V_i$.*

Step 2. *From the set $\{\mathbf{c}_1, \ldots, \mathbf{c}_t\}$, eliminate all $\mathbf{c}_i$ for which $\mathbf{m}^{\mathbf{c}_j} \mid \mathbf{m}^{\mathbf{c}_i}$ for some $j \neq i$. We call the remaining set, without loss of generality, $\{\mathbf{c}_1, \ldots, \mathbf{c}_u\}$.*

Step 3. *Set $V = \{\mathbf{c}_1, \ldots, \mathbf{c}_u\}$.*

Define $\mathsf{L}^r(I)$ to be the induced subcomplex of L_q^r on the vertex set V.

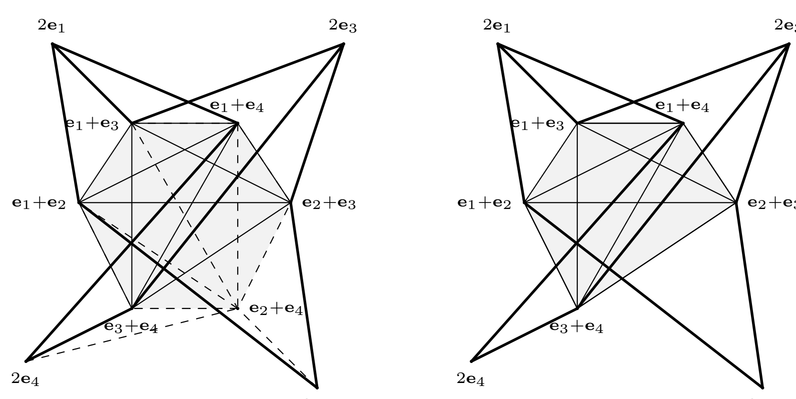
The complex $\mathsf{L}^r(I)$ is a subcomplex of Taylor(I^r). While $\mathsf{L}^r(I)$ is dependent on the choices made when building its vertex set V, we will abuse the notation and ignore these choices, as they do not have an impact on our results. Also note that the set of monomials $\{\mathbf{m}^{\mathbf{c}_1}, \dots, \mathbf{m}^{\mathbf{c}_u}\}$ forms a minimal monomial generating set for the ideal I^r. There are known classes of ideals, e.g., square-free monomial ideals of projective dimension 1 [4, Proposition 4.1], where the generating set $\{\mathbf{m}^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{N}_q^r\}$ does not contain redundancies, in which case $\mathsf{L}_q^r = \mathsf{L}^r(I)$. However, in general, $\mathsf{L}^r(I)$ will be a proper subcomplex of L_q^r. Information about known redundancies in the set $\{\mathbf{m}^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{N}_q^r\}$ can be used to produce $\mathsf{L}^r(I)$. As an example, consider edge ideals of graphs. For such ideals, the redundancies in $\{\mathbf{m}^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{N}_q^r\}$ are encoded in the ideal of equations of the fiber cone of the ideal, whose generators correspond to primitive even closed walks in the graph (see [21] for relevant definitions and details).

Example 5.2. Let $S = \mathsf{k}[x, y, z, w]$ and $I = (xy, yz, zw, xw) = (m_1, m_2, m_3, m_4)$. By Proposition 4.3, the facets of L_4^2 are a 5-simplex B^2 and four tetrahedra G_i^2 for $1 \leqslant i \leqslant 4$, depicted on the left in the figure below.

By Definition 5.1, since $m_1 m_3 = m_2 m_4$ is the only equation determining an equivalence class with more than one element, we select the vertex $\mathbf{e}_1 + \mathbf{e}_3$ to represent this equivalence class and form $\mathsf{L}^2(I)$. Then $\mathsf{L}^2(I)$ consists of a 4-simplex on the vertices

$$\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4$$

together with two triangles and two tetrahedra depicted on the right in the figure below. Note that vertex $\mathbf{e}_2 + \mathbf{e}_4$ has been removed. The edges depicted by dotted lines in the figure of L_4^2 do not appear in $\mathsf{L}^2(I)$ and higher dimensional faces of L_4^2 containing $\mathbf{e}_2 + \mathbf{e}_4$ have also been deleted.



L_4^2 $\qquad\qquad$ $\mathsf{L}^2(I)$

Next, consider I^3. The equation $m_1 m_3 = m_2 m_4$ produces four non-trivial equivalence classes from Definition 5.1, namely the sets

$$\{2\mathbf{e}_1{+}\mathbf{e}_3, \mathbf{e}_1{+}\mathbf{e}_2{+}\mathbf{e}_4\},\ \{\mathbf{e}_1{+}\mathbf{e}_2{+}\mathbf{e}_3, 2\mathbf{e}_2{+}\mathbf{e}_4\},\ \{\mathbf{e}_1{+}2\mathbf{e}_3, \mathbf{e}_2{+}\mathbf{e}_3{+}\mathbf{e}_4\},\ \{\mathbf{e}_1{+}\mathbf{e}_3{+}\mathbf{e}_4, \mathbf{e}_2{+}2\mathbf{e}_4\}.$$

One can check that the above relations are the only ones. Since $r = 3$, we have $s = 2$ and so all the above duplicated vertices are in B^3. In particular, $\mathsf{L}^3(I)$ will be the induced subcomplex of L_4^3 on a vertex set V with 16 vertices. The sets below are two different possible sets of vertices V for $\mathsf{L}^3(I)$:

$$\begin{aligned} V = \{&3\mathbf{e}_1, 2\mathbf{e}_1 + \mathbf{e}_2, 2\mathbf{e}_1 + \mathbf{e}_4, 3\mathbf{e}_2, 2\mathbf{e}_2 + \mathbf{e}_1, 2\mathbf{e}_2 + \mathbf{e}_3, 3\mathbf{e}_3, 2\mathbf{e}_3 + \mathbf{e}_2, 2\mathbf{e}_3 + \mathbf{e}_4, 3\mathbf{e}_4, \\ &2\mathbf{e}_4 + \mathbf{e}_1, 2\mathbf{e}_4 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4, \mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4\} \end{aligned}$$

or

$$\begin{aligned} V = \{&3\mathbf{e}_1, 2\mathbf{e}_1+\mathbf{e}_2, 2\mathbf{e}_1+\mathbf{e}_3, 2\mathbf{e}_1+\mathbf{e}_4, 3\mathbf{e}_2, 2\mathbf{e}_2+\mathbf{e}_1, 2\mathbf{e}_2+\mathbf{e}_3, 2\mathbf{e}_2+\mathbf{e}_4, 3\mathbf{e}_3,\\ &2\mathbf{e}_3+\mathbf{e}_1, 2\mathbf{e}_3+\mathbf{e}_2, 2\mathbf{e}_3+\mathbf{e}_4, 3\mathbf{e}_4, 2\mathbf{e}_4+\mathbf{e}_1, 2\mathbf{e}_4+\mathbf{e}_2, 2\mathbf{e}_4+\mathbf{e}_3\}\end{aligned}$$

Example 5.3. Let $S = \mathsf{k}[a,b,c,d,e,f,x,y,z]$ and

$$I = (xyz, abc, def, xza, xzb, xyc, xyd, yze, yzf) = (m_1, m_2, \ldots, m_9).$$

We have the following relation:

$$m_1{}^4 m_2 m_3 = m_4 m_5 m_6 m_7 m_8 m_9\,.$$

For $r = 6$ we have $s = 3$, and so

$$4\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \in F_1^6 \smallsetminus B^6 \quad \text{but} \quad \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8 + \mathbf{e}_9 \in B^6.$$

Therefore, the induced subcomplex $\mathsf{L}^6(I)$ would not contain the vertex $4\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ corresponding to $m_1{}^4 m_2 m_3$.

While the examples above show that, in general, $\mathsf{L}^r(I)$ is smaller than L_q^r, there are also cases when the two complexes are the same. In Section 7 we identify an ideal $\mathcal{E}_q$ with $\mathsf{L}^r(\mathcal{E}_q) = \mathsf{L}_q^r$ for each q. The two complexes are also equal for all I with $q \leqslant 3$, as shown below.

Proposition 5.4. *Let I be an ideal minimally generated by square-free monomials $m_1, \ldots, m_q$ in the polynomial ring S. If $q \leqslant 3$ then $\mathsf{L}^r(I) = \mathsf{L}_q^r$ for all $r \geqslant 1$.*

Proof. If $q = 1$, then $I^r = (m_1{}^r)$ for all r, and both $\mathsf{L}^r(I)$ and L_q^r consist of a single point, so the equality holds.

Assume $1 < q \leqslant 3$. By Definition 5.1 it suffices to show that $\mathsf{L}^r(I)$ and L_q^r have the same vertex set, or in other words

$$\mathbf{m}^{\mathbf{a}} \mid \mathbf{m}^{\mathbf{b}} \implies \mathbf{a} = \mathbf{b} \quad \text{for all} \quad \mathbf{a}, \mathbf{b} \in \mathcal{N}_q^r.$$

Suppose $q = 2$ and $\mathbf{m}^{\mathbf{a}} \mid \mathbf{m}^{\mathbf{b}}$ with

$$\mathbf{a} = (a_1, a_2), \quad \mathbf{b} = (b_1, b_2), \quad \text{and} \quad a_1 + a_2 = b_1 + b_2 = r.$$

If $\mathbf{a} \neq \mathbf{b}$, then we may assume $a_1 > b_1$ and $a_2 < b_2$. Since the monomials m_i are square-free, we have

$$m_1{}^{a_1} m_2{}^{a_2} \mid m_1{}^{b_1} m_2{}^{b_2} \implies m_1{}^{a_1 - b_1} \mid m_2{}^{b_2 - a_2} \implies m_1 \mid m_2\,,$$

which contradicts the minimality of the generators of I.

Suppose $q = 3$ and $\mathbf{m}^{\mathbf{a}} \mid \mathbf{m}^{\mathbf{b}}$ with

$$\mathbf{a} = (a_1, a_2, a_3), \quad \mathbf{b} = (b_1, b_2, b_3) \quad \text{and} \quad a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = r.$$

Assume $\mathbf{a} \neq \mathbf{b}$. If $a_i = b_i$ for some i then we can reduce to the case $q = 2$, so we may assume $a_i \neq b_i$ for all i. Without loss of generality, assume $a_1 > b_1$.

