ALGEBRAIC COMBINATORICS

Yue Ren

On intersections and stable intersections of tropical hypersurfaces

Volume 7, issue 1 (2024), p. 9-15.
https://doi.org/10.5802/alco.327

Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

Algebraic Combinatorics
Volume 7, issue 1 (2024), p. 9–15
https://doi.org/10.5802/alco.327

On intersections and stable intersections of tropical hypersurfaces

Yue Ren

Abstract We prove that every connected component of an intersection of tropical hypersurfaces contains a point of their stable intersection unless that stable intersection is empty. This is done by studying algebraic hypersurfaces that tropicalize to them and the tropicalization of their intersection.

1. Introduction

Tropical varieties are commonly described as combinatorial shadows of algebraic varieties. The tropicalization of an algebraic variety shares many common properties with its algebraic counterpart, such as its dimension [8, Structure Theorem 3.3.5]. This has prominently been used to show finiteness of central configurations in four and five body problems of celestial mechanics [5, 3], as well as central configurations with restrictions on their geometry rather than the number of bodies [4, 2]. In these proofs, the authors rely on the fact that their central configurations satisfy certain algebraic equations and thus lie on an algebraic variety. They then show that the tropicalization of the algebraic variety is zero-dimensional by intersecting tropical hypersurfaces and eliminating all resulting positive-dimensional polyhedra. Intersections of tropical hypersurfaces are also used in the study of Lagrangian rational homology spheres in Calabi-Yau threefolds [10].

Computing an intersection of tropical hypersurfaces can however be an incredibly challenging task. Tropical hypersurfaces may have many maximal polyhedra and the computation requires intersecting all combinations thereof. While parallelisation and a clever choice of intersection order can lead to significant improvements in performance [6], there is no definite way to avoid the exponential blowup in the number of intersections required. This circumstance is especially unsatisfying when one expects the final intersection to be very small, such as in the aforementioned applications.

One alternative would be to compute the intersection "bottom-up": if one can identify at least one point in every connected component of the intersection, then the remaining points can be obtained using a traversal as in the computation of

Manuscript received 8th April 2023, revised and accepted 31st July 2023.

Keywords. tropical geometry, tropical hypersurfaces, stable intersections.

Acknowledgements. The work was partially done during Collaborate@ICERM "*Numerical Algebraic Geometry and Tropical Geometry*" with Tianran Chen (Auburn University), Paul Helminck (Durham University), Anders Jensen (Aarhus University), Anton Leykin (Georgia Tech) and Josephine Yu (Georgia Tech). The author would like to thank them for helpful discussions, and the institute for its hospitality. The author is supported by UK Research and Innovation under the Future Leaders Fellowship programme (MR/S034463/2).

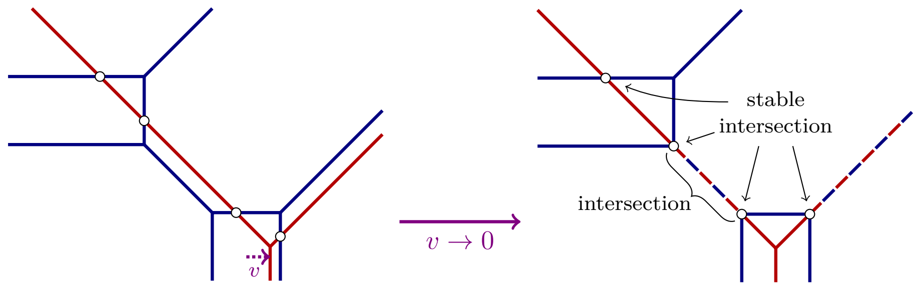

Figure 1. The intersection and stable intersection of two tropical plane curves. Notice how every connected component of the intersection contains a point in the stable intersection.

tropical varieties [1, 11]. When the number of hypersurfaces does not exceed the ambient dimension, a natural candidate for such a set of starting points is their stable intersection. The stable intersection can be computed quickly using techniques such as tropical homotopy continuation [7]. And, due to the similarity in their definitions, it is not unreasonable to expect every connected component of the intersection to contain a point of the stable intersection, see Figure 1. However, until now there has been no known proof of that statement. This paper aims to close that gap:

Theorem 3.3. *Let $\Sigma_1, \dots \Sigma_k$ be tropical hypersurfaces in $\mathbb{R}^n$ with a non-empty stable intersection $\bigwedge_{i=1}^k \Sigma_i \neq \varnothing$. Then, every connected component of their intersection $\bigcap_{i=1}^k |\Sigma_i|$ contains a point in the support of their stable intersection $|\bigwedge_{i=1}^k \Sigma_i|$.*

Despite the obvious combinatorial nature of the statement, our proof relies on an algebro-geometric result by Josephine Yu on generic polynomials generating prime ideals [12], and the structural properties of tropicalizations of irreducible varieties.

2. Background

In this article, we closely follow the notation of [8]. In particular:

Notation 2.1. For the remainder of the paper, we will fix an algebraically closed field K of characteristic 0 with non-trivial valuation $\mathrm{val}\colon K^* \to \mathbb{R}$. Let $K[x^{\pm 1}] \coloneqq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a multivariate (Laurent) polynomial ring thereover.

For the sake of brevity, we will abbreviate "pure, weighted, balanced polyhedral complex" by "balanced polyhedral complex", and we will consider tropical hypersurfaces as balanced polyhedral complexes instead of supports thereof. Additionally, we will denote the stable intersection by "$\wedge$" for better inline formatting.

We will further assume some familiarity with the basic concepts in Sections 2 and 3 of [8], such as the duality between tropical hypersurfaces and regular subdivisions of Newton polytopes, tropicalizations of algebraic varieties, stable intersections of balanced polyhedral complexes, and mixed volumes of polytopes. In particular, we will be using the following results:

Theorem 2.2 ([8, Theorem 3.3.5] Structure Theorem for Tropical Varieties). *Let X be an irreducible variety in $(K^*)^n$ of dimension d. Then, $\mathrm{trop}(X)$ is the support of a balanced polyhedral complex in $\mathbb{R}^n$ of dimension d that is connected in codimension 1.*

Theorem 2.3 ([8, Theorem 3.6.1]). *Let Σ_1, Σ_2 be two balanced polyhedral complexes in $\mathbb{R}^n$ whose support is the tropicalization of two varieties $X_1, X_2 \subseteq (K^*)^n$. Then,*

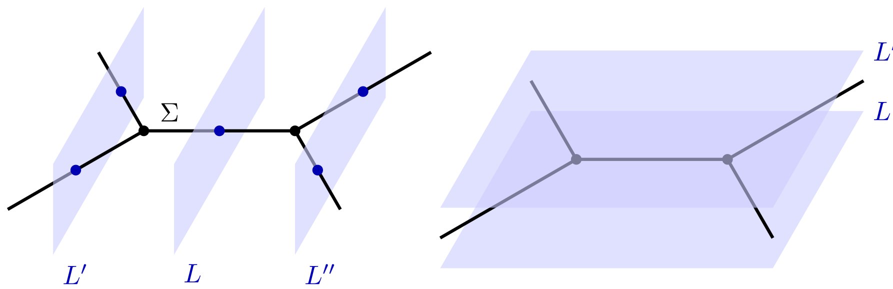

Figure 2. A fixed balanced polyhedral complex Σ with different translates of an affine subspace. Notice how the stable intersection is either always non-empty (left) or empty (right), but not both.

there is a Zariski dense subset $U \subseteq (K^*)^n$ consisting of elements with componentwise valuation 0 such that

$$|\Sigma_1 \wedge \Sigma_2| = \operatorname{trop}(X_1 \cap z \cdot X_2) \qquad \text{for all } z \in U.$$

Theorem 2.4 ([8, Theorem 3.6.10]). *Let Σ_1, Σ_2 be two balanced polyhedral complexes in $\mathbb{R}^n$ of codimension d and e, respectively. If the stable intersection $\Sigma_1 \wedge \Sigma_2$ is non-empty, then it is a balanced polyhedral complex of codimension $d + e$.*

Additionally, we will require the following theorem by Yu. Whilst [12, Theorem 3] is more general and also covers fields of positive characteristic, we will only need and state its specialisation to fields of characteristic 0. In the theorem, "general polynomials" means polynomials whose coefficients lie in a fixed Zariski open, dense set of the coefficient space.