We have three cases:

(i) Suppose $a_2 > b_2$. In this case, we must also have $a_3 < b_3$. Then

$$m_1{}^{a_1} m_2{}^{a_2} m_3{}^{a_3} \mid m_1{}^{b_1} m_2{}^{b_2} m_3{}^{b_3} \implies m_1{}^{a_1 - b_1} m_2{}^{a_2 - b_2} \mid m_3{}^{b_3 - a_3} \implies m_1{}^{a_1 - b_1} \mid m_3{}^{b_3 - a_3}.$$

Since m_1 and m_3 are square-free, this implies that $m_1 \mid m_3$ which gives a contradiction.

(ii) Suppose $a_2 < b_2$ and $a_3 < b_3$. We have

$$m_1{}^{a_1} m_2{}^{a_2} m_3{}^{a_3} \mid m_1{}^{b_1} m_2{}^{b_2} m_3{}^{b_3} \implies m_1{}^{a_1 - b_1} \mid m_2{}^{b_2 - a_2} m_3{}^{b_3 - a_3}.$$

Let x be a variable such that $x \mid m_1$. It follows that $x \mid m_2$ or $x \mid m_3$. Assume $x \nmid m_3$. We then have $x^{a_1-b_1} \mid {m_2}^{b_2-a_2}$. Since m_2 is square-free, it follows that $b_2 - a_2 \geqslant a_1 - b_1$. This is a contradiction, since

$$a_1 - b_1 = b_2 - a_2 + b_3 - a_3 > b_2 - a_2.$$

A similar contradiction is obtained when $x \nmid m_2$. We conclude that for every $x \mid m_1$, we must have $x \mid m_2$ and $x \mid m_3$. This implies $m_1 \mid m_2$ and $m_1 \mid m_3$, which is a contradiction.

(iii) Suppose $a_2 < b_2$ and $a_3 > b_3$. After relabeling, this case reduces to Case (i).

Remark 5.5. Note that the square-free assumption is necessary in Proposition 5.4, for if $I = (x^2y, yz^2, xyz)$, then $m_1m_2 \mid {m_3}^2$ in I^2, but $(1,1,0) \neq (0,0,2)$ in $\mathcal{N}_3^2$.

The next proposition shows that the vertices labeled by $r\mathbf{e}_i$ belong to the induced subcomplex $\mathsf{L}^r(I)$ for all $i \in [q]$, regardless of the choices made in Definition 5.1. Moreover, if $q \geqslant 2$, then for each $i \in [q]$ there exists some $j \in [q] \setminus \{i\}$ such that the vertex labeled by $(r-1)\mathbf{e}_i + \mathbf{e}_j$ belongs to $\mathsf{L}^r(I)$.

In what follows we use the standard notation $\mathrm{LCM}(I^r)$ to denote the **lcm lattice** of I^r, which is the set of all least common multiples of subsets of the minimal monomial generating set of I^r, partially ordered by division.

Proposition 5.6. *Let $r \geqslant 1$, I an ideal in S minimally generated by square-free monomials $m_1, \ldots, m_q$, and $i \in [q]$.*

(i) *If $\mathbf{m}^{\mathbf{a}} \mid {m_i}^r$, then $\mathbf{a} = r\mathbf{e}_i$, for any $\mathbf{a} \in \mathcal{N}_q^r$.*

(ii) *If $q \geqslant 2$ and ${m_i}^{r-1} \mid M$ for some monomial $M \neq {m_i}^r$ with $M \in \mathrm{LCM}(I^r)$, then there exists $j \in [q] \setminus \{i\}$ such that $m_j \mid M$ and, for every $\mathbf{a} \in \mathcal{N}_q^r$,*

$$\mathbf{m}^{\mathbf{a}} \mid {m_i}^{r-1}m_j \iff \mathbf{m}^{\mathbf{a}} = {m_i}^{r-1}m_j \iff \mathbf{a} = (r-1)\mathbf{e}_i + \mathbf{e}_j.$$

In particular, for all i, and all j as in (ii),

$$r\mathbf{e}_i \quad \text{and} \quad (r-1)\mathbf{e}_i + \mathbf{e}_j$$

are vertices of $\mathsf{L}^r(I)$, and

$${m_i}^r \quad \text{and} \quad {m_i}^{r-1}m_j$$

are minimal monomial generators of I^r.

Proof. We first observe that, for $g, h > 0$ and $u, v \in [q]$, we have:

$$ {m_u}^h \mid {m_v}^g \quad \Longrightarrow \quad u = v \tag{3}$$

Indeed, since m_v is square-free, if m_u^h divides m_v^g, then $m_u \mid m_v$, and we conclude $u = v$ because m_u, m_v are minimal generators.

Let $\mathbf{a} = (a_1, \ldots, a_q) \in \mathcal{N}_q^r$, and suppose $\mathbf{m}^{\mathbf{a}} \mid {m_i}^r$. If, for some $j \in [q]$, $a_j \neq 0$ then ${m_j}^{a_j} \mid {m_i}^r$, which by (3) implies that $j = i$, which results in $\mathbf{a} = r\mathbf{e}_i$. Thus (i) holds.

We now prove (ii). If $r = 1$ the statement is trivial, so assume $r, q \geqslant 2$ and $M \in \mathrm{LCM}(I^r)$ satisfies ${m_i}^{r-1} \mid M$ but $M \neq {m_i}^r$. Then there exists $k \in [q]$ with $k \neq i$ and $m_k \mid M$. Let

$$A = \{k \in [q] \colon m_k \mid M\}.$$

Choose $j \in A \setminus \{i\}$ such that m_j has the fewest number of variables not dividing m_i.

Now suppose for some $\mathbf{a} = (a_1, \ldots, a_q) \in \mathcal{N}_q^r$,

$$\mathbf{m}^{\mathbf{a}} = {m_1}^{a_1} \cdots {m_q}^{a_q} \mid {m_i}^{r-1}m_j \tag{4}$$

If $a_j \geqslant 2$, by canceling a copy of m_j in (4) we will have $m_j^{a_j-1} \mid {m_i}^{r-1}$ which by (3) implies that $i = j$, a contradiction. Similarly if $a_i = r$ then by canceling a copy of

${m_i}^{r-1}$ in (4) we obtain $m_i \mid m_j$, again a contradiction since m_i and m_j are minimal generators. So we must have $a_j \leqslant 1$ and $a_i \leqslant r-1$.

Now we claim that we must have $a_j = 1$ or $a_i = r-1$. Otherwise, if $a_j = 0$ and $a_i \leqslant r-2$, since $a_1 + \cdots + a_q = r$, there must exist $c, d \in [q] \setminus \{i, j\}$ with $a_c, a_d > 0$ (c and d could be equal). In particular by (4)

$$m_c m_d \mid {m_i}^{r-1} m_j. \tag{5}$$

Let

$$m_c = \gcd(m_c, m_i)\cdot m'_c, \quad m_d = \gcd(m_d, m_i)\cdot m'_d \quad \text{and} \quad m_j = \gcd(m_j, m_i)\cdot m'_j. \tag{6}$$

From (6) one can see that m'_c and m'_d do not share any variables with m_i, and so by (5)

$$m'_c m'_d \mid m'_j \tag{7}$$

and also

$$m'_c \mid m'_j \mid m_j \mid M \quad \text{and} \quad m'_d \mid m'_j \mid m_j \mid M. \tag{8}$$

On the other hand $m_i \mid M$, so (6) and (8), together with the fact that m_c and m_d are square-free, imply that

$$m_c \mid M \quad \text{and} \quad m_d \mid M,$$

which in turn implies that $c, d \in A$. Assume without loss of generality that $\deg m'_c \leqslant \deg m'_d$. The fact that $j \in A$ was picked so that m_j has the fewest number of variables outside m_i and (7) together imply

$$\deg m'_c + \deg m'_d \leqslant \deg m'_j \leqslant \deg m'_c \leqslant \deg m'_d;$$

a contradiction since m'_c, m'_d and m'_j all have positive degrees.

So the only possibilities are $a_j = 1$ or $a_i = r-1$. If $a_j = 1$ then by (i) we have $\mathbf{m}^{\mathbf{a}} = {m_i}^{r-1} m_j$, and we are done. If $a_i = r-1$ we have $\mathbf{m}^{\mathbf{a}} = {m_i}^{r-1} m_k \mid {m_i}^{r-1} m_j$ for some $k \in [q]$, hence by cancellation $k = j$ and $\mathbf{m}^{\mathbf{a}} = {m_i}^{r-1} m_j$.

The remaining statements now follow immediately from (i), (ii) and Definition 5.1.

One additional lemma is necessary before the statement of our main result.

Lemma 5.7. *Using Notation* 4.1, *let* $i, j \in [q]$ *with* $i \neq j$, *and let* $\mathbf{a} = (a_1, \ldots, a_q)$ *and* $\mathbf{b} = (b_1, \ldots, b_q)$ *be in* $\mathcal{N}_q^r$. *If* α *is a non-negative integer such that*

$$a_i \geqslant \alpha \quad \textit{and} \quad b_j > r - \alpha$$

Then

(i) $\mathbf{m}^{\mathbf{b}} \nmid \mathbf{m}^{\mathbf{a}}$;
(ii) ${m_i}^{\alpha} {m_j}^{a_i-\alpha} \frac{\mathbf{m}^{\mathbf{a}}}{{m_i}^{a_i}} \mid \operatorname{lcm}(\mathbf{m}^{\mathbf{a}}, \mathbf{m}^{\mathbf{b}})$;
(iii) $\deg({m_i}^{\alpha} {m_j}^{a_i-\alpha} \frac{\mathbf{m}^{\mathbf{a}}}{{m_i}^{a_i}}) \leqslant \deg(\mathbf{m}^{\mathbf{a}}) \iff \deg(m_i) \geqslant \deg(m_j)$ *or* $a_i = \alpha$.

Proof. To prove (i), assume $\mathbf{m}^{\mathbf{b}} \mid \mathbf{m}^{\mathbf{a}}$. Set

$$\mathbf{a}' = \mathbf{a} - a_i \mathbf{e}_i \in \mathcal{N}_q^{r-a_i}$$

so that

$$\mathbf{m}^{\mathbf{a}} = \mathbf{m}^{\mathbf{a}'} {m_i}^{a_i}.$$

Let x be a variable such that $x \mid m_j$. Then $x^{b_j} \mid \mathbf{m}^{\mathbf{b}}$ and hence $x^{b_j} \mid \mathbf{m}^{\mathbf{a}}$. Suppose that x does not divide m_i. Then $x^{b_j} \mid \mathbf{m}^{\mathbf{a}'}$. Since $\mathbf{m}^{\mathbf{a}'}$ is a product of $r - a_i$ square-free monomials, we have then

$$b_j \leqslant r - a_i.$$

The hypothesis on a_i and b_j gives

$$r - a_i \leqslant r - \alpha < b_j\,.$$

The contradiction that arises shows $x \mid m_i$. We conclude $m_j \mid m_i$, hence $i = j$, a contradiction.

Now we prove (ii). Set

$$\mathbf{a}' = \mathbf{a} - a_i\mathbf{e}_i \in \mathcal{N}_q^{r-a_i} \quad \text{ and } \quad \mathbf{b}' = \mathbf{b} - b_j\mathbf{e}_j \in \mathcal{N}_q^{r-b_j}$$

so that

$$\mathbf{m}^{\mathbf{a}} = \mathbf{m}^{\mathbf{a}'}{m_i}^{a_i} \quad \text{ and } \quad \mathbf{m}^{\mathbf{b}} = \mathbf{m}^{\mathbf{b}'}{m_j}^{b_j}\,.$$

It will be shown that

$$\deg_x({m_i}^{\alpha}{m_j}^{a_i-\alpha}\mathbf{m}^{\mathbf{a}'}) \leqslant \max\{\deg_x(\mathbf{m}^{\mathbf{a}}),\, \deg_x(\mathbf{m}^{\mathbf{b}})\}$$

for all variables x, where for a monomial M, $\deg_x(M) = \max\{t\colon x^t \mid M\}$.

Let $u = \deg_x(\mathbf{m}^{\mathbf{a}'})$ and $v = \deg_x(\mathbf{m}^{\mathbf{b}'})$. Since $\mathbf{m}^{\mathbf{a}'}$ is a product of $r - a_i$ square-free monomials, and $\mathbf{m}^{\mathbf{b}'}$ is a product of $r - b_j$ square-free monomials, we have

$$0 \leqslant u \leqslant r - a_i \quad \text{ and } \quad 0 \leqslant v \leqslant r - b_j. \tag{9}$$

Also note that

$$\deg_x({m_i}^{\alpha}{m_j}^{a_i-\alpha}\mathbf{m}^{\mathbf{a}'}) = \alpha \cdot \deg_x(m_i) + (a_i - \alpha)\cdot \deg_x(m_j) + u.$$

- If $x \mid m_i$, then

$$\begin{aligned}
\deg_x({m_i}^{\alpha}{m_j}^{a_i-\alpha}\mathbf{m}^{\mathbf{a}'}) &= \alpha + (a_i - \alpha)\deg_x(m_j) + u\\
&\leqslant \alpha + (a_i - \alpha) + u = a_i + u\\
&= \deg_x(\mathbf{m}^{\mathbf{a}})\\
&\leqslant \max\{\deg_x(\mathbf{m}^{\mathbf{a}}), \deg_x(\mathbf{m}^{\mathbf{b}})\}.
\end{aligned}$$

- If $x \nmid m_i$, but $x \mid m_j$, then

$$\begin{aligned}
\deg_x({m_i}^{\alpha}{m_j}^{a_i-\alpha}\mathbf{m}^{\mathbf{a}'}) &= (a_i - \alpha) + u\\
&\leqslant (a_i - \alpha) + (r - a_i) \quad \text{by (9)}\\
&= r - \alpha < b_j \leqslant b_j + v\\
&= \deg_x(\mathbf{m}^{\mathbf{b}})\\
&\leqslant \max\{\deg_x(\mathbf{m}^{\mathbf{a}}), \deg_x(\mathbf{m}^{\mathbf{b}})\}.
\end{aligned}$$

- If $x \nmid m_i$ and $x \nmid m_j$, then

$$\deg_x({m_i}^{\alpha}{m_j}^{a_i-\alpha}\mathbf{m}^{\mathbf{a}'}) = \deg_x(\mathbf{m}^{\mathbf{a}'}) = \deg_x(\mathbf{m}^{\mathbf{a}}) \leqslant \max\{\deg_x(\mathbf{m}^{\mathbf{a}}), \deg_x(\mathbf{m}^{\mathbf{b}})\}.$$

This finishes the proof of (*ii*).

Finally, (iii) follows directly from the fact that

$$\begin{aligned}
\deg({m_i}^{\alpha}{m_j}^{a_i-\alpha}\mathbf{m}^{\mathbf{a}'}) &= \alpha\cdot \deg m_i + (a_i - \alpha)\cdot \deg m_j + \deg(\mathbf{m}^{\mathbf{a}'})\\
&= \alpha\cdot \deg m_i + (a_i - \alpha)\cdot \deg m_j + \deg(\mathbf{m}^{\mathbf{a}}) - a_i \cdot \deg m_i\\
&= \deg(\mathbf{m}^{\mathbf{a}}) - (a_i - \alpha)(\deg(m_i) - \deg(m_j)).
\end{aligned}$$

Remark 5.8. Recall that $s = \left\lceil \frac{r}{2} \right\rceil$. When $\alpha = s$, the condition on b_j in Lemma 5.7 can be translated as follows:

$$b_j > r - s \iff \begin{cases} b_j \geqslant s & \text{when } r \text{ is odd}\\ b_j > s & \text{when } r \text{ is even.}\end{cases}$$

Recall that, by Theorem 3.6, if Δ is a quasi-tree with vertices labeled with the monomial generating set of a monomial ideal $I \subseteq \mathsf{k}[x_1, \dots, x_n]$, then Δ supports a resolution of I if and only if Δ_M is connected for every monomial M. It is easy to check that the connectivity need only be verified for M in $\mathrm{LCM}(I)$. We now use this criterion to prove the main result of the paper.

Theorem 5.9 (**Main Result**). *Let $q > 1$ and let I be a monomial ideal minimally generated by square-free monomials $m_1, \dots, m_q$. By labeling each vertex $\mathbf{a}$ of the simplicial complexes below with the monomial $\mathbf{m}^{\mathbf{a}}$, the following hold for any $r > 1$:*

(i) *L^r_q supports a free resolution of I^r;*
(ii) *$\mathsf{L}^r(I)$ supports a free resolution of I^r.*

Proof. In this proof, we let L denote either L^r_q or $\mathsf{L}^r(I)$. Fix L as one of these two choices. Let V denote the set of vertices of L and let $\{m_1, \dots, m_q\}$ be the minimal monomial generating set of I. Following Notation 4.1, for $\mathbf{a} = (a_1, \dots, a_q) \in \mathcal{N}^r_q$ we let $\mathbf{m}^{\mathbf{a}} = {m_1}^{a_1} \cdots {m_q}^{a_q}$ and $\mathbf{e}_1, \dots, \mathbf{e}_q$ denote the standard basis vectors for R^q.

We will show that for every M in $\mathrm{LCM}(I^r)$, L_M is empty or connected, where L_M is the induced subcomplex of the complex L on the set

$$V_M = \{\mathbf{a} \in V \colon \mathbf{m}^{\mathbf{a}} \mid M\}\,.$$

By Proposition 4.6, L^r_q is a quasi-tree and by Proposition 3.3, $\mathsf{L}^r(I)$ is a quasi-forest, which is connected and thus a quasi-tree. In view of Theorem 3.6, we can then conclude that L supports a free resolution of I^r.

Suppose $M \in \mathrm{LCM}(I^r)$ and $\mathsf{L}_M \neq ?$. If $M = {m_i}^r$ for some $i \in [q]$, then L_M is a point by Proposition 5.6 (i), and hence is connected.

Assume $M \neq {m_i}^r$ for all $i \in [q]$. Note that this implies $q > 1$.

The facets of L_M are the maximal sets, under inclusion, among the sets

$$F^r_1 \cap V_M, \dots, F^r_q \cap V_M, G^r_1 \cap V_M, \dots, G^r_q \cap V_M. \tag{10}$$

Note that not all these sets are facets of L_M, but all facets of L_M are among those listed in (10). We refer to the facets of L_M of form $F^r_i \cap V_M$ as *the first layer*, and those of the form $G^r_i \cap V_M$ as *the second layer*. We refer to $B^r \cap V_M$ as the *base* of L_M. The base $B^r \cap V_M$ could become empty, depending on the choice of M.

We use the faces in (10) to argue the connectedness of L_M as follows: Claim 1 below shows that any facet of L_M in the second layer is connected to a facet in the first layer. Claim 2 implies that any two facets of L_M that are in the first layer connect through the nonempty base. The combination of these two facts implies that L_M is connected, which will end our proof.

Claim 1: (The second layer facets of L_M intersect the first layer facets). *For any $i \in [q]$ we have:*

$$G^r_i \cap V_M \neq ? \implies G^r_i \cap F^r_i \cap V_M \neq ?.$$

Proof of Claim 1. Assume $G^r_i \cap V_M \neq ?$ for some $i \in [q]$. By Definition 4.2, there exists $a \in [q]$ such that

$$\mathbf{d} = (r-1)\mathbf{e}_i + \mathbf{e}_a \in G^r_i \cap V_M\,.$$

If $i \neq a$, then $\mathbf{d} \in F^r_i$ as well, and hence $G^r_i \cap F^r_i \cap V_M \neq ?$ as desired.

Assume $a = i$. Then $\mathbf{d} \in V_M$ implies ${m_i}^r \mid M$. Since $M \neq {m_i}^r$, Proposition 5.6 guarantees that there exists $j \in [q]$ with $j \neq i$ and $m_j \mid M$ so that $(r-1)\mathbf{e}_i + \mathbf{e}_j \in V$.