Theorem 2.5 ([12, Theorem 3]). *Let $A_1, \dots, A_k \subseteq \mathbb{Z}^n$, $k \leqslant n$, and $0 \in A_i$ for all $i = 1, \dots, k$. General polynomials in $K[x]$ with monomial supports $A_1, \dots, A_k$ generate a proper ideal whose radical is prime if and only if for every $\varnothing \neq J \subseteq [k]$ one of the following holds:*

(1) $\dim \operatorname{Span} \bigcup_{j \in J} A_j > |J|$, *or*
(2) $\dim \operatorname{Span} \bigcup_{j \in J} A_j = |J|$ *and* $\operatorname{MV}(\operatorname{Conv}(A_j) \mid j \in J) = 1$.

3. Main Theorem

In this section, we prove Theorem 3.3, beginning with two lemmas. In the first lemma, we regard affine subspaces as balanced polyhedral complexes consisting of a single polyhedron. We show that if an affine subspace has an empty stable intersection with a balanced polyhedral complex, then so do its translates, see Figure 2.

Lemma 3.1. *Let Σ be a balanced polyhedral complex in $\mathbb{R}^n$. Let L, L' be two parallel affine subspaces in $\mathbb{R}^n$, i.e., $L = \operatorname{Ker}(A) + v$ and $L' = \operatorname{Ker}(A) + v'$ for some matrix $A \in \mathbb{R}^{k \times n}$ and vectors $v, v' \in \mathbb{R}^n$. Then*

$$L \wedge \Sigma = \varnothing \quad \Longleftrightarrow \quad L' \wedge \Sigma = \varnothing.$$

Proof. Without loss of generality, we may assume that L is a linear space, i.e., $0 \in L$. Note that both L and L' are supports of stable intersections of affine hyperplanes, i.e., $L = |H_1 \wedge \cdots \wedge H_k|$ and $L' = |(H_1 + v') \wedge H_2 \wedge \cdots \wedge H_k|$ for suitable hyperplanes $H_1, \dots, H_k$ and $v' \in \mathbb{R}^n$. Since H_1 and $H_1 + v'$ remain parallel affine subspaces, and $H_2 \wedge \cdots \wedge H_k \wedge \Sigma$ remains a balanced polyhedral complex by Theorem 2.4, we may assume without loss of generality that L is a hyperplane and L' is an affine hyperplane.

Consider the projection of $|\Sigma|$ to the one-dimensional orthogonal complement $L^\perp$. The image of the projection remains the support of a balanced polyhedral complex, which means it is either finite or the entire complement. If the projection is finite, then Σ is contained in a union of affine hyperplanes parallel to L and L', and we have $L \wedge \Sigma = \varnothing = L' \wedge \Sigma$. If the projection is the entire complement, then we have $L \wedge \Sigma \neq \varnothing \neq L' \wedge \Sigma$. □

In the next lemma, the connected component of a balanced polyhedral complex Σ is a maximal polyhedral subset $\Gamma \subseteq \Sigma$ such that $|\Gamma|$ is a connected component of $|\Sigma|$. Note that Γ is naturally a balanced polyhedral complex. We use Lemma 3.1 to prove a similar statement but for connected components of a stable intersection of tropical hypersurfaces instead of translates of an affine subspace.