Since ${m_i}^r \mid M$ and $m_j \mid M$, we see that ${m_i}^{r-1} m_j \mid M$. Indeed, we set $M = {m_i}^r M'$ and, since m_j is square-free, we have $m_j \mid m_i M'$, hence ${m_i}^{r-1} m_j \mid {m_i}^r M' = M$. Thus we have

$$(r-1)\mathbf{e}_i + \mathbf{e}_j \in G^r_i \cap F^r_i \cap V_M.$$

Claim 2: (The first layer facets connect through the base of L_M). *For any $i, j \in [q]$ with $i \neq j$ we have:*

$$\text{if} \quad (F_i^r \cup G_i^r) \cap V_M \neq \emptyset \quad \text{and} \quad (F_j^r \cup G_j^r) \cap V_M \neq \emptyset \quad \text{then} \quad B^r \cap V_M \neq \emptyset .$$

Proof of Claim 2. Assume for some $i, j \in [q]$ with $i \neq j$ that

$$(F_i^r \cup G_i^r) \cap V_M \neq \emptyset \quad \text{and} \quad (F_j^r \cup G_j^r) \cap V_M \neq \emptyset .$$

Without loss of generality, assume $\deg(m_i) \geqslant \deg(m_j)$. We choose

$$\mathbf{a} = (a_1, \ldots, a_q) \in (F_i^r \cup G_i^r) \cap V_M \quad \text{and} \quad \mathbf{b} = (b_1, \ldots, b_q) \in (F_j^r \cup G_j^r) \cap V_M$$

such that

$$\deg(\mathbf{m}^{\mathbf{a}}) = \min\{\deg(\mathbf{m}^{\mathbf{d}}) \mid \ \mathbf{d} \in (F_i^r \cup G_i^r) \cap V_M\} . \tag{11}$$

Assume, by way of contradiction, that $B^r \cap V_M = \emptyset$. Since $\mathbf{a}, \mathbf{b} \in V_M$, we have then $\mathbf{a}, \mathbf{b} \notin B^r$. Therefore,

$$s < a_i, b_j \leqslant r \quad \text{and} \quad \mathbf{m}^{\mathbf{a}}, \mathbf{m}^{\mathbf{b}} \mid M.$$

Set

$$\mathbf{a}' = \mathbf{a} - a_i \mathbf{e}_i \in \mathcal{N}_q^{r-a_i} \text{ and } \mathbf{c} := (c_1, \ldots, c_q) = s\mathbf{e}_i + (a_i - s)\mathbf{e}_j + \sum_{1 \leqslant k \leqslant q, k \neq i} a_k \mathbf{e}_k \in \mathcal{N}_q^r .$$

Since $b_j > s \geqslant r - s$, by Lemma 5.7 (ii) and Remark 5.8,

$$\mathbf{m}^{\mathbf{c}} = {m_i}^s {m_j}^{a_i - s} \mathbf{m}^{\mathbf{a}'} \mid \operatorname{lcm}(\mathbf{m}^{\mathbf{a}}, \mathbf{m}^{\mathbf{b}}) \mid M. \tag{12}$$

Moreover $\mathbf{c} \in B^r$, because

$$\begin{aligned} c_i &= s \\ c_j &= a_i - s + a_j \leqslant a_i - s + (r - a_i) = r - s \leqslant s && \text{since } a_i + a_j \leqslant r \\ c_k &= a_k \leqslant r - a_i < r - s \leqslant s && \text{for all } k \neq i, j. \end{aligned}$$

If $\mathsf{L} = \mathsf{L}_q^r$, then $V = \mathcal{N}_q^r$. Since (12) shows $\mathbf{c} \in V_M$, we conclude $\mathbf{c} \in B^r \cap V_M$, a contradiction. This finishes the proof of part (i).

Assume now $\mathsf{L} = \mathsf{L}^r(I)$. Since we assumed $B^r \cap V_M = \emptyset$, we must have $\mathbf{c} \notin V$. Definition 5.1 implies then

$$\mathbf{m}^{\mathbf{c}'} \mid \mathbf{m}^{\mathbf{c}} \quad \text{for some} \quad \mathbf{c}' := (c'_1, \ldots, c'_q) \in V.$$

Note that $\mathbf{m}^{\mathbf{c}'} \mid M$ by (12) and hence $\mathbf{c}' \in V_M$. Since $B^r \cap V_M = \emptyset$, we must have $\mathbf{c}' \notin B^r$. Using the notation in Definition 5.1, let V_a be the equivalence class of which $\mathbf{c}'$ is a representative. If $\mathbf{m}^{\mathbf{c}'} = \mathbf{m}^{\mathbf{c}}$, then $\mathbf{c} \in V_a$ as well, and hence $V_a \cap B^r \neq \emptyset$. Then Step 1 in Definition 5.1 implies $\mathbf{c}' \in B^r$, a contradiction. Hence $\mathbf{m}^{\mathbf{c}'} \neq \mathbf{m}^{\mathbf{c}}$.

Since $\mathbf{c}' \notin B^r$, there exists $k \in [q]$ such that $c'_k > s \geqslant r - s$. Since $c_i = s$ and $\mathbf{m}^{\mathbf{c}'} \mid \mathbf{m}^{\mathbf{c}}$, Lemma 5.7 (i) implies $k = i$. In particular, $c'_i > s$, and thus

$$\mathbf{c}' \in (F_i^r \cup G_i^r) \cap V_M .$$

Since $\mathbf{m}^{\mathbf{c}'}$ strictly divides $\mathbf{m}^{\mathbf{c}} = {m_i}^s {m_j}^{a_i - s} \mathbf{m}^{\mathbf{a}'}$, using Lemma 5.7 (iii), we have

$$\deg(\mathbf{m}^{\mathbf{c}'}) < \deg(\mathbf{m}^{\mathbf{c}}) \leqslant \deg(\mathbf{m}^{\mathbf{a}}),$$

contradicting the assumption made in (11). This finishes the proof of part (ii).

Remark 5.10. One may feel that part (i) of the statement above is weaker than (ii), since $\mathsf{L}^r(I) \subseteq \mathsf{L}^r_q$. However, the remarkable aspect of (i) is that, before labeling, L^r_q does not depend on the ideal I. Thus the same topological structure, depending only on r and q, supports a free resolution of the r^{th} power of **any** ideal generated by q square-free monomials. The idea is that L^r_q provides an alternative notion to the Taylor complex for powers of I: when $r = 1$, L^r_q is the Taylor simplex, and when $r > 1$, L^r_q is significantly smaller than the Taylor simplex but still supports a resolution of I^r.

6. Bounds on Betti numbers

One of the key applications of Theorem 5.9 is that we are able to improve the bounds on Betti numbers for powers of ideals from that given by the standard Taylor resolution of I^r since

$$\mathsf{L}^r(I) \subseteq \mathsf{L}^r_q \subseteq \mathrm{Taylor}(I^r).$$

This section contains computations that illustrate the extent to which our results improve the Taylor bounds.

When $r = 2$, we were able to provide a concrete formula for the number of faces of $\mathsf{L}^r(I)$ in [3], and as a result we provided bounds for Betti numbers of I^2, which are shown in Proposition 7.11 to be sharp. When $r > 2$, however, such formulas are not as easy to write and the numbers are very large even for small examples. In this case we shift our attention to L^r_q. While the bound on the Betti numbers stemming from $\mathsf{L}^r(I)$ is dependent on the relations among the generators of I, one can use the structure of L^r_q to get a general, albeit weaker, bound on $\beta_t(I^r)$.

Theorem 6.1 (**Bounds on Betti numbers of I^r**). *If I is a square-free monomial ideal with q minimal generators, then the Betti numbers of I^r for $r \geqslant 2$ are bounded above by*

$$\beta_t(I^r) \leqslant q\left(\binom{q-1}{t} + \binom{f}{t+1} - \binom{b}{t+1}\right) + \binom{b}{t+1}$$

where $t \geqslant 0$, b is the coefficient of x^r in the expansion of

$$(1 + x + x^2 + \cdots + x^{\lceil \frac{r}{2} \rceil})^q,$$

and

$$f = \frac{\binom{q+r-1}{r} - b - q}{q} + b.$$

In particular, $\mathrm{pd}(I^r) \leqslant \max\{q-1, f-1\}$.

Proof. Let $s = \left\lceil \frac{r}{2} \right\rceil$. Note that $\beta_t(I^r)$ is bounded above by the number of faces of dimension t of L^r_q. To count the faces of dimension t, we use the sets B^r, F^r_i, and G^r_i from Definition 4.2.

Observe that the coefficient b of x^r in the expansion of

$$(1 + x + x^2 + \cdots + x^s)^q$$

is exactly the number of q-tuples $(a_1, \ldots, a_q)$ with $a_i \leqslant s$ and $a_1 + \cdots + a_q = r$, in other words $b = |B^r|$.

Note that the vertices of L^r_q are formed by selecting r of the original q generators, so using the standard formula for counting with repetition, we have

$$|V(\mathsf{L}^r_q)| = \binom{q+r-1}{r}.$$

Now $|F^r_i| = |F^r_j|$ for all i, j, so let $f = |F^r_i|$. Since $|G^r_i \setminus F^r_i| = 1$, there are $|V(\mathsf{L}^r_q)| - q$ vertices in $\cup_{i=1}^q F^r_i$. Also $F^r_i \cap F^r_j = B^r$ for all i, j such that $i \neq j$, so we have

$$(13)\qquad f = |F_i^r| = \frac{1}{q}\big((|V(\mathsf{L}_q^r)| - q) - |B^r|\big) + |B^r| = \frac{\binom{q+r-1}{r} - q - b}{q} + b.$$

To count the number of faces of dimension t, that is, faces with $t+1$ vertices, in L_q^r, we separate the faces into three distinct types.

(i) Faces containing ${m_i}^r$ for some i: by the definition of L_q^r, faces of this type must be contained in G_i^r. Since the vertex corresponding to ${m_i}^r$ has been fixed, t additional vertices of G_i^r are needed. There are $q-1$ such vertices since all vertices of G_i^r have the form $(r-1)\mathbf{e}_i + \mathbf{e}_j$ where $1 \leqslant j \leqslant q$. Since there are q choices for i, there are
$$q\binom{q-1}{t}$$
such faces.

(ii) Faces contained in B^r: setting $b = |B^r|$ as above, there are
$$\binom{b}{t+1}$$
such faces.

(iii) Faces contained in F_i^r but not in B^r: recalling that $f = |F_i^r|$ and $B^r \subseteq F_i^r$, there are
$$q\left(\binom{f}{t+1} - \binom{b}{t+1}\right)$$
such faces.

Combining the three cases, we see that there are
$$q\left(\binom{q-1}{t} + \binom{f}{t+1} - \binom{b}{t+1}\right) + \binom{b}{t+1}$$
faces of L_q^r of dimension t. Thus for I^r,
$$\beta_t(I^r) \leqslant q\left(\binom{q-1}{t} + \binom{f}{t+1} - \binom{b}{t+1}\right) + \binom{b}{t+1}.$$

In particular, if $t > q-1$ and $t+1 > f$, we must have $\beta_t(I^r) = 0$. Thus
$$(14)\qquad \operatorname{pd}(I^r) \leqslant \max\{q-1, f-1\}.$$

Corollary 6.2. *If an ideal I is minimally generated by q square-free monomials, then the Betti numbers of I^r are bounded by*
$$\beta_t(I^r) \leqslant q\binom{q-1}{t} + \binom{b}{t+1}$$
for $r = 2, 3$, where b is as defined in Theorem 6.1. *In the case where $r = 2$, $b = \binom{q}{2}$ and the bound reduces to the bound given in* [3].

Proof. When $r = 2$ or $r = 3$, then $F_i^r = B^r$ for all i, so $b = f$ and the result follows immediately from simplifying the formula in Theorem 6.1. Moreover, when $r = 2$, the coefficient of x^2 in the binomial expansion of $(1+x)^q$ is $\binom{q}{2}$ as stated. The reduction is then evident.

Notice that when $r = 2$, the bound is known to be sharp; it agrees with the result in [3]. In Proposition 7.11 we characterize the values of r and q for which these bounds are sharp, i.e. can be realized by some ideal.

Example 6.3. As a first example, we examine cases where r or q is small. We first consider b and f for small values of q.

- For $q = 2$, computing $(1 + x + x^2 + \cdots + x^s)^2$ shows that $b = 1$ when $r = 2s$ is even and $b = 2$ when $r = 2s - 1$ is odd. If r is even, we then have

$$f = \frac{\binom{2+r-1}{r} - 2 - 1}{2} + 1 = \frac{r + 1 - 2 - 1}{2} + 1 = \frac{r}{2} = s. \tag{15}$$

  A similar computation shows that when r is odd, $f = s$ as well.
- If $q = 3$ and $r = 3$ or $r = 4$ (so that $s = 2$), then computing $(1 + x + x^2)^3$ yields $b = 7$ and $f = 7$ when $r = 3$ and $b = 6$ and $f = 8$ when $r = 4$.

Applying the equations from Theorem 6.1 yields:

- For $r = 2$, any q, and any t,
$$\beta_t(I^r) \ 6 \ q\binom{q-1}{t} + \binom{\frac{1}{2}q(q-1)}{t+1}.$$
- For any r, $q = 2$, and any $t > 2$, if $s = \lceil r/2 \rceil$,
$$\beta_t(I^r) \ 6 \ 2\binom{s}{t+1}.$$
- For $r = q = 3$ and any t,
$$\beta_t(I^r) \ 6 \ 3\binom{2}{t} + \binom{7}{t+1}.$$
- For $r = 4$, $q = 3$ and any $t > 3$,
$$\beta_t(I^r) \ 6 \ 3\binom{8}{t+1} - 2\binom{6}{t+1}.$$

Remark 6.4. In (14) it is useful to understand which of the integers $q - 1$, $f - 1$ achieves the maximum. For small values of q and r, it is possible to have $q > f$. However, the opposite holds in most cases. More precisely, we show below that if r, q satisfy any of the following assumptions:

(i) $q = 2$ and $r > 5$;
(ii) $q = 3$ and $r > 3$;
(iii) $q > 4$ and $r > 2$,

then $f > q$, and hence
$$\mathrm{pd}(I^r) \ 6 \ f - 1 = \dim \mathsf{L}^r_q\,.$$

In the case $q = 2$, (15) shows $f = s = \lceil \frac{r}{2} \rceil$. If $r > 5$, we must have $s > 2$, thus $f = s > 2 = q$. This settles (i).

Now suppose $q > 2$. If $r = 2$, then Corollary 6.2 shows that $f = b = \binom{q}{2}$. When $q > 4$, we have $f = \binom{q}{2} > q$. Thus, to show (ii) and (iii) it remains to consider the case when $q > 3$ and $r > 3$. Observe that when $r > 3$, $s > 2$. In this scenario, since b is the coefficient of x^r in $(1 + x + \cdots + x^s)^q$ and $q > 2$, then $b > 2$ and $bq - b > q$.

To see that $f > q$ holds, we will show $f - q > 0$, which by (13) amounts to
$$\frac{\binom{q+r-1}{r} - q - b}{q} + b - q = \frac{\binom{q+r-1}{r} - q - b + bq - q^2}{q} > 0\,.$$

It is sufficient to show the numerator is positive, so we observe the following, where the first inequality results from $bq - b > q$ and the second follows since $r > 3$:

$$\begin{aligned}\binom{r+q-1}{q-1} - q - b + bq - q^2 &> \binom{r+q-1}{q-1} - q^2 \\ &> \binom{q+2}{q-1} - q^2 \\ &= \frac{(q+2)(q+1)q}{6} - q^2 = \frac{q(q-1)(q-2)}{6} > 0\,.\end{aligned}$$

This ends our argument.

Example 6.5. In general, the bounds from Theorem 6.1 will be considerably smaller than those provided by the Taylor complex, which is a simplex, and the bound on the projective dimension will also decrease significantly. We display this phenomenon in the table below.

Bound Comparisons						
	$q = 2$, $r = 3$		$q = 3$, $r = 3$		$q = 3$, $r = 4$	
	Thm. 6.1	3-simplex	Thm. 6.1	9-simplex	Thm. 6.1	14-simplex
$\beta_0(I^r)$ 6	4	4	10	10	15	15
$\beta_1(I^r)$ 6	3	6	27	45	60	105
$\beta_2(I^r)$ 6	0	4	38	120	131	455
$\mathrm{pd}(I^r)$ 6	1	3	6	9	7	14

The bounds on Betti numbers and projective dimension given by the complex L^r_q in Theorem 6.1 are most effective when the generating set $\{\mathbf{m}^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{N}^r_q\}$ does not contain redundancies. When there are redundancies, using $\mathsf{L}^r(I)$ will yield improved bounds. We illustrate how to use this improvement by continuing Example 5.2.

Example 6.6. Let $I = (xy, yz, zw, xw) = (m_1, m_2, m_3, m_4)$ as in Example 5.2. Counting faces of size i in the complex $\mathsf{L}^2(I)$ provides a bound on the i^{th} Betti number of I^2. Note that in general, these improvements in the Betti numbers follow from knowledge of the redundancies in $\{\mathbf{m}^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{N}^r_q\}$ and can often be computed from that information using the equivalence classes without explicitly constructing $\mathsf{L}^r(I)$. For instance, a comparison of the bounds is summarized in the table below for I^2 and for I^3 using the first of the two vertex sets for $\mathsf{L}^3(I)$ given in Example 5.2.

Bound Comparisons								
	$r = 2$				$r = 3$			
	$\mathsf{L}^2(I)$	8-simplex	L^2_4 Thm. 6.1	9-simplex	$\mathsf{L}^3(I)$	15-simplex	L^3_4 Thm. 6.1	19-simplex
$\beta_0(I^r)$ 6	9	9	10	10	16	16	20	20
$\beta_1(I^r)$ 6	20	36	27	45	74	120	132	190
$\beta_2(I^r)$ 6	18	84	32	120	224	560	572	1,140
$\mathrm{pd}(I^r)$ 6	4	8	5	9	11	15	15	19

7. Extremal Ideals: When does L^r_q support a minimal resolution?

Based upon the above work, a natural question that arises is the following: for given r and q, can one find ideals I for which I^r has a *minimal* free resolution supported on L^r_q itself? When $r = 1$, L^r_q is the Taylor complex, which one can easily see always supports a minimal free resolution of the ideal generated by q variables $(x_1, x_2, \ldots, x_q)$. In the case where $r > 1$, Proposition 7.11 describes exactly when the bounds for Betti numbers in Theorem 6.1 and Corollary 6.2 are sharp, in the sense that there exist ideals for which equality is attained. We call the ideals which realize these bounds *q-extremal ideals* and denote them by $\mathcal{E}_q$. In fact we can show a much stronger statement in Theorem 7.9: the powers ${\mathcal{E}_q}^r$ have maximal Betti numbers among the ideals I^r where I is generated by q square-free monomials.

Definition 7.1 (**Extremal ideals**). *Let q be a positive integer. For every set A with $\varnothing \neq A \subsetneq [q]$, introduce a variable x_A, and consider the polynomial ring*

$$S_{\mathcal{E}} = \mathsf{k}\big[x_A \colon \varnothing \neq A \subseteq [q]\big]$$

over a field k. For each $i \in [q]$ define a square-free monomial ϵ_i in $S_{\mathcal{E}}$ as

$$\epsilon_i = \prod_{\substack{A \subseteq [q] \\ i \in A}} x_A.$$

*The square-free monomial ideal $\mathcal{E}_q = (\epsilon_1, \ldots, \epsilon_q)$ is called a **q-extremal ideal**.*

When it is unlikely to lead to confusion, we will simplify the notation by writing x_1 for $x_{\{1\}}$, x_{12} for $x_{\{1,2\}}$, etc., and refer to a q-extremal ideal simply as an extremal ideal. The ring $S_{\mathcal{E}}$ has $2^q - 1$ variables, corresponding to the power set of $[q]$ (excluding $\varnothing$), and each ϵ_i is the product of 2^{q-1} variables; that is, those corresponding to the subsets of $[q]$ that contain i. The following example illustrates how this works for $\mathcal{E}_4$.

Example 7.2. When $q = 4$, the ideal $\mathcal{E}_4$ is generated by the monomials

$$\begin{aligned}
\epsilon_1 &= x_1 x_{12} x_{13} x_{14} x_{123} x_{124} x_{134} x_{1234};\\
\epsilon_2 &= x_2 x_{12} x_{23} x_{24} x_{123} x_{124} x_{234} x_{1234};\\
\epsilon_3 &= x_3 x_{13} x_{23} x_{34} x_{123} x_{134} x_{234} x_{1234};\\
\epsilon_4 &= x_4 x_{14} x_{24} x_{34} x_{124} x_{134} x_{234} x_{1234}
\end{aligned}$$

in $\mathsf{k}[x_1, x_2, x_3, x_4, x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, x_{123}, x_{124}, x_{134}, x_{234}, x_{1234}]$.

Using the terminology of [18], it naturally follows that $\mathcal{E}_q$ is precisely the *nearly Scarf ideal* of a $(q-1)$-simplex: we see this by matching the variable $x_{[q] \smallsetminus B}$ with the face B of the simplex.