Lemma 3.2. *Let $\Sigma_1, \ldots, \Sigma_k, \Sigma_{k+1}$ be tropical hypersurfaces in $\mathbb{R}^n$ with $\bigwedge_{i=1}^k \Sigma_i \neq \varnothing$. Then for any two connected components $\Gamma, \Gamma' \subseteq \bigwedge_{i=1}^k \Sigma_i$ we have*

$$\Gamma \wedge \Sigma_{k+1} = \varnothing \quad \iff \quad \Gamma' \wedge \Sigma_{k+1} = \varnothing.$$

Proof. Let $A_1, \ldots, A_k \subseteq \mathbb{Z}^n$ be monomial supports of polynomials with tropical hypersurfaces $\Sigma_1, \ldots, \Sigma_k$ and $0 \in A_i$ for all i. We distinguish between two cases:

Case 1: If $A_1, \ldots, A_k$ do not satisfy Conditions (1) or (2) of Theorem 2.5, then there is a subset $J \subseteq [k]$ such that $\dim \operatorname{Span}(\bigcup_{j\in J} A_j) \leqslant |J|$. Note that the lineality space of $\bigwedge_{j\in J} \Sigma_j$ contains the orthogonal complement of $\operatorname{Span}(\bigcup_{j\in J} A_j)$. Therefore, the lineality space of $\bigwedge_{j\in J} \Sigma_j$ has codimension at most $|J|$. And since $\bigwedge_{j\in J} \Sigma_j \supseteq \bigwedge_{i=1}^k \Sigma_i \neq \varnothing$, Theorem 2.4 implies that $\bigwedge_{j\in J} \Sigma_j$ has codimension exactly $|J|$. Hence, $|\bigwedge_{j\in J} \Sigma_j| = L_1 \cup \cdots \cup L_r$, where $L_1, \ldots, L_r$ are affine subspaces parallel to its lineality space. The claim follows by applying Lemma 3.1 to $\Sigma \coloneqq \bigwedge_{i\in[k+1]\smallsetminus J} \Sigma_i$ and the affine subspaces $L_1, \ldots, L_r$.

Case 2: If $A_1, \ldots, A_k$ satisfy Conditions (1) or (2) of Theorem 2.5, consider

- $K \cdot x^A \coloneqq \bigoplus_{i=1}^k \bigoplus_{\alpha\in A_i} K \cdot x^\alpha$, the vector space of polynomial tuples $(f_1, \ldots, f_k)$ where f_i has monomial support A_i, and
- $K^{|A|} \coloneqq K^{|A_1|+\cdots+|A_k|}$, their coefficient space.

In particular, any choice of coefficients $c = (c_{i,\alpha})_{i\in[k],\alpha\in A_i} \in K^{|A|}$ defines a tuple of polynomials $f(c) \coloneqq (f_i(c))_{i\in[k]} \in K \cdot x^A$ with $f_i(c) \coloneqq \sum_{\alpha\in A_i} c_{i,\alpha} x^\alpha$ and vice versa. By Theorem 2.5, there is a Zariski open, dense subset $U \subseteq K^{|A|}$ such that any $c \in U$ yields an $f(c)$ whose entries generate an ideal whose radical is prime. By Theorem 2.2, the tropicalization $\operatorname{trop}(V(f(c))) \coloneqq \operatorname{trop}(V(\langle f_1(c), \ldots, f_k(c)\rangle))$ will be connected for any $c \in U$. We will now show that there is a $c_0 \in U$ for which $\operatorname{trop}(V(f(c_0))) = |\bigwedge_{i=1}^k \Sigma_i|$, so that the claim holds trivially as there is only one connected component.

Pick any $c \in U$. By Theorem 2.3, combined with a simple induction on k, there is a Zariski dense $S \subseteq ((K^*)^n)^{k-1}$ with $\operatorname{trop}(V(f_1(c)) \cap z_2 V(f_2(c)) \cap \cdots \cap z_k V(f_k(c))) = |\bigwedge_{i=1}^k \Sigma_i|$ for all $(z_2, \ldots, z_k) \in S$. Then for any $z = (z_2, \ldots, z_k) \in ((K^*)^n)^{k-1}$, let $z^{-1} \cdot c \in K^{|A|}$ denote the coefficients of the polynomials $f_1(z^{-1} \cdot c) = \sum_{\alpha\in A_1} c_{1,\alpha} x^\alpha$ and $f_i(z^{-1} \cdot c) = \sum_{\alpha\in A_i} c_{i,\alpha} z_i^{-\alpha} x^\alpha$ for $i > 1$, so that $V(f_1(z^{-1}c)) = V(f_1(c))$ and $V(f_i(z^{-1}c)) = z_i V(f_i(c))$ for $i > 1$. Consider the rational map