Following Notation 4.1, let r and q be positive integers and I an ideal generated by square-free monomials $m_1, \ldots, m_q$. If $\mathbf{a} = (a_1, \ldots, a_q) \in \mathsf{N}^q$, set

$$\mathbf{m}^{\mathbf{a}} = {m_1}^{a_1} \cdots {m_q}^{a_q} \quad \text{and} \quad \boldsymbol{\epsilon}^{\mathbf{a}} = {\epsilon_1}^{a_1} \cdots {\epsilon_q}^{a_q}.$$

Observe

$$\boldsymbol{\epsilon}^{\mathbf{a}} = \prod_{j \in [q]} \prod_{\substack{A \subseteq [q] \\ j \in A}} {x_A}^{a_j} = \prod_{\varnothing \neq A \subseteq [q]} (x_A)^{\sum_{j \in A} a_j}. \tag{16}$$

The r^{th} powers I^r and ${\mathcal{E}_q}^r$ are generated by monomials of the form $\mathbf{m}^{\mathbf{a}}$ and $\boldsymbol{\epsilon}^{\mathbf{a}}$, respectively, with $\mathbf{a} \in \mathcal{N}_q^r$. Proposition 7.3 demonstrates that for ${\mathcal{E}_q}^r$, the elements $\boldsymbol{\epsilon}^{\mathbf{a}}$ with $\mathbf{a} \in \mathcal{N}_q^r$ form a minimal generating set. In fact, all faces of L_q^r are necessary in $\mathsf{L}^r(\mathcal{E}_q)$; i.e. none may be removed when constructing $\mathsf{L}^r(\mathcal{E}_q)$.

Proposition 7.3. *Let r and q be positive integers, and $\mathbf{a} = (a_1, \ldots, a_q), \mathbf{b} = (b_1, \ldots, b_q) \in \mathsf{N}^q$. Then*

(i) $\boldsymbol{\epsilon}^{\mathbf{b}} \mid \boldsymbol{\epsilon}^{\mathbf{a}} \iff b_i \leqslant a_i$ *for every $i \in [q]$.*
(ii) *If $\mathbf{a}, \mathbf{b} \in \mathcal{N}_q^r$, then* $\boldsymbol{\epsilon}^{\mathbf{b}} \mid \boldsymbol{\epsilon}^{\mathbf{a}} \iff \mathbf{b} = \mathbf{a}$.

In particular, if $\mathbf{a}, \mathbf{b} \in \mathcal{N}_q^r$, then then $\mathsf{L}_q^r = \mathsf{L}^r(\mathcal{E}_q)$.

Proof. To prove (i), use (16) to justify the first equivalence below.

$$\boldsymbol{\epsilon}^{\mathbf{b}} \mid \boldsymbol{\epsilon}^{\mathbf{a}} \iff \sum_{j\in A} b_j \leqslant \sum_{j \in A} a_j \text{ for each } \emptyset \neq A \subseteq [q] \iff b_j \leqslant a_j \text{ for each } j \in [q].$$

Now (ii) follows from (i) and the added assumption

$$b_1 + \cdots + b_q = a_1 + \cdots + a_q = r.$$

The final claim follows directly from Definition 5.1.

In general, given the discussions above (see also [13]), for every r and q we have

$$\mathsf{L}^r(\mathcal{E}_q) = \mathsf{L}^r_q \subseteq \mathrm{Taylor}({\mathcal{E}_q}^r), \tag{17}$$

where we assume that every vertex $\mathbf{a}$ of L^r_q is labeled with a generator $\boldsymbol{\epsilon}^{\mathbf{a}}$ of ${\mathcal{E}_q}^r$, and each face σ is labeled with the lcm of the labels of its vertices, denoted M_σ. The following observation will be useful when working with the monomial label M_σ of $\sigma \in \mathrm{Taylor}({\mathcal{E}_q}^r)$.

Remark 7.4. Let $\sigma = \{\boldsymbol{\epsilon}^{\mathbf{a}_1}, \ldots, \boldsymbol{\epsilon}^{\mathbf{a}_t}\} \in \mathrm{Taylor}({\mathcal{E}_q}^r)$ and set $\mathbf{a}_i = (a_{i1}, a_{i2}, \ldots, a_{iq})$ for $i \in [t]$. Using (16), we then have

$$M_\sigma = \mathrm{lcm}\Big(\prod_{\emptyset \neq A \subseteq [q]} (x_A)^{\sum_{j\in A} a_{ij}} : i \in [t] \Big) = \prod_{\emptyset \neq A \subseteq [q]} (x_A)^{\max\limits_{1 \leqslant i \leqslant t} \sum_{j \in A} a_{ij}} . \tag{18}$$

Furthermore, if $\mathbf{c} = (c_1, \ldots, c_q) \in \mathcal{N}^r_q$, (16) and (18) give

$$\boldsymbol{\epsilon}^{\mathbf{c}} \mid M_\sigma \iff \sum_{j \in A} c_j \leqslant \max_{1 \leqslant i \leqslant t} \sum_{j \in A} a_{ij} \qquad \text{for all} \quad \emptyset \neq A \subseteq [q]. \tag{19}$$

We show in Theorem 7.9 that powers of extremal ideals attain maximal Betti numbers among powers of all square-free monomial ideals with the same number of generators. To prove this, we first define a ring homomorphism $S_{\mathcal{E}} \to S$ and discuss its properties.

Definition 7.5 (**The ring homomorphism ψ_I**). *Let I be an ideal of the polynomial ring $S = \mathsf{k}[x_1, \ldots, x_n]$ minimally generated by square-free monomials $m_1, \ldots, m_q$. For each $k \in [n]$, set*

$$A_k = \{ j \in [q] \colon x_k \mid m_j \} .$$

We have thus

$$j \in A_k \iff x_k \mid m_j .$$

Define ψ_I to be the ring homomorphism

$$\psi_I \colon S_{\mathcal{E}} \to S \qquad \text{where} \qquad \psi_I(x_A) = \begin{cases} \displaystyle\prod_{\substack{k \in [n] \\ A = A_k}} x_k & \text{if } A = A_k \text{ for some } k \in [n], \\ 1 & \text{otherwise.} \end{cases}$$

Before proceeding directly with our work on extremal ideals, we illustrate how the map ψ_I works in a sample case where there are four generators and seven variables.

Example 7.6. Let I be the ideal of $\mathsf{k}[x_1, \ldots, x_7]$ generated by the square-free monomials

$$m_1 = x_1x_2x_5x_7, \quad m_2 = x_2x_3x_7, \quad m_3 = x_3x_4x_6, \quad m_4 = x_1x_4.$$

Since $n = 7$ and $q = 4$, it follows that

$$A_1 = \{1,4\},\ A_2 = \{1,2\},\ A_3 = \{2,3\},\ A_4 = \{3,4\},\ A_5 = \{1\},\ A_6 = \{3\},\ A_7 = \{1,2\}.$$

The function

$$\psi_I : \mathsf{k}\big[x_A \colon \emptyset \neq A \subseteq [q]\big] \to \mathsf{k}[x_1, \ldots, x_7]$$

maps

$$\psi_I(x_{\{1,4\}}) = x_1,\, \psi_I(x_{\{1,2\}}) = x_2x_7,\, \psi_I(x_{\{2,3\}}) = x_3,\, \psi_I(x_{\{3,4\}}) = x_4,$$

$$\psi_I(x_{\{1\}}) = x_5, \quad \psi_I(x_{\{3\}}) = x_6, \qquad \text{and} \qquad \psi_I(x_A) = 1 \text{ otherwise.}$$

Observe that

$$\begin{aligned}\psi_I(\epsilon_1) &= \psi_I\Big(\prod_{\substack{A\subseteq[q]\\1\in A}} x_A\Big) = \prod_{\substack{A\subseteq[q]\\1\in A}} \psi_I(x_A)\\ &= \psi_I(x_{\{1,4\}})\psi_I(x_{\{1,2\}})\psi_I(x_{\{1\}}) = x_1(x_2x_7)x_5 = m_1\,.\end{aligned}$$

Generalizing the properties of ψ_I from the example, we arrive at the following lemma.

Lemma 7.7. *Let I be an ideal of the polynomial ring $S = \mathsf{k}[x_1,\dots,x_n]$, minimally generated by square-free monomials $m_1,\dots,m_q$. Then*

(i) $\psi_I(\boldsymbol{\epsilon}^{\mathbf{a}}) = \mathbf{m}^{\mathbf{a}}$ *for each* $\mathbf{a} \in \mathcal{N}_q^r$*;*
(ii) $\psi_I({\mathcal{E}_q}^r)S = I^r$ *for every* $r > 0$*;*
(iii) ψ_I *preserves least common multiples, that is:*

$$\psi_I(\operatorname{lcm}(\boldsymbol{\epsilon}^{\mathbf{a}_1},\dots,\boldsymbol{\epsilon}^{\mathbf{a}_t})) = \operatorname{lcm}(\mathbf{m}^{\mathbf{a}_1},\dots,\mathbf{m}^{\mathbf{a}_t}) \quad \textit{for all} \quad \mathbf{a}_1,\dots,\mathbf{a}_t \in \mathcal{N}_q^r, \quad t > 1.$$

Proof. By definitions 7.1 and 7.5, for every $j \in [q]$

$$\psi_I(\epsilon_j) = \psi_I\Big(\prod_{\substack{A\subseteq[q]\\j\in A}} x_A\Big) = \prod_{\substack{A\subseteq[q]\\j\in A}} \psi_I(x_A) = \prod_{\substack{A\subseteq[q]\\j\in A}} \prod_{\substack{k\in[n]\\A=A_k}} x_k = \prod_{\substack{k\in[n]\\j\in A_k}} x_k = \prod_{\substack{k\in[n]\\x_k|m_j}} x_k = m_j.$$

It follows that $\psi_I(\boldsymbol{\epsilon}^{\mathbf{a}}) = \mathbf{m}^{\mathbf{a}}$ for all $\mathbf{a} \in \mathcal{N}_q^r$, which establishes (i) and also (ii). It remains to show (iii).

Set $\mathbf{a}_i = (a_{i1}, a_{i2},\dots,a_{iq})$, for $i \in [t]$. Using (18) in the first equality below, we have:

$$\begin{aligned}\psi_I(\operatorname{lcm}(\boldsymbol{\epsilon}^{\mathbf{a}_1},\dots,\boldsymbol{\epsilon}^{\mathbf{a}_t})) &= \psi_I\Big(\prod_{? \neq A\subseteq[q]} (x_A)^{\max\limits_{1\,6\,i\,6\,t} \sum_{j\in A} a_{ij}}\Big)\\ &= \prod_{? \neq A\subseteq[q]} \Big(\prod_{\substack{k\in[n]\\A=A_k}} x_k\Big)^{\max\limits_{1\,6\,i\,6\,t} \sum_{j\in A} a_{ij}}\\ &= \prod_{k\in[n],A_k\neq ?} (x_k)^{\max\limits_{1\,6\,i\,6\,t} \sum_{j\in A_k} a_{ij}}\\ &= \operatorname{lcm}\Big(\prod_{k\in[n],A_k\neq ?} (x_k)^{\sum_{j\in A_k} a_{1j}},\dots,\prod_{k\in[n],A_k\neq ?} (x_k)^{\sum_{j\in A_k} a_{tj}}\Big)\\ &= \operatorname{lcm}\Big(\prod_{j\in[q]}\prod_{\substack{k\in[n]\\j\in A_k}} (x_k)^{a_{1j}},\dots,\prod_{j\in[q]}\prod_{\substack{k\in[n]\\j\in A_k}} (x_k)^{a_{tj}}\Big)\\ &= \operatorname{lcm}\Big(\prod_{j\in[q]} m_j^{a_{1j}},\dots,\prod_{j\in[q]} m_j^{a_{tj}}\Big)\\ &= \operatorname{lcm}(\mathbf{m}^{\mathbf{a}_1},\dots,\mathbf{m}^{\mathbf{a}_t}).\end{aligned}$$

Theorem 7.9 below demonstrates why the ideals from Definition 7.5 are called *extremal*: they have the greatest Betti numbers among all ideals minimally generated by q square-free monomials. The following lemma provides the technical preliminaries necessary for the proof of Theorem 7.9.

Lemma 7.8. *Let I be an ideal minimally generated by q square-free monomials in a polynomial ring $\mathsf{k}[x_1, \dots, x_n]$. With notation as in Definition 7.5, if S is viewed as an $S_{\mathcal{E}}$-module via the ring homomorphism $\psi_I \colon S_{\mathcal{E}} \to S$, then*

$$S_{\mathcal{E}}/{\mathcal{E}_q}^r \otimes_{S_{\mathcal{E}}} S \cong S/I^r \quad \textit{and} \quad \operatorname{Tor}_i^{S_{\mathcal{E}}}(S_{\mathcal{E}}/{\mathcal{E}_q}^r, S) = 0 \quad \textit{for all } i > 0.$$

Proof. First, note that

$$\frac{S_{\mathcal{E}}}{{\mathcal{E}_q}^r} \otimes_{S_{\mathcal{E}}} S \cong \frac{S}{({\mathcal{E}_q}^r)S} = \frac{S}{\psi_I({\mathcal{E}_q}^r)S} = \frac{S}{I^r}\,.$$

To compute $\operatorname{Tor}_i^{S_{\mathcal{E}}}(S_{\mathcal{E}}/{\mathcal{E}_q}^r, S)$, use the Taylor complex $\operatorname{Taylor}({\mathcal{E}_q}^r)$, which supports a free resolution F of $S_{\mathcal{E}}/{\mathcal{E}_q}^r$; see (1) for a description of the differentials of F. Since the homomorphism ψ_I changes the labels $\boldsymbol{\epsilon}^{\mathbf{a}}$ to the labels $\mathbf{m}^{\mathbf{a}}$ and preserves least common multiples, the chain complex $\mathsf{F} \otimes_{S_{\mathcal{E}}} S$ is isomorphic to a homogenization of the chain complex associated to the simplex with vertices corresponding to $\mathbf{a} \in \mathcal{N}_q^r$ and labeled with the monomials $\mathbf{m}^{\mathbf{a}}$. This is a version of the Taylor resolution of S/I^r defined starting with a possibly non-minimal generating set of I^r. Such a non-minimal version is a free resolution of S/I^r as well, hence $\operatorname{Tor}_i^{S_{\mathcal{E}}}(S_{\mathcal{E}}/{\mathcal{E}_q}^r, S) = 0$ when $i > 0$. (See Remark 2.6.)

Theorem 7.9 (**Powers of extremal ideals have maximal Betti numbers**). *Given positive integers r and q,*

$$\beta_i^S(I^r) \leqslant \beta_i^{S_{\mathcal{E}}}({\mathcal{E}_q}^r)$$

for any $i \geqslant 0$ and any ideal I minimally generated by q square-free monomials in a polynomial ring S.

Proof. Let F be a minimal free resolution of $S_{\mathcal{E}}/{\mathcal{E}_q}^r$ over $S_{\mathcal{E}}$. Then Lemma 7.8 shows that $\mathsf{F} \otimes_{S_{\mathcal{E}}} S$ is a free resolution of S/I^r over S. Consequently,

$$\beta_i^{S_{\mathcal{E}}}({\mathcal{E}_q}^r) = \operatorname{rank}_{S_{\mathcal{E}}}(\mathsf{F}_{i+1}) = \operatorname{rank}_S(\mathsf{F}_{i+1} \otimes_{S_{\mathcal{E}}} S) \geqslant \beta_i^S(I^r).$$

In view of Theorem 7.9, the homogenized chain complex of any (cell) complex that supports a minimal free resolution of ${\mathcal{E}_q}^r$ can be thought of as an upper bound for the minimal free resolution of the r^{th} power of any ideal minimally generated by q square-free monomials. Proposition 7.10 establishes when our simplicial complex L_q^r supports a minimal free resolution of ${\mathcal{E}_q}^r$.

Proposition 7.10 (**When L_q^r supports a minimal free resolution of ${\mathcal{E}_q}^r$**). *Let r and q be positive integers. The following statements are equivalent:*

(i) *The simplicial complex L_q^r supports a minimal free resolution of the ideal ${\mathcal{E}_q}^r$.*
(ii) $\operatorname{pd}_{S_{\mathcal{E}}}({\mathcal{E}_q}^r) = \dim \mathsf{L}_q^r$.
(iii) *One of the following conditions holds:*
- $q = 1$ *and* $r \geqslant 1$;
- $q = 2$ *and* $1 \leqslant r \leqslant 4$;
- $q \geqslant 3$ *and* $1 \leqslant r \leqslant 2$.

Proof. (i) $\Longrightarrow$ (ii): This implication is clear.

(ii) $\Longrightarrow$ (iii): Assume that $\operatorname{pd}_{S_{\mathcal{E}}}({\mathcal{E}_q}^r) = \dim \mathsf{L}_q^r$ and the values of r and q do not satisfy the conditions in (iii). In particular, we must have $q \geqslant 2$ and $r \geqslant 5$, or $q \geqslant 3$ and $r = 3, 4$. By Remark 6.4 and (13), for all $i, j \in [q]$,

$$|F_i^r| = f > q = |G_j^r| \tag{20}$$

and so $F_1^r, \ldots, F_q^r$ are the facets of L_q^r of highest dimension, with the caveat that $F_1^r = \cdots = F_q^r$ when $r = 3 = q$, and thus $\dim \mathsf{L}_q^r = f - 1$. Since we assumed $\operatorname{pd}_{S_{\mathcal{E}}}({\mathcal{E}_q}^r) = \dim \mathsf{L}_q^r$, we have $\operatorname{pd}_{S_{\mathcal{E}}}({\mathcal{E}_q}^r) = f - 1$ as well. Let F denote the free resolution supported on L_q^r, and let ∂ denote its differential, which is described in (1). Since $\operatorname{pd}_{S_{\mathcal{E}}}({\mathcal{E}_q}^r) = f - 1$ and a minimal free resolution of ${\mathcal{E}_q}^r$ is isomorphic to a direct summand of F, there must exist a basis element $e \in \mathsf{F}_{f-1} \smallsetminus (S_{\mathcal{E}})_{\geqslant 1}\mathsf{F}_{f-1}$ such that

$$\partial(e) \in (S_{\mathcal{E}})_{\geqslant 1}\mathsf{F}_{f-2}\,. \tag{21}$$

As in (1), let e_σ denote the basis element in F corresponding to $\sigma \in \mathsf{L}_q^r$. We write

$$e = \sum_{F \in \mathsf{L}_q^r, |F| = f} \alpha_F e_F$$

with $\alpha_F \in S_{\mathcal{E}}$. The assumption $e \notin (S_{\mathcal{E}})_{\geqslant 1}\mathsf{F}_{f-1}$ implies that α_F is a unit for some $F \in \mathsf{L}_q^r$ with $|F| = f$. By (20), we see that $F = F_i^r$ for some $i \in [q]$. Without loss of generality, assume $i = 1$.

Recall that M_σ denotes the lcm of the monomial labels of the vertices in $\sigma \in \mathsf{L}_q^r$.

Claim. *There exists* $\mathbf{c} \in F_1^r \smallsetminus G_1^r$ *such that* $M_{F_1^r} = M_{F_1^r \smallsetminus \{\mathbf{c}\}}$.

Proof of Claim. Assume first $r > 4$, and $q \geqslant 2$, so that $s = \left\lceil \frac{r}{2} \right\rceil \geqslant 3$. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{N}_q^r$ be such that

$$\boldsymbol{\epsilon}^{\mathbf{a}} = {\epsilon_1}^{r-s}{\epsilon_2}^{s}, \quad \boldsymbol{\epsilon}^{\mathbf{b}} = {\epsilon_1}^{r-s+2}{\epsilon_2}^{s-2} \quad \text{and} \quad \boldsymbol{\epsilon}^{\mathbf{c}} = {\epsilon_1}^{r-s+1}{\epsilon_2}^{s-1},$$

so that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are distinct vertices of L_q^r. Note that $\boldsymbol{\epsilon}^{\mathbf{c}} \mid \operatorname{lcm}(\boldsymbol{\epsilon}^{\mathbf{a}}, \boldsymbol{\epsilon}^{\mathbf{b}})$. Indeed, after removing the common factors, this divisibility is equivalent to

$$\epsilon_1\epsilon_2 \mid \operatorname{lcm}(\epsilon_1^2, \epsilon_2^2)\,,$$

and can be verified using (19).