$$\varphi\colon \quad ((K^*)^n)^{k-1} \to K^{|A|}, \qquad z \mapsto z^{-1} \cdot c.$$

The preimage $\varphi^{-1}(U)$ is a Zariski open set, and, since $\varphi(1, \ldots, 1) = c \in U$, $\varphi^{-1}(U)$ is non-empty. Thus $\varphi^{-1}(U)$ has to intersect the Zariski dense set S, and any point in the image of the intersection gives us the desired $c_0 \in U$. □

Using Lemma 3.2 we can now prove:

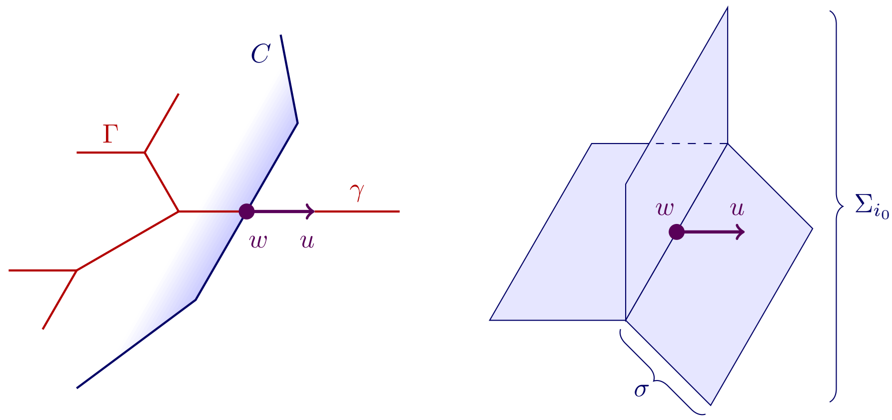

Figure 3. Illustrations for the proof of Theorem 3.3. C is a connected component of $\bigcap_{i=1}^{k}|\Sigma_i|$, Γ is a connected component of $\bigwedge_{j\in J}\Sigma_j$ assumed to be not contained in C, and Σ_{i_0} is a tropical hypersurface that contributes to Γ not being contained in C.

Theorem 3.3. *Let $\Sigma_1, \dots \Sigma_k$ be tropical hypersurfaces in $\mathbb{R}^n$ with a non-empty stable intersection $\bigwedge_{i=1}^{k}\Sigma_i \neq \varnothing$. Then every connected component of their intersection $\bigcap_{i=1}^{k}|\Sigma_i|$ contains a point in the support of their stable intersection $|\bigwedge_{i=1}^{k}\Sigma_i|$.*

Proof. Let C be a connected component of $\bigcap_{i=1}^{k}|\Sigma_i|$ and assume that C contains no point of the stable intersection $\bigwedge_{i=1}^{k}\Sigma_i$. Let $J \subsetneq [k]$ be a maximal subset such that C contains a point of the stable intersection $\bigwedge_{j\in J}\Sigma_j$. Let $\Gamma \subseteq |\bigwedge_{j\in J}\Sigma_j|$ be the connected component containing said point. Note that, since $\bigwedge_{i=1}^{k}\Sigma_i$ is non-empty, we have $k \leqslant n$ by Theorem 2.4. And because $J \subsetneq [k]$, Γ has to be positive dimensional again by Theorem 2.4.

If Γ is completely contained in C, then $\Gamma \wedge \Sigma_i = \varnothing$ for any $i \notin J$ due to the maximality of J. Lemma 3.2 then implies $\bigwedge_{i=1}^{k}\Sigma_i = \varnothing$, contradicting the assumptions.

If Γ is not completely contained in C, then there is a point $w \in C \cap \Gamma$ and a direction $u \in \mathbb{R}^n$ such that $w + \varepsilon \cdot u \in \Gamma$ but $w + \varepsilon \cdot u \notin C$ for $\varepsilon > 0$ sufficiently small, see Figure 3 left. Since $w + \varepsilon \cdot u \notin C$, we also have $w + \varepsilon \cdot u \notin \bigcap_{i=1}^{k}|\Sigma_i|$. But because $w + \varepsilon \cdot u \in \bigcap_{j\in J}|\Sigma_j|$, there must be an $i_0 \notin J$ such that $w + \varepsilon \cdot u \notin |\Sigma_{i_0}|$. Let $\gamma \in \Gamma$ be a polyhedron containing $w + \varepsilon \cdot u$. Let $\sigma \in \Sigma_{i_0}$ be a maximal polyhedron containing w and on the boundary of the region of $\mathbb{R}^n \smallsetminus |\Sigma_{i_0}|$ containing $w + \varepsilon \cdot u$, see Figure 3 right. Then $\dim(\gamma + \sigma) = n$ and hence $w \in \gamma \cap \sigma \in \Gamma \wedge \Sigma_{i_0}$ by the definition of stable intersection. As $w \in C$, this contradicts that $J \subseteq [n]$ is maximal. $\square$

4. Open questions

Our approach to prove Theorem 3.3 relies on the fact that each Σ_i is the tropicalization of an algebraic variety and on dimension arguments to show (non-)emptiness. Consequently, it has two key limitations:

First, it cannot deal with balanced polyhedral complexes which do not arise as tropicalizations of algebraic varieties. Such balanced polyhedral complexes are known to exist. In fact, balanced polyhedral complexes that are tropicalizations of algebraic varieties exhibit particularly nice properties such as higher connectivity [9]. Hence it is not clear whether Theorem 3.3 generalises to arbitrary balanced polyhedral complexes:

Question 4.1. *Let $\Sigma_1, \dots, \Sigma_k$ be balanced polyhedral complexes in $\mathbb{R}^n$ with a non-empty stable intersection $\bigwedge_{i=1}^k \Sigma_i \neq \varnothing$. Does every connected component of their intersection $\bigcap_{i=1}^n |\Sigma_i|$ contain a point of their stable intersection $|\bigwedge_{i=1}^k \Sigma_i|$?*

Second, it cannot predict where the stable intersection points lie. Recall that the original motivation was to find a way to identify a point on each connected component of an intersection of tropical hypersurfaces. In case the number of tropical hypersurfaces in $\mathbb{R}^n$ exceeds n, their stable intersection is empty by Theorem 2.4. Hence their stable intersection does not provide an easy way to construct such points. A natural alternative are the stable intersections of any n tropical hypersurfaces, see Figure 4. However, since our techniques do not give any information on where those stable intersection points lie, it cannot provide an answer to the following question:

Question 4.2. *Let $\Sigma_1, \dots, \Sigma_k$ be tropical hypersurfaces in $\mathbb{R}^n$ with $k > n$ and non-empty stable intersections $\bigwedge_{j \in J} \Sigma_j \neq \varnothing$ for all $J \in \binom{[k]}{n}$. Does every connected component of their intersection $\bigcap_{i=1}^k |\Sigma_i|$ contain a point of a stable intersection $|\bigwedge_{j \in J} \Sigma_j|$ for some $J \in \binom{[k]}{n}$?*