Now let $r = 3$ or $r = 4$, and $q \geqslant 3$. Then $s = 2$, and if one sets

$$\boldsymbol{\epsilon}^{\mathbf{a}} = {\epsilon_1}^{r-2}{\epsilon_2}^{2}, \quad \boldsymbol{\epsilon}^{\mathbf{b}} = {\epsilon_1}^{r-1}\epsilon_3 \quad \text{and} \quad \boldsymbol{\epsilon}^{\mathbf{c}} = {\epsilon_1}^{r-2}\epsilon_2\epsilon_3,$$

then we see that $\boldsymbol{\epsilon}^{\mathbf{c}} \mid \operatorname{lcm}(\boldsymbol{\epsilon}^{\mathbf{a}}, \boldsymbol{\epsilon}^{\mathbf{b}})$. Indeed, after removing the common factors, this divisibility is equivalent to

$$\epsilon_2\epsilon_3 \mid \operatorname{lcm}(\epsilon_2^2, \epsilon_1\epsilon_3)\,,$$

and can be verified using (19).

In both cases, we have $\mathbf{a}, \mathbf{b}, \mathbf{c} \in F_1^r$ and $\mathbf{c} \notin G_1^r$. The divisibility $\boldsymbol{\epsilon}^{\mathbf{c}} \mid \operatorname{lcm}(\boldsymbol{\epsilon}^{\mathbf{a}}, \boldsymbol{\epsilon}^{\mathbf{b}})$ establishes the conclusion of the Claim.

We finish the proof of (ii) $\Longrightarrow$ (iii) as follows. We have

$$\partial(e) = \sum_{F \in \mathsf{L}_q^r, |F| = f} \alpha_F \partial(e_F) = \sum_{F \in \mathsf{L}_q^r, |F| = f} \alpha_F \left(\sum_{\mathbf{c}' \in F} \pm \frac{M_F}{M_{F \smallsetminus \{\mathbf{c}'\}}} e_{F \smallsetminus \{\mathbf{c}'\}} \right). \tag{22}$$

Let $\mathbf{c}$ be as in the Claim and let $F \in \mathsf{L}_q^r$ with $|F| = f$. By (20), we have $F = F_j^r$ for some j. If $\mathbf{c}' \in F$, observe

$$F_1^r \smallsetminus \{\mathbf{c}\} = F \smallsetminus \{\mathbf{c}'\} \iff F_1^r = F \text{ and } \mathbf{c} = \mathbf{c}'. \tag{23}$$

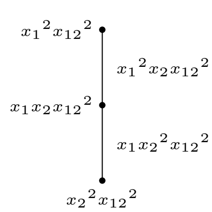
To prove this statement, we refer to Section 4 for basic properties of the sets F_i^r, G_i^r and B^r. Indeed, (23) is clear when $r = 3$, since $F_1^r = B^r = F_j^r$ for all $j \in [q]$ in this case. Assume now $r > 3$, and recall that $q \geqslant 2$. Assuming $F_1^r \setminus \{\mathbf{c}\} = F_j^r \setminus \{\mathbf{c}'\}$ and $j \neq 1$, we use $F_1^r \cap F_j^r = B^r$, and conclude $F_1^r = B^r \cup \{\mathbf{c}\}$. Since $B^r \cap G_1^r = \emptyset$ and $\mathbf{c} \notin G_1^r$, this contradicts the fact that $F_1^r \cap G_1^r \neq \emptyset$ when $r > 3$ and $q \geqslant 2$. Thus, (23) must hold.

In view of the Claim, (23) and (22), we see that the coefficient of $e_{F_1^r \setminus \{\mathbf{c}\}}$ in $\partial(e)$ is equal to $\pm\alpha_{F_1^r}$ hence it is a unit. This contradicts (21).

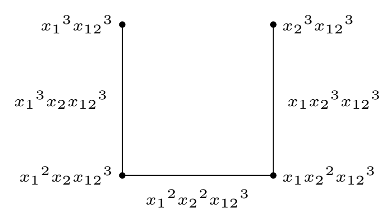
(iii) $\Longrightarrow$ (i): Theorem 5.9 shows that L_q^r supports a free resolution of ${\mathcal{E}_q}^r$ for all $r, q \geqslant 1$. To show minimality, according to Theorem 3.1, it suffices to show that $M_\sigma \neq M_{\sigma'}$ for any faces $\sigma, \sigma' \in \mathsf{L}_q^r$ with $\sigma \neq \sigma'$, or, in other words, that each monomial label appears only once.

If $q = 1$, then $I^r = ({m_1}^r)$ for all r, and all complexes in (17) are one point, so each supports a minimal resolution by default.

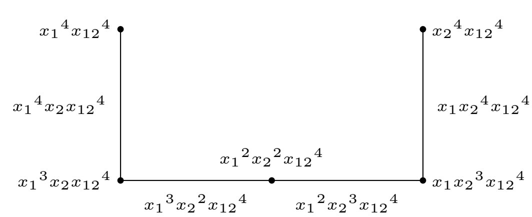
If $q = 2$, then $\mathcal{E}_2 = (x_1x_{12}, x_2x_{12})$, and $\mathsf{L}^2(\mathcal{E}_2)$, $\mathsf{L}^3(\mathcal{E}_2)$, and $\mathsf{L}^4(\mathcal{E}_2)$ are shown below. Observe that each monomial label appears once in each complex, hence $\mathsf{L}^2(\mathcal{E}_2)$, $\mathsf{L}^3(\mathcal{E}_2)$, and $\mathsf{L}^4(\mathcal{E}_2)$ support a minimal free resolution of ${\mathcal{E}_2}^2$, ${\mathcal{E}_2}^3$ and ${\mathcal{E}_2}^4$, respectively.



$\mathsf{L}^2(\mathcal{E}_2)$



$\mathsf{L}^3(\mathcal{E}_2)$



$\mathsf{L}^4(\mathcal{E}_2)$

Now assume $q \geqslant 3$ and $1 \leqslant r \leqslant 2$. We will show that, for every $\mathbf{c} \in \mathcal{N}_q^r$ and $\sigma \in \mathsf{L}_q^r$,

$$\boldsymbol{\epsilon}^{\mathbf{c}} \mid M_\sigma \iff \mathbf{c} \in \sigma. \tag{24}$$

When $r = 1$ observe that for every $i \in [q]$,

$$\epsilon_i \nmid \operatorname{lcm}(\epsilon_{k_1}, \dots, \epsilon_{k_p}) \quad \text{for all} \quad i \in [q], \quad k_1, \dots, k_p \in [q] \setminus \{i\}. \tag{25}$$

This can be seen by noting that the right-hand side of (19) becomes $1 \leqslant 0$ for $A = \{i\}$, $\boldsymbol{\epsilon}^{\mathbf{c}} = \epsilon_i$ and $\sigma = \{\epsilon_{k_1}, \dots, \epsilon_{k_p}\}$. Therefore $\mathsf{L}_q^1 = \mathsf{L}^1(\mathcal{E}_q) = \operatorname{Taylor}(\mathcal{E}_q)$, and (24) follows immediately.

Now let $r = 2$, $q \geqslant 3$, $\sigma \in \mathsf{L}_q^2$ and $\mathbf{c} \in \mathcal{N}_q^2$, with $\boldsymbol{\epsilon}^{\mathbf{c}} \mid M_\sigma$. Since $r = 2$, $\boldsymbol{\epsilon}^{\mathbf{c}} = \epsilon_i\epsilon_j$ for some $i, j \in [q]$. Pick $A = \{i, j\}$ in the right-hand side of (19).

- If $i = j$, then $\boldsymbol{\epsilon}^{\mathbf{c}} = {\epsilon_i}^2$ and $A = \{i\}$. By (19) we must have $\mathbf{c} \in \sigma$.
- If $i \neq j$, then by (19) there exists $\mathbf{b} \in \sigma$ with

$$\boldsymbol{\epsilon}^{\mathbf{b}} = \epsilon_i\epsilon_j, \ {\epsilon_i}^2 \text{ or } {\epsilon_j}^2.$$

  If $\boldsymbol{\epsilon}^{\mathbf{b}} = \epsilon_i\epsilon_j$ then $\mathbf{c} = \mathbf{b} \in \sigma$, as desired. Suppose $\boldsymbol{\epsilon}^{\mathbf{c}} \notin \sigma$, so without loss of generality $\boldsymbol{\epsilon}^{\mathbf{b}} = {\epsilon_i}^2$. As $\sigma \in \mathsf{L}_q^2$, we must have $\sigma \subseteq G_i^2$. So there exist $k_1, \dots, k_p$ in $[q] \setminus \{i, j\}$ such that

$$M_\sigma = \operatorname{lcm}({\epsilon_i}^2, \epsilon_i\epsilon_{k_1}, \dots, \epsilon_i\epsilon_{k_p}) = \epsilon_i \operatorname{lcm}(\epsilon_i, \epsilon_{k_1}, \dots, \epsilon_{k_p}).$$

Since $\epsilon_i\epsilon_j \mid M_\sigma$, it follows that $\epsilon_j \mid \operatorname{lcm}(\epsilon_i, \epsilon_{k_1}, \dots, \epsilon_{k_p})$, which contradicts (25). Thus $\mathbf{c} \in \sigma$.

We have now proved the statement in (24), and by Theorem 3.1 we conclude that L_q^2 supports a minimal free resolution of ${\mathcal{E}_q}^2$ for every $q \geqslant 3$.

A direct consequence of our work in Proposition 7.10 is the following statement.

Proposition 7.11 (**When L_q^r supports a minimal resolution of some I^r**). *If r and q are positive integers, then L_q^r supports a minimal free resolution of I^r for some ideal I minimally generated by q square-free monomials if and only if one of the following holds*

- $q = 1$ *and* $r \geqslant 1$*;*
- $q = 2$ *and* $1 \leqslant r \leqslant 4$*;*
- $q \geqslant 3$ *and* $1 \leqslant r \leqslant 2$*.*

Proof. For any square-free monomial ideal I with q minimal generators the Betti numbers of I^r are bounded above by the Betti numbers of ${\mathcal{E}_q}^r$ by Theorem 7.9. So the question is reduced to when $\mathsf{L}_q^r = \mathsf{L}^r(\mathcal{E}_q)$ supports a minimal free resolution of ${\mathcal{E}_q}^r$. The rest follows from Proposition 7.10.

When $r = 2$, the fact that these bounds are sharp had been previously announced in [3]. The search for sharp(er) bounds when $r > 2$ is continued in [7].

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