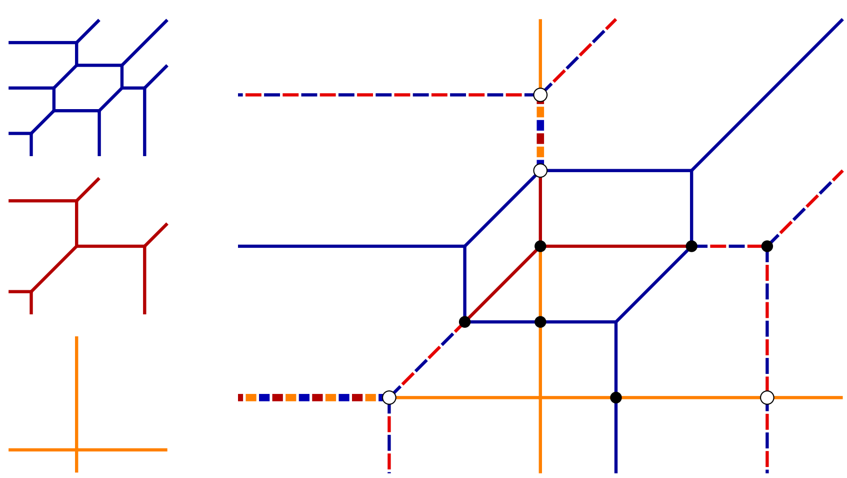

Figure 4. Three tropical plane curves and the stable intersection points of any two curves thereof. Stable intersection points which lie in the intersection of all three curves are highlighted in white.

References

[1] T. Bogart, A. N. Jensen, D. Speyer, B. Sturmfels, and R. R. Thomas, *Computing tropical varieties*, J. Symbolic Comput. **42** (2007), no. 1-2, 54–73, `https://doi.org/10.1016/j.jsc.2006.02.004`.

[2] Yiyang Deng and Marshall Hampton, *Equilateral chains and cyclic central configurations of the planar five-body problem*, J. Nonlinear Sci. **33** (2023), no. 1, article no. 4 (18 pages), `https://doi.org/10.1007/s00332-022-09864-z`.

[3] Marshall Hampton and Anders Jensen, *Finiteness of spatial central configurations in the five-body problem*, Celestial Mech. Dynam. Astronom. **109** (2011), no. 4, 321–332, `https://doi.org/10.1007/s10569-010-9328-9`.

[4] Marshall Hampton and Anders Nedergaard Jensen, *Finiteness of relative equilibria in the planar generalized N-body problem with fixed subconfigurations*, J. Geom. Mech. **7** (2015), no. 1, 35–42, `https://doi.org/10.3934/jgm.2015.7.35`.

[5] Marshall Hampton and Richard Moeckel, *Finiteness of relative equilibria of the four-body problem*, Invent. Math. **163** (2006), no. 2, 289–312, `https://doi.org/10.1007/s00222-005-0461-0`.

[6] Anders Jensen, Jeff Sommars, and Jan Verschelde, *Computing tropical prevarieties in parallel*, PASCO 2017—International Workshop on Parallel Symbolic Computation, ACM, New York, 2017, https://doi.org/10.1145/3115936.3115945.
[7] Anders Nedergaard Jensen, *Tropical homotopy continuation*, 2016, https://arxiv.org/abs/1601.02818.
[8] Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015, https://doi.org/10.1090/gsm/161.
[9] Diane Maclagan and Josephine Yu, *Higher connectivity of tropicalizations*, Math. Ann. **384** (2022), no. 1-2, 775–788, https://doi.org/10.1007/s00208-021-02281-9.
[10] Cheuk Yu Mak and Helge Ruddat, *Tropically constructed Lagrangians in mirror quintic threefolds*, Forum Math. Sigma **8** (2020), article no. e58 (55 pages), https://doi.org/10.1017/fms.2020.54.
[11] Thomas Markwig and Yue Ren, *Computing tropical varieties over fields with valuation*, Found. Comput. Math. **20** (2020), no. 4, 783–800, https://doi.org/10.1007/s10208-019-09430-2.
[12] Josephine Yu, *Do most polynomials generate a prime ideal?*, J. Algebra **459** (2016), 468–474, https://doi.org/10.1016/j.jalgebra.2016.03.050.

Yue Ren, Department of Mathematics, Durham University, Upper Mountjoy Campus, Durham DH1 3LE, United Kingdom
E-mail : yue.ren2@durham.ac.uk
Url : https://yueren.de